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Stochastic Partial Differential Equations with Two Reflecting Walls

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Abstract. We study stochastic partial differential equations (SPDEs) driven by space-time white noise with two reflecting smooth walls h_1 and h_2. If the solution stays in the open interval $(h_1(x,t), h_2(x,t))$, the dynamics obeys a usual type of SPDEs, and at a point where the value of the solution is h_1 or h_2, we add forces in order to prevent it from exiting the interval $[h_1, h_2]$. We will first show the existence and uniqueness of the solutions, and secondly study the stationary distribution of the dynamics and corresponding Dirichlet forms.

1. Introduction

Stochastic partial differential equations of parabolic type are often considered in the context of fluctuation dispersion phenomena in (non-equilibrium) statistical mechanics. For instance, if there are several distinct pure phases coexist, interfaces are formed and separate them. SPDEs can be regarded as to describe its mesoscopic time evolutions of such (random) interfaces. In this paper, we will consider a case that such an interface is formed in a small rubber hose, namely, the interface never sticks to the walls, which repulse the interface. That is, we will study one-dimensional SPDEs driven by space-time white noise $\dot{W}$ with two reflecting (deterministic) walls h_1 and h_2 assuming they are smooth.

Namely, we will study an SPDE of the following type:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t;u(x,t)) + \dot{W} \tag{1}$$

in $x \in (0,1)$ and $t > 0$ while $h_1(x,t) < u(x,t) < h_2(x,t)$. If $u(x,t)$ hits $h_1(x,t)$ or $h_2(x,t)$, we add additional forces in order to prevent u from

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exiting $[h_1, h_2]$. Such an effect will be expressed by adding extra (unknown) terms ξ and η in (1) which play a similar role to the local time in the usual Skorokhod–Tanaka equation constructing Brownian motions with reflecting boundaries. Such a method was first introduced by Nualart–Pardoux[7]. Their case corresponds to the situation that $h_1 = 0$ and $h_2 = \infty$. In this case they used a comparison argument effectively, namely, the interface with a wall occupies a high position than that without a wall almost surely. Our situation, however, does not have this type of comparison. Therefore we treat the walls one by one. The precise formulation will be given in Section 2 and the existence of the solution and its uniqueness will be proved there. After that we will study its stationary distribution in Section 3. Finally, following Zambotti[11], we will give a Dirichlet form with which our dynamics is associated.

We will put the following assumptions on the coefficients throughout the present paper. The smooth walls $h_i(x,t)$, $i = 1, 2$, are continuous functions satisfying $h_1(0,t) \le a$, $h_1(1,t) \le b$, $h_2(0,t) \ge a$, and $h_2(1,t) \ge b$ for some $a, b \in \mathbb{R}$, and

(H1) $h_1(x,t) < h_2(x,t)$ for $x \in (0,1)$ and $t \ge 0$;

(H2) $\partial h_i/\partial t + \partial^2 h_i/\partial x^2 \in L^2([0,1] \times [0,T])$, where $\partial/\partial t$ and $\partial^2/\partial x^2$ are interpreted as distributions' sense;

(H3) $\frac{\partial}{\partial t} h_i(0,t) = \frac{\partial}{\partial t} h_i(1,t) = 0$ for $t \ge 0$;

(H4) $\frac{\partial}{\partial t}(h_2 - h_1) \ge 0$.

We also assume that an external force $f : [0,1] \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies for every $T > 0$,

(F1) $f(\cdot, \cdot; 0) \in L^2([0,1] \times [0,T])$;

(F2) there exists $K_T > 0$ such that $|f(x,t;z) - f(x,t;\bar{z})| \le K_T |z - \bar{z}|$ for every $x \in [0,1]$ and $t \in [0,T]$.

For the sake of simplicity, $f(x,t;u(x,t))$ will be sometimes abbreviated as $f(u)$.

Before leaving this introduction, we summarize a preliminary result related to evolutionary variational inequalities needed in the paper. We omit the proof since it can be easily done using the arguments in [1, pp. 243–263, pp. 287–290].

Proposition 1.1. *Let* $u_0 \in H^2(0,1)$ *satisfy* $h_1(x,0) \le u_0(x) \le h_2(x,0)$, $u_0(0) = a$, *and* $u_0(1) = b$. *Then there exists a unique* $u \in C(0,T;C([0,1])) \cap L^2(0,T;H^2(0,1))$, $\frac{du}{dt} \in L^2(0,T;L^2(0,1))$, $u(0,t) = a$, *and* $u(1,t) = b$ *such that, for every* $t \in (0,T)$,

$$\left(\frac{\partial u}{\partial t}, v-u\right) + \frac{1}{2}\left(\frac{\partial u}{\partial x}, \frac{\partial (v-u)}{\partial x}\right) + (f(u), v-u) \ge 0 \tag{2}$$

is satisfied for every $v \in H^1$ *with* $h_1(x,t) \le v(x) \le h_2(x,t)$, $v(0) = a$ *and* $v(1) = b$, *where* $(\cdot,\cdot)$ *denotes the usual* L^2*-inner product.*

Remark 1.1. Formally speaking, (2) can be rewritten as

$$(p(u), v-u) := \left(\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + f(u), v-u\right) \ge 0.$$

It means that u obeys $p(u) = 0$ while $h_1 < u < h_2$. When $u = h_1$, since $v - u \ge 0$, we have $p(u) \ge 0$, and when $u = h_2$, we have $p(u) \le 0$.

For the sake of simplicity of notations, we will denote $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ by u_t and Δu, respectively, if there is no possibility of confusions.

2. Existing of the Dynamics

In the present paper, we will consider the following SPDE for a pair of a function and two measures (u,η,ξ):

$$u_t = \frac{1}{2}\Delta u - f(x,t;u(x,t)) + \eta - \xi + \dot{W}, \quad x \in (0,1),\ t > 0, \tag{3}$$

under conditions

$$\begin{cases} u(0,t) = a,\ u(1,t) = b \text{ for } t \ge 0; \\ u(x,0) = u_0(x) \in C([0,1]); \\ h_1(x,t) \le u(x,t) \le h_2(x,t) \text{ for } (x,t) \in [0,1]\times[0,\infty); \\ \displaystyle\int_0^\infty \int_0^1 (u(x,t) - h_1(x,t))\,\eta(dx,dt) \\ \displaystyle\qquad = \int_0^\infty \int_0^1 (h_2(x,t) - u(x,t))\,\xi(dx,dt) = 0. \end{cases}$$

Here the assumptions on f, and h_i $(i = 1, 2)$ are same with those of the previous section.

2.1. Definition of the solution

We call a pair of a function and two measures (u, η, ξ) defined on a filtered probability space $(\Omega, \mathcal{B}, P; \{\mathcal{F}_t\})$ a solution to the SPDE (3) (called a double reflection problem and denoted by $(u_0; a, b; f; h_1, h_2)$) if it satisfies the followings.

(1) $u = \{u(x,t); (x,t) \in [0,1] \times [0,\infty)\}$ is a continuous and adapted (for each $T \geq 0$, $u(\cdot, T)$ is $\mathcal{F}_T$-measurable) function satisfying $h_1(x,t) \leq u(x,t) \leq h_2(x,t)$, $u(0,t) = a$, and $u(1,t) = b$ almost surely.

(2) $\eta(dx, dt)$ and $\xi(dx, dt)$ are positive and adapted (for each $T \geq 0$, $\eta(\cdot, [0,T])$ and $\xi(\cdot, [0,T])$ are $\mathcal{F}_T$-measurable Borel) measures on $(0,1) \times [0,\infty)$ satisfying

$$\eta((\delta, 1-\delta) \times [0,T]) < \infty, \quad \xi((\delta, 1-\delta) \times [0,T]) < \infty$$

for every small $\delta > 0$ and $T > 0$ almost surely.

(3) (u, η, ξ) satisfies the following stochastic integral equation:

$$(4) \quad (u(t), \phi) - (u_0, \phi) = \frac{1}{2}\int_0^t (u(s), \phi'')\, ds - \int_0^t (f(u(s)), \phi)\, ds + \int_0^t \int_0^1 \phi(x)\eta(dx, ds) - \int_0^t \int_0^1 \phi(x)\xi(dx, ds) + \int_0^t (\phi, dW(s))$$

for every $\phi \in C_0^\infty(0,1)$, the set of smooth functions on $[0,1]$ supported on compact subsets in $(0,1)$, and $t > 0$ almost surely, where the last term is a stochastic integral with respect to $\mathcal{F}_t$-adapted white noise process W.

(4) (u, η, ξ) satisfies

$$\int_0^\infty \int_0^1 (u(x,t) - h_1(x,t))\eta(dx, dt) = \int_0^\infty \int_0^1 (h_2(x,t) - u(x,t))\xi(dx, dt) = 0$$

almost surely.

Then the main object of this section is to prove the followin theorem.

THEOREM 2.1. *Under the hypotheses (H1)–(H4), (F1)–(F2), and* u_0 *enjoying the same properties of the Proposition 1.1, there exists a unique solution to a double reflection problem* $(u_0; a, b; f; h_1, h_2)$.

Before giving the proof, we shall prepare some fundamental properties of solutions to single reflecting problems.

2.2. The case of one reflecting wall

We now consider a reflecting problem with one smooth wall (which will be called a single reflection problem) that was studied by Nualart–Pardoux[7] (denoted by $(v_0; a, b; f; h)$):

$$
(5) \qquad \begin{cases} v_t = \dfrac{1}{2}\Delta v - f(x,t; v(x,t)) + \eta + \dot{W}, \\ v(x,0) = v_0(x), \quad v(0,t) = a, \quad v(1,t) = b, \\ v(x,t) \geq h(x,t). \end{cases}
$$

We define the solution to (5) in a similar manner of the two reflecting case. In the case where the reflecting wall h has the same regularity properties with h_i ((H2) and (H3)), we can show the existence and uniqueness result by tracing the method of [7]. That is, let us consider a penalized equation:

$$
(6) \qquad \begin{cases} v_t^\varepsilon = \dfrac{1}{2}\Delta v^\varepsilon - f(x,t; v^\varepsilon(x,t)) + \dfrac{1}{\varepsilon}(v^\varepsilon(x,t) - h(x,t))^- + \dot{W}, \\ v^\varepsilon(x,0) = v_0(x), \quad v^\varepsilon(0,t) = a, \quad v^\varepsilon(1,t) = b, \end{cases}
$$

where z^- denotes the negative part of z, namely $z^- := -\min(z, 0)$. Then $v^\varepsilon(x,t)$ is monotone increasing as $\varepsilon \downarrow 0$, and converges (uniformly) to $v(x,t)$ and

$$
\eta(dx, dt) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(v^\varepsilon(x,t) - h(x,t))^- \, dx \, dt
$$

as a positive Radon measure.

LEMMA 2.2 (comparison). *Let* (v_1, η_1) *and* (v_2, η_2) *be unique solutions to single reflection problems* $(v_{0,1}; a_1, b_1; f_1; h_1)$ *and* $(v_{0,2}; a_2, b_2; f_2; h_2)$, *respectively. If the coefficients satisfy* $a_1 \geq a_2$, $b_1 \geq b_2$, $v_{0,1} \geq v_{0,2}$, $f_1(x,t;z) \leq f_2(x,t;z)$, *and* $h_1(x,t) \geq h_2(x,t)$ *for every* $x \in [0,1]$, $t \in [0,\infty)$, *and* $z \in \mathbb{R}$, *we have* $v_1(x,t) \geq v_2(x,t)$ *almost surely.*

PROOF. Let us consider penalized equations (6) for $(v_{0,i}; a_i, b_i; f_i; h_i)$, $i = 1, 2$, of which solutions are denoted by $v_1^\varepsilon(x,t)$ and $v_2^\varepsilon(x,t)$, respectively. Then the standard comparison theorem for SPDEs ([9]) asserts that $v_1^\varepsilon(x,t) \geq v_2^\varepsilon(x,t)$ almost surely for every $\varepsilon > 0$. Hence the lemma follows immediately by taking limit $\varepsilon \downarrow 0$. □

LEMMA 2.3. *Let v and $\hat{v}$ be given continuous functions and let $z^{\varepsilon,\delta}$ be a unique solution to the following deterministic PDE:*

$$(7) \qquad \begin{cases} z_t^{\varepsilon,\delta} = \dfrac{1}{2}\Delta z^{\varepsilon,\delta} - f(z^{\varepsilon,\delta} + v) + \dfrac{1}{\delta}(z^{\varepsilon,\delta} + v - h_1)^- \\ \qquad\qquad - \dfrac{1}{\varepsilon}(z^{\varepsilon,\delta} + v - h_2)^+, \\ z^{\varepsilon,\delta}(0,t) = z^{\varepsilon,\delta}(1,t) = 0, \\ z^{\varepsilon,\delta}(x,0) = 0, \end{cases}$$

where $z^+ := \max(z, 0)$. We also denote by $\hat{z}^{\varepsilon,\delta}$ the solution to the above PDE replacing v by $\hat{v}$. Then we have, for some constant $C > 0$, $\|z^{\varepsilon,\delta} - \hat{z}^{\varepsilon,\delta}\|_{T,\infty} \leq C\|v - \hat{v}\|_{T,\infty}$, where $\|w\|_{T,\infty} := \sup_{0 \leq t \leq T, 0 \leq x \leq 1} |w(x,t)|$.

PROOF. Set $g(x,t;z) := K_T z + e^{-K_T t} f(x,t; e^{K_T t} z)$. Then it is easily checked that $g(z)$ is monotone increasing. We denote by $z_g^{\varepsilon,\delta}$ the unique solution to (7) replacing f by g, and v, h_1 and h_2 as well. Then we have $z^{\varepsilon,\delta} = e^{K_T t} z_g^{\varepsilon,\delta}$. Therefore it is sufficient to prove the lemma under an additional assumption that $f(x,t;z)$ itself is monotone increasing with respect to z.

Define $k := \|v - \hat{v}\|_{T,\infty}$ and $w(x,t) := (z^{\varepsilon,\delta}(x,t) - \hat{z}^{\varepsilon,\delta}(x,t)) - k$. Then w enjoys the following PDE:

$$\begin{aligned} w_t = \frac{1}{2}\Delta w &- \left(f(z^{\varepsilon,\delta} + v) - f(\hat{z}^{\varepsilon,\delta} + \hat{v})\right) \\ &+ \frac{1}{\delta}\left((z^{\varepsilon,\delta} + v - h_1)^- - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h_1)^-\right) \\ &- \frac{1}{\varepsilon}\left((z^{\varepsilon,\delta} + v - h_2)^+ - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h_2)^+\right). \end{aligned}$$

Now we note that, if $w(x,t) \geq 0$, we have $z^{\varepsilon,\delta} + v \geq \hat{z}^{\varepsilon,\delta} + \hat{v}$. Hence we have $\left((z^{\varepsilon,\delta} + v - h_1)^- - (\hat{z}^{\varepsilon,\delta} + \hat{v} - h_1)^-\right)/\delta \leq 0$ and $\big((z^{\varepsilon,\delta} + v - h_2)^+ -$

$(\hat{z}^{\varepsilon,\delta} + \hat{v} - h_2)^+) / \varepsilon \geq 0$ on $\{(x,t); w(x,t) \geq 0\}$. Combining with the monotonicity of f and taking an inner product with $w(x,t)^+$ in $L^2(0,1)$ immediately lead us to $\frac{1}{2}\frac{d}{dt}\|w^+\|_{L^2}^2 \leq 0$. Hence $w(x,t) \leq 0$, and by changing the role of v and $\hat{v}$, we have the conclusion. □

The next lemma is a straight consequence of the above lemma.

LEMMA 2.4. *Let v and $\hat{v}$ be given continuous functions and let $(z^\varepsilon, \eta^\varepsilon)$ and $(\hat{z}^\varepsilon, \hat{\eta}^\varepsilon)$ be the unique solutions to single reflection problems* $(0; 0, 0; f + (\cdot + v - h_2)^+/\varepsilon; h_1)$ *and* $(0; 0, 0; f + (\cdot + \hat{v} - h_2)^+/\varepsilon; h_1)$, *respectively. Then we have* $\|z^\varepsilon - \hat{z}^\varepsilon\|_{T,\infty} \leq C\|v - \hat{v}\|_{T,\infty}$.

2.3. Proof of Theorem 2.1

Let us consider a single reflection problem $(u_0; a, b; f + (\cdot - h_2)^+/\varepsilon; h_1)$ and denote by $(u^\varepsilon, \eta^\varepsilon)$ its unique solution. By virtue of the comparison lemma (Lemma 2.2), $u^\varepsilon(x,t)$ is a decreasing sequence bounded from below by $h_1(x,t)$. Hence $\lim_{\varepsilon\downarrow 0} u^\varepsilon(x,t)$ exists for every (x,t) and it is easily seen that the convergence is also in $L^p(0,1)$ $(1 < p < \infty)$. We assert that $u(x,t) := \lim_{\varepsilon\downarrow 0} u^\varepsilon(x,t)$ is a part of the solution to the double reflection problem $(u_0; a, b; f; h_1, h_2)$.

Now let us consider a stochastic heat equation with same initial-boundary conditions:

$$\begin{cases} w_t = \dfrac{1}{2}\Delta w + \dot{W}, \\ w(0,t) = a, \quad w(1,t) = b, \quad w(x,0) = u_0(x), \end{cases}$$

and put $z^\varepsilon(x,t) := u^\varepsilon(x,t) - w(x,t)$. Then $z^\varepsilon(x,t)$ satisfies

$$\begin{cases} z^\varepsilon_t = \dfrac{1}{2}\Delta z^\varepsilon - f(z^\varepsilon + w) + \eta^\varepsilon - \dfrac{1}{\varepsilon}\left((z^\varepsilon + w - h_2)^+\right), \\ z^\varepsilon(0,t) = z^\varepsilon(1,t) = 0, \quad z^\varepsilon(x,0) = 0, \\ z^\varepsilon(x,t) + w(x,t) \geq h_1(x,t), \\ \displaystyle\int_0^\infty \int_0^1 (z^\varepsilon + w - h_1)\eta^\varepsilon(dx,dt) = 0. \end{cases} \tag{8}$$

From the argument of Nualart–Pardoux[7], the unique solution z^ε to (8) is obtained by taking limit $\delta \downarrow 0$ in the unique solution $z^{\varepsilon,\delta}$ to the PDE (7). We denote by $z_n^{\varepsilon,\delta}$ the unique solution to the PDE (7) replacing w by a smooth function w_n. Then, from a standard argument of variational inequalities (see, e.g., [1]), $z_n(x,t) := \lim_{\varepsilon,\delta\downarrow 0} z_n^{\varepsilon,\delta}(x,t)$ exists and is a unique solution to an evolutionary variational inequality with two obstacles (Proposition 1.1). However we know $\|z_n^{\varepsilon,\delta} - z^{\varepsilon,\delta}\|_{T,\infty} \le C\|w_n - w\|_{T,\infty}$ by Lemma 2.3. Hence we can conclude that $\|z_n - z\|_{T,\infty} \le C\|w_n - w\|_{T,\infty}$, where $z(x,t) := u(x,t) - w(x,t)$. Letting $\|w_n - w\|_{T,\infty} \to 0$ as $n \to \infty$ proves the continuity of z.

Now, for $\psi \in C_0^\infty((0,1)\times[0,\infty))$, z^ε fulfills the following integral equation:

$$
\begin{aligned}
(9) \qquad -\int_0^\infty (z^\varepsilon(t), \psi_t(t))\,dt = \frac{1}{2}\int_0^\infty (z^\varepsilon(t), \Delta\psi(t))\,dt \\
- \int_0^\infty (f(z^\varepsilon + w), \psi(t))\,dt \\
+ \int_0^\infty \int_0^1 \psi(x,t)(\eta^\varepsilon(dx,dt) - \xi^\varepsilon(dx,dt)),
\end{aligned}
$$

where we put $\xi^\varepsilon(dx,dt) := (z^\varepsilon(x,t) + w(x,t) - h_2(x,t))^+/\varepsilon\, dxdt$. Then it is clear that, under the limit $\varepsilon \downarrow 0$, $\lim_{\varepsilon\downarrow 0} \eta^\varepsilon - \xi^\varepsilon$ exists in the sense of Schwartz distribution. From the hypothesis (H1), we have $\operatorname{supp}\xi^\varepsilon \cap \operatorname{supp}\eta^\varepsilon \cap ([\delta, 1-\delta]\times[0,T]) = \emptyset$ for every $\varepsilon, \delta > 0$. Moreover, $\operatorname{supp}\xi^\varepsilon$ decreases and $\operatorname{supp}\eta^\varepsilon$ increases as ε decreases. Hence, by choosing the support of ψ cleverly, it can be shown that both ξ^ε and η^ε converge to positive distribution ξ and η, respectively.

It is clear, by multiplying both sides of (9) by ε, that $h_1(z,t) \le z(x,t) + w(x,t) \le h_2(x,t)$. Finally, we can show $\int_0^T \int_0^1 (u^\varepsilon(x,t) - h_2(x,t))\xi(dx,dt) \le 0$ for every $T > 0$, which shows $\int_0^\infty \int_0^1 (u(x,t) - h_2(x,t))\xi(dx,dt) = 0$. $\int_0^\infty \int_0^1 (u(x,t) - h_1(x,t))\eta(dx,dt) = 0$ is easy. By taking $\psi \in C_0^\infty((0,1)\times[0,T])$ such that $\psi = 1$ on $\operatorname{supp}\eta$ and $\psi = 0$ on $\operatorname{supp}\xi$, we can conclude $\eta([\delta,1-\delta]\times[0,T]) < \infty$ and similarly $\xi([\delta,1-\delta]\times[0,T]) < \infty$ for every small $\delta > 0$.

To show the uniqueness, it follows from a routine argument and a mollifier technique used in [7], we only sketch of the proof. Let (u,η,ξ) and $(\bar u, \bar\eta, \bar\xi)$ be solutions to a double reflection problem $(u_0; a, b; f; h_1, h_2)$. Putting $w(x,t) := u(x,t) - \bar u(x,t)$, w formally obeys the following PDE:

$$
(10) \qquad w_t = \frac{1}{2}\Delta w - (f(u) - f(\bar u)) + (\eta - \bar\eta) - (\xi - \bar\xi).
$$

We will multiply both sides of (10) by $w(x,t)\phi(x)^2$, $\phi \in C_0^\infty(0,1)$, and integrate them over $[0,1] \times [0,T]$. Then we have $\|w(T)\phi\|_{L^2}^2 \le \frac{1}{2}\int_0^T \int_0^1 w(x,t)^2(\phi^2(x))''\,dx\,dt$. It is not difficult to show $w(T) = 0$ from this relation, see [7]. Hence the uniqueness comes from (10). □

2.4. Some properties of the dynamics

PROPOSITION 2.5. *Suppose that $h_1(0,t) < h_2(0,t)$ and $h_1(1,t) < h_2(1,t)$. Let (u,η,ξ) be the solution to the double reflection problem $(u_0;a,b;f;h_1,h_2)$. Then we have*

$$\int_0^T \int_0^1 x(1-x)\eta(dx,dt) < \infty, \text{ and } \int_0^T \int_0^1 x(1-x)\xi(dx,dt) < \infty.$$

PROOF. We first note that $\int_0^T \int_0^1 x(1-x)(\eta-\xi)(dx,dt) < \infty$. The proof of this assertion goes the same as Nualart–Pardoux did in [7]. However, we already know that $\eta([\delta,1-\delta]\times[0,T]) < \infty$ and $\xi([\delta,1-\delta]\times[0,T]) < \infty$ for every $\delta > 0$. Hence, since $h_1(0,t) < h_2(0,t)$ and $h_1(1,t) < h_2(1,t)$, we can take δ small enough so that $\operatorname{supp}\eta \cap ([0,\delta]\times[0,T]) = \emptyset$ and $\operatorname{supp}\eta\cap([1-\delta,1]\times[0,T]) = \emptyset$, or $\operatorname{supp}\xi\cap([0,\delta]\times[0,T]) = \emptyset$ and $\operatorname{supp}\xi\cap([1-\delta,1]\times[0,T]) = \emptyset$. □

LEMMA 2.6 (comparison). *Let (u_1,η_1,ξ_1) and (u_2,η_2,ξ_2) be solutions to double reflection problems $(u_{0,1};a_1,b_1;f_1;h_{1,1},h_{2,1})$ and $(u_{0,2};a_2,b_2;f_2;h_{1,2},h_{2,2})$, respectively. Suppose that $u_{0,1}(x) \le u_{0,2}(x)$, $a_1 \le a_2$, $b_1 \le b_2$, $f_1(x,t;z) \ge f_2(x,t;z)$, $h_{1,1}(x,t) \le h_{1,2}(x,t)$, and $h_{2,1}(x,t) \le h_{2,2}(x,t)$. Then we have $u_1(x,t) \le u_2(x,t)$ almost surely.*

PROOF. The proof comes immediately from Lemma 2.2 and the construction of the solution. □

PROPOSITION 2.7. *Let $(u_n,\eta_n,\xi_n)_{n\ge1}$ be unique solutions to the respectively double reflection problems $(u_0;a,b;f;h_0,h_n)$ and (u_0,η_0) be a unique solution to a single reflection problem $(u_0;a,b;f;h_0)$. Suppose that $\lim_{n\to\infty} h_n(x,t) = +\infty$ for every (x,t). Then $u_n \to u_0$ almost surely as $n \to \infty$.*

PROOF. We may assume, taking a subsequence if necessary, that $h_n(x,t)$ is monotone increasing with respect to n. From Lemma 2.2, however, we have $u_n(x,t) \le u_0(x,t)$, while $u_n(x,t)$ is monotone increasing from Lemma 2.6. It is clear that $\bar{u}(x,t) := \lim_{n\to\infty} u_n(x,t)$ and $\bar{\eta} := \lim_{n\to\infty} \eta_n$ must be a solution to the single reflection problem $(u_0; a, b; f, h_0)$, of which solution is unique. □

3. Stationary Distribution

Let (u, η, ξ) be a unique solution to a double reflection problem $(u_0; a, b; f; h_1, h_2)$. In this section, we are concerned with stationary (reversible) distribution for the time evolution determined by $S \equiv C([0,1])$-valued diffusion process $u(t)$. Here we say a probability measure μ on S is stationary or reversible if $E_\mu[\Phi(u(0))\Psi(u(t))] = E_\mu[\Phi(u(t))\Psi(u(0))]$ is satisfied for test functions Φ and Ψ from $\mathscr{F} := \{\Psi : S \to \mathbb{R}; \Psi(w) \equiv \psi((w,\phi_1),\ldots,(w,\phi_n))$ for some $n \ge 1, \psi \in C_b^\infty(\mathbb{R}^n,\mathbb{R}),\ \phi_1,\ldots,\phi_n \in C_0^\infty(0,1)\}$. We denoted by E_μ the expectation on our measurable space $(\Omega, \mathscr{B})$ with respect to P_μ which is a probability measure such that $u(t)$ is a Markov process with initial distribution μ on $C([0,1])$.

Generally speaking, if the external factors involving the dynamics vary with a lapse of time, it does not have stationary distribution. Therefore we further assume, in the sequel of the paper, that they do not depend upon time variables, namely $f(x,t;z) \equiv f(x;z)$, $h_1(x,t) \equiv h_1(x)$, and $h_2(x,t) \equiv h_2(x)$. In addition, in the sequel of the paper, we assume that

(F3) there exists an $L^2(0,1)$-function $\alpha(\cdot)$ and a constant $\lambda < \pi^2/2$ such that

$$-\alpha(x) - \lambda|z| \le f(x;z). \tag{11}$$

First, let us recall that stationary distribution β^f of $X(t)$ which solves the following SPDE:

$$\begin{cases} X_t(x,t) = \dfrac{1}{2}\Delta X(x,t) - f(x;X(x,t)) + \dot{W}, \\ X(0,t) = a, \quad X(1,t) = b, \end{cases}$$

is given by the following formula:

$$\beta^f(dw) = \frac{1}{Z^f}\exp\left\{-2\int_0^1 F(x,w(x))\,dx\right\}\beta(dw), \tag{12}$$

where $F(x,z)$ is a potential function such that $\partial F(x,z)/\partial z = f(x;z)$, and β is the law on S induced by Brownian bridge w such that $w(0) = a$ and $w(1) = b$. $Z \equiv Z^f$ denotes the normalizing constant in the sequel and varies in the context. Note that $Z^f < \infty$ by virtue of (11).

We also introduce some probability measures on S by $\beta_1(dw) := \beta(dw \mid w(x) \geq h_1(x),\ 0 \leq \forall x \leq 1)$, and $\beta_{1,2}(dw) := \beta(dw \mid h_1(x) \leq w(x) \leq h_2(x),\ 0 \leq \forall x \leq 1)$. We also define β_1^f and $\beta_{1,2}^f$ similarly to (12). Such measures are naturally defined when neither $w(0)$ nor $w(1)$ touch the wall. In the case they do, they are defined through limiting arguments, see [4].

LEMMA 3.1. *Let (u,η) be a unique solution to a single reflection problem $(\cdot; a, b; f; h_1)$. Then, β_1^f is a reversible measure for $u(t)$.*

PROOF. This assertion is essentially proved in [8, 10]. Let us consider (6) (we denote the solution by u^ε). Then, the reversible measure for $u^\varepsilon(t)$ is given by

$$\mu^\varepsilon(dw) := \frac{1}{Z^\varepsilon} \exp\left\{ -\frac{1}{\varepsilon} \int_0^1 ((w(x) - h_1(x)) \wedge 0)^2 \, dx \right\} \beta^f(dw).$$

Assume first that $h_1(0) < a$, $h_1(1) < b$. Then it is clear that $\mu^\varepsilon(dw) \to \beta_1^f(dw)$ and the reversibility comes from Lebesgue's dominated convergence theorem.

If $h_1(0) = a$ or $h_1(1) = b$, we shall take sequences such that $a_n \downarrow a$ or $b_n \downarrow b$, $a_n > h_1(0)$ and $b_n > h_1(1)$. Note that, since f is Lipschitz continuous, we can easily obtain

$$\|u^\varepsilon(t) - \bar{u}^\varepsilon(t)\|_{L^2} \leq e^{K_T t} \|u_0 - \bar{u}_0\|_{L^2}, \tag{13}$$

where $\bar{u}^\varepsilon$ is the corresponding solution to (6) with initial condition $\bar{u}^\varepsilon(0) = \bar{u}_0$. This equi-continuity with respect to initial conditions still holds for $u(t)$. Recall that β_1^f in this case is the weak limit of such measures that are defined similarly using a_n and b_n. Then the assertion comes immediately (cf. [5], [6, p. 64], and [10]). □

THEOREM 3.2. *Let (u,η,ξ) be a unique solution to a double reflection problem $(\cdot; a, b; f; h_1, h_2)$. Then, $\beta_{1,2}^f$ is reversible under $u(t)$.*

PROOF. Let us consider a single reflection problem $(u_0; a, b; f + (\cdot - h_2)^+/\varepsilon; h_1)$ which has a unique solution $(u^\varepsilon, \eta^\varepsilon)$, and $u^\varepsilon(t)$ converges to $u(t)$ locally uniformly and monotonely. From the above lemma, we know that

$$\mu_\varepsilon(dw) := \frac{1}{Z^\varepsilon} \exp\left\{ -\frac{1}{\varepsilon} \int_0^1 ((w(x) - h_2(x)) \vee 0)^2 \, dx \right\} \beta_1^f(dw)$$

is reversible under $u^\varepsilon(t)$. Similarly to the above lemma, we have the conclusion. □

LEMMA 3.3. *Let (u, ξ, η) and $(\bar{u}, \bar{\eta}, \bar{\xi})$ be unique solutions to double reflection problems $(u_0; a, b; f; h_1, h_2)$ and $(\bar{u}_0; a, b; f; h_1, h_2)$, respectively. Suppose that f fulfills a convexity condition such that $\inf_{\iota \le z, \bar{z} \le \sigma}(f(x; z) - f(x; \bar{z}))/(z - \bar{z}) > -\pi^2/2$, where $\iota := \inf h_1(x)$ and $\sigma := \sup h_2(x)$. Then there exists a constant $c > 0$ such that $\|u(t) - \bar{u}(t)\|_{L^2} \le e^{-ct}\|u_0 - \bar{u}_0\|_{L^2}$.*

PROOF. It is a routine task to show the desired inequality when we consider a penalized equation (i.e., we replace η and ξ by $(u^\varepsilon - h_1)^-/\varepsilon$ and $(u^\varepsilon - h_2)^+/\varepsilon$, respectively, cf. (7)), with suitably extending the domain of f to $\mathbb{R}$. Recall that, as an operator satisfying Dirichlet boundary conditions, $\Delta \le -\pi^2$. Hence the assertion comes immediately. □

COROLLARY 3.4. *Suppose that f enjoys the criterion of the above lemma. Then the stationary measure for $u(t)$ is unique.*

4. Dirichlet Form

The final section of the present paper is devoted to presenting a Dirichlet form determined by a diffusion process $u(t)$. The corresponding result for single reflection problems $(\cdot; a, b; f; 0)$, $a, b \ge 0$, was obtained by Zambotti[11]. In the case of double reflection problems, we can prove a similar result by tracing him with the help of a divergence formula recently obtained by Funaki and Ishitani[4].

Throughout this section, we further assume that the boundaries never touch the wall, namely, $h_1(0) < a < h_2(0)$ and $h_1(1) < b < h_2(1)$.

For $\mathscr{F} \ni \Psi(\cdot) = \psi((\cdot, \phi_1), \ldots, (\cdot, \phi_n))$, we define its Fréchet derivative $\nabla\Psi$ by $(\nabla\Psi(w))(x) := \sum_{i=1}^n \partial_i \psi((w, \phi_1), \ldots, (w, \phi_n))\phi_i(x)$. Then a usual inner product $(\nabla\Phi(w), \nabla\Psi(w))$ is well-defined in $L^2(0, 1)$. We denote by ∇^2

the second Fréchet differential operator on $\mathscr{F}$ and by Δ_x the usual Laplace operator $\partial^2/\partial x^2$, that is, $\Delta_x \nabla\Phi(w)(x) := d^2/dx^2(\nabla\Phi(w)(x))$. We also define an Ornstein–Uhlenbeck operator $\mathscr{L}^{\mathrm{OU}}$ on $\mathscr{F}$ by

$$\mathscr{L}^{\mathrm{OU}}\Phi(w) := \frac{1}{2}\left(\mathrm{Tr}\left[\nabla^2\Phi(w)\right] + (w, \Delta_x\nabla\Phi(w))\right).$$

It is well known that a unique solution to an $\mathscr{L}^{\mathrm{OU}}$-local martingale problem on $\mathscr{F}$ coincides with the (law of) unique solution to a stochastic heat equation driven by space-time white noise.

Let us introduce a set $K_{1,2} := \{w \in C([0,1]); h_1(x) \le w(x) \le h_2(x), \forall x \in (0,1)\}$ which is a support of $\beta_{1,2}$. Then a kind of (infinite dimensional) Gauss' divergence formula holds on $K_{1,2}$.

Theorem 4.1 (Funaki–Ishitani[4]). *There exists a family of measures* $\{\nu(r,\cdot)\}_{r\in[0,1]}$ *such that, for every* $\Phi \in C_b^1(K_{1,2})$ *and* $h \in H_0^1(0,1)\cap H^2(0,1)$,

$$\begin{aligned}\int_{K_{1,2}} (\nabla\Phi(w), h)\, \beta_{1,2}(dw) \\ = -\int_{K_{1,2}} \Phi(w)(h'', w)\beta_{1,2}(dw) + \int_0^1 h(r)\, dr \int \Phi(w)\nu(r, dw).\end{aligned}$$

The family of measures ν describes the *metrical* boundary of $K_{1,2}$ with respect to $\beta_{1,2}$. Such measures are supported on the set of paths which hit h_1 or h_2 exactly once (note that the topological boundary of $K_{1,2}$ is the paths that may hit h_1 or h_2 many times). We omit to give the measure ν in the statement precisely since the formula itself is not important here and rather complicated.

Let us define $\Xi_\varepsilon(w) := \exp\left\{-\frac{1}{\varepsilon}\left(\|(w-h_1)^-\|_{L^2}^2 + \|(w-h_2)^+\|_{L^2}^2\right)\right\}$ and $\zeta_\varepsilon(w;x) := \frac{2}{\varepsilon}\left((w(x)-h_1(x))^- - (w(x)-h_2(x))^+\right)$. Note that $\zeta_\varepsilon(w;x)\Xi_\varepsilon(w) = \nabla\Xi_\varepsilon(w)(x)$. Straightforward computations lead us to

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_S \Phi(w)(h, \zeta_\varepsilon(w;\cdot))\Xi_\varepsilon(w)\beta(dw) \\ = -\int_0^1 dr \int h(r)\Phi(w)\nu(r, dw)\end{aligned} \tag{14}$$

for every $\Phi \in C_b(K_{1,2})$, and

$$(15) \quad \int_{K_{1,2}} (\nabla\Phi(w), \nabla\Psi(w))\beta_{1,2}(dw)$$
$$= -2\int_{K_{1,2}} \Phi(w)\mathscr{L}^{\mathrm{OU}}\Psi(w)\beta_{1,2}(dw) + \int_0^1 dr \int \nabla\Psi(w)(r)\,\Phi(w)\nu(r, dw)$$

for every $\Phi \in C_b^1(K_{1,2})$ and $\Psi \in \mathscr{F}$.

Now, let us define a bilinear form by

$$(16) \qquad \mathscr{E}(\Phi, \Psi) := \frac{1}{2}\int_{K_{1,2}} (\nabla\Phi(w), \nabla\Psi(w))\beta_{1,2}^f(dw)$$

for $\Phi, \Psi \in \mathscr{F}$.

THEOREM 4.2. *$(\mathscr{E}, \mathscr{F})$ is a closable bilinear form and its closure determines a quasi-regular Dirichlet form with which $u(t)$ is associated.*

PROOF. The proof will be done by tracing Zambotti's method[11].

Let us start the proof with defining

$$\mathscr{L}^\varepsilon\Phi(w) := \mathscr{L}^{\mathrm{OU}}\Phi(w) + \frac{1}{2}(\zeta_\varepsilon(w;\cdot), \nabla\Phi(w))$$

for $\Phi \in \mathscr{F}$, and $\mu^\varepsilon(dw) := \Xi_\varepsilon(w)\beta^f(dw)/Z^\varepsilon$. Then $\mathscr{L}^\varepsilon$ is essentially self-adjoint on $L^2(S;\mu^\varepsilon) \equiv L^2(\mu^\varepsilon)$. We denote by $(\mathscr{L}^\varepsilon, D(\mathscr{L}^\varepsilon))$ its closure.

Using the probability measure μ^ε, we define another bilinear form by

$$\mathscr{E}^\varepsilon(\Phi, \Psi) := \frac{1}{2}\int_S (\nabla\Phi(w), \nabla\Psi(w))\mu^\varepsilon(dw)$$

for $\Phi, \Psi \in \mathscr{F}$. Then it is easily shown that

$$(17) \qquad \mathscr{E}^\varepsilon(\Phi, \Psi) = -\int_S \Phi(w)\mathscr{L}^\varepsilon\Psi(w)\mu^\varepsilon(dw)$$

for $\Phi \in D(\mathscr{E}^\varepsilon) \equiv D((-\mathscr{L}^\varepsilon)^{1/2})$ and $\Psi \in D(\mathscr{L}^\varepsilon)$. Note that it holds especially for $\Phi, \Psi \in \mathscr{F}$. Let us define a norm $\|\cdot\|_{W_\varepsilon^{1,2}}$ on $\mathscr{F}$ by $\|\Phi\|_{W_\varepsilon^{1,2}}^2 := \|\Phi\|_{L^2(d\mu^\varepsilon)}^2 + \|\nabla\Phi\|_{L^2([0,1]\times S; dx\times d\mu^\varepsilon)}^2$. We denote by $\bar{\mathscr{F}}_\varepsilon$ the completion of $\mathscr{F}$ by $\|\cdot\|_{W_\varepsilon^{1,2}}$. Then, (17) still holds for $\Phi \in \bar{\mathscr{F}}_\varepsilon$ and $\Psi \in \mathscr{F}$. Note that, it is clear that $W_\varepsilon^{1,2}$, the space of all functionals on S with finite $\|\cdot\|_{W^{1,2}_\varepsilon}$-norm, is contained in $D(\mathscr{E}^\varepsilon)$. Now we see that $(\mathscr{E}^\varepsilon, \mathscr{F})$ is a closable symmetric bilinear form and $(\mathscr{E}^\varepsilon, \bar{\mathscr{F}}_\varepsilon)$ is a Dirichlet form which determines a diffusion process satisfying

$$u_t^\varepsilon(x,t) = \frac{1}{2}\Delta_x u^\varepsilon(x,t) - f(x,t;u^\varepsilon(x,t)) + \frac{1}{2}\zeta_\varepsilon(u^\varepsilon(x,t);x) + \dot{W},$$

see Funaki[3] for detail. Hence, the semigroup $P_t^\varepsilon\Phi(w) := E_w[\Phi(u^\varepsilon(t))]$ and the resolvent $G_\alpha^\varepsilon\Phi(w) := \int_0^\infty e^{-\alpha t}P_t^\varepsilon\Phi(w)\,dt$ correspond to $(\mathscr{E}, \bar{\mathscr{F}}_\varepsilon)$. It means that $G_\alpha(L^2(\mu^\varepsilon)) \subset \bar{\mathscr{F}}_\varepsilon$ and

$$\mathscr{E}_\alpha^\varepsilon(G_\alpha^\varepsilon\Phi, \Psi) := \mathscr{E}^\varepsilon(G_\alpha^\varepsilon\Phi, \Psi) + \alpha(G_\alpha^\varepsilon\Phi, \Psi)_{L^2(\mu^\varepsilon)} = (\Phi, \Psi)_{L^2(\mu^\varepsilon)} \tag{18}$$

for every $\Phi \in L^2(\mu^\varepsilon)$ and $\Psi \in \bar{\mathscr{F}}_\varepsilon$.

Now let us define a semigroup and a resolvent determined by the diffusion process $u(t)$ by $P_t\Phi(w) := E_w[\Phi(u(t))]$ and $G_\alpha\Phi(w) := \int_0^\infty e^{-\alpha t}P_t\Phi(w)\,dt$, respectively. Then it is easy to see that $P_t^\varepsilon\Phi(w) \to P_t\Phi(w)$ and $G_\alpha^\varepsilon\Phi(w) \to G_\alpha\Phi(w)$ as $\varepsilon \downarrow 0$. Hence we have, by using (13) and combining with (14), that

$$\int_S G_\alpha^\varepsilon\Phi(w)(\zeta_\varepsilon, \nabla\Psi(w))\mu^\varepsilon(dw) \to -\int_0^1 \nabla\Psi(w)(r)\,dr \int G_\alpha\Phi(w)\nu(r,dw).$$

It means, by comparing with (15), that $\lim_{\varepsilon\downarrow 0}\mathscr{E}^\varepsilon(G_\alpha^\varepsilon\Phi, \Psi) = \mathscr{E}(G_\alpha\Phi, \Psi)$ for every $\Phi \in \mathscr{F}$ and $\Psi \in C_b(K_{1,2})$.

On the other hand, from (18), we have $\mathscr{E}(G_\alpha\Phi, \Psi) = -(\alpha G_\alpha\Phi - \Phi, \Psi)_{L^2(\beta_{1,2}^f)}$. Since G_α is a resolvent associated with $u(t)$, there exists a Dirichlet form $(\tilde{\mathscr{E}}, D(\tilde{\mathscr{E}}))$, $G_\alpha(L^2(\beta_{1,2}^f)) \subset D(\tilde{\mathscr{E}})$, such that $\mathscr{E}(G_\alpha\Phi, \Psi) = \tilde{\mathscr{E}}(G_\alpha\Phi, \Psi)$ for every $\Phi \in L^2(\beta_{1,2}^f)$ and $\Psi \in \mathscr{F}$, which means that, $\mathscr{E}(\Phi, \Psi) = \tilde{\mathscr{E}}(\Phi, \Psi)$ for every $\Phi \in G_\alpha(\mathscr{F})$ and $\Psi \in \mathscr{F}$. The right hand side is defined for $\Phi \in G_\alpha(L^2(\beta_{1,2}^f))$ and $\Psi \in D(\tilde{\mathscr{E}})$, that is, $(\mathscr{E}, G_\alpha(\mathscr{F}))$ has a closed extension.

Finally, let us consider an approximating bilinear form $\tilde{\mathscr{E}}^{(\alpha)}(\Phi, \Psi) := \alpha(\Phi - \alpha G_\alpha\Phi, \Psi)_{L^2(\beta_{1,2}^f)}$ for $\tilde{\mathscr{E}}$ defined for $\Phi, \Psi \in L^2(\beta_{1,2}^f)$. It is well known that ([2, Lemma 1.3.4]) $\tilde{\mathscr{E}}^{(\alpha)}(\Phi, \Phi)$ is monotone increasing with respect to α, $\tilde{\mathscr{E}}(\Phi, \Psi) = \lim_{\alpha\to\infty}\tilde{\mathscr{E}}^{(\alpha)}(\Phi, \Psi)$, and $D(\tilde{\mathscr{E}}) = \{\Phi \in L^2(\beta_{1,2}^f); \lim_{\alpha\to\infty}\tilde{\mathscr{E}}^{(\alpha)}(\Phi, \Phi) < \infty\}$. Since $\tilde{\mathscr{E}}^{(\alpha)}(\Phi, \Psi) = \alpha\mathscr{E}(G_\alpha\Phi, \Psi)$, we can conclude that $D(\mathscr{E}) = D(\tilde{\mathscr{E}}) = W^{1,2}(\beta_{1,2}^f)$. Now we proved that G_α is a resolvent associated with $(\mathscr{E}, D(\mathscr{E}))$, with which the diffusion process $u(t)$ is associated. Hence $(\mathscr{E}, D(\mathscr{E}))$ is a quasi-regular Dirichlet form. □

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