

## *On the Étale Cohomology of Algebraic Varieties with Totally Degenerate Reduction over $p$ -adic Fields*

*to J. Tate*

By Wayne RASKIND and Xavier XARLES

**Abstract.** Let  $K$  be a field of characteristic zero that is complete with respect to a discrete valuation, with perfect residue field of characteristic  $p > 0$ . We formulate the notion of *totally degenerate reduction* for a smooth projective variety  $X$  over  $K$ . We show that for all prime numbers  $\ell$ , the  $\mathbb{Q}_\ell$ -étale cohomology of such a variety is (after passing to a suitable finite unramified extension of  $K$ ) a successive extension of direct sums of Galois modules of the form  $\mathbb{Q}_\ell(r)$ . More precisely, this cohomology has an increasing filtration whose  $r$ -th graded quotient is of the form  $V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(r)$ , where  $V$  is a finite dimensional  $\mathbb{Q}$ -vector space that is independent of  $\ell$ , with an unramified action of the absolute Galois group of  $K$ .

### Introduction

Let  $K$  be a field of characteristic zero that is complete with respect to a discrete valuation, with perfect residue field of characteristic  $p > 0$ , and let  $A$  be an abelian variety of dimension  $d$  over  $K$ . Let  $\overline{K}$  be an algebraic closure of  $K$ ,  $\ell$  be a prime number and  $A[\ell^n]$  be the group of points of  $A$  of order  $\ell^n$  over  $\overline{K}$ . We set

$$T_\ell(A) = \varinjlim_n A[\ell^n].$$

Assume that  $A$  has totally split multiplicative reduction, or equivalently, that the connected component of the identity of the special fibre of the Néron model of  $A$  over the ring of integers of  $K$  is a split torus. Then the

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theory of  $p$ -adic uniformization of Tate (unpublished), Mumford [Mu2] and Raynaud [Ray1] implies that  $A$  can be realized as the rigid analytic quotient of a multiplicative torus by a lattice  $\Lambda$ . This implies easily that for *every*  $\ell$ , there is an extension of Galois modules:

$$0 \rightarrow \Lambda' \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}(1) \rightarrow T_{\ell}(A) \rightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell} \rightarrow 0,$$

where  $\Lambda'$  is a lattice that is canonically isogenous to  $\Lambda$ . In this paper, we generalize these results to describe in such terms the étale cohomology of a class of varieties having a type of reduction that we call *totally degenerate*. We employ purely algebro-geometric methods, mainly usage of the comparison theorem between  $p$ -adic étale cohomology and log-crystalline cohomology (the semi-stable conjecture  $C_{st}$  proven by Tsuji [Tsu]), to prove our results. While many examples of these varieties have uniformizations, one does not expect that all such do, and we feel that our methods will complement uniformization methods even for the case of abelian varieties as above.

This is the first in a series of papers where we study closely the cohomology of varieties with totally degenerate reduction. In [RX], we apply the results of this paper to define and study “ $p$ -adic intermediate Jacobians.” In [R2], we formulate a conjecture of Hodge-Tate type for such varieties, which would describe the coniveau filtration on  $p$ -adic cohomology in terms of the kernels of “enriched monodromy operators.” In [R1], we prove a form of this conjecture for divisors. The second author and Infante are also studying complex analogues of some of the results in this paper.

To describe more precisely the contents of this paper, let  $X$  be a smooth, projective, geometrically connected variety over  $K$ . Let  $\overline{X}$  denote base extension of  $X$  to  $\overline{K}$ . We begin in §2 by formulating the notion of totally degenerate reduction in our mixed characteristic situation. While this term is used often, we have not seen a precise general definition, and formulating such is one important part of this paper. We view it as a set of conditions that are to be satisfied by the components and their various intersections of the special fibre of a suitable regular proper model of  $X$  over the valuation ring  $R$  of  $K$  with strictly semi-stable reduction (we assume that such a model exists). Roughly speaking, the conditions say that these intersections are very simple cohomologically. Our main result (see Corollary 1 of §4 and Theorem 3 of §6) says that for *all*  $\ell$ , the étale cohomology groups

$H^*(\overline{X}, \mathbb{Q}_\ell)$  have an increasing filtration whose  $r$ -th graded quotient is of the form  $V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(r)$ , where  $V$  is a finite dimensional  $\mathbb{Q}$ -vector space that is independent of  $\ell$ , with an unramified action of the absolute Galois group of  $K$ . These  $\mathbb{Q}$ -vector spaces are given in a natural way by the cohomology of the *Chow complex* that is formed from the Chow groups of the components of the special fibre and their intersections (see §3 for the definition of this complex).

For  $\ell \neq p$ , our main result is essentially known, as it follows without too much difficulty from the work of Rapoport-Zink [RZ1], although the result is not explicitly stated there (see §4 for more details). It is the  $p$ -adic cohomology part that requires a careful analysis of several different filtrations and their eventual coincidence. To do this, we discuss in §5 the monodromy filtration defined by Mokrane [Mok] on the log-crystalline cohomology of the special fibre of a regular proper model of  $X$  over  $R$  with semi-stable reduction, as defined by Hyodo-Kato [HK]. Then, in §6, we use work of Hyodo [H] and Tsuji [Tsu] to “lift” this filtration to a monodromy filtration on the  $p$ -adic étale cohomology of  $\overline{X}$ . We note that there is no “simple” monodromy filtration on  $p$ -adic cohomology, in general, as was first pointed out by Jannsen [Ja1], and we can only expect to get such a filtration on  $\ell$ -adic cohomology for all  $\ell$  for some classes of varieties like those considered here. We feel that the existence of such a filtration on  $p$ -adic cohomology is a very important and useful result, which is the foundation for all of our work in this direction.

Examples of varieties to which the methods of this paper may be applied include abelian varieties with totally multiplicative reduction and products of Mumford curves [Mu1] or other  $p$ -adically uniformizable varieties, such as Drinfeld modular varieties [Mus] and some unitary Shimura varieties (see e.g. [La], [Z], [RZ2] and [Va]). Unfortunately, these assumptions only include rather simple varieties  $X$  with good reduction, as in that case the étale cohomology groups  $H^*(\overline{X}, \mathbb{Q}_\ell)$  are pure Galois modules for  $\ell \neq p$  by Deligne’s theorem (Riemann hypothesis [De2]). Also, there are some inequalities for the Hodge numbers that must be satisfied for a variety with totally degenerate reduction and this excludes certain types of varieties (see Remark 6(ii) below for more details). As pointed out to us by Fontaine, there should be many more examples if one considers *motives* with totally degenerate reduction.

In [De4], Deligne considers “very degenerating” variations of Hodge structures that are similar to the cohomology of the varieties studied in this paper. He establishes an equivalence of categories between such variations  $(V_{\mathbb{Z}}, W, F)$  and pairs consisting of a mixed Tate nilpotent orbit  $(V_{\mathbb{Z}}, W, F_{nilp})$  and an element  $E \in \text{End}(Gr^W(V_{\mathbb{Z}}))_{-2} \otimes \mathcal{O}_{mer}^*(D^{*m})$ , where  $D^{*m}$  is the  $m$ -fold product of the punctured disc and  $\mathcal{O}_{mer}^*(D^{*m})$  is the group of invertible holomorphic functions on  $D^{*m}$  that are meromorphic along the  $s_j = 0$  (see loc. cit., Theorem 11). It is not completely clear to us at the moment how to translate this into the mixed characteristic situation that we consider here.

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## 1. Notation and Preliminaries

Let  $K$  be a field of characteristic zero and complete with respect to a discrete valuation, with valuation ring  $R$  and perfect residue field  $F$  of characteristic  $p > 0$ . We denote by  $\overline{K}$  an algebraic closure of  $K$ , by  $\overline{F}$  the residue field of  $\overline{K}$ , which is an algebraic closure of  $F$ , by  $W(F)$  the ring of Witt vectors of  $F$ , by  $W = W(\overline{F})$  the ring of Witt vectors of  $\overline{F}$ , and by  $W_n = W_n(\overline{F})$  the ring of Witt vectors of length  $n$ . Denote by  $K_0$  the fraction field of  $W(F)$  and by  $L$  the fraction field of  $W$ . The absolute Galois group  $\text{Gal}(\overline{K}/K)$  will be denoted by  $G_K$ . Note that we have a natural epimorphism  $G_K \rightarrow \text{Gal}(\overline{F}/F)$ , so any  $\text{Gal}(\overline{F}/F)$ -module is naturally a  $G$ -module.

For  $X$  a smooth projective variety over  $K$  we denote by  $\overline{X}$  the variety over  $\overline{K}$  given by  $X \times_K \overline{K}$ , and in the same way for a scheme  $Z$  over  $F$  we denote by  $\overline{Z}$  the scheme  $Z \times_F \overline{F}$ . For  $\ell$  a prime number and  $r$  a nonnegative integer, we denote by  $\mathbb{Q}_{\ell}(r)$  the Galois module  $\mathbb{Q}_{\ell}$ , twisted  $r$  times by the cyclotomic character. If  $r$  is negative, then  $\mathbb{Q}_{\ell}(r) = \text{Hom}(\mathbb{Q}_{\ell}(-r), \mathbb{Q}_{\ell})$ .

We denote by  $B_{DR}$  the ring of  $p$ -adic periods of Fontaine. Fix a uniformizer  $\pi$  of  $K$  and the extension of the  $p$ -adic logarithm to  $\overline{K}^*$  with  $\log(\pi) = 0$ . These choices determine an embedding of  $B_{st}$ , the ring of periods of semi-stable varieties, in  $B_{DR}$ . See ([I1], §1.2) for definitions of these rings and more details.

Let  $S$  be any domain with field of fractions  $\text{frac}(S)$  of characteristic zero. If  $M$  is an  $S$ -module, let  $M_{tors}$  be the torsion subgroup of  $M$ . We denote by  $M/tors$  the torsion free quotient of  $M$ ; that is,  $M/M_{tors}$ . We say that a map between  $S$ -modules is an isomorphism modulo torsion if it induces an isomorphism between the torsion free quotients. If  $M$  and  $M'$  are torsion free  $S$ -modules, we say that a map  $\phi: M \rightarrow M'$  is an isogeny if it is injective with cokernel of finite exponent (as abelian group). In this case, there exists a unique map  $\psi: M' \rightarrow M$ , the dual isogeny, such that  $\phi \circ \psi = [e]$  and  $\psi \circ \phi = [e]$ , where  $e$  is the exponent of the cokernel of  $\phi$  and  $[e]$  denotes the map multiplication by  $e$ . In general, if  $M$  and  $M'$  are  $S$ -modules, we say that a morphism  $\psi: M' \rightarrow M$  is an isogeny if the induced map on the torsion free quotients is an isogeny.

Observe that if  $M$  and  $M'$  are torsion free  $S$ -modules together with an action of a group  $G$  (respectively with a fixed endomorphism or with a fixed semilinear endomorphism) and  $\phi: M \rightarrow M'$  is an isogeny that preserves the action of  $G$  (respectively the fixed endomorphism or the fixed semilinear endomorphism), then the dual isogeny also preserves it. This is due to the fact that an isomorphism of  $\text{frac}(S)$ -vector spaces that preserves this additional structure has inverse that also preserves this additional structure, together with the fact that the map  $[e]$  preserves this type of additional structures.

We will use the following basic construction in various places in the paper. Let  $M$ ,  $M'$  and  $M''$  be torsion free  $S$ -modules, and suppose we have isogenies  $\psi: M' \rightarrow M$  and  $\psi': M'' \rightarrow M$ . Then we have a canonical isogeny  $\varphi: M' \rightarrow M''$  which is defined to be the composition of  $\psi$  and the dual isogeny  $\phi'$  of  $\psi'$ . Observe that the isomorphism  $\varphi \otimes \text{frac}(S): M' \otimes \text{frac}(S) \rightarrow M'' \otimes \text{frac}(S)$  does not in general coincide with the composition of  $\psi \otimes \text{frac}(S)$  with the inverse of the isomorphism  $\psi' \otimes \text{frac}(S)$ . If  $e$  denotes the exponent of the cokernel of  $\psi'$ , then the former coincides with the latter multiplied by  $e$ .

We will use subindices for increasing filtrations and superindices for de-

creasing filtrations. If  $M_\bullet$  is an increasing filtration on an abelian group  $H$  (respectively  $F^\bullet$  a decreasing filtration on  $H$ ), we will denote by  $Gr_i^M(H)$  the quotient  $M_i(H)/M_{i-1}(H)$  (respectively,  $Gr_F^i(H)$  the quotient  $F^i(H)/F^{i+1}(H)$ ). Observe that, if  $M_\bullet$  is an increasing filtration, then  $F^i(H) := M_{-i}(H)$  is a decreasing filtration.

Let  $X$  be a smooth projective geometrically connected variety over  $K$  of dimension  $d$ . We assume that  $X$  has a regular proper model  $\mathcal{X}$  over  $R$  which is strictly semi-stable, which means that the following conditions hold:

(\*) Let  $Y$  be the special fibre of  $\mathcal{X}$ . Then  $Y$  is reduced; write

$$Y = \bigcup_{i=1}^n Y_i,$$

with each  $Y_i$  irreducible. For each nonempty subset  $I = \{i_1, \dots, i_k\}$  of  $\{1, \dots, n\}$ , we set

$$Y_I = Y_{i_1} \cap \dots \cap Y_{i_k},$$

scheme theoretically. Then  $Y_I$  is smooth over  $F$  of pure codimension  $|I|$  in  $\mathcal{X}$  if it is nonempty. See ([DeJ] 2.16 and [Ku1], §1.9, 1.10) for a clear summary of these conditions, as well as comparison with other notions of semi-stability.

## 2. Totally Degenerate Varieties and Totally Degenerate Reduction

DEFINITION 1. With notation as above, we say that  $Y$  is *totally degenerate over  $F$*  if there exists an embedding of  $Y$  into a projective space such that the following conditions are satisfied for each  $Y_I$  (where  $d_I = \dim(Y_I)$ ):

- a) For every  $i = 0, \dots, d_I$ , the Chow groups  $CH^i(\overline{Y}_I)$  are finitely generated abelian groups.

The groups  $CH_{\mathbb{Q}}^i(\overline{Y}_I)$  satisfy the Hodge index theorem: let

$$\xi : CH^i(\overline{Y}_I) \rightarrow CH^{i+1}(\overline{Y}_I)$$

be the map given by intersecting with the class of a hyperplane section. For every  $i \leq \frac{d_I}{2}$  and  $x \in CH_{\mathbb{Q}}^i(\overline{Y}_I)$  such that  $\xi^{d_I-2i+1}(x) = 0$ , we have

that  $(-1)^i \deg(x\xi^{d_I-2i}(x)) \geq 0$ , and equality holds if and only if  $x = 0$ . Here  $\deg$  denotes the degree map

$$\deg : CH_{\mathbb{Q}}^{d_I}(\overline{Y}_I) \rightarrow \mathbb{Q}$$

(note that  $\overline{Y}_I$  need not be connected, but since it is smooth, the components do not intersect, and the degree map is that on each connected component).

- b) For every prime number  $\ell$  different from  $p$ , the étale cohomology groups  $H^{2i+1}(\overline{Y}_I, \mathbb{Z}_{\ell})$  are torsion, and the cycle map induces an isomorphism

$$CH^i(\overline{Y}_I)/tors \otimes \mathbb{Z}_{\ell} \cong H^{2i}(\overline{Y}_I, \mathbb{Z}_{\ell}(i))/tors.$$

Note that this is compatible with the action of the absolute Galois group  $Gal(\overline{F}/F)$ .

- c) Denote by  $H_{\text{crys}}^*(\overline{Y}_I/W)$  the crystalline cohomology groups of  $\overline{Y}_I$ . Then  $H_{\text{crys}}^{2i+1}(\overline{Y}_I/W)$  are torsion, and

$$CH^i(\overline{Y}_I) \otimes W(-i)/tors \cong H_{\text{crys}}^{2i}(\overline{Y}_I/W)/tors$$

via the cycle map. Here  $W(-i)$  is  $W$  with the action of Frobenius multiplied by  $p^i$ .

- d)  $Y$  is ordinary, in that  $H^r(\overline{Y}, B\omega^s) = 0$  for all  $r$  and  $s$ . Here  $B\omega$  is the subcomplex of exact forms in the logarithmic de Rham complex on  $\overline{Y}$  (see e.g. [I3], Définition 1.4). By [H] and ([I2], Proposition 1.10), this is implied by the  $\overline{Y}_I$  being ordinary in the usual sense, in that  $H^r(\overline{Y}_I, d\Omega^s) = 0$  for all  $I, r$  and  $s$ . For more on the condition of ordinary, see ([I2], Appendice and [BK], Proposition 7.3)

If the natural maps

$$CH^i(Y_I) \rightarrow CH^i(\overline{Y}_I)$$

are isomorphisms modulo torsion for all  $I$ , we shall say that  $Y$  is *split totally degenerate*. Since the  $CH^i(\overline{Y}_I)$  are all finitely generated abelian groups and there are a finite number of them, there is a finite extension of  $F$  where all the cycle classes in  $CH^i(\overline{Y}_I)$  modulo torsion are defined. So, after a finite extension, any totally degenerate variety becomes split totally degenerate.

REMARK 1.

- (i) For totally degenerate  $Y$ , the Chow groups  $CH^i(\overline{Y}_I)_{\mathbb{Q}} := CH^i(\overline{Y}_I) \otimes \mathbb{Q}$  satisfy the hard Lefschetz Theorem. That is, if  $L$  is the class of a hyperplane section in  $CH_{\mathbb{Q}}^1(\overline{Y}_I)$  and

$$\xi : CH_{\mathbb{Q}}^i(\overline{Y}_I) \rightarrow CH_{\mathbb{Q}}^{i+1}(\overline{Y}_I)$$

denotes the Lefschetz operator associated to  $L$ , then

$$\xi^{d_I-2i} : CH_{\mathbb{Q}}^i(\overline{Y}_I) \rightarrow CH_{\mathbb{Q}}^{d_I-i}(\overline{Y}_I)$$

is an isomorphism for all  $i \leq \frac{d_I}{2}$ . This follows from the Hard Lefschetz Theorem in  $\ell$ -adic étale cohomology, as proved by Deligne [De3] and the bijectivity of the cycle map modulo torsion.

- (ii) Note that condition c) implies that for  $Y_I$ , we have an isomorphism as  $K_0$ -vector spaces

$$(CH^i(\overline{Y}_I) \otimes_{\mathbb{Z}} L)^{Gal(\overline{F}/F)} \rightarrow H^{2i}(Y_I/W(F)) \otimes_{W(F)} K_0(i)$$

where  $K_0$  and  $L$  are the fraction fields of  $W(F)$  and  $W$  respectively, and we are considering the diagonal action by  $Gal(\overline{F}/F)$  on  $CH^i(\overline{Y}_I) \otimes_{\mathbb{Z}} L$ .

Using the well-known equivalence of categories between the category of  $p$ -adic representations  $V$  of  $Gal(\overline{F}/F)$  and the category of finite dimensional  $K_0$ -vector spaces  $D$  with a semilinear endomorphism  $\varphi$  whose slopes are all zero, the above isomorphism implies that the  $p$ -adic representation  $CH^i(\overline{Y}_I) \otimes_{\mathbb{Z}} \mathbb{Q}_p$  of  $Gal(\overline{F}/F)$  corresponds to the  $\varphi$ -module  $H^{2i}(Y_I/W(F)) \otimes_{W(F)} K_0(i)$  via this equivalence of categories.

We will say that  $X$  has *totally degenerate reduction* if it has a regular proper model  $\mathcal{X}$  over  $R$  which is strictly semi-stable and whose special fibre  $Y$  is totally degenerate over  $F$ . We find the name *totally degenerate reduction* a bit pejorative, because one of the main themes of this paper is that “bad reduction is good.” However, since this terminology is well-established, at least for dimension one, we decided to continue with it.



REMARK 2. The condition that the cycle map be an isomorphism modulo torsion can be weakened to say that this map is an isomorphism when tensored with  $\mathbb{Q}_\ell$  for all  $\ell$ , and the same with the crystalline cycle map. The results of this paper would be weakened to only give  $\mathbb{Z}$ -structures to the graded quotients up to isogeny, which are in fact isomorphisms for almost all  $\ell$  using the following lemma.

LEMMA 1. *Let  $Z$  be a smooth, projective, irreducible variety of dimension  $d$  over a separably closed field. Assume that the Chow groups are finitely generated abelian groups and that the cycle map is an isomorphism when tensored with  $\mathbb{Q}_\ell$  for all  $\ell$  different from the characteristic of the field. Then for almost all  $\ell$ , the integral cycle map:*

$$c_\ell : CH^i(Z) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow H^{2i}(Z, \mathbb{Z}_\ell(i))$$

*is an isomorphism.*

PROOF. From the conditions, we know that the kernel and cokernel of the cycle map are finite groups. Since the Chow groups are assumed to be finitely generated abelian groups, the  $\ell$ -torsion is zero for almost all  $\ell$ , and hence the map is injective for almost all  $\ell$ . As for the cokernel, by a theorem of Gabber [G], the étale cohomology groups  $H^j(Z, \mathbb{Z}_\ell)$  are torsion free for almost all  $\ell$ . Consider the following commutative diagram of pairings of finitely generated  $\mathbb{Z}_\ell$ -modules:

$$\begin{array}{ccccc} CH^i(Z) \otimes \mathbb{Z}_\ell & \times & CH^{d-i}(Z) \otimes \mathbb{Z}_\ell & \rightarrow & \mathbb{Z}_\ell \\ \downarrow & & \downarrow & & \downarrow \\ H^{2i}(Z, \mathbb{Z}_\ell(i)) & \times & H^{2d-2i}(Z, \mathbb{Z}_\ell(d-i)) & \rightarrow & \mathbb{Z}_\ell. \end{array}$$

Here the top row is the intersection pairing, tensored with  $\mathbb{Z}_\ell$ , and the bottom pairing is cup-product on cohomology. The vertical maps are the cycle maps. By our assumptions on  $Z$  and Poincaré duality, the intersection pairing on the (integral) Chow groups is perfect when tensored with  $\mathbb{Q}$ , and so its determinant is a non-zero integer, say  $m$ . Let  $S$  be the finite set of prime numbers consisting of those  $\ell$  such that  $H^{2i}(Z, \mathbb{Z}_\ell)$  or  $H^{2d-2i}(Z, \mathbb{Z}_\ell)$  has torsion,  $\ell$  divides  $m$ , or  $CH^i(Z)$  or  $CH^{d-i}(Z)$  has  $\ell$ -torsion. Then for  $\ell \notin S$ , the top pairing of the diagram is perfect. By Poincaré duality, the

second pairing is perfect for all  $\ell \notin S$ . The commutativity of the diagram above implies that the diagram:

$$\begin{array}{ccc} CH^i(Z) \otimes \mathbb{Z}_\ell & \rightarrow & \text{Hom}(CH^{d-i}(Z) \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell) \\ \downarrow & & \uparrow \\ H^{2i}(Z, \mathbb{Z}_\ell(i)) & \rightarrow & \text{Hom}(H^{2d-2i}(Z, \mathbb{Z}_\ell(d-i)), \mathbb{Z}_\ell) \end{array}$$

is commutative. Hence for  $\ell \notin S$ , the horizontal maps are isomorphisms, and so the right vertical map is surjective (note that this does not immediately follow from the injectivity of the cycle map since the functor  $\text{Hom}(-, \mathbb{Z}_\ell)$  is not exact). But the right vertical map is also injective since the cokernel of the cycle map is torsion. Thus the left vertical map must be surjective and all maps in the diagram are isomorphisms. This completes the proof of the lemma.  $\square$

*Example 1.*

- i) Assume that  $\bar{Y}$  is projective and that  $\bar{Y}_T$  are smooth projective toric varieties. Then  $\bar{Y}$  is totally degenerate. Conditions a), b) and c) follow from ([FMSS], Corollary to Theorem 2) and ([Ku1], Proof of Theorem 6.13); see also ([D], Theorems 10.8 and 12.11), or ([F], Section 5.2, Theorem on p. 102 and the argument on p. 103). Note that the étalé cohomology groups  $H^i(\bar{Z}, \mathbb{Z}/\ell\mathbb{Z})$  for  $i$  odd of a toric variety  $Z$  are trivial (for example, by using the same argument as in the proof of Theorem 2.1 in [ES]), so the integral cohomology  $H^i(\bar{Z}, \mathbb{Z}_\ell)$  groups with  $i$  even are torsion free (using Theorem 2 of [FMSS]). To show that a smooth toric variety is ordinary, one can use the fact that any smooth proper variety  $Z$  over  $F$  which admits a lifting to a smooth proper scheme over  $W_2(F)$  together with the Frobenius is ordinary. But toric varieties admit such a lifting even to  $W(F)$ , as is easily seen.
- ii) Let  $A$  be an abelian variety over  $K$ . Let  $\mathcal{A}$  be the Néron model of  $A$  over the ring of integers of  $K$ . Assume that the connected component of identity of the special fibre of  $\mathcal{A}$  over  $F$  is a split torus. We will say that  $A$  has *completely split toric reduction*. By a theorem of Künnemann ([Ku1], Theorem 4.6(iii)), after passing to a finite extension of  $K$ , if necessary,  $A$  has a regular projective semi-stable model  $\tilde{\mathcal{A}}$  whose special fibre is a reduced divisor with smooth components,

which implies that it has strictly normal crossings. In this case, the components of the special fibre are smooth projective toric varieties. The intersection of any number of components is also a smooth projective toric variety. Thus  $A$  has totally degenerate reduction.

- iii) Let  $\widehat{\Omega}_K^d$  be the Drinfeld upper half space obtained by removing the  $K$ -rational hyperplanes from  $\mathbb{P}_K^d$ . This is a rigid analytic space, and its quotient by a cocompact torsion free discrete subgroup  $\Gamma \subset PGL_{d+1}(K)$  has the structure of a smooth projective variety (see [It] for a summary of the results in this direction, and [Mus], Theorem 4.1 and [Mu1], Theorem 2.2.5 for the original proofs). Ito proves the Hodge index theorem for the groups  $CH^*(Y_I)_{\mathbb{Q}}$  (see [It], Conjecture 2.6 and Proposition 3.4). He also proves some of the results in sections §§3-5 of this paper. Note that in this example, the components of the special fibre and their intersections are successive blow-ups of projective spaces along closed linear subschemes, and these are ordinary (see [I2], proposition 1.6). Therefore, the results of this paper apply to these Drinfeld modular varieties.
- iv) There are Shimura varieties that can be  $p$ -adically uniformized by analytic spaces other than Drinfeld upper half planes (see e.g. [Rap], [RZ2] and [Va]). As pointed out in [La] and the introduction to [RZ2], the bad reduction of a Shimura variety is only rarely totally degenerate. In any case, when they can be uniformized, Shimura varieties should have totally degenerate reduction. Unfortunately, it seems difficult to construct the good models we need for this paper (but see the next remark).

REMARK 3. Using the results of de Jong [DeJ], it should be possible to prove that for any variety  $X$  which “looks like” it should have totally degenerate reduction, there is an alteration  $Y \rightarrow X$  such that  $Y$  has strict semi-stable reduction and a motive for numerical equivalence on  $Y$  whose cohomology is isomorphic to that of  $X$ . For example, in the theory of uniformization of Shimura varieties (see e.g. [RZ2], [Va]), one constructs models of Shimura varieties over  $p$ -adic integer rings which are often not even regular, and it is not known at this time how to provide models such as that required in this paper. But the special fibres of such models are

often given by possibly singular toric varieties, and hence the generic fibre should have totally degenerate reduction.

**3. Definition of the Chow Complexes  $C_j^i(\overline{Y})$**

In this section we define *Chow complexes*  $C_j^i(\overline{Y})$  for each  $j = 0, \dots, d := \dim(Y)$ . These will give us the  $\mathbb{Z}$ -structures we require later. The definition is inspired by the work of Deligne [De1], Steenbrink [St] and Guillén/Navarro-Aznar [GN] on mixed Hodge theory, Rapoport and Zink on the monodromy filtration [RZ1], Bloch, Gillet and Soulé [BGS] and Consani [Con] on the monodromy filtration and Euler factors of L-functions. Such  $\mathbb{Z}$ -structures have also been studied in some cases by André [A].

Consider  $R^{sh}$  the strict henselization of  $R$ , which is a possibly ramified extension of  $W$  and an unramified extension of  $R$ . The assumption that  $X$  has a regular proper model  $\mathcal{X}$  over  $R$  which is strictly semi-stable gives us a regular proper scheme  $\overline{\mathcal{X}} := \mathcal{X} \otimes_R R^{sh}$  over  $R^{sh}$  with strictly semi-stable reduction.

As in §1, we write  $\overline{Y} = \bigcup_{i=1}^n \overline{Y}_i$  with each  $\overline{Y}_i$  irreducible, and for a subset  $I = \{i_1, \dots, i_m\}$  of  $\{1, \dots, n\}$  with  $i_1 < \dots < i_m$ , denote

$$\overline{Y}_I = \overline{Y}_{i_1} \cap \dots \cap \overline{Y}_{i_m}.$$

By assumption,  $\overline{Y}_I$  is either empty or a reduced closed subscheme of codimension  $m$  in  $\overline{\mathcal{X}}$ .

We then write  $\overline{Y}^{(m)} := \bigsqcup \overline{Y}_I$ , where the disjoint union is taken over all subsets  $I$  of  $\{1, \dots, n\}$  with  $\#I = m$ .

For each triple of integers  $(i, j, k)$  we define

$$C_j^{i,k} := C_j^{i,k}(\overline{Y}) = CH^{i+j-k}(\overline{Y}^{(2k-i+1)})$$

if  $k \geq \max\{0, i\}$  and 0 if  $k < \max\{0, i\}$ . Now, for each pair of integers  $(i, j)$  we define

$$C_j^i = C_j^i(\overline{Y}) = \bigoplus_k C_j^{i,k} = \bigoplus_{k \geq \max\{0, i\}} CH^{i+j-k}(\overline{Y}^{(2k-i+1)}).$$

Note that there are only a finite number of summands here, because  $k$  runs from  $\max\{i, 0\}$  to  $i+j$ . Note also that  $C_j^i$  can be non-zero only if  $i = -d, \dots, d$  and  $j = -i, \dots, d - i$ .

*Example 2.*  $C_0^i = CH^0(\overline{Y}^{(i+1)})$ ,  $C_d^i = CH^{d+i}(\overline{Y}^{(1-i)})$ ,  $C_{-i}^i = CH^0(\overline{Y}^{(1-i)})$  and  $C_{d-i}^i = CH^{d-i}(\overline{Y}^{(i+1)})$ .

OBSERVATION 1. For the convenience of the reader, we compare our notation with that used in [GN] and [BGS]. Our  $CH^q(\overline{Y}_I)$  is one example of the groups denoted by  $A^q(\overline{Y}_I)$  in the axioms of [BGS]. For the groups  $K^{i,j}$  used in [GN], we have

$$K^{i,j} = C_{\frac{-i+j+d}{2}}^i$$

and

$$C_j^i = K^{i,i+2j-d}.$$

Note that  $K^{i,j} = 0$  if  $-i + j + d$  is odd by definition.

Denote by  $\rho_r : \overline{Y}_{i_1 \dots i_{m+1}} \rightarrow \overline{Y}_{i_1 \dots \widehat{i}_r \dots i_{m+1}}$  the inclusion maps. Here, as usual,  $\overline{Y}_{i_1 \dots \widehat{i}_r \dots i_{m+1}}$  denotes the intersection of the subvarieties  $\overline{Y}_{i_j}$  for  $j = 1, \dots, r - 1, r + 1, \dots, m + 1$  (delete  $\overline{Y}_{i_r}$ ).

Now, for all  $i$  and  $m = 1, \dots, d$  we have maps

$$\theta_{i,m} : CH^i(\overline{Y}^{(m)}) \rightarrow CH^i(\overline{Y}^{(m+1)})$$

defined by  $\theta_{i,m} = \sum_{r=1}^{m+1} (-1)^{r-1} \rho_r^*$ , where  $\rho_r^*$  is the restriction map. Let  $\rho_{r*}$  be the Gysin map

$$\rho_{r*} : CH^i(\overline{Y}_{i_1 \dots i_{m+1}}) \rightarrow CH^{i+1}(\overline{Y}_{i_1 \dots \widehat{i}_r \dots i_{m+1}}),$$

and let

$$\delta_{i,m} : CH^i(\overline{Y}^{(m+1)}) \rightarrow CH^{i+1}(\overline{Y}^{(m)})$$

be the map defined by  $\delta_{i,m} = \sum_{r=1}^{m+1} (-1)^r \rho_{r*}$ . Let

$$d' = \bigoplus_{k \geq \max\{0,i\}} \theta_{i+j-k, 2k-i+1}$$

and

$$d'' = \bigoplus_{k \geq \max\{0,i\}} \delta_{i+j-k, 2k-i}.$$

Then define maps  $d_j^i : C_j^i \rightarrow C_j^{i+1}$  by  $d_j^i = d' + d''$ .

LEMMA 2. *We have that  $d_j^i d_j^{i-1} = 0$ , and so we get a complex for each  $j = 0, \dots, \dim(\bar{Y})$ .*

PROOF. The statement of the lemma is easily deduced from the following facts: For every  $i$  and  $m$ ,  $\delta_{i+1,m-1} \delta_{i,m} = 0$ ,  $\theta_{i,m+1} \theta_{i,m} = 0$  and  $\theta_{i+1,m} \delta_{i,m} + \delta_{i,m} \theta_{i,m+1} = 0$ . The first two equalities are easy and the last one is proved in [BGS], Lemma 1. Note that it is crucial for this proof that  $Y$  is a reduced principal divisor with normal crossings in the regular scheme  $\mathcal{X}$ .  $\square$

Define  $T_j^i = \text{Ker } d_j^i / \text{Im } d_j^{i-1}$ , the homology in degree  $i$  of the complex  $C_j^i$  defined above.

We define pairings:

$$C_j^{i,k} \times C_{d-j}^{-i,k-i} \rightarrow C_{d-k}^{i,k} = CH^{d-(2k-i)}(\bar{Y}^{(2k-i+1)}) \xrightarrow{(-1)^{i+j} \deg} \mathbb{Z}$$

as the intersection pairing. This allows us to define pairings

$$(\ , \ ) : C_j^i \times C_{d-j}^{-i} \rightarrow \mathbb{Z},$$

by pairing each summand in  $C_j^i$  with the appropriate summand in  $C_{d-j}^{-i}$ . Summands not complementary pair to zero. By the projection formula, these pairings are compatible with the differentials:  $(d'x, y) = (x, d''y)$  and  $(d''x, y) = (x, d'y)$ . Hence they induce pairings

$$(\ , \ ) : T_j^i \times T_{d-j}^{-i} \rightarrow \mathbb{Z}.$$

The monodromy operator  $N : C_j^i \rightarrow C_{j-1}^{i+2}$  is defined as the identity map on the summands in common, and the zero map on different summands (observe that  $C_j^{i,k}$  and  $C_{j-1}^{i+2,k+1}$  are equal). Clearly,  $N$  commutes with the differentials, and so induces an operator on the  $T_j^i$ , which we also denote by  $N$ . We have that  $N^i$  is the identity on  $C_j^i$  for  $i \geq 0$ . This is proved by showing that the summands in a given  $C_j^i$  persist throughout, and the others occurring in subsequent groups eventually disappear.

The following result is a direct consequence of a result of Guillen and Navarro ([GN] Prop. 2.9 and Théorème 5.2 or [BGS], Lemma 1.5 and Theorem 2), using the fact that the Chow groups of the components of  $\bar{Y}$  in the

case of totally degenerate reduction satisfy the hard Lefschetz theorem and the Hodge index theorem. It is a crucial result for this paper.

PROPOSITION 1. *Suppose that  $\overline{Y}$  satisfies the conditions in Definition 1. Then the  $i$ -th power  $N^i$  of the monodromy operator  $N$  induces an isogeny:*

$$N^i : T_{j+i}^{-i} \rightarrow T_j^i$$

for all  $i \geq 0$  and  $j$ . Moreover the pairings

$$(\ , \ ) : T_j^i \times T_{d-j}^{-i} \rightarrow \mathbb{Z}$$

are nondegenerate on the torsion free quotients.

#### 4. The Monodromy Filtration in the Case $\ell \neq p$

In this section, we study the monodromy filtration on the étale cohomology  $H^*(\overline{X}, \mathbb{Q}_\ell)$  using the techniques developed in §3. The goal is to establish an isomorphism between the graded quotients for the monodromy filtration and an appropriate twist of  $T_j^i \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$  for any prime number  $\ell \neq p$ . In §5 and §6, we will consider the case  $\ell = p$ .

Recall Grothendieck’s monodromy theorem in this situation (see [SGA VII, I] or [ST], Appendix for the statement, and [RZ1] for the proof): the restriction to the inertia group  $I$  of the  $\ell$ -adic representation associated to the étale cohomology  $H^i(\overline{X}, \mathbb{Q}_\ell)$  of a smooth projective variety  $X$  with semistable reduction over a complete discretely valued field  $K$  is unipotent.

Thus we have the monodromy operator  $N : H^i(\overline{X}, \mathbb{Q}_\ell)(1) \rightarrow H^i(\overline{X}, \mathbb{Q}_\ell)$ , characterized by the fact that the restriction of the representation to the maximal pro- $\ell$  quotient of the inertia group  $I$  is defined by the composition  $\exp \circ N \circ t_\ell$ , where  $t_\ell : I \rightarrow \mathbb{Z}_\ell(1)$  is the natural map. Using this monodromy operator, we can construct ([De3], 1.6.1, 1.6.13) the monodromy filtration,  $M_\bullet$ , which is the unique increasing filtration

$$0 = M_{-i-1} \subseteq M_{-i} \subseteq \dots \subseteq M_{i-1} \subseteq M_i = H^i(\overline{X}, \mathbb{Q}_\ell),$$

such that

$$NM_j(1) \subseteq M_{j-2}$$

and

$$N^j \text{ induces an isomorphism } \text{Gr}_j^M(j) \cong \text{Gr}_{-j}^M.$$

The construction of the monodromy filtration in *ibid.* shows that  $I$  acts trivially on the graded quotients  $\text{Gr}_j^M$  and  $M^I \subseteq M_0$ . If the residue field is finite and the weights of Frobenius are integers, then we also have the weight filtration ([De3], Proposition-Définition 1.7.5): the unique increasing filtration on  $H^i(\overline{X}, \mathbb{Q}_\ell)$  whose graded quotients are pure  $\text{Gal}(\overline{F}/F)$ -modules. In the general case one can also define a so-called weight filtration by using the weight spectral sequence, see below. Recall also the monodromy-weight conjecture ([De1], Principe 8.1; [RZ], Introduction), which can be stated in two equivalent forms:

- (1)  $\text{Gr}_j^M$  is a pure  $\text{Gal}(\overline{F}/F)$ -module of weight  $i + j$ .
- (2) The weight filtration has the properties mentioned above that characterize the monodromy filtration.

This conjecture in its first form was proved by Deligne in the equi-characteristic case ([De2], Théorème 1.8.4), and in its second form by Rapoport and Zink in the case where  $X$  is of dimension 2 and has a model  $\mathcal{X}$  with special fibre  $Y = \sum n_i Y_i$  (as cycle), with  $n_i$  prime to  $l$  for each  $i$  (see [RZ], Satz 2.13).

Observe that when  $\ell = p$ , there is in general no “simple” monodromy filtration, as is pointed out in ([Ja1], page 345). That is the reason for treating this case separately in the next two sections.

Consider the weight spectral sequence of Rapoport-Zink ([RZ], Satz 2.10; see also [Ja2]), in its two forms:

$$E_1^{i,j} = H^{i+j}(\overline{Y}, \text{Gr}_{-i}^W(R\Psi(\mathbb{Z}_\ell))) \implies H^{i+j}(\overline{X}, \mathbb{Z}_\ell).$$

and

$$E_1^{i,j} = \bigoplus_{k \geq \max\{0, i\}} H^{j+2i-2k}(\overline{Y}^{(2k+1-i)}, \mathbb{Z}_\ell(i-k)) \implies H^{i+j}(\overline{X}, \mathbb{Z}_\ell).$$

By Deligne’s weight purity theorem (Riemann hypothesis for smooth projective varieties over finite fields), this spectral sequence tensored by  $\mathbb{Q}_\ell$  degenerates at  $E_2$  when the residue field is finite, and Nakayama proved the degeneration in the general case [Na]. Deligne’s weight purity theorem also shows that the filtration associated to the weight spectral sequence is the weight filtration if the residue field is finite. One defines the weight filtration in general as the filtration associated to this spectral sequence (which



in principle depends on the strictly semi-stable model  $\mathcal{X}$  over  $R$  we take). We denote by  $E_r^{i,j}/tors$  the quotient of  $E_r^{i,j}$  by its torsion subgroup.

We write the two forms of this spectral sequence because the first one has simple indexing, and the groups in the second are very similar to the ones considered in the last section, as we now see:

PROPOSITION 2. *Suppose that  $Y$  satisfies condition b) in Definition 1. Then we have:*

$$E_1^{i,2j}/tors \cong C_j^i(\overline{Y})/tors \otimes \mathbb{Z}_\ell(-j)$$

$$E_1^{i,2j+1}/tors = 0.$$

These are compatible with the differentials on both sides, the differentials on the right hand side being those defined in §3, tensored with  $\mathbb{Z}_\ell$ . Thus we have a natural map induced from the cycle map on étale cohomology:

$$T_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell \rightarrow E_2^{i,2j}.$$

PROOF. The two equalities are clear from our assumptions on  $\overline{Y}$ . The compatibility between differentials is deduced directly from the results of Rapoport and Zink ([RZ], §2, especially Satz 2.10; see also [Ja2], §3).  $\square$

Let us denote by  $W_\bullet$  the filtration on  $H^n(\overline{X}, \mathbb{Z}_\ell)$  induced by the weight spectral sequence, and by  $M_\bullet$  the filtration defined by  $M_i(H^n) := W_{i+n}(H^n)$ . Then the filtration  $W_\bullet$  gives us the weight filtration on  $H^n(\overline{X}, \mathbb{Q}_\ell)$ , so the monodromy conjecture says that the filtration  $M_\bullet$  is the monodromy filtration.

COROLLARY 1. *With assumptions as in Proposition 2, we have canonical isogenies compatible with the action of the Galois group  $G$*

$$T_j^i(\overline{Y})/tors \otimes \mathbb{Z}_\ell(-j) \rightarrow Gr_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Z}_\ell)/tors,$$

and  $Gr_{-i}^M H^{i+2j+1}(\overline{X}, \mathbb{Z}_\ell)$  is torsion.

Moreover, if  $Y$  also satisfies condition a) in Definition 1, these isogenies are isomorphisms for almost all  $\ell$ . Hence  $H^n(\overline{X}, \mathbb{Z}_\ell)$  and  $Gr_i^M H^n(\overline{X}, \mathbb{Z}_\ell)$ ,  $n, i \in \mathbb{Z}$ , are torsion free for almost all  $\ell$ .

PROOF. By Theorem 0.1 in [Na], we know that the weight spectral sequence degenerates at  $E_2$  after tensoring by  $\mathbb{Q}_\ell$ . This implies that  $E_\infty^{i,j}/tors$  is naturally isomorphic to a submodule in  $E_2^{i,j}/tors$  of finite index. So, the natural maps

$$\begin{aligned} Gr_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Z}_\ell) &= Gr_{2j}^W H^{i+2j}(\overline{X}, \mathbb{Z}_\ell) \\ &\cong E_\infty^{i,2j} \rightarrow E_2^{i,2j}/tors \leftarrow T_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell(-j) \end{aligned}$$

are isomorphisms after tensoring by  $\mathbb{Q}_\ell$  and so the induced maps on the torsion free quotients are isogenies. It also implies that  $Gr_{-i}^M H^{i+2j+1}(\overline{X}, \mathbb{Q}_\ell) = 0$ . The canonical isogeny in the statement of the corollary is the composition of the second one with the dual of the first (see §1 for more on isogenies).

Lemma 1 implies that

$$\begin{aligned} E_1^{i,2j} &\cong C_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell(-j) \\ E_1^{i,2j+1} &= 0 \end{aligned}$$

for almost all  $\ell$ , and using the compatibility of the differentials, we have for these primes  $\ell$ ,

$$\begin{aligned} E_2^{i,2j} &\cong T_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell(-j) \\ E_2^{i,2j+1} &= 0. \end{aligned}$$

To see the assertion about torsion freeness, we only need to recall that the groups  $T_j^i(\overline{Y})$  are finitely generated abelian groups, so that  $T_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell$  is torsion free for almost all  $\ell$ . Considering the primes satisfying both conditions, we get that  $E_2^{i,2j}$  are torsion free and this implies that the spectral sequence degenerates before tensoring by  $\mathbb{Q}_\ell$ .  $\square$

COROLLARY 2. *Suppose that  $Y$  satisfies conditions a) and b) in Definition 1. Then the filtration  $M_\bullet \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  is the monodromy filtration and the induced maps*

$$Gr_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{N} Gr_{-i-2}^M H^{i+2j}(\overline{X}, \mathbb{Q}_\ell)(-1)$$

*coincide under the natural isomorphism*

$$T_j^i \otimes \mathbb{Q}_\ell(-j) \cong Gr_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Q}_\ell)$$

with the monodromy operator defined in §3.

PROOF. It is not difficult to see that  $NM_i(1) \subseteq M_{i-2}$ . Thus one only has to show that the  $i$ -th power of the monodromy induces an isomorphism between  $\text{Gr}_i^M H^n \cong \text{Gr}_{-i}^M H^n(-i)$ . Using the corollary above, this is reduced to showing that  $N^i$  induces an isomorphism

$$T_{j+i}^{-i} \otimes \mathbb{Q}_\ell(-j-i) \rightarrow T_j^i \otimes \mathbb{Q}_\ell(-j-i).$$

In ([RZ], §2), it is shown that the map

$$\begin{aligned} T_j^i \otimes \mathbb{Q}_\ell(-j) &\cong \text{Gr}_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Q}_\ell) \\ &\xrightarrow{N} \text{Gr}_{-i-2}^M H^{i+2j}(\overline{X}, \mathbb{Q}_\ell)(-1) \cong T_{j-1}^{i+2} \otimes \mathbb{Q}_\ell(-j) \end{aligned}$$

is the same as the map that we constructed in §3, and hence we can deduce the result from Proposition 1.  $\square$

REMARK 4. Observe that the filtration  $M_\bullet$  coincides with the monodromy filtration on  $H^n(\overline{X}, \mathbb{Z}_\ell)$  before tensoring by  $\mathbb{Q}_\ell$  for almost all  $\ell$ . Indeed, by Corollary 1,  $T_j^i(\overline{Y})$  and  $\text{Gr}_i^M H^n(\overline{X}, \mathbb{Z}_\ell)$ ,  $n, i, j \in \mathbb{Z}$ , are torsion free for almost all  $\ell$ , and we have natural isomorphisms

$$T_j^i(\overline{Y}) \otimes \mathbb{Z}_\ell(-j) \rightarrow \text{Gr}_{-i}^M H^{i+2j}(\overline{X}, \mathbb{Z}_\ell)$$

for almost all  $\ell$ . Furthermore, by Proposition 1, the maps

$$N^i : T_{j+i}^{-i} \otimes \mathbb{Z}_\ell \rightarrow T_j^i \otimes \mathbb{Z}_\ell$$

are isomorphisms for almost all  $\ell$ . Hence, for almost all  $\ell$ , the maps

$$N^i : \text{Gr}_i^M H^n(\overline{X}, \mathbb{Z}_\ell)(i) \rightarrow \text{Gr}_{-i}^M H^n(\overline{X}, \mathbb{Z}_\ell)$$

are isomorphisms, and hence  $M_\bullet H^n(\overline{X}, \mathbb{Z}_\ell)$  is the monodromy filtration.

REMARK 5. In papers of Ito [It] and de Shalit [dS], the monodromy and weight filtrations are considered in  $\ell$ -adic and log-crystalline cohomology, respectively. The appropriate versions of the monodromy-weight conjecture are proved for varieties uniformized by the Drinfeld upper half space. We compare and contrast our results with theirs. Whereas we assume the Hodge

index theorem for the Chow groups of intersections of components of the special fibre of a regular proper model of our  $X$ , Ito proves this for the varieties he considers. Thus Ito's paper provides us with more examples to which the methods of this paper may be applied. de Shalit proves the  $p$ -adic version of the monodromy-weight conjecture by doing harmonic analysis and combinatorics on the Bruhat-Tits building of  $PGL_{d+1}(K)$ . On the other hand, their results and methods do not give a monodromy filtration on the  $p$ -adic cohomology of  $\overline{X}$ , as we do here.

## 5. The Monodromy Filtration on Log-crystalline Cohomology

Recall the notations we introduced in §1:  $F$  is a perfect field of characteristic  $p > 0$ ,  $W(F)$  is the ring of Witt vectors with coefficients in  $F$ , and  $K_0$  is the field of fractions of  $W(F)$ . We will denote  $W(\overline{F})$  by  $W$  and denote its fraction field by  $L$ .

Let  $\mathcal{L}$  be the log structure (in the sense of Fontaine, Illusie and Kato (see [Ka])) on  $F$  given by the map  $\mathbb{N} \rightarrow F$  that sends 1 to 0. Let  $(Y, \mathcal{M})$  be a smooth  $(F, \mathcal{L})$ -log-scheme, i.e.  $Y$  is a scheme over  $F$  and  $\mathcal{M}$  is a fine log-structure with a smooth map of log schemes  $f : (Y, \mathcal{M}) \rightarrow (F, \mathcal{L})$ . We assume also that  $(Y, \mathcal{M})$  is semistable in the sense of ([Mok], Définition 2.4.1). Hyodo and Kato [HK] have defined under these conditions the log-crystalline cohomology of  $Y$ , which we will denote by  $H^i(Y^\times/W(F)^\times)$ . It is a  $W(F)$ -module of finite type, with a Frobenius operator  $\Phi$  that is bijective after tensoring by  $K_0$  and semilinear with respect to the Frobenius in  $W(F)$ , and a monodromy operator  $N$  that is nilpotent and satisfies  $N\Phi = p\Phi N$ . This operator  $N$  then determines a filtration on  $H^i(Y^\times/W(F)^\times)$  that we call the *monodromy filtration*. Recall that the log-crystalline cohomology can be computed as the inverse limit of the hypercohomology of the de Rham-Witt complex  $W_n\omega_Y^\bullet$  of level  $n$ .

Suppose now that  $Y$  is a proper variety. Then we have the spectral sequence of Mokrane ([Mok], 3.23):

$$\begin{aligned} E_1^{i,j}(Y) &:= \bigoplus_{k \geq \max\{0,i\}} H^{j+2i-2k}(Y^{(2k+1-i)}/W(F))(i-k) \\ &\implies H^{i+j}(Y^\times/W(F)^\times), \end{aligned}$$

where the twist by  $i-k$  means to multiply the Frobenius operator by  $p^{k-i}$ . We denote by  $E_1^{i,j}/tors$  the torsion-free quotient of the terms of this spectral

sequence. The associated filtration on the abutment is called the *weight filtration*, and as before we will denote it by  $W_\bullet$ . We will also denote by  $M_\bullet$  the filtration defined by  $M_i(H^n) := W_{i+n}(H^n)$ .

We can consider also the log-crystalline cohomology  $H^n(\overline{Y}^\times/W^\times)$  of  $\overline{Y}$ , which is a  $W$ -module of finite type, with a Frobenius operator  $\Phi$  and a monodromy operator, and which also has an action of the absolute Galois group  $Gal(\overline{F}/F)$ . We then have a canonical isomorphism as  $W(F)$ -modules with a Frobenius

$$H^n(Y^\times/W(F)^\times) \cong H^n(\overline{Y}^\times/W^\times)^{Gal(\overline{F}/F)}.$$

Moreover, Mokrane’s spectral sequence

$$E_1^{i,j}(\overline{Y}) := \bigoplus_{k \geq \max\{0,i\}} H^{j+2i-2k}(\overline{Y}^{(2k+1-i)}/W)(i-k) \implies H^{i+j}(\overline{Y}^\times/W^\times)$$

is also compatible with the Galois action and the spectral sequence for  $Y$  above is canonically isomorphic to the  $Gal(\overline{F}/F)$ -invariant part of this spectral sequence.

PROPOSITION 3. *With notation and assumptions as above, suppose further that  $Y$  satisfies the assumption c) of Definition 1. Then we have a canonical isomorphism compatible with the action of  $Gal(\overline{F}/F)$  and with the Frobenius*

$$E_1^{i,2j}(\overline{Y})/tors \cong C_j^i(\overline{Y})/tors \otimes W(-j)$$

$$E_1^{i,2j+1}(\overline{Y})/tors = 0.$$

*These are compatible with the differentials on both sides, the differentials on the right hand side being those defined in §3.*

*We also have that  $E_1^{i,2j}(Y)(j) \otimes_{W(F)} K_0$  corresponds to the  $p$ -adic representation  $C_j^i(\overline{Y}) \otimes_{\mathbb{Z}_p}$  of  $Gal(\overline{F}/F)$  by the correspondence explained in Remark 1 (ii), and  $E_1^{i,2j+1}(Y) \otimes_{W(F)} K_0 = 0$ .*

PROOF. The first two equalities are clear by condition c) of §1, and the compatibility between the differentials is clear by using the results of Mokrane ([Mok], §3), who proves that the spectral sequence degenerates at  $E_2$  modulo torsion for any  $Y$  ([Mok], 3.32(2)). For the last assertion, apply Remark 1, (ii) of §2.  $\square$

COROLLARY 3. *Suppose that  $Y$  satisfies c) in definition 1. Then, after tensoring with  $L = \text{frac}(W)$ , the spectral sequence of Mokrane degenerates at  $E_2$  and there is a canonical isogeny compatible with the actions of  $\text{Gal}(\overline{F}/F)$  and the Frobenius*

$$T_j^i(\overline{Y})/\text{tors} \otimes W(-j) \rightarrow \text{Gr}_{-i}^M H^{i+2j}(\overline{Y}^\times/W^\times)/\text{tors},$$

and  $\text{Gr}_{-i}^M H^{i+2j+1}(\overline{Y}^\times/W^\times)$  is torsion. Moreover, the monodromy map on the graded quotients agrees with the map defined in §3.

We also have that the  $\varphi$ -module  $\text{Gr}_{-i}^M H^{i+2j}(Y^\times/W(F)^\times) \otimes_{W(F)} K_0(j)$  corresponds to the  $p$ -adic representation  $T_j^i(\overline{Y}) \otimes \mathbb{Q}_p$  of  $\text{Gal}(\overline{F}/F)$ .

Moreover, if  $Y$  also satisfies a) in definition 1, then the weight filtration induces the monodromy filtration.

PROOF. The degeneration of the spectral sequence is deduced using a slope argument. Observe that for any  $i$  and any  $k$ , the cohomology groups

$$H^{2k}(\overline{Y}^{(i)}/W)(k)/\text{tors}$$

are of pure slope 0 because they are generated by algebraic cycles by our conditions in §2, c). So, the  $E_1^{i,2j}$ -terms are of pure slope  $j$  modulo torsion, as are any of their subquotients, such as the  $E_2^{i,2j}$  terms. So, we have that  $\text{Hom}(E_2^{r,s}, E_2^{r+t, s-t+1}) = 0$  modulo torsion for all  $r$  and  $s$  and  $t \geq 2$ , because these are of different slopes. Hence  $E_t^{r,s}/\text{tors} = E_2^{r,s}/\text{tors}$  for all  $r, s$  and  $t \geq 2$ , and by an easy induction the differentials are zero on the  $E_t^{r,s}$ -terms modulo torsion for all  $r, s$  and  $t \geq 2$ .

The assertion about the coincidence of the weight filtration and the monodromy filtration is deduced from Proposition 1 by using ([Mok], Proposition 3.18) as in the proof of Corollary 2.

The other assertions are deduced easily from Proposition 3, using Remark 1, (ii) of §2.  $\square$

Now, also assume that  $Y$  is ordinary (see condition d) of §1). Note that this condition is stable under base change to  $\overline{F}$ . Then there exists a canonical decomposition as  $W$ -modules with a Frobenius (see [I3], Proposition 1.5. (b))

$$H^n(Y^\times/W^\times) \cong \bigoplus_{i+j=n} H^j(Y, W\omega^i)(-i).$$

We will identify the two sides of the isomorphism.

The existence of this canonical decomposition allows us to define the filtration by *increasing slopes*  $U_j$  as

$$U_j(H^n(Y^\times/W^\times)) := \bigoplus_{r \leq j} H^{n-r}(Y, W\omega^r)(-r),$$

which has the property that

$$\text{Gr}_j^U H^n(Y^\times/W^\times) \cong H^{n-j}(Y, W\omega^j)(-j).$$

This filtration is *opposed* to the usual *slope filtration*, which is given by decreasing slopes. We can do the same thing for  $\bar{Y}$ .

**COROLLARY 4.** *Suppose that  $Y$  satisfies c) and d) in Definition 1. We have then that*

$$\begin{aligned} M_{2j-n}(H^n(\bar{Y}^\times/W^\times)/tors) &\subseteq M_{2j-n+1}(H^n(\bar{Y}^\times/W^\times)/tors) \\ &\subseteq U_j(H^n(\bar{Y}^\times/W^\times)/tors) \end{aligned}$$

with cokernel of finite exponent. Thus if we tensor with  $L$  we have equalities, and this holds also when  $\bar{Y}$  is replaced by  $Y$  and we tensor by  $K_0$ .

As a consequence we have canonical isogenies

$$T_j^{i-j}(\bar{Y})/tors \otimes W \rightarrow H^i(\bar{Y}, W\omega_{\bar{Y}}^j)/tors$$

(see §1 for more on isogenies).

**PROOF.** Observe first that the filtration by increasing slopes has a splitting and hence the torsion free part of  $\text{Gr}_j^U H^{i+2j}(\bar{Y}^\times/W^\times)$  is equal to  $\text{Gr}_j^U (H^{i+2j}(\bar{Y}^\times/W^\times)/tors)$ .

By Proposition 3 and Corollary 3, we have that  $\text{Gr}_{-i}^M H^{i+2j}(\bar{Y}^\times/W^\times)/tors$  is of pure slope  $j$ , and  $\text{Gr}_{-i}^M H^{i+2j+1}(\bar{Y}^\times/W^\times)$  is torsion, hence the first assertion is true.

From these facts, we have that  $\text{Gr}_{-i}^M H^{i+2j}(\bar{Y}^\times/W^\times)/tors$  is canonically isogenous to

$$\text{Gr}_j^U H^{i+2j}(\bar{Y}^\times/W^\times)/tors \cong H^{i+j}(\bar{Y}, W\omega^j)/tors(-j).$$

Composing with the canonical isogeny

$$T_j^i(\overline{Y})/tors \otimes W(-j) \rightarrow \mathrm{Gr}_{-i}^M H^{i+2j}(\overline{Y}^\times/W^\times)/tors,$$

we get a canonical isogeny

$$T_j^i(\overline{Y})/tors \otimes W \rightarrow H^{i+j}(\overline{Y}, W\omega^j)/tors. \quad \square$$

Now let's consider the logarithmic Hodge-Witt pro-sheaves  $W\omega_{\overline{Y},log}^j$  on  $\overline{Y}$ , defined as the kernel of  $1 - F$  on the pro-sheaves  $W\omega_{\overline{Y}}^j$  for the étale site.

To prove the next result we will use Proposition 2.3 in [I3], which says that

$$H^i(\overline{Y}, W\omega_{\overline{Y},log}^j) \otimes_{\mathbb{Z}_p} W \cong H^i(\overline{Y}, W\omega_{\overline{Y}}^j).$$

**COROLLARY 5.** *Suppose that  $Y$  satisfies c) and d) in Definition 1. Then we have a canonical isogeny*

$$T_j^{i-j}(\overline{Y})/tors \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow H^i(\overline{Y}, W\omega_{\overline{Y},log}^j)/tors$$

*which is compatible with the action of  $\mathrm{Gal}(\overline{F}/F)$ . See §1 for more on isogenies.*

**PROOF.** By Corollary 4 we have a canonical isogeny

$$T_j^{i-j}(\overline{Y}) \otimes_{\mathbb{Z}} W \rightarrow H^i(\overline{Y}, W\omega_{\overline{Y}}^j)$$

compatible with the actions of  $\mathrm{Gal}(\overline{F}/F)$  and the Frobenius automorphisms. By taking the fixed part  $Frob = 1$  of both sides, we obtain a canonical isogeny

$$T_j^{i-j}(\overline{Y}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow H^i(\overline{Y}, W\omega_{\overline{Y},log}^j)$$

compatible with the action of  $\mathrm{Gal}(\overline{F}/F)$ , using the fact that the action of the Frobenius on  $H^i(\overline{Y}, W\omega_{\overline{Y},log}^j)$  is multiplication by 1.  $\square$



### 6. The Monodromy Filtration on $p$ -adic Cohomology

The goal of this section is to show that the filtration on the  $p$ -adic cohomology of  $X$  induced by the vanishing cycles spectral sequence has all of the properties that a monodromy filtration should have when  $X$  has totally degenerate reduction. We assume that  $\mathcal{X}$  has special fibre that satisfies the assumptions of §1.

First of all, recall that a  $p$ -adic representation  $V$  of  $G_K$  is *ordinary* if there exists a filtration  $(\text{Fil}^i V)_{i \in \mathbb{Z}}$  of  $V$ , stable by the action of  $G_K$ , such that the inertia subgroup  $I_K$  acts on  $\text{Fil}^i V / \text{Fil}^{i+1} V$  via  $\chi^i$ , where  $\chi$  denotes the cyclotomic character. It is easy to see that this filtration is unique. The next theorem is the main result in [H] (see also [I3], Théorème 2.5 and Corollaire 2.7).

**THEOREM 1.** *(Hyodo) Assume that  $Y$  is ordinary. Then the  $p$ -adic representation  $H^m(\overline{X}, \mathbb{Q}_p)$  is ordinary. Moreover, the vanishing cycles spectral sequence*

$$E_2^{i,j} = H^i(\overline{Y}, R^j \Psi(\mathbb{Z}_p)) \implies H^{i+j}(\overline{X}, \mathbb{Z}_p)$$

*degenerates at  $E_2$  modulo torsion, and if  $F^\bullet$  denotes the corresponding filtration on  $H^{i+j}(\overline{X}, \mathbb{Z}_p)$ , one has canonical isogenies as  $G_K$ -modules*

$$\text{Gr}_F^i H^{i+j}(\overline{X}, \mathbb{Z}_p) / \text{tors} \rightarrow H^i(\overline{Y}, W\omega_{Y, \log}^j) / \text{tors}(-j)$$

*(see §1 for more on isogenies).*

As a consequence of Corollary 5 in §5, we get the following result.

**COROLLARY 6.** *Assume that  $Y$  satisfies c) and d) in Definition 1. Then, if  $F^\bullet$  denotes the filtration obtained from the vanishing cycles spectral sequence, one has canonical isogenies as  $G_K$ -modules*

$$T_j^i(\overline{Y}) / \text{tors} \otimes \mathbb{Z}_p(-j) \rightarrow \text{Gr}_F^{i+j} H^{i+2j}(\overline{X}, \mathbb{Z}_p) / \text{tors}$$

We have then that the filtration  $F^\bullet$  has the same type of graded quotients as the monodromy filtration in  $\ell$ -adic cohomology (modulo isogeny). In

fact we will prove that this filtration can be obtained from the monodromy filtration on the log-crystalline cohomology by taking the functor  $V_{st}$ .

Recall that a filtered  $(\Phi, N)$ -module  $H$  is a  $K_0$ -vector space of finite dimension  $H$  with a Frobenius  $\Phi$  which is a semi-linear automorphism and a monodromy map  $N$  which is a  $K_0$ -linear endomorphism satisfying that  $N\Phi = p\Phi N$  and a decreasing filtration  $Fil^\bullet$  on  $H \otimes_{K_0} K$  by  $K$ -subspaces which is exhaustive and separated. Given an admissible filtered  $(\Phi, N)$ -module  $H$ , we define

$$V_{st}(H) := (B_{st} \otimes_{K_0} H)^{N=0, \Phi=1} \cap Fil^0(B_{dR} \otimes_K (H \otimes_{K_0} K)).$$

It is a finite dimensional  $\mathbb{Q}_p$ -vector space with a continuous action of  $G_K$ . Recall from §1 that we fixed a choice of uniformizer  $\pi$  of  $K$ , which determined the embedding of  $B_{st}$  in  $B_{DR}$ , and so this  $p$ -adic representation depends on that choice.

Consider the log-crystalline cohomology  $H^n(Y^\times/W^\times)$  we used in the last section. Hyodo and Kato proved in [HK], §5, that we have an isomorphism

$$\rho_\pi: H^n(Y^\times/W^\times) \otimes_W K \cong H_{dR}^n(X/K)$$

depending on the uniformizer  $\pi$  that we have chosen. Using this isomorphism, we get a structure of filtered  $(\Phi, N)$ -module on the log-crystalline cohomology of the log-scheme  $Y$ .

Now assume only that  $Y$  is ordinary, and consider the filtration by increasing slopes  $U_\bullet$  that we discussed in §5. Then the induced  $(\Phi, N)$ -module structure on the  $U_i$  via the Hyodo-Kato isomorphism  $\rho_\pi$  gives a filtration by filtered  $(\Phi, N)$ -modules. We will show below that  $U_i$  is admissible for all  $i$ . Then applying the functor  $V_{st}$ , and using the main result of Tsuji ([Ts], p. 235) on the conjecture  $C_{st}$

$$H^n(\overline{X}, \mathbb{Q}_p) \cong V_{st}(H^n(Y^\times/W^\times)),$$

we get a filtration on the  $p$ -adic cohomology  $H^n(\overline{X}, \mathbb{Q}_p)$ .

**THEOREM 2.** *Assume only that  $Y$  is ordinary. Then the filtrations  $V_{st}(U_i)$  and  $F^{n-i}$  on  $H^n(\overline{X}, \mathbb{Q}_p)$  are the same.*

**PROOF.** We are grateful to the referee for the following argument, which clarifies and improves our original one. For a finite dimensional

semi-stable  $p$ -adic representation  $V$  of  $G_K$ , let  $D_{st}(V) = \text{Hom}_{\mathbb{Q}_p}(V, B_{st})^{G_K}$ . Then  $D_{st}(V)$  is a filtered  $(\Phi, N)$ -module. Set  $V^n = H^n(\overline{X}, \mathbb{Q}_p)$  and  $D^n = H^n(Y^\times/W^\times) \cong V_{st}(V^n)$ . It is well-known that semi-stable representations remain so under taking sub and quotient objects, and that the functor  $D_{st}$  is exact. Hence  $F^i V^n$  and  $Gr_F^i V^n$  are semi-stable representations. Define the filtration  $F^i$  on  $D^n$  to be the image of  $D_{st}(F^i V^n) \hookrightarrow D_{st}(V^n) \cong D^n$ . Note that the map is strictly compatible with the filtrations. Then we have an isomorphism:

$$Gr_F^i D_{st}(V^n) \cong D_{st}(Gr_F^i V^n).$$

By Theorem 1, we have an isomorphism

$$Gr_F^i V^n \cong H^i(\overline{Y}, W\omega_{\overline{Y}, \log}^{n-i}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-n+i)$$

that is compatible with the action of  $G_K$ . Since the action of  $G_K$  on  $H^i(\overline{Y}, W\omega_{\overline{Y}, \log}^{n-i})$  factors through  $Gal(\overline{F}/F)$ , the action of  $G_K$  on  $Gr_F^i V^n(n-i)$  is unramified. For an unramified representation  $p$ -adic representation  $V$  of  $G_K$ , the slope of Frobenius of  $D_{st}(V)$  is 0. Hence the slopes of

$$Gr_F^i D^n \cong D_{st}(Gr_F^i V^n) \cong D_{st}(Gr_F^i V^n(n-i)) \otimes D_{st}(\mathbb{Q}_p(-n+i))$$

are all equal to  $n-i$  because the slope of  $D_{st}(\mathbb{Q}_p(-n+i))$  is  $n-i$ . Thus we have  $F^i D = U_{n-i}$ . This also shows that the  $U_i$  are admissible.  $\square$

REMARK 6.

- (i) For  $X$  with ordinary reduction, there is another structure of  $(\Phi, N)$ -filtered module on the de Rham cohomology  $H_{dR}^n(X/K)$  given by the result of Hyodo [H] (see also [I3], Corollaire 2.6 (c)). This structure is not the right one for applying Tsuji's comparison theorem, although the filtration by increasing slopes we get (or, equivalently, the monodromy filtration) is also opposed to the Hodge filtration and so they all have isomorphic graded quotients.
- (ii) We have canonical isomorphisms

$$T_j^{i-j}(\overline{Y}) \otimes_{\mathbb{Z}} K \cong H^i(X, \Omega_X^j)$$

because the filtration by increasing slopes is opposed to the Hodge filtration. Now, proposition 1 implies that the monodromy map  $N: T_j^i \rightarrow$

$T_{j-1}^{i+2}$  is, after tensoring by  $\mathbb{Q}$ , injective if  $i < 0$  and surjective if  $i \geq 0$ . This implies that we have a monodromy map

$$N: H^i(X, \Omega_X^j) \rightarrow H^{i+1}(X, \Omega_X^{j-1})$$

which is injective if  $i < j$  and surjective if  $i \geq j$ . So, a necessary condition for a variety to have totally degenerate reduction is that the dimensions  $h^{j,i} := \dim_K H^i(X, \Omega_X^j)$  satisfy that, if  $n = 2i$ ,

$$h^{n,0} \leq h^{n-1,1} \leq \dots \leq h^{i,i}$$

and if  $n = 2i - 1$ ,

$$h^{n,0} \leq h^{n-1,1} \leq \dots \leq h^{i,i-1}$$

(recall that  $h^{j,i} = h^{i,j}$  by Hodge theory).

We summarize our discussion in the following:

**THEOREM 3.** *For  $X$  with totally degenerate reduction, consider the filtration  $M_\bullet$  on  $H^n = H^n(\bar{X}, \mathbb{Q}_p)$  defined as  $M_i(H^n) := F^j(H^n)$ , where  $j = \frac{n-i}{2}$  or  $j = \frac{n-i+1}{2}$  depending on the parity of  $n - i$ . Then we have*

$$T_j^i(\bar{Y}) \otimes \mathbb{Q}_p(-j) \cong Gr_{-i}^M H^{i+2j}(\bar{X}, \mathbb{Q}_p),$$

and the maps on these graded quotients induced by  $N$  on the  $T_j^i$  coincide after applying the functor  $V_{st}$  with the maps  $N$  defined in §3. Moreover, they verify that  $N^i$  induces an isomorphism  $Gr_i^M \cong Gr_{-i}^M(-i)$ . The other graded quotients are zero.

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