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On Logarithmic Hodge-Witt Cohomology of Regular Schemes

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Abstract. In this paper, we prove the purity of the logarithmic Hodge-Witt cohomology for an excellent regular pair of characteristic $p > 0$ and the Gersten-type conjecture for the p-primary part of the Kato complex (the arithmetic Bloch-Ogus complex) of the spectrum of an excellent regular local ring of characteristic $p > 0$. They are generalizations of results of Gros and Suwa to regular schemes which are not necessarily smooth over a perfect field.

Contents

1. Introduction

Let p be a prime. In this paper, we prove the following two theorems concerning the logarithmic Hodge-Witt cohomology of regular schemes: First, we prove the purity for the logarithmic Hodge-Witt cohomology of an excellent regular pair $Z \hookrightarrow X$ of characteristic p. Second, we prove the Gersten-type conjecture for the p-primary part of the Kato complex (the arithmetic Bloch-Ogus complex in [Kat3]) of the spectrum of an excellent regular local ring of characteristic p. The first theorem is proved by Gros and Suwa ([G], [Su]) in the case of smooth pairs over a perfect field and the second theorem is also essentially proved by them ([G-Su]) when the spectrum in consideration is the localization of a smooth scheme over a perfect field of

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characteristic $p > 0$. (See also a recent work of Jannsen-Saito-Sato [J-Sai-Sat].) So, our result is a generalization of their results and the proof is done by reducing to them.

Let us explain our theorems briefly. Fix a non-negative integer N and let X be an equidimensional excellent regular scheme of characteristic p such that $[\kappa(x) : \kappa(x)^p] = p^N$ holds for any generic point x of X. Then we can define the logarithmic Hodge-Witt sheaf $W_m\Omega^j_{X,\log}$ (see Section 2). It is expected that there exists the following canonical isomorphism for a regular closed immersion $Z \hookrightarrow X$ of pure codimension r

$$H^q(Z, W_m\Omega^{i-r}_{Z,\log}) \xrightarrow{\sim} H^{q+r}_Z(X, W_m\Omega^i_{X,\log})$$

in the case $q = 0$ or $q > 0, i = N$. (This does not hold in the case $q > 0, i \neq N$. See [G, p.45, p.48].) In this paper, we prove this expectation (which is called purity) is true. It is known when $Z \hookrightarrow X$ is a smooth pair over a perfect field (Gros [G, II,Thm 3.5.8, (3.5.19)], Suwa [Su, Cor 2.6]). Note also that the similar theorem in the case of l-adic etale cohomology is already known ([SGA 4] in smooth case, [F1] in equicharacteristic case and [F2] in general case).

Next we explain the Gersten-type conjecture for Kato complex. For a field k of characteristic p, we have the logarithmic Hodge-Witt sheaf $W_m\Omega^i_{\mathrm{Spec}\,k,\log}$. On the other hand, let $K_i^{\mathrm{M}}(k)$ be the Milnor K-group of k. Then, by Bloch-Gabber-Kato theorem, the symbol map induces the isomorphism

$$K_i^{\mathrm{M}}(k)/p^m \xrightarrow{\sim} H^0(k, W_m\Omega^i_{\mathrm{Spec}\,k,\log}).$$

Moreover, there is an interpretation of the first cohmology group $H^1(k, W_m\Omega^i_{\mathrm{Spec}\,k,\log})$ in terms of the typical part of K-groups introduced by Bloch. ([Bl], [Kat1]. See also [J-Sai-Sat].) So there is a close relation between the logarithmic Hodge-Witt cohomologies and K-groups.

Let $m \in \mathbb{N}, s, i \in \mathbb{Z}$ and let X be an excellent scheme over $\mathbb{F}_p$ satisfying the following condition:

(∗) When $s = i + 1$ holds, we have $[\kappa(x) : \kappa(x)^p] \leq p^i$ for any closed point $x \in X$.

For such X, Kato defined in [Kat3] the complex of the form

$$
\begin{aligned}
C^{s,i}_{p^m,X}: \quad \cdots \longrightarrow \bigoplus_{x \in X_j} H^{s-i}(x, W_m\Omega^{i+j}_{x,\log}) \longrightarrow \cdots \\
\longrightarrow \bigoplus_{x \in X_1} H^{s-i}(x, W_m\Omega^{i+1}_{x,\log}) \longrightarrow \bigoplus_{x \in X_0} H^{s-i}(x, W_m\Omega^{i}_{x,\log}) \longrightarrow 0
\end{aligned}
$$

(where the last non-zero term is sitting at degree 0) by using K-theoretic method. This complex can be regarded as a generalization of the p-primary part of the Hasse principle for a funtion field to higher dimensional case. (In fact, Kato defines the complex $C^{s,i}_{n,X}$ for any $n \in \mathbb{N}$ and for any excellent scheme X satisfying the condition similar to $(*)$.) He gave conjectures on the cohomology of the complex $C^{s,i}_{n,X}$ (particularly in the case $s = i+1$) and proved them in certain cases. (For the precise form of the conjectures and the known results, see [Kat3], [CT-Sa-So], [CT], [Sai2], [J-Sai] and a recent work of Jannsen-Saito.)

Now let us fix non-negative integers n, N with $n \leq N$ and assume moreover that X is of pure dimension n satisfying $[\kappa(x) : \kappa(x)^p] = p^N$ for any generic point x of X. In this case, we denote the shift by $-n$ of the complex $C^{i-n+q,i-n}_{p^m,X}$ by $C^{q,i}_m(X)^\bullet$. (By the condition $(*)$, it is defined and non-zero only if $q = 0$ or $(q,i) = (1,N)$.) The complex $C^{q,i}_m(X)^\bullet$ is regarded as the analogue of Brown-Gersten-Quillen complex in algebraic K-theory. So, as an analogue of Gersten conjecture, one can expect that the following claim is true: Assume moreover that X is the spectrum of an excellent regular local ring. Then we have

$$
H^r(C^{q,i}_m(X)^\bullet) = \begin{cases} H^q(X, W_m\Omega^i_{X,\log}), & r = 0, \\ 0, & r > 0. \end{cases}
$$

In this paper, we prove that this claim is true (when $q = 0$ or $(q,i) = (1,N)$ holds).

Let us explain the method of the proof. The key ingredient for the proof of the purity is the following two propositions: The first one is a result of Popescu ([Po1], [Po2], [Po3], [Ogo]) which says that any regular local ring of characteristic $p > 0$ can be written as a filtering inductive limit of finitely generated smooth algebras over $\mathbb{F}_p$. (This is used by Panin [Pa] to prove the equicharacteristic case of the Gersten conjecture for K-theory.)

The second one is the proposition which claims that a regular scheme X such that the absolute Frobenius $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}$ is finite for any point $x \in X$ is, flat relatively perfect locally, isomorphic to an affine space over $\mathbb{F}_p$. (See Definition 2.14 and Remark 2.16.) Using the first proposition, we can reduce the proof of purity for $q = 0$ to the smooth case. With some more calculation using the second proposition, we can prove the purity for $q > 0, i = N$.

The key to the proof of the Gersten-type conjecture for Kato complex is the Bloch-Ogus complex (denoted by $B_m^{q,i}(X)^\bullet$)

$$\begin{aligned} 0 \longrightarrow \bigoplus_{x \in X^0} H_x^q(X, W_m\Omega^i_{X,\log}) &\longrightarrow \bigoplus_{x \in X^1} H_x^{q+1}(X, W_m\Omega^i_{X,\log}) \longrightarrow \cdots \\ \longrightarrow \bigoplus_{x \in X^s} H_x^{q+s}(X, W_m\Omega^i_{X,\log}) &\longrightarrow \cdots, \end{aligned}$$

which is defined as the complex of $E_1^{\bullet,q}$-terms of the coniveau spectral sequence ([Bl-Ogu], see also [CT-Ho-Kah])

$$E_1^{s,t} = \bigoplus_{x \in X^s} H_x^{s+t}(X, W_m\Omega^i_{X,\log}) \Longrightarrow E^{s+t} = H^{s+t}(X, W_m\Omega^i_{X,\log}).$$

First we prove that the Bloch-Ogus complex satisfies the Gersten-type conjecture: In the case where X is a localization of a smooth scheme over a perfect field, it is due to Gros-Suwa ([G-Su, Thm 1.4]). In general case, we prove it by using a technique of Panin in [Pa]. Via the purity theorem which we already proved, each term of Kato complex $C_m^{q,i}(X)^\bullet$ is isomorphic to that of Bloch-Ogus complex $B_m^{q,i}(X)^\bullet$. So we expect that the purity isomorphism induces an isomorphism of complexes $C_m^{q,i}(X)^\bullet \xrightarrow{\sim} B_m^{q,i}(X)^\bullet$ up to sign. (If it is true, then the Gersten conjecute for Kato complex is true.) This expectation means the coincidence between the K-theoretically defined complex $C_m^{q,i}(X)^\bullet$ and the sheaf-theoretically defined complex $B_m^{q,i}(X)^\bullet$, and so we think it is interesting itself. Gros-Suwa ([G-Su, Rem 4.19]) and Suwa ([Su, Rem 1.3, 2.12]) claimed that this is true for smooth X, but their proofs seem to be incomplete. Recently, Jansenn-Saito-Sato have given a complete proof for smooth X, by using the trace map for logarithmic de Rham-Witt cohomology developped by Ekedahl ([E]) and Gros ([G]). In this paper, we prove that this expectation is true for $m = 1$ and excellent regular X, by using the trace map for generalized residual complex developped by Hartshorne ([Ha],

[Ha2]). This result is sufficient to deduce the Gersten-type conjecture for the Kato complex. If we can develop a satisfactory theory of trace maps for logarithmic Hodge-Witt cohomology for regular schemes, we will be able to prove the coincidence (up to explicit sign) of the Bloch-Kato complex and the Kato complex for arbitrary m and excellent regular X. We hope to do it in a future paper.

The results in this paper seem to be useful if one would like to study the arithmetic of the spectrum of excellent regular rings of characteristic $p > 0$ or smooth schemes over them. In fact, it seems to the author that our results were already used, for example, in [Sai1]. They are used also in [Ma].

The content of each section is as follows: In Section 2, we give a review of the de Rham-Witt complex and the logarithmic Hodge-Witt sheaf. We extend some basic properties of them to the case of regular schemes. In Section 3, we give a proof of the purity. In Section 4, we give a proof of the Gersten-type conjecture for the Bloch-Ogus complex. In Section 5, we compare the Bloch-Ogus complex and the Kato complex and deduce the Gersten-type conjecture for the Kato complex.

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Notation. Throughout this paper, p will be a fixed prime, unless otherwise stated. For integers a, b, we denote the set $\{n \in \mathbb{Z} \,|\, a \leq n \leq b\}$ simply by $[a, b]$. For a scheme X, we denote the set of points of codimension i (resp. dimension i) by X^i (resp. X_i). For a scheme X over $\mathbb{F}_p$, we denote the differential module $\Omega^i_{X/\mathbb{F}_p}$ simply by Ω^i_X and we denote

$\mathrm{Ker}(d : \Omega^i_X \longrightarrow \Omega^{i+1}_X)$ simply by $Z\Omega^i_X$. For a complex $C := (C^\bullet, d^\bullet)$, we denote the complex $(C^{\bullet+n}, (-1)^n d^{\bullet+n})$ by $C[n]$ and the complex $(C^{\bullet+n}, d^{\bullet+n})$ by $C\{n\}$. A diagram of (sheaves of) abelian groups of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{g'} & D \end{array}$$

is said to be $(-1)^n$-commutative ($n \in \mathbb{Z}$) if we have $g \circ f(a) = (-1)^n g' \circ f'(a)$ for any $a \in A$. Sheaves are considered on small etale site unless otherwise stated. (Note that the exceptions occur in Section 5.)

2. Preliminaries

In this section, we give the definition of the logarithmic Hodge-Witt sheaf for a regular scheme of characteristic p and prove some basic properties of the de Rham-Witt complexes and the logarithmic Hodge-Witt sheaves of regular schemes. The results in this section are known in the case of smooth schemes over a perfect field. So our task is to extend them to the regular case by reducing to the smooth case.

First, let us recall two results which are important when we try to reduce the propositions to the smooth case. The first one is the following theorem of Popescu ([Po1], [Po3], [Ogo]):

THEOREM 2.1 (Popescu). *Any regular local ring R of characteristic p can be written as a filtering inductive limit $\varinjlim_\lambda R_\lambda$ of finitely generated smooth algebras over $\mathbb{F}_p$.*

The second one is the following theorem of Grothendieck ([SGA 4, VII,Thm 5.7],[Pa, Thm 6.6]):

THEOREM 2.2. *Let X be a Noetherian scheme and let $\{X_i\}_{i\in I}$ be a filtering projective system of Noetherian schemes such that each transition morphism is affine and that $X = \varprojlim_{i\in I} X_i$ holds. Let us denote the canonical projection $X \longrightarrow X_i$ by φ_i. Let $\{F_i\}$ be a compatible system of sheaves on $\{X_{i,\tau}\}$ (where $\tau =$ Zar or et), and put $F := \varinjlim_{i\in I} \varphi_i^{-1}(F_i)$. Then we* *have the isomorphism*

$$H^j(X_\tau, F) \overset{\cong}{=} \varinjlim_{i\in I} H^j(X_{i,\tau}, F_i).$$

For a scheme X of characteristic p, let $W_m\Omega_X^\bullet$ be the de Rham-Witt complex of X. The degree 0 part $W_m\Omega_X^0$ is written by $W_m\mathcal{O}_X$ and it is called the sheaf of rings of Witt vectors of $\mathcal{O}_X$. Let W_mX be the Witt scheme (that is, the ringed space $(|X|, W_m\mathcal{O}_X)$). It is known that $W_m\Omega_X^i$ is a quasi-coherent $W_m\mathcal{O}_X$-module for any i. For $x \in \mathcal{O}_X$, let $\underline{x} := (x, 0, \cdots, 0) \in W_m\mathcal{O}_X$ be the Teichmüller representative.

Denote the differential $W_m\Omega_X^i \longrightarrow W_m\Omega_X^{i+1}$ by d and let

$$\begin{aligned} &F : W_m\Omega_X^\bullet \longrightarrow W_{m-1}\Omega_X^\bullet, \quad V : W_m\Omega_X^\bullet \longrightarrow W_{m+1}\Omega_X^\bullet, \\ &R : W_m\Omega_X^\bullet \longrightarrow W_{m-1}\Omega_X^\bullet \end{aligned}$$

be Frobenius, Verschiebung and the projection of $W_\bullet\Omega_X^\bullet$, respectively. The Frobenius operator induces the endomorphism $W_m\mathcal{O}_X \longrightarrow W_m\mathcal{O}_X$, which we denote also by F. For precise definition and the basic properties of $W_m\mathcal{O}_X$, $W_m\Omega_X^\bullet, F, V, R$, see [I].

For $m, n \in \mathbb{N}$, the canonical filtration $\mathrm{Fil}^n W_m\Omega_X^\bullet$ of $W_m\Omega_X^\bullet$ is defined in the following way:

$$\mathrm{Fil}^n W_m\Omega_X^\bullet := \begin{cases} W_m\Omega_X^\bullet, & \text{if } n = 0, \\ \mathrm{Ker}(R^{m-n} : W_m\Omega_X^\bullet \longrightarrow W_n\Omega_X^\bullet), & \text{if } 1 \le n < m, \\ 0. & \text{if } n \ge m. \end{cases}$$

Then we have the following proposition (cf. [I, I.3.2]).

Proposition 2.3. *Let X be a regular scheme over $\mathbb{F}_p$. Then we have the equality*

$$\mathrm{Fil}^n W_m\Omega_X^i = V^n W_{m-n}\Omega_X^i + dV^n W_{m-n}\Omega_X^{i-1}.$$

Proof. It is easy to see that the right hand side is contained in the left hand side. Let us prove the converse. To prove it, we may assume that

X is strictly local. Let $\{X_j\}_{j\in J}$ be a projective system of affine schemes which are smooth over $\mathbb{F}_p$ satisfying $X \cong \varprojlim_{j\in J} X_j$. (The existence of such a system is assured by Theorem 2.1.) Then, by Theorem 2.2, we have

$$\begin{aligned} H^0(X, \mathrm{Fil}^n W_m\Omega^i_X) &\cong \varinjlim_{j\in J} H^0(X_j, \mathrm{Fil}^n W_m\Omega^i_{X_j}) \\ &\cong \varinjlim_{j\in J} H^0(X_j, V^n W_{m-n}\Omega^i_{X_j} + dV^n W_{m-n}\Omega^{i-1}_{X_j}) \\ &\cong H^0(X, V^n W_{m-n}\Omega^i_X + dV^n W_{m-n}\Omega^{i-1}_X), \end{aligned}$$

since the assertion is known in smooth case ([I, I.3.2]). Hence we obtain the assertion. □

We put $\mathrm{gr}^n W_m\Omega^i_X := \mathrm{Fil}^n W_m\Omega^i_X / \mathrm{Fil}^{n+1} W_m\Omega^i_X$. Then it is easy to see the equality $\mathrm{gr}^n W_m\Omega^i_X = \mathrm{Fil}^n W_{n+1}\Omega^i_X$. Concerning the structure of it, let us recall the following result of Illusie ([I, I.3.9]):

Proposition 2.4. *Let X be a smooth scheme over a perfect field and let $n, i \in \mathbb{N}$. Let us regard $\mathrm{gr}^n W_m\Omega^i_X = \mathrm{Fil}^n W_{n+1}\Omega^i_X$ and $\mathrm{Fil}^n W_{n+1}\Omega^i_X / dV^n\Omega^{i-1}_X$ as $\mathcal{O}_X$-modules by $(a, \omega) \mapsto \varphi(a)\omega$, where φ is the composite $\mathcal{O}_X = W_{n+1}\mathcal{O}_X / VW_{n+1}\mathcal{O}_X \xrightarrow{F} W_{n+1}\mathcal{O}_X / pW_{n+1}\mathcal{O}_X$. Then they are locally free $\mathcal{O}_X$-module of finite type.*

Later in this section, we extend the above proposition to certain regular schemes. (See Proposition 2.20.)

Next, for a scheme X over $\mathbb{F}_p$, let $C^{-1} : \Omega^\bullet_X \longrightarrow \underline{H}^\bullet(\Omega^\bullet_X)$ be the Cartier inverse homomorphism, that is, the homomorphism of graded algebras characterized by the following properties:

(1) $C^{-1}(a\omega) = a^p C^{-1}(\omega)$ $(a \in \mathcal{O}_X, \omega \in \Omega^\bullet_X)$.

(2) $C^{-1}(dx) := [x^{p-1}dx]$ $(x \in \mathcal{O}_X)$, where [?] denotes the class of ?.

In the case where X is smooth over a perfect field, it is well-known that C^{-1} is an isomorphism. We can extend it to the regular case:

Proposition 2.5. *If X is a regular scheme over $\mathbb{F}_p$, the Cartier inverse homomorphism C^{-1} is an isomorphism.*

Proof. We may assume that X is strictly local, and in this case, we can reduce to the smooth case by using Theorems 2.1, 2.2. The detail is left to the reader. □

For a regular scheme X over $\mathbb{F}_p$, we define the Cartier homomorphism $C : \underline{H}^\bullet(\Omega^\bullet_X) \longrightarrow \Omega^\bullet_X$ by $C := (C^{-1})^{-1}$. We denote the composite

$$Z\Omega^i_X \longrightarrow \underline{H}^i(\Omega^\bullet_X) \xrightarrow{C} \Omega^i_X$$

also by C, by abuse of notation.

Next we recall the definition of the logarithmic Hodge-Witt sheaf (the logarithmic part of the de Rham-Witt sheaf) and prove some exact sequences which we need later.

Definition 2.6. Let X be a scheme over $\mathbb{F}_p$ and let $i, m \in \mathbb{N}$. Then we define the logarithmic Hodge-Witt sheaf $W_m\Omega^i_{X,\log}$ by

$$W_m\Omega^i_{X,\log} := \mathrm{Im}(s : (\mathcal{O}^\times_X)^{\otimes i} \longrightarrow W_m\Omega^i_X),$$

where s is defined by

$$s(x_1 \otimes \cdots \otimes x_i) := \mathrm{dlog}\, \underline{x}_1 \wedge \cdots \wedge \mathrm{dlog}\, \underline{x}_i.$$

(Here Im is considered in the category of sheaves on X_{et}.) We denote $W_1\Omega^i_{X,\log}$ simply by $\Omega^i_{X,\log}$.

This definition is the naive generalization of that in [I], where the logarithmic Hodge-Witt sheaves are studied in the case of smooth schemes over a perfect field.

The following lemma is a generalization of [I, I.3.3] to the regular case:

Lemma 2.7. *Let X be a regular scheme over $\mathbb{F}_p$. Then the Frobenius operator $F : W_{m+1}\Omega^i_X \longrightarrow W_m\Omega^i_X$ induces the homomorphism $F : W_m\Omega^i_X \longrightarrow W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X$. (We will denote this homomorphism also by F.)*

Proof. This is clear by Proposition 2.3 and the equations

$$FV^m\Omega^i_X = pV^{m-1}\Omega^i_X = 0, \quad FdV^m\Omega^{i-1}_X = dV^{m-1}\Omega^{i-1}_X. \ \square$$

The following exact sequence is a generalization of [CT-Sa-So, §1, Lemme 2] to the regular case:

Proposition 2.8. *Let X be a regular scheme over $\mathbb{F}_p$ and let $i, m \in \mathbb{N}$. Then we have the exact sequence*

$$0 \longrightarrow W_m\Omega^i_{X,\log} \longrightarrow W_m\Omega^i_X \xrightarrow{1-F} W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X \longrightarrow 0.$$

Proof. We may assume that X is strictly local, and in this case, we can reduce to the smooth case by using Theorems 2.1, 2.2. In smooth case, the proposition is proved in [CT-Sa-So, §1, Lemme 2]. □

Let $W_\bullet\Omega^i_{X,\log}$, $W_\bullet\Omega^i_X$ be the pro-objects $\{W_n\Omega^i_{X,\log}, R\}_{n\in\mathbb{N}}$, $\{W_n\Omega^i_X, R\}_{n\in\mathbb{N}}$, respectively. Then we have the following corollary:

Corollary 2.9. *With the notation above, the following sequence of pro-sheaves is exact*:

$$0 \longrightarrow W_\bullet\Omega^i_{X,\log} \longrightarrow W_\bullet\Omega^i_X \xrightarrow{R-F} W_\bullet\Omega^i_X \longrightarrow 0.$$

Proof. The assertion follows from the proposition and the fact that the natural projection $W_\bullet\Omega^i_X \longrightarrow W_\bullet\Omega^i_X/dV^{\bullet-1}\Omega^{i-1}_X$ is an isomorphism as a homomorphism of pro-sheaves. □

The following exact sequence, which is well-known in the smooth case ([I]), is also useful:

Proposition 2.10. *Let X be a regular scheme over $\mathbb{F}_p$. Then we have the following exact sequence*:

$$0 \longrightarrow \Omega^i_{X,\log} \longrightarrow Z\Omega^i_X \xrightarrow{C-1} \Omega^i_X \longrightarrow 0.$$

Proof. We may assume that X is strictly local, and in this case, we can reduce to the smooth case by using Theorems 2.1, 2.2. □

Remark 2.11. As for the relation between the exact sequence in Proposition 2.8 and that in Proposition 2.10, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^i_{X,\log} & \longrightarrow & \Omega^i_X & \xrightarrow{1-F} & \Omega^i_X/d\Omega^{i-1}_X & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle C^{-1}} & & \\ 0 & \longrightarrow & \Omega^i_{X,\log} & \longrightarrow & Z\Omega^i_X & \xrightarrow{C-1} & \Omega^i_X & \longrightarrow & 0, \end{array}$$

where the upper (resp. the lower) horizontal line is the exact sequence in Proposition 2.8 for $m = 1$ (resp. Proposition 2.10) and the middle vertical arrow is the canonical inclusion.

We also need the following exact sequence (cf. [CT-Sa-So, Lemme 3]):

Proposition 2.12. *Let X be a regular scheme over $\mathbb{F}_p$ and let n, m be positive integers. Then the multiplication by p^m $W_{n+m}\Omega^i_{X,\log} \longrightarrow W_{n+m}\Omega^i_{X,\log}$ induces a homomorphism $\underline{p}^m : W_n\Omega^i_{X,\log} \longrightarrow W_{n+m}\Omega^i_{X,\log}$ and the following sequence is exact*:

$$0 \longrightarrow W_n\Omega^i_{X,\log} \xrightarrow{\underline{p}^m} W_{n+m}\Omega^i_{X,\log} \xrightarrow{R^n} W_m\Omega^i_{X,\log} \longrightarrow 0.$$

Proof. All the assertions can be reduced to the smooth case and they are proved in [CT-Sa-So, Lemme 3] in smooth case. □

Corollary 2.13. *Let X be a regular scheme over $\mathbb{F}_p$. Then the following sequence of pro-sheaves is exact*:

$$0 \longrightarrow W_\bullet\Omega^i_{X,\log} \xrightarrow{\underline{p}^m} W_\bullet\Omega^i_{X,\log} \xrightarrow{\text{proj.}} W_m\Omega^i_{X,\log} \longrightarrow 0.$$

Proof. Immediate. □

Let $\mathcal{C}$ be the category of regular schemes of characteristic p such that, for any $x \in X$, the absolute Frobenius $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is finite. We would like to extend Proposition 2.4 to the schemes in the category $\mathcal{C}$. To do this, we recall the definition and basic properties of the relatively perfect morphism of schemes.

Definition 2.14. A morphism $f : X \longrightarrow Y$ of schemes over $\mathbb{F}_p$ is said to be relatively perfect if the following diagram is Cartesian:

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow{F_Y} & Y, \end{array}$$

where F_X, F_Y are the absolute Frobenius morphisms.

The following facts are known: An etale morphism is relatively perfect, and a relatively perfect morphism is formally etale [EGA IV, (0.21.2.7)]. A relatively perfect morphism $X \longrightarrow Y$ with Y locally noetherian and regular is flat (Gabber, [Kat4, Prop 5.2]).

We give typical examples of relatively perfect morphisms which we use later.

Proposition 2.15.

(1) *Let k be a field of characteristic $p > 0$ with $[k : k^p] = p^i$ and let $x_1, \cdots, x_i$ be a p-basis of k. Then, the morphism*

$$\operatorname{Spec} k[[x_{i+1}, \cdots, x_n]] \longrightarrow \operatorname{Spec} \mathbb{F}_p[t_1, \cdots, t_n]$$

induced by $t_j \mapsto x_j \, (1 \leq j \leq n)$ is flat relatively perfect.

(2) *Let $X = \operatorname{Spec} A$ be a scheme in the category $\mathcal{C}$ with A local. Let I be an ideal of A and let $\hat{A}$ be the I-adic completion of A. If we put $Y := \operatorname{Spec} \hat{A}$, the natural morphism $Y \longrightarrow X$ is flat relatively perfect.*

Proof. The assertion (1) is easy and the proof is left to the reader. Let us prove the assertion (2). Let M be the A-module A on which the structure of the A-module is defined by $A \times M \longrightarrow M$, $(a, m) \mapsto a^p m$. Since X is in the category $\mathcal{C}$, M is a finite A-module. So we have $\hat{M} \cong \hat{A} \otimes_A M$, where $\hat{}$ denotes the I-adic completion. Let $I^{(p)}$ be the ideal of A generated by the elements $x^p \, (x \in I)$. Then, by definition, we have

$\hat{M} \cong \varprojlim_n A/(I^{(p)})^n$. Since the system $\{(I^{(p)})^n\}_n$ is cofinal with the system $\{I^n\}_n$, $\hat{M}$ is isomorphic to $\hat{A}$. So the homomorphism of rings $\hat{A} \otimes_{A,F} A \xrightarrow{F \otimes i} \hat{A}$ (where $i : A \to \hat{A}$ is the natural homomorphism) is an isomorphism. So we are done. □

REMARK 2.16. The above proposition shows that a scheme X in the category $\mathcal{C}$ is isomorphic to an affine space over $\mathbb{F}_p$ 'flat relatively perfect locally'. Indeed, for each $x \in X$, the map $\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$ is flat relatively perfect and the map $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \longrightarrow \operatorname{Spec} \mathcal{O}_{X,x}$ is flat relatively perfect by Proposition 2.15 (2). So the map $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \longrightarrow X$ is flat relatively perfect. On the other hand, $\hat{\mathcal{O}}_{X,x}$ has the form $\kappa(x)[[x_1, \cdots, x_n]]$ for some n. So we have a flat relatively perfect morphism of the form $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \longrightarrow \mathbb{A}^m_{\mathbb{F}_p}$ for some m, by Proposition 2.15 (1).

The following property of the relatively perfect morphism is important:

PROPOSITION 2.17.

(1) *Let $f : X \longrightarrow Y$ be a flat relatively perfect morphism of schemes over $\mathbb{F}_p$. Then the induced morphism $W_m f : W_m X \longrightarrow W_m Y$ is formally etale and flat for any m and the following diagrams are Cartesian:*

$$\begin{array}{ccc} W_{m-1}X & \xrightarrow{R^*} & W_mX \\ {\scriptstyle W_{m-1}f}\downarrow & & \downarrow{\scriptstyle W_mf} \\ W_{m-1}Y & \xrightarrow{R^*} & W_mY, \end{array} \qquad \begin{array}{ccc} W_mX & \xrightarrow{F^*} & W_mX \\ {\scriptstyle W_mf}\downarrow & & \downarrow{\scriptstyle W_mf} \\ W_mY & \xrightarrow{F^*} & W_mY, \end{array}$$

where R^* (resp. F^*) is the morphism induced by $R : W_m? \longrightarrow W_{m-1}?$ (resp. $F : W_m? \longrightarrow W_m?$) $(? = \mathcal{O}_X, \mathcal{O}_Y)$.

(2) *Let $f : X \longrightarrow Y$ be a flat relatively perfect morphism of schemes over $\mathbb{F}_p$, and let $g : Z \longrightarrow Y$ be any morphism. Then the following diagram is Cartesian:*

$$\begin{array}{ccc} W_m(X \times_Y Z) & \xrightarrow{W_m(f \times \mathrm{id})} & W_mZ \\ {\scriptstyle W_m(\mathrm{id} \times g)}\downarrow & & \downarrow{\scriptstyle W_mg} \\ W_mX & \xrightarrow{W_mf} & W_mY. \end{array}$$

Proof. The formal etaleness and flatness of the morphism $W_m f$ is proved in [Kat2, Lemma 2]. One can verify the other assertions by using the proof of [Kat2, Lemma 1]. The detail is left to the reader. □

As a consequence, one can check that the construction of the de Rham-Witt complex is compatible with flat relatively perfect morphism:

Proposition 2.18. *Let $f : X \longrightarrow Y$ be a flat relatively perfect morphism of schemes over $\mathbb{F}_p$. Then the natural homomorphism*

$$W_m\mathcal{O}_X \otimes_{W_m\mathcal{O}_Y} W_m\Omega^{\bullet}_Y \longrightarrow W_m\Omega^{\bullet}_X$$

is an isomorphism.

Proof. One can prove the proposition in the same way as Proposition 1.14 in [I]. (Indeed, he only uses the properties in Proposition 2.17 to prove the assertion of this proposition.) □

Proposition 2.19. *Let $f : X \longrightarrow Y$ be a flat relatively perfect morphism over $\mathbb{F}_p$. Let us regard $\mathrm{gr}^n W_m\Omega^i_? = \mathrm{Fil}^n W_{n+1}\Omega^i_?$ and $\mathrm{Fil}^n W_{n+1}\Omega^i_?/dV^n\Omega^{i-1}_?$ as $\mathcal{O}_?$-modules by $(a,\omega) \mapsto \varphi(a)\omega$, where φ is the composite $\mathcal{O}_? = W_{n+1}\mathcal{O}_?/VW_{n+1}\mathcal{O}_? \xrightarrow{F} W_{n+1}\mathcal{O}_?/pW_{n+1}\mathcal{O}_?$ $(? = X, Y)$. Then there exist canonical isomorphisms*

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{gr}^n W_m\Omega^i_Y \overset{\sim}{=} \mathrm{gr}^n W_m\Omega^i_X,$$

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{Fil}^n W_{n+1}\Omega^i_Y/dV^n\Omega^{i-1}_Y \overset{\sim}{=} \mathrm{Fil}^n W_{n+1}\Omega^i_X/dV^n\Omega^{i-1}_X.$$

Proof. First, by Proposition 2.18 and the flatness of $W_m f : W_m X \longrightarrow W_m Y$, we have the isomorphism

$$W_m\mathcal{O}_X \otimes_{W_m\mathcal{O}_Y} \mathrm{gr}^n W_m\Omega^i_Y \overset{\sim}{=} \mathrm{gr}^n W_m\Omega^i_X, \tag{2.1}$$

where we regard $\mathrm{gr}^n W_m\Omega^i_?$ $(? = X, Y)$ as $W_m\mathcal{O}_?$-module in the canonical way. By using the Cartesian diagrams in Proposition 2.17 (1), one can see that the left hand side (resp. the right hand side) of the isomorphism (2.1) is isomorphic to $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{gr}^n W_m\Omega^i_Y$ (resp. $\mathrm{gr}^n W_m\Omega^i_X$), where we regard $\mathrm{gr}^n W_m\Omega^i_?$ $(? = X, Y)$ as $\mathcal{O}_?$-modules. So we obtain the first isomorphism.

Let $(\Omega_?^{i-1})'$ $(? = X, Y)$ be the $\mathcal{O}_?$-module $\Omega_?^{i-1}$ on which the structure of $\mathcal{O}_?$-module is defined by $(a, \omega) \mapsto a^{p^{n+1}}\omega\,(a \in \mathcal{O}_?, \omega \in \Omega_?^{i-1})$. Then, Proposition 2.18 for $m = 1$ and the flat relative perfectness of f imply the isomorphism

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} (\Omega_Y^{i-1})' \overset{\sim}{=} (\Omega_X^{i-1})'. \tag{2.2}$$

On the other hand, the homomorphism

$$dV^n : (\Omega_?^{i-1})' \longrightarrow \mathrm{Fil}^n W_{n+1}\Omega_?^i \quad (? = X, Y)$$

is $\mathcal{O}_?$-linear. Hence the first isomorphism and the isomorphism (2.2) imply the second isomorphism. So we are done. □

Now we extend Proposition 2.4 to the schemes in the category $\mathcal{C}$:

PROPOSITION 2.20. *Let X be a scheme in the category $\mathcal{C}$. Let us regard $\mathrm{gr}^n W_m\Omega_X^i = \mathrm{Fil}^n W_{n+1}\Omega_X^i$ and $\mathrm{Fil}^n W_{n+1}\Omega_X^i / dV^n\Omega_X^{i-1}$ as $\mathcal{O}_X$-modules as in Proposition 2.19. Then they are locally free $\mathcal{O}_X$-module of finite type.*

PROOF. For a scheme S, let M_S be $\mathrm{gr}^n W_m\Omega_S^i$ or $\mathrm{Fil}^n W_{n+1}\Omega_S^i / dV^n\Omega_S^{i-1}$. To prove the assertion, we may assume X is local. Then, by Proposition 2.15, we can take the diagram of the form $X \overset{f}{\leftarrow} Y \overset{g}{\rightarrow} Z = \mathbb{A}_{\mathbb{F}_p}^N$ in the category $\mathcal{C}$, where f and g are flat relatively perfect and f is faithfully flat. (See Remark 2.16.) Then we have, by Proposition 2.19, the isomorphisms

$$\mathcal{O}_Y \otimes_{\mathcal{O}_X} M_X \overset{\sim}{=} M_Y, \quad \mathcal{O}_Y \otimes_{\mathcal{O}_Z} M_Z \overset{\sim}{=} M_Z.$$

So M_X is locally free of finite type if and only if so is M_Y, and M_Y is locally free of finite type if so is M_Z. So we can reduce to the smooth case, and it is nothing but Proposition 2.4. Hence the proof is finished. □

REMARK 2.21. Let $X = \mathrm{Spec}\, A$ be a scheme in the category $\mathcal{C}$ with A local. Then Ω_X^1 is a free A-module by the above proposition. Let $\mathfrak{m}$ be the maximal ideal of A. Let $x_1, \cdots, x_r \in A$ be a lift of p-basis of $k := A/\mathfrak{m}$ and let $x_{r+1}, \cdots, x_n$ be a regular parameter of A. Here we remark that the elements $dx_i\,(1 \leq i \leq n)$ forms a free basis of Ω_X^1. Indeed, to prove it, we may replace A by the completion $\hat{A}$ of A by Proposition 2.19 and in the complete case, it is easy to see the assertion because we have the isomorphism $A \cong k[[x_{r+1}, \cdots, x_n]]$ such that the images of x_i $(1 \leq i \leq r)$ forms a p-basis of k.

Finally, we give a simple equivalent condition for a regular scheme X to be in the category $\mathcal{C}$. Recall that a Noetherian ring A is called excellent if A is a universally catenary G-ring such that, for any finitely generated A-algebra B, the regular locus of $\operatorname{Spec} B$ is an open subset. (See [M2, p.260], [M1, (34.A)], [EGA IV, (7.8.2)].) A locally Noetherian scheme X is called excellent if it is covered by spectra of excellent rings ([EGA IV, (7.8.5)]).

Proposition 2.22. *Let X be a regular scheme. Then the following are equivalent*:

(1) *X is in the category $\mathcal{C}$.*

(2) *For any $x \in X$, $\mathcal{O}_{X,x}$ is excellent and $[\kappa(x) : \kappa(x)^p] < \infty$ holds.*

(3) *For any $x \in X_0$, $\mathcal{O}_{X,x}$ is excellent and $[\kappa(x) : \kappa(x)^p] < \infty$ holds.*

Proof. Note that X is in the category $\mathcal{C}$ if and only if $\operatorname{Spec} \mathcal{O}_{X,x}$ is in the category $\mathcal{C}$ for any x. Then the equivalence of (1) and (2) follows immediately from [Ku, Cor 2.6].

It is obvious that (2) implies (3). So it suffices to prove (2) assuming (3). Let x be a point of X and let y be a closed point of X contained in $\overline{\{x\}}$. Then the excellence of $\mathcal{O}_{X,y}$ implies that of $\mathcal{O}_{X,x}$ and, by [Ku, Cor 2.6, 2.7], the finiteness of $[\kappa(y) : \kappa(y)^p]$ implies that of $[\kappa(x) : \kappa(x)^p]$. So we have the assertion (2). Hence the proposition is proved. □

In particular, any excellent regular scheme X over $\mathbb{F}_p$ satisfying $[\kappa(x) : \kappa(x)^p] < \infty$ for any $x \in X_0$ belongs to the category $\mathcal{C}$. Note that, by [Ku, Cor 2.6, 2.7], the condition '$[\kappa(x) : \kappa(x)^p] < \infty$ for any $x \in X_0$' can be replaced by the condition '$[\kappa(x) : \kappa(x)^p] < \infty$ for any $x \in X^0$'.

3. Purity

Let k be a perfect field of characteristic p. Let X, Z be smooth schemes over k with X of pure dimension n, and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r. Then the following theorem, which is usually called the purity for the logarithmic Hodge-Witt cohomology, is proved by Gros ([G, II,Thm 3.5.8, (3.5.19)]) in the case $q = 0$ and by Suwa ([Su, Cor 1.6]) in the case $q > 0, i = n$:

THEOREM 3.1 (Gros, Suwa). *Let the situation be as above and let $m \in \mathbb{N}$. Then, there exists a canonical isomorphism*

$$\theta^{q,i,\log}_{\iota,m} : H^q(Z, W_m\Omega^{i-r}_{Z,\log}) \xrightarrow{\cong} H^{q+r}_Z(X, W_m\Omega^i_{X,\log})$$

for $q = 0$ or $q > 0, i = n$.

In this section, we generalize the above theorem to the case of certain regular schemes. Namely, we prove the following theorem:

THEOREM 3.2 (Purity in regular case). *Let X, Z be regular schemes and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r. Assume moreover that we have $[\kappa(x) : \kappa(x)^p] = p^N$ for any $x \in X^0$. Then there exists a canonical isomorphism*

$$\theta^{q,i,\log}_{\iota,m} : H^q(Z, W_m\Omega^{i-r}_{Z,\log}) \xrightarrow{\cong} H^{q+r}_Z(X, W_m\Omega^i_{X,\log})$$

if $q = 0$ holds or if $q > 0, i = N$ holds and X is in the category $\mathcal{C}$.

We recall some preliminary facts and give the definition of $\theta^{q,i,\log}_{\iota,m}$.

Let X, Z be regular schemes over $\mathbb{F}_p$ and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r defined by $t_1, \cdots, t_r \in \mathcal{O}_X$. Let U be $X - Z$ and for $I \subset [1, r]$, let U_I be the open subscheme of X on which the elements $t_i\,(i \in I)$ are invertible. Denote the open immersion $U_I \longrightarrow X$ by j_I. (In particular, we have $U_\emptyset = X$ and $j_\emptyset = \mathrm{id}$.) For a quasi-coherent sheaf $\mathcal{F}$ on $X(\simeq W_mX)$ and an integer s, let us define $C^s(\mathcal{F})$ by $C^s(\mathcal{F}) := \bigoplus_{|I|=s} j_{I,*}j_I^*\mathcal{F}$. Then, $C^s(\mathcal{F})$'s form a complex $C^\bullet(\mathcal{F})$ in natural way (the differential $C^s(\mathcal{F}) \longrightarrow C^{s+1}(\mathcal{F}); (f_I)_I \mapsto (g_I)_I$ is given by $g_{i_1,\dots,i_{s+1}} := \sum_{j=1}^{s+1}(-1)^{j+1} f_{i_1,\dots,\check{i}_j,\dots,i_{s+1}}$) and one can check that there exists a quasi-isomorphism

$$R\underline{\Gamma}_Z(X, \mathcal{F}) \simeq C^\bullet(\mathcal{F}). \tag{3.1}$$

In particular, we have a quasi-isomorphism $R\underline{\Gamma}_Z(X, W_m\Omega^i_X) \simeq C^\bullet(W_m\Omega^i_X)$, hence an isomorphism

$$\underline{H}^r_Z(X, W_m\Omega^i_X) = j_* W_m\Omega^i_U / \sum_{|I|=r-1} j_{I,*} W_m\Omega^i_{U_I}. \tag{3.2}$$

The following lemma is a generalization of [G, (3.3.20)]:

Lemma 3.3. *Let X, Z be regular schemes over $\mathbb{F}_p$ and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r. Then we have $\underline{H}^j_Z(X, W_m\Omega^i_X) = 0$, $\underline{H}^j_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X) = 0$ for $j \neq r$.*

Proof. It suffices to prove the vanishing $H^j_Z(X, W_m\Omega^i_X) = 0$, $H^j_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X) = 0$ for $j \neq r$ under the assumption that X is local. In this case, there exist elements $t_1, \cdots, t_r \in \mathcal{O}_X$ which defines the closed immersion ι. Let $\{X_k\}_{k\in K}$ be a projective system of affine schemes smooth over $\mathbb{F}_p$ such that $X = \varprojlim_{k\in K} X_k$ holds. By localizing X_k's and replacing K, we may assume that each X_k is a localization of a smooth scheme over $\mathbb{F}_p$ and that $t_1, \cdots, t_r$ define a regular closed immersion $Z_k \hookrightarrow X_k$ of codimension r with Z_k smooth over $\mathbb{F}_p$. Then, by Theorem 2.2, we have the isomorphisms

$$H^j_Z(X, W_m\Omega^i_X) = \varinjlim_{k\in K} H^j_{Z_k}(X_k, W_m\Omega^i_{X_k}),$$

$$H^j_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X) = \varinjlim_{k\in K} H^j_{Z_k}(X_k, W_m\Omega^i_{X_k}/dV^{m-1}\Omega^{i-1}_{X_k}).$$

So we may assume that X, Z are localizations of smooth schemes over $\mathbb{F}_p$ and that there exist elements $t_1, \cdots, t_r$ which define the closed immersion $Z \hookrightarrow X$.

In this case, the lemma is proved in [G, (3.3.20)]: it suffices to show the vanishing $H^j(X, C^\bullet(W_m\Omega^i_X)), H^j(X, C^\bullet(W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X))\ (j \neq r)$ and by using Proposition 2.4, we can reduce to showing the vanishing $H^j(X, C^\bullet(\mathcal{O}_X))\ (j \neq r)$, which we can check directly. □

Corollary 3.4. *Let the notations be as above. Then we have $\underline{H}^j_Z(X, W_m\Omega^i_{X,\log}) = 0$ for $j \neq r, r+1$.*

Proof. It is immediate from the previous lemma and the long exact sequence associated to the exact sequence

$$0 \longrightarrow W_m\Omega^i_{X,\log} \longrightarrow W_m\Omega^i_X \xrightarrow{1-F} W_m\Omega^i_X/dV^{m-1}\Omega^i_X \longrightarrow 0. \ \square$$

Let X, Z be affine regular schemes over $\mathbb{F}_p$ and let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension 1 defined by $t \in \mathcal{O}_X$. Then we have the following lemma:

LEMMA 3.5. *Let the notations be as above. Then, for an element* $\omega \in W_m\Omega_Z^{i-1}$, *there exists a lift* $\tilde{\omega} \in W_m\Omega_X^{i-1}$ *of* ω, *and the image of* $\operatorname{dlog} \underline{t} \wedge \tilde{\omega} \in j_*W_m\Omega_U^i$ *in* $j_*W_m\Omega_U^i/W_m\Omega_X^i$ *is independent of the choice of the lift* $\tilde{\omega}$. (*Hence we will sometimes denote the image by* $\operatorname{dlog} \underline{t} \wedge \omega$, *by abuse of notation.*)

PROOF. First, let us recall that, for a scheme Y over $\mathbb{F}_p$, there exists a functorial surjective homomorphism of differential graded algebras $\pi_Y : \Omega^\bullet_{W_mY} \longrightarrow W_m\Omega^\bullet_Y$ over $W_m\mathcal{O}_Y$. So there exists the following commutative diagram:

$$\begin{array}{ccc} \Omega^{i-1}_{W_mX} & \xrightarrow{(W_m\iota)^*} & \Omega^{i-1}_{W_mZ} \\ \pi_X \downarrow & & \pi_Z \downarrow \\ W_m\Omega^{i-1}_X & \xrightarrow{W_m(\iota^*)} & W_m\Omega^{i-1}_Z. \end{array}$$

Since $(W_m\iota)^*$ and π_Z are surjective, $W_m(\iota^*)$ is a surjection of quasi-coherent $W_m\mathcal{O}_X$-modules. Hence there exists a lift $\tilde{\omega} \in W_m\Omega_X^{i-1}$ of ω.

Next we prove the independence of the image of the element $\operatorname{dlog} \underline{t} \wedge \tilde{\omega}$ in $j_*W_m\Omega_U^i/W_m\Omega_X^i$. To prove this, we may assume that X is local. Then, X can be written as the projective limit $X = \varprojlim_{j \in J} X_j$ of affine schemes X_j which are smooth over $\mathbb{F}_p$. We may assume moreover that t defines a regular closed immersion $Z_j \hookrightarrow X_j$ of codimension 1 with Z_j smooth over $\mathbb{F}_p$ for any $j \in J$. Then it suffices to prove the desired independence for $Z_j \hookrightarrow X_j$. So we can reduce the proof to the smooth case. Then, since we may work etale locally, we may assume that the closed immersion $Z \hookrightarrow X$ has a section $s : X \longrightarrow Z$.

Now we prove the desired independence under the assumption that the closed immersion $Z \hookrightarrow X$ admits a section s. The claim in this case is used in [G, p.40] (in more generalized form), but here we give a detailed proof because the proof is omitted there. Since π_X is surjective, it suffices to prove the following claim:

(∗) For any $\eta \in \mathrm{Ker}(W_m(\iota^*) \circ \pi_X)$, we have $\mathrm{dlog}\,\underline{t} \wedge \pi_X(\eta) \in W_m\Omega^i_X$.

Let us consider the following commutative diagram:

$$\begin{array}{ccccc} \Omega^{i-1}_{W_mZ} & \xrightarrow{(W_ms)^*} & \Omega^{i-1}_{W_mX} & \xrightarrow{(W_m\iota)^*} & \Omega^{i-1}_{W_mZ} \\ {\scriptstyle\pi_Z}\downarrow & & {\scriptstyle\pi_X}\downarrow & & {\scriptstyle\pi_Z}\downarrow \\ W_m\Omega^{i-1}_Z & \xrightarrow{W_m(s^*)} & W_m\Omega^{i-1}_X & \xrightarrow{W_m(\iota^*)} & W_m\Omega^{i-1}_Z. \end{array}$$

Since $s \circ \iota = \mathrm{id}$ holds, the composites of the horizontal arrows are equal to identity. For $\eta \in \mathrm{Ker}(W_m(\iota^*) \circ \pi_X)$, define η_1 and η_2 by $\eta_1 := (W_ms)^* \circ (W_m\iota)^*(\eta), \eta_2 := \eta - \eta_1$. Then we have

$$\begin{aligned} \pi_X(\eta_1) = \pi_X \circ (W_ms)^* \circ (W_m\iota)^*(\eta) &= W_m(s^*) \circ \pi_Z \circ (W_m\iota)^*(\eta) \\ &= W_m(s^*) \circ W_m(\iota^*) \circ \pi_X(\eta) = 0. \end{aligned}$$

Hence we have $\mathrm{dlog}\,\underline{t} \wedge \pi_X(\eta) = \mathrm{dlog}\,\underline{t} \wedge \pi_X(\eta_2)$. On the other hand, we have

$$(W_m\iota)^*(\eta_2) = (W_m\iota)^*(\eta) - (W_m\iota)^* \circ (W_ms)^* \circ (W_m\iota)^*(\eta) = 0.$$

Hence it suffices to show the following claim:

(∗∗) For any $\eta \in \mathrm{Ker}((W_m\iota)^*)$, we have $\mathrm{dlog}\,\underline{t} \wedge \pi_X(\eta) \in W_m\Omega^i_X$.

Now put $\mathcal{I} := t\mathcal{O}_X$ and define $W_m\mathcal{I}$ by $W_m\mathcal{I} := \{\sum_{a=0}^{m-1} V^a\underline{x_a} \,|\, x_a \in \mathcal{I}\}$. Then we have $W_m\mathcal{I} = \mathrm{Ker}(W_m\mathcal{O}_X \longrightarrow W_m\mathcal{O}_Z)$ ([I, 0.1.5.6,(i)]). So one can see that $\mathrm{Ker}((W_m\iota)^*)$ is generated over $W_m\mathcal{O}_X$ by the elements of the following forms:

$$\alpha\eta \ (\alpha \in W_m\mathcal{I}, \eta \in \Omega^{i-1}_{W_mX}), \quad d\alpha \wedge \eta' \ (\alpha \in W_m\mathcal{I}, \eta' \in \Omega^{i-2}_{W_mX}).$$

Hence the claim (∗∗) is reduced to the following:

Claim. For $\alpha \in W_m\mathcal{I}, \eta \in W_m\Omega^{i-1}_X$ and $\eta' \in W_m\Omega^{i-2}_X$, the elements $\mathrm{dlog}\,\underline{t} \wedge \alpha\eta, \mathrm{dlog}\,\underline{t} \wedge d\alpha \wedge \eta'$ are contained in $W_m\Omega^i_X$.

Proof of Claim. We may assume $\alpha = V^a(\underline{xt})$ for some $a \in \mathbb{N}, x \in \mathcal{O}_X$. Then we have the following equations:

$$\begin{aligned} \mathrm{dlog}\,\underline{t} \wedge \alpha\eta &= V^a(\underline{xt})\mathrm{dlog}\,\underline{t} \wedge \eta \\ &= V^a(\underline{xt}\mathrm{dlog}\,\underline{t} \wedge F^a(\eta)) \\ &\in V^a(W_{m-a}\Omega^i_X) \subseteq W_m\Omega^i_X. \end{aligned}$$

$$\begin{aligned}
\operatorname{dlog}\underline{t}\wedge d\alpha\wedge\eta' &= \operatorname{dlog}\underline{t}\wedge dV^a(\underline{xt})\wedge\eta' \\
&= -d(V^a(\underline{xt})\operatorname{dlog}\underline{t}\wedge\eta') - V^a(\underline{xt})\operatorname{dlog}\underline{t}\wedge d\eta' \\
&= -dV^a(\underline{xt}\operatorname{dlog}\underline{t}\wedge F^a(\eta')) - V^a(\underline{xt}\operatorname{dlog}\underline{t}\wedge F^a(d\eta')) \\
&\in dV^a(W_{m-a}\Omega_X^{i-1}) + V^a(W_{m-a}\Omega_X^i) \subseteq W_m\Omega_X^i.
\end{aligned}$$

Hence the claim is proved. □

Since the claim is proved, the proof of the lemma is finished. □

Now, for regular schemes X, Z over $\mathbb{F}_p$ and a regular closed immersion $\iota : Z \hookrightarrow X$ of codimension r, we will define the homomorphism

$$\rho^i_{\iota,m} : W_m\Omega_Z^{i-r} \longrightarrow \underline{H}^r_Z(X, W_m\Omega_X^i)$$

by induction on r. First, we assume $r = 1$ and assume moreover that X is affine and that ι is defined by an element $t \in \mathcal{O}_X$. Then we define the homomorphism $\rho^i_{\iota,t,m} : W_m\Omega_Z^{i-1} \longrightarrow \underline{H}^1_Z(X, W_m\Omega_X^i)$ by the composite

$$W_m\Omega_Z^{i-1} \longrightarrow W_m\Omega^i_{X-Z}/W_m\Omega^i_X \cong \underline{H}^1_Z(X, W_m\Omega^i_X),$$

where the first map is defined as $\omega \mapsto \operatorname{dlog}\underline{t}\wedge\omega$ and the second isomorphism is given by (3.2) (with $r = 1$), using t. Then we have the following:

Lemma 3.6. *The map $\rho^i_{\iota,t,m}$ is independent of the choice of t.*

Proof. Let t' be another element which also induces the closed immersion ι. To prove the lemma, we may assume that X is local. Then, X can be written as the projective limit $X = \varprojlim_{j\in J} X_j$ of affine schemes X_j which are smooth over $\mathbb{F}_p$. We may assume moreover that t, t' define the same regular closed immersion $Z_j \hookrightarrow X_j$ for each $j \in J$. Then we have

$$\rho^i_{\iota,t,m} = \varinjlim_{j\in J}\rho^i_{\iota_j,t,m}, \quad \rho^i_{\iota,t',m} = \varinjlim_{j\in J}\rho^i_{\iota_j,t',m}.$$

So it suffice to prove the claim for regular closed immersion $Z_j \hookrightarrow X_j$, that is, we may reduce to the smooth case.

In smooth case, the lemma is due to Gros: Indeed, in In [G, II(1.2.6), II(3.4.1–3)], Gros defines a homomorphism of sheaves

$$\rho : W_m\Omega_Z^{i-1} \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_X)$$

by using Ekedahl's duality ([E]) without using t and in [G, (II.3.4)], he proves the equality $\rho = \rho^i_{\iota,t,m}$ for any t defining the closed immersion ι. (See also Remark 3.7 below.) Hence $\rho^i_{\iota,t,m}$ is independent of t and so the proof is finished. □

Remark 3.7. In [G, II(3.4)], Gros writes that the map ρ in the proof above has the form $\rho(\omega) = \omega \wedge \mathrm{dlog}\,\underline{t} \in j_*W_m\Omega^i_{X-Z}/W_m\Omega^i_X \cong \underline{H}^1_Z(X, W_m\Omega^i_X)$. However, we remark here that ρ has in fact the form $\rho(\omega) = \mathrm{dlog}\,\underline{t} \wedge \omega$, as we claim in the above proof. His calculation in [G, II(3.4)] depends on the calculation of the Gysin map in de Rham cohomology by Berthelot [Be, VI,Prop 3.1.3], which is described as follows: Let $\mathfrak{i} : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a regular immersion of codimension 1 of schemes smooth over $\operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z}$ defined by $t \in \mathcal{O}_{\mathcal{X}}$. Then the Gysin map

$$\Omega^{i-1}_{\mathcal{Z}} \longrightarrow \underline{H}^1_{\mathcal{Z}}(\mathcal{X}, \Omega^i_{\mathcal{X}})$$

is given by the composite

$$\Omega^{i-1}_{\mathcal{Z}} \longrightarrow \Omega^i_{\mathcal{X}} \otimes \omega_{\mathcal{Z}/\mathcal{X}} \longrightarrow \mathcal{E}\mathrm{xt}^1(\mathcal{O}_{\mathcal{Z}}, \Omega^i_{\mathcal{X}}) \longrightarrow \underline{H}^1_{\mathcal{Z}}(\mathcal{X}, \Omega^i_{\mathcal{X}}).$$

Here $\omega_{\mathcal{Z}/\mathcal{X}}$ is the conormal sheaf $\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}$. The first map is induced by the exact sequence

$$0 \longrightarrow \mathcal{N}_{\mathcal{Z}/\mathcal{X}} \longrightarrow \mathfrak{i}^*\Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{Z}} \longrightarrow 0$$

and so it is written as $\omega \mapsto \tilde{\omega} \wedge dt \otimes t^\vee$ (where $\tilde{\omega}$ is a lift of ω.) The second map is the fundamental local isomorphism of Hartshorne ([Ha, III,7.2]). The third map is the canonical map induced from the map of functors $\mathcal{H}\mathrm{om}(\mathcal{O}_{\mathcal{Z}}, -) \longrightarrow \underline{H}^0_{\mathcal{Z}}(\mathcal{X}, -)$. In conclusion, it is claimed that the Gysin map is given by $\omega \mapsto \tilde{\omega} \wedge \mathrm{dlog}\, t$, if we identify $\underline{H}^1_{\mathcal{Z}}(\mathcal{X}, \Omega^i_{\mathcal{X}})$ with $\Omega^i_{\mathcal{X}-\mathcal{Z}}/\Omega^i_{\mathcal{X}}$. However, Conrad points out in [Co] that, in order that the Gysin map is well-behaved in the duality theory, the fundamental local isomorphism should be corrected to $\omega_{\mathcal{Z}/\mathcal{X}} \otimes \Omega^i_{\mathcal{X}} \longrightarrow \mathcal{E}\mathrm{xt}^1(\mathcal{O}_{\mathcal{Z}}, \Omega^i_{\mathcal{X}})$. Then the Gysin map should be corrected to the composite

$$\Omega^{i-1}_{\mathcal{Z}} \longrightarrow \omega_{\mathcal{Z}/\mathcal{X}} \otimes \Omega^i_{\mathcal{X}} \longrightarrow \mathcal{E}\mathrm{xt}^1(\mathcal{O}_{\mathcal{Z}}, \Omega^i_{\mathcal{X}}) \longrightarrow \underline{H}^1_{\mathcal{Z}}(\mathcal{X}, \Omega^i_{\mathcal{X}}),$$

and so it should be written as $\omega \mapsto \mathrm{dlog}\, t \wedge \tilde{\omega}$. By taking this correction into account and arguing in the same way as [G, II(3.4)], we see that ρ has the form $\rho(\omega) = \mathrm{dlog}\,\underline{t} \wedge \omega$.

Let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension 1 between regular schemes over $\mathbb{F}_p$. Locally on X, X is affine and ι is defined by an element $t \in \mathcal{O}_X$. So we can define the map $\rho^i_{\iota,t,m}$ locally and it is independent of t by Lemma 3.6. Hence we can define the map $\rho^i_{\iota,m} : W_m\Omega^{i-1}_Z \longrightarrow \underline{H}^1_Z(X, W_m\Omega^i_X)$ globally, by glueing the maps $\rho^i_{\iota,t,m}$.

Now let us consider the case that $\iota : Z \hookrightarrow X$ is a regular immersion of codimension r between regular schemes over $\mathbb{F}_p$. Locally on X, we can factorize ι as $Z \overset{\iota'}{\hookrightarrow} Y \overset{\iota''}{\hookrightarrow} X$, where ι' is a regular closed immersion of codimension $r-1$ and ι'' is a regular closed immersion of codimension 1. We define the map $\iota^i_{\iota,Y,m} : W_m\Omega^{i-r}_Z \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_X)$ by the composite

$$W_m\Omega^{i-r}_Z \overset{\rho^{i-1}_{\iota',m}}{\longrightarrow} \underline{H}^{r-1}_Z(Y, W_m\Omega^{i-1}_Y)$$
$$\overset{\underline{H}^{r-1}_Z(\rho^i_{\iota'',m})}{\longrightarrow} \underline{H}^{r-1}_Z(Y, \underline{H}^1_Y(X, W_m\Omega^i_X)) \overset{\cong}{\longrightarrow} \underline{H}^r_Z(X, W_m\Omega^i_X),$$

where the third map is induced by the Leray spectral sequence and Lemma 3.3. Then we have the following:

Lemma 3.8. *The map $\rho^i_{\iota,Y,m}$ is independent of the choice of Y.*

Proof. Let $Z \hookrightarrow Y' \hookrightarrow X$ be another factorization satisfying the same condition as $Z \hookrightarrow Y \hookrightarrow X$. To prove the lemma, we may assume that X is local and so X can be written as the projective limit $X = \varprojlim_{j \in J} X_j$ of affine schemes X_j which are smooth over $\mathbb{F}_p$. We may assume moreover that there are factorizations $Z_j \hookrightarrow Y_j \hookrightarrow X_j, Z_j \hookrightarrow Y'_j \hookrightarrow X_j$ with Y_j, Y'_j smooth over $\mathbb{F}_p$ which induce $Z \hookrightarrow Y \hookrightarrow X, Z \hookrightarrow Y' \hookrightarrow X$ respectively when we pull them back by $X \longrightarrow X_j$. Denote the closed immersion $Z_j \hookrightarrow X_j$ by ι_j. Then we have

$$\rho^i_{\iota,Y,m} = \varinjlim_{j \in J} \rho^i_{\iota_j,Y_j,m}, \quad \rho^i_{\iota,Y',m} = \varinjlim_{j \in J} \rho^i_{\iota_j,Y'_j,m}.$$

So it suffice to prove the claim for $Z_j \hookrightarrow X_j$, that is, we may reduce to the smooth case.

In smooth case, the lemma is again due to Gros: In [G, II(1.2.6), II(3.4.1–3)], Gros defines a homomorphism of sheaves $\rho : W_m\Omega^{i-r}_Z \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_X)$ for any regular closed immersion $Z \hookrightarrow X$ of smooth schemes over $\mathbb{F}_p$ with $\rho = \rho^i_{\iota,m}$ if ι is of codimension 1, and he proves that this map satisfies the transitivity. So, by induction on r, we can prove the equalities

$$(\rho \text{ for } Z_j \hookrightarrow X_j) = \rho^i_{\iota_j,Y_j,m}, \ \ (\rho \text{ for } Z_j \hookrightarrow X_j) = \rho^i_{\iota_j,Y'_j,m}.$$

So we are done. □

By Lemma 3.8, we can define the map $\rho^i_{\iota,m} : W_m\Omega^{i-r}_Z \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_X)$ globally, by glueing the maps $\rho^i_{\iota,Y,m}$.

Remark 3.9. It would be possible to give the definition of $\rho^i_{\iota,m}$ in more explicit way without using induction on r, by using [G, II(3.4)]: However, we would like to adopt the inductive definition given above to avoid calculation involving complicated signs later in this paper.

Let $\iota : Z \hookrightarrow X$ be a regular immersion of codimension r between regular schemes over $\mathbb{F}_p$. Then it is easy to see that the map $\rho^i_{\iota,m}$ we defined above is compatible with respect to m. So the family of maps $\{\rho^i_{\iota,m}\}_m$ induces the homomorphism of pro-sheaves

$$\rho^i_{\iota,\bullet} : W_\bullet\Omega^{i-r}_Z \longrightarrow \underline{H}^r_Z(X, W_\bullet\Omega^i_X).$$

It is easy to see from the definition that $\rho^i_{\iota,\bullet}$ is compatible with $1-F$. So, by the exact sequence

$$0 \longrightarrow W_\bullet\Omega^j_{?,\log} \longrightarrow W_\bullet\Omega^j_? \xrightarrow{1-F} W_\bullet\Omega^j_? \longrightarrow 0$$

for $(?, j) = (Z, i-r), (X, i)$, $\rho^i_{\iota,\bullet}$ induces the homomorphism

$$\rho^{i,\log}_{\iota,\bullet} : W_\bullet\Omega^{i-r}_{Z,\log} \longrightarrow \underline{H}^r_Z(X, W_\bullet\Omega^i_{X,\log}).$$

Then, by the exact sequence

$$0 \longrightarrow W_\bullet\Omega^j_{?,\log} \xrightarrow{p^m} W_\bullet\Omega^j_{?,\log} \longrightarrow W_m\Omega^j_{?,\log} \longrightarrow 0$$

for $(?, j) = (Z, i-r), (X, i)$, $\rho^{i,\log}_{\iota,\bullet}$ induces the homomorphism

$$\rho^{i,\log}_{\iota,m} : W_m\Omega^{i-r}_{Z,\log} \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_{X,\log}).$$

Using $\rho^i_{\iota,m}$ and $\rho^{i,\log}_{\iota,m}$, we define the homomorphisms

$$\theta^i_{\iota,m} : H^0(Z, W_m\Omega^{i-r}_Z) \longrightarrow H^r_Z(X, W_m\Omega^i_X),$$

$$\theta^{q,i,\log}_{\iota,m} : H^q(Z, W_m\Omega^{i-r}_{Z,\log}) \longrightarrow H^{q+r}_Z(X, W_m\Omega^i_{X,\log}),$$

by the composite

$$(3.3) \qquad H^0(Z, W_m\Omega^{i-r}_Z) \stackrel{H^0(\rho^i_{\iota,m})}{\longrightarrow} H^0(Z, \underline{H}^r_Z(X, W_m\Omega^i_X)) \stackrel{\cong}{\longrightarrow} H^r_Z(X, W_m\Omega^i_X),$$

$$(3.4) \qquad H^q(Z, W_m\Omega^{i-r}_{Z,\log}) \stackrel{H^q(\rho^{i,\log}_{\iota,m})}{\longrightarrow} H^q(Z, \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})) \longrightarrow H^{q+r}_Z(X, W_m\Omega^i_{X,\log}),$$

where the second homomorphisms are induced by Leray spectral sequences. (Here we use Lemma 3.3 and Corollary 3.4.) The map $\theta^{q,i,\log}_{\iota,m}$ is the one which appears in the statements in Theorems 3.1, 3.2.

Now we give a proof of Theorem 3.2, that is, we prove that $\theta^{q,i,\log}_{\iota,m}$ is an isomorphism if $q = 0$ holds or if $q > 0, j = N$ holds and X is in the category $\mathcal{C}$. First we prove it in the case $q = 0$:

PROOF OF THEOREM 3.2, *Step* 1: *The case* $q = 0$. In this case, the second map in (3.4) is an isomorphism. So it suffices to prove that $\rho^{i,\log}_{\iota,m}$ is an isomorphism. To show this, we may assume X is local. Then, X can be written as the projective limit $X = \varprojlim_{j\in J} X_j$ of affine schemes X_j which are smooth over $\mathbb{F}_p$. We may assume moreover that there exists a projective system of regular closed immersions $\{\iota_j : Z_j \hookrightarrow X_j\}_j$ with $Z = \varprojlim_{j\in J} Z_j$. Then we have $\rho^{i,\log}_{\iota,m} = \varinjlim_{j\in J} \rho^{i,\log}_{\iota_j,m}$. So the claim is reduced to showing that $\rho^{i,\log}_{\iota,m}$ is an isomorphims in the case where Z, X are smooth over $\mathbb{F}_p$. This is proven by Gros ([G, II,Thm 3.5.8]). So we are done. □

Next we prove Theorem 3.2 in the case $q > 0, i = N$:

PROOF OF THEOREM 3.2, *Step* 2: *The case* $q > 0, i = N$ *and* X *is in the category* $\mathcal{C}$. First let us note that it suffices to prove the following claim:

Claim 1. With the notation of the theorem, we have $\underline{H}_Z^{r+1}(X, W_m\Omega^N_{X,\log}) = 0$.

Indeed, if we have this claim, we see that the second map in (3.4) is an isomorphism. (We also use Corollary 3.4.) Since we have already proved that $\rho^{i,\log}_{\iota,m}$ is an isomorphism, the first map in (3.4) is an isomorphism and so $\theta^{q,i,\log}_{\iota,m}$ is an isomorphism. Hence the theorem is reduced to the claim 1.

To prove claim 1, we may replace X by its strict henselization. In this case, the claim is nothing but the vanishing $H_Z^{r+1}(X, W_m\Omega^N_{X,\log}) = 0$. Now let us prove the following claim:

Claim 2. Let $\hat{X}$ be the completion of X along Z. Then, for any i, j, we have the canonical isomorphism $H^j_Z(X, W_m\Omega^i_{X,\log}) \overset{\simeq}{=} H^j_Z(\hat{X}, W_m\Omega^i_{\hat{X},\log})$.

Proof of Claim 2. By the exact sequences

$$0 \longrightarrow W_\bullet\Omega^i_{?,\log} \overset{p^m}{\longrightarrow} W_\bullet\Omega^i_{?,\log} \longrightarrow W_m\Omega^i_{?,\log} \longrightarrow 0,$$
$$0 \longrightarrow W_\bullet\Omega^i_{?,\log} \longrightarrow W_\bullet\Omega^i_{?} \overset{1-F}{\longrightarrow} W_\bullet\Omega^i_{?} \longrightarrow 0$$

for $? = X, \hat{X}$, it suffices to show that the canonical homomorphism

$$H^j_Z(X, W_m\Omega^i_X) \longrightarrow H^j_Z(\hat{X}, W_m\Omega^i_{\hat{X}})$$

is an isomorphism. Let us take elements $t_1, \cdots, t_r$ defining the closed immersion $\iota : Z \hookrightarrow X$. Then we have the isomorphism

$$H^j_Z(X, W_m\Omega^i_X) \overset{\simeq}{=} H^j(X, C^\bullet(W_m\Omega^i_X)),$$

and the similar isomorphism holds also for $\hat{X}$. Since the morphism $\hat{X} \longrightarrow X$ is flat relatively perfect, the functor $- \otimes_{W_m\mathcal{O}_X} W_m\mathcal{O}_{\hat{X}}$ is compatible with the canonical filtrations of de Rham-Witt sheaves. Hence it suffices to show that the canonial homomorphism

$$H^j(X, C^\bullet(\mathrm{gr}^k W_m\Omega^i_X)) \longrightarrow H^j(\hat{X}, C^\bullet(\mathrm{gr}^k W_m\Omega^i_{\hat{X}}))$$

is an isomorphism.

For a sheaf of $W_{k+1}\mathcal{O}_?$-module $\mathcal{M}$ ($? = X, \hat{X}$), let $\mathcal{M}'$ be be the sheaf $\mathcal{M}$ on which the structure of $\mathcal{O}_?$-modules is defined by $(a, \omega) \mapsto \varphi(a)\omega$, where

φ is the composite $\mathcal{O}_? = W_{k+1}\mathcal{O}_?/VW_{k+1}\mathcal{O}_? \xrightarrow{F} W_{k+1}\mathcal{O}_?/pW_{k+1}\mathcal{O}_?$. Then we have

$$j_{I,*}j_I^*(\mathrm{gr}^k W_m\Omega^i_X)' = (j_{I,*}j_I^*\mathrm{gr}^k W_m\Omega^i_X)'$$

and similar equality holds also for $\hat{X}$. Moreover, by Proposition 2.20 and the flat relative perfectness of $\hat{X} \longrightarrow X$, the sheaf $(\mathrm{gr}^k W_m\Omega^i_X)'$ is a free $\mathcal{O}_X$-module of finite type and that the canonical homomorphism

$$(\mathrm{gr}^k W_m\Omega^i_X)' \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{X}} \longrightarrow (\mathrm{gr}^k W_m\Omega^i_{\hat{X}})'$$

is an isomorphism. Hence it suffices to show that the homomorphism

$$H^j(X, C^\bullet(\mathcal{O}_X)) \longrightarrow H^j(\hat{X}, C^\bullet(\mathcal{O}_{\hat{X}}))$$

is an isomorphism.

When $j \neq r$ holds, both hand sides are equal to zero. Let us consider the case $j = r$. For $l \in \mathbb{N}$, let $I^{(l)}$ be the ideal of $A := \Gamma(X, \mathcal{O}_X)$ generated by $t_1^l, \cdots, t_r^l$. Then we have

$$H^r(X, C^\bullet(\mathcal{O}_X)) = H^0(X, j_*\mathcal{O}_U/\sum_{|I|=r-1} j_{I,*}\mathcal{O}_{U_I}) = \varinjlim_{l\in\mathbb{N}} A/I^{(l)},$$

where the transition map $A/I^{(l)} \longrightarrow A/I^{(l')}$ $(l \leq l')$ is defined by the multiplication by $(t_1 \cdots t_r)^{l'-l}$. Similarly, we have

$$H^r(\hat{X}, C^\bullet(\mathcal{O}_{\hat{X}})) = \varinjlim_{l\in\mathbb{N}} \hat{A}/I^{(l)}\hat{A},$$

where we put $\hat{A} := \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$. Since $(t_1, \cdots, t_r)^{(l-1)r+1} \subset I^{(l)}$ holds, we have the natural isomorphism $A/I^{(l)} \cong \hat{A}/I^{(l)}\hat{A}$. Hence $H^r(X, C^\bullet(\mathcal{O}_X))$ is naturally isomorphic to $H^r(\hat{X}, C^\bullet(\mathcal{O}_{\hat{X}}))$. So the assertion is proved. □

By claim 2, we may assume $X \cong \operatorname{Spec} \mathcal{O}_Z[[t_1, \cdots, t_r]]$ to prove the claim 1. Now we prove claim 1:

Proof of Claim 1. By using the exact sequence

$$0 \longrightarrow W_{m-1}\Omega^N_{X,\log} \longrightarrow W_m\Omega^N_{X,\log} \longrightarrow \Omega^N_{X,\log} \longrightarrow 0,$$

one can reduce to the case $m = 1$. So one can reduce to the following assertion $(*)$:

$(*)$: Let $r, N \in \mathbb{N}, N \geq r$ and let $Z = \operatorname{Spec} A$ be a strictly local regular scheme in $\mathcal{C}$ such that $[\kappa(z) : \kappa(z)^p] = p^{N-r}$ holds for the generic point z of Z. Let $\iota : Z \hookrightarrow X$ be the closed immersion $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A[[x_1, \cdots, x_r]]$. Then $H_Z^{r+1}(X, \Omega_{X,\log}^N) = 0$ holds.

We prove the assertion $(*)$. Put $B := A[[x_1, \cdots, x_r]]$ and let $\mathfrak{m}$ be the maximal ideal of A. In the situation of $(*)$, we can take a local parameter $x_{r+1}, \cdots, x_{r+s}$ of $\mathfrak{m}$ and a p-basis $x_{r+s+1}, \cdots, x_N$ of $A/\mathfrak{m}$ for some s. By the exact sequence

$$0 \longrightarrow \Omega_{X,\log}^N \longrightarrow \Omega_X^N \xrightarrow{C-1} \Omega_X^N \longrightarrow 0,$$

it suffices to show that the homomorphism $C - 1 : H_Z^r(X, \Omega_X^n) \longrightarrow H_Z^r(X, \Omega_X^n)$ is surjective. Let us recall the isomorphism

$$H_Z^r(X, \Omega_X^N) \cong \Omega_B^N[x_1^{-1}, \cdots, x_r^{-1}] / \sum_{j=1}^r \Omega_B^N[x_1^{-1}, \cdots, \check{x}_j^{-1}, \cdots, x_r^{-1}]$$

defined by $x_1, \cdots, x_r$. Denote the right hand side by Ω. Since the above isomorphism is compatible with the action of $C - 1$, it suffices to show that the homomorphism $C - 1 : \Omega \longrightarrow \Omega$ is surjective.

Put $M := \sum_{j=1}^r \Omega_B^N[x_1^{-1}, \cdots, \check{x}_j^{-1}, \cdots, x_r^{-1}]$, and let $\operatorname{dlog} x := \operatorname{dlog} x_1 \wedge \cdots \wedge \operatorname{dlog} x_r, dx := dx_{r+1} \wedge \cdots \wedge dx_N \in \Omega_A^{N-r}$. (Note that Ω_A^{N-r} is isomorphic to the free A-module $A dx$. See Remark 2.21.) For $l \in \mathbb{N}$, let us define $H_l \subset \Omega$ by

$$H_l := \{\omega \in \Omega \,|\, \omega \in \sum_{-l \leq l_1, \cdots, l_r \leq 0} a_{l_1 \cdots l_r} x_1^{l_1} \cdots x_r^{l_r} dx \wedge \operatorname{dlog} x + M \quad \text{(for some } a_{l_1 \cdots l_r} \in A)\}.$$

Then we have $H_l \subseteq H_{l+1}\,(l \in \mathbb{N})$ and $\bigcup_{l \in \mathbb{N}} H_l = \Omega$. For $\omega = a x_1^{l_1} \cdots x_r^{l_r} dx \wedge \operatorname{dlog} x\,(a \in A)$, One can calculate $C(\omega)$ as follows: $C(\omega)$ has the form $b x_1^{l_1/p} \cdots x_r^{l_r/p} dx \wedge \operatorname{dlog} x$ (for some $b \in A$) if $p | l_i$ holds for all $1 \leq i \leq r$ and is equal to 0 otherwise. Hence we have $CH_l \subseteq H_{l-1}$ for $l \geq 1$ and $CH_0 \subseteq H_0$. Hence $C^{l'} H_l \subseteq H_0$ holds for $l' \geq l$. Then, for $\omega \in \Omega$, we have

$$\omega = (C-1)\left(\sum_{i=0}^{l-1} C^i(-\omega)\right) + C^l \omega \in (C-1)\Omega + H_0,$$

for sufficiently large l. Hence it suffices to show the surjectivity of $C - 1 : H_0 \longrightarrow H_0$. Since we have $(C-1)(adx \wedge \mathrm{dlog}\, x) = (C-1)(adx) \wedge \mathrm{dlog}\, x$, it suffices to prove the surjectivity of $C - 1 : \Omega_A^{N-r} \longrightarrow \Omega_A^{N-r}$. Put $x := x_{r+1} \cdots x_N$. For an element $a \in A$, take an element $b \in A$ satisfying $b - b^p x^{p-1} = a$. (Note that such an element $b \in A$ always exists, since A is henselian.) Then we have

$$\begin{aligned}(C-1)(b^p x^{p-1} dx) &= bdx - b^p x^{p-1} dx \\ &= adx.\end{aligned}$$

So $C - 1$ is surjective and the proof of the claim is finished. □

Since we have proved claim 2, the proof of the theorem is finished. □

Remark 3.10. It seems not easy to prove Theorem 3.2, Step 2 directly by reducing to the smooth case, since if we write X as a projective limit of affine smooth schemes $X = \varprojlim_j X_j$, we cannot control the dimension of X_j's.

Remark 3.11. Let X be a regular scheme over $\mathbb{F}_p$ with $[\kappa(y) : \kappa(y)^p] = p^N$ for any $y \in X^0$, let x be a point of codimension r of X and denote the localization of X at x by X_x. Then we have $\underline{H}_x^r(X, \mathcal{F}) = \underline{H}_x^r(X_x, \mathcal{F})$ for any abelian sheaf $\mathcal{F}$. So $\rho^i_{x \hookrightarrow X_x, m}$ has the form $W_m\Omega_x^{i-r} \longrightarrow \underline{H}_x^r(X, W_m\Omega_X^i)$. We denote the map $\rho^i_{x \hookrightarrow X_x, m}$ by $\rho^i_{x \hookrightarrow X, m}$, by abuse of notation. Then the map $\rho^i_{x \hookrightarrow X, m}$ induces the maps

$$\begin{aligned}\rho^{i,\log}_{x \hookrightarrow X, m} &: W_m\Omega^{i-r}_{x,\log} \longrightarrow \underline{H}_x^r(X, W_m\Omega^i_{X,\log}), \\ \theta^i_{x \hookrightarrow X, m} &: H^0(x, W_m\Omega_x^{i-r}) \longrightarrow H_x^r(X, W_m\Omega_X^i), \\ \theta^{q,i,\log}_{x \hookrightarrow X, m} &: H^q(x, W_m\Omega^{i-r}_{x,\log}) \longrightarrow H_x^{q+r}(X, W_m\Omega^i_{X,\log})\end{aligned}$$

in the same way as explained in this section and by Theorem 3.2, the last map is an isomorphism if $q = 0$ holds or if $q = 1, i = N$ holds and X is in the category $\mathcal{C}$.

Finally in this section, we give two remarks on some compatibilities concerning the maps $\theta^i_{\iota,m}, \theta^{q,i,\log}_{\iota,m}$.

Remark 3.12. Let $\iota : Z \hookrightarrow X$ be a regular immersion of codimension r between regular schemes in the category $\mathcal{C}$, and assume that we have $[\kappa(x) : \kappa(x)^p] = p^N$ for any $x \in X^0$. Let

$$\delta_Z^0 : H^0(Z, W_m\Omega_{Z,\log}^{i-r}) \longrightarrow H^0(Z, W_m\Omega_Z^{i-r}),$$

$$\delta_X^0 : H_Z^r(X, W_m\Omega_{X,\log}^{i}) \longrightarrow H_Z^r(X, W_m\Omega_X^{i})$$

be the homomorphisms induced by the inclusion $W_m\Omega_{?,\log}^* \hookrightarrow W_m\Omega_?^*$ for $(?, *) = (Z, i-r), (X, i)$ respectively, and let

$$\delta_Z^1 : H^0(Z, W_m\Omega_Z^{i-r}) \longrightarrow H^1(Z, W_m\Omega_{Z,\log}^{i-r}),$$

$$\delta_X^1 : H_Z^r(X, W_m\Omega_X^{i}) \longrightarrow H_Z^{r+1}(X, W_m\Omega_{X,\log}^{i}),$$

be the composites

$$H^0(Z, W_m\Omega_Z^{i-r}) \longrightarrow H^0(Z, W_m\Omega_Z^{i-r}/dV^{m-1}\Omega_Z^{i-r-1}) \longrightarrow H^1(Z, W_m\Omega_{Z,\log}^{i-r}),$$

$$H_Z^r(X, W_m\Omega_X^{i}) \longrightarrow H_Z^r(X, W_m\Omega_X^{i}/dV^{m-1}\Omega_X^{i-1}) \longrightarrow H_Z^{r+1}(X, W_m\Omega_{X,\log}^{i}),$$

where the first maps are induced by the natural projection $W_m\Omega_?^* \longrightarrow W_m\Omega_?^*/dV^{m-1}\Omega_?^{*-1}$ and the second maps are the connecting homomorphism associated to the exact sequence

$$0 \longrightarrow W_m\Omega_{?,\log}^* \longrightarrow W_m\Omega_?^* \xrightarrow{1-F} W_m\Omega_?^*/dV^{m-1}\Omega_?^{*-1} \longrightarrow 0$$

for $(?, *) = (Z, i-r), (X, i)$, respectively.

Then, as for the compatibility of the maps $\theta_{\iota,m}^i, \theta_{\iota,m}^{q,i,\log}, \delta_?^* \ (* = 0, 1, ? = Z, X)$, we have the following claim:

Claim. The diagram

$$(3.5) \qquad \begin{array}{ccc} H^0(Z, W_m\Omega_{Z,\log}^{i-r}) & \xrightarrow{\theta_{\iota,m}^{0,i,\log}} & H_Z^r(X, W_m\Omega_{X,\log}^{i}) \\ {\scriptstyle \delta_Z^0}\big\downarrow & & {\scriptstyle \delta_X^0}\big\downarrow \\ H^0(Z, W_m\Omega_Z^{i-r}) & \xrightarrow{\theta_{\iota,m}^{i}} & H_Z^r(X, W_m\Omega_X^{i}) \end{array}$$

is commutative and the diagram

$$
(3.6)\qquad
\begin{array}{ccc}
H^0(Z, W_m\Omega_Z^{i-r}) & \xrightarrow{\theta^i_{\iota,m}} & H^r_Z(X, W_m\Omega^i_X) \\
\delta^1_Z \Big\downarrow & & \delta^1_X \Big\downarrow \\
H^1(Z, W_m\Omega^{i-r}_{Z,\log}) & \xrightarrow{\theta^{1,i,\log}_{\iota,m}} & H^{r+1}_Z(X, W_m\Omega^i_{X,\log})
\end{array}
$$

is $(-1)^r$-commutative.

Proof of Claim. First we claim that there exists the following commutative diagram, where the horizontal lines are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_Z^{i-r-1} & \xrightarrow{dV^{m-1}} & W_m\Omega_Z^{i-r} & \longrightarrow & W_m\Omega_Z^{i-r}/dV^{m-1}\Omega_Z^{i-r-1} & \longrightarrow & 0 \\
 & & (-1)^r\rho^{i-1}_{\iota,1}\Big\downarrow & & \rho^i_{\iota,m}\Big\downarrow & & & & \\
0 & \longrightarrow & \underline{H}^r_Z(X, \Omega_X^{i-1}) & \xrightarrow{dV^{m-1}} & \underline{H}^r_Z(X, W_m\Omega^i_X) & \longrightarrow & \underline{H}^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega_X^{i-1}) & \longrightarrow & 0.
\end{array}
$$

Indeed, the exactness of the lower horizontal line follows from Lemma 3.3. The commutativity of the square is reduced to the case $r = 1$ by induction, and in this case, the commutativity follows from the following calculation in $\underline{H}^1_Z(X, W_m\Omega^i_X)$ for $\omega \in \Omega_Z^{n-2}$:

$$
\begin{aligned}
-dV^{m-1}(\operatorname{dlog} t \wedge \tilde{\omega}) &= -dV^{m-1}(F^{m-1}\operatorname{dlog} \underline{t} \wedge \tilde{\omega}) = -d(\operatorname{dlog} \underline{t} \wedge V^{m-1}\tilde{\omega}) \\
&= \operatorname{dlog} \underline{t} \wedge dV^{m-1}\tilde{\omega} = \operatorname{dlog} \underline{t} \wedge d\widetilde{V^{m-1}\omega},
\end{aligned}
$$

where $\tilde{}$ means a lift of elements in $W_*\Omega_Z^{i-2}$ to those in $W_*\Omega_X^{i-2}$ for $* = 1$ or m.

The above diagram allows us to define the morphism

$$
(\rho^i_{\iota,m})' : W_m\Omega_Z^{i-r}/dV^{m-1}\Omega_Z^{i-r-1} \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X)
$$

which is compatible with $\rho^i_{\iota,m}$. Then, as homomorphisms of pro-sheaves, we have the equality $\rho^i_{\iota,\bullet} = (\rho^i_{\iota,\bullet})'$. By this fact and by definition of $\rho^{i,\log}_{\iota,m}$, one obtain the following commutative diagram, where the horizontal lines are exact:

$$
(3.7)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & W_m\Omega^{i-r}_{Z,\log} & \longrightarrow & W_m\Omega_Z^{i-r} & \xrightarrow{1-F} & W_m\Omega_Z^{i-r}/dV^{m-1}\Omega_Z^{i-r-1} & \longrightarrow & 0 \\
 & & \rho^{i,\log}_{\iota,m}\Big\downarrow & & \rho^i_{\iota,m}\Big\downarrow & & (\rho^i_{\iota,m})'\Big\downarrow & & \\
0 & \longrightarrow & \underline{H}^r_Z(X, W_m\Omega^i_{X,\log}) & \longrightarrow & \underline{H}^r_Z(X, W_m\Omega^i_X) & \xrightarrow{1-F} & \underline{H}^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega_X^{n-1}). & &
\end{array}
$$

(The exactness of the lower horizontal line follows from Lemma 3.3.)

Now we prove the commutativity of the diagram (3.5). The diagram (3.5) is factorized as

$$\begin{array}{ccccc}
H^0(Z, W_m\Omega^{i-r}_{Z,\log}) & \xrightarrow{H^0(\rho^{i,\log}_{\iota,m})} & H^0(Z, \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})) & \xrightarrow{\cong} & H^r_Z(X, W_m\Omega^i_{X,\log}) \\
{\scriptstyle \delta^0_Z}\downarrow & & \downarrow & & {\scriptstyle \delta^0_X}\downarrow \\
H^0(Z, W_m\Omega^{i-r}_Z) & \xrightarrow{H^0(\rho^i_{\iota,m})} & H^0(Z, \underline{H}^r_Z(X, W_m\Omega^i_X)) & \xrightarrow{\cong} & H^r_Z(X, W_m\Omega^i_X).
\end{array}$$

The left square is (the transpose of) H^0 of the left square in the diagram (3.7) and so it is commutative. The right square is commutative due to the functoriality of the Leray spectral sequence. So we have shown that (3.5) is commutative.

Next we prove the $(-1)^r$-commutativity of the diagram (3.6). If we denote the map

$$W_m\Omega^{i-r}_{Z,\log} \xrightarrow{\rho^{i,\log}_{\iota,m}} \underline{H}^r_Z(X, W_m\Omega^i_{X,\log}) \longrightarrow \underline{R\Gamma}_Z(X, W_m\Omega^i_{X,\log})[-r]$$

also by $\rho^{i,\log}_{\iota,m}$, then the diagram (3.7) induces the commutative diagram

$$(3.8)\qquad \begin{array}{ccccccc}
0 \longrightarrow & W_m\Omega^{i-r}_{Z,\log} & \longrightarrow & W_m\Omega^{i-r}_Z & \xrightarrow{1-F} & W_m\Omega^{i-r}_Z/dV^{m-1}\Omega^{i-r-1}_Z & \longrightarrow 0 \\
& {\scriptstyle \rho^{i,\log}_{\iota,m}}\downarrow & & {\scriptstyle \rho^i_{\iota,m}}\downarrow & & {\scriptstyle (\rho^i_{\iota,m})'}\downarrow & \\
& \underline{R\Gamma}_Z(X, W_m\Omega^i_{X,\log})[-r] & \longrightarrow & \underline{H}^r_Z(X, W_m\Omega^i_X) & \xrightarrow{1-F} & \underline{H}^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{n-1}_X), &
\end{array}$$

where the lower horizontal arrow is a distinguished triangle. Now let us note that the diagram (3.6) is factorized as

$$\begin{array}{ccccc}
H^0(Z, W_m\Omega^{i-r}_Z) & \xrightarrow{H^0(\rho^i_{\iota,m})} & H^0(Z, \underline{H}^r_Z(X, W_m\Omega^i_X)) & \xrightarrow{\cong} & H^r_Z(X, W_m\Omega^i_X) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(Z, W_m\Omega^{i-r}_Z/dV^{m-1}\Omega^{i-r-1}_Z) & \xrightarrow{H^0((\rho^i_{\iota,m})')} & H^0(Z, \underline{H}^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X)) & \xrightarrow{\cong} & H^r_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X) \\
\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow \\
H^1(Z, W_m\Omega^{i-r}_{Z,\log}) & \xrightarrow{\rho^{i,\log}_{\iota,m}} & H^1(Z, \underline{R\Gamma}_Z(X, W_m\Omega^i_{X,\log})[-r]) & \xrightarrow{\cong} & H^{r+1}(X, W_m\Omega^i_{X,\log}).
\end{array}$$

The upper left square is commutative since it is (the transpose of) H^0 of the right square of the diagram (3.8), and the lower left square is also commutative since it is the connecting homomorphism induced by (3.8). The upper right square is commutative by functoriality and the lower right square is

$(-1)^r$-commutative because the degree of the connecting homomorphisms α, β differs by r. So the diagram (3.6) is $(-1)^r$-commutative. □

Remark 3.13. Let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r between regular schemes over $\mathbb{F}_p$ and assume that ι admits a factorization $Z \overset{\iota'}{\hookrightarrow} Y \overset{\iota''}{\hookrightarrow} X$, where ι', ι'' is a regular closed immersion of codimension r', r'', respectively. In this remark, we prove the transitivity of the map $\rho^{i,\log}_{\iota,m}$: That is, we prove that the composite

$$W_m\Omega^{i-r}_{Z,\log} \xrightarrow{\rho^{i-r'',\log}_{\iota',m}} \underline{H}^{r'}_Z(Y, W_m\Omega^{i-r''}_{Y,\log}) \xrightarrow{\underline{H}^{r'}_Z(\rho^{i,\log}_{\iota'',m})} \underline{H}^{r'}_Z(Y, \underline{H}^{r''}_Y(X, W_m\Omega^i_{X,\log})) \longrightarrow \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})$$

(where the last map is induced by Leray spectral sequence) is equal to $\rho^{i,\log}_{\iota,m}$.

By the commutativity of the left square in the diagram (3.7), the claim is reduced to the corresponding claim for $\rho^i_{\iota,m}$, that is, it suffices to prove that the composite

$$W_m\Omega^{i-r}_{Z} \xrightarrow{\rho^{i-r''}_{\iota',m}} \underline{H}^{r'}_Z(Y, W_m\Omega^{i-r''}_{Y}) \xrightarrow{\underline{H}^{r'}_Z(\rho^{i}_{\iota'',m})} \underline{H}^{r'}_Z(Y, \underline{H}^{r''}_Y(X, W_m\Omega^i_{X})) \xrightarrow{\cong} \underline{H}^r_Z(X, W_m\Omega^i_{X})$$

is equal to $\rho^i_{\iota,m}$. To prove it, we may assume that X is local and then we can reduce to the smooth case using Theorems 2.1, 2.2. In this case, the claim (transitivity) is proved by Gros (see [G, proof of II,Prop 2.1.1, II(3.4.1–3)]). So we are done.

From the above claim, we can deduce that the composite

$$\begin{aligned} H^q(Z, W_m\Omega^{i-r}_{Z,\log}) &\xrightarrow{\theta^{q,i-r'',\log}_{\iota',m}} H^{q+r'}_Z(Y, W_m\Omega^{i-r''}_{Y,\log}) \\ &\xrightarrow{H^{q+r'}_Z(\rho^{i,\log}_{\iota'',m})} H^{q+r'}_Z(Y, \underline{H}^{r''}_Y(X, W_m\Omega^i_{X,\log})) \\ &\longrightarrow H^{q+r}_Z(X, W_m\Omega^i_{X,\log}) \end{aligned}$$

is equal to $\theta^{q,i,\log}_{\iota,m}$.

4. Gersten-type Conjecture (I)

Let X be an equidimensional scheme over $\mathbb{F}_p$. Then we have the coniveau spectral sequence ([Bl-Ogu], see also [CT-Ho-Kah])

$$E_1^{s,t} = \bigoplus_{x \in X^s} H_x^{s+t}(X, W_m\Omega^i_{X,\log}) \Longrightarrow E^{s+t} = H^{s+t}(X, W_m\Omega^i_{X,\log})$$

converging to the logarithmic Hodge-Witt cohomology. We call the complex of $E_1^{\bullet,q}$-terms

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{x\in X^0} H^q_x(X, W_m\Omega^i_{X,\log}) \longrightarrow \bigoplus_{x\in X^1} H^{q+1}_x(X, W_m\Omega^i_{X,\log}) \longrightarrow \cdots \\ &\longrightarrow \bigoplus_{x\in X^s} H^{q+s}_x(X, W_m\Omega^i_{X,\log}) \longrightarrow \cdots \end{aligned}$$

the Bloch-Ogus complex and denote it by $B_m^{q,i}(X)^\bullet$. It is a cohomological analogue of the Brown-Gersten-Quillen complex in algebraic K-theory. As an analogue of Gersten conjecture in algebraic K-theory, it is natural to expect that, if X is the spectrum of a regular local ring over $\mathbb{F}_p$, the complex $B_m^{q,i}(X)^\bullet$ is acyclic in positive degree. In fact, we have the following:

THEOREM 4.1 (Gersten-type conjecture for Bloch-Ogus complex). *Let X be the spectrum of an equidimensional regular local ring over $\mathbb{F}_p$. Then we have*

$$H^n(B_m^{q,i}(X)^\bullet) = \begin{cases} H^q(X, W_m\Omega^i_{X,\log}) & (n=0), \\ 0 & (n>0). \end{cases} \tag{4.1}$$

The purpose of this section is to give a proof of the above theorem. It is proved in the case where X is a localization of a smooth scheme over a perfect field by Gros-Suwa([G-Su, Thm 1.4]). We will reduce the general case to the smooth case by using a technique of Panin ([Pa]). (In the paper [Pa], he proves the Gersten conjecture for K-groups in equicharacteristic case.)

First let us note that, by the argument in the proof of Proposition 3.3 and Corollary 3.4, both $H^q(X, W_m\Omega^i_{X,\log})$ and $B_m^{q,i}(X)^s$ are zero in the case $q > 1$. So the assertion is automatically true when $q > 1$. Moreover, one can prove the theorem for $q = 1$ by an easy diagram chase from the theorem for $q = 0$. So we may assume $q = 0$ to prove the theorem.

We prepare some preliminary results which we need for the proof of Theorem 4.1.

PROPOSITION 4.2. *Let $\iota : Z \hookrightarrow X$ be a regular closed immersion of codimension r between regular schemes. Denote the subcomplex*

$$0 \longrightarrow \bigoplus_{x \in Z^0} H^r_x(X, W_m\Omega^i_{X,\log}) \longrightarrow \bigoplus_{x \in Z^1} H^{r+1}_x(X, W_m\Omega^i_{X,\log}) \longrightarrow \cdots$$

(where the first non-zero term is sitting in degree r) of $B^{0,i}_m(X)^\bullet$ by $B^{0,i}_{m,Z}(X)^\bullet$. Then the isomorphisms

$$\psi_x := \theta^{0,i,\log}_{x \hookrightarrow X,m} \circ (\theta^{0,i-r,\log}_{x \hookrightarrow Z,m})^{-1} : H^s_x(Z, W_m\Omega^{i-r}_{Z,\log}) \longrightarrow H^{s+r}_x(X, W_m\Omega^i_{X,\log}) \qquad (x \in Z^s, s \in \mathbb{N})$$

induce the isomorphism of complexes $B^{0,i-r}_m(Z)^\bullet \xrightarrow{\cong} B^{0,i}_{m,Z}(X)^\bullet[r]$.

PROOF. By Remark 3.13, ψ_x is the composite

$$\text{(4.2)} \qquad H^s_x(Z, W_m\Omega^{i-r}_{Z,\log}) \overset{H^s_x(\rho^{i,\log}_{Z \hookrightarrow X,m})}{\longrightarrow} H^s_x(Z, \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})) \longrightarrow H^{s+r}_x(X, W_m\Omega^i_{X,\log}),$$

where the second arrow is induced by Leray spectral sequence. Take points $y \in Z^{s-1}, x \in Z^s$. Then the (y, x)-component of the boundary map of $B^{0,i}_{m,Z}(X)^\bullet$ is non-zero only if $x \in \overline{\{y\}} =: Y$ holds and in this case, it is the connecting homomorphism in the long exact sequence

$$\text{(4.3)} \qquad \cdots \longrightarrow H^j_{Y_x}(X_x, W_m\Omega^i_{X,\log}) \longrightarrow H^j_y(X_x - x, W_m\Omega^i_{X,\log}) \longrightarrow H^{j+1}_x(X_x, W_m\Omega^i_{X,\log}) \longrightarrow \cdots,$$

where X_x, Y_x denotes the localization of X, Y at x, respectively. (Note that we have $H^j_y(X, W_m\Omega^i_{X,\log}) = H^j_y(X_x - x, W_m\Omega^i_{X,\log})$, $H^{j+1}_x(X, W_m\Omega^i_{X,\log}) = H^{j+1}_x(X_x, W_m\Omega^i_{X,\log})$ by excision.) Similar description is true also for the boundary map of $B_m^{0,i-r}(Z)^\bullet$. So it suffices to prove that, in the diagram

$$\begin{array}{ccccc} H_y^{s-1}(Z, W_m\Omega^{i-r}_{Z,\log}) & \xrightarrow{H_y^{s-1}(\rho^{i,\log}_{Z\hookrightarrow X,m})} & H_y^{s-1}(Z, \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})) & \longrightarrow & H_y^{s+r-1}(X, W_m\Omega^i_{X,\log}), \\ \downarrow & & \downarrow & & \downarrow \\ H_x^{s}(Z, W_m\Omega^{i-r}_{Z,\log}) & \xrightarrow{H_x^{s}(\rho^{i,\log}_{Z\hookrightarrow X,m})} & H_x^{s}(Z, \underline{H}^r_Z(X, W_m\Omega^i_{X,\log})) & \longrightarrow & H_x^{s+r}(X, W_m\Omega^i_{X,\log}) \end{array}$$

(where the horizontal lines are defined as in (4.2) and the vertical lines are the connecting homomorphisms arising from the long exact sequences like (4.3)), the large rectangle is $(-1)^r$-commutative. In fact, it is true because the left square is commutative and the right square is $(-1)^r$-commutative. (The sign $(-1)^r$ arises from the difference of the degrees.) So we are done. □

Lemma 4.3. *Let $X = \operatorname{Spec} A$ be as in Theorem 4.1, let t be a local parameter in A and let us denote the scheme $\operatorname{Spec} A[1/t]$ by X_t. Denote the canonical morphism of sites $X_{\mathrm{et}} \longrightarrow X_{\mathrm{Zar}}$ by α. Then we have $H^n(X_{t,\mathrm{Zar}}, \alpha_* W_m\Omega^i_{X,\log}) = 0$ for $n > 0$.*

Proof. First let us consider the case where X is a localization of a smooth scheme over a perfect field. Let Z be the closed subscheme of X defined by the equation $t = 0$. Then we have the exact sequence of complexes

$$0 \longrightarrow B^{0,i}_{m,Z}(X)^\bullet \longrightarrow B^{0,i}_m(X)^\bullet \longrightarrow B^{0,i}_m(X_t)^\bullet \longrightarrow 0,$$

and by Proposition 4.2, we have the isomorphism $B_m^{0,i-1}(Z)^\bullet \cong B^{0,i}_{m,Z}(X)^\bullet[1]$. By the Gersten-type conjecture in smooth case by Gros-Suwa, we have $H^n(B_m^{0,i-1}(Z)^\bullet) = 0, H^n(B_m^{0,i}(X)^\bullet) = 0$ for $n > 0$. Hence we have $H^n(B_m^{0,i}(X_t)^\bullet) = 0$ for $n > 0$. On the other hand, let $\underline{B}_m^{0,i}(X_t)^\bullet$ be the Zariski sheafification of $B_m^{0,i}(X_t)^\bullet$. Then, we have $B_m^{0,i}(X_t)^\bullet = \Gamma(X_t, \underline{B}_m^{0,i}(X_t)^\bullet)$ and each term of $\underline{B}_m^{0,i}(X_t)$ is flasque. Moreover, by the Gersten-type conjecture in smooth case, the complex $\underline{B}_m^{0,i}(X_t)^\bullet$ is a resolution of the sheaf $\alpha_*\Omega^n_{X_t,\log}$. Hence we have $H^n(X_t, \alpha_* W_m\Omega^i_{X,\log}) = H^n(B_m^{0,i}(X_t)^\bullet) = 0$ for $n > 0$. Hence the assertion is proved in the case where X is a localization of a smooth scheme over a perfect field.

In general case, X can be written as a projective limit of localizations of smooth schemes X_j $(j \in J)$ over $\mathbb{F}_p$ such that t is a local parameter in each $\mathcal{O}_{X_j}$. Then we have

$$H^n(X_t, \alpha_* W_m \Omega^i_{X,\log}) = \varinjlim_{j \in J} H^n(X_{j,t}, \alpha_* W_m \Omega^i_{X_j,\log}) = 0$$

for $n > 0$. □

Lemma 4.4. *With the notation in Lemma 4.3, the sequence*

$$0 \to H^0(X, W_m\Omega^i_{X,\log}) \to H^0(X_t, W_m\Omega^i_{X,\log}) \to H^1_Z(X, W_m\Omega^i_{X,\log}) \to 0$$

induced by the localization sequence is exact.

Proof. It suffices to prove the injectivity of the morphism $H^1(X, W_m\Omega^i_{X,\log}) \longrightarrow H^1(X_t, W_m\Omega^i_{X,\log})$ and the vanishing $H^0_Z(X, W_m\Omega^i_{X,\log}) = 0$. Both assertions can be reduced to the smooth case, by using Theorems 2.1, 2.2. Hence we assume that X is a localization of a smooth scheme over a perfect field. Then, by Theorem 4.1 for smooth case, we have

$$H^1(X_t, W_m\Omega^i_{X,\log}) \subset \bigoplus_{x \in X_t^0} H^1_x(X_t, W_m\Omega^i_{X,\log}),$$
$$H^1(X, W_m\Omega^i_{X,\log}) \subset \bigoplus_{x \in X^0} H^1_x(X, W_m\Omega^i_{X,\log})$$

and the restriction $H^1(X, W_m\Omega^i_{X,\log}) \longrightarrow H^1(X_t, W_m\Omega^i_{X,\log})$ is compatible with the canonical identification

$$\bigoplus_{x \in X^0} H^1_x(X, W_m\Omega^i_{X,\log}) = \bigoplus_{x \in X_t^0} H^1_x(X_t, W_m\Omega^i_{X,\log}).$$

Hence we obtain the first assertion. The second assertion can be proved in the same way as Corollary 3.4. □

Now we give the proof of Theorem 4.1.

Proof of Theorem 4.1. We already have reduced the theorem to the case $q = 0$. So we assume this and prove the theorem by induction on the dimension of X. The case $\dim X = 1$ follows from Lemma 4.4. Let us consider the general case. Let us take a local parameter t and let X_t, Z, α

be as in Lemma 4.3. Then, by Proposition 4.2, we have $B_m^{0,i-1}(Z)^\bullet \cong B_{m,Z}^{0,i}(X)^\bullet[1]$ and we have the exact sequence

$$0 \longrightarrow B_{m,Z}^{0,i}(X)^\bullet \longrightarrow B_m^{0,i}(X)^\bullet \longrightarrow B_m^{0,i}(X_t)^\bullet \longrightarrow 0. \tag{4.4}$$

By induction hypothesis, the theorem is true for Z, that is, we have isomorphisms $H^0(Z, W_m\Omega^{i-1}_{Z,\log}) = H^0(B_m^{0,i-1}(Z)), H^n(B_m^{0,i-1}(Z)) = 0\,(n > 0)$. So we obtain the isomorphisms $H^1_Z(X, W_m\Omega^i_{X,\log}) = H^1(B_{m,Z}^{0,i}(X)^\bullet)$, $H^n(B_{m,Z}^{0,i}(X)) = 0\,(n \geq 2)$. On the other hand, let $\underline{B}_m^{0,i}(X_t)^\bullet$ be the Zariski sheafification of $B_m^{0,i}(X_t)$. Then we have

$$\begin{aligned} H^n(B_m^{0,i}(X_t)^\bullet) &= H^n(H^0(X_{t,\mathrm{Zar}}, \underline{B}_m^{0,i}(X_t)^\bullet)) \\ &= H^n(X_{t,\mathrm{Zar}}, \underline{B}_m^{0,i}(X_t)^\bullet) \quad (\underline{B}_m^{0,i}(X_t)^\bullet \text{ is flasque}) \\ &= H^n(X_{t,\mathrm{Zar}}, \alpha_* W_m\Omega^i_{X,\log}) \quad (\text{induction hypothesis}) \\ &= \begin{cases} H^0(X_t, W_m\Omega^i_{X,\log}), & n = 0 \\ 0, & n > 0 \end{cases} \quad (\text{Lemma 4.3}). \end{aligned}$$

Then, by the exact sequence (4.4), we have $H^n(B_m^{0,i}(X)^\bullet) = 0\,(n \geq 2)$. Moreover, we have the following commutative diagram induced from the localization sequence and the exact sequence (4.4)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(X, W_m\Omega^i_{X,\log}) & \longrightarrow & H^0(X_t, W_m\Omega^i_{X,\log}) & \longrightarrow & H^1_Z(X, W_m\Omega^i_{X,\log}) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & H^0(B_m^{0,i}(X)^\bullet) & \longrightarrow & H^0(B_m^{0,i}(X_t)^\bullet) & \longrightarrow & H^1(B_{m,Z}^{0,i}(X)^\bullet) & \longrightarrow & H^1(B_m^{0,i}(X)^\bullet) & \longrightarrow & 0, \end{array}$$

where the horizontal lines are exact (the exactness of the upper horizontal line follows from Lemma 4.4) and the middle and the right vertical arrows are isomorphisms. From this diagram, one can deduce the equalities $H^0(B_m^{0,i}(X)^\bullet) = H^0(X, W_m\Omega^i_{X,\log})$ and $H^1(B_m^{0,i}(X)^\bullet) = 0$. So we are done. □

5. Gersten-type Conjecture (II)

In the paper [Kat3], Kato constructed a complex similar to the Bloch-Ogus complex (which we call Kato complex) for an excellent scheme satisfying certain condition. In the case where X is of characteristic p, the

p-primary part of the Kato complex is constructed via K-theoretic method. The aim of this section is to prove the Gersten-type conjecture for the p-primary part of the Kato complex for schemes of characteristic p.

Let us roughly recall the definition of the p-primary part of the Kato complex for schemes over $\mathbb{F}_p$. For a field k of characteristic $p > 0$, let us denote the n-th Milnor K-group of k by $K_n^{\mathrm{M}}(k)$. Denote the symbol map $K_n^{\mathrm{M}}(k)/p^m \longrightarrow H^n(\operatorname{Spec} k, W_m\Omega^n_{\operatorname{Spec} k,\log})$, which is characterized by $\{a_1, \cdots, a_n\} \mapsto \operatorname{dlog} \underline{a}_1 \wedge \cdots \wedge \operatorname{dlog} \underline{a}_n$, by h_k^n. By Bloch-Gabber-Kato theorem ([Bl-Kat]), h_k^n is an isomorphism. For a discrete valuation field K with integer ring O and residue field k, let us denote by ∂ the tame symbol $K^{\mathrm{M}}_{n+1}(K) \longrightarrow K_n^{\mathrm{M}}(k)$, which is characterized by $\{t, a_1, \cdots, a_n\} \mapsto \{\overline{a}_1, \cdots, \overline{a}_r\}$ for $a_i \in O^{\times}$ and a uniformizer t (where $\overline{a}_i$ denotes the residue class of a_i).

Let $m \in \mathbb{N}, s, i \in \mathbb{Z}$ and let X be an excellent scheme over $\mathbb{F}_p$ satisfying the following condition:

When $s = i + 1$ holds, we have $[\kappa(x) : \kappa(x)^p] \leq p^i$ for any closed point $x \in X$.

For such X, the Kato complex $C^{s,i}_{p^m,X}$ (or $(C^{s,i,\bullet}_{p^m,X}, d^{\bullet}_{C^{s,i}_{p^m,X}})$) is defined and it has the form

$$\begin{aligned} C^{s,i}_{p^m,X} : \quad \cdots \longrightarrow \bigoplus_{x \in X_j} H^{s-i}(x, W_m\Omega^{i+j}_{x,\log}) \longrightarrow \cdots \\ \longrightarrow \bigoplus_{x \in X_1} H^{s-i}(x, W_m\Omega^{i+1}_{x,\log}) \longrightarrow \bigoplus_{x \in X_0} H^{s-i}(x, W_m\Omega^{i}_{x,\log}) \longrightarrow 0 \end{aligned}$$

(where the last non-zero term is sitting at degree 0). Let us recall the definition of the boundary map of the complex. (Note that it suffices to define in the case $s = i, i+1$ because all the terms are equal to zero otherwise.)

First, let K be a discrete valuation field of characteristic $p > 0$ with residue field k satisfying $[k : k^p] \leq p^i$ if $s = i + 1$. Then we define the homomorphism

$$\partial_{K,k} : H^{s-i}(K, W_m\Omega^{i+1}_{\operatorname{Spec} K,\log}) \longrightarrow H^{s-i}(k, W_m\Omega^{i}_{\operatorname{Spec} k,\log})$$

as follows: When $s=i$ holds, it is defined as the composite

$$H^0(K, W_m\Omega^{i+1}_{\mathrm{Spec}\,K,\log}) \xrightarrow{(h^{i+1}_K)^{-1}} K^{\mathrm{M}}_{i+1}(K)/p^m \xrightarrow{\partial/p^m} K^{\mathrm{M}}_i(k)/p^m \xrightarrow{h^i_k} H^0(k, W_m\Omega^i_{\mathrm{Spec}\,k,\log}).$$

When $s=i+1$ holds, it is defined as the composite

$$\begin{aligned} H^1(K, W_m\Omega^{i+1}_{\mathrm{Spec}\,K,\log}) &\longrightarrow H^1(\hat{K}, W_m\Omega^{i+1}_{\mathrm{Spec}\,\hat{K},\log}) \\ &\cong H^1(k, H^0(\hat{K}^{\mathrm{sh}}, W_m\Omega^{i+1}_{\mathrm{Spec}\,\hat{K}^{\mathrm{sh}},\log})) \\ &\xrightarrow{H^1(\partial_{\hat{K}^{\mathrm{sh}},\overline{k}})} H^1(k, W_m\Omega^i_{\mathrm{Spec}\,k,\log}), \end{aligned}$$

where $\hat{K}$ denotes the completion of K, $\hat{K}^{\mathrm{sh}}$ denotes the maximal unramified extension of $\hat{K}$ and $\overline{k}$ denotes the separable closure of k. Note that the second isomorphism follows from the facts $\mathrm{cd}_p(k) \leq 1$ and $H^1(\hat{K}^{\mathrm{sh}}, W_m\Omega^i_{\hat{K},\log}) = 0$.

Now let X be as above and let $y \in X_{j+1}, x \in X_j$. (Note that we have $[\kappa(y):\kappa(y)^p] \leq p^{i+j+1}$ in the case $s=i+1$ ([Ku, Cor 2.6, 2.7]).) Then the (y,x)-component

$$\partial_{y,x} : H^{s-i}(y, W_m\Omega^{i+j+1}_{y,\log}) \longrightarrow H^{s-i}(x, W_m\Omega^{i+j}_{x,\log})$$

of the boundary map of the complex $C^{s,i}_{p^m,X}$ is defined as follows: If x is not contained in the closure $\overline{\{y\}}$ of y in X, we define $\partial_{y,x} := 0$. If we have $x \in \overline{\{y\}}$, let $\pi : Y \longrightarrow X$ be the normalization of $\overline{\{y\}}$, and put $S := \{v \in Y \mid \pi(v) = x\}$. Then $\partial_{y,x}$ is defined by

$$\partial_{y,x} := \sum_{v\in S} \mathrm{Cor}_{\kappa(v)/\kappa(x)} \circ \partial_{\kappa(y),\kappa(v)},$$

where $\partial_{\kappa(y),\kappa(v)}$ is as in the previous paragraph, and $\mathrm{Cor}_{\kappa(v)/\kappa(x)} : H^{s-i}(v, W_m\Omega^{i+j}_{v,\log}) \longrightarrow H^{s-i}(x, W_m\Omega^{i+j}_{x,\log})$ is the 'corestriction map': In the case $s=i$, it is defined as the composite

$$H^{s-i}(v, W_m\Omega^{i+j}_{v,\log}) \xrightarrow{(h^{i+j}_{\kappa(v)})^{-1}} K^{\mathrm{M}}_{i+j}(\kappa(v))/p^m \xrightarrow{N/p^m} K^{\mathrm{M}}_{i+j}(\kappa(x))/p^m \xrightarrow{h^{i+j}_{\kappa(x)}} H^{s-i}(x, W_m\Omega^{i+j}_{x,\log}),$$

where N/p^m is induced by the norm map $N : K^{\mathrm{M}}_{i+j}(\kappa(v)) \longrightarrow K^{\mathrm{M}}_{i+j}(\kappa(x))$ of Milnor K-groups. In the case $s = i+1$, see the following remark.

Remark 5.1. Let $m, i \in \mathbb{Z}, m > 0, i \geq 0$. For a finite extension $K \subset K'$ of fields of characteristic $p > 0$, Kato defined in [Kat1, p.658] the corestriction map

$$H^1(K', W_m\Omega^i_{\mathrm{Spec}\,K',\log}) \longrightarrow H^1(K, W_m\Omega^i_{\mathrm{Spec}\,K,\log})$$

by using Bloch's theory of typical part of K-groups. (Note that the group $H^1(K, W_m\Omega^i_{\mathrm{Spec}\,K,\log})$ is denoted as $P^i_m(K)$ in [Kat1]. See also [J-Sai-Sat].) Let us denote the above homomorphism by $\mathrm{Cor}^{m,i}_{K'/K}$. Then the homomorphism $\mathrm{Cor}_{\kappa(v)/\kappa(x)}$ used in the definition of Kato complex in the case $s = i+1$ is nothing but the map $\mathrm{Cor}^{m,i+j}_{\kappa(v)/\kappa(x)}$ in this notation.

We omit the definition of $\mathrm{Cor}^{m,i}_{K'/K}$: Here we just give some properties which they satisfy.

(1) Let $\{\ ,\ \}_K : H^1(K, \mathbb{Z}/p^m\mathbb{Z}) \times K_i(K)/p^m \longrightarrow H^1(K, W_m\Omega^i_{\mathrm{Spec}\,K,\log})$ be the composite of the symbol map h^i_K and the cup product, and let $\alpha : H^1(K, \mathbb{Z}/p^m\mathbb{Z}) \longrightarrow H^1(K', \mathbb{Z}/p^m\mathbb{Z})$, $\beta : K_i(K)/p^m \longrightarrow K_i(K')/p^m$ be the maps induced by the inclusion $K \subset K'$ of fields. Then we have

$$\mathrm{Cor}^{m,i}_{K'/K}\{\alpha(x), y\}_{K'} = \{x, N(y)\}_K,$$

$$\mathrm{Cor}^{m,i}_{K'/K}\{x, \beta(y)\}_{K'} = \{\mathrm{Cor}^{m,0}_{K'/K}(x), y\}_K,$$

where N is the norm map of K-groups.

(2) For finite extensions $K \subset K' \subset K''$ of fields of characteristic $p > 0$, we have the transitivity

$$\mathrm{Cor}^{m,i}_{K'/K} \circ \mathrm{Cor}^{m,i}_{K''/K'} = \mathrm{Cor}^{m,i}_{K''/K}.$$

(3) Let us identify $H^1(L, \mathbb{Z}/p^m\mathbb{Z})$ with $W_mL/(1-F)W_mL \quad (L = K, K')$ by the isomorphism induced by the connecting homomorphism of the exact sequence

$$0 \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \longrightarrow W_m\mathcal{O}_{\mathrm{Spec}\,L} \xrightarrow{1-F} W_m\mathcal{O}_{\mathrm{Spec}\,L} \longrightarrow 0.$$

Then, if $K \subset K'$ is separable, $\mathrm{Cor}^{m,0}_{K'/K}$ is identical with the map $W_mK'/(1-F)W_mK' \longrightarrow W_mK/(1-F)W_mK$ induced by the trace map $W_mK' \longrightarrow W_mK$ in classical sense (note that the extension $W_mK \subset W_mK'$ is finite etale).

(4) The following diagram is commutative:

$$\begin{array}{ccc} H^1(K', W_m\Omega^i_{\mathrm{Spec}\,K',\log}) & \xrightarrow{\mathrm{Cor}^{m,i}_{K'/K}} & H^1(K, W_m\Omega^i_{\mathrm{Spec}\,K,\log}) \\ {\scriptstyle R}\downarrow & & {\scriptstyle R}\downarrow \\ H^1(K', W_{m-1}\Omega^i_{\mathrm{Spec}\,K',\log}) & \xrightarrow{\mathrm{Cor}^{m-1,i}_{K'/K}} & H^1(K, W_{m-1}\Omega^i_{\mathrm{Spec}\,K,\log}). \end{array}$$

(The assertion (1) are proved in [Kat1, 3.2, Lem 1], and one can check the assertions (2), (3), (4) by looking at the definition of $\mathrm{Cor}^{m,i}_{K'/K}$ carefully. In this paper, we only use the assertions (1), (2), (3) in the case $m = 1$.)

Let $q, i, m, n \in \mathbb{N}$ and let X be an n-dimensional excellent scheme over $\mathbb{F}_p$ satisfying the following condition:

When $q = 1$ holds, we have $[\kappa(x) : \kappa(x)^p] \leq p^i$ for any $x \in X^0$.

For such X, we denote the complex $C^{i-n+q,i-n}_{p^m,X}\{-n\}$ (that is, the complex $(C^{i-n+q,i-n,\bullet-n}_{p^m,X}, d^{\bullet-n}_{C^{i-n+q,i-n}_{p^m,X}})$) by $C^{q,i}_m(X)^\bullet$ (or $(C^{q,i}_m(X)^\bullet, d^\bullet_{C^{q,i}_m(X)})$) and we also call it the Kato complex. The main result in this section is the following:

Theorem 5.2. (Gersten-type conjecture for Kato complex). *Let X be the spectrum of an excellent regular local ring over $\mathbb{F}_p$ such that $[\kappa(x) : \kappa(x)^p] = p^N$ holds for $x \in X^0$. Let $q, i, m \in \mathbb{N}$ and assume that $i \geq N$ holds in the case $q = 1$. (In this case, the complex $C^{q,i}_m(X)^\bullet$ is defined.) Then we have*

$$H^n(C^{q,i}_m(X)^\bullet) = \begin{cases} H^q(X, W_m\Omega^i_{X,\log}) & (n = 0), \\ 0 & (n > 0). \end{cases} \tag{5.1}$$

Remark 5.3. We can see that both hand sides are zero if $q > 1$ holds or $i > N$ holds. So it suffices to prove the theorem in the case $q = 0$ and the case $(q, i) = (1, N)$.

Let X, q, i, m, N be as in the statement of Theorem 5.2 and suppose either $q = 0$ or $(q, i) = (1, N)$. For $x \in X$, denote the canonical inclusion $x \hookrightarrow X$ by ι_x. Then, for $x \in X^s$, we have the isomorphism of purity

$$\theta^{q,i,\log}_{\iota_x,m} : H^q(x, W_m\Omega^{i-s}_{x,\log}) \xrightarrow{\cong} H^{q+s}_x(X, W_m\Omega^i_{X,\log})$$

and it induces the isomorphism $\theta^s : C^{q,i}_m(X)^s \longrightarrow B^{q,i}_m(X)^s$ of each terms of the complexes $B^{q,i}_m(X)^\bullet, C^{q,i}_m(X)^\bullet$. However, it is not a priori clear whether θ^s's induce the isomorphism of these complexes. The key result for the proof of Theorem 5.2 is the following theorem, which gives a partial answer to the above question:

Theorem 5.4. *Let X be an excellent regular scheme over $\mathbb{F}_p$ such that $[\kappa(x) : \kappa(x)^p] = p^N$ holds for any $x \in X^0$. Let $q, i \in \mathbb{N}$ and assume that either $q = 0$ or $(q, i) = (i, N)$ holds. Then the maps $\theta^s : C^{q,i}_1(X)^s \longrightarrow B^{q,i}_1(X)^s$ $(s \in \mathbb{N})$ defined above induces the isomorphism of complexes*

$$\overline{C}^{q,i}_1(X)^\bullet \longrightarrow B^{q,i}_1(X)^\bullet,$$

where $\overline{C}^{q,i}_1(X)^\bullet$ denotes the complex $(C^{q,i}_1(X)^\bullet, (-1)^{\bullet-1}d^\bullet_{C^{q,i}_1(X)})$.

Remark 5.5. We expect that the maps $\theta^s : C^{q,i}_m(X)^s \longrightarrow B^{q,i}_m(X)^s$ $(s \in \mathbb{N})$ induces the isomorphism of complexes $\overline{C}^{q,i}_m(X)^\bullet \longrightarrow B^{q,i}_m(X)^\bullet$ for any $m \in \mathbb{N}$, where $\overline{C}^{q,i}_m(X)^\bullet$ denotes the complex $(C^{q,i}_m(X)^\bullet, (-1)^{\bullet-1}d^\bullet_{C^{q,i}_m(X)})$. In the case where X is a smooth scheme over a perfect field, this claim (for general m) is stated in [G-Su, Rem 4.19], [Su, Rem 1.3] in the case $q = 0$ and in [Su, Rem 2.12] in the case $(q, i) = (1, N)$. However, the proofs given there seem to be incomplete. In a recent work of Jannsen-Saito-Sato [J-Sai-Sat], they give a complete proof of the claim for general m in the case where X is a smooth scheme over a perfect field, by using a theory of trace map in de Rham-Witt cohomology developped by Ekedahl [E] and Gros [G]. Our proof (for $m = 1$ and X is excellent regular) uses the theory of trace map for generalized residual complex by Hartshorne. It seems that, if we can develop a satisfactory theory of trace map in de Rham-Witt cohomology for regular schemes which are not necessarily smooth over a perfect field, the proof of Jansenn-Saito-Sato or us would be generalized to the case where X is excellent regular and m is arbitrary.

Before the proof of Theorem 5.4, we give a proof of Theorem 5.2 admitting Theorem 5.4:

PROOF OF THEOREM 5.4 $\Longrightarrow$ THEOREM 5.2. By Theorem 4.1 and Theorem 5.4, we see that Theorem 5.2 is true for $m = 1$. Moreover, we have the exact sequence of complexes

$$0 \longrightarrow C_1^{q,i}(X)^\bullet \xrightarrow{p^{m-1}} C_m^{q,i}(X)^\bullet \longrightarrow C_{m-1}^{q,i}(X)^\bullet \longrightarrow 0 :$$

Indeed, the case $q = 0$ follows from Bloch-Gabber-Kato theorem and the case $q = 1$ follows from the case $q = 0$. Then we can prove $H^n(C_m^{q,i}(X)^\bullet) = 0$ for $n > 0$ by induction on m, using the above exact sequence. Moreover, we have a commutative diagram

$$\begin{array}{ccccccccc} & & H^q(X, W_1\Omega^i_{X,\log}) & \xrightarrow{p^{m-1}} & H^q(X, W_m\Omega^i_{X,\log}) & \longrightarrow & H^q(X, W_{m-1}\Omega^i_{X,\log}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^q(C^{0,i}_{m-1}(X)^\bullet) & \xrightarrow{p^{m-1}} & H^0(C^{q,i}_m(X)^\bullet) & \longrightarrow & H^0(C^{q,i}_1(X)^\bullet) & \longrightarrow & 0, \end{array}$$

where the vertical arrows are the maps induced by the restriction $H^q(X, W_m\Omega^i_{X,\log}) \longrightarrow \bigoplus_{x \in X^0} H^q(x, W_m\Omega^i_{x,\log})$ and the horizontal lines are exact. Then, by induction on m, we can also prove the equality $H^0(C_m^{q,i}(X)^\bullet) = H^q(X, W_m\Omega^i_{X,\log})$ (under the condition $q = 0$ or $(q,i) = (1,N)$ holds). So we are done. □

Now we fix some notations for the proof of Theorem 5.4. Fix $r \in \mathbb{N}$ and take points $y \in X^{r-1}$, $z \in X^r$ such that $z \in \overline{\{y\}}$ holds. For $q = 0$ or $(q,i) = (1,N)$, let

$$\partial_B^q : H_y^{q+r-1}(X, \Omega^i_{X,\log}) \longrightarrow H_z^{q+r}(X, \Omega^i_{X,\log}),$$

$$\partial_C^q : H^q(y, \Omega^{i-r+1}_{y,\log}) \longrightarrow H^q(z, \Omega^{i-r}_{z,\log}),$$

be the (y,z)-component of the boundary map of the complex $B_1^{q,i}(X)^\bullet$, $C_1^{q,i}(X)^\bullet$. To prove Theorem 5.4, it suffices to show the equality $(-1)^r\theta^{q,i,\log}_{z\hookrightarrow X,1}\circ\partial^q_C=\partial^q_B\circ\theta^{q,i,\log}_{y\hookrightarrow X,1}$.

Since the homomorphisms $\partial^q_B,\partial^q_C$ are unchanged when we localize X at z, we may assume that X is local with closed point z. Let $\pi:\hat{X}\longrightarrow X$ be the completion of X along z, and put $\pi^{-1}(y)=\{y_1,\cdots,y_l\}$. Let $\pi_i^*: H^q(\kappa(y),W_m\Omega^{N-r+1}_{y,\log})\longrightarrow H^q(\kappa(y_i),W_m\Omega^{N-r+1}_{y_i,\log})\,(1\le i\le l)$ be the homomorphism induced by π. Then one can check that, for $?=B,C$, we have the following equality:

$$(\partial^q_? \text{ for } (X,y,z))=\sum_{i=1}^{l}(\partial^q_? \text{ for } (\hat{X},y_i,z))\circ\pi_i^*.$$

(Here we use claim 2 in the proof of Theorem 3.2, Step 2.) Hence we may assume that X is the spectrum of a complete regular local ring with closed point z. Under this assumption, let us denote the closure of y in X by Y. Then Y is a one-dimensional integral closed subscheme of X.

By definition, the map ∂^q_B is the connecting homomorphism in the long exact sequence

$$\begin{aligned}\cdots\longrightarrow H^j_Y(X,\Omega^i_{X,\log})\longrightarrow H^j_y(X-z,\Omega^i_{X-z,\log})&\\ \longrightarrow H^{j+1}_z(X,\Omega^i_{X,\log})\longrightarrow\cdots.&\end{aligned}$$

We define a related map $\tilde{\partial}_B: H^{r-1}_y(X,\Omega^i_X)\longrightarrow H^r_z(X,\Omega^i_X)$ as the connecting homomorphism in the long exact sequence

$$\cdots\longrightarrow H^j_Y(X,\Omega^i_X)\longrightarrow H^j_y(X-z,\Omega^i_{X-z})\longrightarrow H^{j+1}_z(X,\Omega^i_X)\longrightarrow\cdots.$$

On the other hand, Let us define the maps

$$\begin{aligned}\overline{\partial}^q_C&: H^q(y,\Omega^{i-r+1}_{y,\log})\longrightarrow H^{q+1}_z(Y,\Omega^{i-r+1}_{Y,\log}),\\ \tilde{\partial}_C&: H^0(y,\Omega^{i-r+1}_y)\longrightarrow H^1_z(Y,\Omega^{i-r+1}_Y)\end{aligned}$$

as the connecting homomorphism of the localization sequence for $z\hookrightarrow Y\hookleftarrow Y-z$.

Now let us give a proof of Theorem 5.4 in the case where Y is regular, that is, the case where $\mathcal{O}_Y$ is a discrete valuation ring. First we treat the case $q=0$:

PROOF OF THEOREM 5.4, *Step* 1: *The case* Y *is regular and* $q = 0$. First, let us prove the following claim:

CLAIM. ∂_C^0 is factorized as

$$H^0(y, \Omega^{i-r+1}_{y,\log}) \xrightarrow{-\overline{\partial}_C^0} H^1_z(Y, \Omega^{i-r+1}_{Y,\log}) \xrightarrow{(\theta^{0,i-r+1,\log}_{z\hookrightarrow Y,1})^{-1}} H^0(z, \Omega^{i-r}_{z,\log}).$$

PROOF OF CLAIM. Let us note the following commutative diagrams

$$\begin{array}{ccc} H^0(y, \Omega^{i-r+1}_{y,\log}) & \xrightarrow{\overline{\partial}_C^0} & H^1_z(Y, \Omega^{i-r+1}_{Y,\log}) \\ \downarrow & & \downarrow \\ H^0(y, \Omega^{i-r+1}_{y}) & \xrightarrow{\tilde{\partial}_C} & H^1_z(Y, \Omega^{i-r+1}_{Y}), \end{array} \quad \begin{array}{ccc} H^0(z, \Omega^{i-r}_{z,\log}) & \xrightarrow{\theta^{0,i-r,\log}_{z\hookrightarrow Y,1}} & H^1_z(Y, \Omega^{i-r+1}_{Y,\log}) \\ \downarrow & & \downarrow \\ H^0(z, \Omega^{i-r}_{z}) & \xrightarrow{\theta^{i-r}_{z\hookrightarrow Y,1}} & H^1_z(Y, \Omega^{i-r+1}_{Y}), \end{array}$$

where the vertical arrows are induced by the inclusion $\Omega^*_{?,\log} \hookrightarrow \Omega^*_?$. Noting the definition of $\theta^{i-r+1}_{z\hookrightarrow Y,1}$, we see that the proof is reduced to the following claim:

Let t be a uniformizer of $\mathcal{O}_Y$, let $y_1, \cdots, y_{i-r} \in \mathcal{O}_Y^\times$ and denote $\operatorname{dlog} y_1 \wedge \cdots \wedge \operatorname{dlog} y_{i-r}$ by $\operatorname{dlog} y$. Then we have the equality $\tilde{\partial}_C(\operatorname{dlog} t \wedge \operatorname{dlog} y) = -\operatorname{dlog} t \wedge \operatorname{dlog} y$, if we identify $H^1_z(Y, \Omega^{i-r+1}_Y)$ with $\Omega^{i-r+1}_y / \Omega^{i-r+1}_Y$ by using t (see (3.2)).

We prove this claim. The map $\tilde{\partial}_C$ is induced by the distinguished triangle

$$\underline{R\Gamma}_z(Y, \Omega^{i-r+1}_Y) \longrightarrow \Omega^{i-r+1}_Y \longrightarrow \Omega^{i-r+1}_y \tag{5.2}$$

and the following exact sequence of complexes (of length 2) gives an acyclic resolution of the triangle (5.2):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{i-r+1}_Y & \xrightarrow{\text{incl}} & \Omega^{i-r+1}_Y \oplus \Omega^{i-r+1}_y & \xrightarrow{\text{proj}} & \Omega^{i-r+1}_y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^{i-r+1}_y & \longrightarrow & \Omega^{i-r+1}_y & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Here incl denotes the inclusion into the first factor and proj denotes the projection to the second factor. The left vertical arrow is given by $\omega \mapsto \omega|_y$ and the central vertical arrow is given by $(\omega, \eta) \mapsto \omega|_y - \eta$. By using this resolution and the snake lemma, we see that the map $\tilde{\partial}_C$ is given by $\eta \mapsto -\eta$. So we are done. □

Now let us define the maps

$$\overline{\theta}^{q,i,\log}_{Y\hookrightarrow X,1} : H^{q+1}_z(Y, \Omega^{i-r+1}_{Y,\log}) \longrightarrow H^{q+r}_z(X, \Omega^i_{X,\log}),$$

$$\overline{\theta}^{i}_{Y\hookrightarrow X,1} : H^{1}_z(Y, \Omega^{i-r+1}_{Y}) \longrightarrow H^{r}_z(X, \Omega^i_{X}),$$

by the composite

$$H^{q+1}_z(Y, \Omega^{i-r+1}_{Y,\log}) \overset{H^{q+1}_z(\rho^{i,\log}_{Y\hookrightarrow X,1})}{\longrightarrow} H^{q+1}_z(Y, \underline{H}^r_Y(X, \Omega^i_{X,\log})) \longrightarrow H^{q+r}_z(X, \Omega^i_{X,\log}),$$

$$H^1_z(Y, \Omega^{i-r+1}_{Y}) \overset{H^1_z(\rho^{i}_{Y\hookrightarrow X,1})}{\longrightarrow} H^1_z(Y, \underline{H}^r_Y(X, \Omega^i_{X,\log})) \longrightarrow H^{r+1}_z(X, \Omega^i_{X,\log}),$$

respectively. Then, by Remark 3.13, we have $\overline{\theta}^{q,i,\log}_{Y\hookrightarrow X,1} \circ \theta^{q,i-r+1,\log}_{z\hookrightarrow Y,1} = \theta^{q,i,\log}_{z\hookrightarrow X,1}$. By this fact and the above claim, the theorem is reduced to showing the equality $(-1)^{r-1}\overline{\theta}^{0,i,\log}_{Y\hookrightarrow X,1} \circ \overline{\partial}^0_C = \partial^0_B \circ \theta^{0,i,\log}_{y\hookrightarrow X,1}$ and then it is reduced to the equality $(-1)^{r-1}\overline{\theta}^{i}_{Y\hookrightarrow X,1} \circ \tilde{\partial}_C = \tilde{\partial}_B \circ \theta^{i}_{y\hookrightarrow X,1}$. This equality follows from the following diagram

$$\begin{array}{ccccc}
H^0(y, \Omega^{i-r+1}_y) & \xrightarrow{H^0(y,\rho^i_{y\hookrightarrow X,1})} & H^0(y, \underline{H}^{r-1}_y(X, \Omega^i_X)) & \longrightarrow & H^{r-1}_y(X, \Omega^i_X), \\
\tilde{\partial}_C \downarrow & & \downarrow & & \tilde{\partial}_B \downarrow \\
H^1_z(Y, \Omega^{i-r+1}_Y) & \xrightarrow{H^1_z(Y,\rho^i_{Y\hookrightarrow X,1})} & H^1_z(Y, \underline{H}^{r-1}_Y(X, \Omega^i_X)) & \longrightarrow & H^r_z(X, \Omega^i_X)
\end{array}$$

(where the middle vertical arrow is also induced by localization sequence), where the left square is commutative and the right square is $(-1)^{r-1}$-commutative (the sign $(-1)^{r-1}$ coming from the difference of the degree). So the proof is finished. □

Next we treat the case $(q, i) = (1, N)$:

Proof of Theorem 5.4, *Step* 2: *The case* Y *is regular and* $(q,i) = (1,N)$. We prove the following claim:

Claim. ∂^1_C is factorized as

$$H^1(y,\Omega^{N-r+1}_{y,\log}) \xrightarrow{-\overline{\partial}^1_C} H^2_z(Y,\Omega^{N-r+1}_{Y,\log}) \xrightarrow{(\theta^{1,N-r+1,\log}_{z\hookrightarrow Y,1})^{-1}} H^1(z,\Omega^{N-r}_{z,\log}).$$

Proof of Claim. Let us consider the following diagram:

$$\begin{array}{ccccc} H^1(y,\Omega^{N-r+1}_{y,\log}) & \xrightarrow{\cong} & H^1(z,\underline{H}^0(y,\Omega^{N-r+1}_y)) & \xrightarrow{H^1(z,\partial^0_C)} & H^1(z,\Omega^{N-r}_{z,\log}) \\ {\scriptstyle -\overline{\partial}^1_C}\downarrow & & {\scriptstyle -H^1(z,\overline{\partial}^0_C)}\downarrow & & \| \\ H^2_z(Y,\Omega^{N-r+1}_{Y,\log}) & \xrightarrow{\cong} & H^1(z,\underline{H}^1_z(Y,\Omega^{N-r+1}_{Y,\log})) & \xrightarrow{H^1(z,\rho^{N-r+1,\log}_{z\hookrightarrow Y,1})^{-1}} & H^1(z,\Omega^{N-r}_{z,\log}). \end{array}$$

(Here the left horizontal arrows are the inverse of the map induced by the Hochschild-Serre spectral sequence.) Then the left square is commutative by the functoriality of spectral sequence and the right square is commutative by the claim in the case $q=0$. Moreover, by definition, the map ∂^1_C is equal to the composite of top horizontal arrows and the map $(\theta^{1,N-r+1,\log}_{z\hookrightarrow Y,1})^{-1}$ is equal to the composite of bottom horizontal arrows. So the claim is proved. □

By the above claim, the theorem is reduced to the equality $(-1)^{r-1}\overline{\theta}^{1,N,\log}_{Y\hookrightarrow X,1} \circ \overline{\partial}^1_C = \partial^1_B \circ \theta^{1,N,\log}_{y\hookrightarrow X,1}$. Let us note that, in the diagram

$$\begin{array}{ccccc} H^1(y,\Omega^{N-r+1}_{y,\log}) & \xrightarrow{\overline{\partial}^1_C} & H^2_z(Y,\Omega^{N-r+1}_{Y,\log}) & \xrightarrow{\overline{\theta}^{1,N,\log}_{Y\hookrightarrow X,1}} & H^{r+1}_z(X,\Omega^N_{X,\log}) \\ \uparrow & & \uparrow & & \uparrow \\ H^0(y,\Omega^{N-r+1}_y) & \xrightarrow{\tilde{\partial}_C} & H^1_z(Y,\Omega^{N-r+1}_Y) & \xrightarrow{\overline{\theta}^N_{Y\hookrightarrow X,1}} & H^r_z(X,\Omega^N_X) \end{array}$$

(where the vertical arrows are defined as in δ^1_X in Remark 3.12), the left square is (-1)-commutative and the right square is $(-1)^{r-1}$-commutative. (The former is standard and the latter can be proved in the same way as

Remark 3.12.) On the other hand, in the diagram

$$\begin{array}{ccccc} H^1(y,\Omega^{N-r+1}_{y,\log}) & \xrightarrow{\theta^{1,N,\log}_{y\hookrightarrow X,1}} & H^r_y(X,\Omega^N_{X,\log}) & \xrightarrow{\partial^1_B} & H^{r+1}_z(X,\Omega^N_{X,\log}) \\ \uparrow & & \uparrow & & \uparrow \\ H^0(y,\Omega^{N-r+1}_y) & \xrightarrow{\theta^N_{y\hookrightarrow X,1}} & H^{r-1}_y(X,\Omega^N_X) & \xrightarrow{\tilde{\partial}_B} & H^r_z(X,\Omega^N_X), \end{array}$$

the left square is $(-1)^{r-1}$-commutative by Remark 3.12 and the right square is (-1)-commutative. By these properties and the surjectivity of the map $H^0(y,\Omega^{N-r+1}_y) \longrightarrow H^1(y,\Omega^{N-r+1}_{y,\log})$, the theorem is reduced to the claim $(-1)^{r-1}\overline{\theta}^N_{Y\hookrightarrow X,1}\circ\tilde{\partial}_C = \tilde{\partial}_B\circ\theta^N_{y\hookrightarrow X,1}$ and it is already proved in Step 1. So we are done. □

Remark 5.6.

(1) The above proof works even if we replace $\Omega^*_?$ and $\Omega^*_{?,\log}$ by $W_m\Omega^*_?$ and $W_m\Omega^*_{?,\log}$, respectively.

(2) It seems to the author that, in the papers [G-Su] and [Su], they give a proof of Theorem 5.4 only when Y is regular.

To give a proof of Theorem 5.4 in the remaining cases, we need to develop a theory of trace map for generalized residual complex associated to differential modules, based on [Ha2]. (See also [Ha], [Co].) First we give a brief review on the trace map of generalized residual complexes which is defined in [Ha], [Ha2] (see also [Co]).

For a Noetherian scheme X, let $C^+_c(X)$ be the category of bounded below complexes of sheaves of $\mathcal{O}_X$-modules on X_{Zar} with coherent cohomologies, let $D^+_c(X)$ be the derived category of $C^+_c(X)$ and let $Q : C^+_c(X) \longrightarrow D^+_c(X)$ be the canonical functor. For a smooth morphism $f : X \longrightarrow Y$ of Noetherian schemes of relative dimension n, let $f^\sharp : D^+_c(Y) \longrightarrow D^+_c(X)$ be the functor $\Omega^n_{X/Y}[n]\otimes^{\mathbb{L}} f^*(-)$. For a finite morphism $f : X \longrightarrow Y$ of Noetherian schemes, let $f^\flat : D^+_c(Y) \longrightarrow D^+_c(X)$ be the functor $\overline{f}^*\mathbb{R}\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X,-)$, where $\overline{f}$ is the morphism $(X,\mathcal{O}_X) \longrightarrow (Y,f_*\mathcal{O}_X)$. Fundamental local isomorphism [Ha, III.7.3] says that, if $f : X \longrightarrow Y$ is a regular closed immersion of pure codimension n, there exists the canonical isomorphism of functors $f^\flat \cong \omega_{X/Y}[-n]\otimes^{\mathbb{L}}\mathbb{L}f^*(-)$.

The functors $f^\sharp$ and $f^\flat$ satisfy several compatibilities. Here we recall some of them:

(1) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a morphism of Noetherian schemes such that f and g are smooth (resp. finite), then we have $(g \circ f)^\sharp = g^\sharp \circ f^\sharp$ (resp. $(g \circ f)^\flat = g^\flat \circ f^\flat$).

(2) If $f : X \longrightarrow Y$ is a smooth morphism of Noetherian schemes of relative dimension n which admits a section s, we have the canonical isomorphism $\mathrm{id} \cong s^\flat \circ f^\sharp$. When s is defined by a regular sequence $t_1, \cdots, t_n$, the isomorphism is expressed, via the fundamental local isomorphism, as

$$\begin{aligned}&\mathcal{F} \longrightarrow \omega_{Y/X}[-n] \otimes \Omega^n_{Y/X}[n] \otimes \mathcal{F} = s^\flat \circ f^\sharp \mathcal{F};\\ &x \mapsto (t_1^\vee \wedge \cdots \wedge t_n^\vee) \otimes dt_n \wedge \cdots \wedge dt_1 \otimes x.\end{aligned}$$

(3) If we have the following Cartesian diagram with g smooth and h finite

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{h'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{h} & Z, \end{array}$$

we have the canonical isomorphism $(h')^\flat \circ g^\sharp \xrightarrow{\sim} (g')^\sharp \circ h^\flat$.

(4) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of Noetherian schemes such that f is finite, g is smooth and $h := g \circ f$ is finite, we have the canonical isomorphism of functors $f^\flat \circ g^\sharp \cong h^\flat$. If we denote the graph of f by γ, the above isomorphism is given by the composite

$$f^\flat \circ g^\sharp \overset{(1)}{\cong} \gamma^\flat \circ (h')^\flat \circ g^\sharp \overset{(3)}{\cong} \gamma^\flat \circ (g')^\sharp \circ h^\flat \overset{(2)}{\cong} h^\flat,$$

where g' and h' are as in (3).

For an equidimensional Noetherian scheme X and $\mathcal{F}^\bullet \in D^+_c(X)$, we denote by $E(\mathcal{F}^\bullet)$ the Cousin complex

$$0 \longrightarrow \bigoplus_{x \in X^0} \underline{H}^0_x(X, \mathcal{F}^\bullet) \longrightarrow \bigoplus_{x \in X^1} \underline{H}^1_x(X, \mathcal{F}^\bullet) \longrightarrow \cdots,$$

where $\bigoplus_{x\in X^0} \underline{H}^0_x(X,\mathcal{F}^\bullet)$ is sitting at degree 0. (It is nothing but the complex of $E_1^{\bullet,0}$-terms of the sheafified coniveau spectral sequence.)

A residual complex (resp. generalized residual complex) on a scheme X is a bounded below complex $K^\bullet$ of quasi-coherent flasque $\mathcal{O}_X$-modules with coherent cohomology which admits an isomorphism $\bigoplus_{n\in\mathbb{Z}} K^n \overset{\simeq}{=} \bigoplus_{x\in X} J_x$ (resp. $\bigoplus_{n\in\mathbb{Z}} K^n \overset{\simeq}{=} (\bigoplus_{x\in X} J_x)^r$ for some r), where J_x denotes the sheaf $i_{x*}I_x$, where $i_x : x \hookrightarrow X$ and I_x is the injective hull of $\kappa(x)$ over $\mathcal{O}_{X,x}$. In the case where X is an equidimensional regular scheme, the complex $E(\mathcal{F})[n]$ is a residual complex (resp. generalized residual complex) if $\mathcal{F}$ is an invertible sheaf (resp. a locally free sheaf of finite rank) and $n \in \mathbb{Z}$. We denote the category of generalized residual complexes by $\mathrm{Gres}(X)$. Then, for a morphism of Noetherian schemes of finite type $f : X \longrightarrow Y$ with $\mathrm{Gres}(Y) \neq \emptyset$, we can define the functor

$$f^\Delta : \mathrm{Gres}(Y) \longrightarrow \mathrm{Gres}(X)$$

such that $f^\Delta(K^\bullet) = E(f^\flat QK^\bullet)$ holds if f is finite and $f^\Delta(K^\bullet) = E(f^\sharp QK^\bullet)$ holds if f is smooth. For a finite morphism $f : X \longrightarrow Y$ of Noetherian schemes with $\mathrm{Gres}(Y) \neq \emptyset$, let us define the morphism $\rho_f : f_* f^\Delta(K^\bullet) \longrightarrow K^\bullet$ by the composite

$$f_* f^\Delta(K^\bullet) \overset{\simeq}{=} f_* \overline{f}^* \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K^\bullet) = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K^\bullet) \xrightarrow{\mathrm{ev}} K^\bullet,$$

where ev is the evaluation at 1. Then we have the following theorem ([Ha, VI.4.2, VII.2.1], [Ha2, p.31]):

Theorem 5.7.

(1) *For each morphism $f : X \longrightarrow Y$ of finite type between Noetherian schemes with* $\mathrm{Gres}(Y) \neq \emptyset$*, there exists a morphism*

$$\mathrm{Tr}_f : f_* f^\Delta \longrightarrow 1$$

of functors from $\mathrm{Gres}(Y)$ *to the category of graded $\mathcal{O}_Y$-modules* (*where* 1 *denotes the forgetful functor*) *such that* $\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ g_*\mathrm{Tr}_f$ *holds and that* $\mathrm{Tr}_f = \rho_f$ *holds if f is finite.*

(2) (*Residue Theorem*) *Let $f : X \longrightarrow Y$ be as above and assume moreover that it is proper. Then, for any $K^\bullet \in$* $\mathrm{Gres}(Y)$*, the trace morphism*

$$\mathrm{Tr}_f : f_* f^\Delta K^\bullet \longrightarrow K^\bullet$$

is a homomorphism of complexes.

For a scheme X in the category $\mathcal{C}$ (for definition of $\mathcal{C}$, see Section 2), the complex $E(\Omega^i_X)$ is a generalized residual complex. Now we define, for an lci morphism (for definition, see below) $f : X \longrightarrow Y$ between schemes in $\mathcal{C}$, the trace map of the form $\mathrm{tr}_f : f_*E(\Omega^{i+*}_X)[*] \longrightarrow E(\Omega^i_Y)$ (for some $* \in \mathbb{Z}$), essentially following Hartshorne ([Ha2, II,§2]).

First let is consider the case where $f : X \longrightarrow Y$ is a smooth morphism of relative dimension n between schemes in $\mathcal{C}$. In this case, we have a canonical map $\Omega^{i+n}_X \longrightarrow \Omega^n_{X/Y} \otimes f^*\Omega^i_Y$ induced by the exact sequence $0 \longrightarrow f^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$. Using this map, we define the trace map $\mathrm{tr}_f : f_*E(\Omega^{i+n}_X)[n] \longrightarrow E(\Omega^i_Y)$ as the composite

$$f_*E(\Omega^{i+n}_X)[n] \longrightarrow f_*E(\Omega^n_{X/Y} \otimes f^*\Omega^i_Y)[n] = f_*f^\Delta E(\Omega^i_Y) \xrightarrow{\mathrm{Tr}_f} E(\Omega^i_Y).$$

Next let us consider the case where $f : X \longrightarrow Y$ is a regular closed immersion of codimension n between schemes in $\mathcal{C}$. In this case, we have the canonical map $\Omega^{i-n}_X \longrightarrow \omega_{X/Y} \otimes \Omega^i_Y$ which is locally defined by $\omega \mapsto (t_1^\vee \wedge \cdots \wedge t_n^\vee) \otimes (dt_n \wedge \cdots \wedge dt_1 \wedge \omega)$, using the elements $t_1, \cdots, t_n \in \mathcal{O}_Y$ defining f. Using this map, we define the trace map $\mathrm{tr}_f : f_*E(\Omega^{i-n}_X)[-n] \longrightarrow E(\Omega^i_Y)$ as the composite

$$f_*E(\Omega^{i-n}_X)[-n] \longrightarrow f_*E(\omega_{X/Y} \otimes \Omega^i_Y)[-n] = f_*f^\Delta E(\Omega^i_Y) \xrightarrow{\mathrm{Tr}_f} E(\Omega^i_Y).$$

If we have morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that each of $f, g, g \circ f$ is either a smooth morphism or a regular closed immersion, we have $\mathrm{tr}_{g\circ f} = \mathrm{tr}_g \circ g_*\mathrm{tr}_f$. (This follows from Theorem 5.7 (1).) Moreover, one can see that, if we have a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f smooth and g a regular closed immersion, we have $\mathrm{tr}_g \circ g_*\mathrm{tr}_{f'} = \mathrm{tr}_f \circ f_*\mathrm{tr}_{g'}$.

Now let us given a morphism $f : X \longrightarrow Y$ between schemes in $\mathcal{C}$ which admits locally a factorization $X \overset{i}{\hookrightarrow} Z \xrightarrow{g} Y$, where i is a regular closed immersion of codimension m and g is a smooth morphism of relative dimension n. (Such a morphism is called an lci morphism.) For such f, we define the trace map $\mathrm{tr}_f : f_*E(\Omega_X^{i+n-m})[n-m] \longrightarrow E(\Omega_Y^i)$ by $\mathrm{tr}_f = \mathrm{tr}_g \circ g_*\mathrm{tr}_i$. This definition is independent of the factorization and so the trace map is well-defined: Indeed, if we have another factorization $X \overset{i'}{\hookrightarrow} Z' \overset{g'}{\longrightarrow} Y$ and if we denote the morphisms $X \longrightarrow Z \times_Y Z', Z \times_Y Z' \longrightarrow Z, Z \times_Y Z' \longrightarrow Z'$ by i'', p_1, p_2 respectively, we have

$$\mathrm{tr}_g \circ g_*\mathrm{tr}_i = \mathrm{tr}_g \circ g_*\mathrm{tr}_{p_1} \circ (g \circ p_1)_*\mathrm{tr}_{i''} = \mathrm{tr}_{g'} \circ g'_*\mathrm{tr}_{p_2} \circ (g' \circ p_2)_*\mathrm{tr}_{i''} = \mathrm{tr}_{g'} \circ g'_*\mathrm{tr}_{i'}.$$

Note that the tr_f is also factorized as

$$\begin{aligned} & f_*E(\Omega_X^{i+n-m})[n-m] \longrightarrow f_*E(\omega_{X/Z} \otimes \Omega_Z^{i+n})[n-m] \\ & \longrightarrow f_*E(\omega_{X/Z} \otimes \Omega^n_{Z/Y} \otimes \Omega_Y^i)[n-m] \\ & = f_*i^\Delta g^\Delta E(\Omega_Y^i) = f_*f^\Delta E(\Omega_Y^i) \overset{\mathrm{Tr}_f}{\longrightarrow} E(\Omega_Y^i). \end{aligned}$$

For an lci morphism $f : X \longrightarrow Y$, the trace map tr_f is a map of graded sheaves and if f is proper, it is a map of complexes. (It follows from Theorem 5.7.) We can check also that, if we have a diagram $X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z$ such that f and g are lci morphisms berween regular schemes in $\mathcal{C}$, then so is $g \circ f$ and we have the equality $\mathrm{tr}_{g \circ f} = \mathrm{tr}_g \circ g_*\mathrm{tr}_f$.

To apply the theory of the trace maps to the proof of Theorem 5.4, we need to calculate them in special cases: First, we prove a coincidence between trace map tr_ι and the map $\theta^i_{\iota,1}$ for certain ι:

Proposition 5.8. *Let X be a regular scheme of dimension r in $\mathcal{C}$, let z be a closed point in X and denote the canonical closed immersion $z \longrightarrow X$ by ι. Let us denote the z-component of the trace map* tr_ι

$$H^0(z, \Omega_z^{i-r}) \longrightarrow H^r_z(X, \Omega_X^i)$$

by $\mathrm{tr}_{\iota,z}$. *Then we have the equality* $\mathrm{tr}_{\iota,z} = \theta^i_{\iota,1}$.

Proof. We may assume X is local. We prove the proposition by induction on r.

First consider the case $r = 1$. Let t be a regular element of $\mathcal{O}_X$ defining the closed immersion ι. Then $\mathrm{tr}_{\iota,z}$ is given by the composite

$$\begin{aligned}\Omega_z^{i-1} &\longrightarrow \omega_{z/X} \otimes \Omega_X^i \xrightarrow{\cong} \mathrm{Ext}^1(\mathcal{O}_z, \Omega_X^i) \cong H^1(\mathrm{Hom}^\bullet(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_X^i)) \\ &\cong H^1(\mathrm{Hom}^\bullet(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_{X-z}^i \to \underline{H}_z^1(X, \Omega_X^i))) \\ &\cong H^1(\mathrm{Hom}^\bullet(\mathcal{O}_z, \Omega_{X-z}^i \to \underline{H}_z^1(X, \Omega_X^i))) \\ &= \mathrm{Hom}(\mathcal{O}_z, \underline{H}_z^1(X, \Omega_X^i)) \xrightarrow{\mathrm{ev}} \underline{H}_z^1(X, \Omega_X^i),\end{aligned}$$

where the first map is given by $\omega \mapsto t^\vee \otimes (dt \wedge \omega)$, the second map is the fundamental local isomorphism of Hartshorne (corrected by Conrad), the next three isomorphisms are the standard ones. The composite of the second map and the next isomorphism sends $t^\vee \otimes (dt \wedge \omega)$ to the class of $-dt \wedge \omega \in \Omega_X^i = \mathrm{Hom}^1(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_X^i)$, since the isomorphism of the complxes $[\mathrm{Hom}(\mathcal{O}_X, \Omega_X^i) \xrightarrow{-\circ t} \mathrm{Hom}(\mathcal{O}_X, \Omega_X^i)] = \mathrm{Hom}^\bullet(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_X^i)$ involves the sign -1 at degree 1 (see [Co, (1.3.15,27,28)]). By the next isomorphism, it is sent to the class of $(-dt \wedge \omega, 0) \in \Omega_{X-z}^i \oplus \underline{H}_z^1(X, \Omega_X^i) = \mathrm{Hom}^1(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_{X-z}^i \to \underline{H}_z^1(X, \Omega_X^i))$. Now note that the boundary map $\Omega_{X-z}^i \longrightarrow \Omega_{X-z}^i \oplus \underline{H}_z^1(X, \Omega_X^i)$ of $\mathrm{Hom}^\bullet(\mathcal{O}_X \xrightarrow{t} \mathcal{O}_X, \Omega_{X-z}^i \to \underline{H}_z^1(X, \Omega_X^i))$ is defined by $\eta \mapsto (-t\eta, -\eta)$ (via the identification $\underline{H}_z^1(X, \Omega_X^i) \cong \Omega_{X-z}^i / \Omega_X^i$ using t), because of the sign convention in [Co, p.10] and the description of the boundary map in localization sequence given in the proof of Theorem 5.4, Step 1. So the class of $(-dt \wedge \omega, 0) \in \Omega_{X-z}^i \oplus \underline{H}_z^1(X, \Omega_X^i)$ is the same as the class of $(0, \mathrm{dlog}\, t \wedge \omega)$. Then, by following the above description of $\mathrm{tr}_{\iota,z}$, we can see that the map $\mathrm{tr}_{\iota,z}$ is given by $\omega \mapsto \mathrm{dlog}\, t \wedge \omega$. So the map $\mathrm{tr}_{\iota,z}$ is equal to the map $\theta_{\iota,1}^i$ in the case $r = 1$.

Now let us prove the proposition for general r. Let us take a factorization $z \hookrightarrow Y \hookrightarrow X$ of ι such that the first (resp. the second) map is a regular closed immersion of codimension 1 (resp. $r-1$) and put $y := Y - z$. Then we have the following $(-1)^{r-1}$-commutative diagram

$$\begin{array}{ccc} H^0(y, \Omega_y^{i-r+1}) & \xrightarrow{\tilde{\partial}_C} & H_z^1(Y, \Omega_Y^{i-r+1}) \\ {\scriptstyle \theta_{y\hookrightarrow X,1}^i}\downarrow & & \downarrow{\scriptstyle \overline{\theta}_{Y\hookrightarrow X,1}^i} \\ H_y^{r-1}(X, \Omega_X^i) & \xrightarrow{\tilde{\partial}_B} & H_z^r(X, \Omega_X^i) \end{array}$$

(where the notations are as in the proof of Theorem 5.4, Step 1). On the other hand, we have the $(-1)^{r-1}$-commutative diagram

$$\begin{array}{ccc} H^0(y, \Omega_y^{i-r+1}) & \xrightarrow{\tilde{\partial}_C} & H_z^1(Y, \Omega_Y^{i-r+1}) \\ {\scriptstyle \mathrm{tr}_{Y\hookrightarrow X,y}}\downarrow & & {\scriptstyle \mathrm{tr}_{Y\hookrightarrow X,z}}\downarrow \\ H_y^{r-1}(X, \Omega_X^i) & \xrightarrow{\tilde{\partial}_B} & H_z^r(X, \Omega_X^i) \end{array}$$

(where $\mathrm{tr}_{Y\hookrightarrow X,y}, \mathrm{tr}_{Y\hookrightarrow X,z}$ are the y-component, z-component of the map $\mathrm{tr}_{Y\hookrightarrow X}$, respectively), since $\mathrm{tr}_{Y\hookrightarrow X}$ is a map of complexes $E(\Omega_Y^{i-r+1})[1-r] \longrightarrow E(\Omega_X^i)$. By induction hypothesis, we have $\theta^i_{y\hookrightarrow X,1} = \mathrm{tr}_{Y\hookrightarrow X,y}$. So the above diagrams and the surjectivity of $\tilde{\partial}_C$ implies the equality $\overline{\theta}^i_{Y\hookrightarrow X,1} = \mathrm{tr}_{Y\hookrightarrow X,z}$. Then we can see the equality

$$\theta^i_{z\hookrightarrow X,1} = \overline{\theta}^i_{Y\hookrightarrow X,1} \circ \theta^{i-r+1}_{z\hookrightarrow Y,1} = \mathrm{tr}_{Y\hookrightarrow X,z} \circ \mathrm{tr}_{z\hookrightarrow Y,z} = \mathrm{tr}_{z\hookrightarrow X,z},$$

again by using induction hypothesis. So we have proved the proposition in general case. □

Second, we give an explicit calculation of the trace map for the map of schemes induced by a purely inseparable extension of fields. (The author thinks that this calculation is interesting itself.)

Let k be a field of characteric $p > 0$ with $[k : k^p] = p^n < \infty$ and let k' be a purely inseparable extension field of k with $[k' : k] = p$. We write $k' = k(\alpha), \alpha = x_1^{1/p}$ for some $x_1 \in k$. Let $X := \operatorname{Spec} k', Y := \operatorname{Spec} k[t], Z := \operatorname{Spec} k$ and define the morphism $f : X \longrightarrow Y, g : Y \longrightarrow Z$ by the ones induced by the ring homomorphisms $k[t] \longrightarrow k'; t \mapsto \alpha, k \hookrightarrow k[t]$, respectively. Let us denote the composite $g \circ f$ by h. We would like to compute the trace map $\mathrm{tr}_h : \Omega^i_{k'} \longrightarrow \Omega^i_k$.

By definition, tr_h is defined as the composite

$$\begin{aligned} \Omega_i^{k'} &\xrightarrow{\varphi_1} \omega_{k'/k[t]} \otimes \Omega^{i+1}_{k[t]} \xrightarrow{\varphi_2} \omega_{k'/k[t]} \otimes \Omega^1_{k[t]/k} \otimes \Omega^i_k \\ &= f^\flat \circ g^\sharp \Omega^i_k \overset{(*)}{\cong} h^\flat \Omega^i_k = \mathrm{Hom}_k(k', \Omega^i_k) \xrightarrow{\mathrm{ev}} \Omega^i_k, \end{aligned}$$

where φ_1 is given by $\omega \mapsto (t^p - x_1)^\vee \otimes (-dx_1 \wedge \tilde{\omega})$ (where $\tilde{\omega}$ is a lift of ω), φ_2 is the natural one and ev is the evaluation at 1. The identification $(*)$ is given by the composite

$$\begin{aligned}\omega_{k'/k[t]} \otimes \Omega^1_{k[t]/k} \otimes \Omega^i_k &\overset{\varphi_3}{\cong} k[t]/(t^p - x_1) \otimes \Omega^1_{k[t]/k} \otimes \Omega^i_k \\ &\overset{\varphi_4}{\cong} \mathrm{Ext}^1_{k[t]}(k', \Omega^1_{k[t]/k} \otimes \Omega^i_k) \\ &\overset{\varphi_5}{\cong} \mathrm{Hom}_{k[t]}(k'[t], \Omega^1_{k[t]/k} \otimes \Omega^i_k)/(t-\alpha) \\ &\overset{\varphi_6}{\cong} (\Omega^1_{k'[t]/k'} \otimes_{k'} \mathrm{Hom}_k(k', \Omega^i_k))/(t-\alpha) \\ &\overset{\varphi_7}{\cong} \mathrm{Hom}_k(k', \Omega^i_k),\end{aligned}$$

where φ_3 is the map $(t^p - x_1)^\vee \otimes \omega \otimes \eta \mapsto 1 \otimes \omega \otimes \eta$, φ_4 is the identification via the free resolution

$$0 \longrightarrow k[t] \overset{t^p - x_1}{\longrightarrow} k[t] \longrightarrow k' \longrightarrow 0,$$

φ_5 is the identification via the free resolution

$$0 \longrightarrow k'[t] \overset{t-\alpha}{\longrightarrow} k'[t] \longrightarrow k' \longrightarrow 0,$$

φ_6 is the natural map and φ_7 is the map $dt \otimes \psi \mapsto \psi$.

Let $x_2, \cdots, x_i$ be elements of k and put $\mathrm{dlog}\, x := \mathrm{dlog}\, x_2 \wedge \cdots \wedge \mathrm{dlog}\, x_i$. Then we can explicitly calculate the trace map tr_h as follows:

Proposition 5.9. *Let the notations be as above. Then, for an integer l with $0 \le l \le p-1$, we have*

$$\mathrm{tr}_h(\alpha^l \mathrm{dlog}\, \alpha \wedge \mathrm{dlog}\, x) = \begin{cases} \mathrm{dlog}\, x_1 \wedge \mathrm{dlog}\, x, & l = 0, \\ 0, & l > 0. \end{cases}$$

Proof. By the equality $\alpha^l \mathrm{dlog}\, \alpha \wedge \mathrm{dlog}\, x = (1/x_1)\alpha^{p-1+l} d\alpha \wedge \mathrm{dlog}\, x$, we have $\varphi_1(\alpha^l \mathrm{dlog}\, \alpha \wedge \mathrm{dlog}\, x) = -(t^p - x_1)^\vee \wedge \mathrm{dlog}\, x_1 \wedge t^{p-1+l} dt \wedge \mathrm{dlog}\, x = (t^p - x_1)^\vee \wedge t^{p-1+l} dt \wedge \mathrm{dlog}\, x_1 \wedge \mathrm{dlog}\, x$. It is sent by $\varphi_3 \circ \varphi_2$ to $t^{p-1+l} dt \otimes \mathrm{dlog}\, x_1 \wedge \mathrm{dlog}\, x$. The map $\varphi_5 \circ \varphi_4$ is induced by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k[t] & \overset{t^p - x_1}{\longrightarrow} & k[t] & \longrightarrow & k' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & k'[t] & \overset{t-\alpha}{\longrightarrow} & k'[t] & \longrightarrow & k' & \longrightarrow & 0, \end{array}$$

where the first vertical arrow is given by $\alpha^j \mapsto 0\,(0 \le j \le p-2), \alpha^{p-1} \mapsto 1$ and the second map is given by $\alpha^j \mapsto t^j\,(0 \le j \le p-1)$. So the element $t^{p-1+l}dt \otimes \operatorname{dlog} x_1 \wedge \operatorname{dlog} x$ is sent, by $\varphi_6 \circ \varphi_5 \circ \varphi_4$, to the class of $t^{p-1+l}dt \otimes \psi_{p-1}$, where ψ_k is the map satisfying $\psi_k(\alpha^j) = 0\,(j \not\equiv k \bmod p), \psi_k(\alpha^k) = \operatorname{dlog} x_1 \wedge \operatorname{dlog} x$. It is sent by φ_7 to the element $\alpha^{p-1+l}\psi_{p-1} = \psi_{-l}$ and then sent by ev to 0 when $l \neq 0$ and to $\operatorname{dlog} x_1 \wedge \operatorname{dlog} x$ when $l = 0$. So we are done. □

Remark 5.10. The similar (but much more easier) calculation shows that, for $y \in k$, we have $\operatorname{tr}_h(\operatorname{dlog} y \wedge \operatorname{dlog} x) = 0$. Indeed, we see that $\varphi_2 \circ \varphi_1(\operatorname{dlog} y \wedge \operatorname{dlog} x) = 0$.

Now we can give a proof of Theorem 5.4 in the case where Y is not regular:

Proof of Theorem 5.4, *Step* 3: *The case* Y *is not regular and* $q = 0$. Let $\pi : Y' \longrightarrow Y$ be the normalization of Y. Then, since $\dim Y = 1$, Y' is regular and the morphism π is projective. Hence we can form the following commutative diagram:

$$\begin{array}{ccccc} z' & \longrightarrow & Y' & \longrightarrow & X' \\ \downarrow & & \pi\downarrow & & \downarrow \\ z & \longrightarrow & Y & \longrightarrow & X, \end{array}$$

where $X' := \mathbb{P}^a_X$ for some a, $z' := \pi^{-1}(z)$, and the horizontal arrows are closed immersions. Let y' be $\pi^{-1}(y)$ (note that it is isomorphic to y via π), and put $i' := i + a, r' := r + a$. First, let us note the following claim:

Claim 1. Let us consider the following diagram

$$\begin{array}{ccccccc}
H^0(y', \Omega^{i-r+1}_{y',\log}) & \xrightarrow{\theta^{0,i',\log}_{y' \hookrightarrow X',1}} & H^{r'-1}_{y'}(X', \Omega^{i'}_{X',\log}) & \xrightarrow{(\partial^0_B)'} & H^{r'}_{z'}(X', \Omega^{i'}_{X',\log}) & \xleftarrow{\theta^{0,i',\log}_{z' \hookrightarrow X',1}} & H^0(z', \Omega^{i-r}_{z',\log}) \\
\delta^0_{y'} \downarrow & (\diamondsuit) & \delta^0_{X',y'} \downarrow & (\heartsuit) & \delta^0_{X',z'} \downarrow & (\diamondsuit) & \delta^0_{z'} \downarrow \\
H^0(y', \Omega^{i-r+1}_{y'}) & \xrightarrow{\operatorname{tr}_{y' \hookrightarrow X',y'}} & H^{r'-1}_{y'}(X', \Omega^{i'}_{X'}) & \xrightarrow{(\tilde{\partial}_B)'} & H^{r'}_{z'}(X', \Omega^{i'}_{X'}) & \xleftarrow{\operatorname{tr}_{z' \hookrightarrow X',z'}} & H^0(z', \Omega^{i-r}_{z'}) \\
\operatorname{tr}_{y' \to y,y} \downarrow & (\clubsuit) & \operatorname{tr}_{X' \to X,y} \downarrow & (\spadesuit) & \operatorname{tr}_{X' \to X,z} \downarrow & (\clubsuit) & \operatorname{tr}_{z' \to z,z} \downarrow \\
H^0(y, \Omega^{i-r+1}_{y}) & \xrightarrow{\operatorname{tr}_{y \hookrightarrow X,y}} & H^{r-1}_{y}(X, \Omega^{i}_{X}) & \xrightarrow{\tilde{\partial}_B} & H^{r}_{z}(X, \Omega^{i}_{X}) & \xleftarrow{\operatorname{tr}_{z \hookrightarrow X,z}} & H^0(z, \Omega^{i-r}_{z}) \\
\delta^0_{y} \uparrow & (\diamondsuit) & \delta^0_{X,y} \uparrow & (\heartsuit) & \delta^0_{X,z} \uparrow & (\diamondsuit) & \delta^0_{z} \uparrow \\
H^0(y, \Omega^{i-r+1}_{y,\log}) & \xrightarrow{\theta^{0,i,\log}_{y \hookrightarrow X,1}} & H^{r-1}_{y}(X, \Omega^{i}_{X,\log}) & \xrightarrow{\partial^0_B} & H^{r}_{z}(X, \Omega^{i}_{X,\log}) & \xleftarrow{\theta^{0,i,\log}_{z \hookrightarrow X,1}} & H^0(z, \Omega^{N-r}_{z,\log}),
\end{array}$$

where the maps are defined as follows: The maps $\delta^0_?, \delta^0_{?,??}$ are the ones induced by the inclusion $\Omega^*_{?,\log} \hookrightarrow \Omega^*_?$. For a morphism $Y_1 \longrightarrow Y_2$ and a point $y_2 \in Y_2$, $\mathrm{tr}_{Y_1 \to Y_2, y_2}$ is the map defined as the y_2-component of trace map for $Y_1 \longrightarrow Y_2$. $(\partial^0_B)', (\tilde{\partial}_B)'$ are the maps $\partial^0_B, \tilde{\partial}_B$ for (X', y', z'). Then the squares other than (♠) is commutative and the square (♠) is $(-1)^a$-commutative.

Proof of Claim 1. The commutativity of the squares (♡) is obvious. The commutativity of the squares (♢) is proved as follows (we discuss only the commutativity of the lower right square): By Proposition 5.8, we have $\mathrm{tr}_{z \hookrightarrow X, z} = \theta^i_{z \hookrightarrow X, 1}$, Then the commutativity follows from the claim in Remark 3.12. The commutativity of the squares (♣) follows from the transitivity of trace maps. Finally, The $(-1)^a$-commutativity of the square (♠) follows from the fact that the trace map is a map of Cousin complexes $E(\Omega^{i'}_{X'})[a] \longrightarrow E(\Omega^i_X)$, because $X' \longrightarrow X$ is proper. □

Let us denote the maps $\partial^0_B, \partial^0_C$ for (X', y', z') by $(\partial^0_B)', (\partial^0_C)'$. Since Y' is regular, we have

$$(-1)^{r'} \theta^{0,i',\log}_{z' \hookrightarrow X', 1} \circ (\partial^0_C)' = (\partial^0_B)' \circ \theta^{0,i',\log}_{y' \hookrightarrow X, 1}.$$

By using the diagram in claim 1 and the fact that y' is isomorphic to y, we can deduce the following equality:

$$(5.3) \qquad (-1)^r \mathrm{tr}_{z \hookrightarrow X, z} \circ \mathrm{tr}_{z' \to z, z} \circ \delta^0_{z'} \circ (\partial^0_C)' = \delta^0_{X,z} \circ \partial^0_B \circ \theta^{0,i,\log}_{y \hookrightarrow X, 1}.$$

Now let us admit the validity of the following claim for the moment:

Claim 2. The following diagram is commutative:

$$\begin{array}{ccc} H^0(z', \Omega^{i-r}_{z',\log}) & \xrightarrow{\delta^0_{z'}} & H^0(z', \Omega^{i-r}_{z'}) \\ {\scriptstyle \mathrm{Cor}_{\kappa(z')/\kappa(z)}} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}_{z' \to z, z}} \\ H^0(z, \Omega^{i-r}_{z,\log}) & \xrightarrow{\delta_z} & H^0(z, \Omega^{i-r}_z), \end{array}$$

where $\mathrm{Cor}_{\kappa(z')/\kappa(z)}$ is as in the definition of Kato complex.

Then the left hand side of the equation (5.3) can be rewritten as follows:

$$\begin{aligned} \text{LHS of (5.3)} &= (-1)^r \mathrm{tr}_{z \hookrightarrow X, z} \circ \delta^0_z \circ \mathrm{Cor}_{\kappa(z')/\kappa(z)} \circ (\partial^0_C)' \\ &= (-1)^r \delta^0_{X,z} \circ \theta^{0,i,\log}_{z \hookrightarrow X, 1} \circ \partial^0_C. \end{aligned}$$

So we obtain the equality

$$(-1)^r \theta^{0,i,\log}_{z\hookrightarrow X,1} \circ \partial^0_C = \theta^{0,i,\log}_{y\hookrightarrow X,1} \circ \partial^0_B.$$

Hence it suffices to prove the above claim to finish the proof in this case.

Proof of Claim 2. It suffices to prove the following: Let K be a field of characteristic $p > 0$ and let K' be a finite extension of K. Then the following diagram is commutative:

$$\begin{array}{ccc} H^0(K', \Omega^i_{K',\log}) & \longrightarrow & H^0(K', \Omega^i_{K'}) \\ {\scriptstyle \mathrm{Cor}}\downarrow & & {\scriptstyle \mathrm{tr}}\downarrow \\ H^0(K, \Omega^i_{K,\log}) & \longrightarrow & H^0(K, \Omega^i_K), \end{array}$$

where Cor, tr denotes the corestriction map and the trace map respectively, and the horizontal maps are induced by the inclusion $\Omega^i_{?,\log} \hookrightarrow \Omega^i_?$. Since both Cor and tr satisfy the transitivity, we may assume that K'/K is separable or K'/K is purely inseparable of degree p.

First let us consider the case that the extension K'/K is separable. Then there exists a Galois extension $K \subset L$ such that $K' \otimes_K L \cong \prod_{j=1}^n L$ holds for some $n \in \mathbb{N}$. Note that both Cor and tr are compatible with base change by the morphism $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$ (in the case of Cor, it follows from the compatibility with base change of norm maps of K-groups, and in the case of tr, it follows from the definition). Then we may replace the extension $K \subset K'$ by $L \subset K' \otimes_K L \cong \prod_{j=1}^n L$ to prove the claim in separable case, that is, it suffices to show the commutativity of the following diagram:

$$\begin{array}{ccc} \bigoplus_{j=1}^n H^0(L, \Omega^i_{L,\log}) & \longrightarrow & \bigoplus_{j=1}^n H^0(L, \Omega^i_L) \\ {\scriptstyle \mathrm{Cor}}\downarrow & & {\scriptstyle \mathrm{tr}}\downarrow \\ H^0(L, \Omega^i_{L,\log}) & \longrightarrow & H^0(L, \Omega^i_L). \end{array}$$

But it is trivial, since both Cor and tr are equal to the map 'taking sum'.

Next, let us consider the case where K'/K is purely inseparable of degree p. By the argument in the previous paragraph, we may replace K by its separable closure. In this case, it is known ([Ba-Ta]) that $K^{\mathrm{M}}_i(K')$ is generated by the elements of the form $\{x_1, \cdots, x_i\}$, where $x_1 \in K', x_2, \cdots, x_i \in K$.

By Bloch-Gabber-Kato theorem, it suffices to show the following equality to prove the claim:

$$\mathrm{tr}(\mathrm{dlog}\,\underline{x}_1 \wedge \cdots \wedge \mathrm{dlog}\,\underline{x}_i) = h^i_K \circ N(\{x_1, \cdots, x_i\}),$$

where N is the norm map of K-groups. By Proposition 5.9 and Remark 5.10, the left hand side is equal to $\mathrm{dlog}\,\underline{x}_1^p \wedge \mathrm{dlog}\,\underline{x}_2 \wedge \cdots \wedge \mathrm{dlog}\,\underline{x}_i$, and we can calculate the right hand side as follows:

$$\begin{aligned} h^i_K \circ N(\{x_1, \cdots, x_i\}) &= h^i_K(\{x_1^p, x_2, \cdots, x_i\}) \quad \text{(projection formula for } N) \\ &= \mathrm{dlog}\,\underline{x}_1^p \wedge \mathrm{dlog}\,\underline{x}_2 \wedge \cdots \wedge \mathrm{dlog}\,\underline{x}_i. \end{aligned}$$

Hence the assertion is proved. □

Since the claim is proved, the proof of Theorem 5.4, Step 3 is finished. □

Finally we treat the remaining case.

Proof of Theorem 5.4, *Step* 4: *The case* Y *is not regular and* $(q, i) = (1, N)$. First we prove a technical claim on logarithmic Hodge-Witt cohomology:

Claim 1. Let k be a field of characteristic $p > 0$ with $[k : k^p] = p^i$. Let $x_1, \cdots, x_i$ be a p-basis of k and put $\mathrm{dlog}\, x := \mathrm{dlog}\, x_1 \wedge \cdots \wedge \mathrm{dlog}\, x_i$. Let $a \in \mathbb{N}$ and put $k_0 := k^{p^a}$, $K := k((t))$. Then the homomorphism

$$H^1(k_0, \mathbb{Z}/p^m\mathbb{Z}) \longrightarrow H^1(K, \Omega^{i+1}_{K,\log}); \quad y \mapsto y\mathrm{dlog}\, t \wedge \mathrm{dlog}\, x$$

is surjective.

Proof of Claim 1. Since we have the isomorphism $H^1(K, \Omega^{i+1}_{K,\log}) \cong \Omega^{i+1}_K/(C-1)\Omega^{i+1}_K$, it suffices to prove the following: Any element in Ω^{i+1}_K has the form $y\mathrm{dlog}\, t \wedge \mathrm{dlog}\, x\ (y \in k_0)$ modulo $(C-1)\Omega^{i+1}_K$. (See also Remark 2.11.)

First let us prove the weaker assertion which claims that any element in Ω^{i+1}_K has the form $y\mathrm{dlog}\, t \wedge \mathrm{dlog}\, x\ (y \in k)$ modulo $(C-1)\Omega^{i+1}_K$. For an integer $l \geq -1$, let us define $H_l \subset \Omega^{i+1}_K$ by

$$H_l := \{\sum_{j \geq -l} a_j t^j \mathrm{dlog}\, t \wedge \mathrm{dlog}\, x \mid a_j \in k\}.$$

Then we have $\bigcup_{l\geq -1} H_l = \Omega_K^{i+1}$. Since we have the equality

$$C(\sum_{j\geq -l} a_j t^j \mathrm{dlog}\, t\wedge \mathrm{dlog}\, x) = \sum_{\substack{j\geq -l\\ p|j}} b_j t^{j/p} \mathrm{dlog}\, t\wedge \mathrm{dlog}\, x$$

for some $b_j\in k\,(j\geq -l, p|j)$, we have $CH_l\subset H_{l-1}$ for $l>0$ and $CH_0\subset H_0$. So, for any $\omega\in\Omega_K^{i+1}$, we have

$$\omega = C^l\omega - (C-1)(\sum_{j=1}^{l-1} C^j)\omega \in H_0 + (C-1)\Omega_K^{i+1}$$

for sufficiently large l. Since we have $H_0=\{y\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x \,|\, y\in k\}+H_{-1}$, it suffices to show the inclusion $H_{-1}\subset (C-1)H_{-1}$. Let us prove it. Note that there exists an additive homomorphism $D: k\longrightarrow k$ such that $D(x^p)=x$ holds and that, for any $\omega=\sum_j a_j t^j \mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\in\Omega_K^{i+1}$, we have the equality $(C-1)\omega = \sum_j (D(a_{pj})-a_j)t^j\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x$. For any element $\eta := \sum_{j\geq 1} b_j t^j \mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\in H_{-1}$, let us define $a_j\in k\,(j\geq 1)$ inductively as follows:

$$a_j := \begin{cases} 0, & (j,p)=1,\\ (b_{j/p}+a_{j/p})^p, & p|j.\end{cases}$$

Put $\omega=\sum_{j\geq 1} a_j t^j\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\in H_{-1}$. Then we have

$$\begin{aligned}(C-1)\omega &= \sum_{j\geq 1}(D(a_{pj})-a_j)t^j\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\\ &= \sum_{j\geq 1}((b_j+a_j)-a_j)t^j\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x = \eta.\end{aligned}$$

Hence we have $H_{-1}\subset (C-1)H_{-1}$. So the weaker assertion is proved.

Now we prove the claim. By the result in the previous paragraph, it suffices to prove that any element in Ω_K^{i+1} of the form $z\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\ (z\in k)$ has in fact the form $y\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x\ (y\in k_0)$ modulo $(C-1)\Omega_K^{i+1}$. Write z as

$$z=\sum_{n_1,\cdots,n_i=0}^{p^a-1} b_{n_1\cdots n_i}^{p^a} x_1^{n_1}\cdots x_i^{n_i}\quad (b_{n_1\cdots n_i}\in k)$$

and put

$$b_{00\cdots 0} = \sum_{n_1,\cdots,n_i=0}^{p^a-1} c_{n_1\cdots n_i}^{p^a} x_1^{n_1}\cdots x_i^{n_i} \quad (c_{n_1\cdots n_i}\in k).$$

Let d be $b_{00\cdots 0} - c_{00\cdots 0}^{p^a}$. Then we have

$$\begin{aligned} & z\mathrm{dlog}\, t\wedge \mathrm{dlog}\, x + (C-1)(C^{a-1}+\cdots+C+1)((z+d)\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x) \\ = {} & z\mathrm{dlog}\, t\wedge \mathrm{dlog}\, x + (C^a-1)((z+d)\mathrm{dlog}\, t\wedge\mathrm{dlog}\, \underline{x}) \\ = {} & (z+b_{00\cdots 0}-z-d)\mathrm{dlog}\, t\wedge\mathrm{dlog}\, x = c_{00\cdots 0}^{p^a}\mathrm{dlog}\, t\wedge \mathrm{dlog}\, x. \end{aligned}$$

Hence we obtain the assertion. □

Now let us begin the proof of the theorem. Let the notations be as in Step 3 and put $N' := N+a$. Then we have the following claim:

CLAIM 2. Let us consider the following diagram

$$\begin{array}{ccccccccc}
H^1(y',\Omega^{N-r+1}_{y',\log}) & \overset{\theta^{1,N',\log}_{y'\hookrightarrow X,1}}{\longrightarrow} & H^{r'}_{y'}(X',\Omega^{N'}_{X',\log}) & \overset{(\partial^1_B)'}{\longrightarrow} & H^{r'+1}_{z'}(X',\Omega^{N'}_{X',\log}) & \overset{\theta^{1,N,\log}_{z'\hookrightarrow X',1}}{\longleftarrow} & H^1(z',\Omega^{N-r}_{z',\log}) \\
\delta^1_{y'}\uparrow & (\diamondsuit)_{r'-1} & \delta^1_{X',y'}\uparrow & (\heartsuit) & \delta^1_{X',z'}\uparrow & (\diamondsuit)_{r'} & \delta^1_{z'}\uparrow \\
H^0(y',\Omega^{N-r+1}_{y'}) & \overset{\mathrm{tr}_{y'\hookrightarrow X',y'}}{\longrightarrow} & H^{r'-1}_{y'}(X',\Omega^{N'}_{X'}) & \overset{(\tilde\partial_B)'}{\longrightarrow} & H^{r'}_{z'}(X',\Omega^{N'}_{X'}) & \overset{\mathrm{tr}_{z'\hookrightarrow X',z'}}{\longleftarrow} & H^0(z',\Omega^{N-r}_{z'}) \\
\mathrm{tr}_{y'\to y,y}\downarrow & & \mathrm{tr}_{X'\to X,y}\downarrow & (\spadesuit) & \mathrm{tr}_{X'\to X,z}\downarrow & & \mathrm{tr}_{z'\to z,z}\downarrow \\
H^0(y,\Omega^{N-r+1}_{y}) & \overset{\mathrm{tr}_{y\hookrightarrow X,y}}{\longrightarrow} & H^{r-1}_{y}(X,\Omega^{N}_{X}) & \overset{\tilde\partial_B}{\longrightarrow} & H^{r}_{z}(X,\Omega^{N}_{X}) & \overset{\mathrm{tr}_{z\hookrightarrow X,z}}{\longleftarrow} & H^0(z,\Omega^{N-r}_{z}) \\
\delta^1_{y}\downarrow & (\diamondsuit)_{r-1} & \delta^1_{X,y}\downarrow & (\heartsuit) & \delta^1_{X,z}\downarrow & (\diamondsuit)_{r} & \delta^1_{z}\downarrow \\
H^1(y,\Omega^{N-r+1}_{y,\log}) & \overset{\theta^{1,N,\log}_{y\hookrightarrow X,1}}{\longrightarrow} & H^{r}_{y}(X,\Omega^{N}_{X,\log}) & \overset{\partial^1_B}{\longrightarrow} & H^{r+1}_{z}(X,\Omega^{N}_{X,\log}) & \overset{\theta^{1,N,\log}_{z\hookrightarrow X,1}}{\longleftarrow} & H^0(z,\Omega^{N-r}_{z,\log}),
\end{array}$$

where $(\partial^1_B)'$ is the map ∂^1_B for (X',y',z') and the maps $\delta^1_?, \delta^1_{?,??}$ are the ones defined as the composite of the homomorphism of cohomologies induced by the projection

$$W_m\Omega^*_? \longrightarrow W_m\Omega^*_?/dV^{m-1}\Omega^{*-1}_?$$

and the connecting homomorphism of the exact sequence

$$0\longrightarrow W_m\Omega^*_{?,\log}\longrightarrow W_m\Omega^*_? \overset{1-F}{\longrightarrow} W_m\Omega^*_?/dV^{m-1}\Omega^{*-1}_? \longrightarrow 0.$$

(The other maps are defined as in Step 3.) Then the squares $(\heartsuit)$ are (-1)-commutative, the square $(\diamondsuit)_l$ is $(-1)^l$-commutative $(l=r-1,r,r'-1,r')$, the square $(\spadesuit)$ is $(-1)^a$-commutative and the other squares are commutative.

Proof of Claim 2. The $(-1)^l$-commutativity of the squares $(\diamondsuit)_l$ and the (-1)-commutativity of the squares $(\heartsuit)$ follow from Proposition 5.8, Remark 3.12 and the results in the proof of Theorem 5.4, Step 2. The $(-1)^a$-commutativity of $(\spadesuit)$ and the commutativity of the other squares are proved in Step 3, claim 1. $\square$

Let t be a uniformizer of $\mathcal{O}_{Y'}$. Then we have $\mathcal{O}_{Y'} = \kappa(z')[[t]]$. Let $x_1, \cdots, x_i$ be a p-basis of $\kappa(z')$ and put $\operatorname{dlog} x := \operatorname{dlog} x_1 \wedge \cdots \wedge \operatorname{dlog} x_i$. Denote the separable closure of $\kappa(z)$ in $\kappa(z')$ by k. Now define $V \subset \Omega^{N-r+1}_{y'}$ by $V := \{b\operatorname{dlog} t \wedge \operatorname{dlog} x \mid b \in k\}$. Then, by claim 1, the homomorphism $\delta^1_{y'}|_V : V \longrightarrow H^1(y', \Omega^{N-r+1}_{y',\log})$ is surjective.

Let η be an element in $H^1(y', \Omega^{N-r+1}_{y',\log})(= H^1(y, \Omega^{N-r+1}_{y,\log}))$ and take an element $a \in k$ satisfying $\delta^1_{y'}(a\operatorname{dlog} t \wedge \operatorname{dlog} x) = \eta$. Then, by the results in the proof of Theorem 5.4, Step 2 and Proposition 5.8, we have the following equality:

$$\begin{aligned}
&(\tilde{\partial}_B)' \circ \operatorname{tr}_{y' \hookrightarrow X, y'}(a\operatorname{dlog} t \wedge \operatorname{dlog} x) \\
&\quad = (-1)^{r'-1} \overline{\theta}^N_{Y' \hookrightarrow X',1} \circ (\tilde{\partial}_C)'(a\operatorname{dlog} t \wedge \operatorname{dlog} x) \\
&\quad = (-1)^{r'} \overline{\theta}^N_{Y' \hookrightarrow X',1}(a\operatorname{dlog} t \wedge \operatorname{dlog} x) \\
&\quad = (-1)^{r'} \theta^N_{z' \hookrightarrow X',1}(a\operatorname{dlog} x) = (-1)^{r'} \operatorname{tr}_{z' \hookrightarrow X', z'}(a\operatorname{dlog} x).
\end{aligned}$$

By using the upper three squares of the diagram in the claim 2, we can deduce the equality $(-1)^{r'}(\partial^1_B)' \circ \theta^{1,N',\log}_{y' \hookrightarrow X',1}(\eta) = \theta^{1,N',\log}_{z' \hookrightarrow X',1} \circ \delta^1_{z'}(a\operatorname{dlog} x)$. From this and the main result in Step 2, we deduce the equality

$$\delta^1_{z'}(a\operatorname{dlog} x) = (\partial^1_C)'(\eta). \tag{5.4}$$

On the other hand, by using the diagram in claim and the fact that y' is isomorphic to y, we can deduce the following equality:

$$(-1)^r \partial^1_B \circ \theta^{1,N,\log}_{y \hookrightarrow X,1}(\eta) = \theta^{1,N,\log}_{z \hookrightarrow X,1} \circ \delta^1_z \circ \operatorname{tr}_{z' \to z, z}(a\operatorname{dlog} x). \tag{5.5}$$

Now assume for the moment that we have the equality

$$\delta^1_z \circ \operatorname{tr}_{z' \to z,z}(a\operatorname{dlog} \underline{x}) = \operatorname{Cor}_{\kappa(z')/\kappa(z)} \circ \delta^1_{z'}(a\operatorname{dlog} x),$$

where

$$\operatorname{Cor}_{\kappa(z')/\kappa(z)} : H^1(z', \Omega^{N-r}_{z',\log}) \longrightarrow H^1(z, \Omega^{N-r}_{z,\log})$$

be as in the definition of Kato complex. Then the right hand side in the equation (5.5) can be rewritten as follows:

$$\begin{aligned}\text{RHS of (5.5)} &= \theta^{1,N,\log}_{z\hookrightarrow X,1} \circ \mathrm{Cor}_{\kappa(z')/\kappa(z)} \circ \delta^1_{z'}(a\mathrm{dlog}\, x) \\ &= \theta^{1,N,\log}_{z\hookrightarrow X,1} \circ \mathrm{Cor}_{\kappa(z')/\kappa(z)} \circ (\partial^1_C)'(\eta) = \theta^{1,N,\log}_{z\hookrightarrow X,1} \circ \partial^1_C(\eta).\end{aligned}$$

So we have the desired equality $(-1)^r\partial^1_B \circ \theta^{1,N,\log}_{y\hookrightarrow X,1} = \theta^{1,N,\log}_{z\hookrightarrow X,1} \circ \partial^1_C$. So the proof of the theorem is reduced to the following claim:

Claim 3. Let K be a field of characteristic $p > 0$ with $[K : K^p] = p^i$ and let K' be a finite extension of K. Let K_0 be the separable closure of K in K'. Let us consider the following diagram (Attention: we do not know the commutativity):

$$\begin{array}{ccc} H^0(K', \Omega^i_{K'}) & \xrightarrow{\gamma_{K'}} & H^1(K', \Omega^i_{K',\log}) \\ {\scriptstyle \mathrm{tr}_{K'/K}}\downarrow & & {\scriptstyle \mathrm{Cor}_{K'/K}}\downarrow \\ H^0(K, \Omega^i_K) & \xrightarrow{\gamma_K} & H^1(K, \Omega^i_{K,\log}), \end{array}$$

where $\mathrm{tr}_{K'/K}$, $\mathrm{Cor}_{K'/K}$ denotes the trace map and the corestriction map (the map $\mathrm{Cor}^{1,i}_{K'/K}$ in the notation in Remark 5.1) respectively, and the homomorphism $\gamma_?$ $(? = K, K')$ is defined as the composite of the homomorphism $H^0(?, \Omega^i_?) \longrightarrow H^0(?, \Omega^i_?/d\Omega^{i-1}_?)$ induced by the natural projection and the connecting homomorphism associated to the exact sequence

$$0 \longrightarrow \Omega^i_{?,\log} \longrightarrow \Omega^i_? \xrightarrow{1-F} \Omega^i_?/d\Omega^{i-1}_? \longrightarrow 0.$$

Let $x_1, \cdots, x_i$ be a p-basis of K' and put $\mathrm{dlog}\, x := \mathrm{dlog}\, x_1 \wedge \cdots \wedge \mathrm{dlog}\, x_i$. Then, for $a \in K_0$, we have the equality

$$\gamma_K \circ \mathrm{tr}_{K'/K}(a\mathrm{dlog}\, x) = \mathrm{Cor}_{K'/K} \circ \gamma_{K'}(a\mathrm{dlog}\, x).$$

Proof of Claim 3. It suffices to prove the following two assertions:

(1) When K'/K is purely inseparable, we have the equality

$$\gamma_K \circ \mathrm{tr}_{K'/K}(a\mathrm{dlog}\, x) = \mathrm{Cor}_{K'/K} \circ \gamma_{K'}(a\mathrm{dlog}\, x).$$

(2) When K'/K is separable, we have $\gamma_K \circ \mathrm{tr}_{K'/K} = \mathrm{Cor}_{K'/K} \circ \gamma_{K'}$.

First we prove the assertion (1). We may assume that x_1 is not contained in K. Put $K'' := K(x_1^p, x_2, \cdots, x_i)$. Then $x_1^p, x_2, \cdots, x_i$ forms a p-basis of K''. By induction on $[K' : K]$, it suffices to prove the following two equalities:

$$\mathrm{tr}_{K'/K''}(a\mathrm{dlog}\, x) = a\mathrm{dlog}\, x_1^p \wedge \mathrm{dlog}\, x_2 \wedge \cdots \wedge \mathrm{dlog}\, x_i. \tag{5.6}$$

$$\gamma_{K''} \circ \mathrm{tr}_{K'/K''}(a\mathrm{dlog}\, x) = \mathrm{Cor}_{K'/K''} \circ \gamma_{K'}(a\mathrm{dlog}\, x). \tag{5.7}$$

The equality (5.6) follows from Proposition 5.9 and the K''-linearity of $\mathrm{Tr}_{K'/K''}$. Let us prove the equality (5.7). We denote the class of a in $K''/(1-F)K'' \cong H^1(K'', \mathbb{Z}/p^m\mathbb{Z})$ by $\overline{a}$. Then, in the notation in Remark 5.1, we have

$$\text{LHS of (5.7)} = \{\overline{a}, \{x_1^p, x_2, \cdots, x_i\}\}_{K''}.$$

On the other hand, we can calculate the right hand side as follows:

$$\begin{aligned} \text{RHS of (5.7)} &= \mathrm{Cor}_{K'/K''}\{\alpha(\overline{a}), \{x_1, \cdots, x_i\}\}_{K'} = \{\overline{a}, N\{x_1, \cdots, x_i\}\}_{K''} \\ &= \{\overline{a}, \{x_1^p, x_2, \cdots, x_i\}\}_{K''} \quad \text{(projection formula)}. \end{aligned}$$

Hence the equality (5.7) is proved and the proof of the assertion (1) is finished.

Next let us prove the assertion (2). Let $y_1, \cdots, y_i$ be a p-basis of K and put $\mathrm{dlog}\, y := \mathrm{dlog}\, y_1 \wedge \cdots \wedge \mathrm{dlog}\, y_i$. Then, since $y_1, \cdots, y_i$ forms a p-basis of K', any element of $H^0(K', \Omega^i_{K'})$ has the form $a\mathrm{dlog}\, y$ ($a \in K'$). Let L be the Galois closure of K'/K and put $G := \mathrm{Gal}(L/K), H := \mathrm{Gal}(L/K')$. Then we have $L \otimes_K K' = \prod_{\tau \in G/H} L$. Note that we have the commutative diagram with injective horizontal arrows

$$\begin{array}{ccc} H^0(K', \Omega^i_{K'}) & \longrightarrow & \bigoplus_{\tau \in G/H} H^0(L, \Omega^i_L) \\ {\scriptstyle \mathrm{tr}_{K'/K}} \downarrow & & {\scriptstyle \mathrm{tr}_{(\prod_{\tau \in G/H} L)/L}} \downarrow \\ H^0(K, \Omega^i_K) & \longrightarrow & H^0(L, \Omega^i_L), \end{array}$$

and the homomorphism $\mathrm{tr}_{(\prod_{\tau\in G/H} L)/L}$ is nothing but the summation. Hence we have

$$\mathrm{tr}_{K'/K}(\omega) = \sum_{\tau\in G/H} \tau(\omega). \tag{5.8}$$

So it suffices to prove the equality

$$\mathrm{Cor}_{K'/K}(\gamma_{K'}(a\mathrm{dlog}\, y)) = \gamma_K((\sum_{\tau\in G/H} \tau(a))\mathrm{dlog}\, y). \tag{5.9}$$

In the notation of Remark 5.1, we have $\gamma_{K'}(a\mathrm{dlog}\, y) = \{\overline{a}, \beta(\{y_1, \cdots, y_i\})\}_{K'}$. So we have

$$\begin{aligned}
\text{LHS of (5.9)} &= \mathrm{Cor}_{K'/K}\{\overline{a}, \beta(\{y_1, \cdots, y_i\})\}_{K'} \\
&= \{\mathrm{Cor}^{1,0}_{K'/K}(\overline{a}), \{y_1, \cdots, y_i\}\}_K \\
&= \{\overline{\sum_{\tau\in G/H} \tau(a)}, \{y_1, \cdots, y_i\}\}_K \quad \text{(Remark 5.1)} \\
&= \text{RHS of (5.9)}.
\end{aligned}$$

So the assertion (2) is proved and so the proof of the claim is finished. □

Since the claim 3 is proved, the proof of Theorem 5.4, Step 4 is finished. □

Since we have given the proof of Theorem 5.4 in all the cases, the proof of Theorem 5.4 is now completed.

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