COMPACTIFICATION OF THE HOMEOMORPHISM GROUP OF A GRAPH

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ABSTRACT. Let Γ be a countable locally finite graph and let $\mathcal{H}(\Gamma)$ and $\mathcal{H}_+(\Gamma)$ denote the homeomorphism group of Γ with the compact-open topology and its identity component. These groups can be embedded into the space $\mathrm{Cld}^*_F(\Gamma \times \Gamma)$ of all closed sets of $\Gamma \times \Gamma$ with the Fell topology, which is compact. Taking closure, we have natural compactifications $\overline{\mathcal{H}}_F(\Gamma)$ and $\overline{\mathcal{H}}_{+,F}(\Gamma)$. In this paper, we completely determine the topological type of the pair $(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma))$ and give a necessary and sufficient condition for this pair to be a (Q, s)-manifold. The pair $(\overline{\mathcal{H}}_F(\Gamma), \mathcal{H}(\Gamma))$ is also considered for simple examples and we find that the case where Γ is a circle yields an interesting result. In this investigation we point out a certain inaccuracy in Sakai-Uehara's preceding results on $(\overline{\mathcal{H}}_F(\Gamma), \mathcal{H}_F(\Gamma))$ for finite graphs Γ. Moreover, we show that these pairs cannot be (Q, s)-manifolds if Γ is replaced by a manifold of dimension $\geqq 2$.

1. INTRODUCTION

Let X be a locally compact, locally connected separable metrizable space. Then it is well known that the homeomorphism group $\mathcal{H}(X)$ of X is a topological group with respect to the compact-open topology. The identity component $\mathcal{H}_+(X)$ of $\mathcal{H}(X)$ is a closed normal subgroup of $\mathcal{H}(X)$.[1] Investigating topological properties of $\mathcal{H}(X)$ and $\mathcal{H}_+(X)$ is a difficult problem in general. Even for

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[1]In the literature, the symbol $\mathcal{H}_0(X)$ is often used to denote our $\mathcal{H}_+(X)$. Our notation is set up so that it is compatible with the one in Banakh-Mine-Sakai [3], where the symbol $\mathcal{H}_0(X)$ is used to denote the identity component of $\mathcal{H}(X)$ with respect to a stronger topology called the *Whitney topology*, and $\mathcal{H}_+(X)$ to denote the one with respect to the compact-open topology.

a compact n-manifold M of dimension > 2, we do not know whether $\mathcal{H}(M)$ is an ANR. This is a classical problem known as the Homeomorphism Group Problem (cf. Problems 633 and 958 in [15]).

Every element of $\mathcal{H}(X)$ can be identified with its graph, which is a closed subset of $X \times X$. Consider the hyperspace $\mathrm{Cld}_F^*(X \times X)$ of all closed sets including the empty set with the Fell topology (the definition is given in §2.1), which is compact metrizable and coincides with the Vietoris topology if X is compact. Then the inclusion $\mathcal{H}(X) \hookrightarrow \mathrm{Cld}_F^*(X \times X)$ is an embedding (§2.1). Given a subgroup $\mathcal{H}$ of $\mathcal{H}(X)$, we have a compactification $\overline{\mathcal{H}}_F$ of $\mathcal{H}$ by taking the closure in $\mathrm{Cld}_F^*(X \times X)$. In the case $\mathcal{H} = \mathcal{H}(X)$ and $\mathcal{H} = \mathcal{H}_+(X)$, we write $\overline{\mathcal{H}}_F(X)$ and $\overline{\mathcal{H}}_{+,F}(X)$ to denote $\overline{\mathcal{H}}_F$, respectively.

Hereafter we mainly consider the case where X is a *graph*, that is, a space triangulated as a simplicial complex of dimension $\leqq 1$ with the CW topology. A graph is called *finite*, *locally finite* or *countable* if it can be triangulated by a simplicial complex which is finite, locally finite or countable, respectively. We denote by $Q = [-1, 1]^{\mathbb{N}}$ the Hilbert cube and by s its *pseudo-interior*, that is, $s = (-1, 1)^{\mathbb{N}}$. For a finite graph Γ, it is known that the homeomorphism group $\mathcal{H}(\Gamma)$ is an s-manifold (Anderson [2], see also [6, p.203]). More generally, Banakh-Mine-Sakai [3] has shown that this result is valid for countable locally finite graphs. A similar result is also known for 2-manifolds. Yagasaki [21] has shown that if M is a (separable) 2-manifold, the identity component $\mathcal{H}_+(M)$ of $\mathcal{H}(M)$ is an s-manifold.

Sakai and Uehara [17] has shown that if Γ is the unit closed interval $I = [0, 1]$, then the pair $(\overline{\mathcal{H}}_F(I), \mathcal{H}(I))$ is homeomorphic to (Q, s). Notice that in this case $\overline{\mathcal{H}}_F(I)$ can also be considered as the closure with respect to the Vietoris topology, since I is compact. In this paper, we aim to generalize the results in [17] to locally finite countable graphs. These graphs are precisely the separable metrizable graphs and are, in addition, locally compact.

For a graph Γ, we define $\Gamma^{(0)}$ to be the set of points which have no open neighborhood homeomorphic to $\mathbb{R}$.[2] By $\mathbb{T}$ we mean the unit circle $\{z \in \mathbb{C}\,;\, |z| = 1\}$. A *bouquet* is defined to be a finite graph obtained as a quotient space

$$\{(1,0)\} \cup (\mathbb{T} \times \{1,\ldots,m\})/\{1\} \times \{0,1,\ldots,m\}$$

for some non-negative integer m. Notice that our definition of a bouquet includes the one-point space $*$ and the circle $\mathbb{T}$.

We define cardinals o_Γ, s_Γ, and l_Γ as follows:

$$o_\Gamma = |\{\Gamma' \in \pi_0(\Gamma)\,;\, \Gamma' \approx \mathbb{T}\}|,$$
$$s_\Gamma = |\{\Gamma' \in \pi_0(\Gamma)\,;\, \Gamma' \text{ is homeomorphic to neither } \mathbb{R} \text{ nor a bouquet}\}|,$$
$$l_\Gamma = |\{\Gamma' \in \pi_0(\Gamma)\,;\, \Gamma' \approx \mathbb{R}\}| + |\{E \in \pi_0(\Gamma \setminus \Gamma^{(0)})\,;\, (\mathrm{Cl}\, E, \mathrm{Bd}\, E) \approx (\mathbb{T}, 1)\}|.$$

where $\approx$ means "homeomorphic to", π_0 denotes the set of path components, and Cl, Bd stand for closure, boundary in Γ, respectively. Notice that each of these cardinals does not exceed $\aleph_0$ if Γ is a countable graph. We simply write $\mathbf{1}$ and $-\mathbf{1}$ to denote the points $(1,0,0,\ldots)$ and $(-1,0,0,\ldots)$ in $Q \setminus s$, respectively. We will consider the space $Q/\{\pm\mathbf{1}\}$ obtained by identifying $\mathbf{1}$ and $-\mathbf{1}$ in Q. We can naturally regard s as a subspace of $Q/\{\pm\mathbf{1}\}$.

The next result completely determines the topological type of the compatification of the identity component $\mathcal{H}_+(\Gamma)$.

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[2]A point of $\Gamma^{(0)}$ is called a *topological vertex* of Γ in §4.

Theorem 1.1. *Let Γ be a countable locally finite graph. Then we have the following homeomorphism:*

$$(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma)) \approx (Q^{s_\Gamma + o_\Gamma} \times (Q/\{\pm\mathbf{1}\})^{l_\Gamma}, s^{s_\Gamma+o_\Gamma+l_\Gamma}) \times \mathbb{T}^{o_\Gamma}.$$

Corollary 1.2. *Let Γ be a countable locally finite graph. Then we have:*

(1) *$(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma))$ is a (Q, s)-manifold if and only if $s_\Gamma + o_\Gamma \geqq 1$, $o_\Gamma < \aleph_0$ and $l_\Gamma = 0$.*

(2) *$\overline{\mathcal{H}}_{+,F}(\Gamma)$ is a Q-manifold if and only if $s_\Gamma + o_\Gamma \geqq 1$ and $l_\Gamma + o_\Gamma < \aleph_0$.*[3]

We will prove Theorem 1.1 and Corollary 1.2 in §4.

Remark 1.3. In Sakai-Uehara [17], the pair $(\overline{\mathcal{H}}_F(\Gamma), \mathcal{H}(\Gamma))$ is studied for a finite graph Γ. They claimed that $(\overline{\mathcal{H}}_F(\Gamma), \mathcal{H}(\Gamma))$ is always a (Q, s)-manifold, but this pair seems to be a far more complicated object in general. In fact, this result is not correct even for $\Gamma = \mathbb{T}$; it follows from Theorem 3.18 that $\overline{\mathcal{H}}_F(\mathbb{T})$ is not a Q-manifold.

The result stated in [17] is certainly valid for $\Gamma = I$ (Proposition 3.6 (1)). However, the argument in [17] contains an essential gap when deducing results for an arbitrary finite graph using their result for $\Gamma = I$. As noted in Anderson's manuscript [2], it is easy to construct a compact polyhedron K and a homeomorphism $\mathcal{H}(\Gamma) \approx \mathcal{H}_+(I) \times K$ for each finite graph Γ. Sakai-Uehara claimed to be able to extend this homeomorphism to $(\overline{\mathcal{H}}_F(\Gamma), \mathcal{H}(\Gamma)) \approx (\overline{\mathcal{H}}_{+,F}(I) \times K, \mathcal{H}_+(I) \times K)$, which is not always possible.

[3]For a metrizable space M and its subspace $X \subset M$, the pair (M, X) (resp. the space M) is called a (Q, s)-*manifold* (resp. Q-*manifold*) if for each $p \in M$ there exists an open neighborhood U of p such that the pair $(U, U \cap X)$ (resp. U) is homeomorphic to $(V, V \cap s)$ (resp. V) for some open set V in Q.

It should be noticed that we can show $(\overline{\mathcal{H}}_{+,F}(\mathbb{T}), \mathcal{H}_+(\mathbb{T})) \approx (\overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}, \mathcal{H}_+(I) \times \mathbb{T})$ (Proposition 3.13), even though a natural homeomorphism $\mathcal{H}_+(\mathbb{T}) \approx \mathcal{H}_+(I) \times \mathbb{T}$ cannot be extended to give this homeomorphism (see Lemma 3.11).

In Theorem 1.1, we have many situations where $(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma))$ is a (Q, s)-manifold. However, if we replace Γ by a manifold M of dimension $\geqq 2$, this never happens. That is,

Theorem 1.4. *Let M be a topological manifold and assume that $\dim M \geqq 2$. Then the pair $(\overline{\mathcal{H}}_{+,F}(M), \mathcal{H}_+(M))$ is not a (Q, s)-manifold.*

Theorem 1.4 will be proved in §5 in a generalized form (Theorem 5.1).

In the case where $\Gamma = [0, 1]$, $(0, 1)$ or $\mathbb{T}$, we can determine the compactification $\overline{\mathcal{H}}_F(\Gamma)$ of the whole homeomorphism group $\mathcal{H}(\Gamma)$, not restricted to the identity component. These results will be presented in §3 (Propositions 3.6 (2), 3.9, Theorem 3.18). Here we single out the result for $\Gamma = \mathbb{T}$. Let $\mathbb{D}^2$ be the unit closed disk $\mathbb{D}^2 = \{z \in \mathbb{C}\,;\, |z| \leqq 1\}$ which has $\mathbb{T}$ as the boundary circle.

Theorem 3.18. *Let $Q' = Q \times \mathbb{D}^2$, $T_0 = (\{0\} \times \mathbb{T}) \times \mathbb{T} \subset Q' \times \mathbb{T}$ and $s' = s \times (\mathbb{D}^2 \setminus \mathbb{T})$. Then there exists a homeomorphism*

$$(\overline{\mathcal{H}}_F(\mathbb{T}), \mathcal{H}(\mathbb{T})) \approx (Q' \times \mathbb{T} \cup_H Q' \times \mathbb{T}, s' \times \mathbb{T} \sqcup s' \times \mathbb{T}),$$

where $H\colon T_0 \to T_0$ is the homeomorphism given by $H(0, u, v) = (0, u, u^{-2}v^{-1})$.

The compactification of the group $\mathcal{H}(\mathbb{T}, 1) = \{h \in \mathcal{H}(\mathbb{T})\,;\, h(1) = 1\}$ is also determined, and will be presented in Proposition 3.10 (2).

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2. Preliminaries

Hereafter all (topological) spaces are assumed to be separable and metrizable. In particular, graphs are assumed to be countable and locally finite. By I we understand the unit closed interval $[0,1]$ and by $\mathbb{N}$ we understand the set of positive integers. Let X be a space and A be a subset. The symbols $\operatorname{Cl}_X A = \operatorname{Cl} A$, $\operatorname{Int}_X A = \operatorname{Int} A$ and $\operatorname{Bd}_X A = \operatorname{Bd} A$ denote the closure, the interior and the boundary of A in X, respectively (As an exception, $\operatorname{Int} N$ will denote the interior of N as a manifold in §5). All function spaces are assumed to carry the compact-open topology.

2.1. **Definition and basic properties of Fell topology.** Let Y be a space and by $\operatorname{Cld}(Y)$ denote the set of all nonempty closed sets of Y and let $\operatorname{Cld}^*(Y) = \operatorname{Cld}(Y) \cup \{\emptyset\}$ denote the set of all closed sets of Y. For a compact space Y, there is a well known topology on $\operatorname{Cld}^*(Y)$, namely the Vietoris topology. With this topology, $\operatorname{Cld}^*(Y)$ is also a compact space and $\emptyset$ is an isolated point. For this reason, it is customary to consider the space $\operatorname{Cld}(Y)$ of nonempty closed sets when Y is compact. If Y is non-compact, the Vietoris topology is no longer metrizable (for example, see Illanes-Nadler [12, Theorem 2.4]). This gives an enough reason to consider another topology on $\mathrm{Cld}^*(Y)$ or $\mathrm{Cld}(Y)$ for non-compact Y.

For $S \subset Y$, let S^+ and S^- defined by

$$S^+ = \{A \in \mathrm{Cld}^*(Y)\,;\, A \subset S\},$$

$$S^- = \{A \in \mathrm{Cld}^*(Y)\,;\, A \cap S \neq \emptyset\}.$$

The *Fell topology* on $\mathrm{Cld}^*(Y)$ is the topology generated by the family

$$\{U^-\,;\, U \text{ is open in } Y\} \cup \{(Y \setminus K)^+\,;\, K \text{ is a compact subset of } Y\}.$$

It is clear from the definition that this topology coincides with the Vietoris topology if Y is compact. The space $\mathrm{Cld}^*(Y)$ equipped with the Fell topology is denoted by $\mathrm{Cld}^*_F(Y)$. An important property of the Fell topology is that if Y is locally compact, then $\mathrm{Cld}^*_F(Y)$ is compact (see Beer [5, Theorem 5.1.5]). Moreover, $\emptyset$ is not isolated in $\mathrm{Cld}^*_F(Y)$ unless Y is compact.

It is well known that if X is a locally compact locally connected space, then the homeomorphism group $\mathcal{H}(X)$ equipped with the compact-open topology is a topological group. As is explained in §1, every element $h \in \mathcal{H}(X)$ can be thought of as an element of $\mathrm{Cld}^*_F(X \times X)$ by identifying h with its graph. Let $\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times X \to X$ denote the projections to the first and the second factor.

Lemma 2.1. *Let X be a locally compact, locally connected space. Then the inclusion $i \colon \mathcal{H}(X) \hookrightarrow \mathrm{Cld}^*_F(X \times X)$ is an embedding.*

Proof. Fix an admissible metric d on X. First we show that i is continuous. Take $h \in \mathcal{H}(X)$, an open set $U \subset X \times X$ and a compact set $K \subset X \times X$ such that $i(h) \in U^- \cap ((X \times X) \setminus K)^+$. We shall find neighborhood $\mathcal{U}$ of h such that $g \in \mathcal{U}$ implies $i(g) \in U^- \cap ((X \times X) \setminus K)^+$. Fix a point $p \in X$ such that $(p, h(p)) \in U$ and let $L = \mathrm{pr}_1(K) \cup \{p\} \subset X$. By the compactness of K and the continuity of h, we may choose $\varepsilon > 0$ small enough so that every $g \in \mathcal{H}(X)$ with $d(g|_L, h|_L) < \varepsilon$ satisfies $i(g) \cap K = \emptyset$ and $(p, g(p)) \in U$. Since L is compact, the set $\mathcal{U} = \{g \in \mathcal{H}(X)\,;\, d(g|_L, f|_L) < \varepsilon\}$ is a neighborhood of h in $\mathcal{H}(X)$ with respect to the compact-open topology with the required property. This proves the continuity of i.

Next we show that $i^{-1} \colon i(\mathcal{H}(X)) \to \mathcal{H}(X)$ is continuous. Take $h \in \mathcal{H}(X)$, a compact set $L \subset X$ and an open set $V \subset X$ with $h(L) \subset V$. We shall find a neighborhood $\mathcal{U}$ of $i(h)$ in $\mathrm{Cld}_F^*(X \times X)$ such that if $i(g) \in \mathcal{U}$ then $g(L) \subset V$. Since X is locally compact and locally connected, we may find a compact set L' containing L having only finitely many connected components such that $h(L') \subset V$. Thus it is enough to consider the case where L is connected. Further, we may assume that $\mathrm{Cl}\, V$ is compact and that L contains a nonempty open set U. Consider the neighborhood

$$\mathcal{V} = (U \times V)^- \cap ((X \times X) \setminus (L \times \mathrm{Bd}\, V))^+$$

of $i(h)$ in $\mathrm{Cld}_F^*(X \times X)$. Then $i(g) \in \mathcal{V}$ implies $g(L) \subset V$ because of the connectedness of L. □

The following lemmas are easy to prove and are stated without proofs.

Lemma 2.2. *Let X be a space and let $(X_\lambda)_{\lambda \in \Lambda}$ be a locally finite family of closed sets in X. Then the map $\Sigma \colon \prod_{\lambda \in \Lambda} \mathrm{Cld}_F^*(X_\lambda) \to \mathrm{Cld}_F^*(X)$ defined by $\Sigma((A_\lambda)_{\lambda \in \Lambda}) = \bigcup_{\lambda \in \Lambda} A_\lambda \; (A_\lambda \in \mathrm{Cld}_F^*(X_\lambda),\ \lambda \in \Lambda)$ is continuous.* □

Lemma 2.3. *Let X be a space and $U \subset X$ be an open set. Then the map $r_U\colon \mathrm{Cld}^*_F(X) \to \mathrm{Cld}^*_F(U)$ defined by $r_U(A) = A \cap U\,(A \in \mathrm{Cld}^*_F(X))$ is continuous.* □

A *perfect map* is a closed, continuous map with compact fibers. A perfect map $f\colon X \to Y$ is always proper, that is, $f^{-1}(K)$ is compact for every compact subset K of Y ([10, Theorem 3.7.2]).

Lemma 2.4. *Let $f\colon X \to Y$ be a perfect map. Then the map $f_*\colon \mathrm{Cld}^*_F(X) \to \mathrm{Cld}^*_F(Y)$ defined by $f_*(A) = f(A)\,(A \in \mathrm{Cld}^*_F(X))$ is continuous.* □

2.2. **Cap-sets in Q-manifolds and compact sets in pseudo-boundary.** The subset $Q \setminus s$ of Q is called the *pseudo-boundary* of Q. To prove most results in this paper, we have to move various kinds of compact sets in the pseudo-boundary $Q \setminus s$ (or in $(Q \setminus s) \times \mathbb{T}$) to the "right" place by a homeomorphism of Q (or $Q \times \mathbb{T}$) that preserves s (or $s \times \mathbb{T}$). This is made possible by the theory of cap-sets in Q-manifolds, which we will quickly review. Basic references are Chapman's paper [7] and van Mill's monographs [13, 14].

A closed subset A of a metric space (M, d) is called a *Z-set* if every continuous map $f\colon Q \to M$ and $\varepsilon > 0$, there is a continuous map $g\colon Q \to M \setminus A$ with $d(g, f) < \varepsilon$. Then every closed subset of a Z-set is a Z-set, and the union of two Z-sets is again a Z-set. A *Z_σ-set* is a countable union of Z-sets. An important property of Z-sets in Q-manifolds is the following (see van Mill [13, Theorem 7.4.9]):

Theorem 2.5 (Z-set unknotting theorem)**.** *Let M be a compact Q-manifold and let $A, B \subset M$ be Z-sets. Suppose that there exists a homeomorphism*

$h\colon A \to B$ which is homotopic to id_A in M. Then, h can be extended to a homeomorphism $\tilde{h}\colon M \to M$. □

A subset X of a metric space (M, d) is called a *cap-set* (abbreviation for *set with compact absorbing property*) if it is (a Z_σ-set and) expressed as the union of increasing sequence $(X_i)_{i=1}^\infty$ of Z-sets of M with the following property: "for each compact Z-set $K \subset M$, $\varepsilon > 0$ and $i \in \mathbb{N}$, there exist $j \in \mathbb{N}$ and an embedding $h\colon K \to X_j$ such that $h|_{K \cap X_i} = \mathrm{id}$ and $d(h, \mathrm{id}_K) < \varepsilon$". The importance of cap-sets consists in the following property, which states that, in particular, a cap-set is placed in a Q-manifold in a topologically unique way:

Theorem 2.6. *Let M be a Q-manifold, $X, Y \subset M$ be cap-sets, and B be a closed subset of M such that $B \cap X = B \cap Y$. Then there exists a homeomorphism $h\colon M \to M$ such that $h(X) = Y$ and $h|_B = \mathrm{id}$.* □

Theorem 2.6 can easily be derived from Chapman [7, Theorem 6.2, Lemma 5.4].

The pseudo-boundary $Q \setminus s$ is a cap-set in Q (van Mill [14, Corollary 5.4.8]). It is known that for every locally compact ANR K, the product $Q \times K$ is a Q-manifold (Chapman [8, Theorem 44.1]).[4] The subset $(Q \setminus s) \times K$ is a cap-set in the Q-manifold $Q \times K$ (Chapman [7, proof of Lemma 5.6]).

Using the above two theorems, we deduce the following:

Proposition 2.7. *Let A and B be compact subsets of $Q \setminus s$ and let $h\colon A \to B$ be a homeomorphism. Then there exists a homeomorphism $\tilde{h}\colon (Q, s) \to (Q, s)$ such that $\tilde{h}|_A = h$.*

[4] We need only the case where K is compact. In van Mill's monograph [13, Theorem 7.8.1], a proof of this special case is presented.

Proof. First notice that both of A and B are Z-sets in Q. Therefore by Theorem 2.5, there exists a homeomorphism $\tilde{h}'\colon Q \to Q$ such that $\tilde{h}'|_A = h$. Then, $\tilde{h}'(Q \setminus s)$ and $Q \setminus s$ are two cap-sets in Q, which contain a Z-set B in common. Thus, by Theorem 2.6, there exists a homeomorphism $\tilde{h}''\colon Q \to Q$ such that $\tilde{h}'' \circ \tilde{h}'(Q \setminus s) = Q \setminus s$ and $\tilde{h}''|_B = \mathrm{id}$. Then, $\tilde{h} = \tilde{h}'' \circ \tilde{h}'\colon Q \to Q$ is the required homeomorphism. □

3. Compactifications of the homeomorphism group of intervals and circles

Let Γ be a graph. To prove Theorem 1.1 announced in §1, we first aim to determine the topological type of the compactification of $\mathcal{H}_+(\Gamma)$ when the graph Γ is an interval or a circle. We devide this step into three cases. The first (§3.1) is the case where $\Gamma = I$ or $\Gamma = [0, 1)$. The second (§3.2) is the case where Γ is the open interval $(0, 1)$ or "Γ is the pair $(\mathbb{T}, 1)$" (that is, we consider $\mathcal{H}(\mathbb{T}, 1) = \{h \in \mathcal{H}(\mathbb{T})\,;\, h(1) = 1\}$ and its identity component). The third (§3.3) is the case where $\Gamma = \mathbb{T}$, which seems to be harder than the others. When studying these cases, we treat not only the compactification of the identity component $\mathcal{H}_+(\Gamma)$ but also of the whole group $\mathcal{H}(\Gamma)$, since it does not require too much additional work.

For a graph Γ, a subset of $\Gamma \times \Gamma$ is frequently regarded as a set-valued function; that is, if $A \subset \Gamma \times \Gamma$ and $x \in \Gamma$, by $A(x)$ we mean the subset $\{y \in \Gamma\,;\, (x, y) \in A\}$ of Γ.

3.1. **Compactifications of the homeomorphism groups of a closed interval and a half-open interval.** We first consider the compactifications $\overline{\mathcal{H}}_F(\Gamma)$ and $\overline{\mathcal{H}}_{+,F}(\Gamma)$ in the case where $\Gamma = I$ or $[0,1)$.

To begin with, we recall the following characterization of the elements of $\overline{\mathcal{H}}_+(I)$ among the closed sets of I^2, which is obtained in the remark after the proof of Lemma 3 in Sakai-Uehara [17].

Lemma 3.1. *A closed set $A \subset I^2$ belongs to $\overline{\mathcal{H}}_{+,F}(I)$ if and only if the following conditions are satisfied:*

(1) $(0,0) \in A, (1,1) \in A$,

(2) *For all $x \in I$, the set $A(x)$ is either a singleton or a closed interval,*

(3) $\max A(x_1) \leqq \min A(x_2)$ *if* $x_1 < x_2$.

□

Remark 3.2. Notice that, if $A \in \overline{\mathcal{H}}_{+,F}(I)$ then $A^{-1} = \{(x,y) \in I^2\,;\,(y,x) \in A\}$ belongs to $\overline{\mathcal{H}}_{+,F}(I)$, although the conditions in Lemma 3.1 are apparently not symmetric with respect to exchange of the coordinates.

The next lemma, which can easily be derived from Lemma 3.1, is an analogue of the "intermediate value theorem" for elements of $\overline{\mathcal{H}}_{+,F}(I)$.

Lemma 3.3. *Assume $A \in \overline{\mathcal{H}}_{+,F}(I)$, $(x,y), (x',y') \in A$, $x \leqq x'$ and $y \leqq y_0 \leqq y'$. Then there exists $x_0 \in [x,x']$ such that $(x_0,y_0) \in A$.* □

The next lemma, which means the "continuity" of the elements of $\overline{\mathcal{H}}_{+,F}(I)$, is also easily proved and useful.

Lemma 3.4. *Let $A \in \overline{\mathcal{H}}_{+,F}(I)$, $x \in I$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $x - \delta < x' < x$ then $A(x') \subset (\min A(x) - \varepsilon, \min A(x)]$ and if $x < x' < x + \delta$ then $A(x') \subset [\max A(x), \max A(x) + \varepsilon)$.* □

Recall that the Fell topology on $\mathrm{Cld}^*(\Gamma \times \Gamma)$ agrees with the Vietoris topology if Γ is compact, in which case $\emptyset$ is an isolated point in $\mathrm{Cld}^*_F(\Gamma \times \Gamma)$. For any subset A of $I^2 = I \times I$, define $R(A) \subset I^2$ by $R(A) = \{(x, 1 - y)\,;\,(x, y) \in A\}$. Then $\mathcal{H}(I)$, whose elements are identified with their graph, consists of two components. These components are $\mathcal{H}_+(I)$ and $R(\mathcal{H}_+(I))$, the latter being the set of all orientation-reversing homeomorphisms of I. Similarly, $\mathcal{H}((0,1))$ is decomposed into two components $\mathcal{H}_+((0,1))$ and $R(\mathcal{H}_+((0,1)))$. Clearly, R is an involution (that is, $R(R(A)) = A$ for each $A \subset I^2$).

Then we observe:

Lemma 3.5. *The space $\overline{\mathcal{H}}_F(I)$ is the disjoint union of two components $\overline{\mathcal{H}}_{+,F}(I)$ and $R(\overline{\mathcal{H}}_{+,F}(I))$.*

Proof. Notice that R is a homeomorphism of $\mathrm{Cld}^*_F(I^2)$ onto itself. Taking the closure of both sides of $\mathcal{H}(I) = \mathcal{H}_+(I) \cup R(\mathcal{H}_+(I))$, we have $\overline{\mathcal{H}}_F(I) = \overline{\mathcal{H}}_{+,F}(I) \cup R(\overline{\mathcal{H}}_{+,F}(I))$. Suppose there exists an element $A \in \overline{\mathcal{H}}_{+,F}(I) \cap R(\overline{\mathcal{H}}_{+,F}(I))$. Then by Lemma 3.1 (1), we have $(0,1), (1,0) \in A$, since $A \in R(\overline{\mathcal{H}}_{+,F}(I))$. This means $\max A(0) = 1 > 0 = \min A(1)$, which contradicts the fact that $A \in \overline{\mathcal{H}}_{+,F}(I)$, by Lemma 3.1 (3). □

Let $Q = [-1,1]^{\mathbb{N}}$ be the Hilbert cube and let $s = (-1,1)^{\mathbb{N}} \subset Q$. The next result is essentially shown in Sakai-Uehara [17]:

Proposition 3.6. *We have the following homeomorphisms:*

(1) $(\overline{\mathcal{H}}_{+,F}(I), \mathcal{H}_+(I)) \approx (Q, s)$,

(2) $(\overline{\mathcal{H}}_F(I), \mathcal{H}(I)) \approx (Q, s) \times \{0, 1\}$.

Proof. The statement (1) is [17, Theorem 4]. The next statement (2) follows from (1) and Lemma 3.5. □

For the homeomorphism group of $[0, 1)$, we have the following:

Proposition 3.7. $\big(\overline{\mathcal{H}}_{+,F}([0, 1)), \mathcal{H}_+([0, 1))\big) = \big(\overline{\mathcal{H}}_F([0, 1)), \mathcal{H}([0, 1))\big) \approx (Q, s)$.

Proof. Since $\mathcal{H}_+([0, 1)) = \mathcal{H}([0, 1))$, the first equality holds. To show the remaining part, by Proposition 3.6 (1), it suffices to show that

$$(\overline{\mathcal{H}}_F([0, 1)), \mathcal{H}([0, 1))) \approx (\overline{\mathcal{H}}_{+,F}(I), \mathcal{H}_+(I)).$$

We can define a homeomorphism $\psi\colon \mathcal{H}_+(I) \to \mathcal{H}([0, 1))$ by $\psi(h) = h|_{[0,1)}$. Then, ψ can be extended to a homeomorphism $\overline{\psi}\colon \overline{\mathcal{H}}_{+,F}(I) \to \overline{\mathcal{H}}_F([0, 1))$. This can be achieved by simply defining

$$\overline{\psi}(A) = A \cap [0, 1)^2, \quad A \in \overline{\mathcal{H}}_{+,F}(I) \subset \mathrm{Cld}^*_F(I^2).$$

Indeed, by Lemma 2.3, $\overline{\psi}\colon \overline{\mathcal{H}}_{+,F}(I) \to \mathrm{Cld}^*_F([0, 1)^2)$ is continuous, and it follows from this continuity and the compactness of $\overline{\mathcal{H}}_{+,F}(I)$ that the image of $\overline{\psi}$ is equal to $\overline{\mathcal{H}}_F([0, 1))$. It now remains to show that $\overline{\psi}$ is injective. Take any $A, B \in \overline{\mathcal{H}}_{+,F}(I)$ such that $\overline{\psi}(A) = \overline{\psi}(B)$. Then both A and B have the properties (1)–(3) in Lemma 3.1. To show $A \subset B$, take any $(x, y) \in A$. If $(x, y) \in [0, 1)^2$, then $(x, y) \in B$ by definition. If $(x, y) \notin [0, 1)^2$, then we have $x = 1$ or $y = 1$. It is enough to consider the case $x = 1$ only, since we can exchange coordinates if necessary. Put $y_0 = \sup \mathrm{pr}_2(\overline{\psi}(A)) = \sup \mathrm{pr}_2(\overline{\psi}(B))$ (these suprema certainly exist, since $(0, 0) \in \overline{\psi}(A) \cap \overline{\psi}(B)$ by the property (1) of A and B). We have $(1, y_0) \in B$ since B is closed in I^2, and we have $y_0 \leqq y$ by Lemma 3.1 (3) applied to A. Again by Lemma 3.1 (1) and (2) applied to B, the set $B(1)$ is connected and $(1, 1) \in B$. Thus, we have $[y_0, 1] \subset B(1)$ and hence $(x, y) = (1, y) \in B$. Consequently, we have $A \subset B$. Since $B \subset A$ can be proved similarly, we have $A = B$ and thus $\overline{\psi}$ is injective. □

3.2. **Compactifications of the homeomorphism groups of an open interval and a pointed circle.** We denote the points $(1, 0, 0, \ldots), (-1, 0, 0, \ldots)$ in Q simply by $\mathbf{1}, -\mathbf{1}$, respectively. The compact space $Q/\{\pm\mathbf{1}\}$ obtained by identifying $\mathbf{1}$ and $-\mathbf{1}$ in Q naturally contains $s = (-1, 1)^{\mathbb{N}}$ as a subspace. It will be convenient to make the following definitions (see Figure 1):

$$L_0 = (I \times \{0\}) \cup (\{1\} \times I), \;\; L_\infty = (\{0\} \times I) \cup (I \times \{1\}),$$

$$L_{0,t} = (\{0\} \times [0, t]) \cup (I \times \{t\}) \cup (\{1\} \times [t, 1]) \quad (t \in I),$$

$$L_{\infty,t} = ([0, t] \times \{0\}) \cup (\{t\} \times I) \cup ([t, 1] \times \{1\}) \quad (t \in I).$$

By Lemma 3.1, all of these sets we have defined are elements of $\overline{\mathcal{H}}_{+,F}(I)$. Note that $L_{0,0} = L_{\infty,1} = L_0$ and $L_{0,1} = L_{\infty,0} = L_\infty$.

We have the following result for the compactification of the group $\mathcal{H}_+((0, 1))$ of orientation-preserving homeomphism of the open interval $(0, 1)$.

Proposition 3.8. $\big(\overline{\mathcal{H}}_{+,F}((0, 1)), \mathcal{H}_+((0, 1))\big) \approx (Q/\{\pm\mathbf{1}\}, s)$.

Proof. As in the proof of Proposition 3.7, a homeomorphism $\psi\colon \mathcal{H}_+(I) \to \mathcal{H}_+((0, 1))$ can be defined by $\psi(h) = h|_{(0,1)}$. Again, this homeomorphism can be extended to a continuous surjective map $\overline{\psi}\colon \overline{\mathcal{H}}_{+,F}(I) \to \overline{\mathcal{H}}_{+,F}((0, 1))$ by

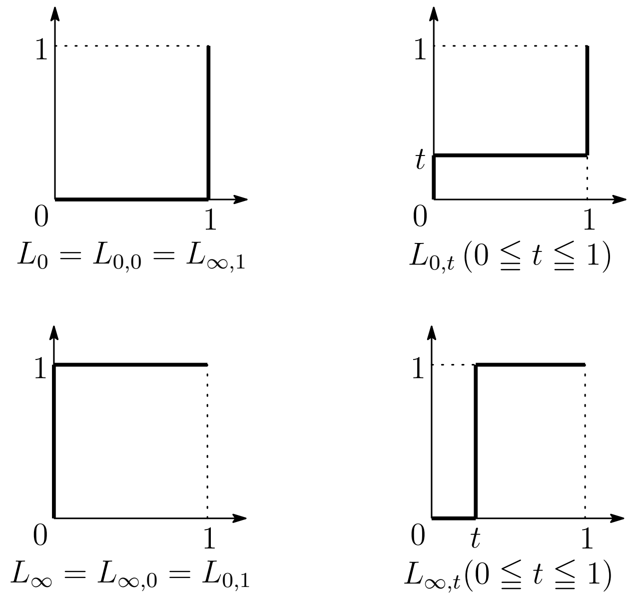

FIGURE 1. Definition of $L_0, L_{0,t}, L_\infty$ and $L_{\infty,t}$

defining $\overline{\psi}(A) = A \cap (0,1)^2$ for each $A \in \overline{\mathcal{H}}_{+,F}(I)$. We assert that $\overline{\psi}$ is "almost injective", that is,

Assertion 1. *For $A, B \in \overline{\mathcal{H}}_{+,F}(I)$, the equality $\overline{\psi}(A) = \overline{\psi}(B)$ holds if and only if $A = B$ or $\{A, B\} = \{L_0, L_\infty\}$.*

Proof of Assertion 1. Take any $A, B \in \overline{\mathcal{H}}_{+,F}(I)$ such that $\overline{\psi}(A) = \overline{\psi}(B)(= C)$. Then, by definition, $C = A \cap (0,1)^2 = B \cap (0,1)^2$. If $C = \emptyset$, we observe from Lemma 3.1 that $\{A, B\} \subset \{L_0, L_\infty\}$.

Next we consider the case $C \neq \emptyset$. To prove $A \subset B$, we take arbitrary $(x, y) \in A \setminus (0,1)^2$. Then, at least one of (i) $x = 0$, (ii) $x = 1$, (iii) $y = 0$ and (iv) $y = 1$ holds. The cases (i) and (ii) can be treated similarly. The case (iii) and (iv) can be reduced to (i) and (ii) by exchanging coordinates, respectively. Thus we have only to consider the case (ii). This case is, however, essentially considered in the proof of Proposition 3.7. Indeed, since $C \neq \emptyset$, we can define

$y_0 = \sup \mathrm{pr}_2(C)$ and we have $(1, y_0) \in B$, $y_0 \leqq y$, $[y_0, 1] \subset B(1)$ and hence $(x, y) = (1, y) \in B$, exactly as in the proof of Proposition 3.7. $\square$

By this assertion and Proposition 3.6 (1) and Proposition 2.7, we have a required homeomorphism. The proof of Proposition 3.8 is completed. $\square$

To state next proposition, we need a space obtained by gluing two copies of $Q/\{\pm\mathbf{1}\}$ in a way described below. We define a disk D_0 contained in Q by

$$D_0 = [-1, 1]^2 \times \{(0, 0, \ldots)\}$$

and let S_0 be the boundary circle of D_0. Then $A_\pm = \{(x_i)_{i\in\mathbb{N}} \in S_0\,;\, \pm x_2 \geqq 0\}$ are arcs contained in S_0 with $A_+ \cap A_- = \{\pm\mathbf{1}\}$ and $A_+ \cup A_- = S_0$. Let $\pi\colon Q \to Q/\{\pm\mathbf{1}\}$ denote the quotient map and let $\overline{S}_0 = \pi(S_0)$. We define $h\colon S_0 \to \overline{S}_0$ by

$$h(x, y, 0, 0, \ldots) = \begin{cases} \pi(x, y, 0, 0, \ldots) & \text{if } (x, y, 0, 0, \ldots) \in A_+, \\ \pi(-x, y, 0, 0, \ldots) & \text{if } (x, y, 0, 0, \ldots) \in A_-. \end{cases}$$

Then h is certainly well-defined and induces a homeomorphism $\bar{h}\colon \overline{S}_0 \to \overline{S}_0$.

Consider the adjunction space $Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}$. This is formally defined as the quotient space of $(Q/\{\pm\mathbf{1}\}) \times \{-1, 1\}$ by the equivalence relation $\sim$ generated by $(x, 1) \sim (\bar{h}(x), -1)$, $x \in \overline{S}_0$. Notice that $s \times \{1, -1\}$, denoted by $s \sqcup s$, is naturally embedded into $Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}$ as a subspace. Finally, notice that $Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}$ can be considered also as the quotient space of $Q \times \{1, -1\}$.

Proposition 3.9. *We have the following homeomorphism:*

$$\big(\overline{\mathcal{H}}_F((0, 1)), \mathcal{H}((0, 1))\big) \approx (Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}, s \sqcup s).$$

Proof. We can define a homeomorphism $\psi\colon \mathcal{H}(I) \to \mathcal{H}((0,1))$ by $\psi(h) = h|_{(0,1)}$. This can be extended to a continuous surjective map $\overline{\psi}\colon \overline{\mathcal{H}}_F(I) \to \overline{\mathcal{H}}_F((0,1))$ by $\overline{\psi}(A) = A \cap (0,1)^2$.

Assertion 2. *For $A, B \in \overline{\mathcal{H}}_F(I)$, the equality $\overline{\psi}(A) = \overline{\psi}(B)$ holds if and only if one of the following holds:*

(a) $A = B$,

(b) $\{A, B\} \subset \{L_0, L_\infty, R(L_0), R(L_\infty)\}$,

(c) $\{A, B\} = \{L_{0,t}, R(L_{0,1-t})\}$ *for some* $t \in (0,1)$,

(d) $\{A, B\} = \{L_{\infty,t}, R(L_{\infty,t})\}$ *for some* $t \in (0,1)$.

Proof of Assertion 2. Clearly any of (a)–(d) implies $\overline{\psi}(A) = \overline{\psi}(B)$. To prove the converse, assume that $\overline{\psi}(A) = \overline{\psi}(B)(= C)$ and $A \neq B$. We shall show that at least one of (b)–(d) holds. We distinguish two cases: (i) A and B are in the same component of $\overline{\mathcal{H}}_F(I)$, and (ii) A and B are in different components of $\overline{\mathcal{H}}_F(I)$. (i) If $A, B \in \overline{\mathcal{H}}_{+,F}(I)$, by the argument in the proof of Proposition 3.8, we have $\{A, B\} = \{L_0, L_\infty\}$. In particular, (b) holds. Similarly if $A, B \in R(\overline{\mathcal{H}}_{+,F}(I))$, we have $\{A, B\} = \{R(L_0), R(L_\infty)\}$ and thus (b) holds.

(ii) We may assume that $A \in \overline{\mathcal{H}}_{+,F}(I)$ and $B \in R(\overline{\mathcal{H}}_{+,F}(I))$. We further distinguish the case (ii-1) $C = \emptyset$ and the case (ii-2) $C \neq \emptyset$.

(ii-1) If $C = \emptyset$, then we observe from Lemma 3.1 that $A \in \{L_0, L_\infty\}$ and $B \in \{R(L_0), R(L_\infty)\}$. Therefore, we obtain (b).

(ii-2) If $C \neq \emptyset$, then $A \cap (0,1)^2 \neq \emptyset$. Therefore, by an application of Lemma 3.1 to A, we see that C has at least two points. Fix such distinct points $(x_0, y_0), (x_1, y_1) \in C$. We further distinguish three subcases.

(ii-2-1) First we suppose $x_0 \neq x_1$ and $y_0 \neq y_1$. Since $A \in \overline{\mathcal{H}}_{+,F}(I)$ and $B \in R(\overline{\mathcal{H}}_{+,F}(I))$, we can apply Lemma 3.1 (3) to A and $R(B)$ to obtain a contradiction.

(ii-2-2) Next we suppose $x_0 = x_1$ and $y_0 \neq y_1$. If there exists a point $(x, y) \in C$ with $x \neq x_0$, similarly to the last subcase, Lemma 3.1 (3) for A or $R(B)$ leads to a contradiction. Thus $C \subset \{x_0\} \times (0, 1)$. Then it is easy to see that $A = R(B) = L_{\infty,x_0}$ using Lemma 3.1, which means (d) by letting $t = x_0$.

(ii-2-3) We are left with the case $x_0 \neq x_1$ and $y_0 = y_1$. This case can be treated similarly as the last subcase and we have $A = L_{0,y_0}$ and $B = R(L_{0,1-y_0})$, which means (c) by letting $t = y_0$. $\square$

Recall that $L_{0,1} = L_{\infty,0} = L_\infty$, $L_{\infty,1} = L_{0,0} = L_0$ and hence $\{L_{0,t}\,;\, t \in I\} \cup \{L_{\infty,t}\,;\, t \in I\}$ is a simple closed curve in $\overline{\mathcal{H}}_{+,F}(I)$. By Propositions 3.6 (1) and 2.7, we have a homeomorphism $\varphi\colon (\overline{\mathcal{H}}_{+,F}(I), \mathcal{H}_+(I)) \to (Q, s)$ such that

- $\varphi(\{L_{\infty,t}\,;\, t \in I\}) = A_+$,
- $\varphi(\{L_{0,t}\,;\, t \in I\}) = A_-$,
- $\varphi(L_{0,t}) = (x, y, 0, 0, \ldots)$ if and only if $\varphi(L_{0,1-t}) = (-x, y, 0, 0, \ldots)$.

Using this homeomorphism, we can define a homeomorphism

$$\tilde{\varphi}\colon (\overline{\mathcal{H}}_F(I), \mathcal{H}(I)) \to (Q, s) \times \{-1, 1\}$$

by $\tilde{\varphi}(A) = (\varphi(A), 1)$ for $A \in \overline{\mathcal{H}}_{+,F}(I)$ and $\tilde{\varphi}(A) = (\varphi(R(A)), -1)$ for $A \in R(\overline{\mathcal{H}}_{+,F}(I))$.

Let $\pi'\colon Q \times \{1, -1\} \to Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}$ denote the quotient map. Using the relation $R(L_{0,t}) = L_{0,1-t}$ and Assertion 2, it is easy to check that the above $\tilde{\varphi}$ induces a homeomorphism $\overline{\varphi}\colon (\overline{\mathcal{H}}_F((0,1)), \mathcal{H}((0,1))) \to (Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}, s \sqcup s)$ with $\overline{\varphi} \circ \overline{\psi} = \pi' \circ \tilde{\varphi}$, which is the desired. $\square$

Next we consider the topological group

$$\mathcal{H}(\mathbb{T},1) = \{h \in \mathcal{H}(\mathbb{T})\,;\, h(1) = 1\}$$

and its identity component $\mathcal{H}_+(\mathbb{T},1)$. Their compactifications turn out to be the same as ones of $\mathcal{H}_+((0,1))$ and $\mathcal{H}((0,1))$.

Proposition 3.10. *We have the following homeomorphisms:*

(1) $(\overline{\mathcal{H}}_{+,F}(\mathbb{T},1), \mathcal{H}_+(\mathbb{T},1)) \approx (Q/\{\pm\mathbf{1}\}, s)$.

(2) $\big(\overline{\mathcal{H}}_F(\mathbb{T},1), \mathcal{H}(\mathbb{T},1)\big) \approx (Q/\{\pm\mathbf{1}\} \cup_{\bar{h}} Q/\{\pm\mathbf{1}\}, s \sqcup s)$.

Proof. Consider the homeomorphism

$$\psi\colon (\mathcal{H}(I), \mathcal{H}_+(I)) \to (\mathcal{H}(\mathbb{T},1), \mathcal{H}_+(\mathbb{T},1))$$

defined by

$$\psi(h)(q(t)) = q(h(t)), \quad t \in [0,1],$$

where $q(t) = e^{2\pi i t}$. This definition can be written also as $\psi(h) = (q \times q)(h)$, regarding each homeomorphism as its graph. This map ψ can be extended to a continuous surjective map $\overline{\psi}\colon \overline{\mathcal{H}}_F(I) \to \overline{\mathcal{H}}_F(\mathbb{T},1)$ defined by $\overline{\psi}(A) = (q \times q)(A)$. Indeed, the continuity of $\overline{\psi}$ comes from Lemma 2.4.

Assertion 3. *We have the following:*

(i) *For $A, B \in \overline{\mathcal{H}}_{+,F}(I)$, $\overline{\psi}(A) = \overline{\psi}(B)$ if and only if $A = B$ or $\{A, B\} = \{L_0, L_\infty\}$ (cf. the proof of Proposition 3.8).*

(ii) *For $A, B \in \overline{\mathcal{H}}_F(I)$, $\overline{\psi}(A) = \overline{\psi}(B)$ if and only if one of the cases (a)–(d) in the proof of Proposition 3.9 holds.*

Proof of Assertion 3. Indeed, the "if" part of (i) and (ii) are directly verified. To prove the "only if" part of (i), assume that $A, B \in \overline{\mathcal{H}}_F(I)$ and $\overline{\psi}(A) = \overline{\psi}(B)$.

Then, by definition, we have $A \cap (0,1)^2 = B \cap (0,1)^2$. Therefore, if $A, B \in \overline{\mathcal{H}}_{+,F}(I)$, by the observation made in the proof of Proposition 3.8, we have $A = B$ or $\{A, B\} = \{L_0, L_\infty\}$. By the reasoning as in the proof of Proposition 3.9, we can prove the "only if" part of (ii) in a similar way. $\square$

Using (i) and (ii) in Assertion 3, we can obtain the required homeomorphisms exactly as in the proof of Propositions 3.8 and 3.9. $\square$

3.3. **Compactification of the homeomorphism groups of the circle.** Here we consider the homeomorphism group of the circle $\mathbb{T}$. Let $q\colon \mathbb{R} \to \mathbb{T}$ be the covering projection $q(x) = e^{2\pi i x}$ and let $\varpi = q \times q\colon \mathbb{R}^2 \to \mathbb{T}^2$. For $u, v \in \mathbb{T}$, we define a homeomorphism $\mu_{u,v}\colon \mathbb{T}^2 \to \mathbb{T}^2$ by $\mu_{u,v}(z, w) = (uz, vw)$. There is a natural homeomorphism $\varphi\colon \mathcal{H}_+(I) \times \mathbb{T} \to \mathcal{H}_+(\mathbb{T})$ defined by

$$\varphi(h, u)(q(s)) = uq(h(s)).$$

An alternative definition of φ is

$$\varphi(h, u) = \mu_{1,u}(\varpi(h)),$$

where h is identified with its graph. However, a natural extension $\overline{\varphi}\colon \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ of φ defined by

$$\overline{\varphi}(A, u) = \mu_{1,u}(\varpi(A))$$

is not injective. For example, we have $\overline{\varphi}(L_0, 1) = \overline{\varphi}(L_\infty, 1)$. More precisely, we can prove the following fact about the map $\overline{\varphi}$:

Lemma 3.11. *The map $\overline{\varphi}\colon \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ defined by $\overline{\varphi}(A, u) = \mu_{1,u}(\varpi(A))$ is continuous and surjective. Moreover, each fiber of $\overline{\varphi}$ is either a singleton or an arc.*

To prove this lemma, we introduce some notation. For $A \in \overline{\mathcal{H}}_{+,F}(I)$, let $k(A) = \max A(0)$ and $l(A) = 1 - \min A(1)$. For $A \in \overline{\mathcal{H}}_{+,F}(I)$ and t such that $-k(A) \leqq t \leqq l(A)$, define $A[t] \in \overline{\mathcal{H}}_{+,F}(I)$ by

$$A[t] = \big(\{0\} \times [0, k(A)+t]\big) \cup \Big(\big(A \setminus (\{0,1\} \times I)\big) + t\mathbf{e}_2\Big) \cup \big(\{1\} \times [1-l(A)+t, 1]\big),$$

where $\mathbf{e}_2$ denotes the unit vector $(0,1) \in \mathbb{R}^2$.

Lemma 3.12. *Let $(A,u), (B,v) \in \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}$. Then, $\overline{\varphi}(B,v) = \overline{\varphi}(A,u)$ if and only if there exists $t \in [-k(A), l(A)]$ such that $B = A[t]$ and $q(t) = v^{-1}u$.*

First we prove Lemma 3.11 assuming Lemma 3.12.

Proof of Lemma 3.11 asuuming Lemma 3.12. Since $\varpi|_{I^2} \colon I^2 \to \mathbb{T}^2$ is a perfect map, $\overline{\varphi}$ is continuous by Lemma 2.4. The surjectivity of $\overline{\varphi}$ follows from the compactness of $\overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}$.

By Lemma 3.12, the fiber of $\overline{\varphi}$ through $(A,u) \in \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}$ is

$$F_{A,u} = \{(A[t], uq(-t))\,;\, -k(A) \leqq t \leqq l(A)\}.$$

Notice the following facts, each of which can easily be verified:

- $A[s] \neq A[t]$ whenever $s \neq t$,
- The map $[-k(A), l(A)] \ni t \mapsto A[t] \in \overline{\mathcal{H}}_{+,F}(I)$ is continuous for each $A \in \overline{\mathcal{H}}_{+,F}(I)$.

By these facts, the map $[-k(A), l(A)] \ni t \mapsto (A[t], uq(-t)) \in F_{A,u}$ is a homeomorphism. It follows that the fiber $F_{A,u}$ is a singleton if $k(A) = l(A) = 0$ and otherwise an arc. $\square$

We prove the remaining Lemma 3.12.

Proof of Lemma 3.12. Take any (A, u) and (B, v) satisfying

$$(\sharp) \qquad \overline{\varphi}(A, u) = \overline{\varphi}(B, v).$$

By Lemma 3.1, there are at most countably many $x \in I$ for which $A(x)$ or $B(x)$ is not a singleton. Therefore, there exists $0 < x_0 < 1$ such that we can write $A(x_0) = \{a_0\}$ and $B(x_0) = \{b_0\}$. We define $t_0 = b_0 - a_0$. By ($\sharp$) and the definition of $\overline{\varphi}$, for each $0 < x < 1$, we have

$$(\natural) \qquad v^{-1}uq(A(x)) = q(B(x)) \subset \mathbb{T}.$$

Take any $0 < x < 1$. Since each of $A(x)$ and $B(x)$ is a singleton or an interval of length $\leqq 1$, the equality ($\natural$) shows that $A(x)$ and $B(x)$ are intervals of the same length (where a singleton is regarded as an interval of length zero). Thus, there exists a unique $t(x) \in \mathbb{R}$ such that $B(x) = A(x) + t(x)$. Letting $x = x_0$, we have $t(x_0) = t_0$. It remains to show that $t_0 \in [-k(A), l(A)]$, $B = A[t_0]$ and $q(t_0) = v^{-1}u$.

Assertion 4. *For each $0 < x < 1$, we have $q(t(x)) = v^{-1}u$. In particular, $q(t_0) = v^{-1}u$.*

Proof of Assertion 4. Let us note that for each arc $A \subset \mathbb{T}$ we can define a preferred endpoint of A, by choosing the endpoint which coincides with $q(\sup \tilde{A})$ for some closed interval $\tilde{A} \subset \mathbb{R}$ such that $q|_{\tilde{A}}$ gives a homeomorphism from $\tilde{A}$ onto A.

Take any $0 < x < 1$. If $A(x)$ has the length < 1 (and so is $B(x)$), both sides of ($\natural$) are an arc in $\mathbb{T}$, whose preferred endpoint is $v^{-1}uq(\max A(x)) = q(\max B(x)) = q(\max A(x) + t(x))$, which in turn means $v^{-1}u = q(t(x))$. If $A(x)$ has the length $= 1$, then $x \neq x_0$, $A(x) = B(x) = I$, $t(x) = 0$ and

$A = B = L_{0,x}$. Then, we have $\{v^{-1}u\} = v^{-1}uq(A(x_0)) = q(B(x_0)) = \{1\}$ and hence $v^{-1}u = 1 = q(t(x))$. $\square$

Assertion 4 means that the open interval $(0, 1)$ can be expressed as the disjoint union of $I_n = \{x \in (0,1)\,;\, t(x) = t_0 + n\}\,(n \in \mathbb{Z})$. (We easily see that $I_n = \emptyset$ at least for $|n| > 2$, but we do not need this fact).

Assertion 5. *For each $n \in \mathbb{Z}$, the set I_n is open.*

Proof of Assertion 5. Take any $x \in I_n$ and express $A(x)$ and $B(x)$ as $A(x) = [a, a']$ and $B(x) = [b, b']$. Then, $b = a + t_0 + n$ and $b' = a' + t_0 + n$. By Lemma 3.4, we can take $\delta > 0$ such that, for any x' with $x - \delta < x' < x$ we have

$$A(x') \subset [a - 1/2, a],\ B(x') \subset [b - 1/2, b], \tag{†}$$

and for any x' with $x < x' < x + \delta$ we have

$$A(x') \subset [a', a' + 1/2],\ B(x') \subset [b', b' + 1/2]. \tag{‡}$$

Take any x' with $|x' - x| < \delta$. If $x' < x$, then by (†) and $A(x') = B(x') + t(x')$, we have $|t(x') - t(x)| = |t(x') - (b - a)| \leqq 1/2 < 1$. As $I = \bigcup_{m\in\mathbb{Z}} I_m$, we see that $|t(x') - t(x)|$ must be an integer and hence is 0, which means $t(x') = t(x)$. If $x < x'$, we similarly obtain $t(x') = t(x)$ using (‡). Consequently $t(x') = t(x) = t_0 + n$, in other words $x' \in I_n$, for all x' with $|x' - x| < \delta$. $\square$

By Assertion 5, the open interval is expressed as the disjoint union of open sets $I_n\,(n \in \mathbb{Z})$. Since $(0, 1)$ is connected and $x_0 \in I_0$, we have $I = I_0$. This means that

$$B \setminus (\{0,1\} \times I) = (A + t_0\mathbf{e}_2) \setminus (\{0,1\} \times I), \tag{★}$$

where $\mathbf{e}_2$ denotes the unit vector $(0,1) \in \mathbb{R}^2$. Then, we have (using Lemma 3.4),

$$\begin{aligned} k(A) + t_0 &= \inf \mathrm{pr}_2(A \cap ((0,1) \times I)) + t_0 \\ &= \inf \mathrm{pr}_2(B \cap ((0,1) \times I)) \\ &\geqq 0, \end{aligned}$$

and hence $-k(A) \leqq t_0$. Similarly, we obtain $t_0 \leqq l(A)$ and thus $t_0 \in [-k(A), l(A)]$. Thus we can define $A[t_0] \in \overline{\mathcal{H}}_{+,F}(I)$. By the equality (★) and Lemma 3.1, we have $B = A[t_0]$. This concludes the proof of Lemma 3.12. □

Proposition 3.13. $(\overline{\mathcal{H}}_{+,F}(\mathbb{T}), \mathcal{H}_+(\mathbb{T})) \approx (Q \times \mathbb{T}, s \times \mathbb{T})$.

Our method to prove Proposition 3.13 is a modification of the proof of Proposition 3.6 in Sakai-Uehara [17, Theorem 4].

We shall use the following characterization of cap-sets in a compact Q-manifold M, which is essentially proved in Chapman [7, Lemma 8.1], where the statement is restricted to the case $M = Q$.

Theorem 3.14. *Let M be a compact Q-manifold. Fix a compatible metric d on M. Then a subset X of M is a cap-set in M if and only if X is a Z_σ-set in M and satisfies the following condition:*

> ($\flat$) *Given compact sets $B \subset A \subset M$ with $B \subset X$ and $\varepsilon > 0$, there exists an embedding $h\colon A \to X$ such that $h|_B = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$.*

Proof. The proof is almost the same as [7, Lemma 8.1]. We have only to make the following modification: In the last line of the proof of [7, Lemma 8.1], use "Theorem 6.6" instead of "Lemma 4.2". □

A subset Y of a space X is called *homotopy dense* if there is a homotopy $h\colon X\times I\to X$ such that $h(X\times(0,1])\subset Y$. An important property of homotopy dense sets is the following, which can be derived from a characterization of ANR's (Hu [11, Chapter IV, Theorem 6.3], see also Banakh-Radul-Zarichnyi [4, §1.2, Exercise 16]):

Lemma 3.15. *If Y is a homotopy dense subset of X, then Y is an ANR if and only if X is an ANR.* □

Lemma 3.16. *$\mathcal{H}_+(\mathbb{T})$ is homotopy dense in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$.*

Proof. We shall define the desired homotopy $h\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}))\times I\to\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ by the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{H}}_{+,F}(I)\times\mathbb{T}\times I \arrow[d, "\overline{\varphi}\times\mathrm{id}"'] \arrow[dr, "h'"] & \\
\overline{\mathcal{H}}_{+,F}(\mathbb{T})\times I \arrow[r, "h"'] & \overline{\mathcal{H}}_{+,F}(\mathbb{T})
\end{tikzcd}$$

Notice that the vertical arrow in this diagram is a quotient map. We define $m\colon \overline{\mathcal{H}}_{+,F}(I)\to[0,1]$ by $m(A)=\max\{y-x\,;\,(x,y)\in A\}$. It is easy to see that m is continuous.

We define an isotopy $(\gamma_t)_{t\in I}$ of $\mathbb{R}^2$ by $\gamma_t(x,y)=(1-t/3)(x,y)+(t/3)(y,x)$. Define $h''\colon \overline{\mathcal{H}}_{+,F}(I)\times I\to \mathrm{Cld}^*_F(\mathbb{R}^2)$ by $h''(A,t)=\gamma_t(A-m(A)\mathbf{e}_2)+m(A)\mathbf{e}_2$, where $\mathbf{e}_2$ denotes the unit vector $(0,1)\in\mathbb{R}^2$. Notice that for each $t\in(0,1]$, the set $h''(A,t)$ is (the graph of) a homeomophism from $[-(t/3)m(A),1-(t/3)m(A)]$ onto $[(t/3)m(A),1+(t/3)m(A)]$ (see Figure 2 and Sakai-Uehara [17, Lemma 3]).

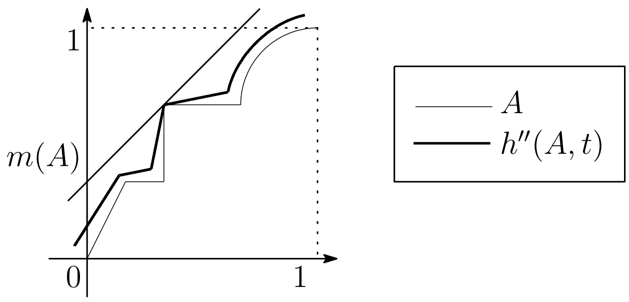

FIGURE 2. Definition of h''

Then we define $h' \colon \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \times I \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ by

$$h'(A,u,t) = \mu_{1,u}(\varpi(h''(A,t))) \in \mathrm{Cld}^*_F(\mathbb{T} \times \mathbb{T}).$$

It is easy to see that $h'(A,u,0) = \overline{\varphi}(A,u) \in \overline{\mathcal{H}}_+(\mathbb{T})$ and that $h'(A,u,t) \in \mathcal{H}_+(\mathbb{T})$ for each $t \in (0,1]$. By using Lemma 3.12, we can check that h' factors through $\overline{\mathcal{H}}_{+,F}(\mathbb{T}) \times I$ as in the last diagram. To this end, notice the following fact: Under the notation in Lemma 3.12, we have $m(A[t]) = m(A) + t$ for each $A \in \overline{\mathcal{H}}_{+,F}(I)$ and $t \in [-k(A), l(A)]$.

Then the induced map $h \colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \times I \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ is the desired homotopy. □

Lemma 3.17. *There exists a homotopy $f \colon \mathcal{H}_+(\mathbb{T}) \times I \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ such that*

- $f_0 = \mathrm{id}$,
- *f is injective,*
- $f_t(\mathcal{H}_+(\mathbb{T})) \subset \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \setminus \mathcal{H}_+(\mathbb{T})$ *for each* $t > 0$.

Proof. Given any $h \in \mathcal{H}_+(\mathbb{T})$ and $t \in I$, choose a lift $\tilde{h} \colon \mathbb{R} \to \mathbb{R}$ with respect to the covering projection $q \colon \mathbb{R} \to \mathbb{T}$. Then we define a closed set $H \subset \mathbb{R}^2$, which is regarded as a set-valued function, by

$$H(x)=\begin{cases}\left\{\tilde{h}(n)+(1-t/2)\left(\tilde{h}(n+(1-t/2)^{-1}(x-n))-\tilde{h}(n)\right)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n\leqq x\leqq n+(1-t/2) \text{ and } n\in\mathbb{Z}, \\ \{\tilde{h}(n+1)-t/2\} \qquad\qquad\quad \text{if } n+(1-t/2)\leqq x<n+1 \text{ and } n\in\mathbb{Z}, \\ [\tilde{h}(x)-t/2,\tilde{h}(x)] \qquad\qquad\; \text{if } x\in\mathbb{Z}.\end{cases}$$

We define $f(h,t)=\varpi(H)$, where $\varpi=q\times q\colon \mathbb{R}^2\to\mathbb{T}^2$. Then $f(h,t)$ belongs to $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ and does not depend on the choice of the lift of $\tilde{h}$. Then, $f\colon \mathcal{H}_+(\mathbb{T})\times I\to\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ satisfies the requirements. $\square$

Now we can prove Proposition 3.13. A continuous surjective map $f\colon X\to Y$ between compact spaces is a *cell-like map* if for every $y\in Y$ the fiber $f^{-1}(y)$ is contractible in every neighborhood in X.

Proof of Proposition 3.13. We know that $\mathcal{H}_+(\mathbb{T})\approx\mathcal{H}_+(I)\times\mathbb{T}\approx s\times\mathbb{T}$ by Proposition 3.6 (1). Notice that $s\times\mathbb{T}$ is an ANR. Thus, by Lemmas 3.16 and 3.15, $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ is an ANR. Since $\overline{\mathcal{H}}_{+,F}(I)\approx Q$ by Proposition 3.6, $\overline{\mathcal{H}}_{+,F}(I)\times\mathbb{T}$ is a Q-manifold. On the other hand, Lemmas 3.11 shows that the map $\overline{\varphi}\colon\overline{\mathcal{H}}_{+,F}(I)\times\mathbb{T}\to\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ is a cell-like map. Therefore, by Toruńczyk's approximation theorem ([18], see also [13, Theorem 7.5.7]), $\overline{\varphi}$ can be approximated by homeomorphisms. In particular, $\overline{\mathcal{H}}_{+,F}(\mathbb{T})\approx\overline{\mathcal{H}}_{+,F}(I)\times\mathbb{T}\approx Q\times\mathbb{T}$ by Proposition 3.6 (1).

Now, by Theorem 3.14, it suffices to prove that $X=\overline{\mathcal{H}}_{+,F}(\mathbb{T})\setminus\mathcal{H}_+(\mathbb{T})$ satisfies the condition ($\flat$) in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. Fix an admissible metric d on $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. Let $B\subset A\subset\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ be compact sets satisfying $B\subset X$ and let $\varepsilon>0$. By Lemma 3.16, we can construct a continuous map $h_0\colon A\to\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ such that $d(h_0,\mathrm{id})<\varepsilon/3$ and $h_0(A\setminus B)\subset\mathcal{H}_+(\mathbb{T})$ and $h_0|_B=\mathrm{id}$. Since $\mathcal{H}_+(\mathbb{T})\approx s\times\mathbb{T}$ is a s-manifold and $B\setminus A$ is completely metrizable, there exists an embedding

$h_1\colon A \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ such that $h_1|_B = \mathrm{id}$ and $h_1|_{A\setminus B}$ is a closed embedding into $\mathcal{H}_+(\mathbb{T})$ satisfying $d(h_1, h_0) < \varepsilon/3$ ([19, Proposition 2.1]). Applying Lemma 3.17, we can construct an embedding $h_2\colon A \to X$ such that $h_2|_B = h_1$ and $h_2(A) \subset X$ and $d(h_2, h_1) < \varepsilon/3$. Then $d(h_2, \mathrm{id}) < \varepsilon$ and $h_2|_B = \mathrm{id}$. This means that X satisfies the condition ($\flat$). $\square$

To describe the topological type of the compatification $\overline{\mathcal{H}}_F(\mathbb{T})$ of the whole homeomorphism group $\mathcal{H}(\mathbb{T})$ of the circle, we introduce the following notation. Let $\mathbb{D}^2 = \{z \in \mathbb{C}\,;\, |z| \leqq 1\}$ be the unit closed disk with the boundary $\mathbb{T}$. For notational convenience, we shall often replace (Q, s) by (Q', s'), where $Q' = Q \times \mathbb{D}^2$ and $s' = s \times (\mathbb{D}^2 \setminus \mathbb{T})$. Let T_0 denote the torus $\{0\} \times \mathbb{T} \times \mathbb{T}$ contained in $Q' \times \mathbb{T}$.

Theorem 3.18. $(\overline{\mathcal{H}}_F(\mathbb{T}), \mathcal{H}(\mathbb{T})) \approx (Q' \times \mathbb{T} \cup_H Q' \times \mathbb{T}, s' \times \mathbb{T} \sqcup s' \times \mathbb{T})$*, where* $H\colon T_0 \to T_0$ *is the homeomorphism given by* $H(0, u, v) = (0, u, u^{-2}v^{-1})$.

In the above, $Q' \times \mathbb{T} \cup_H Q' \times \mathbb{T}$ is formally defined as the quotient space of $Q' \times \mathbb{T} \times \{1, -1\}$ with respect to an equivalence relation $\sim$ generated by $(x, 1) \sim (H(x), -1)$, $x \in T_0$. The image of $s' \times \mathbb{T} \times \{1, -1\}$ in this quotient space is denoted by $s' \times \mathbb{T} \sqcup s' \times \mathbb{T}$. The proof of this theorem is split into several lemmas.

For any subset $S \subset \mathbb{T}^2$ we define $R'(S) \subset \mathbb{T}^2$ by $R'(S) = \{(z, w) \in \mathbb{T}^2\,;\, (z, w^{-1}) \in S\}$. Then $\mathcal{H}(\mathbb{T})$ is the disjoint union of $\mathcal{H}_+(\mathbb{T})$ and $R'(\mathcal{H}_+(\mathbb{T}))$. We have $\overline{\mathcal{H}}_F(\mathbb{T}) = \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \cup R'(\overline{\mathcal{H}}_{+,F}(\mathbb{T}))$, but $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ and $R(\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ intersect in a topological torus as shown in next lemma. For $z, w \in \mathbb{T}^2$, we define

$$C_{z,w} = (\{z\} \times \mathbb{T}) \cup (\mathbb{T} \times \{w\}).$$

Lemma 3.19. *Suppose* $A, B \in \overline{\mathcal{H}}_{+,F}(\mathbb{T})$. *Then* $A = R'(B)$ *holds if and only if there exist* $z, w \in \mathbb{T}$ *such that* $A = C_{z,w}$ *and* $B = C_{z,w^{-1}}$.

Proof. The "if" part is trivial. We shall prove the "only if" part. Suppose $A, B \in \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ and $A = R'(B)$. Clearly, it suffices to show that A can be written as $A = C_{z,w}$ for some $z, w \in \mathbb{T}$. By Lemma 3.11, there exist $(\tilde{A}, u), (\tilde{B}, v) \in \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}$ such that $\overline{\varphi}(\tilde{A}, u) = A$ and $\overline{\varphi}(\tilde{B}, v) = B$. Then it is enough to show that $\tilde{A} = L_{0,t}$ for some $t \in I$ (see §3.2 for definition). The proof of this fact is easily reduced to show that there do *not* exist $0 < x_0 < x_1 < x_2 < 1$ and $0 < y_0 < y_1 < y_2 < 1$ such that $(x_i, y_i) \in \tilde{A}\,(i = 0, 1, 2)$, in view of Lemma 3.3.

Suppose such $x_i, y_i\,(i = 0, 1, 2)$ exist. Since $\overline{\varphi}(\tilde{A}, u) = A$, we see that $(q(x_i), uq(y_i)) \in A$. Since $B = R'(A)$, it follows that $(q(x_i), u^{-1}q(1 - y_i)) \in B\,(i = 0, 1, 2)$. We see from $B = \overline{\varphi}(\tilde{B}, v)$ that there exist $y'_i \in [0, 1]$ such that $(x_i, y'_i) \in \tilde{B}$ and $vq(y'_i) = u^{-1}q(1-y_i)\,(i = 0, 1, 2)$. It follows that $q(1-y_i-y'_i) = uv\,(i = 0, 1, 2)$. Thus $d_{01} = (y_0 + y'_0) - (y_1 + y'_1)$, $d_{02} = (y_0 + y'_0) - (y_2 + y'_2)$ and $d_{12} = (y_1 + y'_1) - (y_2 + y'_2)$ are integers. Clearly, we have $d_{01} + d_{12} = d_{02}$. Since $0 < y_0+y'_0, y_1+y'_1 < 2$ and $d_{01} = (y_0-y_1)+(y'_0-y'_1) < 0+1 = 1$, either $d_{01} = 0$ or $d_{01} = -1$ holds. If $d_{01} = 0$, we have $y'_1 < y'_0$. This contradicts Lemma 3.1, since $x_0 < x_1$, $(x_i, y'_i) \in \tilde{B}\,(i = 0, 1)$ and $\tilde{B} \in \overline{\mathcal{H}}_{+,F}(I)$. Thus we have $d_{01} = -1$. Similarly we obtain $d_{02} = d_{12} = -1$, and hence $-2 = d_{01} + d_{12} = d_{02} = -1$, which is a contradiction. □

Define $\mathcal{C} = \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \cap R'(\overline{\mathcal{H}}_{+,F}(\mathbb{T}))$. By Lemma 3.19, we have

$$\mathcal{C} = \{C_{z,w}\,;\, z, w \in \mathbb{T}\}(\approx \mathbb{T}^2),$$

which is a topological torus. Clearly, $R'(\mathcal{C}) = \mathcal{C}$ and $R'(C_{z,w}) = C_{z,w^{-1}}$ $(z, w \in \mathbb{T})$.

Lemma 3.20. *There exists a homeomorphism* $h\colon (\overline{\mathcal{H}}_{+,F}(\mathbb{T}), \mathcal{C}) \to (Q' \times \mathbb{T}, T_0)$ *such that* $h \circ R' \circ h^{-1}|_{T_0}$ *is given by*

$$(\diamondsuit) \qquad h \circ R' \circ h^{-1}(0, q(t), v) = \begin{cases} (0, q(t), v^{-1}q(-4t)) & \text{if } 0 \leqq t \leqq 1/2, \\ (0, q(t), v^{-1}) & \text{if } 1/2 \leqq t \leqq 1, \end{cases}$$

where $t \in I$ *and* $v \in \mathbb{T}$.

Proof. We have seen that $\overline{\mathcal{H}}_{+,F}(\mathbb{T}) \approx Q \times \mathbb{T} \approx Q' \times \mathbb{T}$ in Proposition 3.13. For $z, w \in \mathbb{T}$, we define $D_{z,w} \in \overline{\mathcal{H}}_{+,F}$ by the following: (see Figure 3.)

$$D_{z,w}(u) = \begin{cases} (uz^{-1})^2(-w) & \text{if } uz^{-1} \in q([-1/4, 1/4]), \\ w & \text{otherwise.} \end{cases}$$

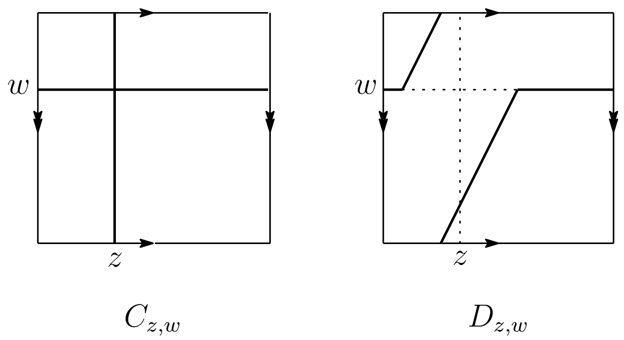

FIGURE 3. $C_{z,w}$ and $D_{z,w}$

Let $\mathcal{D} = \{D_{z,w}\,;\, z, w \in \mathbb{T}\}$. Then $f_1\colon \mathcal{C} \to \mathcal{D}$ defined by $f_1(C_{z,w}) = D_{z,w}$ is a homeomorphism, which is homotopic to $\mathrm{id}_{\mathcal{C}}$ in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. Indeed, there is a homotopy $f_t\colon \mathcal{C} \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ $(t \in I)$ defined by $f_0 = \mathrm{id}_{\mathcal{C}}$ and

$$f_t(C_{z,w})(u) = \begin{cases} (uz^{-1})^{2t}(-w) & \text{if } uz^{-1} \in q([-t/4, t/4]), \\ w & \text{otherwise,} \end{cases}$$

for $t > 0$, which connects $\mathrm{id}_{\mathcal{C}}$ to f_1. By Proposition 3.13, $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ is a Q-manifold. By Proposition 3.6, both $\mathcal{C}$ and $\mathcal{D}$ are Z-sets in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ since $\mathcal{C}, \mathcal{D} \subset \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \setminus \overline{\mathcal{H}}_{+}(\mathbb{T})$. Therefore by Theorem 2.5, there is a homeomorphism $h_1 \colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ extending f_1. We define an embedding $\gamma \colon \mathbb{T} \to \overline{\mathcal{H}}_{+,F}(I) \setminus \mathcal{H}_{+}(I)$ by the following: (see Figure 4.)

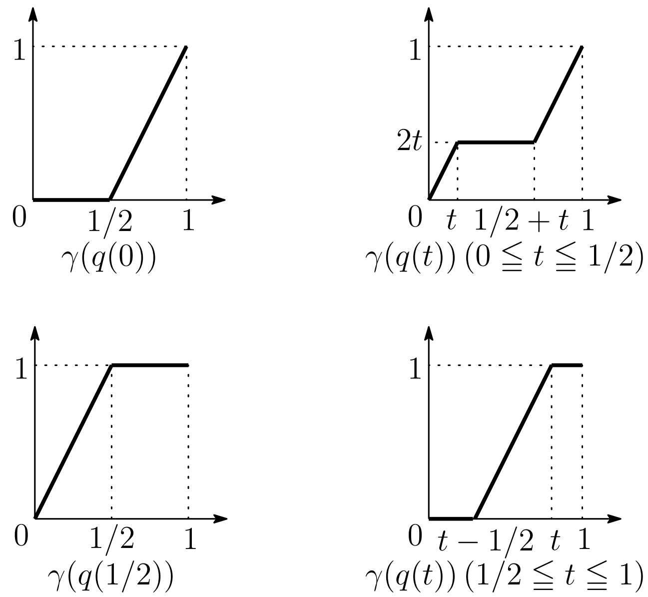

FIGURE 4. Definition of $\gamma \colon \mathbb{T} \to \overline{\mathcal{H}}_{+,F} \setminus \mathcal{H}_{+}(I)$

$$\gamma(q(t))(s) = \begin{cases} 2s & \text{if } 0 \leqq t \leqq 1/2 \text{ and } 0 \leqq s \leqq t, \\ 2t & \text{if } 0 \leqq t \leqq 1/2 \text{ and } t \leqq s \leqq 1/2 + t, \\ 2s - 1 & \text{if } 0 \leqq t \leqq 1/2 \text{ and } 1/2 + t \leqq s \leqq 1, \\ 0 & \text{if } 1/2 \leqq t \leqq 1 \text{ and } 0 \leqq s \leqq t - 1/2, \\ 2s + 2t - 1 & \text{if } 1/2 \leqq t \leqq 1 \text{ and } t - 1/2 \leqq s \leqq t, \\ 1 & \text{if } 1/2 \leqq t \leqq 1 \text{ and } t \leqq s \leqq 1. \end{cases}$$

By the definition of $\overline{\varphi}$, we have $\overline{\varphi}^{-1}(\mathcal{D}) = \gamma(\mathbb{T}) \times \mathbb{T}$ and $\overline{\varphi}|_{\gamma(\mathbb{T}) \times \mathbb{T}} : \gamma(\mathbb{T}) \times \mathbb{T} \to \mathcal{D}$ is a homeomorphism.

By Propositions 2.7 and 3.6 (1), there is a homeomorphism $h_2' \colon \overline{\mathcal{H}}_{+,F}(I) \to Q'$ such that $h_2'(\gamma(u)) = (0, u) \in Q \times \mathbb{D}^2 = Q'$ for each $u \in \mathbb{T}$. Then $h_2 =$

$h_2' \times \mathrm{id}_{\mathbb{T}}$ gives a homeomorphism $(\overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T}, \gamma(\mathbb{T}) \times \mathbb{T}) \approx (Q' \times \mathbb{T}, T_0)$. As noted in the proof of Proposition 3.13, $\overline{\varphi}\colon \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ can be approximated by homeomorphisms. Choose a homeomorphism $\overline{\varphi}'\colon \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ sufficiently closely approximating $\overline{\varphi}$ so that $\overline{\varphi}' \circ \overline{\varphi}^{-1}|_{\mathcal{D}}\colon \mathcal{D} \to \overline{\varphi}'(\gamma(\mathbb{T}) \times \mathbb{T})$ is homotopic to $\mathrm{id}_{\mathcal{D}}$ in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. Since $\gamma(\mathbb{T}) \subset \overline{\mathcal{H}}_{+,F}(I) \setminus \mathcal{H}_+(I)$, by Proposition 3.6, $\overline{\varphi}'(\gamma(\mathbb{T}) \times \mathbb{T})$ is a Z-set in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. By Theorem 2.5, we have a homeomorphism $h_3\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ with $h_3|_{\mathcal{D}} = \overline{\varphi}' \circ \overline{\varphi}^{-1}|_{\mathcal{D}}$. Consequently, we have obtained homeomorphisms as in the following diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "f_1"', "\approx"] & \subset & \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \arrow[d, "\approx"', "h_1"] \\
\mathcal{D} \arrow[d, "\overline{\varphi}' \circ \overline{\varphi}^{-1}"', "\approx"] & \subset & \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \arrow[d, "\approx"', "h_3"] \\
\overline{\varphi}'(\gamma(\mathbb{T}) \times \mathbb{T}) \arrow[d, "\overline{\varphi}'^{-1}"', "\approx"] & \subset & \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \arrow[d, "\approx"', "\overline{\varphi}'^{-1}"] \\
\gamma(\mathbb{T}) \times \mathbb{T} \arrow[d, "\gamma^{-1} \times \mathrm{id}"', "\approx"] & \subset & \overline{\mathcal{H}}_{+,F}(I) \times \mathbb{T} \arrow[d, "\approx"', "h_2"] \\
T_0 & \subset & Q' \times \mathbb{T}
\end{tikzcd}$$

We define $h\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to Q' \times \mathbb{T}$ as the composition of the right column in the diagram, that is, $h = h_2 \circ \overline{\varphi}'^{-1} \circ h_3 \circ h_1$.

Then we compute

$$\begin{aligned}
h^{-1}(0, q(t), v) &= f_1^{-1}(\overline{\varphi}(\gamma(q(t)), v)) \\
&= \begin{cases} C_{q(t-1/4), vq(2t)} & \text{if } 0 \leqq t \leqq 1/2, \\ C_{q(t-1/4), v} & \text{if } 1/2 \leqq t \leqq 1. \end{cases}
\end{aligned}$$

Since $R'(C_{z,w}) = C_{z,w^{-1}}$ for $z, w \in \mathbb{C}$, we obtain

$$R' \circ h^{-1}(0, q(t), v) = \begin{cases} C_{q(t-1/4), v^{-1}q(-2t)} & \text{if } 0 \leqq t \leqq 1/2, \\ C_{q(t-1/4), v^{-1}} & \text{if } 1/2 \leqq t \leqq 1, \end{cases}$$
$$= \begin{cases} h^{-1}(0, q(t), v^{-1}q(-4t)) & \text{if } 0 \leqq t \leqq 1/2, \\ h^{-1}(0, q(t), v^{-1}) & \text{if } 1/2 \leqq t \leqq 1, \end{cases}$$

which means the required formula ($\diamondsuit$). $\square$

Now we can complete the proof of Theorem 3.18.

Proof of Theorem 3.18. Let $h\colon (\overline{\mathcal{H}}_{+,F}(\mathbb{T}), \mathcal{C}) \to (Q' \times \mathbb{T}, T_0)$ be the homeomorphism obtained in Lemma 3.20. Then this h may or may not satisfy $h(\mathcal{H}_+(\mathbb{T})) = s' \times \mathbb{T}$. We shall first show that h can be modified to h' so that $h'|_{\mathcal{C}} = h|_{\mathcal{C}}$ and $h'(\mathcal{H}_+(\mathbb{T})) = s' \times \mathbb{T}$.

Since $X = (Q' \setminus s') \times \mathbb{T}$ is a cap-set in $Q' \times \mathbb{T}$ (see §2.1), $X' = h^{-1}((Q' \setminus s') \times \mathbb{T})$ is a cap-set in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$. On the other hand, $X'' = \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \setminus \mathcal{H}_+(\mathbb{T})$ is a cap-set in $\overline{\mathcal{H}}_{+,F}(\mathbb{T})$ by Proposition 3.13. Since $\mathcal{C} \subset X' \cap X''$, we can use Theorem 2.6 to obtain a homeomorphism $h''\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to \overline{\mathcal{H}}_{+,F}(\mathbb{T})$ with $h''(X'') = X'$ and $h''|_{\mathcal{C}} = \mathrm{id}$. Then $h' = h \circ h''\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to Q' \times \mathbb{T}$ satisfies $h'(X'') = (Q' \setminus s') \times \mathbb{T}$, and hence $h'(\mathcal{H}_+(\mathbb{T})) = s' \times \mathbb{T}$. This h' is the desired modification of h, since $h'|_{\mathcal{C}} = h|_{\mathcal{C}}$.

The formula ($\diamondsuit$) shows that $h' \circ R' \circ h'^{-1}|_{T_0} = h \circ R \circ h^{-1}|_{T_0}$ is isotopic to $H\colon T_0 \to T_0$ given in the statement of Theorem 3.18. Thus, $H \circ (h \circ R' \circ h^{-1}|_{T_0})^{-1}$ is a self-homeomorphism of T_0 isotopic to the identity, and hence can be extended to the solid torus $\{0\} \times \mathbb{D}^2 \times \mathbb{T}$. This extension is further extended to a self-homeomorphism H' of $Q \times \mathbb{D}^2 \times \mathbb{T} = Q' \times \mathbb{T}$ satisfying $H'(s' \times \mathbb{T}) = s' \times \mathbb{T}$ in a trivial way.

Then, $h'\colon \overline{\mathcal{H}}_{+,F}(\mathbb{T}) \to Q' \times \mathbb{T}$ and $H' \circ h' \circ R'^{-1}\colon R'(\overline{\mathcal{H}}_{+,F}(\mathbb{T})) \to Q' \times \mathbb{T}$ fit together to give a homeomorphism $\bar{h}\colon \overline{\mathcal{H}}_F(\mathbb{T}) \to Q' \times \mathbb{T} \cup_H Q' \times \mathbb{T}$. Finally, $\bar{h}$ satisfies $\bar{h}(\mathcal{H}(\mathbb{T})) = s' \times \mathbb{T} \sqcup s' \times \mathbb{T}$, since h' satisfies $h'(\mathcal{H}_+(\mathbb{T})) = s' \times \mathbb{T}$. □

4. Compactification of the identity component of the homeomorphism group of a graph

Let Γ be a (countable locally finite) graph. Following Banakh-Mine-Sakai [3] we say that $v \in \Gamma$ is a *topological vertex* of Γ if it has no neighborhood homeomorphic to an open subset of $\mathbb{R}$. The set of all topological vertices of Γ is denoted by $\Gamma^{(0)}$. We write the complement $\Gamma \backslash \Gamma^{(0)}$ as the disjoint union $\bigcup_{\lambda\in\Lambda} E_\lambda$ of connected components $E_\lambda\, (\lambda \in \Lambda)$ and let $X_\lambda = \operatorname{Cl}_\Gamma E_\lambda$ and $F_\lambda = \operatorname{Bd}_\Gamma E_\lambda$ for each $\lambda \in \Lambda$. Observe that for each $\lambda \in \Lambda$, the pair (X_λ, F_λ) is homeomorphic to one of $\mathbb{T}, \mathbb{R}, (\mathbb{T}, 1), (I, \{0, 1\})$ and $([0, 1), 0)$.[5] Consequently the index set Λ is decomposed into $\Lambda_\mathbb{T}, \Lambda_\mathbb{R}, \Lambda_{(\mathbb{T},1)}, \Lambda_{(I,\{0,1\})}$, and $\Lambda_{([0,1),0)}$ according as the topological type of (X_λ, F_λ).

For a space X and $S \subset X$, the group of homeomorphisms of X which are identity on S is denoted by $\mathcal{H}(X, S)$ and its identity component is denoted by $\mathcal{H}_+(X, S)$. The closure of $\mathcal{H}_+(X, S)$ in $\operatorname{Cld}^*_F(X \times X)$ is denoted by $\overline{\mathcal{H}}_{+,F}(X, S)$.

Lemma 4.1. *Let Γ be a countable, locally finite graph and let $(X_\lambda)_{\lambda\in\Lambda}$ and $(F_\lambda)_{\lambda\in\Lambda}$ be defined as above.*

Then we have a homeomorphism

$$\Phi\colon (\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma)) \to \left(\prod_{\lambda\in\Lambda} \overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda), \prod_{\lambda\in\Lambda} \mathcal{H}_+(X_\lambda, F_\lambda)\right)$$

[5] For each space X and $x \in X$, the pair $(X, \emptyset)$ is identified with X and the pair $(X, \{x\})$ is denoted by (X, x).

of pairs defined by $\Phi(B) = ((B \cap (X_\lambda \times X_\lambda))^d)_{\lambda \in \Lambda}$*, where* Y^d *means the derived set of* Y*.*[6]

Proof. Observe that

- $h(X_\lambda) = X_\lambda$ for all $h \in \mathcal{H}_+(\Gamma)$ and that
- $h|_{F_\lambda} = \mathrm{id}$ for all $h \in \mathcal{H}_+(\Gamma)$.

From these facts, it follows that $\Phi(\mathcal{H}_+(\Gamma)) \subset \prod_{\lambda \in \Lambda} \mathcal{H}_+(X_\lambda, F_\lambda)$ and $\Phi(h) = (h|_{X_\lambda})_{\lambda \in \Lambda}$ for each $h \in \mathcal{H}_+(\Gamma)$. Consequently, the restriction $\Phi|_{\mathcal{H}_+(\Gamma)}$ is continuous by the definition of compact-open topology.

Let $\Gamma_*^{(0)}$ denote the set of all isolated points of Γ. The map

$$\Psi\colon \prod_{\lambda \in \Lambda} \overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda) \to \mathrm{Cld}_F^*(X \times X)$$

defined by $\Psi((A_\lambda)_{\lambda \in \Lambda}) = \bigcup_{\lambda \in \Lambda} A_\lambda \cup \{(v, v)\,;\, v \in \Gamma_*^{(0)}\}$ is continuous, by Lemma 2.2. Then it is easy to see that $(\Psi|_{\prod_{\lambda \in \Lambda} \mathcal{H}_+(X_\lambda, F_\lambda)}) \circ (\Phi|_{\mathcal{H}_+(\Gamma)}) = \mathrm{id}_{\mathcal{H}_+(\Gamma)}$, and hence $\Psi(\prod_{\lambda \in \Lambda} \mathcal{H}_+(X_\lambda, F_\lambda)) = \mathcal{H}_+(\Gamma)$. Since $\prod_{\lambda \in \Lambda} \overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda)$ is compact, the map Ψ is a closed map onto $\overline{\mathcal{H}}_{+,F}(\Gamma)$. Now it suffices to prove $\Phi \circ \Psi = \mathrm{id}$. Take any element $(A_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} \overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda)$ and any $\mu \in \Lambda$. Clearly, we have

$$A_\mu \subset \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) \cap (X_\mu \times X_\mu) \subset A_\mu \cup (F_\mu \times F_\mu).$$

Now A_μ has no isolated points, since it is the limit of a sequence of sets with no isolated points in $\mathrm{Cld}_F^*(X_\mu \times X_\mu)$. Since $F_\mu \times F_\mu$ is finite, we have

$$\left(\left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) \cap (X_\mu \times X_\mu) \right)^d = A_\lambda,$$

which proves $\Phi \circ \Psi = \mathrm{id}$. □

[6]The *derived set* of a space Y is the set of all non-isolated points in Y.

Example 4.2. In the definition of Φ in Lemma 4.1, the operation ^{d} cannot be removed. Let Γ be the quotient space of $I \times \{0,1,2\}$, where $\{0,1,2\}$ has the discrete topology, by the equivalence relation $\sim$ defined by

$$(t,i) \sim (s,j) \iff (t,i) = (s,j) \text{ or } t = s \in \{0,1,2\}.$$

Let $\pi\colon I \times \{0,1,2\} \to \Gamma$ be the quotient map and let $v_0 = \pi(0,0)$ and $v_1 = \pi(1,0)$. Then the space Γ is a finite graph with $\Gamma^{(0)} = \{v_0, v_1\}$. The complement $\Gamma \setminus \Gamma^{(0)}$ has three components $E_i = \pi((0,1) \times \{i\})$ $(i = 0,1,2)$ and we have $X_i = \operatorname{Cl} E_i = \pi(I \times \{i\})$ and $F_i = \operatorname{Bd} E_i = \{v_0, v_1\}$ $(i = 0,1,2)$.

Define $\Phi'\colon \overline{\mathcal{H}}_{+,F}(\Gamma) \to \prod_{i=0}^{2} \operatorname{Cld}_F^*(X_i \times X_i)$ by $\Phi'(A) = (A \cap (X_i \times X_i))_{i=0}^2$. (The definition of Φ' is almost the same as that of Φ, except for the absence of the operation ^{d}.) We show that there exists $A \in \overline{\mathcal{H}}_{+,F}(\Gamma)$ such that $\Phi'(A) \notin \prod_{i=0}^{2} \overline{\mathcal{H}}_{+,F}(X_i, F_i)$.

We define $A_i \in \overline{\mathcal{H}}_{+,F}(\Gamma)$ by

$$A = \pi((L_0 \times \{0\}) \cup (\Delta \times \{1,2\}),$$

where $L_0 = (I \times \{0\}) \cup (\{1\} \times I)$ and Δ is the diagonal $\{(t,s) \in I^2\,;\, t = s\} \subset I^2$. Then, $\Phi'(A) = (A_i)_{i=0}^2 \in \prod_{i=0}^{2} \operatorname{Cld}_F^*(X_i \times X_i)$, where

$$A_0 = \pi(L_0 \times \{0\}), \quad A_i = \pi(\Delta \times \{i\}) \cup \{(v_1, v_0)\} \ (i = 1,2).$$

Then, for $i = 1,2$, A_i has an isolated point (v_1, v_0) and hence does not belong to $\overline{\mathcal{H}}_{+,F}(X_i, F_i)$. □

Now we can prove Theorem 1.1:

Proof of Theorem 1.1. For each $\lambda \in \Lambda$, the topological type of the pair

$$(\overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda), \mathcal{H}_+(X_\lambda, F_\lambda))$$

is determined by Propositions 3.6, 3.7, 3.8, and 3.10 as follows:

(♣)
$$(\overline{\mathcal{H}}_{+,F}(X_\lambda, F_\lambda), \mathcal{H}_+(X_\lambda, F_\lambda)) \approx \begin{cases} (Q, s) & \text{if } \lambda \in \Lambda_{(I,\{0,1\})} \cup \Lambda_{([0,1),0)}, \\ (Q, s) \times \mathbb{T} & \text{if } \lambda \in \Lambda_{\mathbb{T}}, \\ (Q/\{\pm\mathbf{1}\}, s) & \text{if } \lambda \in \Lambda_{\mathbb{R}} \cup \Lambda_{(\mathbb{T},1)}. \end{cases}$$

Recall from §1 that o_Γ (resp. s_Γ) is the number of components of Γ that are homeomorphic to $\mathbb{T}$ (resp. *not* homeomorphic to $\mathbb{R}$ or a bouquet), and that l_Γ is the number of components of Γ homeomorphic to $\mathbb{R}$ plus the number of indices $\lambda \in \Lambda$ for which $(X_\lambda, F_\lambda) \approx (\mathbb{T}, 1)$. We observe:

(♡)
$$\begin{aligned} &|\Lambda_{\mathbb{T}}| = o_\Gamma, \\ &|\Lambda_{\mathbb{R}} \cup \Lambda_{(\mathbb{T},1)}| = l_\Gamma, \\ &|\Lambda_{(I,\{0,1\})} \cup \Lambda_{([0,1),0)}| \cdot \aleph_0 = s_\Gamma \cdot \aleph_0. \end{aligned}$$

Finally, by Lemma 4.1, (♣), (♡), $Q^{\aleph_0} = Q$ and $s^{\aleph_0} = s$, we have a homeomorphism

$$(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma)) \approx (Q^{s_\Gamma + o_\Gamma} \times (Q/\{\pm\mathbf{1}\})^{l_\Gamma}, s^{s_\Gamma + o_\Gamma + l_\Gamma}) \times \mathbb{T}^{o_\Gamma}.$$

The proof of Theorem 1.1 is completed. □

To derive Corollary 1.2, we introduce the notion of relative LC^0-ness. We say that a subset Y of a space X is *relatively* LC^0 if for each $x \in X$ and each neighborhood U of x in X there exists a smaller neighborhood V such that every two points in $V \cap Y$ can be joined by a path in $U \cap Y$.

If (M, Y) is a (Q, s)-manifold, then Y must be relatively LC^0 in M. Indeed, in such a case M is locally path-connected and Y is homotopy dense in M.[7]

[7]The homotopy denseness can be derived as follows. (i) The complement $M \setminus Y$ is a cap-set in M ([7, Lemma 5.4, Theorem 6.5]). (ii) $M \times (Q \setminus s)$ is a cap-set in $M \times Q$ ([7, proof

Proof of Corollary 1.2 (1). The "if" part is clear from Theorem 1.1. To show the "only if" part, we prove that $(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma))$ is not a (Q, s)-manifold provided one of the following three cases occur: (i) $s_\Gamma + o_\Gamma = 0$ and $l_\Gamma = 0$, (ii) $o_\Gamma = \aleph_0$, (iii) $l_\Gamma > 0$. If (i) holds, then by Theorem 1.1, $\overline{\mathcal{H}}_{+,F}(\Gamma)$ is merely a point and thus not a Q-manifold. If (ii) holds, then by Theorem 1.1, $\overline{\mathcal{H}}_{+,F}(\Gamma)$ is not locally simply connected and thus not a Q-manifold. Finally if (iii) holds, then by Theorem 1.1 $\mathcal{H}_+(\Gamma)$ is not relatively LC^0 in $\overline{\mathcal{H}}_{+,F}(\Gamma)$. This shows that the pair $(\overline{\mathcal{H}}_{+,F}(\Gamma), \mathcal{H}_+(\Gamma))$ is not a (Q, s)-manifold by the previous paragraph. $\square$

Proof of Corollary 1.2 (2). The proof of the "only if" part is essentially contained in that of (1). To prove the "if" part, assume that $s_\Gamma + o_\Gamma \geqq 1$ and $l_\Gamma + o_\Gamma < \aleph_0$. It is easy to see that the space $Q/\{\pm\mathbf{1}\}$ is an ANR (this is the union of two AR closed sets with the intersection an ANR). Thus, $(Q/\{\pm\mathbf{1}\})^{l_\Gamma}$ is an ANR as a finite product of ANR's. By the same reason, $\mathbb{T}^{o_\Gamma}$ is an ANR and hence $(Q/\{\pm\mathbf{1}\})^{l_\Gamma} \times \mathbb{T}^{o_\Gamma}$ is also an ANR. Since $s_\Gamma + o_\Gamma \geqq 1$, we have $Q^{s_\Gamma + o_\Gamma} \approx Q$. Therefore, by [13, Theorem 7.8.1], the product

$$Q^{s_\Gamma + o_\Gamma} \times ((Q/\{\pm\mathbf{1}\})^{l_\Gamma} \times \mathbb{T}^{o_\Gamma}),$$

is a Q-manifold. By Theorem 1.1, this product is homeomorphic to $\overline{\mathcal{H}}_{+,F}(\Gamma)$. $\square$

of Lemma 5.6]). (iii) M is homeomorphic to $M \times Q$ ([13, Theorem 7.5.6]). By (i)–(iii) and Theorem 2.6, we have $(M, M \setminus Y) \approx (M \times Q, M \times (Q \setminus s))$, that is, $(M, Y) \approx (M \times Q, M \times s)$. Then it is clear that Y is homotopy dense in M.

5. A remark on higher dimensional cases

In this section, for a manifold N, the symbols $\operatorname{Int} N$ and ∂N mean the interior and boundary as a manifold, respectively. Here we prove Theorem 1.4 in a generalized form.

Theorem 5.1. *Let M be a locally compact, locally connected (separable metrizable) space and let $D \subset M$ be a closed subset homeomorphic to the closed n-ball with $n \geqq 2$ such that $\operatorname{Int} D$ is an open set in M. Let $\mathcal{H}$ be a closed subgroup of $\mathcal{H}(M)$ containing all homeomorphisms of M supported on D. Then, $\mathcal{H}$ is not relatively LC^0 (see §4) in the closure $\overline{\mathcal{H}}_F$ of $\mathcal{H}$ in $\mathrm{Cld}^*_F(M \times M)$. In particular, $(\overline{\mathcal{H}}_F, \mathcal{H})$ is not a (Q, s)-manifold.*

Proof of Theorem 5.1. Let $D \subset M$ and $\mathcal{H} \subset \mathcal{H}(M)$ be as in the hypothesis. Let $\mathbf{D} = 100\mathbb{D}^2$ and $\mathbf{T} = \mathbf{D} \times \mathbb{T}^{n-2}$. Fix an embedding $\mathbf{T} \hookrightarrow \operatorname{Int} D$ and we think of $\mathbf{T}$ as a subset of M. Let $\mathcal{H}_\partial(\mathbf{T})$ denote the group of homeomorphisms of $\mathbf{T}$ which is identity on the boundary $\partial\mathbf{T}$. Every element h of $\mathcal{H}_\partial(\mathbf{T})$ can be identified with the homeomorphism of M by extending h via identity. Thus, by our assumption on $\mathcal{H}$, the group $\mathcal{H}_\partial(\mathbf{T})$ can be embedded into $\mathcal{H}$ as a subgroup.

For $m \in \mathbb{N}$, let $\gamma_m \colon [0, \infty) \to \mathbb{R}$ be a continuous map defined by

$$\gamma_m(r) = \begin{cases} 2m\pi & \text{if } 0 \leqq r \leqq 1 + (2m\pi)^{-1} \\ (r-1)^{-1} & \text{if } 1 + (2m\pi)^{-1} \leqq r \leqq 1 + (2\pi)^{-1} \\ 2\pi & \text{if } r \geqq 1 + (2\pi)^{-1}. \end{cases}$$

Then we define $h_m \in \mathcal{H}_\partial(\mathbf{T}) \subset \mathcal{H}$ $(m \in \mathbb{N})$ by

$$h_m(re^{i\theta}, x) = (re^{i(\theta + \gamma_m(r))}, x), \quad r \in [0, 100],\ \theta \in \mathbb{R}, \text{ and } x \in \mathbb{T}^{n-2}.$$

Let us make the following definitions:

$$e = (1, \dots, 1) \in \mathbb{T} \times \cdots \times \mathbb{T} = \mathbb{T}^{n-2},$$
$$T = \{0\} \times \mathbb{T}^{n-2} \subset \mathbf{D} \times \mathbb{T}^{n-2} = \mathbf{T}.$$

In the next assertion, the closed interval $[1/2, 2]$ is regarded as a subset of the real axis in $\mathbb{C}$ (and hence as a subset of the disk $\mathbf{D} = 100\mathbb{D}^2$); for example, $(1/2, e)$ denotes a point in $\mathbf{T} = \mathbf{D} \times \mathbb{T}^{n-2}$. Let $\mathrm{pr}_{\mathbf{D}} \colon \mathbf{T} \to \mathbf{D}$ denote the projection.

Assertion 6. *Let $l, m \in \mathbb{N}$. Suppose that there exists a path $\Phi\colon [0,1] \to \mathcal{H}$ with $\Phi(0) = h_l$, $\Phi(1) = h_m$ such that for each $t \in [0,1]$ we have*

(i) $\Phi(t)([1/2, 2] \times \{e\}) \subset \mathbf{T}$,

(ii) $\Phi(t)([1/2, 2] \times \{e\}) \cap T = \emptyset$,

(iii) $|\,\mathrm{pr}_{\mathbf{D}}(\Phi(t)(2, e)) - 2| < 2$,

(iv) $|\,\mathrm{pr}_{\mathbf{D}}(\Phi(t)(1/2, e)) - 1/2| < 1/2$.

Then, $l = m$ holds.

This assertion will be proved later.

We define $H \in \mathrm{Cld}^*_F(M \times M)$, regarded as a set-valued function, by

$$H(re^{i\theta}, x) = \begin{cases} \{(re^{i\theta}, x)\} & \text{if } r < 1 \text{ or } r \geqq 1 + (2\pi)^{-1} \\ \mathbb{T} \times \{x\} & \text{if } r = 1 \\ \{(re^{i(\theta + (r-1)^{-1})}, x)\} & \text{if } 1 < r \leqq 1 + (2\pi)^{-1}, \end{cases}$$
$$H(p) = \{p\} \quad \text{if } p \in M \setminus \mathbf{T},$$

where $r \in [0, 100]$, $\theta \in \mathbb{R}$ and $x \in \mathbb{T}^{n-2}$. Then H belongs to $\overline{\mathcal{H}}_F \setminus \mathcal{H}$, since it is the limit of the sequence $(h_m)_{m\in\mathbb{N}}$. We define a disk $\Delta \subset \mathbf{T}$ and a submanifold

S of $\mathbf{T}$ by

$$\Delta = (1/4\mathbb{D}^2) \times \{e\},$$
$$S = (1/4\mathbb{D}^2) \times \mathbb{T}^{n-2}.$$

The proof of the next assertion is left to the reader.

Assertion 7. *There exists an open neighborhood $\mathcal{U}$ of H in $\overline{\mathcal{H}}_F$ such that for all $h \in \mathcal{U} \cap \mathcal{H}$ the following (1)–(6) hold:*

(1) $h(\partial S) \cap T = \emptyset$,

(2) $h(\partial \Delta) \subset (\operatorname{Int} D) \setminus T$,

(3) $h|_{\partial\Delta} \colon \partial\Delta \to (\operatorname{Int} D) \setminus T$ *is homotopic in* $(\operatorname{Int} D) \setminus T$ *to the inclusion map,*

(4) $h([1/2, 2] \times \{e\}) \subset \mathbf{T}$,

(5) $|\operatorname{pr}_{\mathbf{D}}(h(2, e)) - 2| < 2$,

(6) $|\operatorname{pr}_{\mathbf{D}}(h(1/2, e)) - 1/2| < 1/2$.

□

The next assertion will also be proved later.

Assertion 8. *Let $\mathcal{U}$ be an open neighborhood of H in $\overline{\mathcal{H}}_F$ satisfying the conditions in Assertion 7. Then, for each $h \in \mathcal{U} \cap \mathcal{H}$ we have*

$$h([1/2, 2] \times \{e\}) \cap T = \emptyset.$$

Then, we can complete the proof of Theorem 5.1 as follows. Choose a neighborhood $\mathcal{U}$ of H as in Assertion 7 and take any neighborhood $\mathcal{V}$ of H in $\overline{\mathcal{H}}_F$ with $\mathcal{V} \subset \mathcal{U}$. Then there exists $l, m \in \mathbb{N}$ with $l \neq m$ such that $h_l, h_m \in \mathcal{V}$. Suppose that h_l and h_m can be joined by a path Φ in $\mathcal{U} \cap \mathcal{H}$. Since $\mathcal{U}$ satisfy the conditions in Assertion 7, the path Φ satisfies the hypotheses (i),(iii) and (iv) in Assertion 6. Furthermore, by Assertion 8, the path Φ satisfies the hypothesis (ii) in Assertion 6. Thus, by Assertion 6, we have $l = m$, which is a contradiction. This means that $\mathcal{H}$ is not relatively LC^0 in $\overline{\mathcal{H}}_F$. The proof of Theorem 5.1 is complete. $\square$

Finally, we prove the remaining two assertions.

Proof of Assertion 6. Let $l, m \in \mathbb{N}$ and let $\Phi\colon [0,1] \to \mathcal{H}$ as in the hypothesis of the assertion. By the condition (i) and (ii), we may define $f\colon [1/2,2] \times [0,1] \to \mathbb{T}$ by

$$f(r,t) = \mathrm{pr}_{\mathbf{D}}(\Phi(t)(r,e))/|\,\mathrm{pr}_{\mathbf{D}}(\Phi(t)(r,e))|.$$

Since $\Phi(0) = h_l$ and $\Phi(1) = h_m$, we have $f(2,0) = f(2,1) = 1 \in \mathbb{T}$. Let $\tilde{f}\colon [1/2,2] \times [0,1] \to \mathbb{R}$ denote the unique lift of f satisfying $\tilde{f}(2,0) = 0$ with respect to the covering projection $q\colon \mathbb{R} \to \mathbb{T}$ defined by $q(x) = e^{2\pi i x}$.

By the condition (iii), we have

$$f(\{2\} \times [0,1]) \subset \{z \in \mathbb{T}\,;\, \mathrm{Re}(z) > 0\}.$$

It follows that $\tilde{f}(\{2\} \times [0,1])$ contains at most one integer, and hence $\tilde{f}(2,1) = 0$. Since $\Phi(0) = h_l$ and $\Phi(1) = h_m$, we have $\tilde{f}(1/2,0) = l$ and $\tilde{f}(1/2,1) = m$ by the definitions of f, h_l and h_m.

By the condition (iv), we also have

$$f(\{1/2\} \times [0,1]) \subset \{z \in \mathbb{T}\,;\, \mathrm{Re}(z) > 0\}.$$

It follows that $\tilde{f}(\{1/2\} \times [0,1])$ contains at most exactly one integer, which means $l = \tilde{f}(1/2,0) = \tilde{f}(1/2,1) = m$. $\square$

Proof of Assertion 8. Take any $h \in \mathcal{U} \cap \mathcal{H}$. By the choice of $\mathcal{U}$, $h|_{\partial\Delta}\colon \partial\Delta \to (\operatorname{Int} D) \setminus T$ is homotopic to the inclusion into $(\operatorname{Int} D) \setminus T$. We observe from a Mayer-Vietoris sequence that $\partial\Delta$ represents a nontrivial homology class in $H_1((\operatorname{Int} D) \setminus T)$. Thus, the inclusion $\partial\Delta \hookrightarrow (\operatorname{Int} D) \setminus T$ is not null-homotopic, and hence by Assertion 7 (3), $h|_{\partial\Delta}$ is also not null-homotopic in $(\operatorname{Int} D) \setminus T$. This implies that $h(\Delta) \cap T \neq \emptyset$.

Since $\Delta \subset S$, we have $h(S) \cap T \neq \emptyset$. Assertion 7 (1) means that

$$h(S) \cap T = h(\operatorname{Int} S) \cap T,$$

and hence $h(S) \cap T$ is closed and open in T. Since T is connected, we have $T \subset h(S)$, which in turn implies $h([1/2, 2] \times \{e\}) \cap T = \emptyset$, since $[1/2, 2] \times \{e\}$ does not intersect $S = 1/4\mathbb{D}^2 \times \mathbb{T}^{n-2}$. $\square$

Proof of Theorem 1.4. This is a special case of Theorem 5.1. $\square$

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