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Massera Criterion for Linear Functional Equations in a Framework of Hyperfunctions

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Abstract. We introduce a sheaf of bounded hyperfunctions at infinity in one variable, and consider periodic linear functional equations. We show that a Massera type theorem holds in this framework. Moreover we consider its vector-valued variants and give applications to partial differential equations periodic with respect to one variable.

1. Introduction

In [13], Massera studied the existence of periodic solutions to periodic ordinary differential equations in several situations. In the linear setting, the following result was given ([13, Theorem 4]).

THEOREM 1.1. *Consider a system of ordinary differential equations*

$$\frac{d}{dt}x = a(t)x + b(t),$$

where $a(t)$ and $b(t)$ are $\mathbb{R}^{m\times m}$-valued and $\mathbb{R}^m$-valued continuous functions. Assume that $a(t)$ and $b(t)$ are 1-periodic. Then the existence of a solution which is bounded in the future implies the existence of a 1-periodic solution.

This criterion "the existence of a solution bounded in the future" for the existence of a periodic solution is called the Massera criterion, and has been studied by many authors for a variety of equations.

Concerning to linear equations, we refer to Chow-Hale [2] for functional differential equations with retarded type, to Hino-Murakami [5] and [4] for equations with infinite delay, to Li-Lin-Li [11] for equations with advance and delay, and to Li-Cong-Lin-Liu [10] for evolution equations. Refer also to

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Shin-Naito [18] for the abstract functional differential equations. Zubelevich [19] gave a similar result for discrete dynamical systems in locally convex spaces with the Montel property, as well as for those in reflexive Banach spaces, with some interesting applications.

In this article, we are interested in a similar property for linear functional equations in a framework of hyperfunctions.

The notion of hyperfunction was introduced by Sato [15] and [16], and studied by many researchers, mainly in the context of linear ordinary and partial differential equations. When we study a Massera type criterion in hyperfunctions, the main obstacle seems to come from the lack of a notion of boundedness for usual hyperfunctions.

A notion of boundedness for hyperfunctions was first introduced by Chung-Kim-Lee [3]. They defined, using duality, the spaces of hyperfunctions with L^p growth $(1 < p \leq \infty)$ in several variables, which can be embedded into the space of globally defined Fourier hyperfunctions, and they studied bounded hyperfunctions using the heat kernel method. On the other hand, to study a Massera type criterion, we need a notion of boundedness for hyperfunctions in a neighborhood of infinity in terms of defining functions (in the sense of Sato [15]), which we can apply to ordinary differential operators directly. Therefore, we introduce here the sheaf of (univariate) bounded hyperfunctions at infinity, in a similar manner to the original cohomological definitions of hyperfunctions and Fourier hyperfunctions given in the one-dimensional case in Sato [14].

Though our sheaf is defined only in the one-dimensional case, we also consider its vector-valued variants applicable to partial differential equations. We study the properties of our bounded hyperfunctions, define some classes of differential operators and of functional differential operators which act on them, and give our main result, that is, a Massera type theorem (Theorem 4.3) in a framework of hyperfunctions. Since Theorem 4.3 has somewhat an abstract nature, we also give some applications to partial differential equations.

The plan of this paper is as follows. We define the sheaf of bounded hyperfunctions at infinity and study the properties and the relation with Chung-Kim-Lee's space of bounded hyperfunctions in section 2. We also define its vector-valued variants in a parallel manner. Then we define some classes of operators acting on bounded hyperfunctions in section 3. Nonlocal operators such as difference operators are also considered as well as local operators. The notion of periodicity for hyperfunctions and bounded hyperfunctions is also studied in this section, since we define the periodicity using difference equations. In section 4, after recalling the notion of the Montel property for locally convex spaces, we give our main result. In the final section 5, we give applications to partial differential equations periodic with respect to one of the variables. There we study solutions in the space of hyperfunctions with real analytic parameters and those with holomorphic parameters.

2. Bounded Hyperfunctions at Infinity

In this section, we define the sheaf of bounded hyperfunctions at infinity. We give its canonical embedding into the sheaf of Fourier hyperfunctions and show that the global sections coincide with Chung-Kim-Lee's bounded hyperfunctions. As was already mentioned, we follow the argument in Sato [14]. Refer also to Sato [15], Kawai [8], Sato-Kawai-Kashiwara [17], and Kaneko [7], for hyperfunctions, Fourier hyperfunctions, and related topics. For the vector-valued hyperfunctions, Ion-Kawai [6] made a comprehensive study. But here we use an elementary method valid only in one-dimensional case, since it is unclear to the author, at this moment, if their approach is applicable to our situation under the boundedness condition at infinity.

2.1. Sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions

At first, we briefly recall the notion of Fourier hyperfunctions in the one-dimensional case. Let $\mathbb{D}^1 = [-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$ be a compactification of $\mathbb{R}$, and we identify $\mathbb{C}$ with the subset $\mathbb{R} + i\mathbb{R}$ in $\mathbb{D}^1 + i\mathbb{R}$. We take a coordinate t for $\mathbb{R}$ or $\mathbb{D}^1$ and w for $\mathbb{C}$. We denote by $\tilde{\mathscr{O}}$ the sheaf of slowly increasing holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$, where $\tilde{\mathscr{O}}(U)$ on an open set $U \subset \mathbb{D}^1 + i\mathbb{R}$ is given by

$$\{f \in \mathscr{O}(U \cap \mathbb{C});\ \text{for any compact subset } K \subset U \text{ and for any } \varepsilon > 0, \\ \sup_{w \in K \cap \mathbb{C}} |f(w)| e^{-\varepsilon|\mathrm{Re}\, w|} < +\infty\}.$$

The sheaf $\mathscr{Q}$ of Fourier hyperfunctions on $\mathbb{D}^1$ is defined by $\mathscr{Q} := \mathscr{H}^1_{\mathbb{D}^1}(\tilde{\mathscr{O}})$. We list up several properties of $\mathscr{Q}$ and introduce some notations.

(i) $\mathcal{Q}$ becomes a flabby sheaf on $\mathbb{D}^1$, and the restriction of $\mathcal{Q}$ on $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}$ of hyperfunctions. Therefore the canonical map $\mathcal{Q}(\mathbb{D}^1) \to \mathcal{B}(\mathbb{R})$ is surjective, but it is not injective.

(ii) The sections are given by

$$\mathcal{Q}(\Omega) = \varinjlim_{U} \frac{\tilde{\mathcal{O}}(U \setminus \Omega)}{\tilde{\mathcal{O}}(U)},$$

where U runs through open sets in $\mathbb{D}^1 + i\mathbb{R}$ including Ω as a closed subset. We call such U a complex neighborhood of Ω. Moreover the Fourier hyperfunction represented by a function $f \in \tilde{\mathcal{O}}(U \setminus \Omega)$ is denoted by $[f]$ or $f(t+i0) - f(t-i0)$ and f is called a defining function of $[f]$. Note that the notion of "defining function of a Fourier hyperfunction" used in [3] is different from that in this paper.

(iii) A function $f \in L^1_{\mathrm{loc}}(\Omega \cap \mathbb{R})$ satisfying the estimate

$$\forall \varepsilon > 0, \forall K \Subset \Omega, \operatorname*{ess\,sup}_{t \in K \cap \mathbb{R}} |f(t)| e^{-\varepsilon |t|} < +\infty$$

can be naturally regarded as a section in $\mathcal{Q}(\Omega)$. Especially, $f = 0$ in $L^1_{\mathrm{loc}}(\Omega \cap \mathbb{R})$ if its embedded image in $\mathcal{Q}(\Omega)$ is zero. When the closure of $\operatorname{supp} f$ in Ω is compact, the image of f can be given, for example, by a defining function

$$\tilde{f}(w) := -\frac{1}{2\pi i} \int_{\Omega \cap \mathbb{R}} f(s) \frac{e^{-(w-s)^2}}{w-s} ds \in \tilde{\mathcal{O}}(\mathbb{D}^1 + i\mathbb{R} \setminus \mathbb{D}^1).$$

Note that this is always the case when $\Omega = \mathbb{D}^1$.

Now we introduce the notion of bounded hyperfunction at infinity. We define the sheaf $\mathcal{O}_{L^\infty}$ of bounded holomorphic functions at infinity on $\mathbb{D}^1 + i\mathbb{R}$ by

$$\mathcal{O}_{L^\infty}(U) := \{f \in \mathcal{O}(U \cap \mathbb{C});\ \text{for any compact subset } K \subset U,$$
$$\|f\|_K := \sup_{w \in K \cap \mathbb{C}} |f(w)| < +\infty\},$$

for any open set $U \subset \mathbb{D}^1 + i\mathbb{R}$. It is easily seen that the correspondence $U \mapsto \mathcal{O}_{L^\infty}(U)$ becomes actually a sheaf and the restriction of $\mathcal{O}_{L^\infty}$ on $\mathbb{C}$ is nothing but the sheaf $\mathcal{O}$ of holomorphic functions.

Note that a section $f \in \mathcal{O}_{L^\infty}(U)$, which is a function on $U \cap \mathbb{C}$, is not in general bounded on the whole $U \cap \mathbb{C}$. For example, consider the case when U is a tube domain $\mathbb{D}^1 + i]a, b[$. A function $f \in \mathcal{O}(\mathbb{R} + i]a, b[)$ belongs to $\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i]a, b[)$ if and only if it is bounded on $\mathbb{R} + i]a + \delta, b - \delta[$ for any $\delta > 0$. The space $\mathcal{O}_{L^\infty}(U)$ is endowed with the Fréchet topology by the family of seminorms $\|\cdot\|_K$ $(K \Subset U)$.

We also give the notion of vector-valued bounded holomorphic functions. Throughout this paper, E denotes a sequentially complete separated locally convex space over $\mathbb{C}$. We define the sheaf ${}^E\mathcal{O}_{L^\infty}$ of E-valued bounded holomorphic functions at infinity on $\mathbb{D}^1 + i\mathbb{R}$ by

$$
\begin{aligned}
{}^E\mathcal{O}_{L^\infty}(U) := \{f \in {}^E\mathcal{O}(U \cap \mathbb{C});\ & \text{for any compact subset } K \subset U, \\
& \text{and for any continuous seminorm } p \text{ of } E, \\
& \|f\|_{K,p} := \sup_{w \in K \cap \mathbb{C}} p(f(w)) < +\infty\}.
\end{aligned}
$$

Here we denote by ${}^E\mathcal{O}$ the sheaf of E-valued holomorphic functions on $\mathbb{C}$. We refer to Bochnak-Siciak [1] for the several properties of holomorphic functions taking values in a locally convex space. We endow ${}^E\mathcal{O}_{L^\infty}(U)$ with a locally convex topology by the family of seminorms $\|\cdot\|_{K,p}$.

Now we give,

Definition 2.1 (Sheaf of bounded hyperfunctions at infinity). We define the sheaf $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions at infinity on $\mathbb{D}^1$ as the sheaf associated with the presheaf

$$
\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{\mathcal{O}_{L^\infty}(U \setminus \Omega)}{\mathcal{O}_{L^\infty}(U)}, \tag{2.1}
$$

where U runs through complex neighborhoods of Ω. Similarly we define ${}^E\mathcal{B}_{L^\infty}$ as the sheaf associated with the presheaf

$$
\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{{}^E\mathcal{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathcal{O}_{L^\infty}(U)}. \tag{2.2}
$$

It is an immediate consequence of this definition that the restriction of $\mathcal{B}_{L^\infty}$ on $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}$ of hyperfunctions.

REMARK 2.2. Chung-Kim-Lee [3] used the symbol $\mathcal{B}_{L^\infty}$ for the space of bounded hyperfunctions defined via duality in several variables. We introduced here the similar symbol $\mathscr{B}_{L^\infty}$, not for a space but for a sheaf, which we hope would not cause a confusion. We will show later that their space $\mathcal{B}_{L^\infty}$ in the sense of [3] in one-dimensional case can be identified with the space $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ of the global sections of our sheaf $\mathscr{B}_{L^\infty}$.

As for vector-valued variants, Ion-Kawai [6] introduced the sheaf ${}^E\mathscr{B}$ of E-valued hyperfunctions for a Fréchet space E on real analytic manifolds. When E is Fréchet, the restriction of our sheaf ${}^E\mathscr{B}_{L^\infty}$ on $\mathbb{R}$ coincides with ${}^E\mathscr{B}$ for the one-dimensional euclidean case.

2.2. Fundamental properties of $\mathscr{B}_{L^\infty}$

In the $\mathbb{C}$-valued case, the presheaf given by (2.1) itself becomes a flabby sheaf and a section on Ω is an equivalent class $[f]$ of $f \in \mathscr{O}_{L^\infty}(U \setminus \Omega)$ with a complex neighborhood U of Ω, as we will see later. In the general case, we will show that a section on a compact set can be represented in a similar manner. In the sequel, we use the conventions $B_d :=]-d, d[$ and $\dot{B}_d := B_d \setminus \{0\}$ for $d > 0$.

PROPOSITION 2.3. *Let $\Omega \subset \mathbb{D}^1$ be an open set with a complex neighborhood U. Then the canonical map*

$$\frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)} \to {}^E\mathscr{B}_{L^\infty}(\Omega) \tag{2.3}$$

is always injective.

On the other hand, let K be a compact set in $\mathbb{D}^1$. Then we have

$$ {}^E\mathscr{B}_{L^\infty}(K) = \varinjlim_{\Omega, d>0} \frac{{}^E\mathscr{O}_{L^\infty}(\Omega + i\dot{B}_d)}{{}^E\mathscr{O}_{L^\infty}(\Omega + iB_d)}, \tag{2.4}$$

$$\Gamma_K(\mathbb{D}^1, {}^E\mathscr{B}_{L^\infty}) = \varinjlim_{d>0} \frac{{}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + iB_d \setminus K)}{{}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + iB_d)}, \tag{2.5}$$

where Ω in the inductive limit in (2.4) *runs through open neighborhoods of K in $\mathbb{D}^1$.*

PROOF. Let $\mathscr{F}$ denote the presheaf (2.2), that is, $\mathscr{F}(\Omega) := \varinjlim_U \frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)}$. Then, we can easily see the following two properties:

(i) the map $\frac{{}^E\mathcal{O}_{L^\infty}(U\setminus\Omega)}{{}^E\mathcal{O}_{L^\infty}(U)} \to \mathcal{F}(\Omega)$ is always injective.

(ii) for any open covering $\Omega = \bigcup_\lambda \Omega_\lambda$ and for any $u \in \mathcal{F}(\Omega)$,

$$\forall\lambda, u|_{\Omega_\lambda} = 0 \text{ implies } u = 0. \tag{2.6}$$

The property (ii) implies the injectivity of $\mathcal{F}(\Omega) \to {}^E\mathcal{B}_{L^\infty}(\Omega)$, which, together with (i), proves the injectivity of (2.3). Moreover, from the property (ii), we can see that a section u of ${}^E\mathcal{B}_{L^\infty}$ on an open set Ω is given by a family $(u_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda}\mathcal{F}(\Omega_\lambda)$ with respect to an open covering $\Omega = \bigcup_\lambda \Omega_\lambda$, which satisfies the compatibility condition

$$\forall\lambda, \forall\mu \in \Lambda, u_\lambda|_{\Omega_\lambda\cap\Omega_\mu} - u_\mu|_{\Omega_\lambda\cap\Omega_\mu} = 0 \text{ in } \mathcal{F}(\Omega_\lambda\cap\Omega_\mu).$$

Thus, by a standard argument, the equality (2.4) can be reduced to the following

LEMMA 2.4. *Consider two sections $f_j \in \mathcal{F}(\Omega_j)$ on connected open sets $\Omega_j \subset \mathbb{D}^1$, $(j = 1, 2)$. Assume that $f_1|_{\Omega_1\cap\Omega_2} = f_2|_{\Omega_1\cap\Omega_2}$ in $\mathcal{F}(\Omega_1\cap\Omega_2)$. Then there exists $h \in \mathcal{F}(\Omega)$, $(\Omega := \Omega_1 \cup \Omega_2)$, such that $h|_{\Omega_j} = f_j$, $(j = 1, 2)$.*

PROOF. The conclusion is trivial if Ω coincides with Ω_1 or Ω_2. Thus we may assume from the beginning that there exists $a \in \Omega_1 \cap \Omega_2 \cap \mathbb{R}$ such that

$$(\Omega_1 \cap [-\infty, a]) \cup (\Omega_2 \cap [a, +\infty]) = \Omega_1 \cup \Omega_2.$$

Moreover since $\mathcal{F}$ satisfies (2.6), it suffices to construct a section $h \in \mathcal{F}(\Omega)$ such that $h = f_1$ on $\Omega_1 \cap [-\infty, a]$ and $h = f_2$ on $\Omega_2 \cap [a, +\infty]$.

We take defining functions $\tilde{f}_j \in {}^E\mathcal{O}_{L^\infty}(U_j\setminus\Omega_j)$ of f_j $(j = 1, 2)$ with some complex neighborhoods U_j of Ω_j, satisfying $U_j \setminus \Omega_j = U_j \setminus \mathbb{D}^1$. Then, from the assumption, we can take $d > 0$ such that $\{w \in \mathbb{C}; |w-a| < 2d\} \subset U_1\cap U_2$ and that $\tilde{f}_1 - \tilde{f}_2$ extends to a section $g \in {}^E\mathcal{O}_{L^\infty}(\{w \in \mathbb{C}; |w - a| < 2d\})$. We define two functions $g_1(w)$ on $\operatorname{Re} w < a$, $|\operatorname{Im} w| < d$ and $g_2(w)$ on $\operatorname{Re} w > a$, $|\operatorname{Im} w| < d$ by

$$g_j(w) = \frac{1}{2\pi i}\int_{a-id}^{a+id} \frac{g(s)}{s-w}ds,$$

where the path of integration is the line segment with the endpoints $a \pm id$. Then by a contour deformation, the function g_1 extends holomorphically from $\{\mathrm{Re}\, w < a, |\mathrm{Im}\, w| < d\}$ to $\{\mathrm{Re}\, w < a + d, |\mathrm{Im}\, w| < d\}$ and defines a section $g_1 \in {}^E\mathcal{O}_{L^\infty}(V_1)$, $(V_1 := [-\infty, a + d[+ iB_d)$. Similarly g_2 extends and defines a section $g_2 \in {}^E\mathcal{O}_{L^\infty}(V_2)$, $(V_2 :=]a - d, +\infty] + iB_d)$. Moreover we have

$$g_1(w) - g_2(w) = g(w) \text{ on } V_1 \cap V_2.$$

Now we consider two sections $\tilde{f}_j - g_j$ of ${}^E\mathcal{O}_{L^\infty}$ on $(U_j \cap V_j) \setminus \mathbb{D}^1$, $(j = 1, 2)$. They coincide on the common domain of definition and define a global section $\tilde{h} \in {}^E\mathcal{O}_{L^\infty}((U_1 \cap V_1) \cup (U_2 \cap V_2) \setminus \mathbb{D}^1)$. Since $(U_1 \cap V_1) \cup (U_2 \cap V_2)$ is a complex neighborhood of Ω, $h := [\tilde{h}] \in \mathcal{F}(\Omega)$ becomes a desired section. □

To complete the proof of Proposition 2.3, we note that the representation (2.5) follows from (2.4) and from the definition of the support. □

As for the $\mathbb{C}$-valued case, we give

THEOREM 2.5. *A $\mathbb{C}$-valued bounded hyperfunction on an open set is given by the right hand side of* (2.1), *that is,*

$$\mathcal{B}_{L^\infty}(\Omega) = \varinjlim_{U} \frac{\mathcal{O}_{L^\infty}(U \setminus \Omega)}{\mathcal{O}_{L^\infty}(U)}. \tag{2.7}$$

Moreover the sheaf $\mathcal{B}_{L^\infty}$ is flabby.

PROOF. We will prove (2.7), equivalently, $\mathcal{B}_{L^\infty}(\Omega) = \mathcal{F}(\Omega)$, where $\mathcal{F}$ denotes the presheaf given by (2.1). That is, $\mathcal{F}$ is the same presheaf as in the proof of Proposition 2.3 under $E = \mathbb{C}$. Since $\mathcal{B}_{L^\infty}|_{\mathbb{R}} = \mathcal{B}$, and since $\mathcal{B}(\Omega) = \frac{\mathcal{O}(U \setminus \Omega)}{\mathcal{O}(U)}$ for any open set $\Omega \subset \mathbb{R}$ and for any complex neighborhood U of Ω, it suffices to show that $\mathcal{B}_{L^\infty}(\Omega) = \mathcal{F}(\Omega)$ in the case that Ω contains $+\infty$ or $-\infty$. Assume at first that $+\infty \in \Omega$ and $-\infty \notin \Omega$. Then we can take a compact subset $K := [a, +\infty] \subset \Omega$ and define $\Omega_1 := \Omega \cap]-\infty, a + 1[$. Let $u \in \mathcal{B}_{L^\infty}(\Omega)$. Applying (2.4) to $u|_K$, we can take an open neighborhood $\Omega_2 =]a - \delta, +\infty]$ of K such that $u|_{\Omega_2} \in \mathcal{F}(\Omega_2)$. On the other hand, $u|_{\Omega_1} \in \mathcal{F}(\Omega_1)$ since $\Omega_1 \subset \mathbb{R}$. Then $u|_{\Omega_1}$ and $u|_{\Omega_2}$ satisfy the assumption of Lemma 2.4, and we can take $h \in \mathcal{F}(\Omega)$ which must be equal to u. In the case

$-\infty \in \Omega$ and $+\infty \notin \Omega$, the argument is the same. In the case $\pm\infty \in \Omega$, take compact neighborhoods K_1 of $-\infty$ and K_3 of $+\infty$ in Ω, and also take an open set $\Omega_2 \subset \mathbb{R}$ so that $\Omega = K_1 \cup \Omega_2 \cup K_3$. Then we can use the similar argument twice.

The flabbiness of $\mathcal{B}_{L^\infty}$ is reduced to the surjectivity of the restriction map

$$\mathcal{B}_{L^\infty}(\mathbb{D}^1) \to \mathcal{B}(\mathbb{R}). \tag{2.8}$$

In fact, let u be a section in $\mathcal{B}_{L^\infty}(\Omega)$ for an open set $\Omega \subset \mathbb{D}^1$. Then since $\mathcal{B}_{L^\infty}|_{\mathbb{R}} = \mathcal{B}$ is flabby, there exists an extension $u_1 \in \mathcal{B}(\mathbb{R})$ of $u|_{\Omega\cap\mathbb{R}}$. Assume that there exists an extension $u_2 \in \mathcal{B}_{L^\infty}(\mathbb{D}^1)$ of u_1. Then $\Omega' := \mathbb{R}\cup(\{\pm\infty\}\setminus\Omega)$ becomes an open set satisfying $\Omega\cup\Omega' = \mathbb{D}^1$ and $\Omega\cap\Omega' = \Omega\cap\mathbb{R}$. The two sections $u \in \mathcal{B}_{L^\infty}(\Omega)$ and $u' := u_2|_{\Omega'} \in \mathcal{B}_{L^\infty}(\Omega')$ coincide on $\Omega\cap\Omega'$ and defines a global section $\tilde{u} \in \mathcal{B}_{L^\infty}(\mathbb{D}^1)$, which is an extension of u.

The surjectivity of (2.8) was given already in Sato [14, §10]. Also see Theorem 8.4.4 and Corollary 8.4.5 of Kaneko [7] with their proof, for a little bit more explicit argument. □

The sheaf $\mathcal{O}_{L^\infty}$ is a subsheaf of $\tilde{\mathcal{O}}$, as is easily seen from their definitions. The embedding $\mathcal{O}_{L^\infty} \hookrightarrow \tilde{\mathcal{O}}$ defines the standard morphism $\mathcal{B}_{L^\infty} \to \mathcal{Q}$. From the next proposition, we can regard $\mathcal{B}_{L^\infty}$ as a subsheaf of $\mathcal{Q}$.

Proposition 2.6. *The standard morphism $\mathcal{B}_{L^\infty} \to \mathcal{Q}$ is injective.*

Proof. Since they coincide on $\mathbb{R}$, it suffices to show that the maps between the stalks at $\pm\infty$ are injective.

Let u be a germ of $\mathcal{B}_{L^\infty}$ at $+\infty$. There exists a defining function $f \in \mathcal{O}_{L^\infty}(]a, +\infty] + i\dot{B}_d)$ with some $a \in \mathbb{R}$ and $d > 0$. Assume that u is zero as a germ of $\mathcal{Q}$ at $+\infty$. Then f belongs to $\tilde{\mathcal{O}}(]a', +\infty] + iB_{d'})$ with some $a' \geq a$ and $0 < d' \leq d$. Under these conditions, we want to show that $[f]$ vanishes in a neighborhoods of $+\infty$, that is, f belongs to $\mathcal{O}_{L^\infty}(]a'', +\infty] + iB_{d''})$ with some $a'' \geq a'$ and $0 < d'' \leq d'$. Thus, it suffices to prove that $f \in \mathcal{O}_{L^\infty}(]a, +\infty] + i\dot{B}_d) \cap \tilde{\mathcal{O}}(]a, +\infty] + iB_d)$ implies $f \in \mathcal{O}_{L^\infty}(]a, +\infty] + iB_d)$.

Now we consider a holomorphic function $f_\varepsilon(w) := f(w)e^{-\varepsilon w^2}$ on $U_\delta :=]a + \delta, +\infty[+ iB_{d-\delta} \subset \mathbb{C}$ for arbitrarily fixed $\delta > 0$ and $\varepsilon > 0$. Since

$f \in \tilde{\mathcal{O}}(]a,+\infty] + iB_d)$, f_ε is bounded on U_δ. Therefore, by the Phragmén-Lindelöf principle, we have

$$\sup_{w\in U_\delta} |f_\varepsilon(w)| = \sup_{w\in \partial U_\delta} |f_\varepsilon(w)|.$$

On the other hand, since $f \in \mathcal{O}_{L^\infty}(]a,+\infty]+i\dot{B}_d)$ and since f is holomorphic in a neighborhood of $a+\delta$, f is bounded on ∂U_δ and satisfies

$$\sup_{w\in \partial U_\delta} |f_\varepsilon(w)| \le \sup_{w\in \partial U_\delta} |f(w)| \cdot \sup_{w\in \partial U_\delta} e^{-\varepsilon \operatorname{Re} w^2} \le c_\delta e^{\varepsilon(d-\delta)^2},$$

where $c_\delta := \sup_{w\in\partial U_\delta}|f(w)| < +\infty$. From these estimates, we have the estimate $\sup_{w\in U_\delta}|f_\varepsilon(w)| \le c_\delta e^{\varepsilon(d-\delta)^2}$, or equivalently,

$$|f(w)| \le c_\delta \exp \varepsilon\big[(d-\delta)^2 + \operatorname{Re} w^2\big] \text{ for any } w \in U_\delta.$$

By taking the limit $\varepsilon \downarrow 0$, we obtain

$$\sup_{w\in U_\delta} |f(w)| \le c_\delta.$$

Since $\delta > 0$ was arbitrary, f belongs to $\mathcal{O}_{L^\infty}(]a,+\infty] + iB_d)$. □

Bounded measurable functions can be naturally considered as a section of $\mathcal{B}_{L^\infty}$.

PROPOSITION 2.7. *Let $\Omega \subset \mathbb{D}^1$ be a connected open set.*

(i) *A function $f(t) \in L^\infty(\Omega\cap\mathbb{R})$ can be naturally regarded as a section in $\mathcal{B}_{L^\infty}(\Omega)$. The image of f is given by a defining function*

$$-\frac{1}{2\pi i}\int_{\Omega\cap\mathbb{R}} f(s)\frac{e^{-(w-s)^2}}{w-s}ds \tag{2.9}$$

which belongs to $\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\mathbb{R}\setminus\mathbb{D}^1)$. This map $L^\infty(\Omega\cap\mathbb{R}) \to \mathcal{B}_{L^\infty}(\Omega)$ is injective.

(ii) *A bounded continuous map $f : \Omega\cap\mathbb{R} \to E$ can be naturally regarded as a section in ${}^E\mathcal{B}_{L^\infty}(\Omega)$. The image of f is again given by a defining function* (2.9), *which in this case belongs to ${}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\mathbb{R}\setminus\mathbb{D}^1)$. This correspondence is injective.*

Proof. (i) The fact that the convolution integral (2.9) belongs to $\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\mathbb{R} \setminus \mathbb{D}^1)$ can be seen easily. The injectivity follows from the injectivity of the composition map $L^\infty(\Omega \cap \mathbb{R}) \to \mathscr{B}_{L^\infty}(\Omega) \to \mathscr{B}(\Omega \cap \mathbb{R})$, where the second map is the restriction.

(ii) The well-definedness is proved in a similar manner. Then we can show that the integral over $\Omega \cap \mathbb{R}$ is holomorphic outside $\overline{\Omega \cap \mathbb{R}}$. Thus to show the injectivity, it suffices to prove that for an open interval $I \Subset \Omega \cap \mathbb{R}$, if the section $[g]$ given by

$$g(w) := -\frac{1}{2\pi i}\int_{\overline{I}} f(s)\frac{e^{-(w-s)^2}}{w-s}ds$$

is 0 on I, then $f = 0$ on I.

From the assumption, $g \in {}^{E}\mathscr{O}(\mathbb{C} \setminus \partial I)$. For any $h \in E'$, we have

$$h(g(w)) = -\frac{1}{2\pi i}\int_{\overline{I}} h(f(s))\frac{e^{-(w-s)^2}}{w-s}ds,$$

which implies that $h \circ g$ is a defining function of $h \circ f$, and belongs to $\mathscr{O}(\mathbb{C} \setminus \partial I)$. Now the conclusion follows from (i) and the Hahn-Banach theorem. □

Remark 2.8. We also give another embedding of not only bounded functions but also locally integrable functions satisfying some "boundedness" conditions at infinity as follows. A function $f(t) \in L^1_{\mathrm{loc}}(]a, +\infty[)$ satisfying the condition

$$\sup_{b \geq a} \|f(t)\|_{L^1(]b,b+1[)} < +\infty \tag{2.10}$$

can be naturally regarded as a section in $\mathscr{B}_{L^\infty}(]a, +\infty])$. The image of f is given by the same convolution integral (2.9) with the domain of integration $\Omega \cap \mathbb{R}$ replaced by $]a, +\infty[$.

2.3. Relation between $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ and $\mathcal{B}_{L^\infty}$

We recall the notion of Chung-Kim-Lee's bounded hyperfunctions in [3]. They introduced the space $\mathcal{B}_{L^p}$ $(1 < p \leq \infty)$ of hyperfunctions of L^p growth as the dual space of the locally convex space $\mathcal{A}_{L^q}$ with $1/p + 1/q = 1$. Here the spaces $\mathcal{A}_{L^q}$ $(1 \leq q < \infty)$ are defined by

$$\mathcal{A}_{L^q} := \varinjlim_{h>0} A_{q,h},$$

$$A_{q,h} := \{\varphi \in C^\infty(\mathbb{R}^n); \|\varphi\|_{L^q,h} := \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^q(\mathbb{R}^n)}}{h^{|\alpha|}\alpha!} < +\infty\},$$

and endowed with locally convex inductive limit topologies. The space $\mathcal{B}_{L^\infty}$ for the case $p = \infty$, (that is, $q = 1$), was called the space of bounded hyperfunctions.

We will now prove that the space $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is isomorphic to $\mathcal{B}_{L^\infty}$. In the first part, we give another description of $\mathcal{A}_{L^1}$ by a similar argument used in Kim-Chung-Kim [9, §1].

LEMMA 2.9. *A function $\varphi(t) \in A_{q,h}$ is real analytic and extends holomorphically to the tube domain $\{w \in \mathbb{C}^n; \max_j|\operatorname{Im} w_j| < 1/h\}$.*

PROOF. Using the Sobolev embedding theorem, we have

$$\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)} \le c \sum_{|\beta|\le N} \|\partial^{\alpha+\beta}\varphi\|_{L^q(\mathbb{R}^n)} \le c\|\varphi\|_{L^q,h} \sum_{|\beta|\le N} h^{|\alpha+\beta|}(\alpha+\beta)!$$

for some constant c and N depending only on n and q. Thus for any $h' > h$,

$$\frac{\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)}}{h'^{|\alpha|}\alpha!} \le c\|\varphi\|_{L^q,h} \sum_{|\beta|\le N} \beta! h^{|\beta|} \binom{\alpha+\beta}{\alpha} \left(\frac{h}{h'}\right)^{|\alpha|}.$$

Using the estimate $\sup_\alpha \binom{\alpha+\beta}{\alpha} x^\alpha \le \sum_\alpha \binom{\alpha+\beta}{\alpha} x^\alpha = \prod_{j=1}^n (1-x_j)^{-\beta_j-1}$ for $x \in [0,1[^n$, we obtain

$$\sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^n)}}{h'^{|\alpha|}\alpha!} \le \frac{cc_h}{(1-h/h')^{N+n}} \|\varphi\|_{L^q,h}, \tag{2.11}$$

where $c_h = \sum_{|\beta|\le N} \beta! h^{|\beta|}$ is a constant depending only on n, N and h. Therefore φ extends holomorphically to $\{\max_j|\operatorname{Im} w_j| < 1/h'\}$. Since $h' > h$ was arbitrary, this completes the proof. □

LEMMA 2.10. *Let $\varphi(t) \in A_{q,h}$ and denote by $\varphi(w)$ its holomorphic extension to $\{\max_j|\operatorname{Im} w_j| < 1/h\}$. Then for any fixed $s \in \mathbb{R}^n$ with $\max_j|s_j| < 1/h$, the function $t \mapsto \varphi(t+is)$ belongs to $L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Moreover for any $0 < d < 1/h$, there exists a constant c'_d independent of φ such that*

$$\sup_{s\in[-d,d]^n} \|\varphi(\cdot + is)\|_{L^q(\mathbb{R}^n)} \le c'_d \|\varphi\|_{L^q,h}, \tag{2.12}$$

$$\sup_{s\in[-d,d]^n} \|\varphi(\cdot + is)\|_{L^\infty(\mathbb{R}^n)} \le c'_d \|\varphi\|_{L^q,h}. \tag{2.13}$$

PROOF. Since $\varphi(t+is) = \sum_\alpha \frac{\partial^\alpha \varphi(t)}{\alpha!}(is)^\alpha$, we have, for $s \in [-d,d]^n$,

$$\begin{aligned}\|\varphi(\cdot+is)\|_{L^q(\mathbb{R}^n)} &\le \sum_\alpha \frac{\|\partial^\alpha\varphi\|_{L^q(\mathbb{R}^n)}}{\alpha!}|s^\alpha| = \sum_\alpha \frac{\|\partial^\alpha\varphi\|_{L^q(\mathbb{R}^n)}}{h^{|\alpha|}\alpha!}|(hs)^\alpha| \\ &\le \|\varphi\|_{L^q,h}\prod_{j=1}^n \frac{1}{1-h|s_j|} \le \frac{\|\varphi\|_{L^q,h}}{(1-hd)^n}.\end{aligned}$$

As for the L^∞-norm, it follows from (2.11) that

$$\begin{aligned}\|\varphi(\cdot+is)\|_{L^\infty(\mathbb{R}^n)} &\le \sum_\alpha \frac{\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)}}{\alpha!}|s^\alpha| \le \frac{cc_h\|\varphi\|_{L^q,h}}{(1-h/h')^{N+n}}\sum_\alpha |(h's)^\alpha| \\ &\le \frac{cc_h\|\varphi\|_{L^q,h}}{(1-h/h')^{N+n}(1-h'd)^n},\end{aligned}$$

where h' is a fixed number satisfying $h < h' < 1/d$. Then the conclusion follows if we take

$$c'_d := \max\{(1-hd)^{-n}, cc_h(1-h/h')^{-N-n}(1-h'd)^{-n}\}. \square$$

Consider the case $q = 1$. We denote by $A^{1,d}$ the space of functions φ, holomorphic in the tube domain $\{\max_j|\operatorname{Im} w_j| < d\}$ and continuous in its closure $\{\max_j|\operatorname{Im} w_j| \le d\}$, satisfying

$$\|\varphi\|_{A^{1,d}} := \sup_{s\in[-d,d]^n}\|\varphi(\cdot+is)\|_{L^1(\mathbb{R}^n)} + \sup_{s\in[-d,d]^n}\|\varphi(\cdot+is)\|_{L^\infty(\mathbb{R}^n)} < +\infty.$$

Note that $\sup_{s\in[-d,d]^n}\|\varphi(\cdot+is)\|_{L^\infty(\mathbb{R}^n)} = \|\varphi\|_{L^\infty(\{\max_j|\operatorname{Im} w_j|\le d\})}$, and that the space $A^{1,d}$ endowed with the norm $\|\cdot\|_{A^{1,d}}$ is a Banach space.

LEMMA 2.11. *$\mathcal{A}_{L^1}$ is topologically isomorphic to $\varinjlim_{d>0} A^{1,d}$.*

PROOF. Lemma 2.10 shows that the inclusion $A_{1,h} \to A^{1,d}$ is continuous if $d < 1/h$.

Conversely, let φ be a function in $A^{1,d}$. We have from the Cauchy formula that, for $0 < r < d$,

$$|\partial^\alpha\varphi(t)| \le \frac{\alpha!}{r^{|\alpha|}}\int_{\{\forall j, |w_j|=r\}}|\varphi(t+w)|\frac{|dw_1|\cdots|dw_n|}{(2\pi r)^n}.$$

We put $h := 1/r > 1/d$. By integrating in the t variable and using Fubini's theorem, we have

$$\begin{aligned}\frac{\|\partial^\alpha \varphi\|_{L^1(\mathbb{R}^n)}}{h^{|\alpha|}\alpha!} &\le \int_{\{\forall j, |w_j|=r\}} \|\varphi(\cdot + w)\|_{L^1(\mathbb{R}^n)} \frac{|dw_1|\cdots|dw_n|}{(2\pi r)^n} \\ &\le \sup_{\max_j |s_j| \le r} \|\varphi(\cdot + is)\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

which implies that the inclusion $A^{1,d} \to A_{1,h}$ is continuous if $h > 1/d$. □

Assume moreover $n = 1$. We define a pairing between $\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ and $A^{1,d}$ by the formula

$$\langle f, \varphi \rangle := \int_{\gamma(s^+)-\gamma(s^-)} f(w)\varphi(w)dw \tag{2.14}$$

for $\varphi \in A^{1,d}$ and $f \in \mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$, where $s^\pm$ are arbitrary constants with

$$-d < s^- < 0 < s^+ < d \tag{2.15}$$

and $\gamma(s)$ denotes the contour $\mathbb{R} \ni t \mapsto t + is \in \mathbb{C}$.

LEMMA 2.12. *The right hand side of* (2.14) *converges and does not depend on the choice of* $s^\pm$. *Therefore* $\langle f, \cdot \rangle$ *defines a continuous linear functional of* $A^{1,d}$.

PROOF. The convergence of the integral follows from the facts $f(\cdot + is^\pm) \in L^\infty(\mathbb{R})$ and $\varphi(\cdot + is^\pm) \in L^1(\mathbb{R})$.

From the same facts and the Lebesgue convergence theorem, we have the equality

$$\int_{\gamma(s^\pm)} f(w)\varphi(w)dw = \lim_{\varepsilon \downarrow 0} \int_{\gamma(s^\pm)} f(w)\varphi(w)e^{-\varepsilon w^2}dw.$$

Since $\varphi(\cdot + is^\pm) \in L^\infty(\mathbb{R})$, a usual contour deformation argument proves that each $\int_{\gamma(s^\pm)} f(w)\varphi(w)e^{-\varepsilon w^2}dw$ is independent of $s^\pm$, for fixed $\varepsilon > 0$.

Once the well-definedness is established, the estimate

$$\begin{aligned}|\langle f, \varphi \rangle| \le\ & \|f(\cdot + is^+)\|_{L^\infty(\mathbb{R})}\|\varphi(\cdot + is^+)\|_{L^1(\mathbb{R})} \\ &+ \|f(\cdot + is^-)\|_{L^\infty(\mathbb{R})}\|\varphi(\cdot + is^-)\|_{L^1(\mathbb{R})}\end{aligned}$$

for fixed $s^\pm$ shows the continuity of $\langle f, \cdot\rangle$. $\square$

Now we give

THEOREM 2.13. *The pairing* (2.14) *induces a pairing between* $\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ *and* $\mathcal{A}_{L^1}$. *Moreover* $\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ *is isomorphic to the dual space* $\mathcal{A}'_{L^1}$.

PROOF. For $f \in \mathcal{O}_{L^\infty}(\mathbb{D}^1 + iB_d)$ and $\varphi \in A^{1,d}$, we have

$$\int_{\gamma(s^+)-\gamma(s^-)} f(w)\varphi(w)ds = \lim_{\varepsilon\downarrow 0}\int_{\gamma(s^+)-\gamma(s^-)} f(w)\varphi(w)e^{-\varepsilon w^2}ds$$

and the integral in the right hand side vanishes from a usual contour deformation argument, which proves $\langle f, \cdot\rangle = 0$ on $A^{1,d}$. By taking the inductive limit in $d > 0$, it is proved that $\langle\cdot,\cdot\rangle : \mathcal{B}_{L^\infty}(\mathbb{D}^1) \times \mathcal{A}_{L^1} \to \mathbb{C}$ is well-defined. The continuity of $\langle [f], \cdot\rangle$ on $\mathcal{A}_{L^1}$ follows from the continuity of $\langle f, \cdot\rangle$ on every $A^{1,d'}$ $(0 < d' < d)$.

In this way, we have constructed the linear map

$$\mathcal{B}_{L^\infty}(\mathbb{D}^1) \ni [f] \mapsto \langle [f], \cdot\rangle \in \mathcal{A}'_{L^1}. \tag{2.16}$$

We will prove its bijectivity.

To show the injectivity of (2.16), assume that $f \in \mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ satisfies $\langle [f], \cdot\rangle = 0$ on $\mathcal{A}_{L^1}$. Then fix s with $0 < s < d$ and consider the integral

$$g_s(w) := -\frac{1}{2\pi i}\int_{\gamma(s)-\gamma(-s)} f(w')\frac{e^{-(w-w')^2}}{w-w'}dw' \tag{2.17}$$

for $\operatorname{Im} w \neq \pm s$. If $|\operatorname{Im} w| > s$, the integral is equal to $\langle f, -\frac{1}{2\pi i}\frac{e^{-(w-\cdot)^2}}{w-\cdot}\rangle$ and vanishes from the assumption. In fact, in this case $-\frac{1}{2\pi i}\frac{e^{-(w-\cdot)^2}}{w-\cdot} \in A^{1,d'}$ for $s < d' < |\operatorname{Im} w|$. When $0 < \operatorname{Im} w < s$, we take another s' with $0 < s' < \operatorname{Im} w$ so that $g_{s'}(w) = 0$, and obtain

$$g_s(w) = g_s(w) - g_{s'}(w) = -\frac{1}{2\pi i}\int_{\gamma(s)-\gamma(s')} - \frac{1}{2\pi i}\int_{-\gamma(-s)+\gamma(-s')}.$$

Since the integrand is holomorphic outside $\{w\}\cup\mathbb{R}$, and decays exponentially with $\operatorname{Re} w' \to \pm\infty$ locally uniformly in $\operatorname{Im} w'$, the second term vanishes and the first term is equal to $\mathrm{Res}_{w'=w}\Big[f(w')\frac{e^{-(w-w')^2}}{w-w'}\Big] = f(w)$. We can similarly argue also in the case $-s < \operatorname{Im} w < 0$ and get

$$g_s(w) = f(w) \text{ for } 0 < |\operatorname{Im} w| < s.$$

On the other hand, in view of (2.17), we can easily see that $g_s(w)$ is holomorphic on $\{|\operatorname{Im} w| < s\}$ and bounded on each domain $\{|\operatorname{Im} w| < s'\}$ for any $s' < s$. Therefore g_s gives an extension of $f \in \mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ to $\mathcal{O}_{L^\infty}(\mathbb{D}^1 + iB_s)$, which proves $[f] = 0$ in $\mathcal{B}_{L^\infty}(\mathbb{D}^1)$.

To show the surjectivity of (2.16), consider an element $\psi \in \mathcal{A}'_{L^1}$ and define

$$f(w) := -\frac{1}{2\pi i}\psi\Big(\frac{e^{-(w-\cdot)^2}}{w-\cdot}\Big)$$

for $w \in \mathbb{C} \setminus \mathbb{R}$. Note that for fixed $d > 0$, the map $\{|\operatorname{Im} w| > d\} \ni w \mapsto \frac{e^{-(w-\cdot)^2}}{w-\cdot} \in A^{1,d}$ belongs to ${}^{A^{1,d}}\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i(\mathbb{R}\setminus[-d,d]))$, i.e., it is $A^{1,d}$-valued holomorphic function and the family $\{\frac{e^{-(w-\cdot)^2}}{w-\cdot}\}_{|\operatorname{Im} w|>d'}$ is bounded for every $d' > d$. Thus $f(w) \in \mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\mathbb{R}\setminus\mathbb{D}^1)$ and represents a global section $[f] \in \mathcal{B}_{L^\infty}(\mathbb{D}^1)$. It suffices to show that $\langle [f], \varphi\rangle = \psi(\varphi)$ for any $\varphi \in \mathcal{A}_{L^1}$. When $\varphi \in A^{1,d}$, we take a contour $\gamma_s := \gamma(s) - \gamma(-s)$ with some $0 < s < d$, and calculate as follows.

$$\begin{aligned}\langle f, \varphi\rangle &= \int_{\gamma_s} f(w)\varphi(w)dw \overset{\spadesuit}{=} \lim_{\varepsilon\downarrow 0}\int_{\gamma_s} f(w)\varphi(w)e^{-\varepsilon w^2}dw\\ &= -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{\gamma_s}\psi\Big(\frac{e^{-(w-\cdot)^2}}{w-\cdot}\Big)\varphi(w)e^{-\varepsilon w^2}dw\\ &\overset{\clubsuit}{=} \lim_{\varepsilon\downarrow 0}\psi\Big(-\frac{1}{2\pi i}\int_{\gamma_s}\frac{e^{-(w-\cdot)^2}}{w-\cdot}\varphi(w)e^{-\varepsilon w^2}dw\Big)\\ &= \lim_{\varepsilon\downarrow 0}\psi\Big(\varphi(\cdot)e^{-\varepsilon(\cdot)^2}\Big) \overset{\heartsuit}{=} \psi(\varphi).\end{aligned}$$

Here we used the Lebesgue convergence theorem at ♠, the convergence of $\int_{\gamma_s}\frac{e^{-(w-\cdot)^2}}{w-\cdot}\varphi(w)e^{-\varepsilon w^2}dw$ in $A^{1,d'}$ for some $0 < d' < s$ at ♣, and the convergence $\lim_{\varepsilon\downarrow 0}\varphi(\cdot)e^{-\varepsilon(\cdot)^2} = \varphi(\cdot)$ in $A^{1,d'}$ for $0 < d' < d$ at ♡. □

3. Operators for Bounded Hyperfunctions

In this section we consider some classes of operators acting on bounded hyperfunctions. Some are sheaf morphisms, typically some differential operators, while others are families of linear maps $P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + K) \to {}^E\mathscr{B}_{L^\infty}(\Omega)$ for $\Omega \subset \mathbb{D}^1$ with a closed interval $K \subset \mathbb{R}$, typically some difference operators. Here $\Omega + K$ denotes the set $\{t + s; t \in \Omega, s \in K\}$ with convention $\pm\infty + s = \pm\infty$ for $s \in \mathbb{R}$. Note also that for an open set $U \subset \mathbb{D}^1 + i\mathbb{R}$, we denote by $U + K$ the set $\{w + s; w \in U, s \in K\}$ under similar conventions $w + s = w$ for $\operatorname{Re} w = \pm\infty$, $s \in \mathbb{R}$.

Then we introduce the notion of periodicity for hyperfunctions, bounded hyperfunctions and operators acting on bounded hyperfunctions, using difference operators.

3.1. Operators commuting with restrictions

Let I be an open set in $\mathbb{D}^1$ with a complex neighborhood U. A sheaf endomorphism P of ${}^E\mathscr{O}_{L^\infty}$ (resp. ${}^E\mathscr{O}_{L^\infty}|_U$) canonically induces a sheaf endomorphism of ${}^E\mathscr{B}_{L^\infty}$ (resp. ${}^E\mathscr{B}_{L^\infty}|_I$).

Example 3.1. We denote by $\mathcal{L}_b(E)$ the space of the linear continuous operators $E \to E$ endowed with the topology of uniform convergence on bounded subsets. That is, the topology of $\mathcal{L}_b(E)$ is given by the family of seminorms $q_{S,p}$ defined by

$$q_{S,p}(L) := \sup_{x\in S} p(Lx) \quad \text{for } L \in \mathcal{L}_b(E),$$

where S runs through bounded subsets in E and p runs through continuous seminorms of E. A section $P \in {}^{\mathcal{L}_b(E)}\mathscr{O}_{L^\infty}(U)$ defines a sheaf endomorphism of ${}^E\mathscr{O}_{L^\infty}|_U$, and therefore a sheaf endomorphism of ${}^E\mathscr{B}_{L^\infty}|_I$. In particular, in the case $E = \mathbb{C}$, a section $a(t) \in \mathscr{O}_{L^\infty}(\mathbb{D}^1)$ defines a multiplier $a\cdot$ on $\mathscr{B}_{L^\infty}$, and in the case $E = \mathbb{C}^m$ (thus, $\mathcal{L}_b(E) \simeq \mathbb{C}^{m^2}$), a matrix $A(t) = (a_{ij}(t))_{1\le i,j\le m}$ with entries in $\mathscr{O}_{L^\infty}(\mathbb{D}^1)$ defines $A\cdot$ on $(\mathscr{B}_{L^\infty})^m$.

Example 3.2. The differentiation $\partial_w = d/dw$ defines a sheaf endomorphism of ${}^E\mathscr{O}_{L^\infty}$, and therefore a sheaf endomorphism ∂_t of ${}^E\mathscr{B}_{L^\infty}$. Moreover for $k \in \Gamma_{\{0\}}(\mathbb{R}, \mathscr{B})$, consider the endomorphism $f \mapsto k * f$ of ${}^E\mathscr{O}$ given by the convolution

$$(k * f)(w) := -2\pi i \operatorname{Res}_{w'=0}\bigl[\tilde{k}(w')f(w - w')\bigr], \tag{3.1}$$

where $\tilde{k}$ is a defining function of k. The operator $k*$ does not depend on the choice of $\tilde{k}$, and extends to an endomorphism of ${}^E\mathcal{O}_{L^\infty}$, and that of ${}^E\mathcal{B}_{L^\infty}$. These operators $k*$ cover differential operators with constant coefficients, not only of finite order but also of infinite order.

Now we define sheaves of ordinary differential operators with vector-valued holomorphic coefficients bounded at infinity.

DEFINITION 3.3. Let F be a $\mathbb{C}$-algebra endowed with a sequentially complete separated locally convex topology which admits a continuous homomorphism

$$F \to \mathcal{L}_b(E).$$

We define the sheaf ${}^F\mathcal{D}_{L^\infty}$ of ordinary differential operators with ${}^F\mathcal{O}_{L^\infty}$ coefficients on $\mathbb{D}^1 + i\mathbb{R}$ as the sheaf associated with the presheaf

$$U \mapsto \{P(w, \partial_w) := \sum_{j=0}^{m} a_j(w)\partial_w^j; m \in \mathbb{N}, a_j \in {}^F\mathcal{O}_{L^\infty}(U)\}. \tag{3.2}$$

The presheaf (3.2) is not a sheaf, but the right hand side of (3.2) coincides with ${}^F\mathcal{D}_{L^\infty}(U)$ when U consists of finite connected components. The sheaf ${}^F\mathcal{D}_{L^\infty}$ is endowed with the usual Hörmander-Leibniz product, and becomes a sheaf of $\mathbb{C}$-algebra. Moreover, an operator $P \in {}^F\mathcal{D}_{L^\infty}(U)$ acts continuously on every ${}^E\mathcal{O}_{L^\infty}(V)$, $V \subset U$, and also defines a sheaf endomorphism on ${}^E\mathcal{B}_{L^\infty}|_{U\cap\mathbb{D}^1}$.

When $F = \mathbb{C}$, we usually omit F and denote it by $\mathcal{D}_{L^\infty}$. The restriction $\mathcal{D}_{L^\infty}|_{\mathbb{C}}$ coincides with the sheaf $\mathcal{D}$ of ordinary differential operators with holomorphic coefficients.

Example 3.4. We can also consider a convolution $k*$ with kernel hyperfunction $k \in \mathcal{B}(\mathbb{R})$ with compact support. If $\operatorname{supp} k \subset -K := \{t \in \mathbb{R}; -t \in K\}$ with some compact set K, then $k*$ defines a continuous linear map

$${}^E\mathcal{O}_{L^\infty}(U + K) \to {}^E\mathcal{O}_{L^\infty}(U)$$

for any $U \subset \mathbb{D}^1 + i\mathbb{R}$. Therefore $k*$ induces a family of linear maps

$${}^E\mathcal{B}_{L^\infty}(\Omega + K) \to {}^E\mathcal{B}_{L^\infty}(\Omega), \quad \Omega \overset{\text{open}}{\subset} \mathbb{D}^1.$$

As typical examples other than differential operators, we give a translation operator

$$\delta_{-\omega}* : {}^E\mathscr{B}_{L^\infty}(\Omega + \omega) \ni u \mapsto u(\cdot + \omega) \in {}^E\mathscr{B}_{L^\infty}(\Omega),$$

and a difference operator

$$(\delta_{-\omega} - \delta)* : {}^E\mathscr{B}_{L^\infty}((\Omega + \omega) \cup \Omega) \ni u \mapsto u(\cdot + \omega) - u(\cdot) \in {}^E\mathscr{B}_{L^\infty}(\Omega),$$

where ω is a real constant and $\delta_{-\omega} = \delta(\cdot + \omega) \in \Gamma_{\{-\omega\}}(\mathbb{R}, \mathscr{B})$ denotes the Dirac delta distribution at $-\omega$. We denote the operator $\delta_{-\omega}*$ by T_ω, and the operator $(\delta_{-\omega} - \delta)*$ by $T_\omega - 1$.

Let F be as in Definition 3.3. For a compact set in $K \subset \mathbb{R}$ and an open set $U \subset \mathbb{D}^1 + i\mathbb{R}$, consider a linear combination of convolution operators with coefficients in ${}^F\mathscr{O}_{L^\infty}(U)$:

$$P := \sum_{j}^{\text{finite}} a_j(w) k_j*, \quad a_j \in {}^F\mathscr{O}_{L^\infty}(U),\ k_j \in \Gamma_{-K}(\mathbb{R}, \mathscr{B}). \tag{3.3}$$

Then P defines a family of continuous linear maps

$$P_V : {}^E\mathscr{O}_{L^\infty}(V + K) \to {}^E\mathscr{O}_{L^\infty}(V) \quad \text{for } V \subset U,$$

and a family of linear maps

$$P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + K) \to {}^E\mathscr{B}_{L^\infty}(\Omega) \quad \text{for } \Omega \subset U \cap \mathbb{D}^1.$$

These families of maps are not sheaf endomorphisms, but satisfy a commutativity condition with restrictions. That is, we have the commutative diagrams, where the vertical arrows are the restrictions: for $V_2 \subset V_1$ in U,

$$\begin{array}{ccc} {}^E\mathscr{O}_{L^\infty}(V_1 + K) & \xrightarrow{P_{V_1}} & {}^E\mathscr{O}_{L^\infty}(V_1) \\ \downarrow & & \downarrow \\ {}^E\mathscr{O}_{L^\infty}(V_2 + K) & \xrightarrow{P_{V_2}} & {}^E\mathscr{O}_{L^\infty}(V_2), \end{array} \tag{3.4}$$

and for $\Omega_2 \subset \Omega_1$ in $U \cap \mathbb{D}^1$,

$$\begin{array}{ccc} {}^E\mathscr{B}_{L^\infty}(\Omega_1 + K) & \xrightarrow{P_{\Omega_1}} & {}^E\mathscr{B}_{L^\infty}(\Omega_1) \\ \downarrow & & \downarrow \\ {}^E\mathscr{B}_{L^\infty}(\Omega_2 + K) & \xrightarrow{P_{\Omega_2}} & {}^E\mathscr{B}_{L^\infty}(\Omega_2). \end{array} \tag{3.5}$$

More generally, we give

DEFINITION 3.5. Let $K \subset \mathbb{R}$ be a compact set and $U \subset \mathbb{D}^1 + i\mathbb{R}$ an open set. A family $P = \{P_V\}_{V \subset U}$ of linear maps $P_V : {}^E\mathcal{O}_{L^\infty}(V+K) \to {}^E\mathcal{O}_{L^\infty}(V)$ is called an operator of type K on U for ${}^E\mathcal{O}_{L^\infty}$, if each P_V is continuous and each diagram (3.4) is commutative for any $V_2 \subset V_1$.

An operator P of type K automatically induces a family $\{P_\Omega\}_\Omega$ of linear maps with commutative diagrams (3.5). For $f \in {}^E\mathcal{O}_{L^\infty}(V)$, we say that u is an ${}^E\mathcal{O}_{L^\infty}$-solution to the equation $Pu = f$ on V if $u \in {}^E\mathcal{O}_{L^\infty}(V+K)$ and if $P_V u = f$. Similarly, for $f \in {}^E\mathcal{B}_{L^\infty}(\Omega)$, we say that u is an ${}^E\mathcal{B}_{L^\infty}$-solution to $Pu = f$ on Ω if $u \in {}^E\mathcal{B}_{L^\infty}(\Omega + K)$ and $P_\Omega u = f$. The commutativity condition implies that the restriction of a solution on Ω_1 to $\Omega_2 + K$ ($\Omega_2 \subset \Omega_1$) is a solution on Ω_2. Therefore, when P is an operator of type K on a neighborhood U of $+\infty$, and f is a germ of ${}^E\mathcal{B}_{L^\infty}$ at $+\infty$, it makes sense to consider an $({}^E\mathcal{B}_{L^\infty})_{+\infty}$-solution to an equation $Pu = f$.

Let F be as in Definition 3.3. Then $P \in {}^F\mathcal{D}_{L^\infty}(U)$ is an operator of type $\{0\}$ on U for ${}^E\mathcal{O}_{L^\infty}$. From now on, whenever we consider operators of type K, we take K as a closed interval, so that $\Omega + K$ becomes an interval for any interval Ω.

3.2. Periodic hyperfunctions and periodic operators

Now we consider periodic real analytic functions and periodic hyperfunctions with period $\omega > 0$. Let $I \subset \mathbb{R}$ be an open interval and K the closed interval $[0, \omega]$. First note that if an E-valued holomorphic function $f(w)$ defined in a complex neighborhood $I + K + iB_d$ of $I + K$ satisfies the periodicity condition $f(t + \omega) = f(t)$ for any $t \in I$, then it extends holomorphically to a tube domain $\mathbb{R} + iB_d$ and satisfies $f(w + \omega) = f(w)$ for any $w \in \mathbb{R} + iB_d$, which implies $f \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + iB_d)$. In other words, a local ${}^E\mathcal{O}$-solution to an equation $(T_\omega - 1)f = 0$ always extends to a global ${}^E\mathcal{O}_{L^\infty}$-solution on a tube domain. Moreover we give also a local solvability result in ${}^E\mathcal{O}$ of an equation $(T_\omega - 1)u = f$.

LEMMA 3.6. *Let $L = [a, b] + i\overline{B_d}$ be a compact rectangle. For any $f \in {}^E\mathcal{O}(L)$, there exists $u \in {}^E\mathcal{O}(L + K)$ satisfying $(T_\omega - 1)u = f$ on L.*

PROOF. Assume that f is defined on $]a - 3\delta, b + 3\delta[+ iB_{d+3\delta}$ with some $\delta > 0$. We define f_1 on $U_1 :=]-\infty, b + \delta[+ iB_{d+\delta}$ and f_2 on $U_2 :=$

$]a-\delta, +\infty[\, + iB_{d+\delta}$ by

$$f_j(w) = \frac{1}{2\pi i}\int_{\gamma_j} f(s)\frac{e^{-(s-w)^2}}{s-w}ds,$$

where γ_2 is the line segment from $a-2\delta+i(d+2\delta)$ to $a-2\delta-i(d+2\delta)$ and $\gamma_1 = \partial(]a-2\delta, b+2\delta[\,+iB_{d+2\delta}) - \gamma_2$. Therefore it follows that

$$|f_j(w)| \le ce^{-|\operatorname{Re} w|^2}$$

on U_j respectively, and that $f_1+f_2 = f$ on $U_1 \cap U_2 = \,]a-\delta, b+\delta[\,+iB_{d+\delta}$. Then $u_1 := \sum_{k=1}^{\infty} T_{-k\omega} f_1$ converges in ${}^E\mathcal{O}(U_1+K)$ and solves an equation $(T_\omega - 1)u_1 = f_1$, and $u_2 := \sum_{k=0}^{\infty} T_{k\omega} f_2$ converges in ${}^E\mathcal{O}(U_2)$ and solves $(T_\omega-1)u_2 = f_2$. Since $L+K \subset (U_1+K)\cap U_2$, $u := u_1+u_2$ is a desired solution. □

As for periodic hyperfunctions, we define

Definition 3.7. Let ω be a positive constant. We say that a hyperfunction $f \in {}^E\mathcal{B}(\mathbb{R})$ is ω-periodic if f is a solution to the equation $(T_\omega - 1)f = 0$.

We can also consider globally defined ω-periodic bounded hyperfunctions and locally defined ω-periodic hyperfunctions. Periodic hyperfunctions are also studied in Chung-Kim-Lee [3] in several variables in $\mathbb{C}$-valued case. We give here the following

Proposition 3.8. *Let $\Omega \subset \mathbb{R}$ be an open interval and K the closed interval $[0,\omega]$. The restriction maps ${}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1) \to {}^E\mathcal{B}(\mathbb{R})$ and ${}^E\mathcal{B}(\mathbb{R}) \to {}^E\mathcal{B}(\Omega+K)$ induce the following isomorphisms respectively.*

$$\{f \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1); (T_\omega-1)f = 0\} \to \{f \in {}^E\mathcal{B}(\mathbb{R}); (T_\omega-1)f=0\}, \tag{3.6}$$

$$\{f \in {}^E\mathcal{B}(\mathbb{R}); (T_\omega-1)f = 0\} \to \{f \in {}^E\mathcal{B}(\Omega+K); (T_\omega-1)f=0\}. \tag{3.7}$$

Moreover, any ω-periodic hyperfunction $g \in {}^E\mathcal{B}(\mathbb{R})$ has an ω-periodic defining function $f \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$ with some $d>0$.

Proof. The well-definedness and the linearity of these maps are clear.

The bijectivity of (3.7) follows from the fact that ${}^E\mathcal{B}$ is a sheaf. In fact, let $f \in {}^E\mathcal{B}(\mathbb{R})$ be an ω-periodic hyperfunction which satisfies $f|_{\Omega+K} = 0$. Then we have $f|_{\Omega+K-j\omega} = (T_{j\omega}f)|_{\Omega+K-j\omega} = T_{j\omega}(f|_{\Omega+K}) = 0$ for any $j \in \mathbb{Z}$ and $f = 0$. Similarly we assume that g is an element of the right hand side of (3.7), that is, $g \in {}^E\mathcal{B}(\Omega + K)$ satisfies $(T_\omega - 1)g = 0$ on Ω. Then the sections $T_{j\omega}g \in {}^E\mathcal{B}(\Omega + K - j\omega)$ for $j \in \mathbb{Z}$ can be patched together and defines an extension $f \in {}^E\mathcal{B}(\mathbb{R})$ of g, satisfying $(T_\omega - 1)f = 0$.

In order to show the surjectivity of (3.6), assume that we are given an arbitrary $g \in {}^E\mathcal{B}(\mathbb{R})$ satisfying $(T_\omega - 1)g = 0$. We fix an open interval $\Omega =]a, b[$ and take a local boundary value representation $g = [\tilde{g}]$ on a neighborhood of $\overline{\Omega} + K$, using Proposition 2.3. That is, we may assume that $\tilde{g} \in {}^E\mathcal{O}(]a - d, b + d + \omega[+ i\dot{B}_{2d})$ for some $d > 0$ and that $g = [\tilde{g}]$ on $]a - d, b + d + \omega[$. The ω-periodicity of g implies that $(T_\omega - 1)\tilde{g}$ extends to a section $h \in {}^E\mathcal{O}(]a - d, b + d[+ iB_{2d})$. We use Lemma 3.6 to take a solution $u \in {}^E\mathcal{O}(\Omega + K + iB_d)$ to $(T_\omega - 1)u = h$, and define $f := \tilde{g} - u \in {}^E\mathcal{O}(\Omega + K + i\dot{B}_d)$. From the ω-periodicity of f, it extends holomorphically to $\mathbb{R} + i\dot{B}_d$ and defines a section in ${}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)$, which we again denote by f. Since $[u] = 0$ on $\Omega + K$, $[f]$ gives an ω-periodic extension of $g|_{\Omega+K}$. Therefore we proved the surjectivity of the composition of (3.6) and (3.7), the latter of which is bijective. Note that the existence of f for g also assures the last statement.

In order to show the injectivity of (3.6), assume that $u \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ is ω-periodic and that $u|_{\mathbb{R}} = 0$. We take, again using Proposition 2.3, a global defining function $f \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_{2d})$ of u. Then $u|_{\mathbb{R}} = 0$ implies that $f \in {}^E\mathcal{O}(\mathbb{R} + iB_{2d})$, and the ω-periodicity of u implies that $g := (T_\omega - 1)f \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + iB_{2d})$. If we show that $f \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + iB_d)$, then the conclusion holds. Now fix an arbitrary continuous seminorm p on E. Since $C_p := \|g\|_{\mathbb{D}^1 + i\overline{B_d}, p} < \infty$ and since $f(w + \omega) = f(w) + g(w)$, we have the estimate

$$\sup_{w \in [j\omega, (j+1)\omega] + i\overline{B_d}} p(f(w)) \le C_p|j| + \sup_{w \in [0,\omega] + i\overline{B_d}} p(f(w)),$$

or,

$$p(f(w)) \le c_{p,1}|\mathrm{Re}\, w| + c_{p,2}, \quad \text{if } |\mathrm{Im}\, w| \le d,$$

for some positive constants $c_{p,1}$ and $c_{p,2}$. Once we have this global a priori estimate for f, the remainder of the proof is almost a repetition of the

Phragmén-Lindelöf argument in the proof of Proposition 2.6, and we omit the details. Note that the maximum principle for holomorphic functions holds also in E-valued case if we replace $|\cdot|$ on $\mathbb{C}$ by an arbitrarily fixed continuous seminorm on E. □

We also define the notion of periodic operator.

DEFINITION 3.9. Let ω be a positive constant, K a closed interval in $\mathbb{R}$, I a connected open neighborhood of 0 in $\mathbb{R}$, and $U = \mathbb{D}^1 + iI$ a tube domain in $\mathbb{D}^1 + i\mathbb{R}$. An operator $P = \{P_V\}_{V \subset U}$ of type K on U for ${}^E\mathcal{O}_{L^\infty}$ is called ω-periodic, if it commutes with T_ω, that is, the diagram

$$\begin{array}{ccc} {}^E\mathcal{O}_{L^\infty}(V+\omega+K) & \xrightarrow{P_{V+\omega}} & {}^E\mathcal{O}_{L^\infty}(V+\omega) \\ {\scriptstyle T_\omega}\downarrow & & \downarrow{\scriptstyle T_\omega} \\ {}^E\mathcal{O}_{L^\infty}(V+K) & \xrightarrow{P_V} & {}^E\mathcal{O}_{L^\infty}(V), \end{array} \tag{3.8}$$

commutes for any $V \subset U$.

The commutative diagrams (3.8) induce a commutative diagram

$$\begin{array}{ccc} {}^E\mathcal{B}_{L^\infty}(\Omega+\omega+K) & \xrightarrow{P_{\Omega+\omega}} & {}^E\mathcal{B}_{L^\infty}(\Omega+\omega) \\ {\scriptstyle T_\omega}\downarrow & & \downarrow{\scriptstyle T_\omega} \\ {}^E\mathcal{B}_{L^\infty}(\Omega+K) & \xrightarrow{P_\Omega} & {}^E\mathcal{B}_{L^\infty}(\Omega), \end{array} \tag{3.9}$$

for any $\Omega \subset \mathbb{D}^1$. Note that ω-periodic operators preserve the ω-periodicity of their operands.

Example 3.10. Let F be as in Definition 3.3, and $U := \mathbb{D}^1 + iB_d$ a tube domain. Consider $P(w, \partial_w) \in {}^F\mathcal{D}_{L^\infty}(U)$ given by

$$P(w, \partial_w) = \sum_{j=0}^{m} a_j(w)\partial_w^j,$$

where a_j are ω-periodic. Then P becomes an ω-periodic operator of type $\{0\}$.

Similarly, consider an operator P of the form (3.3) on a tube domain U, where the coefficients a_j are ω-periodic. Then P becomes an ω-periodic operator of type K.

The isomorphism (3.6) in Proposition 3.8 plays an important role in the study of ω-periodic solutions to ω-periodic equations. Consider an ω-periodic equation

$$Pu = f \quad \text{on } \mathbb{R}, \tag{3.10}$$

where P is an ω-periodic operator of type K and $f \in {}^E\mathcal{B}(\mathbb{R})$ is an ω-periodic E-valued hyperfunction. From Proposition 3.8, f has naturally an ω-periodic extension $\tilde{f} \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ and we can also consider an ω-periodic equation

$$P\tilde{u} = \tilde{f} \quad \text{on } \mathbb{D}^1. \tag{3.11}$$

If we have an ω-periodic solution $\tilde{u} \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ to (3.11), then the restriction $u := \tilde{u}|_{\mathbb{R}}$ becomes an ω-periodic solution to (3.10). On the other hand, assume that we are given an ω-periodic solution $u \in {}^E\mathcal{B}(\mathbb{R})$ to (3.10). Then, again from Proposition 3.8, u admits an ω-periodic extension $\tilde{u} \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1)$. Now we claim that $\tilde{u}$ is an ω-periodic solution to (3.11). In fact, since $P\tilde{u} - \tilde{f} \in {}^E\mathcal{B}_{L^\infty}(\mathbb{D}^1)$ is ω-periodic and vanishes on $\mathbb{R}$, the injectivity of (3.6) implies that $P\tilde{u} - \tilde{f}$ vanishes also on $\mathbb{D}^1$.

4. Massera Type Theorem

In this section, we give our main result after a preparation of a Montel type theorem.

Let E be a sequentially complete separated locally convex space. We consider the following Montel property for E as in Zubelevich [19].

(M) Any bounded sequence in E has a convergent subsequence.

Theorem 4.1. *Assume that E satisfies the Montel property* (M)*. Then for any bounded sequence $(f_j)_j$ in ${}^E\mathcal{O}_{L^\infty}(U)$, we can take a subsequence $(f_{j_k})_k$ which converges in ${}^E\mathcal{O}(U \cap \mathbb{C})$. The limit $f \in {}^E\mathcal{O}(U \cap \mathbb{C})$ of such a convergent subsequence belongs to ${}^E\mathcal{O}_{L^\infty}(U)$.*

Proof. Note that the second statement follows immediately from the first one. Moreover, since a bounded sequence in ${}^E\mathcal{O}_{L^\infty}(U)$ is bounded in ${}^E\mathcal{O}(U \cap \mathbb{C})$, it suffices to show the existence of a convergent subsequence in the case $U \subset \mathbb{C}$. That is, what we should prove is that any bounded sequence in ${}^E\mathcal{O}(U)$ has a convergent subsequence.

The proof goes in almost the same way as in the $\mathbb{C}$-valued case, but we give it here, in order to clarify that the topology of E may be given by uncountably many seminorms, and that the sequential completeness and the Montel property of E are enough.

First we fix a countable dense subset Q of U. Then from the assumption (M) for E, we can take a subsequence $(f_{j_k})_k$ such that $(f_{j_k}(w))_k$ is convergent for any $w \in Q$, by a standard diagonal method. It suffices to show that $(f_{j_k})_k$ is a convergent sequence. Thus we may assume, from the beginning, that $(f_j(w))_j$ converges for any $w \in Q$. In what follows we will prove that $(f_j)_j$ converges in ${}^E\mathcal{O}(U)$.

For any compact $K \subset U$ and any continuous seminorm p of E, the boundedness of $(f_j)_j$ implies the uniform boundedness of $(f_j)_j$ on K:

$$p(f_j(w)) \le C_{K,p} \quad \text{for } w \in K,\ j \in \mathbb{N},$$

where $C_{K,p} := \sup_j \|f_j\|_{K,p} < +\infty$. We can also show that $\sup_j \|\partial f_j\|_{K,p}$ is finite. In fact, we can take $d > 0$ such that $K_d := \{w \in \mathbb{C}; \mathrm{dist}(w, K) \le d\} \Subset U$. Then from the Cauchy estimate, it follows that

$$\sup_j \|\partial f_j\|_{K,p} \le \sup_j d^{-1} \|f_j\|_{K_d,p} \le d^{-1} C_{K_d,p}.$$

We choose such $d_K := d$ for each K and denote $d^{-1}C_{K_d,p}$ by $C'_{K,p}$. Note that for any $w \in K$ and $w' \in \mathbb{C}$ with $|w - w'| < d_K$, the line segment connecting them is included in K_{d_K}, from which follows the uniform equicontinuity of $(f_j)_j$ on K:

$$p(f_j(w) - f_j(w')) \le C'_{K,p}|w - w'| \quad \text{for } w \in K,\ |w - w'| < d_K,\ j \in \mathbb{N}.$$

Let $K \Subset U$ be again an arbitrary compact set, on which we want to show the uniform convergence of $(f_j)_j$. Replacing K by a neighborhood of K, we may assume that $K \cap Q$ is dense in K. From the pointwise convergence of $(f_j)_j$ on $K \cap Q$ and the equicontinuity of $(f_j)_j$ on K, we can show, using the Ascoli-Arzelà argument, that for any continuous seminorm p of E and for any $\varepsilon > 0$, there exists a number N such that

$$p(f_j(w) - f_k(w)) < \varepsilon \quad \text{for } j, k \geq N,\, w \in K. \tag{4.1}$$

Then it follows from (4.1) and the sequential completeness of E that $(f_j)_j$ converges uniformly on K. This concludes the proof. □

REMARK 4.2. Under the hypothesis of the theorem, $(f_j)_j$ may not have a convergent subsequence in ${}^E\mathcal{O}_{L^\infty}(U)$, even in the $\mathbb{C}$-valued case.

Consider, in fact, a sequence $(f_j)_j$ in $\mathcal{O}_{L^\infty}(U)$ given by $f_j(w) := e^{iw/j}$ for the case $E = \mathbb{C}$ and $U = \mathbb{D}^1 + i]0, \infty[$. It follows from the estimate

$$\sup_{\operatorname{Im} w > 0} |f_j(w)| = 1$$

that $f_j \in \mathcal{O}_{L^\infty}(U)$ and they are uniformly bounded. We can easily see that $(f_j)_j$ converges to 1 uniformly on any compact subset in the upper half plane. Therefore, if some subsequence $(f_{j_k})_k$ converges in $\mathcal{O}_{L^\infty}(U)$, then the limit must be 1 and the estimate

$$\lim_{k\to\infty} \sup_{1/N \leq \operatorname{Im} w \leq N} |f_{j_k}(w) - 1| = 0.$$

must hold for any N. But, since

$$\sup_{1/N \leq \operatorname{Im} w \leq N} |f_j(w) - 1| \geq |f_j(\pi j + i) - 1| = |-e^{-1/j} - 1| \xrightarrow{j\to\infty} 2,$$

no subsequence can converge to 1.

Now we state our main theorem.

THEOREM 4.3. *Let E be a sequentially complete separated locally convex space, K a closed interval in $\mathbb{R}$, and ω a positive number. Consider an ω-periodic operator P of type K on $\mathbb{D}^1 + iB_d$ for ${}^E\mathcal{O}_{L^\infty}$ with some $d > 0$ and an ω-periodic E-valued hyperfunction f. Assume that E satisfies the Montel property* (M). *Then the equation $Pu = f$ has an ω-periodic E-valued hyperfunction solution if and only if it has an ${}^E\mathcal{B}_{L^\infty}$-solution in a neighborhood of $+\infty$.*

PROOF. In view of Proposition 3.8, the necessity is trivial. Thus we will prove the sufficiency.

Assume that the equation $Pu = f$ has an ${}^E\mathscr{B}_{L^\infty}$-solution u on a neighborhood of $+\infty$. Then, also from Proposition 3.8, we can take $\tilde{u} \in {}^E\mathcal{O}_{L^\infty}(\dot{U} + K)$, $\tilde{f} \in {}^E\mathcal{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_{d'})$, and $g \in {}^E\mathcal{O}_{L^\infty}(U)$ such that $\tilde{f}$ is an ω-periodic defining function of f and that

$$P\tilde{u} - \tilde{f} = g \quad \text{in } {}^E\mathcal{O}_{L^\infty}(\dot{U}),$$

where U and $\dot{U}$ are given by

$$U :=]a, +\infty] + iB_{d'}, \quad \dot{U} :=]a, +\infty] + i\dot{B}_{d'} = U \setminus \mathbb{D}^1,$$

with some $a \in \mathbb{R}$ and $0 < d' < d$. Since P and $\tilde{f}$ are ω-periodic, we have

$$P(T_{j\omega}\tilde{u}) - \tilde{f} = T_{j\omega}g \tag{4.2}$$

for any $j \in \mathbb{N}$ as sections in ${}^E\mathcal{O}_{L^\infty}(\dot{U} + (-j\omega))$, but the commutativity of P with restrictions asserts that the equality (4.2) holds in ${}^E\mathcal{O}_{L^\infty}(\dot{U})$. We will continue using the commutativity of operators of type K with restrictions without an explicit mention each time.

Now we consider the seminorms $\|\cdot\|_{L,p}$ of ${}^E\mathcal{O}_{L^\infty}(\dot{U}+K)$ (resp. ${}^E\mathcal{O}_{L^\infty}(U)$), where L runs through compact sets of the form

$$L = \{t' + is; {}^\exists t, t + is \in L_0, t \leq t' \leq +\infty\} \tag{4.3}$$

with some $L_0 \Subset \dot{U} + K$ (resp. $L_0 \Subset U$), and p runs through continuous seminorms of E. Since L satisfies $L_0 \subset L \Subset \dot{U} + K$ (resp. $L_0 \subset L \Subset U$) and $L + (-\omega) \supset L$, these seminorms defines the topology of ${}^E\mathcal{O}_{L^\infty}(\dot{U} + K)$ (resp. ${}^E\mathcal{O}_{L^\infty}(U)$), and the translation operators $T_{j\omega}$ ($j \in \mathbb{N}$) admit

$$\|T_{j\omega}\tilde{u}\|_{L,p} \leq \|\tilde{u}\|_{L,p}, \quad \|T_{j\omega}g\|_{L,p} \leq \|g\|_{L,p}. \tag{4.4}$$

We define $S_k = \frac{1}{k}\sum_{j=0}^{k-1} T_{j\omega}$ and consider the sequences $(S_k\tilde{u})_{k\geq 1}$ in ${}^E\mathcal{O}_{L^\infty}(\dot{U} + K)$ and $(S_kg)_{k\geq 1}$ in ${}^E\mathcal{O}_{L^\infty}(U)$. Then it follows from (4.4) that they are bounded, and the equalities (4.2) for $j = 0, \ldots, k-1$ directly yield

$$PS_k\tilde{u} - \tilde{f} = S_kg \quad \text{in } {}^E\mathcal{O}_{L^\infty}(\dot{U}),$$

and also in ${}^{E}\mathcal{O}(\dot{U} \cap \mathbb{C})$.

By applying Theorem 4.1 twice, we can obtain a subsequence $(k(l))_l$ of $(k)_k$ such that $(S_{k(l)}\tilde{u})_l$ converges to a section $v \in {}^{E}\mathcal{O}_{L^\infty}(\dot{U} + K)$ in the topology of ${}^{E}\mathcal{O}((\dot{U} + K) \cap \mathbb{C})$, and that $(S_{k(l)}g)_l$ converges to a section $h \in {}^{E}\mathcal{O}_{L^\infty}(U)$ in the topology of ${}^{E}\mathcal{O}(U \cap \mathbb{C})$. The continuity of the action of P on ${}^{E}\mathcal{O}((\dot{U} + K) \cap \mathbb{C})$ gives us

$$Pv - \tilde{f} = h \quad \text{in } {}^{E}\mathcal{O}(\dot{U} \cap \mathbb{C}),$$

which implies the same equality in ${}^{E}\mathcal{O}_{L^\infty}(\dot{U})$. In particular, $[v]$ becomes another ${}^{E}\mathcal{B}_{L^\infty}(]a, +\infty])$-solution to the equation $Pu = f$.

Finally we show the periodicity of $[v]$. We have by a direct calculation that $(T_\omega - 1)S_k\tilde{u} = k^{-1}(T_{k\omega} - 1)\tilde{u}$, which yields

$$\|(T_\omega - 1)S_{k(l)}\tilde{u}\|_{L,p} \leq \frac{2\|\tilde{u}\|_{L,p}}{k(l)} \to 0 \quad \text{as } l \to \infty \tag{4.5}$$

for any $L \Subset \dot{U}$ of the form (4.3) and any continuous seminorm p of E. Therefore (4.5) holds also for any $L \Subset \dot{U} \cap \mathbb{C}$, and $(T_\omega - 1)v = 0$ follows by taking the limit in ${}^{E}\mathcal{O}(\dot{U} \cap \mathbb{C})$. This concludes the proof in view of Proposition 3.8. □

We give a remark.

REMARK 4.4. Let F be as in Definition 3.3 and $P(w, \partial_w) \in {}^{F}\mathcal{D}_{L^\infty}(\mathbb{D}^1 + iB_d)$ an ω-periodic operator of the form

$$P(w, \partial_w) = \sum_{j=0}^{m} a_j(w)\partial_w^j.$$

Assume that the equation $Pu = f$ as in Theorem 4.3 admits a classical bounded solution u_0, that is, a bounded C^m-class map $u_0 :]a, +\infty[\to E$ satisfying $\sum_{j=0}^{m} a_j(t)\partial_t^j u_0 = f(t)$. Then from Proposition 2.7, u_0 can be embedded into ${}^{E}\mathcal{B}_{L^\infty}(]a, +\infty])$ and the assumptions of Theorem 4.3 are fulfilled.

On the other hand, ω-periodic hyperfunction solutions to ω-periodic differential equations may be singular generalized functions or unbounded functions. For example,

- $\sum_{n\in\mathbb{Z}}\delta(t+2n)$ is 2-periodic and solves $(\sin\pi t)u=0$.
- $\tan(t+i0)$ is π-periodic and solves $(\cos^2 t)\partial_t u=1$.
- $\exp\tan(t+i0)$ is π-periodic and solves $((\cos^2 t)\partial_t-1)u=0$.

In the classical Massera theorem 1.1, solutions are necessarily C^1-class since the differential equation is non-singular. We will see in the examples given in the next section that some additional ellipticity assumptions for partial differential equations enable our main theorem to give results about classical solutions.

5. Applications

We give some applications of Theorem 4.3 to partial differential equations. In what follows, we often use the conventions $\mathscr{O}(\Omega;E)={}^E\mathscr{O}(\Omega)$, $\mathscr{O}_{L^\infty}(\Omega;E)={}^E\mathscr{O}_{L^\infty}(\Omega)$, $\mathscr{B}(\Omega;E)={}^E\mathscr{B}(\Omega)$, $\mathscr{B}_{L^\infty}(\Omega;E)={}^E\mathscr{B}_{L^\infty}(\Omega)$ and so on, in order to avoid the use of a heavy pre-superscript.

Let us consider the spaces $\mathbb{R}^n$ with coordinates $x=(x_1,\dots,x_n)$ and $\mathbb{C}^n$ with coordinates $z=(z_1,\dots,z_n)$. We study the sheaf ${}^E\mathscr{B}_{L^\infty}$ in the case $E=\mathscr{A}(V)$ for some open domain $V\subset\mathbb{R}^n$, and in the case $E=\mathscr{O}(V)$ for some open domain $V\subset\mathbb{C}^n$.

The space $\mathscr{O}(V)$ is endowed with the usual Fréchet-Schwartz topology, and then the completeness and the Montel property for $\mathscr{O}(V)$ are clear.

As for $\mathscr{A}(V)$, we endow it with a locally convex topology by a formula

$$\mathscr{A}(V)=\varprojlim_{K\Subset V}\mathscr{A}(K), \tag{5.1}$$

where K runs through compact subsets in V and each $\mathscr{A}(K)$ is endowed with the usual dual Fréchet-Schwartz topology. It is well-known that $\mathscr{A}(V)$ also admits an inductive limit representation $\mathscr{A}(V)\simeq\varinjlim_{W\supset V}\mathscr{O}(W)$ as a locally convex space, where W runs through complex neighborhoods of V, but we use only the fact that $\varinjlim_{W\supset V}\mathscr{O}(W)\to\mathscr{A}(V)$ is continuous, which follows immediately from the definition. Though $\mathscr{A}(V)$ is not a Fréchet space, we can characterize, from (5.1) and the continuity mentioned above, a bounded set X in $\mathscr{A}(V)$ as the image of a bounded set in $\mathscr{O}(W)$ with a complex neighborhood W of V. Therefore the sequential completeness and the Montel property for $\mathscr{A}(V)$ are clear.

5.1. $\mathscr{BA}$-solutions to partial differential equations

We denote by $\mathscr{BA}$ the sheaf of hyperfunctions with real analytic parameter x defined on $\mathbb{R} \times \mathbb{R}^n$, and by $\mathscr{OA}$ the sheaf $\mathscr{O}|_{\mathbb{C}\times\mathbb{R}^n}$. Let $V \subset \mathbb{R}^n$ be an open domain. First we study the relation between $\mathscr{B}(\Omega; \mathscr{A}(V))$ and $\mathscr{BA}(\Omega \times V)$ for $\Omega \subset \mathbb{R}$. Note that for an open set $U \subset \mathbb{C}$, there exists a standard isomorphism $\mathscr{O}(U; \mathscr{A}(V)) \xrightarrow{\sim} \mathscr{OA}(U \times V)$.

PROPOSITION 5.1. (i) *The family of maps* $\mathscr{O}(U; \mathscr{A}(V)) \to \mathscr{OA}(U \times V)$ *for* $U \subset \mathbb{C}$ *induces a standard isomorphism*

$$\imath : \mathscr{B}(\Omega; \mathscr{A}(V)) \xrightarrow{\sim} \mathscr{BA}(\Omega \times V) \tag{5.2}$$

for $\Omega \subset \mathbb{R}$, *and also an isomorphism*

$$\imath : {}^{\mathscr{A}(V)}\mathscr{B} \xrightarrow{\sim} p_*(\mathscr{BA}|_{\mathbb{R}\times V}) \tag{5.3}$$

between sheaves on $\mathbb{R}$, *where* $p : \mathbb{R} \times V \to \mathbb{R}$ *denotes the projection.*

(ii) *If* $f \in \mathscr{B}(\Omega; \mathscr{A}(V))$ *satisfies* $\imath(f) \in \mathscr{A}(\Omega \times V)$, *then* f *is actually a* C^∞*-class map from* Ω *to* $\mathscr{A}(V)$, *that is,* $f \in C^\infty(\Omega; \mathscr{A}(V))$.

PROOF. (i) First we consider a section $f \in \mathscr{B}(\Omega; \mathscr{A}(V))$ which admits a defining function $\tilde{f}(w, x) \in \mathscr{O}(\Omega + i\dot{B}_d; \mathscr{A}(V)) \simeq \mathscr{OA}((\Omega + i\dot{B}_d) \times V)$ with some $d > 0$. From a local version of Bochner's tube theorem, each $\tilde{f}|_{\pm \operatorname{Im} w > 0}$ extends holmorphically to a wedge domain

$$\bigcup_{K \Subset \Omega \times V} \{(t + is, x + iy) \in \mathbb{C} \times \mathbb{C}^n; (t, x) \in K, c_K|y| < \pm s < d/2\},$$

where c_K are positive constants depending on K. Therefore their boundary values make sense and define a section $g := \tilde{f}(t + i0, x) - \tilde{f}(t - i0, x) \in \mathscr{BA}(\Omega \times V)$. The section g does not depend on the choice of a defining function, and the correspondence $f \mapsto g$ defines the map (5.2) and the sheaf morphism (5.3) in view of Proposition 2.3.

Assume that $\imath(f) = 0$ in $\Omega' \times V$ with $\Omega' \Subset \Omega$. Then a defining function $\tilde{f}$ of f on Ω' extends holomorphically to a neighborhood of $\Omega' \times V$. Here we used the fact that the convex hull of the union of the two cones $\{(s, y) \in \mathbb{R} \times \mathbb{R}^n; c_K|y| < \pm s\} \subset \mathbb{R} \times \mathbb{R}^n$ is the whole space $\mathbb{R} \times \mathbb{R}^n$. Therefore, $f = 0$ holds on Ω', from which follows the injectivity of (5.3).

Let us prove the surjectivity of the maps between stalks. Assume that we are given a germ $f_{\dot{t}}$ of $p_*(\mathcal{BA}|_{\mathbb{R}\times V})$ at a point $\dot{t} \in \Omega$ represented by a section $f(t,x) \in \mathcal{BA}(\Omega \times V)$. We fix open intervals I_j $(j = 0,1,2)$ with $\dot{t} \in I_0 \Subset I_1 \Subset I_2 \Subset \Omega$. Then, from the flabbiness of the sheaf of microfunctions and the exact sequence

$$0 \to \mathcal{A}(\Omega \times V) \to \mathcal{B}(\Omega \times V) \to \mathcal{C}((\Omega \times V) \times \dot{\mathbb{R}}^{1+n}) \to 0,$$

where $\dot{\mathbb{R}}^{1+n}$ denotes $\mathbb{R}^{1+n} \setminus \{0\}$, we can find a hyperfunction $g \in \mathcal{B}(\Omega \times V)$ such that

$$(f-g)|_{I_1 \times V} \in \mathcal{A}(I_1 \times V) \tag{5.4}$$

and that

$$\mathrm{WF}_A(g) \subset \overline{\mathrm{WF}_A(f) \cap ((I_1 \times V) \times \dot{\mathbb{R}}^{1+n})}, \tag{5.5}$$

where the closure is taken in $(\Omega \times V) \times \dot{\mathbb{R}}^{1+n}$, and $\mathrm{WF}_A(f)$ denotes the analytic wave front set of f. The estimate (5.5) immediately implies that

$$g|_{(\Omega \setminus \overline{I_1}) \times V} \in \mathcal{A}((\Omega \setminus \overline{I_1}) \times V). \tag{5.6}$$

Since the right hand side of (5.5) does not meet

$$\{(t,x;0,\xi); (t,x) \in \Omega \times V, \xi \neq 0\},$$

we have $g \in \mathcal{BA}(\Omega \times V)$. Now we denote by χ_j the characteristic functions of I_j, $(j = 0,2)$. From (5.4) and (5.6), the multiplications in the right hand side of

$$h := (f-g)\chi_0 + g\chi_2$$

are well-defined, and h satisfies

$$h \in \Gamma_{\overline{I_2} \times V}(\Omega \times V, \mathcal{BA}), \quad h|_{I_0 \times V} = f|_{I_0 \times V}.$$

That is, $f_{\dot{t}}$ has another representative $h \in \mathcal{BA}(\Omega \times V)$ supported in $\overline{I_2} \times V$.

Therefore, we may assume, from the beginning, that $f(t,x) \in \Gamma_{K \times V}(\mathbb{R} \times V, \mathcal{BA})$ with some compact interval $K \subset \mathbb{R}$, and it suffices to show that there exists a function $\tilde{f}(w,x) \in \mathcal{OA}((U \setminus \mathbb{R}) \times V) \simeq \mathcal{O}(U \setminus \mathbb{R}; \mathcal{A}(V))$ with some complex neighborhood U of $\mathbb{R}$, such that $f = \tilde{f}(t+i0,x) - \tilde{f}(t-i0,x)$.

Consider the function

$$\tilde{f}(w,x) := -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t,x)}{w-t}dt \in \mathcal{O}\mathcal{A}((\mathbb{C}\setminus K)\times V). \tag{5.7}$$

Using again a local version of Bochner's tube theorem, each $\tilde{f}|_{\pm \operatorname{Im} w>0}$ extends holomorphically to a suitable wedge domain and defines

$$g(t,x) := \tilde{f}(t+i0,x) - \tilde{f}(t-i0,x) \in \mathcal{B}\mathcal{A}(\mathbb{R}\times V). \tag{5.8}$$

In order to establish the equality $f = g$ in $\mathcal{B}\mathcal{A}(\mathbb{R}\times V)$, it suffices to show $f|_{x=\dot{x}} = g|_{x=\dot{x}}$ for each $\dot{x}$. The equality $g|_{x=\dot{x}} = \tilde{f}(t+i0,\dot{x}) - \tilde{f}(t-i0,\dot{x})$ follows from (5.8). On the other hand, applying the restriction $(\cdot)|_{x=\dot{x}}$ to (5.7), it follows that $\tilde{f}(w,\dot{x})$ is the standard defining function of $f|_{x=\dot{x}} \in \Gamma_K(\mathbb{R},\mathcal{B})$, since $(\cdot)|_{x=\dot{x}}$ commutes with $\int(\cdot)dt$ for hyperfunctions supported in $\{t \in K\}$. Therefore the equality $f|_{x=\dot{x}} = g|_{x=\dot{x}}$ holds. The fact that $f|_{x=\dot{x}} = g|_{x=\dot{x}}$ for any $\dot{x} \in V$ implies $f = g$ was originally proved by Oshima and Kataoka, and written as Theorem 4.4.7' in Kaneko [7]. See also Liess-Okada-Tose [12] for related results.

(ii) Since the statement is local in the t variable, we may assume, from the beginning, that $f \in \mathcal{B}(\Omega;\mathcal{A}(V))$ has a global defining function $\tilde{f} \in \mathcal{O}(\Omega + i\dot{B}_d;\mathcal{A}(V))$ with some $d > 0$. If $\imath(f) \in \mathcal{A}(\Omega\times V)$, then each $\tilde{f}|_{\pm \operatorname{Im} w>0}$ extends holomorphically to a neighborhood of $\Omega\times V$. We denote each extension by $g_\pm(w,z)$. For any open $\Omega' \Subset \Omega$ and for any open $V' \Subset V$, we can take a constant $d' > 0$ such that $g_\pm(w,z)$ are holomorphic in $\Omega'_{d'} \times V'_{d'}$, where $\Omega'_{d'}$ and $V'_{d'}$ are the complex d'-neighborhoods of Ω' and V' respectively. Therefore, we can show that the convergence in

$$\partial_t^j g_\pm(t,z) = \lim_{s\downarrow 0} \partial_t^j \tilde{f}(t \pm is, z) \quad \text{for } (t,z) \in \Omega' \times V'_{d'},\ j \in \mathbb{N},$$

is uniform in the (t,z) variables, which implies $g_\pm \in C^\infty(\Omega;\mathcal{A}(V))$. Since f coincides with the embedded image of $g_+ - g_-$, the conclusion follows. □

We consider, as an example, the following class of differential operators.

PROPOSITION 5.2. *We consider a partial differential operator*

$$P(t,x,\partial_t,\partial_x) = \sum_{(j,\alpha)\in\mathbb{N}^{n+1}}^{\text{finite}} a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha$$

where the coefficients $a_{j,\alpha}(t,x)$ *extend holomorphically to* $a_{j,\alpha}(w,z) \in \mathcal{O}((\mathbb{R}+iB_d)\times W)$ *with some* $d>0$ *and some complex neighborhood* W *of* V. *Assume that* $a_{j,\alpha}$ *is* ω*-periodic in the* w *variable. Then* P *induces an* ω*-periodic operator of type* $\{0\}$ *on* $\mathbb{D}^1 + iB_d$ *for* ${}^{\mathscr{A}(V)}\mathcal{O}_{L^\infty}$.

Proof. We denote by $F[m]$ the space of differential operators in the z variable with $\mathcal{O}(W)$-coefficients of order at most m, endowed with a sequentially complete separated locally convex topology by

$$F[m] = \bigoplus_{|\alpha|\le m} \mathcal{O}(W)\partial_z^\alpha.$$

Then we can easily see that $F[m]$ can be continuously embedded to $\mathcal{L}_b(\mathcal{O}(W))$. We write

$$P = \sum_j P_j(w,z,\partial_z)\partial_w^j, \quad P_j(w,z,\partial_z) := \sum_\alpha a_{j,\alpha}(w,z)\partial_z^\alpha$$

and take $m := \max_j \operatorname{ord}(P_j)$. Then the correspondences $w \mapsto P_j(w,z,\partial_w)$ become ω-periodic sections in $\mathcal{O}(\mathbb{R}+iB_d; F[m])$, and P becomes an ω-periodic section in $\mathscr{D}_{L^\infty}(\mathbb{D}^1 + iB_d; F[m])$. Therefore, from Example 3.10, P induces an ω-periodic operator of type $\{0\}$ on $\mathbb{D}^1 + iB_d$ for ${}^{\mathcal{O}(W)}\mathcal{O}_{L^\infty}$.

Replacing W by W' with $V \subset W' \subset W$ and taking the inductive limit in W', the conclusion follows. □

Corollary 5.3. *Let* $V \subset \mathbb{R}^n$ *be an open set,* P *an operator as in Proposition 5.2,* $f \in \mathscr{B}\mathscr{A}(\mathbb{R}\times V)$ *a hyperfunction with real analytic parameter which is* ω*-periodic in the* t *variable. Then the equation*

$$Pu = f$$

admits a $\mathscr{B}\mathscr{A}(\mathbb{R}\times V)$*-solution* ω*-periodic in the* t *variable if and only if it admits an* $\mathscr{A}(V)$*-valued bounded hyperfunction solution in a neighborhood of* $+\infty$.

Proof. Since we have $\mathscr{B}(\mathbb{R};\mathscr{A}(V)) \xrightarrow{\sim} \mathscr{B}\mathscr{A}(\mathbb{R}\times V)$ from Proposition 5.1, the existence of a $\mathscr{B}\mathscr{A}(\mathbb{R}\times V)$-solution ω-periodic in the t variable is equivalent to the existence of an ω-periodic $\mathscr{B}(\mathbb{R};\mathscr{A}(V))$-solution. Therefore the conclusion follows from Theorem 4.3. □

We also give a result for classical solutions.

COROLLARY 5.4. *We pose the same assumptions as in the previous corollary. Assume moreover that P is of order m and that the real characteristic variety of P is included in $\{(t,x;0,\xi); t\in\mathbb{R}, x\in V, \xi\in\mathbb{R}^n\}$. If the equation $Pu=f$ admits a bounded solution in $C^m(]a,+\infty[;\mathscr{A}(V))$ with some $a\in\mathbb{R}$, then there exists a solution in $C^m(\mathbb{R}\times V)$ real analytic in x and ω-periodic in the t variable.*

Note that the assumption about the real characteristic variety implies that $\mathscr{B}\mathscr{A}$-solutions to the homogeneous equation necessarily belong to $\mathscr{A}$.

As a topological vector space E, we can take a direct sum of $\mathscr{A}(V)$. Then, as an example corresponding to Corollary 5.4, we give

Example 5.5. Let $Q=(Q_{ij})$ be a square matrix of differential operators $Q_{ij}(x,\partial_x)$ in the x variable with $\mathscr{A}(V)$-coefficients. Assume that Q is essentially of order less than 1, that is, there exists a positive integer m such that $\operatorname{ord}(Q^m)<m$. Then Corollary 5.4 is applicable to the operator $P=\partial_t-Q$.

In fact, since Q commutes with ∂_t, the characteristic variety of ∂_t-Q is included in that of $\partial_t^m-Q^m$, which is equal to $\{(t,x;\tau,\xi);\tau=0\}$.

5.2. $\mathscr{B}\mathscr{O}$-solutions to partial differential equations

We denote by $\mathscr{B}\mathscr{O}$ the sheaf of hyperfunctions with holomorphic parameter on $\mathbb{R}\times\mathbb{C}^n$, and by $\mathscr{A}\mathscr{O}$ the sheaf $\mathscr{O}|_{\mathbb{R}\times\mathbb{C}^n}$. Let $V\subset\mathbb{C}^n$ be an open domain, and we study the relation between $\mathscr{B}(\Omega;\mathscr{O}(V))$ and $\mathscr{B}\mathscr{O}(\Omega\times V)$. In this setting, there exists a standard isomorphism $\mathscr{O}(U;\mathscr{O}(V))\to\mathscr{O}(U\times V)$ for an open set $U\subset\mathbb{C}$.

PROPOSITION 5.6. (i) *The family of maps $\mathscr{O}(U;\mathscr{O}(V))\to\mathscr{O}(U\times V)$ for $U\subset\mathbb{C}$ induces the standard embeddings*

$$\imath:\mathscr{B}(\Omega;\mathscr{O}(V))\hookrightarrow\mathscr{B}\mathscr{O}(\Omega\times V).$$

for $\Omega\subset\mathbb{R}$, and

$$\imath:{}^{\mathscr{O}(V)}\mathscr{B}\hookrightarrow p_*(\mathscr{B}\mathscr{O}|_{\mathbb{R}\times V})$$

between sheaves on $\mathbb{R}$, where $p : \mathbb{R} \times V \to \mathbb{R}$ denotes the projection. Moreover, if V is Stein, then these embeddings become isomorphisms.

(ii) *If $f \in \mathcal{B}(\Omega; \mathcal{O}(V))$ satisfies $\imath(f) \in \mathcal{A}\mathcal{O}(\Omega \times V)$, then f is actually a C^∞-class map from Ω to $\mathcal{O}(V)$, that is, $f \in C^\infty(\Omega; \mathcal{O}(V))$.*

Proof. (i) The proof goes in a similar and simpler way as in that of Proposition 5.1(i). Note that since $\mathcal{O}(V)$ is a Fréchet space, the equality

$$\mathcal{B}(\Omega; \mathcal{O}(V)) = \frac{\mathcal{O}(U \setminus \Omega; \mathcal{O}(V))}{\mathcal{O}(U; \mathcal{O}(V))} = \frac{\mathcal{O}((U \setminus \Omega) \times V)}{\mathcal{O}(U \times V)}$$

holds. (See Ion-Kawai [6, §3].) Also note that if V is Stein, $p_*(\mathcal{B}\mathcal{O}|_{\mathbb{R}\times V})$ is flabby. (See Kaneko [7, Theorem 7.4.4].)

(ii) Also a similar argument as in Proposition 5.1(ii) proves this case. □

In this case, we can apply our main theorem to a class of partial differential operators with coefficients ω-periodic in the t variable. The proof is already given in the proof of Proposition 5.2.

Proposition 5.7. *We consider a partial differential operator*

$$P(t, z, \partial_t, \partial_z) = \sum_{(j,\alpha)\in\mathbb{N}^{n+1}}^{\textit{finite}} a_{j,\alpha}(t, z)\partial_t^j \partial_z^\alpha,$$

where the coefficients $a_{j,\alpha}(t, z)$ extend holomorphically to $a_{j,\alpha}(w, z) \in \mathcal{O}((\mathbb{R}+ iB_d)\times V)$ with some $d > 0$. Assume that $a_{j,\alpha}$ is ω-periodic in the w variable. Then the usual action of P on the sheaf $\mathcal{B}\mathcal{O}$ induces an ω-periodic operator of type $\{0\}$ on $\mathbb{D}^1 + iB_d$ for ${}^{\mathcal{O}(V)}\mathcal{O}_{L^\infty}$.

Now we give a result corresponding to Corollary 5.3 in the $\mathcal{B}\mathcal{A}$ case.

Corollary 5.8. *Let $V \subset \mathbb{C}^n$ be a Stein domain, P a partial differential operator as in Proposition 5.7, and $f \in \mathcal{B}\mathcal{O}(\mathbb{R} \times V)$ a hyperfunction with holomorphic parameter which is ω-periodic in the t variable. Then, the equation*

$$Pu = f$$

admits a $\mathscr{BO}(\mathbb{R}\times V)$*-solution* ω*-periodic in the* t *variable if and only if it admits an* $\mathscr{O}(V)$*-valued bounded hyperfunction solution in a neighborhood of* $+\infty$*.*

If hyperplanes $\{t = \text{const.}\}$ are always non-characteristic for the operator P, then $\mathscr{BO}$-solutions to the homogeneous equation necessarily belong to $\mathscr{AO}$. Therefore we can give a result for classical solutions.

COROLLARY 5.9. *We pose the same assumptions as in the previous corollary. Assume moreover that the operator* P *is of order* m *and that*

$$a_{m,0}(t,z) \neq 0 \quad \textit{for } (t,z) \in \mathbb{R}\times V.$$

If the equation $Pu = f$ *admits a solution in* $C^m(]a,+\infty[\times V)$ *with some* $a \in \mathbb{R}$*, holomorphic in the* z *variable and bounded on* $]a,+\infty[\times K$ *for every* $K \Subset V$*, then there exists a solution in* $C^m(\mathbb{R}\times V)$ *holomorphic in the* z *variable and* ω*-periodic in the* t *variable.*

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References

[1] Bochnak, J. and J. Siciak, Analytic functions in topological vector spaces, Studia Math. **39** (1971), 77–112.

[2] Chow, S. N. and J. K. Hale, Strongly limit-compact maps, Funkcial. Ekvac. **17** (1974), 31–38.

[3] Chung, S.-Y., Kim, D. and E. G. Lee, Periodic hyperfunctions and Fourier series, Proc. Amer. Math. Soc. **128**(8) (2000), 2421–2430.

[4] Hino, Y. and S. Murakami, Favard's property for linear retarded equations with infinite delay, Funkcial. Ekvac. **29**(1) (1986), 11–17.

[5] Hino, Y. and S. Murakami, Periodic solutions of a linear Volterra system, In *Differential equations (Xanthi, 1987)*, volume 118 of *Lecture Notes in Pure and Appl. Math.*, pages 319–326, Dekker, New York, 1989.

[6] Ion, P. D. F. and T. Kawai, Theory of vector-valued hyperfunctions, Publ. Res. Inst. Math. Sci. **11**(1) (1975/76), 1–19.

[7] Kaneko, A., *Introduction to hyperfunctions*, volume 3 of *Mathematics and its Applications (Japanese Series)*, Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Japanese by Y. Yamamoto.

[8] Kawai, T., On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **17** (1970), 467–517.
[9] Kim, K. W., Chung, S.-Y. and D. Kim, Fourier hyperfunctions as the boundary values of smooth solutions of heat equations, Publ. Res. Inst. Math. Sci. **29**(2) (1993), 289–300.
[10] Li, Y., Cong, F., Lin, Z. and W. Liu, Periodic solutions for evolution equations, Nonlinear Anal. **36**(3, Ser. A: Theory Methods) (1999), 275–293.
[11] Li, Y., Lin, Z. and Z. Li, A Massera type criterion for linear functional-differential equations with advance and delay, J. Math. Anal. Appl. **200**(3) (1996), 717–725.
[12] Liess, O., Okada, Y. and N. Tose, Hartogs' phenomena for microfunctions with holomorphic parameters, Publ. Res. Inst. Math. Sci. **37**(2) (2001), 221–238.
[13] Massera, J. L., The existence of periodic solutions of systems of differential equations, Duke Math. J. **17** (1950), 457–475.
[14] Sato, M., Chôkansû no riron, Sûgaku **10** (1958), 1–27 (in Japanese).
[15] Sato, M., Theory of hyperfunctions. I, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1959), 139–193.
[16] Sato, M., Theory of hyperfunctions. II, J. Fac. Sci. Univ. Tokyo Sect. I **8** (1960), 387–437.
[17] Sato, M., Kawai, T. and M. Kashiwara, Microfunctions and pseudo-differential equations, In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pages 265–529, Lecture Notes in Math., Vol. 287, Springer, Berlin, 1973.
[18] Shin, J. S. and T. Naito, Semi-Fredholm operators and periodic solutions for linear functional-differential equations, J. Differential Equations **153**(2) (1999), 407–441.
[19] Zubelevich, O., A note on theorem of Massera, Regul. Chaotic Dyn. **11**(4) (2006), 475–481.

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