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Homogeneous Law-Invariant Coherent Multiperiod Value Measures and their Limits

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Abstract. The authors introduce a new notion, homogeneous law invariant coherent multiperiod value measures, and give some basic properties. Also, they give certain limit theorems on these value measures in two models, Brownian-Poisson filtration and collective risk.

1. Introduction

The concept of coherent risk measures was introduced by Artzner et al. [1], and a characterization theorem was given by Delbaen [6]. Recently coherent multiperiod risk measures were introduced in [2], and many studies have already appeared (e.g. [3], [4], [5]). On the other hand, the concept of law invariant coherent risk measures was given in [9]. In the present paper, we extend this idea to multiperiod ones. The basic tool is the concept of conditional law invariant coherent risk measures. We remark that such a kind of idea is not new (c.f. Gerber [7]). We also study continuous limits of such risk measures.

Note that ϕ is called a value measure, if $-\phi$ is a risk measure. We state our results by using notions of value measures instead of risk measures, since value measures have preferable properties. Let us state our main results in the rest of this section.

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space. We denote $L^p(\Omega, \mathcal{F}, P)$ by L^p, $1 \leqq p \leqq \infty$.

Definition 1. We say that a map $\phi : L^\infty \to \mathbf{R}$ is a coherent value measure, if the following are satisfied.

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(1) If $X \geqq 0$, then $\phi(X) \geqq 0$.
(2) Superadditivity : $\phi(X_1 + X_2) \geqq \phi(X_1) + \phi(X_2)$.
(3) Positive homogeneity : for $\lambda > 0$ we have $\phi(\lambda X) = \lambda\phi(X)$.
(4) For every constant c we have $\phi(X + c) = \phi(X) + c$.

Then Delbaen [6] proved the following.

Theorem 2. *For $\phi : L^\infty \to \mathbf{R}$, the following conditions are equivalent.*
(1) *There is a (closed convex) set of probability measures $\mathcal{Q}$ such that any $Q \in \mathcal{Q}$ is absolutely continuous with respect to P and for $X \in L^\infty$*

$$\phi(X) = \inf\{E^Q[X];\ Q \in \mathcal{Q}\}.$$

(2) *ϕ is a coherent value measure and satisfies the Fatou property, i.e., if $\{X_n\}_{n=1}^\infty \subset L^\infty$ is uniformly bounded and converging to X in probability, then*

$$\phi(X) \geqq \limsup \phi(X_n).$$

(3) *ϕ is a coherent value measure and satisfies the following property. If X_n is a uniformly bounded sequence that decreases to X, then $\phi(X_n)$ tends to $\phi(X)$.*

Now we introduce the following notion.

Definition 3. We say that a map $\phi : L^\infty \to \mathbf{R}$ is law invariant, if $\phi(X) = \phi(Y)$ whenever $X, Y \in L^\infty$ have the same probability law.

Let $\mathcal{L}$ denote the set of probability measures on $\mathbf{R}$, $\mathcal{L}_p$, $p \in [1,\infty)$, denote the set of probability measures ν on $\mathbf{R}$ such that $\int_{\mathbf{R}} |x|^p \nu(dx) < \infty$, and $\mathcal{L}_\infty$ denote the set of probability measures ν on $\mathbf{R}$ such that $\nu(\mathbf{R}\setminus[-M,M]) = 0$ for some $M > 0$. Also, $\mathcal{M}_{[0,1]}$ be the set of probability measures on $[0,1]$.

For $\nu \in \mathcal{L}$, let F_ν be the distribution function of ν, i.e., $F_\nu(z) = \nu((-\infty, z])$, $z \in \mathbf{R}$. Let us define $Z : [0,1) \times \mathcal{L} \to \mathbf{R}$ by

$$Z(x,\nu) = \inf\{z; F_\nu(z) > x\}, \qquad x \in [0,1),\ \nu \in \mathcal{L}.$$

Then $Z(\cdot,\nu) : [0,1) \to \mathbf{R}$ is non-decreasing and right continuous, and the probability law of $Z(\cdot,\nu)$ under Lebesgue measure on $[0,1)$ is ν (c.f.[10]). For any random variable X, we denote by μ_X the probability law of X.

For each $\alpha \in (0,1]$, let $\eta_\alpha : \mathcal{L}_1 \to \mathbf{R}$ be given by

$$\eta_\alpha(\nu) = \alpha^{-1} \int_0^\alpha Z(x,\nu)dx, \qquad \nu \in \mathcal{L}_1.$$

Also, we define $\eta_0 : \mathcal{L}_1 \to [-\infty, \infty)$ by

$$\eta_0(\nu) = \inf\{x \in \mathbf{R}; \nu((-\infty, x]) > 0\} \qquad \nu \in \mathcal{L}_1.$$

Then we have the following (cf. [9]).

THEOREM 4. *Assume that $(\Omega, \mathcal{F}, P)$ is a standard probability space and P is non-atomic. Let $\phi : L^\infty \to \mathbf{R}$. Then the following conditions are equivalent.*
(1) *There is a (compact convex) subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that*

$$\phi(X) = \inf\{\int_0^1 \eta_\alpha(\mu_X)m(d\alpha);\ m \in \mathcal{M}_0\}, \qquad X \in L^\infty.$$

(2) *ϕ is a law invariant coherent value measure with the Fatou property.*

DEFINITION 5. We say that a map $\eta : \mathcal{L}_\infty \to \mathbf{R}$ is a mild value measure (abbreviated to MVM), if there is a subset $\mathcal{M}_0$ of $\mathcal{M}_{[0,1]}$ such that

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu)m(d\alpha);\ m \in \mathcal{M}_0\}, \qquad \nu \in \mathcal{L}_\infty.$$

For any MVM η, we define a subset $\mathcal{M}(\eta)$ of $\mathcal{M}_{[0,1]}$ by

$$\mathcal{M}(\eta) = \{m \in \mathcal{M};\ \eta(\nu) \leqq \int_0^1 \eta_\alpha(\nu)m(d\alpha) \text{ for all } \nu \in \mathcal{L}_\infty\}.$$

For any $\nu \in \mathcal{L}_1$, we see that $\eta_\alpha(\nu) \leqq \eta_1(\nu)$, $\alpha \in [0,1]$. So any MVM η can be extended to a map from $\mathcal{L}_1$ to $[-\infty, \infty)$ by

$$\eta(\nu) = \inf\{\int_0^1 \eta_\alpha(\nu)m(d\alpha);\ m \in \mathcal{M}(\eta)\}, \qquad \nu \in \mathcal{L}_1.$$

We denote this map by the same symbol η.

Definition 6. Let η be an MVM and $(\Omega, \mathcal{F}, P)$ be a standard probability space.
(1) For any integrable random variable X and any sub-σ-algebra $\mathcal{G}$, we define a $\mathcal{G}$-measurable random variable $\eta(X|\mathcal{G})$ by

$$\eta(X|\mathcal{G}) = \eta(P(X \in dx|\mathcal{G})),$$

where $P(X \in dx|\mathcal{G})$ is a regular conditional probability law of X given a sub-σ-algebra $\mathcal{G}$. We call $\eta(X|\mathcal{G})$ a conditional value measure of a random variable X with respect to a sub-σ-algebra $\mathcal{G}$.

We denote $\eta(X|\{\emptyset, \Omega\})$ simply by $\eta(X)$.
(2) For any integrable random variable X and any filtration $\{\mathcal{F}_k\}_{k=0}^n$, we define an adapted process $\{Z_k\}_{k=0}^n$ inductively by

$$Z_n = \eta(X|\mathcal{F}_n),$$

$$Z_{k-1} = \eta(Z_k|\mathcal{F}_{k-1}), \qquad k = n, n-1, \ldots, 1.$$

We denote an $\mathcal{F}_0$-measurable random variable Z_0 by $\eta(X|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure of a random variable X with respect to a filtration $\{\mathcal{F}_k\}_{k=0}^n$.
(3) For any filtration $\{\mathcal{F}_k\}_{k=0}^n$ and any integrable adapted process $\{X_k\}_{k=0}^n$, we define an adapted process $\{Y_k\}_{k=0}^n$ inductively by

$$Y_n = X_n,$$

$$Y_{k-1} = X_{k-1} \wedge \eta(Y_k|\mathcal{F}_{k-1}), \qquad k = n, n-1, \ldots, 1.$$

We denote an $\mathcal{F}_0$-measurable random variable Y_0 by $\eta(\{X_k\}_{k=0}^n|\{\mathcal{F}_k\}_{k=0}^n)$, and call it a homogeneous filtered value measure of an adapted process $\{X_k\}_{k=0}^n$ with respect to a filtration $\{\mathcal{F}_k\}_{k=0}^n$.

In this paper, we consider two kinds of limit theorem for homogeneous filtered value measures. Let us introduce the following notion. For any MVM η and $p \in [1, \infty)$, let

$$\Delta_p(\eta) = \sup\{\int_0^1 (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha}) m(d\alpha);\ m \in \mathcal{M}(\eta)\}.$$

1.1. Brownian-Poisson filtration

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{B(t); t \in [0, \infty)\}$ be a d-dimensional Brownian motion and $\{N_i(t); t \in [0, \infty)\}$, $i = 1, \dots, \ell$, be Poisson processes with an intensity λ_i. We assume that they are independent. Let $\lambda = \sum_{i=1}^{\ell} \lambda_i$, and let $\mathcal{F}_t = \sigma\{B(s), N_i(s); s \leqq t, i = 1, \dots, \ell\}$, $t \geqq 0$.

Let η_n, $n = 1, 2, \dots$, be MVM's. We assume the following.

(A-1) There is a constant $C > 0$ such that $\Delta_2(\eta_n) \leqq C 2^{-n/2}$, $n = 1, 2, \dots$.

Let $F_0(y; \alpha, \beta)$, $y \in \mathbf{R}^\ell$, $0 \leqq \alpha \leqq \beta \leqq 1$, be given by

$$\begin{aligned} F_0(y; \alpha, \beta) &= \inf\{\int_0^\gamma Z(x, \lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i}) dx;\ \alpha \leqq \gamma \leqq \beta\} \\ &= \inf\{\gamma \eta_\gamma (\lambda^{-1} \sum_{i=1}^{\ell} \lambda_i \delta_{y_i});\ \alpha \leqq \gamma \leqq \beta\}, \end{aligned}$$

and let $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$, $n = 1, 2, \dots$, be given by

$$\begin{aligned} b_n(x, y) = \ & \inf\{|x| 2^{n/2} (\int_0^1 \eta_\alpha(\mu_0) m(d\alpha)) \\ & + \lambda (\int_0^1 m(d\alpha) \alpha^{-1} F_0(y; 0 \vee (1 - 2^n \lambda^{-1}(1 - \alpha)), 1 \wedge 2^n \lambda^{-1} \alpha)); \\ & \qquad m \in \mathcal{M}(\eta_n)\}. \end{aligned}$$

Here μ_0 is a standard normal distribution.

Then $b_n : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$ is concave,

$$b_n(sx, sy) = s b_n(x, y), \qquad x \in \mathbf{R}^d, \quad y \in \mathbf{R}^\ell, \quad s \geqq 0,$$

and

$$b_n(x, y_1, \dots, y_\ell) \leqq b_n(x', y'_1, \dots, y'_\ell),$$

if $|x| \geqq |x'|$, $y_1 \leqq y'_1$, $\dots$, $y_\ell \leqq y'_\ell$.

Let us assume the following furthermore.

(A-2) There is a continuous function $b : \mathbf{R}^d \times \mathbf{R}^\ell \to \mathbf{R}$ such that $b_n \to b$, $n \to \infty$, uniformly on compacts in $\mathbf{R}^d \times \mathbf{R}^\ell$.

Let K be a compact convex set in $\mathbf{R}^d \times \mathbf{R}^\ell$ given by

$$K = \{(z,w) \in \mathbf{R}^d \times [0,\infty)^\ell;\ b(x,y) \leqq x \cdot z + \sum_{i=1}^{\ell} \lambda_i y_i w_i \quad \text{for all } (x,y) \in \mathbf{R}^d \times \mathbf{R}^\ell\}.$$

Also, let $\mathcal{K}$ be a set of martingales $\rho(t)$ such that there are predictable processes $\varphi : [0,\infty) \times \Omega \to \mathbf{R}^d$, $\psi_i : [0,\infty) \times \Omega \to [0,\infty)$, $i = 1, \dots, \ell$, for which

$$P((\varphi(t), \psi_1(t), \dots, \psi_\ell(t)) \in K \text{ for any } t \in [0,T]) = 1$$

and

$$\rho(t) = \prod_{i=1}^{\ell} \Big(\prod_{s \in (0,t], \Delta N_i(s) \neq 0} \psi_i(s) \Big) \times \exp\Big(\int_0^t \varphi(s) dB(s) - \frac{1}{2} \int_0^t |\varphi(s)|^2 ds - \sum_{i=1}^{\ell} \lambda_i \int_0^t (\psi_i(s) - 1) ds \Big),$$

$t \geqq 0$.

Then we have the following.

Theorem 7. *Under the assumption* (A-1) *and* (A-2), *we have the following.*

For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, $T > 0$,

$$\lim_{n \to \infty} \eta_n(X | \{\mathcal{F}_{2^{-n}k}\}_{k=0}^{2^{2n}}) = \inf\{E[\rho(T)X]; \rho \in \mathcal{K}\}.$$

Note that

$$\eta_n(X | \{\mathcal{F}_{2^{-n}k}\}_{k=0}^{2^{2n}}) = \eta_n(X | \{\mathcal{F}_{2^{-n}k}\}_{k=0}^{[2^n T]+1})$$

if n is sufficiently large.

We prove this theorem in Section 5 via a nonlinear partial differential equation.

1.2. Collective risk

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $K \geqq 1$, $p \in (1, \infty)$, $p_k \in \mathbf{R}$, $\lambda_k > 0$, and $\nu_k \in \mathcal{L}_p$, $k = 1, \dots, K$. Let $Z_i^{(k)}$, $\tau_i^{(k)}$, $k = 1, \dots, K$, $i = 1, 2, \dots$, be independent random variables such that the distribution of $Z_i^{(k)}$ is ν_k, and $P(\tau_i^{(k)} > t) = \exp(-\lambda_k t)$, $t \geqq 0$, for $k = 1, \dots, K$, $i = 1, 2, \dots$. Let $N_i^{(k)}(t) = 1_{\{\tau_i^{(k)} \leqq t\}}$, and $X_i^{(k)}(t) = Z_i^{(k)} N_i^{(k)}(t) + p_k(\tau_i^{(k)} \wedge t)$ for $t \geqq 0$, $k = 1, \dots, K$, $i = 1, 2, \dots$.

Let $\mathcal{F}_t = \sigma\{X_i^{(k)}(s);\ s \in [0, t], k = 1, \dots, K, i = 1, 2, \dots\}$, $t \geqq 0$. Also, let

$$X(t; m_1, \dots, m_K) = \sum_{k=1}^{K} \sum_{i=1}^{m_k} X_i^{(k)}(t)$$

for any $t \geqq 0$, and any $m_1, \dots, m_K \in \mathbf{Z}_{\geqq 0}$. Here $\mathbf{Z}_{\geqq 0}$ denotes the set of non-negative integers.

Theorem 8. *Let η be MVM. Assume that $\Delta_p(\eta) < \infty$. Let $\Phi : [0, \infty)^K \times \mathbf{R}^K \to \mathbf{R}$ be given by*

$$\Phi(x, \xi) = \eta\Big(\sum_{\ell_1, \dots, \ell_K = 0}^{\infty} \Big(\prod_{k=1}^{K} (\exp(-\lambda_k x_k) \frac{(\lambda_k x_k)^{\ell_k}}{\ell_k!} \Big)\Big)$$
$$\times (\nu_1 - \xi_1)^{*\ell_1} * \cdots * (\nu_K - \xi_K)^{*\ell_K}) + \sum_{k=1}^{K} p_k x_k,$$

for $x \in [0, \infty)^K$, $\xi \in \mathbf{R}^K$. Here $*$ stands for the convolution and $\nu + a$ denotes a probability measure on $\mathbf{R}$ given by the following for any probability measure ν on $\mathbf{R}$ and $a \in \mathbf{R}$.

$$(\nu + a)(A) = \nu(\{x \in \mathbf{R};\ x - a \in A\}) \text{ for any Borel set } A \text{ in } \mathbf{R}.$$

Assume that there is a C^1 fucntion $u : [0, \infty) \times [0, \infty)^K \to \mathbf{R}$ such that $u(0, x) = 0$, $x \in [0, \infty)^K$, and satisfies the following Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} u(t, x) = \Phi(x, \frac{\partial}{\partial x^1} u(t, x), \dots, \frac{\partial}{\partial x^K} u(t, x)), \qquad (t, x) \in [0, \infty) \times [0, \infty)^K.$$

Then we have the following.

$$\sup\{|h\eta(X(t; m_1, \dots, m_K)|\{\mathcal{F}_{jh}\}_{j=0}^{[h^{-2}]}) - u(t, m_1 h, \dots, m_K h)|;$$
$$t, m_1 h, \dots, m_K h \in [0, R], m_1, \dots, m_K \in \mathbf{Z}_{\geqq 0}\} \to 0,$$

as $h \downarrow 0$, *for any* $R > 0$.

2. Basic Estimates

Proposition 9. (1) *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Then for any* $\alpha \in (0,1]$ *and* $X \in L^1(\Omega, \mathcal{F}, P)$,

$$\begin{aligned}&\inf\{E[\rho X];\ \rho \in L^\infty, 0 \leqq \rho \leqq 1, E[\rho] = \alpha\}\\ =\ &\inf\{\int_{\mathbf{R}} x f(x)\mu_X(dx);\ 0 \leqq f \leqq 1, \int_{\mathbf{R}} f(x)\mu_X(dx) = \alpha\}.\end{aligned}$$

(2) *For any* $\alpha \in (0,1]$ *and* $\mu \in \mathcal{L}_1$

$$\begin{aligned}&\int_0^\alpha Z(x;\mu)dx\\ =\ &\inf\{\int_{\mathbf{R}} x f(x)\mu(dx);\ 0 \leqq f \leqq 1, \int_{\mathbf{R}} f(x)\mu(dx) = \alpha\}.\end{aligned}$$

Proof. Let $\rho \in L^\infty$ with $0 \leqq \rho \leqq 1$ and $E[\rho] = \alpha$. Then there is a Borel measurable function $h : \mathbf{R} \to \mathbf{R}$ such that $h(X) = E[\rho|\sigma\{X\}]$ *a.s.* Let $f = (h \vee 0) \wedge 1$. Then we have

$$E[\rho X] = E[XE[\rho|\sigma\{X\}] = E[Xf(X)] = \int_{\mathbf{R}} x f(x)\mu_X(dx)$$

and

$$\int_{\mathbf{R}} f(x)\mu_X(dx) = E[f(X)] = E[\rho] = \alpha.$$

These imply the assertion (1).

Let $\mu \in \mathcal{L}_\infty$. Then taking $([0,1), \mathcal{B}([0,1)), dx)$ as a probability space, we have from the assertion (1),

$$\begin{aligned}&\inf\{\int_{\mathbf{R}} x f(x)\mu(dx);\ 0 \leqq f \leqq 1, \int_{\mathbf{R}} f(x)\mu(dx) = \alpha\}\\ =\ &\inf\{\int_0^1 Z(x;\mu)\rho(x)dx;\ 0 \leqq \rho \leqq 1, \int_0^1 \rho(x)dx = \alpha\}\\ =\ &\int_0^\alpha Z(x;\mu)dx.\end{aligned}$$

This implies the assertion (2).

This completes the proof. □

PROPOSITION 10. *Let ν be a probability measure on $\mathbf{R}^2$ such that $\int_{\mathbf{R}^2}(|x| + |y|)\ \nu(dx, dy) < \infty$. Let $\mu_1(dx)$, $\mu_2(dy)$ be the marginal distributions of ν, and let μ_3 be the probability law of $x+y$ under $\nu(dx, dy)$. Then for any MVM η,*

$$\eta(\mu_3) \geqq \eta(\mu_1) + \eta(\mu_2).$$

PROOF. First we show that

$$\eta_\alpha(\mu_3) \geqq \eta_\alpha(\mu_1) + \eta_\alpha(\mu_2) \qquad \alpha \in (0, 1]. \tag{1}$$

There are $\rho_1(\cdot, y), \rho_2(\cdot, x) \in \mathcal{L}$, $x, y \in \mathbf{R}$, such that

$$\nu(dx, dy) = \rho_1(dx, y)\mu_2(dy) = \rho_2(dy, x)\mu_1(dx).$$

Then we have for any measurable function $f : \mathbf{R} \to [0, 1]$

$$\begin{aligned} &\int_{\mathbf{R}} zf(z)\mu_3(dz) = \int_{\mathbf{R}} (x + y)f(x + y)\nu(dx, dy) \\ &= \int_{\mathbf{R}} x(\int_{\mathbf{R}} f(x + y)\rho_2(dy, x))\mu_1(dx) + \int_{\mathbf{R}} y(\int_{\mathbf{R}} f(x + y)\rho_1(dx, y))\mu_2(dy). \end{aligned}$$

Note that

$$0 \leqq \int_{\mathbf{R}} f(x + y)\rho_2(dy, x) \leqq 1, \qquad 0 \leqq \int_{\mathbf{R}} f(x + y)\rho_1(dx, y) \leqq 1,$$

and

$$\begin{aligned} &\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x + y)\rho_2(dy, x))\mu_1(dx) \\ &= \int_{\mathbf{R}} (\int_{\mathbf{R}} f(x + y)\rho_1(dx, y))\mu_2(dy) = \int_{\mathbf{R}} f(z)\mu_3(dz). \end{aligned}$$

This and Proposition 9 imply Equation (1).

Our assertion is an easy consequence of Equation (1). □

Proposition 11. (1) *For any* $\alpha \in (0,1]$, *and* $\nu \in \mathcal{L}_p$, $p \in [1,\infty)$,

$$|\eta_\alpha(\nu)| \leqq \alpha^{-1/p}(\int_{\mathbf{R}} |x|^p \nu(dx))^{1/p}.$$

(2) *For any* $\alpha \in (0,1]$, *and* $\nu \in \mathcal{L}_p$, $p \in [1,\infty)$, *with* $\int_{\mathbf{R}} x\nu(dx) = 0$,

$$|\eta_\alpha(\nu)| \leqq \frac{(1-\alpha)^{1-1/p}}{\alpha}(\int_{\mathbf{R}} |x|^p \nu(dx))^{1/p}.$$

(3) *For any MVM* η, $p \in [1,\infty)$ *and* $\nu \in \mathcal{L}_p$, *with* $\int_{\mathbf{R}} x\nu(dx) = 0$,

$$|\eta(\nu)| \leqq \Delta_p(\eta)(\int_{\mathbf{R}} |x|^p \nu(dx))^{1/p}.$$

In other words, for any $X \in L^p$,

$$E[X] - \Delta_p(\eta)E[|X - E[X]|^p]^{1/p} \leqq \eta(\mu_X) \leqq E[X].$$

Proof. The assertion (1) follows from

$$|\eta_\alpha(\nu)| = \frac{1}{\alpha}|\int_0^\alpha Z(x;\nu)dx| \leqq \frac{1}{\alpha}\alpha^{1-1/p}(\int_0^\alpha |Z(x;\nu)|^p dx)^{1/p}.$$

The assertion (2) follows from

$$|\eta_\alpha(\nu)| = \frac{1}{\alpha}|\int_\alpha^1 Z(x;\nu)dx| \leqq \frac{1}{\alpha}(1-\alpha)^{1-1/p}(\int_\alpha^1 |Z(x;\nu)|^p dx)^{1/p}.$$

The assertions (1) and (2) imply that

$$|\eta_\alpha(\nu)| \leqq (\alpha^{-1/p} \wedge \frac{(1-\alpha)^{1-1/p}}{\alpha})(\int_{\mathbf{R}} |x|^p \nu(dx))^{1/p}$$

for any $\alpha \in (0,1]$, and $\nu \in \mathcal{L}_p$, $p \in [1,\infty)$, with $\int_{\mathbf{R}} x\nu(dx) = 0$. This implies the assertion (3).

This completes the proof. □

Proposition 12. *Let* X, X_1, X_2 *be integrable random variables.*
(1) *If* $X \geqq 0$, *then* $\eta(X|\mathcal{G}) \geqq 0$.

(2) $\eta(X_1 + X_2|\mathcal{G}) \geqq \eta(X_1|\mathcal{G}) + \eta(X_2|\mathcal{G})$.
(3) *For any $\mathcal{G}$-measurable bounded nonnegative random variable Z, we have* $\eta(ZX|\mathcal{G}) = Z\eta(X|\mathcal{G})$.
(4) *For any $\mathcal{G}$-measurable integrable random variable Y,* $\eta(X + Y|\mathcal{G}) = \eta(X|\mathcal{G}) + Y$.
(5) *For any $p \in (1, \infty)$,*

$$E[X|\mathcal{G}] - \Delta_p(\eta)E[|X - E[X|\mathcal{G}]|^p|\mathcal{G}]^{1/p} \leqq \eta(X|\mathcal{G}) \leqq E[X|\mathcal{G}]$$

Proof. Since the proofs are similar, we only prove the assertion (2). Let ν_ω be a regular conditional probability measure of (X_1, X_2) under $\mathcal{G}$. Let $\mu_{1,\omega}$, $\mu_{2,\omega}$ and $\mu_{3,\omega}$ be regular conditional probability measures of X_1, X_2 and $X_1 + X_2$ under $\mathcal{G}$, respectively. Then we have

$$\mu_{1,\omega}(A) = \nu_\omega(A, \mathbf{R}), \qquad \mu_{2,\omega}(A) = \nu_\omega(\mathbf{R}, A) \quad a.s.$$

and

$$\mu_{3,\omega}(A) = \nu_\omega(\{(x, y); x + y \in A\}) \quad a.s.$$

for any Borel set A. So by Proposition 10, we see that

$$\eta(\mu_{3,\omega}) \geqq \eta(\mu_{1,\omega}) + \eta(\mu_{2,\omega}).$$

This completes the proof. □

Proposition 13. *For any $X_1, X_2 \in L^p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$,*

$$|\eta(X_1|\mathcal{G}) - \eta(X_2|\mathcal{G})| \leqq |\eta(X_1 - X_2|\mathcal{G})| \vee |\eta(X_2 - X_1|\mathcal{G})|.$$

In particular,

$$\begin{aligned}
&|\eta(X_1|\mathcal{G}) - \eta(X_2|\mathcal{G})| \\
&\leqq \Delta_p(\eta)E[|X_1 - X_2|^p|\mathcal{G}]^{1/p} + (1 + \Delta_p(\eta))|E[X_1 - X_2|\mathcal{G}]| \\
&\leqq 2\Delta_p(\eta)E[|X_1 - X_2|^p|\mathcal{G}]^{1/p} + |E[X_1 - X_2|\mathcal{G}]|.
\end{aligned}$$

Proof. Since

$$\eta(X_i|\mathcal{G}) - \eta(X_j|\mathcal{G}) \leqq -\eta(X_j - X_i|\mathcal{G}),$$

we have the first assertion. By Proposition 12, we have

$$|\eta(X|\mathcal{G})| \leqq \Delta_p(\eta)E[|X|^p|\mathcal{G}]^{1/p} + (1+\Delta_p(\eta))|E[X|\mathcal{G}]|.$$

By this and the first assertion, we have the second assertion. □

Proposition 14. *Suppose that $\Delta_2(\eta) \leqq 1/2$. Then we have the following.*

(1) *For any integrable random variable X,*

$$|\eta(X|\mathcal{G})|^2 \leqq (1+2\Delta_2(\eta)^2)E[|X|^2|\mathcal{G}].$$

(2) *For any $X_1, X_2 \in L^2(\Omega, \mathcal{F}, P)$,*

$$|\eta(X_1|\mathcal{G}) - \eta(X_2|\mathcal{G})|^2 \leqq (1+2\Delta_2(\eta)^2)E[|X_1 - X_2|^2|\mathcal{G}].$$

Proof. Let $Y = X - E[X|\mathcal{G}]$. By Proposition 12 we have

$$\begin{aligned}
\eta(X|\mathcal{G})^2 &= (E[X|\mathcal{G}] + \eta(Y|\mathcal{G}))^2 \\
&= E[X^2|\mathcal{G}] - E[Y^2|\mathcal{G}] + 2E[X|\mathcal{G}]\eta(Y|\mathcal{G}) + \eta(Y|\mathcal{G})^2 \\
&\leqq E[X^2|\mathcal{G}] - E[Y^2|\mathcal{G}] + 2\Delta_2(\eta)E[X^2|\mathcal{G}]^{1/2}E[Y^2|\mathcal{G}]^{1/2} \\
&+ \Delta_2(\eta)^2E[Y^2|\mathcal{G}] \\
&\leqq E[X^2|\mathcal{G}] - \frac{1}{2}E[Y^2|\mathcal{G}] + 2\Delta_2(\eta)E[X^2|\mathcal{G}]^{1/2}E[Y^2|\mathcal{G}]^{1/2}. \\
&= E[X^2|\mathcal{G}] - \frac{1}{2}(E[Y^2|\mathcal{G}]^{1/2} - 2\Delta_2(\eta)E[X^2|\mathcal{G}]^{1/2})^2 \\
&+ 2\Delta_2(\eta)^2E[X^2|\mathcal{G}].
\end{aligned}$$

So we see that

$$\eta(X|\mathcal{G})^2 \leqq E[X^2|\mathcal{G}] + 2\Delta_2(\eta)^2E[X^2|\mathcal{G}].$$

This implies the assertion (1).

The assertion (1) and Proposition 13 imply the assertion (2).

This completes the proof. □

Also we have the following.

Proposition 15. *Let X, X_1, X_2 be integrable random variables.*
(1) *If $X \geqq 0$, then $\eta(X|\{\mathcal{F}_k\}_{k=0}^n) \geqq 0$.*
(2) $\eta(X_1 + X_2|\{\mathcal{F}_k\}_{k=0}^n\}) \geqq \eta(X_1|\{\mathcal{F}_k\}_{k=0}^n) + \eta(X_2|\{\mathcal{F}_k\}_{k=0}^n)$.
(3) *For any $\mathcal{F}_0$-measurable bounded nonnegative random variable Z, we have*

$$\eta(ZX|\{\mathcal{F}_k\}_{k=0}^n) = Z\eta(X|\{\mathcal{F}_k\}_{k=0}^n).$$

(4) *For any $\mathcal{F}_0$-measurable integrable random variable Y,*

$$\eta(X + Y|\{\mathcal{F}_k\}_{k=0}^n) = \eta(X|\{\mathcal{F}_k\}_{k=0}^n) + Y.$$

(5) *Suppose that $\Delta_2(\eta) \leqq 1/2$. For any $X_1, X_2 \in L^2(\Omega, \mathcal{F}, P)$,*

$$|\eta(X_1|\{\mathcal{F}_k\}_{k=0}^n) - \eta(X_2|\{\mathcal{F}_k\}_{k=0}^n)|^2 \leqq (1 + 2\Delta_2(\eta)^2)^{n+1} E[|X_1 - X_2|^2|\mathcal{F}_0].$$

Proof. All assertions are proved by induction. Since the proofs are similar, we only prove the assertion (5).

The case when $n = 0$ follows from Proposition 14. If the assertion is valid for n, we see that

$$\begin{aligned}
&|\eta(X_1|\{\mathcal{F}_k\}_{k=0}^{n+1}) - \eta(X_2|\{\mathcal{F}_k\}_{k=0}^{n+1})|^2 \\
= &|\eta(\eta(X_1|\{\mathcal{F}_k\}_{k=1}^{n+1})|\mathcal{F}_0) - \eta(\eta(X_2|\{\mathcal{F}_k\}_{k=1}^{n+1})|\mathcal{F}_0)|^2 \\
\leqq &(1 + 2\Delta_2(\eta)^2) E[|\eta(X_1|\{\mathcal{F}_k\}_{k=1}^{n+1}) - \eta(X_2|\{\mathcal{F}_k\}_{k=1}^{n+1})|^2|\mathcal{F}_0] \\
\leqq &(1 + 2\Delta_2(\eta)^2)^{2+n} E[E[|X_1 - X_2|^2|\mathcal{F}_1]|\mathcal{F}_0].
\end{aligned}$$

So we have the assertion (5) by induction.

This completes the proof. □

3. Some Estimates

Proposition 16. *Let $\mu, \nu \in \mathcal{L}$. Then for any $\lambda, \alpha \in [0, 1]$, we have*

$$\begin{aligned}
&\int_0^\alpha Z(x; \lambda\nu + (1 - \lambda)\mu) dx \\
= &\inf\{\lambda \int_0^\beta Z(x; \nu) dx + (1 - \lambda) \int_0^\gamma Z(x; \mu) dx; \\
&\qquad \lambda\beta + (1 - \lambda)\gamma = \alpha, \quad \beta, \gamma \in [0, 1]\}.
\end{aligned}$$

PROOF. Let us think of a random variable X defined on $[0,1)$ given by

$$X = Z(\lambda^{-1}x;\nu)1_{[0,\lambda)}(x) + Z((1-\lambda)^{-1}(x-\lambda);\mu)1_{[\lambda,1)}(x), \qquad x \in [0,1).$$

Then the distribution law of X under Lebesgue measure is $\lambda\nu + (1-\lambda)\mu$. Note that

$$\int_0^\alpha Z(x;\lambda\nu+(1-\lambda)\mu)dx = \inf\{\int_A X(x)dx;\ A \in \mathcal{B}([0,1)), \int_A dx = \alpha\}$$
$$= \inf\{\int_0^a Z(\lambda^{-1}x;\nu)dx + \int_\lambda^{\lambda+b} Z((1-\lambda)^{-1}(x-\lambda);\mu)dx;$$
$$a \in [0,\lambda), b \in [0,1-\lambda), a+b=\alpha\}.$$

This implies our assertion. □

PROPOSITION 17. *Let $\ell \geqq 1$, $\varepsilon \in (0,1]$, $c \geqq 0$ and $\lambda_i > 0$, $i = 1,\dots,\ell$. Let $\lambda = \sum_{i=1}^\ell \lambda_i$, and assume that $\lambda\varepsilon^2 < 1/2$. Let $X, Z_1,\dots,Z_\ell, \xi_1,\dots,\xi_\ell$ be independent random variables such that the distribution of X is the standard normal distribution and*

$$P(\xi_i = 1) = 1 - P(\xi_i = 0) = \lambda_i\varepsilon^2, \qquad i = 1,\dots,\ell.$$

Let ν_i, $i = 1,\dots,\ell$, be the probability law of Z_i. and $\nu = \lambda^{-1}\sum_{i=1}^\ell \lambda_i\nu_i$. Also let

$$F(\nu;\alpha,\beta) = \inf\{\int_0^\gamma Z(x;\nu)dx;\ \alpha \leqq \gamma \leqq \beta\}, \qquad 0 \leqq \alpha \leqq \beta \leqq 1.$$

Then for any MVM η with $\Delta_2(\eta) < \infty$,

$$|\eta(c\varepsilon X + \sum_{i=1}^\ell \xi_i Z_i) - \inf\{|c|\varepsilon(\int_0^1 \eta_\alpha(\mu_0)m(d\alpha))$$
$$+ \lambda\varepsilon^2(\int_0^1 m(d\alpha)\alpha^{-1}F(\nu;0\vee(1-(\lambda\varepsilon^2)^{-1}(1-\alpha)), 1\wedge(\lambda\varepsilon^2)^{-1}\alpha));$$
$$m \in \mathcal{M}(\eta)\}|$$
$$\leqq 2\Delta_2(\eta)(c(2\lambda^{1/2}\varepsilon^2 + 3\varepsilon(2\lambda\varepsilon^2)^{2/5} + \lambda\varepsilon^3) + 2\lambda\varepsilon^2(\int_{\mathbf{R}}|x|^2 d\nu)^{1/2}))$$
$$+ 2c(\lambda\varepsilon^3 + 3\varepsilon(2\lambda\varepsilon^2)^{9/10}) + 2\lambda^2\varepsilon^4(\int_{\mathbf{R}}|x|d\nu),$$

where μ_0 is the standard normal distribution.

Proof. Let $(\Omega_k, \mathcal{F}_k, P_k)$, $k = 0, 1$, be copies of $(\Omega, \mathcal{F}, P)$. Let $\pi_k : \Omega_0 \times \Omega_1 \to \Omega$, $k = 0, 1$, be maps given by $\pi_k(\omega_0, \omega_1) = \omega_k$. Let us think of a probability space $(\Omega_0 \times \Omega_1, \mathcal{F}_0 \otimes \mathcal{F}_1, P_0 \otimes P_1)$.

Let $A_0 = \{\xi_1 = \cdots = \xi_\ell = 0\}$, $A_i = \{\xi_i = 1, \xi_j = 0, j \neq i\}$, $i = 1, \ldots, \ell$. Then

$$P(A_0) = \prod_{j=1}^{\ell}(1 - \lambda_j \varepsilon^2) = 1 - \sum_{i=1}^{\ell} \lambda_i \varepsilon^2 \prod_{j=i+1}^{\ell}(1 - \lambda_j \varepsilon^2) \geqq 1 - \lambda \varepsilon^2,$$

and

$$P(A_i) = \lambda_i \varepsilon^2 \prod_{j \neq i}(1 - \lambda_j \varepsilon^2) \leqq \lambda_i \varepsilon^2, \qquad i = 1, \ldots, \ell.$$

So there are mutually disjoint sets A'_i, $i = 0, 1, \ldots, \ell$, in $\mathcal{F}$ such that $A_0 \supset A'_0$, $A_i \subset A'_i$, $i = 1, \ldots, \ell$, $P(A'_0) = 1 - \lambda\varepsilon^2$, $P(A'_i) = \lambda_i \varepsilon^2$, $i = 1, \ldots, \ell$, and $\cup_{i=0}^{\ell} A'_i = \Omega$. Let $\xi'_i = 1_{A'_i}$, $i = 0, 1, \ldots, \ell$. Then we have

$$\begin{aligned} E[|\xi'_i - \xi_i|^p] = P(A'_i - A_i) &= \lambda_i \varepsilon^2 (1 - \prod_{j \neq i}(1 - \lambda_j \varepsilon^2)) \\ &\leqq \lambda_i \varepsilon^2 (\prod_{j \neq i}(1 + \lambda_j \varepsilon^2) - 1) \\ &\leqq \lambda_i \varepsilon^2 (\exp(\sum_{j=1}^{\ell} \lambda_j \varepsilon^2) - 1) \leqq 2\lambda_i \lambda \varepsilon^4, \end{aligned}$$

for $i = 1, \ldots, \ell$. Note that the probability law of $c\varepsilon X + \sum_{1=1}^{\ell} \xi_i Z_i$ under P and that of $c\varepsilon(X \circ \pi_0) + \sum_{1=1}^{\ell}(Z_i \circ \pi_0)(\xi_i \circ \pi_1)$ under $P_0 \otimes P_1$ are the same. Also, we see that

$$\begin{aligned} &E[|(c\varepsilon(X \circ \pi_0) + \sum_{i=1}^{\ell}(Z_i \circ \pi_0)(\xi_i \circ \pi_1)) \\ &\quad - (c\varepsilon(X \circ \pi_0)(\xi'_0 \circ \pi_1) + \sum_{i=1}^{\ell}(Z_i \circ \pi_0)(\xi'_i \circ \pi_1))|^p]^{1/p} \\ &\leqq c\varepsilon \parallel X \parallel_{L^p} \parallel 1 - \xi'_0 \parallel_{L^p} + E[\sum_{i=1}^{\ell} |Z_i|^p |\xi'_i - \xi_i|^p]^{1/p} \end{aligned}$$

$$\leqq c\varepsilon(2\lambda\varepsilon^2)^{1/p} \parallel X \parallel_{L^p} +(2\lambda\varepsilon^4)^{1/p}E[\sum_{i=1}^{\ell}\lambda_i|Z_i|^p]^{1/p}$$
$$= c\varepsilon(2\lambda\varepsilon^2)^{1/p}(\int_{\mathbf{R}}|x|^p d\mu_0)^{1/p} + (2\lambda^2\varepsilon^4)^{1/p}(\int_{\mathbf{R}}|x|^p d\nu)^{1/p}$$

The probability law of $c\varepsilon(X \circ \pi_0)(\xi_0' \circ \pi_1) + \sum_{i=1}^{\ell}(Z_i \circ \pi_0)(\xi_i' \circ \pi_1)$ is $(1-\lambda\varepsilon^2)\mu+\lambda\varepsilon^2\nu$, where μ is the normal distribution of mean 0 and variance $c^2\varepsilon^2$.

So we have

$$(2) \quad |\eta(c\varepsilon X + \sum_{1=1}^{\ell}\xi_i Z_i) - \eta((1-\lambda\varepsilon^2)\mu + \lambda\varepsilon^2\nu)|$$
$$\leqq 2\Delta_2(\eta)(2c\lambda^{1/2}\varepsilon^2 + 2\lambda\varepsilon^2(\int_{\mathbf{R}}|x|^2 d\nu)^{1/2}) + (2c\lambda\varepsilon^3 + 2\lambda^2\varepsilon^4(\int_{\mathbf{R}}|x|d\nu)).$$

By Proposition 16, we have

$$\int_0^\alpha Z(x;(1-\lambda\varepsilon^2)\mu + \lambda\varepsilon^2\nu)dx$$
$$= \inf\{(1-\lambda\varepsilon^2)\int_0^\beta Z(x;\mu)dx + \lambda\varepsilon^2\int_0^\gamma Z(x;\nu)dx;$$
$$(1-\lambda\varepsilon^2)\beta + \lambda\varepsilon^2\gamma = \alpha, \quad \beta,\gamma \in [0,1]\}.$$

If $(1-\lambda\varepsilon^2)\beta + \lambda\varepsilon^2\gamma = \alpha$, $\beta,\gamma \in [0,1]$, then

$$\beta \leqq (1-\lambda\varepsilon^2)^{-1}\alpha \leqq \alpha + 2\lambda\varepsilon^2\alpha, \text{ and } \beta \geqq (\alpha - \lambda\varepsilon^2) \vee 0,$$

and so

$$|\int_\alpha^\beta Z(x;\mu)dx| \leqq |\alpha-\beta|^{9/10}|\int_\alpha^\beta |Z(x;\mu)|^{10}dx|^{1/10}$$
$$\leqq 3c\varepsilon|\alpha-\beta|^{9/10} \leqq 3c\varepsilon((2\lambda\varepsilon^2)\wedge\alpha)^{9/10}.$$

So we have

$$|\int_0^\alpha Z(x;(1-\lambda\varepsilon^2)\mu + \lambda\varepsilon^2\nu)dx$$
$$- \{(1-\lambda\varepsilon^2)\int_0^\alpha Z(x;\mu)dx$$
$$+ \lambda\varepsilon^2 F(\nu; 0 \vee (1-(\lambda\varepsilon^2)^{-1}(1-\alpha)), 1 \wedge ((\lambda\varepsilon^2)^{-1}\alpha))\}|$$
$$\leqq 3c\varepsilon((2\lambda\varepsilon^2)\wedge\alpha)^{9/10},$$

which implies

$$
\begin{aligned}
&|\eta_\alpha((1-\lambda\varepsilon^2)\mu+\lambda\varepsilon^2\nu) \\
&\quad -\{c\varepsilon\eta_\alpha(\mu_0)+\lambda\varepsilon^2\alpha^{-1}F(\nu;0\vee(1-(\lambda\varepsilon^2)^{-1}(1-\alpha),1\wedge(\lambda\varepsilon^2)^{-1}\alpha)\}| \\
&\leqq 3c\varepsilon\alpha^{-1}((2\lambda\varepsilon^2)\wedge\alpha)^{9/10}+c\lambda\varepsilon^3|\eta_\alpha(\mu_0)|.
\end{aligned}
$$

Therefore, if $m\in\mathcal{M}(\eta)$, then

$$
\begin{aligned}
&|\int_0^1 m(d\alpha)\eta_\alpha((1-\lambda\varepsilon^2)\mu+\lambda\varepsilon^2\nu) \\
&\quad -\{c\varepsilon(\int_0^1\eta_\alpha(\mu_0)m(d\alpha)) \\
&\quad +\lambda\varepsilon^2(\int_0^1 m(d\alpha)\alpha^{-1}F(\nu;0\vee(1-(\lambda\varepsilon^2)^{-1}(1-\alpha),1\wedge(\lambda\varepsilon^2)^{-1}\alpha))\}| \\
&\leqq 3c\varepsilon(2\lambda\varepsilon^2)^{2/5}\int_{[0,1/2]}\alpha^{-1/2}m(d\alpha)+6c\varepsilon(2\lambda\varepsilon^2)^{9/10}\int_{(1/2,1]}m(d\alpha) \\
&\quad +c\lambda\varepsilon^3|\int_{[0,1]}\eta_\alpha(\mu_0)m(d\alpha)| \\
&\leqq 3c\varepsilon\Delta_2(\eta)(2\lambda\varepsilon^2)^{2/5}+6c\varepsilon(2\lambda\varepsilon^2)^{9/10}+c\lambda\varepsilon^3\Delta_2(\eta).
\end{aligned}
$$

Here we use that

$$
0\leqq-\int_0^1\eta_\alpha(\mu_0)m(d\alpha)\leqq-\eta(\mu_0)\leqq\Delta_2(\eta).
$$

So we have

$$
\begin{aligned}
&|\eta((1-\lambda\varepsilon^2)\mu+\lambda\varepsilon^2\nu)-\inf\{c\varepsilon(\int_0^1\eta_\alpha(\mu_0)m(d\alpha)) \\
&\quad +\lambda\varepsilon^2(\int_0^1 m(d\alpha)\alpha^{-1}F(\nu;0\vee(1-(\lambda\varepsilon^2)^{-1}(1-\alpha),1\wedge(\lambda\varepsilon^2)^{-1}\alpha)); \\
&\hspace{20em} m\in\mathcal{M}(\eta)\}| \\
&\leqq\Delta_2(\eta)(3c\varepsilon(2\lambda\varepsilon^2)^{2/5}+c\lambda\varepsilon^3)+6c\varepsilon(2\lambda\varepsilon^2)^{9/10}.
\end{aligned}
$$

This and Equation (2) imply our assertion. □

4. Nonlinear PDE

DEFINITION 18. $\mathcal{C}^0$ is defined to be the set of functions $u : [0,\infty) \times \mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0} \to \mathbf{R}$ such that
(1) $u(t,\cdot,y) : \mathbf{R}^d \to \mathbf{R}$ is continuous for any $(t,y) \in [0,\infty) \times \mathbf{Z}^\ell_{\geqq 0}$,
(2) $\| u \|_{0,T} = \sup\{|u(t,x,y)|;\ (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0}\} < \infty$ for any $T > 0$,
and
(3) $\sup\{|u(t,x,y)|;\ t \in [0,T]\} \to 0$, as $|x| + |y| \to \infty$ for any $T > 0$.

DEFINITION 19. $\mathcal{C}^1$ is defined to be the set of functions $u : [0,\infty) \times \mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0} \to \mathbf{R}$ such that
(1) $u \in \mathcal{C}^0$,
(2) $u(t,\cdot,y) : \mathbf{R}^d \to \mathbf{R}$ is C^1 for any $(t,y) \in [0,\infty) \times \mathbf{Z}^\ell_{\geqq 0}$,
and
(3) $\frac{\partial}{\partial x^i} u \in \mathcal{C}^0$, $i = 1, \dots, d$.

We define $\| u \|_{1,T}$, $u \in \mathcal{C}^1$, $T > 0$, by

$$\| u \|_{1,T} = \| u \|_{0,T} + \sum_{i=1}^d \| \frac{\partial}{\partial x^i} u \|_{0,T} .$$

For $u \in \mathcal{C}^1$, let

$$\nabla_x u(t,x,y) = (\frac{\partial u}{\partial x^i}(t,x,y))_{i=1,\dots,d} \in \mathbf{R}^d,$$

and

$$D_y u(t,x,y) = (u(t,x,y+e_i) - u(t,x,y))_{i=1,\dots,\ell} \in \mathbf{R}^\ell,$$

where $e_i = (\overbrace{0,\dots,0}^{i-1},1,0,\dots,0) \in \mathbf{R}^\ell$.

For $f \in C_b(\mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0})$, let $P_t f : \mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0} \to \mathbf{R}$, $t > 0$, be given by

$$(P_t f)(x,y) = \int_{\mathbf{R}^d} (\frac{1}{2\pi t})^{d/2} \exp(-\frac{|x-x'|^2}{2t}) f(x',y) dx',\ x \in \mathbf{R}^d, y \in \mathbf{Z}^\ell_{\geqq 0}.$$

PROPOSITION 20. *For $f \in \mathcal{C}^0$, let*

$$u(t,x,y) = \int_0^t (P_{t-s}f(s,\cdot,*))(x,y)ds, \qquad (t,x,y) \in [0,\infty) \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}.$$

Then $u \in \mathcal{C}^1$ and

$$\| u \|_{1,T} \leqq (d + \sqrt{T}) \int_0^T (T-s)^{-1/2} \| f \|_{0,s} \, ds.$$

PROOF. Suppose that $f \in \mathcal{C}^0 \cap C^{\infty}([0,\infty) \times \mathbf{R}^d \times \mathbf{R}^{\ell})$.
Then we have

$$\begin{aligned}
&\frac{\partial u}{\partial x^i}(t,x,y) \\
&= -\int_0^t ds \int_{\mathbf{R}^d} (\frac{1}{2\pi(t-s)})^{d/2} \, (\frac{x^i - z^i}{t-s}) \, \exp(-\frac{|x-z|^2}{2(t-s)}) f(s,z,y)dz.
\end{aligned}$$

So we have

$$\begin{aligned}
|\frac{\partial u}{\partial x^i}(t,x,y)| &\leqq \int_0^t \frac{1}{\sqrt{t-s}} \| f \|_{0,s} \, ds \\
&= \int_{T-t}^T \frac{1}{\sqrt{T-s}} \| f \|_{0,s-(T-t)} \, ds, \ t \in [0,T].
\end{aligned}$$

Since $\mathcal{C}^0 \cap C^{\infty}([0,\infty) \times \mathbf{R}^d \times \mathbf{R}^{\ell})$ is dense in $\mathcal{C}^0$, we have our assertion. $\square$

PROPOSITION 21. *For $f \in \mathcal{C}^0$ and $g \in \mathcal{C}^1$, let*

$$\begin{aligned}
u(t,x,y) &= (P_t g(0,\cdot,*))(x,y) \\
&\quad + \int_0^t (P_{t-s}f(s,\cdot,*))(x,y)ds, (t,x,y) \in [0,\infty) \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}.
\end{aligned}$$

Then $u \in \mathcal{C}^1$ and

$$\begin{aligned}
&u(T-t, x+B(t), y+N(t)) - u(T,x,y) \\
&= -\int_0^t f(T-s, x+B(s), y+N(s))ds \\
&\quad + \int_0^t \nabla_x u(T-s, x+B(s), y+N(s-))dB(s) \\
&\quad + \sum_{0<s\leqq t} D_y u(T-s, x+B(s), y+N(s-) \cdot (N(s)-N(s-))
\end{aligned}$$

for $(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}$.

PROOF. Suppose that $f \in \mathcal{C}^0 \cap C^\infty([0,\infty) \times \mathbf{R}^d \times \mathbf{R}^\ell)$. Then we see that $u \in \mathcal{C}^1 \cap C^\infty((0,T] \times \mathbf{R}^d \times \mathbf{R}^\ell)$, and

$$-\frac{\partial}{\partial t} u(T-t,x,y) = \frac{1}{2} \triangle_x u(T-t,x,y) + f(T-t,x,y).$$

So we have our assertion by Ito's formula. Since $\mathcal{C}^0 \cap C^\infty([0,\infty) \times \mathbf{R}^d \times \mathbf{R}^\ell)$ is dense in $\mathcal{C}^0$, we have our assertion. □

THEOREM 22. *Let* $b : \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0} \to \mathbf{R}$ *and* $C > 0$, *and assume that*

$$|b(x,y) - b(x',y')| \leqq C(|x-x'| + |y-y'|), \quad x,x' \in \mathbf{R}^d, \quad y,y' \in \mathbf{Z}^{\ell}_{\geqq 0}.$$

Then for any $f \in \mathcal{C}^1$, *there is a unique* $u \in \mathcal{C}^1$ *such that*

$$u(t,x,y) = f(t,x,y) + \int_0^t P_{t-s}(b(\nabla_x u(s,\cdot,*), D_y u(s,\cdot,*)))(x,y)ds,$$

for $(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}$.

PROOF. For any $v \in \mathcal{C}^1$, let

$$\Phi(v)(t,x,y) = f(t,x,y) + \int_0^t P_{t-s}(b(\nabla_x v(s,\cdot,*), D_y v(s,\cdot,*)))(x,y)ds,$$

$(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}$. Then by Proposition 21 we see that $\Phi(v) \in \mathcal{C}^1$ for any $v \in \mathcal{C}^1$, and

$$\begin{aligned} &\| \Phi(v_1) - \Phi(v_2) \|_{1,t} \\ &\leqq C(1+2^\ell)(d+\sqrt{T}) \int_0^t (t-s)^{-1/2} \| v_1 - v_2 \|_{1,s} \, ds, \ t \in [0,T]. \end{aligned}$$

So we have our assertion by usual argument based on Picard's iteration method and Gronwall's inequality. □

5. Proof of Theorem 7

We prove Theorem 7 in this section. Let us think of the situation in Theorem 7. We may assume that $\Delta_2(\eta_n) \leqq 1/2$, $n \geqq 1$.

PROPOSITION 23. *There is a constant $C > 0$ such that*

$$|\eta_n(x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i N_i(2^{-n}) | \mathcal{F}_0) - 2^{-n} b_n(x, y)| \leqq C 2^{-(7/5)n} (1+\lambda)^2 (|x| + |y|)$$

for any $x \in \mathbf{R}^d$, $y \in \mathbf{R}^\ell$ and $n \geqq 1$ with $2^{-n+1}\lambda < 1$.

PROOF. Let $t_{i,n} = -\lambda_i^{-1} \log(1 - \lambda_i 2^{-n})$, $i = 1, \dots, \ell$. Since $-x^2 \leqq \log(1 - x) + x \leqq 0$, $x \in [0, 1/2]$, we see that $2^{-n} \leqq t_{i,n} \leqq 2^{-n} + \lambda_i 2^{-2n} \leqq 2^{-n+1}$, for $n \geqq 1$ with $\lambda 2^{-n} \leqq 1/2$. So we see that

$$\begin{aligned}
&E[|N_i(t_{i,n}) \wedge 1 - N_i(2^{-n})|] \\
&\leqq E[N_i(t_{i,n}), N_i(t_{i,n}) \geqq 2] + E[(N_i(t_{i,n}) - N_i(2^{-n}))] \\
&= \exp(-\lambda_i t_{i,n}) (\sum_{k=2}^{\infty} \frac{(\lambda_i t_{i,n})^k}{(k-1)!}) + \lambda_i (t_{i,n} - 2^{-n}) \\
&\leqq (\lambda_i t_{i,n})^2 + \lambda_i (t_{i,n} - 2^{-n}) \leqq 2\lambda_i^2 2^{-2n}.
\end{aligned}$$

and

$$\begin{aligned}
&E[|N_i(t_{i,n}) \wedge 1 - N_i(2^{-n})|^2]^{1/2} \\
&\leqq E[2N_i(t_{i,n})(N_i(t_{i,n}) - 1), N_i(t_{i,n}) \geqq 2]^{1/2} + E[(N_i(t_{i,n}) - N_i(2^{-n}))^2]^{1/2} \\
&= (2(\lambda_i t_{i,n})^2)^{1/2} + (\lambda_i (t_{i,n} - 2^{-n}) + (\lambda_i (t_{i,n} - 2^{-n}))^2)^{1/2} \\
&\leqq 2\lambda_i t_{i,n} + 2(\lambda_i (t_{i,n} - 2^{-n}))^{1/2} \leqq 6\lambda_i 2^{-n}.
\end{aligned}$$

Therefore we see that

$$\begin{aligned}
&|E[(x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i N_i(2^{-n})) - (x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i (N_i(t_{i,n}) \wedge 1))]| \\
&\leqq 2\lambda^2 2^{-2n} |y|.
\end{aligned}$$

and

$$\begin{aligned} &E[|(x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i N_i(2^{-n})) \\ &\quad - (x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i (N_i(t_{i,n}) \wedge 1))|^2]^{1/2} \\ &\leqq 6\lambda 2^{-n}|y|. \end{aligned}$$

So we have by Proposition 13

$$\begin{aligned} &|\eta_n((x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i N^i(2^{-n/2})|\mathcal{F}_0) \\ &\quad - \eta_n(x \cdot B(2^{-n}) + \sum_{i=1}^{\ell} y_i (N^i(t_{i,n}) \wedge 1)|\mathcal{F}_0)| \\ &\leqq 6\lambda\Delta_2(\eta_n)\lambda 2^{-n}|y| + 2(1+\Delta_2(\eta_n))\lambda^2 2^{-2n}|y|. \end{aligned}$$

Note that

$$P(N_i(t_{i,n}) \wedge 1 = 1) = 1 - P(N_i(t_{i,n}) \wedge 1 = 0) = \lambda_i 2^{-n}.$$

So by Proposition 17 and the definition of b_n, we have our assertion.

This completes the proof. □

Theorem 24. *For any $f \in \mathcal{C}^1$, there is a unique $u \in \mathcal{C}^1$ such that*

$$u(t,x,y) = (P_t f(0,\cdot,*))(x,y) + \int_0^t P_{t-s}(b(\nabla_x u(s,\cdot,*), D_y u(s,\cdot,*)))(x,y)ds,$$

$(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0}$ by virtue of Theorem 22. Then we have

$$|\eta_n(f(0,x+B(T),y+N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) - u(T,x,y)| \to 0, \qquad n \to \infty,$$

for any $T > 0$, $x \in \mathbf{R}^d$ and $y \in \mathbf{R}^{\ell}$.

Proof. Step 1. We first think of the case that there are integers $K, L \geqq 1$ such that $T = K2^{-L}$. Let us recall the assumption (A-1). So $\Delta_2(\eta_n) \leqq C2^{-n/2}$.

By Proposition 21, we see that

$$u(T-t, x+B(t), y+N(t)) - u(T,x,y) \tag{3}$$

$$
\begin{aligned}
= & -\int_0^t b(T-s, \nabla_x u(T-s, x+B(s), y+N(s)), \\
& \qquad D_y u(T-s, x+B(s), y+N(s))ds \\
& + \int_0^t \nabla_x u(T-s, x+B(s), y+N(s-))dB(s) \\
& + \sum_{0<s \leqq t} D_y u(T-s, x+B(s), y+N(s-) \cdot (N(s)-N(s-)).
\end{aligned}
$$

Let

$$
v_{n,k} = \nabla_x u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n}))
$$

and

$$
w_{n,k} = D_y u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n})),
$$

for $n \geqq 1$, $k = 0, 1, 2, \ldots$. Then we see that

$$
\begin{aligned}
& u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n})) \\
& - u(T-(k-1)2^{-n}, x+B((k-1)2^{-n}), y+N((k-1)2^{-n})) \\
= & -2^{-n} b_n(v_{n,k-1}, w_{n,k-1}) + v_{n,k-1}(B(k2^{-n}) - B((k-1)2^{-n})) \\
& + w_{n,k-1}(N(k2^{-n}) - N((k-1)2^{-n}) + R_{n,k},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{n,k} = & -\int_{(k-1)2^{-n}}^{k2^{-n}} (b(\nabla_x u(T-s, x+B(s), y+N(s)), \\
& \qquad D_y u(T-s, x+B(s), y+N(s)) - b_n(v_{n,k-1}, w_{n,k-1}))ds \\
& + \int_{(k-1)2^{-n}}^{k2^{-n}} (\nabla_x u(T-s, x+B(s), y+N(s-)) - v_{n,k-1})dB(s) \\
& + \sum_{(k-1)2^{-n} < s \leqq k2^{-n}} (D_y u(T-s, x+B(s), y+N(s-) - w_{n,k-1}) \\
& \qquad \cdot (N(s) - N(s-))
\end{aligned}
$$

So we see that there are C_n, $n = L, L+1, , \ldots$, such that $C_n \to 0$, as $n \to \infty$, and that

$$
E[|E[R_{n,k}|\mathcal{F}_{(k-1)2^{-n}}]|^2]^{1/2} \leqq C_n 2^{-n},
$$

and

$$E[|R_{n,k}|^2]^{1/2} \leqq C_n 2^{-n/2}, \qquad n \geqq L,\ k = 1, \dots, K2^{n-L}.$$

So we have by Proposition 13

$$\begin{aligned}
&E[|\eta_n(u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n}))|\mathcal{F}_{(k-1)2^{-n}})\\
&\quad - u(T-(k-1)2^{-n}, x+B((k-1)2^{-n}), y+N((k-1)2^{-n}))|^2]^{1/2}\\
&\leqq E[|\eta_n(v_{n,k-1}\cdot(B(k2^{-n})-B((k-1)2^{-n}))\\
&\quad + w_{n,k-1}\cdot(N(k2^{-n})-N((k-1)2^{-n}))|\mathcal{F}_{(k-1)2^{-n}})\\
&\quad - 2^{-n}b_n(v_{n,k-1}, w_{n,k-1})|^2]^{1/2}\\
&\quad + 2\Delta_2(\eta_n)E[|R_{n,k}|^2]^{1/2} + E[|E[R_{n,k}|\mathcal{F}_{(k-1)2^{-n}}]|^2]^{1/2}.
\end{aligned}$$

Therefore by Proposition 23, we see that there are C'_n, $n = L, L+1, \dots$, such that $C'_n \to 0$, as $n \to \infty$, and that

$$\begin{aligned}
&E[|\eta_n(u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n})|\mathcal{F}_{(k-1)2^{-n}})\\
&\quad - u(T-(k-1)2^{-n}, x+B((k-1)2^{-n}), y+N((k-1)2^{-n}))|^2]^{1/2}\\
&\leqq C'_n 2^{-n} \qquad n \geqq L,\ k = 1, \dots, K2^{n-L}.
\end{aligned}$$

So by Proposition 15(5), we have

$$\begin{aligned}
&|\eta_n(u(T-k2^{-n}, x+B(k2^{-n}), y+N(k2^{-n})|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{K2^{n-L}})\\
&\quad - \eta_n(u(T-(k-1)2^{-n}, x+B((k-1)2^{-n}),\\
&\qquad\qquad y+N((k-1)2^{-n}))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{K2^{n-L}})|\\
&\leqq (1+2\Delta_2(\eta_n)^2)^{K2^{n-L}} C'_n 2^{-n}, \qquad n \geqq L.
\end{aligned}$$

This implies that

$$\begin{aligned}
&|\eta_n(f(0, x+B(T), y+N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{K2^(n-L)}) - u(T,x,y)|\\
&\leqq (1+2C^2 2^{-n})^{K2^{n-L}} C'_n T \leqq C'_n \exp(2C^2T)T, \qquad n \geqq 1
\end{aligned}$$

This implies our assertion for $T = K2^{-L}$, $K, L \geqq 1$.

Step 2. For any $T > 0$, let $T_m = 2^{-m}[2^m T]$, $m = 1, 2, \dots$. Then we see by Proposition 15(5) that

$$\begin{aligned}
&|\eta_n(f(0, x+B(T), y+N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}})\\
&\quad - \eta_n(f(0, x+B(T_m), y+N(T_m))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}})|
\end{aligned}$$

$$\leqq (1+2\Delta_2(\eta_n))^{2^{n-m}([2^m T]+1)} E[|f(0,x+B(T),y+N(T)) - f(0,x+B(T_m),y+N(T_m))|^2]^{1/2}.$$

So we see that

$$\begin{aligned}&\limsup_{n\to\infty} |\eta_n(f(0,x+B(T),y+N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) - u(T,x,y)| \\ \leqq\ & \exp(2C^2(T+2^{-m})E[|f(0,x+B(T),y+N(T)) \\ & - f(0,x+B(T_m),y+N(T_m))|^2]^{1/2} \\ & + |u(T,x,y)-u(T_m,x,y)|.\end{aligned}$$

Taking $m\to\infty$, we have our Theorem.

This completes the proof. $\square$

PROPOSITION 25. *For any $T>0$ and $f\in\mathcal{C}^1$,*

$$\begin{aligned}&\lim_{n\to\infty}\eta_n(f(0,x+B(T),y+N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) \\ =\ & \inf\{E[\rho(T)f(0,x+B(T),y+N(T))];\ \rho\in\mathcal{K}\}.\end{aligned}$$

PROOF. Suppose that $\rho(t)$ is a martingale satisfying

$$\rho(t) = 1+\int_0^t \rho(s-)\varphi(s)dB(s) + \sum_{i=1}^{\ell}(\int_0^t \rho(s-)(\psi_i(s)-1)(dN_i(s)-\lambda_i ds).$$

Now let $u\in\mathcal{C}^1$ be as in Theorem 24 for f. Then by Equation (3) and Ito's formula, we have

$$\begin{aligned}&E[\rho(T)f(0,x+B(T),y+N(T))] - u(T,x,y) \\ =\ & E[\int_0^T \rho(t-)(-b(\nabla_x u(T-t,x+B(t),y+N(t-)), \\ &\qquad D_y u(T-t,x+B(t),y+N(t-))) \\ &+\varphi(t)\cdot\nabla_x u(T-t,x+B(t),y+N(t-)) \\ &+\sum_{i=1}^{\ell}\lambda_i\psi_i(t)D_y u(T-t,x+B(t),y+N(t-))_i)dt].\end{aligned}$$

So we see that

$$E[\rho(T)f(0,x+B(T),y+N(T))]-u(T,x,y) \geqq 0,$$

if $(\varphi(t),\psi_1(t),\ldots,\psi_\ell(t)) \in K$, $t \in [0,T]$, almost surely.

Since $b:\mathbf{R}^d \times \mathbf{R}^\ell$ is convex and positive homogeneous, we see that for any $(x,y) \in \mathbf{R}^d \times \mathbf{R}^\ell$ there is a $(z,w) \in K$ such that $b(x,y) = z\cdot x + w\cdot y$. Then by measurable selection theorem, we see that there is a measurable map $k:\mathbf{R}^d \times \mathbf{R}^\ell \to K$ such that $b(x,y) = (x,y)\cdot k(x,y)$ for all $(x,y) \in \mathbf{R}^d \times \mathbf{R}^\ell$. Let

$$\begin{aligned}
&(\varphi(t),\psi_1(t),\ldots,\psi_\ell(t)) \\
&= k(\nabla_x u(T-t,x+B(t),y+N(t-)), D_y u(T-t,x+B(t),y+N(t-))).
\end{aligned}$$

Then $(\varphi(t),\psi_1(t),\ldots,\psi_\ell(t))$ is a predictable process taking values in K such that

$$\begin{aligned}
&\varphi(t)\cdot\nabla_x u(T-t,x+B(t),y+N(t-)) \\
&\quad + \sum_{i=1}^{\ell} \lambda_i \phi_i(t) D_y u(T-t,x+B(t),y+N(t-))_i) \\
&= b(\nabla_x u(T-t,x+B(t),y+N(t-)), D_y u(T-t,x+B(t),y+N(t-))),
\end{aligned}$$

$t \in [0,T]$, almost surely. These imply our assetion. $\square$

Corollary 26. *For any $T > 0$, and any bounded measurable function $g:\mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0} \to \mathbf{R}$ we have*

$$\begin{aligned}
&\lim_{n\to\infty} \eta_n(g(B(T),N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&= \inf\{E[\rho(T)g(B(T),N(T))];\ \rho \in \mathcal{K}\}.
\end{aligned}$$

Proof. We can find a sequence $f_n \in \mathcal{C}^1$, $n = 1,2,\ldots$, such that

$$E[|g(B(T),N(T)) - f_n(0,B(T),N(T))|^2] \to 0, \qquad n \to \infty.$$

Then by virtue of Proposition 15(5) and Assumption (A-1), we have

$$\begin{aligned}
&|\eta_n(g(B(T),N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) - \eta_n(f_n(0,B(T),N(T))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}})| \\
&\leqq (1+C2^{-n})^{(T+1)2^n} E[|g(B(T),N(T)) - f_n(0,B(T),N(T))|^2]^{1/2} \\
&\to 0, \qquad n \to \infty.
\end{aligned}$$

Note that

$$\sup\{E[\rho(T+1)^2];\ \rho \in \mathcal{K}\} < \infty.$$

So we see that

$$\begin{aligned}
&|\inf\{E[\rho(T)g(B(T),N(T))];\ \rho \in \mathcal{K}\}. \\
&\quad - \inf\{E[\rho(T)f_n(0,B(T),N(T))];\ \rho \in \mathcal{K}\}| \\
&\leqq \sup\{E[\rho(T)|g(B(T),N(T)) - f_n(0,B(T),N(T))|];\ \rho \in \mathcal{K}\} \\
&\to 0, \qquad n \to \infty.
\end{aligned}$$

Thus we have our assertion from the previous Proposition. □

Proposition 27. *For any* $m \geqq 1$, $M \geqq 1$, *and bounded measurable functions* $f : (\mathbf{R}^d \times \mathbf{Z}^\ell_{\geqq 0})^M \to \mathbf{R}$ *we have*

$$\begin{aligned}
&\lim_{n\to\infty} \eta_n(f(B(2^{-m}),N(2^{-m}),B(2\cdot 2^{-m}),N(2\cdot 2^{-m}),\dots, \\
&\qquad\qquad B(M2^{-m}),N(M2^{-m}))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&= \inf\{E[\rho(M2^{-m})f(B(2^{-m}),N(2^{-m}),\dots, \\
&\qquad\qquad B(M2^{-m}),N(M2^{-m}))];\ \rho \in \mathcal{K}\}.
\end{aligned}$$

Proof. We prove our assertion by induction in M. Our assertion is valid for $M = 1$ by Corollary 26. Let us assume that our assertion is valid for M. Then again by Corollary 26 we have

$$\begin{aligned}
&\lim_{n\to\infty} \eta_n(f(B(2^{-m}),N(2^{-m}),\dots \\
&\qquad B((M+1)2^{-m}),N((M+1)2^{-m}))|\{\mathcal{F}_{M2^{-m}+k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&= \tilde{f}(B(2^{-m}),N(2^{-m}),\dots B(M2^{-m}),N(M2^{-m})) \qquad a.s.,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}(x_1,y_1,\dots,x_M,y_M) = \inf\{&E[\rho(2^{-m})f(x_1,y_1,\dots,x_M,y_M,x_M \\
&+ B(2^{-m}),y_M + N(2^{-m}))];\ \rho \in \mathcal{K}\}.
\end{aligned}$$

Since $\mathcal{K}$ is multiplicative, we see that

$$\begin{aligned}
&\tilde{f}(B(2^{-m}),N(2^{-m}),\dots,B(M2^{-m}),N(M2^{-m})) \\
&= \inf\{\rho(M2^{-m})^{-1}E[\rho((M+1)2^{-m})f(B(2^{-m}),N(2^{-m}),\dots, \\
&\qquad B((M+1)2^{-m}),N((M+1)2^{-m}))|\mathcal{F}_{M2^{-m}}];\ \rho \in \mathcal{K}\}.
\end{aligned}$$

By Proposition 15(5) and Assumption (A-1), we see that

$$\begin{aligned}
&E[|\eta_n(f(B(2^{-m}), N(2^{-m}), \dots \\
&\qquad B((M+1)2^{-m}), N((M+1)2^{-m}))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&\quad - \eta_n(\tilde{f}(B(2^{-m}), N(2^{-m}), \dots B(M2^{-m}), N(M2^{-m}))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}})|^2] \\
&\leqq (1 + C2^{-n})^{M2^{n-m}} E[|\eta_n(f(B(2^{-m}), \dots, \\
&\qquad\qquad N((M+1)2^{-m}))|\{\mathcal{F}_{M2^{-m}+k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&\quad - \tilde{f}(B(2^{-m}), N(2^{-m}), \dots B(M2^{-m}), N(M2^{-m}))|^2]
\end{aligned}$$

So from the assumption in induction, we have

$$\begin{aligned}
&\lim_{n\to\infty} \eta_n(f(B(2^{-m}), N(2^{-m}), \dots \\
&\qquad B((M+1)2^{-m}), N((M+1)2^{-m}))|\{\mathcal{F}_{k2^{-n}}\}_{k=0}^{2^{2n}}) \\
&= \inf\{E[\rho(M2^{-m})\tilde{f}(B(2^{-m}), N(2^{-m}), \dots B(M2^{-m}), N(M2^{-m}))];\ \rho \in \mathcal{K}\}. \\
&= \inf\{E[\rho((M+1)2^{-m})f(B(2^{-m}), N(2^{-m}), \dots \\
&\qquad B((M+1)2^{-m}), N((M+1)2^{-m}))];\ \rho \in \mathcal{K}\}.
\end{aligned}$$

Thus we have our assertion for $M+1$.

This completes the proof. □

Now let us prove Theorem 7.

Let $K = [T] + 1$. Then there exists $f_m : (\mathbf{R}^d \times \mathbf{Z}^{\ell}_{\geqq 0})^{K2^m} \to \mathbf{R}$, $m = 1, 2 \dots$, such that

$$\begin{aligned}
&E[|X - f_m(B(2^{-m}), N(2^{-m}), \dots, B((K2^m)2^{-m}), N((K2^m)2^{-m}))|^2] \\
&\to 0, \qquad m \to 0.
\end{aligned}$$

Then we have our assertion in a similar proof of Corollary 26 by using Proposition 27. This completes the proof of Theorem 7.

6. Proof of Theorem 8

Let us think of the situation in Subsection 1.2. Let $\rho_h : [0,\infty) \to [0,\infty)$, $h > 0$, be given by

$$\rho_h(t) = hk \text{ for } t \geqq 0 \text{ with } h(k-1) \leqq t < hk,\ k = 1, 2, \dots .$$

Let

$$Y_h(n, m_1, \dots, m_K) = \sum_{k=1}^{K} \sum_{i=1}^{m_k} (Z_i^{(k)} N_i^{(k)}(nh) + p_k(\rho_h(\tau_i^{(k)}) \wedge hn))$$

for any $n, m_1, \dots, m_K \geqq 0$. Then we have

$$|X(nh; m_1, \dots, m_K) - Y_h(n; m_1, \dots, m_K)| \leqq h \sum_{k=1}^{K} m_k p_k.$$

Let

$$\pi_\ell(m, \lambda; h) = \begin{cases} \binom{m}{\ell}(1 - \exp(-\lambda h))^\ell \exp(-\lambda h(m - \ell)), & 0 \leqq \ell \leqq m \\ 0, & \ell > m, \end{cases}$$

and

$$q_\ell(x, \lambda) = \exp(-\lambda x) \frac{(\lambda x)^\ell}{\ell!}.$$

Proposition 28. *Let η is MVM. Let $h > 0$, and $F_h : \mathbf{Z}_{\geqq 0} \times \mathbf{Z}_{\geqq 0}^K$ be a function inductively defined by*

$$\begin{aligned} & F_h(0; m_1, \dots, m_K) = 0, \\ & F_h(j+1; m_1, \dots, m_K) \\ = \eta(& \sum_{\ell_1, \dots \ell_K = 0}^{\infty} \prod_{k=1}^{K} \pi_{\ell_k}(m_k, \lambda_k; h)(\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K} \\ & + F_h(j; m_1 - \ell_1, \dots, m_K - \ell_K)) + h \sum_{k=1}^{K} m_k p_k \end{aligned}$$

for $j \geqq 0$, $m_1, \dots, m_K \geqq 0$. Then we have

$$\eta(Y_h(N; m_1, \dots, m_K) | \{\mathcal{F}_{jh}\}_{j=0}^N) = F_h(N; m_1, \dots, m_K)$$

for $N \geqq 0$, $m_1, \dots, m_K \geqq 0$.

Proof. Let us fix $m_1, \dots, m_K \geqq 0$. Let

$$V_N = Y_h(N; m_1, \dots, m_K)$$

and

$$V_{j-1} = \eta(V_j|\mathcal{F}_{(j-1)h}), \qquad j = N, N-1, \dots, 1.$$

Let

$$I(j;k) = \{i \in \{1, \dots, m_k\};\ N_i^{(k)}(jh) = 0\}$$

and $a(j;k) = \#(I(j;k))$, $k = 1, \dots, K$, $j \geqq 0$. Here $\#(I)$ denotes the cardinal of a set I. Then it is sufficient to prove the following.

Claim. For $j = N, N-1, \dots, 0$,

$$V_j = F_h(N-j; a(j;1), \dots, a(j,K)) + Y_h(j; m_1, \dots, m_K).$$

We show this Claim by induction. The assertion is obvious when $j = N$.

Suppose that the assertion is valid for $j \leqq N$. For any $I'_k \subset I_k \subset \{1, \dots, m_k\}$, and Borel sets $B_i^{(k)}$, $i = 1, \dots, m_k$, $k = 1, \dots, K$, and $A \in \mathcal{F}_{(j-1)h}$, we have

$$\begin{aligned} &P(\{I(j;k) = I'_k,\ Z_i^{(k)} \in B_i^{(k)},\ i \in I_k \setminus I'_k, I(j-1;k) = I_k, \\ &\qquad\qquad k = 1, \dots, K|\} \cap A) \\ &= \prod_{k=1}^{K}((\prod_{i \in I_k \setminus I'_k} \nu_k(B_i^{(k)}))(1 - \exp(-\lambda_k h))^{\#(I_k \setminus I'_k)} \exp(-\lambda_k h \#(I'_k)) \\ &\qquad \times P(\{I(j-1;k) = I_k,\ k = 1, \dots, K\} \cap A) \end{aligned}$$

So we have for any Borel set B and $0 \leqq b'_k \leqq b_k \leqq m_k$, $k = 1, \dots, K$,

$$\begin{aligned} &P(\{Y_h(j; m_1, \dots, m_K) - Y_h(j-1; m_1, \dots, m_K) - h\sum_{k=1}^{K} b_k p_k \in B\} \\ &\cap \{a(j;k) = b'_k, a(j-1;k) = b_k,\ k = 1, \dots, K\} \cap A) \\ &= \nu_1^{*(b_1 - b'_1)} * \cdots * \nu_K^{*(b_K - b'_K)}(B) \\ &\times \prod_{k=1}^{K} \binom{b_k}{b'_k} (1 - \exp(-\lambda_k h))^{b_k - b'_k} \exp(-\lambda_k h b'_k) \\ &\times P(\{a(j-1;k) = b_k,\ k = 1, \dots, K\} \cap A). \end{aligned}$$

Since

$$V_{j-1} = Y_h(j-1; m_1, \dots, m_K), \eta(F_h(N-j; a(j;1), \dots, a(j,K)) \\ + (Y_h(j; m_1, \dots, m_K) - Y_h(j-1; m_1, \dots, m_K)) | \mathcal{F}_{(j-1)h}),$$

we see that the assertion is valid for $j-1$.

This completes the proof. □

PROPOSITION 29. *For any $R > 1$, and $\lambda_1, \dots, \lambda_K > 0$,*

$$\sup\{ \sum_{\ell_1, \dots, \ell_K = 0}^{\infty} R^{\ell_1 + \dots + \ell_K} | \prod_{k=1}^{K} \pi_{\ell_k}(m_k, \lambda_k; h) - \prod_{k=1}^{K} q_{\ell_k}(m_k h, \lambda_k)|; \\ 0 \leqq m_1, \dots, m_K \leqq h^{-1} R\} \to 0$$

as $h \downarrow 0$.

PROOF. Let $G(\cdot; m, \lambda; h) : \mathbf{C} \to \mathbf{C}$, $F(\cdot; x, \lambda) : \mathbf{C} \to \mathbf{C}$ be an entire function given by

$$G(z; m, \lambda; h) = \sum_{\ell=0}^{\infty} \pi_\ell(m, \lambda; h) z^\ell = \exp(-\lambda h m)\{1 + z(\exp(\lambda h) - 1)\}^m,$$

$$F(z; x, \lambda) = \sum_{\ell=0}^{\infty} q_\ell(x, \lambda) z^\ell = \exp(-\lambda x + \lambda x z).$$

Let $\varepsilon_{R,h}$ be given by

$$\varepsilon_{R,h} = \sup\{| \prod_{k=1}^{K} G(z_k; m_k, \lambda_k; h) - \prod_{k=1}^{K} F(z_k; m_k h, \lambda_k)|; \\ z_k \in \mathbf{C}, |z_k| \leqq R^2, 0 \leqq m_k \leqq h^{-1} R^2, k = 1, \dots, K\}$$

Then it is easy to see that $\varepsilon_{R,h} \to 0$, as $h \downarrow 0$, for any $R > 1$. We see by Cauchy's theorem that

$$\prod_{k=1}^{K} \pi_{\ell_k}(m_k, \lambda_k; h) - \prod_{k=1}^{K} q_{\ell_k}(m_k h, \lambda_k) \\ = (\frac{1}{2\pi i})^K \int_{|z_1| = R^2} \cdots \int_{|z_K| = R^2} \\ \times \frac{\prod_{k=1}^{K} G(z_k; m_k, \lambda_k; h) - \prod_{k=1}^{K} F(z_k; m_k h, \lambda_k)}{z_1^{\ell_1 + 1} \cdots z_K^{\ell_K + 1}} dz_1 \cdots dz_K$$

So we have

$$\sum_{\ell_1,\dots,\ell_K=0}^{\infty} R^{\ell_1+\cdots+\ell_K} | \prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h) - \prod_{k=1}^K q_{\ell_k}(m_k h,\lambda_k)|$$
$$\leqq \varepsilon_{2R,h} \sum_{\ell_1,\dots,\ell_K=0}^{\infty} R^{-(\ell_1+\cdots+\ell_K)}.$$

This implies our assertion. □

PROPOSITION 30. *For any MVM η, $p \in [1,\infty)$, and $\nu_1,\nu_2,\dots,\nu_K \in \mathcal{L}_p$, we have*

$$|\eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} \prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h)\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K})$$
$$- \eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} \prod_{k=1}^K q_{\ell_k}(m_k h,\lambda_k)\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K})|$$
$$\leqq 2^K(1+2\Delta_p(\eta)) \sum_{\ell_1,\dots,\ell_K=0}^{\infty} | \prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h) - \prod_{k=1}^K q_{\ell_k}(m_k h,\lambda_k)|$$
$$\{2(1+\sum_{k=1}^K (\int_{\mathbf{R}} |x|^p \nu_k(dx))^{1/p})\}^{p(\ell_1+\cdots+\ell_K)}.$$

PROOF. Let Y, $Z_i^{(k)}$, $k=1,\dots,K$, $i=1,2,\dots$, be independent random variables such that Y is uniformly distributed and the probability law of $Z_i^{(k)}$ is ν_k. We can take disjoint Borel subsets $A_{\ell_1,\dots,\ell_K}$, $\ell_1,\dots,\ell_K \geqq 0$, in $[0,1]$ such that the Lebesgue measure of $A_{\ell_1,\dots,\ell_K}$ is $\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h) \wedge \prod_{k=1}^K q_{\ell_k}(m_k h,\lambda_k)$. Then we can take disjoint Borel subsets $B_{\ell_1,\dots,\ell_K}$, $\ell_1,\dots,\ell_K \geqq 0$, in $[0,1]$ such that $A_{\ell_1,\dots,\ell_K} \subset B_{\ell_1,\dots,m_K}$ and the Lebesgue measure of $B_{\ell_1,\dots,\ell_K}$ is $\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h)$. Also we can take disjoint Borel subsets $C_{\ell_1,\dots,\ell_K}$, $\ell_1,\dots,\ell_K \geqq 0$, in $[0,1]$ such that $A_{\ell_1,\dots,\ell_K} \subset C_{\ell_1,\dots,m_K}$ and the Lebesgue measure of $C_{\ell_1,\dots,\ell_K}$ is $\prod_{k=1}^K q_{\ell_k}(m_k h,\lambda_k)$. Let

$$X_1 = \sum_{\ell_1,\dots,\ell_K=0}^{\infty} 1_{B_{\ell_1,\dots,\ell_K}}(Y)(\sum_{k=1}^K \sum_{i=1}^{\ell_k} Z_i^{(k)}),$$

and

$$X_2 = \sum_{\ell_1,\dots,\ell_K=0}^{\infty} 1_{C_{\ell_1,\dots,\ell_K}}(Y)(\sum_{k=1}^{K}\sum_{i=1}^{\ell_k} Z_i^{(k)}),$$

Note that the probability laws of X_1 and X_2 are $\sum_{\ell_1,\dots,\ell_K=0}^{\infty}\prod_{k=1}^{K}\cdot \pi_{\ell_k}(m_k,\lambda_k;h)\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K}$ and $\sum_{\ell_1,\dots,\ell_K=0}^{\infty}\prod_{k=1}^{K} q_{\ell_k}(m_k h,\lambda_k)\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K}$, respectively. Then we have

$$|\eta(\mu_{X_1}) - \eta(\mu_{X_2})| \leqq 2\Delta_p(\eta)E[|X_1 - X_2|^p]^{1/p} + E[|X_1 - X_2|]$$

$$\leqq (1+2\Delta_p(\eta)) \sum_{m_1,\dots,m_K=1}^{\infty} E[|1_{B_{\ell_1,\dots,\ell_K}}(Y) - 1_{C_{\ell_1,\dots,\ell_K}}(Y)|^p]^{1/p}$$

$$\times (\sum_{k=1}^{K}\sum_{i=1}^{\ell_k} E[|Z_i^{(k)}|^p]^{1/p})$$

$$= (1+2\Delta_p(\eta)) \sum_{\ell_1,\dots,\ell_K=0}^{\infty} 2^{-(\ell_1+\cdot+\ell_K)}2^{\ell_1+\cdot+\ell_K}$$

$$\times |\prod_{k=1}^{K}\pi_{\ell_k}(m_k,\lambda_k;h) - \prod_{k=1}^{K} q_{\ell_k}(m_k h,\lambda_k)|^{1/p}$$

$$\times (1+\sum_{k=1}^{K} E[|Z_1^{(k)}|^p]^{1/p})^{\ell_1+\cdots+\ell_K}$$

$$\leqq 2^K(1+2\Delta_p(\eta)) \sum_{\ell_1,\dots,\ell_K=1}^{\infty} 2^{(p-1)(m_1+\ldots+m_K)}$$

$$\times |\prod_{k=1}^{K}\pi_{\ell}(m_k,\lambda_k;h) - \prod_{k=1}^{K} q_{\ell}(m_k h,\lambda_k)|$$

$$\times (1+\sum_{k=1}^{K} E[|Z_1^{(k)}|^p]^{1/p})^{p(\ell_1+\cdots+\ell_K)}$$

This implies our assertion. □

PROPOSITION 31. *For any $R > 0$, there is a $C > 0$ such that 5*

$$|\Phi(x,\xi) - \Phi(x,\xi')| \leqq C|\xi - \xi'|$$

for any $x, \xi, \xi' \in \mathbf{R}^N$ *with* $|x| \leqq R$ *and* $|\xi - \xi'| \leqq 1$.

PROOF. Since we have

$$\begin{aligned} &|\Phi(x,\xi) - \Phi(x,\xi')| \\ &\leqq (1+2\Delta_p(\eta))\{\sum_{\ell_1,\dots,\ell_K=1}^{\infty} \prod_{k=1}^{K} (\exp(-\lambda_k x_k)\frac{(\lambda_k x_k)^{\ell_k}}{\ell_k!})(\sum_{k=1}^{K} \ell_k |\xi_k - \xi'_k|)^p\}^{1/p} \end{aligned}$$

we have our assertion. □

Suppose that $u : [0,\infty) \times [0,\infty)^K \to \mathbf{R}$ is a C^1 function and satisfies

$$u(0,x) = 0, \qquad x \in [0,\infty)^K$$

and

$$\frac{\partial}{\partial t} u(t,x) = \Phi(x, \nabla_x u(t,x)), \qquad t \geqq 0,\ x \in [0,\infty)^K.$$

PROPOSITION 32. *Let*

$$\begin{aligned} \varepsilon_h(R) = \sup\{&|h^{-1}u(t+h, m_1 h, \dots, m_K h) - h\sum_{k=1}^{K} m_k p_k \\ &- \eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^{K} \pi_{\ell_k}(m_k, \lambda_k; h))(\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K} \\ &\qquad + h^{-1}u(t, m_1 h - \ell_1 h, \dots, m_K h - \ell_K h))|; \\ &\qquad t \in [0,R], m_1, \cdots, m_K \in \mathbf{Z}_{\geqq 0} \\ &\qquad \text{with } m_k h \in [0,R], k = 1, \dots, K\}. \end{aligned}$$

Then $\varepsilon_h(R) \to 0$ *as* $h \downarrow 0$, *for any* $R > 0$.

PROOF. For any $a > 1$ and $\varepsilon \in (0,1)$, let

$$\begin{aligned} \delta(\varepsilon; a) = \sup\{&|\frac{\partial}{\partial t}u(t,x) - \frac{\partial}{\partial t}u(s,y)| + |\nabla_x u(t,x) - \nabla_x u(s,y)|; \\ &t, s \in [0,a], x, y \in [0,a]^K, |t-s| + |x-y| \leqq \varepsilon\}. \end{aligned}$$

Then we see that $\delta(\varepsilon; a) \to 0$, $\varepsilon \to 0$, for any $a > 1$. Since

$$u(t+h, x_1, \dots, x_K) - u(t, x_1, \dots, x_K) = h\int_0^1 \frac{\partial u}{\partial t}(t+sh, x_1, \dots, x_K)ds$$

$$= h\int_0^1 \Phi(x_1, \dots, x_K, \nabla u(t+sh, x_1, \dots, x_K))ds,$$

we have

$$|u(t+h, x) - u(t, x) - h\Phi(x, \nabla u(t, x))| \leqq h\delta(h; R)$$

for any $t, x \in [0, R]$.

Note that

$$\begin{aligned}&\Phi(x, \nabla u(t, x)) - \sum_{k=1}^{K} x_k p_k \\ &= \eta(\sum_{\ell_1, \dots, \ell_K = 0}^{\infty} (\prod_{k=1}^{K} q_{\ell_k}(x_k, \lambda_k))((\nu_1 - \frac{\partial u}{\partial x_1}(t, x))^{*\ell_1} * \cdots \\ &\qquad * ((\nu_K - \frac{\partial u}{\partial x_K}(t, x))^{*\ell_K})).\end{aligned}$$

So we see that by Proposition 29

$$\begin{aligned}C_h = \sup\{&|\Phi(m_1 h, \dots, m_K h, \nabla u(t, m_1 h, \dots, m_K h)) - h\sum_{k=1}^{K} m_k p_k \\ &- \eta(\sum_{\ell_1, \dots, \ell_K = 0}^{\infty} (\prod_{k=1}^{K} \pi_{\ell_k}(m_k, \lambda_k; h)) \\ &\qquad \times ((\nu_1 - \frac{\partial u}{\partial x_1}(t, m_1 h, \dots, m_K h)^{*\ell_1} * \cdots \\ &\qquad * ((\nu_K - \frac{\partial u}{\partial x_K}(t, m_1 h, \dots, m_K h)^{*\ell_K}))|; \\ &\qquad t \in [0, R], m_1, \dots m_K \in \mathbf{Z}_{\geqq 0} \\ &\qquad \text{with } m_k h \in [0, R], k = 1, \dots, K\} \to 0\end{aligned}$$

Note also that

$$u(t, x_1, \dots, x_K) - u(t, x_1 - y_1, \dots, x_K - y_K)$$

$$= -\int_0^1 (\sum_{k=1}^K y_k \frac{\partial u}{\partial x_k}(t, x_1 - sy_1, \dots, x_K - sy_K) ds.$$

So we have

$$\begin{aligned} &|h^{-1}(u(t,x) - u(t, x-hy)) - (\sum_{k=1}^K y_k \frac{\partial u}{\partial x_k}(t,x))| \\ &\leqq \delta(y_1 + \cdots + y_K; R)(\sum_{k=1}^K y_k) \end{aligned}$$

for $t, x_1, \dots, x_K \in [0, R]$, $0 \leqq y_k \leqq x_k, k = 1, \dots, K$.

Therefore we have

$$\begin{aligned} &|\eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k, \lambda_k; h))((\nu_1 - \frac{\partial u}{\partial x_1}(t, m_1 h, \dots, m_K h)^{*\ell_1} * \cdots \\ &\qquad * ((\nu_K - \frac{\partial u}{\partial x_K}(t, m_1 h, \dots, m_K h)^{*\ell_K})) \\ &- \eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k, \lambda_k; h))(\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K} \\ &- h^{-1}(u(t, m_1 h, \dots, m_K h) - u(t, m_1 h - \ell_1 h, \dots, m_K h - \ell_K h)))| \\ &\leqq \Delta_p(\eta) C_{p,h} \end{aligned}$$

for any $t \in [0, R], m_1, \cdots, m_K \in \mathbf{Z}_{\geqq 0}$ with $m_k h \in [0, R], k = 1, \dots, K$. Here

$$\begin{aligned} &C_{p,h} \\ &= \sup\{\{ \sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k, \lambda_k; h))(\delta((\ell_1 + \cdots + \ell_K)h; R)(\sum_{k=1}^K \ell_k))^p\}^{1/p}; \\ &\qquad m_1, \cdots, m_K \in \mathbf{Z}_{\geqq 0} \text{ with } m_k h \leqq R,\ k = 1, \dots, K\} \end{aligned}$$

Note that

$$\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k, \lambda_k; h))(\delta((\ell_1 + \cdots + \ell_K)h; R)(\sum_{k=1}^K \ell_k))^p$$

$$
\begin{aligned}
&\leqq \delta(h^{1/2};R)^p \sum_{\ell_1+\cdots+\ell_K \leqq h^{-1/2}} (\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h))(\sum_{k=1}^K \ell_k))^p \\
&\quad + \delta(KR;R) \sum_{\ell_1+\cdots+\ell_K > h^{-1/2}} (\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h))(\sum_{k=1}^K \ell_k))^p \\
&\leqq \delta(h^{1/2};R)^p \sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h))(\prod_{k=1}^K 2^{\ell_k})^p \\
&\quad + 2^{-h^{-1/2}}\delta(KR;R) \sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h))(\prod_{k=1}^K 2^{\ell_k})^{p+1} \\
&= \delta(h^{1/2};R)^p \prod_{k=1}^K (\exp(-\lambda_k h) + 2^p(1-\exp(-\lambda_k h))^{m_k} \\
&\quad + 2^{-h^{-1/2}}\delta(KR;R) \prod_{k=1}^K (\exp(-\lambda_k h) + 2^{p+1}(1-\exp(-\lambda_k h))^{m_k} \\
&\leqq \delta(h^{1/2};R)^p \prod_{k=1}^K (1+2^p\lambda_k h))^{m_k} + 2^{-h^{-1/2}}\delta(KR;R) \prod_{k=1}^K (1+2^{p+1}\lambda_k h)^{m_k} \\
&\leqq \delta(h^{1/2};R)^p \prod_{k=1}^K \exp(2^p\lambda_k R) + 2^{-h^{-1/2}}\delta(KR;R) \prod_{k=1}^K \exp(2^{p+1}\lambda_k R)
\end{aligned}
$$

So we have $C_{p,h} \to 0$, as $h \downarrow 0$.

Thus we have

$$
\begin{aligned}
&|h^{-1}u(t+h,m_1h,\dots,m_Kh) - h\sum_{k=1}^K m_k p_k \\
&\quad - \eta(\sum_{\ell_1,\dots,\ell_K=0}^{\infty} (\prod_{k=1}^K \pi_{\ell_k}(m_k,\lambda_k;h))(\nu_1^{*\ell_1} * \cdots * \nu_K^{*\ell_K} \\
&\qquad\qquad + h^{-1}u(t,m_1h-\ell_1h,\dots,m_Kh-\ell_Kh))| \\
&\leqq C_h + \Delta_p(\eta)C_{p,h} + \delta(h;R)
\end{aligned}
$$

for any $t \in [0,R], m_1,\cdots,m_K \in \mathbf{Z}_{\geqq 0}$ with $m_kh \in [0,R], k=1,\dots,K$.

This implies our assertion. □

Now let us prove Theorem 8. Let us take $R>0$ and fix it. Let

$$a_n = \sup\{|F_h(n;m_1,\dots,m_K) - h^{-1}u(nh,m_1h,\dots,m_Kh)|;\\ m_1,\dots,m_K\in \mathbf{Z}_{\geqq 0} \text{ with } m_1h,\dots,m_Kh\leqq R\},\qquad n=0,1,\dots.$$

Then we see that $a_0=0$. Also, we have by Proposition 32

$$a_{n+1}\leqq a_n+\varepsilon_h(R),\qquad n=0,\dots,[h^{-1}R].$$

Therefore we see that

$$ha_n\leqq R\varepsilon_h(R),\qquad n=0,\dots,[h^{-1}R],$$

and so we have

$$\sup\{|hF_h([h^{-1}t];[h^{-1}x_1],\dots,[h^{-1}x_K]) - u(t,x_1,\dots,x_K)|;\\ t,x_1,\dots,x_K\in[0,R]\}\to 0 \tag{4}$$

as $h\downarrow 0$ for any $R>0$.

Note that

$$|X(T,m_1,\dots,m_K)-X(h[h^{-1}T],m_1,\dots,m_K)\\ -\sum_{k=1}^K\sum_{i=1}^{m_k}Z^{(k)}(N_i^{(k)}(T)-N_i^{(k)}([h^{-1}T]h))|\leqq h\sum_{k=1}^K m_kp_k.$$

So we have

$$|\eta(X(T,m_1,\dots,m_K)-X(h[h^{-1}T],m_1,\dots,m_K)|\mathcal{F}_{[h^{-1}T]})\\ -\eta(\sum_{k=1}^K\sum_{i=1}^{m_k}Z^{(k)}(N_i^{(k)}(T)-N_i^{(k)}([h^{-1}T]h))|\mathcal{F}_{[h^{-1}T]})|\leqq h\sum_{k=1}^K m_kp_k.$$

Also we have

$$|\eta(\sum_{k=1}^K\sum_{i=1}^{m_k}Z^{(k)}(N_i^{(k)}(T)-N_i^{(k)}([h^{-1}T]h))|\mathcal{F}_{[h^{-1}T]})|\\ \leqq(1+2\Delta_p(\eta))\sum_{k=1}^K\sum_{i=1}^{m_k}E[|Z^{(k)}(N_i^{(k)}(T)-N_i^{(k)}([h^{-1}T]h))|^p|\mathcal{F}_{[h^{-1}T]}]^{1/p}\\ \leqq(1+2\Delta_p(\eta))h\sum_{k=1}^K m_k(\int_{\mathbf{R}}|x|^p\nu_k(dx))^{1/p}.$$

So we see that

$$\| \eta(X(T, m_1, \dots, m_K)|\mathcal{F}_{[h^{-1}T]}) - Y_h([h^{-1}T], m_1, \dots, m_K) \|_{L^\infty} .$$
$$\leqq (1 + 2\Delta_p(\eta))h \sum_{k=1}^{K} m_k(\int_{\mathbf{R}} |x|^p \nu_k(dx))^{1/p} + 2h \sum_{k=1}^{K} m_k p_k.$$

Therefore we have

$$|\eta(X(T, m_1, \dots, m_K)|\{\mathcal{F}_{nh}\}_{n=0}^{[h^{-1}T]+1}) - F_h([h^{-1}T], m_1, \dots, m_K)|$$
$$\leqq (1 + 2\Delta_p(\eta))h \sum_{k=1}^{K} m_k(\int_{\mathbf{R}} |x|^p \nu_k(dx))^{1/p} + 2h \sum_{k=1}^{K} m_k p_k.$$

This and Equation (4) imply our Theorem.

References

[1] Artzner, P., F. Delbaen, J. M. Eber, and D. Heath, Coherent Measures of Risk, Math. Finance **9** (1999), 203–228.

[2] Artzner, P., F. Delbaen, J. M. Eber, and D. Heath, Multiperiod Risk and Coherent Multiperiod Risk Measurement, Preprint 2003.

[3] Artzner, P., F. Delbaen, J. M. Eber, D. Heath, and H. Ku, Coherent Muliperiod Risk Measurement, Preprint 2003.

[4] Artzner, P., F. Delbaen, J. M. Eber, D. Heath, and H. Ku, Coherent Multiperiod Risk Adjusted Values and Bellman's Priciple, Preprint 2003.

[5] Cheridito,P., F. Delbaen, and M. Kupper, Coherent and Convex Risk Measure for Bounded Cádlág processes, Stoch. Proc. its Appl. **112** (2004), 1–22.

[6] Delbaen, F., Coherent Risk Measures on General Probability Spaces, Advance in finance and stochastics, pp. 1–37, Springer, Berlin, 2002.

[7] Gerber, H., An Introduction to Mathematical Risk Theory, *Huebner Foundation Monogragh Series* No. 8, 1980.

[8] Inoue, A., On the worst conditional expectation, J. Math. Anal. Appl. **286** (2003), 83–95.

[9] Kusuoka, S., Law Invariant Coherent Risk Measures, Adv. Math. Econ. **3** (2001), 83–95.

[10] Williams, D., *Probability with Martingales*, Cambridge University Press 1991, Cambridge.

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