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p-*Adic Weight Spectral Sequences of Log Varieties*

By Yukiyoshi Nakkajima

Abstract. We prove the E_2-degeneration of the p-adic weight spectral sequence of a proper simple normal crossing log variety over a log point whose underlying scheme is the spectrum of a perfect field of characteristic $p > 0$. We also show some properties of the p-adic weight spectral sequence and those of p-adic monodromy operators. The former ones complete the construction of the p-adic Steenbrink complex in [M1]; the latter ones complete the proof of the interpretation of the p-adic monodromy operator in [HK] by a corrected operator of the p-adic Steenbrink complex in [M1]. We also complete some fundamental facts in [HK].

Contents

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1. Introduction

In [Nak] Nakayama has proved the E_2-degeneration of the l-adic weight spectral sequence of a proper simple normal crossing log variety over a log point. As a result we see that the graded pieces of the l-adic weight filtration on the l-adic log cohomology of the log variety above are easily described subquotients of direct sums of usual l-adic cohomologies (with Tate twists) of intersections of the irreducible components of the log variety.

In the Part I of this paper, we prove one of Mokrane's conjectures in [M1]: we prove the E_2-degeneration of the p-adic weight spectral sequence of a proper simple normal crossing log variety over a log point whose underlying scheme is the spectrum of a perfect field κ of characteristic $p > 0$. We also prove the E_2-degeneration of the p-adic weight spectral sequence of an open smooth variety which is the complement of a simple normal crossing divisor in a proper smooth variety over κ. The former result and the latter are generalizations of results in [M1] and [M2], respectively, in which κ is assumed to be a finite field.

In the l-adic case, the cospecialization map is a key ingredient for the proof of the degeneration. In the p-adic case, it seems that there does not exist the canonical cospecialization map. However there is a method of a specialization argument due to Deligne-Illusie in [I1] influenced by the general method of the reduction of geometric problems to arithmetic problems due to Grothendieck ([EGA IV-3], [Gr]). This specialization argument does a job in the p-adic case, and we obtain the E_2-degenerations by a somewhat tricky argument.

In [CL2] Chiarellotto and Le Stum have constructed the p-adic weight spectral sequence of the open variety above by the method of rigid cohomologies, and they have proved the E_2-degeneration of the spectral sequence when the base field is finite. We generalize this result for any field of characteristic p by using Shiho's comparison theorem ([Sh]).

As applications of the E_2-degenerations of the p-adic weight spectral sequences, we prove the E_2-degenerations of the weight spectral sequences of the log Hodge-Witt sheaves of a proper simple normal crossing log variety and an open variety.

In the Part II of this paper, we make some proofs for statements of (idealized) log de Rham-Witt complexes perfect and make some statements and some proofs about p-adic weight spectral sequences by the method of log de Rham-Witt complexes perfect. Of course, we use ideas in published papers and we do not claim that we reconstruct theory of log de Rham-Witt complexes and theory of the p-adic weight spectral sequences from the beginning; some published papers ([Hy1], [Hy2], [HK], [M1], [M2], [Lo]) and this paper are necessary for the theories, and some published papers (e.g., [Sa], [Oc], [Nakk2], [KH]) have already used the theories.

In §6 we give theory of formal de Rham-Witt complexes. As a corollary of this theory, I give a precise proof of the fact that the log de Rham-Witt complex with compact support of a smooth scheme with a normal crossing divisor is compatible with the canonical filtration as a complex over the Cartier-Dieudonné-Raynaud algebra because I have an anxiety in the sketchy proof in [M1].

In §7 we complete the proof of a fundamental fact in [HK] which tells us that the log crystalline cohomology of a log smooth scheme of Cartier type over a perfect field of characteristic p with a fine log structure is canonically isomorphic to the cohomology of the log de Rham-Witt complex of it. We also correct a proposition in [HK] which plays an important role in the proof of the Hyodo-Kato's isomorphism.

In §8, §9, §10 and §11, we correct and complete many results in [M1]. In §8 we prove some fundamental properties of projections of log de Rham-Witt complexes. The compatibility in the paragraph before the previous one, the fundamental fact above claimed in [HK] and the fundamental properties of projections are necessary for the construction of the p-adic Steenbrink complex in [M1] as used in [loc. cit.]. In §9 we prove a complicated compatibility of Poincaré residue isomorphisms with the Frobenius, which has not been proved in [M1]. In §10 we give a right proof of the description of the boundary morphism between the E_1-terms of the p-adic weight spectral sequence in [M1] of a proper simple normal crossing log variety, and correct the sign of the boundary morphism in [M1]. In §11 we prove the coincidence of the p-adic monodromy operator in [HK] and that in [Hy2]. This establishes a relation between the p-adic monodromy operator in [HK] and an operator ν which will be defined in §11; in the same section, we point out mistakes in ν in [M1] of the p-adic Steenbrink double and single complexes. Thanks to results in §11, we can apply the method of Steenbrink-Rapoport-Zink ([St1], [RZ]) to the p-adic monodromy-weight conjecture ([M1]), which has been used for the proof of the p-adic monodromy-weight conjecture for a proper simple normal crossing log curve over a log point ([M1]), for a proper semistable family of surfaces over a complete discrete valuation ring with simple normal crossing special fiber ([Nakk4]), and for other cases ([Nakk3], [Nakk4]).

In a future paper, I shall write the construction of the p-adic weight spectral sequence of a family of open simple normal crossing log varieties (cf. [SZ]) by a different, more natural method.

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Notation. (1) For a log scheme X in the sense of Fontaine-Illusie-Kato ([Ka2]), we denote by $\overset{\circ}{X}$ the underlying scheme of X.

(2) For an affine scheme $\mathrm{Spec}(A)$ and a commutative monoid P with unit element e, we denote by $(\mathrm{Spec}(A), P \oplus A^*)$ a log scheme whose underlying scheme is $\mathrm{Spec}(A)$ and whose log structure is the association of a morphism $P \ni x \longmapsto 0 \in A$ $(x \neq e)$.

(3) Following Friedman [Fr], for a morphism $X \longrightarrow S$ of log schemes, we denote by $\Lambda^i_{X/S}$ $(= \omega^i_{X/S}$ in [Ka2]) the sheaf of the relative logarithmic

differential forms on X/S of degree i ($i \in \mathbb{N}$); the small letter ω is not suitable for the symbol of a sheaf of differential forms.

(4) (S)NCL=(simple) normal crossing log, (S)NCD=(simple) normal crossing divisor.

Conventions. We make the following conventions about signs (cf. [BBM], [Co]).

Let $\mathcal{A}$ be an exact additive category.

(1) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^{\bullet+n}, d^{\bullet+n})$ or $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the following complex:

$$\cdots \longrightarrow \underset{q-1}{E^{q-1+n}} \overset{d^{q-1+n}}{\longrightarrow} \underset{q}{E^{q+n}} \overset{d^{q+n}}{\longrightarrow} \underset{q+1}{E^{q+1+n}} \overset{d^{q+1+n}}{\longrightarrow} \cdots .$$

Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

For a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ of complexes of objects in $\mathcal{A}$, $f\{n\}$ denotes a natural morphism $(E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ induced by f. This operation is well-defined in the derived category: for a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ in the derived category $\mathrm{D}^\star(\mathcal{A})$ ($\star =$ b, $+, -$, nothing) of the complexes of objects in $\mathcal{A}$, there exists a naturally induced morphism $f\{n\}\colon (E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ in $\mathrm{D}^\star(\mathcal{A})$.

(2) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^\bullet[n], d^\bullet[n])$ denotes the following complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphism $d^\bullet[n] = (-1)^n d^{\bullet+n}$.

For a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ of complexes of objects in $\mathcal{A}$, $f[n]$ denotes a natural morphism $(E^\bullet[n], d^\bullet_E[n]) \longrightarrow (F^\bullet[n], d^\bullet_F[n])$ induced by f without change of signs. This operation is well-defined in the derived category as in (1).

(3) ([BBM, 0.3.2], [Co, (1.3.2)]) For a short exact sequence

$$0 \longrightarrow (E^\bullet, d^\bullet_E) \overset{f}{\longrightarrow} (F^\bullet, d^\bullet_F) \overset{g}{\longrightarrow} (G^\bullet, d^\bullet_G) \longrightarrow 0$$

of bounded below complexes of objects in $\mathcal{A}$, let $(E^\bullet[1], d^\bullet_E[1]) \oplus (F^\bullet, d^\bullet_F)$ be the mapping cone of f. We fix an isomorphism "$(E^\bullet[1], d^\bullet_E[1]) \oplus (F^\bullet, d^\bullet_F) \ni (x, y) \longmapsto g(y) \in (G^\bullet, d^\bullet_G)$" in the derived category $\mathrm{D}^+(\mathcal{A})$.

(4) ([BBM, 0.3.2], [Co, (1.3.3)]) Under the situation (3), the boundary morphism $(G^\bullet, d^\bullet_G) \longrightarrow (E^\bullet[1], d^\bullet_E[1])$ in $\mathrm{D}^+(\mathcal{A})$ is, by definition, the following composite morphism

$$(G^\bullet, d^\bullet_G) \overset{\sim}{\longleftarrow} (E^\bullet[1], d^\bullet_E[1]) \oplus (F^\bullet, d^\bullet_F) \overset{\mathrm{proj}}{\longrightarrow} (E^\bullet[1], d^\bullet_E[1]) \overset{(-1)\times}{\longrightarrow} (E^\bullet[1], d^\bullet_E[1]).$$

(5) Assume that $\mathcal{A}$ is an abelian category with enough injectives. Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then, under the situation (3), the boundary morphism $\partial\colon R^q\mathcal{F}((G^\bullet, d_G^\bullet)) \longrightarrow R^{q+1}\mathcal{F}((E^\bullet, d_E^\bullet))$ of cohomologies is, by definition, the induced morphism by the morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in (4). By taking injective resolutions $(I^\bullet, d_I^\bullet)$, $(J^\bullet, d_J^\bullet)$ and $(K^\bullet, d_K^\bullet)$ of $(E^\bullet, d_E^\bullet)$, $(F^\bullet, d_F^\bullet)$ and $(G^\bullet, d_G^\bullet)$, respectively, which fit into the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (I^\bullet, d_I^\bullet) & \longrightarrow & (J^\bullet, d_J^\bullet) & \longrightarrow & (K^\bullet, d_K^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (E^\bullet, d_E^\bullet) & \longrightarrow & (F^\bullet, d_F^\bullet) & \longrightarrow & (G^\bullet, d_G^\bullet) & \longrightarrow & 0
\end{array}
\tag{1.0.1}
$$

of complexes of objects in $\mathcal{A}$ such that the upper horizontal sequence is exact, it is easy to check that the boundary morphism ∂ above is equal to the usual boundary morphism obtained by the upper short exact sequence of (1.0.1). (For a short exact sequence in (3), the existence of the commutative diagram (1.0.1) has been proved in, e.g., [NS, (2.7)] as a very special case.)

(6) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$, the identity id: $E^q \longrightarrow E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $\mathcal{H}^q((E^\bullet, -d^\bullet)) \overset{\sim}{\longrightarrow} \mathcal{H}^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies. We sometimes use this convention.

(7) We often denote a complex $(E^\bullet, d^\bullet)$ simply by $(E^\bullet, d)$ or $E^\bullet$ as usual when there is no risk of confusion.

Part I. Degenerations at E_2 of p-Adic Weight Spectral Sequences

2. Brief review on the p-adic weight spectral sequence of an SNCL variety

Let κ be a perfect field of characteristic $p > 0$ and $s = (\mathrm{Spec}(\kappa), \mathbb{N} \oplus \kappa^*)$ a log point. Let W be the Witt ring of κ and K_0 the fraction field of W. Let X/s be a proper SNCL variety of pure dimension (see [Nakk2, §2] for the definition; in this paper, we do not assume that $\overset{\circ}{X}$ is geometrically connected, nor that the irreducible components of $\overset{\circ}{X}$ are geometrically irreducible.). For a positive integer j, let $\overset{\circ}{X}{}^{(j)}$ be the disjoint union of all j-fold intersections of the distinct irreducible components of $\overset{\circ}{X}$. Denote by $H^h_{\text{log-crys}}(X/W)$ $(h \in \mathbb{N})$ the log crystalline cohomology of X/s over W

([Ka2, §5, §6]). Let us recall the p-adic weight spectral sequence of X/s ([M1, 3.23]; see (2.2) (1) and (2) below):

$$(2.0.1)\qquad E_1^{-k,h+k}(X/W) = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\text{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k) \Longrightarrow H^h_{\text{log-crys}}(X/W).$$

The following has been conjectured in [M1, 3.24] (under the assumption of the projectivity of $\overset{\circ}{X}$):

Conjecture 2.1. *The spectral sequence* (2.0.1) *degenerates at* E_2 *modulo torsion.*

If κ is a finite field, (2.1) is true ([M1, 3.32 (2)]) by the purity of the weight of the crystalline cohomology of a proper smooth variety over κ ([CL1, (1.2)] or (2.2) (4) below). In the Part I of this paper we prove (2.1) for any perfect field of characteristic $p > 0$.

Remark 2.2. (1) There are gaps and mistakes in the construction of the p-adic Steenbrink complex in [M1, §3] and in fundamental properties of it; we fill and correct them in §8, §9, §10 and §11 below.

Let $\star$ be a positive integer n or nothing. Let $W_\star\Lambda^\bullet_X$ be a complex of $W_\star(\mathcal{O}_X)$-modules defined in [HK] (cf. [Hy2]) and denoted by $W_\star\omega^\bullet_X$ in [loc. cit.]. Then there is a gap in the proof of [HK, (4.19)] which claims, as a special case, that there exists a canonical morphism

$$H^h_{\text{log-crys}}(X/W_\star)\longrightarrow H^h(X, W_\star\Lambda^\bullet_X).$$

[HK, (4.19)] is necessary for the construction of (2.0.1). We shall fill the gap in (7.19) below.

(2) Let $\star$ be a positive integer n or nothing. Let $\theta_\star \in \Gamma(X, W_\star\widetilde{\Lambda}^1_X)$ ($W_\star\widetilde{\Lambda}^\bullet_X = W_\star\widetilde{\omega}^\bullet_X$ in [Hy2, (1.2.2)] and [M1]) be a global section constructed in [M1, 3.4 (3)]. By convention, we consider only the following wedge product $(-1)^i\theta_\star\wedge\colon W_\star A^{ij}_X \longrightarrow W_\star A^{i,j+1}_X$ in the construction of the p-adic Steenbrink complex, while Mokrane has considered a wedge product $\wedge\theta_\star$ in [loc. cit.]; by following [RZ, p. 29], our boundary morphisms of the double

complex $W_\star A_X^{\bullet\bullet}$ are as follows:

$$
(2.2.1;\star) \qquad \begin{array}{ccc} W_\star A_X^{i,j+1} & & \\ {\scriptstyle (-1)^i\theta_\star\wedge}\uparrow & & \\ W_\star A_X^{ij} & \xrightarrow{(-1)^{j+1}d} & W_\star A_X^{i+1,j}. \end{array}
$$

By this convention, the left wedge product $\theta_\star\wedge\colon W_\star\Lambda_X^\bullet \longrightarrow W_\star A_X^\bullet$ is a morphism of complexes.

In (10.1.2;$\star$) below, we shall describe the boundary morphisms between the E_1-terms of (2.0.1). See also another change in (11.8.1;$\star$) below.

Though we give no motivic statement in this paper, we need to give statements of certain theorems motivically (e,g., [Nakk4], [Nakk5]). This is a reason why we consider only the wedge product $(-1)^i\theta_\star\wedge$. Evidently, something p-adic should not be studied only for itself. Indeed, as in [Nakk5], if we give motivic formulas of the zeta-functions of the h-th ($h \in \mathbb{N}$) log etale and crystalline cohomologies of X over a finite field where the l-adic and p-adic monodromy operators are killed respectively, we are obliged to take the corresponding boundary morphisms of the l-adic and p-adic Steenbrink complexes.

If one considers the right wedge product $\wedge\theta_\star$ and if one wishes to give motivic statements, one has to change numerous morphisms of complexes in [RZ] (and in [SZ]) in order that they are compatible with the right wedge product $\wedge\theta_\star$.

The following boundary morphisms

$$
(2.2.2;\star) \qquad \begin{array}{ccc} W_\star A_X^{i,j+1} & & \\ {\scriptstyle \theta_\star\wedge}\uparrow & & \\ W_\star A_X^{ij} & \xrightarrow{d} & W_\star A_X^{i+1,j} \end{array}
$$

are the analogues of the boundary morphisms in the ∞-adic case in [St1, (4.14)]. In an application for the interpretation of the p-adic monodromy operator by a morphism of a p-adic Steenbrink complex, the p-adic Steenbrink complex with boundary morphism (2.2.1;$\star$) is useful for the *left* monodromy operator, while the p-adic Steenbrink complex with (2.2.2;$\star$) is useful for the *right* monodromy operator. See (11.8.1;$\star$) and (11.12.3) below for the detail: note the difference between the left and the right wedge products of

$\theta_\star$ in them. Note also that the analogue of a morphism in [St1, (4.16)] is equal to the right wedge product without signs

$$W_\star\Lambda^\bullet_X \ni \omega \longmapsto \omega \wedge \theta_\star \in (W_\star A^\bullet_X, d + (\theta \wedge *)).$$

By virtue of a canonical isomorphism stated in (11.12.7) below, the left monodromy operator can be identified with the right monodromy operator.

Mokrane's p-adic double Steenbrink complex with the following boundary morphisms

$$(2.2.3;\star) \qquad \begin{array}{ccc} W_\star A_X^{i,j+1} & & \\ {\scriptstyle d'':=\wedge\theta_\star}\big\uparrow & & \\ W_\star A_X^{ij} & \xrightarrow{d':=(-1)^j d} & W_\star A_X^{i+1,j} \end{array}$$

is useful for the right monodromy ((11.9.1; $\star$) below); however it obliges us to change ν in [M1, 3.13]. See (11.9) below for the detail.

Furthermore, by considering the anti-symmetry of the boundary morphisms of double complexes with signs, one can also consider the Steenbrink complex with the following boundary morphisms

$$(2.2.4;\star) \qquad \begin{array}{ccc} W_\star A_X^{i,j+1} & & \\ {\scriptstyle d'':=(-1)^i(\wedge\theta_\star)}\big\uparrow & & \\ W_\star A_X^{ij} & \xrightarrow{d':=d} & W_\star A_X^{i+1,j}. \end{array}$$

However I do not think about the boundary morphisms (2.2.4; $\star$) any more.

(3) C. Nakayama has proved the E_2-degeneration of the l-adic weight spectral sequence of a proper SNCL variety ([Nak, (2.1)]).

(4) We can show the purity of the Frobenius eigenvalues of the crystalline cohomology of a proper smooth variety over a finite field in [CL1, (1.2)] easily without using [Ch1, I. Theorem 2.2] as follows (cf. [dJ, p. 52, 53]). Assume that κ is a finite field $\mathbb{F}_q$, and let Y be a proper smooth variety over $\mathbb{F}_q$. By [dJ, (4.1)], there exists a surjective morphism $f\colon Z \longrightarrow Y$ from a projective smooth variety. By [Kl, (1.2.4)], the induced morphism $f^*\colon H^h_{\rm crys}(Y/W)\otimes_W K_0 \longrightarrow H^h_{\rm crys}(Z/W)\otimes_W K_0$ ($h \in \mathbb{Z}$) is an injection (In the proof of [Kl, (1.2.4)] in the case of crystalline cohomologies, we need the definition of the cycle class of a singular integral closed subscheme in

the crystalline cohomology of a proper smooth variety over a perfect field of finite characteristic ([Gs, II §4], [GM]) and we need the formula in [Gs, II (4.2.3)], [GM, II (1.1.3)].). Hence [CL1, (1.2)] immediately follows from [KM, Corollary 1 2)] or [Fa, (5.2)]. As a consequence, as in [KM, Theorem 1, Corollary 1 1)] (cf. [CL1, (1.3), (1.4)]), we see that

$$\det(1-tF^*|H^h_{\mathrm{crys}}(Y/W)\otimes_W K_0) = \det(1-tF^*|H^h_{\mathrm{et}}(\overline{Y},\mathbb{Q}_l)), \tag{2.2.5}$$

in particular,

$$\dim_{K_0} H^h_{\mathrm{crys}}(Y/W)\otimes_W K_0 = \dim_{\mathbb{Q}_l} H^h_{\mathrm{et}}(\overline{Y},\mathbb{Q}_l). \tag{2.2.6}$$

Here $F\colon Y \longrightarrow Y$ on the left hand side of (2.2.5) is the q-th power endomorphism and $F\colon \overline{Y} \longrightarrow \overline{Y}$ on the right hand side of (2.2.5) is the usual geometric Frobenius morphism relative to $\mathbb{F}_q$.

Let κ be any field of characteristic $p>0$ and W a Cohen ring of κ. Let Y be a proper smooth variety over κ. Then we obtain the formula (2.2.6) for Y/κ by the perfection of κ, by a standard deformation theory (e.g., [I1, (3.10)], cf. §3 below), by a standard specialization argument on l-adic cohomologies (e.g., [Nak, (2.3)]) and by a specialization argument of Deligne-Illusie ([I1, (3.10)], cf. (3.4) below).

3. Specialization argument in log crystalline cohomology

Following [I1, (3.10)], we give a specialization argument in log crystalline cohomologies, and we apply it to the solution of (2.1). This specialization argument is an example of Grothendieck's idea for the reduction of geometric problems to arithmetic problems ([EGA IV-3 §9], [Gr, Part II, VII 5]).

We keep the notations in §2. By [Nak, (2.2)], there exist a finitely generated subring A_1 of κ over $\mathbb{F}_p$ and a proper SNCL scheme ([Nakk3, §2]) $\mathcal{X}$ over $S_1 := (\mathrm{Spec}(A_1), \mathbb{N}\oplus A_1^*)$ with a natural morphism $s \longrightarrow S_1$ such that $\mathcal{X}\times_{S_1} s = X$. By [EGA IV-2, (6.12.5)], by [EGA IV-4, (17.15.2)] and by the proof of [Ha2, II (8.16)], we can assume that A_1 is a smooth algebra over a finite field $\mathbb{F}_q$. Let A be a p-adically formally smooth algebra over $W(\mathbb{F}_q)$ which is a lift of A_1. Endow $\mathrm{Spf}(A)$ with a log structure $\mathbb{N}\oplus A^*$, and let S be the resulting log formal scheme over $\mathrm{Spf}(W(\mathbb{F}_q)) = (\mathrm{Spf}(W(\mathbb{F}_q)), W(\mathbb{F}_q)^*)$. The log formal scheme S has a PD-ideal $p\mathcal{O}_S$, which defines an exact closed immersion $S_1 \overset{\subset}{\longrightarrow} S$.

For a log formal affine open subscheme T of S, set $T_1 := T \otimes_{W(\mathbb{F}_q)} \mathbb{F}_q$. Take a closed point $\overset{\circ}{t}$ of $\overset{\circ}{T}_1$. The point $\overset{\circ}{t}$ is the spectrum of a finite field κ_t. We fix a lift $F_T\colon \overset{\circ}{T} \longrightarrow \overset{\circ}{T}$ of the Frobenius endomorphism(=p-th power endomorphism) of $\overset{\circ}{T}_1$. Then we have the Teichmüller lift $\mathcal{O}_T \longrightarrow W(\kappa_t)$ (resp. $\mathcal{O}_T \longrightarrow W$) of the morphism $\mathcal{O}_{T_1} \longrightarrow \kappa_t$ (resp. $\mathcal{O}_{T_1} \longrightarrow \kappa$) (e.g., [I2, 0. 1.3]). Here, by abuse of notation, we denote the global section $\Gamma(T, \mathcal{O}_T)$ (resp. $\Gamma(T_1, \mathcal{O}_{T_1})$) by $\mathcal{O}_T$ (resp. $\mathcal{O}_{T_1}$). The rings $W(\kappa_t)$ and W become $\mathcal{O}_T$-algebras by these lifts. Endow $\mathrm{Spf}(W(\kappa_t))$ and $\overset{\circ}{t}$ with the inverse images of the log structure of T. Set $\mathcal{X}_{T_1} := \mathcal{X} \times_{S_1} T_1$ and $\mathcal{X}_t := \mathcal{X}_{T_1} \times_{T_1} t$.

Let R be a p-adically separated and complete flat noetherian $W(\mathbb{F}_q)$-algebra. The ring R has a PD-ideal pR. Let M be a fine log structure on $\mathrm{Spf}(R)$ and set $\mathrm{Spf}(R)^{\log} := (\mathrm{Spf}(R), M)$. Set $R_1 := R/pR$, and denote by $\mathrm{Spec}(R_1)^{\log}$ the log scheme whose underlying scheme is $\mathrm{Spec}(R_1)$ and whose log structure is the pull-back of M. For a log formal affine open subscheme T of S and for a morphism $\mathrm{Spf}(R)^{\log} \longrightarrow T$ of log formal schemes over $\mathrm{Spf}(W(\mathbb{F}_q))$, set $\mathcal{X}_{R_1} := \mathcal{X}_{T_1} \times_{T_1} \mathrm{Spec}(R_1)^{\log}$.

We start with the following:

Lemma 3.1. *Let L be a finitely generated $\mathcal{O}_S$-module. Then there exists a log formal affine open subscheme T of S such that* $\mathrm{Tor}_1^{\mathcal{O}_T}(L|_T, R) = 0$.

Proof. By Deligne's remark in [I1, (3.10)], we may assume that $L|_T \simeq \mathcal{O}_T$ or $L|_T \simeq \mathcal{O}_T/p^m\mathcal{O}_T$ ($m \in \mathbb{N}$) by shrinking T. In the former case, (3.1) is obvious. In the latter case, (3.1) immediately follows from an exact sequence

$$0 \longrightarrow \mathcal{O}_T \xrightarrow{p^m} \mathcal{O}_T \longrightarrow \mathcal{O}_T/p^m\mathcal{O}_T \longrightarrow 0$$

and from the injectivity of $p^m\colon R \longrightarrow R$. (In fact, $\mathrm{Tor}_r^{\mathcal{O}_T}(L|_T, R) = 0$ ($\forall r \in \mathbb{Z}_{\geq 1}$).) $\square$

Proposition 3.2. *There exists a log formal affine open subscheme T of S such that the canonical morphisms $H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \otimes_{\mathcal{O}_T} R \longrightarrow H^h_{\text{log-crys}}(\mathcal{X}_{R_1}/R)$ for all $h \in \mathbb{N}$ are isomorphisms of R-modules.*

Proof. Let $\Gamma^\bullet := (\Gamma^\bullet, d^\bullet)$ be a strictly perfect complex of $\mathcal{O}_T$-modules representing $R\Gamma(\mathcal{X}_{T_1}/T)$ (cf. [BO, 7.14 Definition, 7.24.3 Theorem]). By

(3.1), shrinking T if necessary, we can assume that $\mathrm{Tor}_1^{\mathcal{O}_T}(L|_T, R) = 0$ for $L = H^j(\Gamma^\bullet)$ and $L = \mathrm{Im}(d^j)$ $(\forall j \in \mathbb{Z})$ since L is a finitely generated $\mathcal{O}_S$-module in either case. Hence $H^h(\Gamma^\bullet \otimes_{\mathcal{O}_T} R) = H^h(\Gamma^\bullet) \otimes_{\mathcal{O}_T} R$. Because the morphism $\mathcal{X}_{T_1} \longrightarrow T_1$ is integral by [Ka2, (4.4) (ii)], the base change theorem $R\Gamma(\mathcal{X}_{T_1}/T) \otimes^L_{\mathcal{O}_T} R = R\Gamma(\mathcal{X}_{R_1}/R)$ holds ([Ka2, (6.10)], cf. [Og1, (3.3)]). By taking the cohomologies of the both hands of the equality above, we obtain (3.2). □

REMARK 3.3. Set $K := R \otimes_{\mathbb{Z}} \mathbb{Q}$. If one wishes only to prove (3.6) below, one may replace (3.2) with the following weaker proposition:

$$H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \otimes_{\mathcal{O}_T} K = H^h_{\text{log-crys}}(\mathcal{X}_{R_1}/R) \otimes_R K \quad (\forall h \in \mathbb{Z}).$$

This follows from the existence of a strictly perfect complex of $\mathcal{O}_T$-modules representing $R\Gamma(\mathcal{X}_{T_1}/T)$ and the flatness of $H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \otimes_{\mathbb{Z}} \mathbb{Q}$ as an $\mathcal{O}_T \otimes_{\mathbb{Z}} \mathbb{Q}$-module. By shrinking T if necessary, we can easily check this flatness by Deligne's remark [I1, (3.10)]. In fact, this shrinkage is not necessary by [Og2, Lemma 36]; $H^h_{\text{log-crys}}(\mathcal{X}/S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a locally free $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathbb{Q}$-module.

The following has been suggested by L. Illusie:

COROLLARY 3.4. *There exists a log formal affine open subscheme T of S such that the canonical morphism*

$$H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \otimes_{\mathcal{O}_T} W(\kappa_t) \longrightarrow H^h_{\text{log-crys}}(\mathcal{X}_t/W(\kappa_t)) \tag{3.4.1}$$

for all $h \in \mathbb{N}$ and any closed point $\mathring{t}$ of $\mathring{T}_1$

$$(\textit{resp.}\ (3.4.2) \qquad H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \otimes_{\mathcal{O}_T} W \longrightarrow H^h_{\text{log-crys}}(X/W))$$

is an isomorphism of $W(\kappa_t)$-modules (resp. W-modules). There also exists the following (non-canonical) isomorphism of $W \otimes_{W(\mathbb{F}_q)} W(\kappa_t) \simeq W(\kappa_t) \otimes_{W(\mathbb{F}_q)} W$-modules:

$$\begin{aligned} &H^h_{\text{log-crys}}(X/W) \otimes_{W(\mathbb{F}_q)} W(\kappa_t) \\ &\quad \xrightarrow{\sim} H^h_{\text{log-crys}}(\mathcal{X}_t/W(\kappa_t)) \otimes_{W(\mathbb{F}_q)} W \quad (\forall h \in \mathbb{N}). \end{aligned} \tag{3.4.3}$$

Proof. (3.4.1) and (3.4.2) follow from (3.2). By Deligne's remark [I1, (3.10)], there exists a finitely generated $W(\mathbb{F}_q)$-module M such that $H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T) \simeq M\otimes_{W(\mathbb{F}_q)}\mathcal{O}_T$ for a small log formal affine open subscheme T of S. Hence we have the following isomorphism:

$$\begin{aligned}&H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T)\otimes_{\mathcal{O}_T}W\otimes_{W(\mathbb{F}_q)}W(\kappa_t)\\ &\qquad\simeq H^h_{\text{log-crys}}(\mathcal{X}_{T_1}/T)\otimes_{\mathcal{O}_T}W(\kappa_t)\otimes_{W(\mathbb{F}_q)}W.\end{aligned}$$

Thus (3.4.3) follows from (3.4.1) and (3.4.2). □

Proposition 3.5. *There exists a log formal affine open subscheme T of S such that, for any closed point $\overset{\circ}{t}$ of $\overset{\circ}{T}_1$ and for all k, $h \in \mathbb{Z}$, there exists a (non-canonical) isomorphism*

$$E_2^{-k,h+k}(X/W)\otimes_{W(\mathbb{F}_q)}W(\kappa_t) \xrightarrow{\sim} E_2^{-k,h+k}(\mathcal{X}_t/W(\kappa_t))\otimes_{W(\mathbb{F}_q)}W$$

of $W\otimes_{W(\mathbb{F}_q)}W(\kappa_t) \simeq W(\kappa_t)\otimes_{W(\mathbb{F}_q)}W$-modules.

Proof. Because we consider no (relative) Frobenius action on (log) crystalline cohomologies in this proof, we ignore the Tate twists in (2.0.1). By [I1, (3.10)] (cf. the proof of (3.2)), there exists a log formal affine open subscheme T of S such that, for all $j \in \mathbb{Z}_{\geq 1}$,

$$H^h_{\text{crys}}(\overset{\circ}{X}{}^{(j)}/W) = H^h_{\text{crys}}(\overset{\circ}{\mathcal{X}}{}^{(j)}_{T_1}/\overset{\circ}{T})\otimes_{\mathcal{O}_T}W$$

and

$$H^h_{\text{crys}}(\overset{\circ}{\mathcal{X}}_t{}^{(j)}/W(\kappa_t)) = H^h_{\text{crys}}(\overset{\circ}{\mathcal{X}}{}^{(j)}_{T_1}/\overset{\circ}{T})\otimes_{\mathcal{O}_T}W(\kappa_t).$$

Let $d_1^{\bullet\bullet}(X)$ (resp. $d_1^{\bullet\bullet}(\mathcal{X}_t)$) be a boundary morphism between E_1-terms of (2.0.1) for X (resp. $\mathcal{X}_t$). The morphisms $d_1^{\bullet\bullet}(X)$ and $d_1^{\bullet\bullet}(\mathcal{X}_t)$ are the sums (with signs) of the induced morphisms of closed immersions and Gysin morphisms by (10.1) below; there are mistakes in [M1, 4.14]; see (10.2) (4) below for the corrections and see also (10.2) (5) below. By [B1, VI (3.3.10)], there exists the Gysin morphism for a smooth pair over any PD scheme. Set

$$F^{-k,h+k} := \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\text{crys}}(\overset{\circ}{\mathcal{X}}{}^{(2j+k+1)}_{T_1}/\overset{\circ}{T})$$

and

$$G^{-k,h+k} := \mathrm{Ker}(F^{-k,h+k} \longrightarrow F^{-k+1,h+k})/\mathrm{Im}(F^{-k-1,h+k} \longrightarrow F^{-k,h+k}).$$

Here the two morphisms on the right hand side in the definition of $G^{-k,h+k}$ are the sums (with signs) of the induced morphisms of closed immersions and Gysin morphisms as in (10.1.2;$\star$). By the base change of Gysin morphisms ([B1, VI Theorem 4.3.12]) and by (3.1), we obtain

$$E_2^{-k,h+k}(X/W) = G^{-k,h+k} \otimes_{\mathcal{O}_T} W$$

and

$$E_2^{-k,h+k}(\mathcal{X}_t/W(\kappa_t)) = G^{-k,h+k} \otimes_{\mathcal{O}_T} W(\kappa_t)$$

for all k, h by shrinking T if necessary. By Deligne's remark, there exists a finitely generated $W(\mathbb{F}_q)$-module $M^{-k,h+k}$ such that $M^{-k,h+k} \otimes_{W(\mathbb{F}_q)} \mathcal{O}_T \simeq G^{-k,h+k}$ by shrinking T. Hence, as in (3.4.3), there exists an isomorphism

$$E_2^{-k,h+k}(X/W) \otimes_{W(\mathbb{F}_q)} W(\kappa_t) \xrightarrow{\sim} E_2^{-k,h+k}(\mathcal{X}_t/W(\kappa_t)) \otimes_{W(\mathbb{F}_q)} W. \ \square$$

THEOREM 3.6. (2.1) *is true.*

PROOF. Let the notations be as in (3.4). Let $K_0(\kappa_t)$ be the fraction field of $W(\kappa_t)$. By (3.5), we obtain

$$\text{(3.6.1)} \qquad \dim_{K_0}(E_2^{-k,h+k}(X/W) \otimes_W K_0) = \dim_{K_0(\kappa_t)}(E_2^{-k,h+k}(\mathcal{X}_t/W(\kappa_t)) \otimes_{W(\kappa_t)} K_0(\kappa_t)).$$

By the purity of the weight ([CL1, (1.2)] or (2.2) (4)) and the yoga of weight, (2.1) is true for $\mathcal{X}_t$. By (3.6.1) and (3.4.3), we obtain (3.6). $\square$

REMARK 3.7. Because we do not construct the p-adic weight spectral sequence for $\mathcal{X}_{T_1}/T$ in this paper, we cannot consider a boundary morphism $d_r^{\bullet\bullet}(\mathcal{X}_{T_1})$ of E_r-terms ($r \geq 2$) of the non-constructed spectral sequence for $\mathcal{X}_{T_1}/T$ a priori; in this paper we cannot reduce the vanishing of $d_r^{\bullet\bullet}(X)$ ($r \geq 2$) modulo torsion to that of $d_r^{\bullet\bullet}(\mathcal{X}_{T_1})$ and then to that of $d_r^{\bullet\bullet}(\mathcal{X}_t)$ in a more standard way. But it is possible to construct the p-adic weight spectral sequence of $\mathcal{X}_{T_1}/T$ by using a (log) crystalline method. We shall discuss this in another paper.

4. The (pre)weight spectral sequences of the log Hodge-Witt sheaves of an SNCL variety

Let the notations be as in §2. In this section we prove the E_2-degeneration of the weight spectral sequence of the log Hodge-Witt sheaves on X as an application of (3.6).

Following [Fr], let us denote by $\Lambda^i_{X/\kappa}$ the sheaf of log differential forms on X/κ of degree i. Let n be a positive integer and let $W_n\Lambda^i_X(= W_n\omega^i_X$ in [HK] and [M1]) be the log Hodge-Witt sheaf of the log differential forms on X of degree i.

Theorem 4.1. *Let i be a non-negative fixed integer. Then there exists the following spectral sequences*:

(4.1.1;n)
$$E_1^{-k,h+k}(W_n\Lambda^i_X) = \bigoplus_{j\geq \max\{-k,0\}} H^{h-i-j}(\overset{\circ}{X}{}^{(2j+k+1)}, W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \Longrightarrow H^h(X, W_n\Lambda^i_X\{-i\}) = H^{h-i}(X, W_n\Lambda^i_X),$$

(*Recall the notation* $\{-i\}$ *in the Convention* (1).)

(4.1.1)
$$E_1^{-k,h+k}(W\Lambda^i_X) = \bigoplus_{j\geq \max\{-k,0\}} H^{h-i-j}(\overset{\circ}{X}{}^{(2j+k+1)}, W\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \Longrightarrow H^h(X, W\Lambda^i_X\{-i\}) = H^{h-i}(X, W\Lambda^i_X).$$

Proof. In this proof we use the notation $W_n\widetilde{\Lambda}^\bullet_X$ instead of the notation $W_n\widetilde{\omega}^\bullet_X$ in [Hy2] and [M1, §2].

As in [M1, 3.8], set $W_nA^{ij}_X = W_n\widetilde{\Lambda}^{i+j+1}_X/P_jW_n\widetilde{\Lambda}^{i+j+1}_X$, where P_j is the filtration defined in [M1, 3.5].

By [M1, 3.15] (cf. (6.28) (9), (6.29) (1) below), the following sequence

(4.1.2)
$$0 \longrightarrow W_n\Lambda^i_X \overset{\theta_n\wedge}{\longrightarrow} W_nA^{i0}_X \overset{(-1)^i\theta_n\wedge}{\longrightarrow} W_nA^{i1}_X \overset{(-1)^i\theta_n\wedge}{\longrightarrow} W_nA^{i2}_X \overset{(-1)^i\theta_n\wedge}{\longrightarrow} \cdots$$

is exact. (Note that the side and the sign of the wedge product of θ_n in (4.1.2) are different from those in [M1, 3.8, 3.15] (cf. (2.2) (2)).) Let us consider a single complex

$$(4.1.3)\quad W_nA_X^{i\bullet} := (\cdots \overset{(-1)^i\theta_n\wedge}{\longrightarrow} W_n\widetilde{\Lambda}_X^{i+j+1}/P_jW_n\widetilde{\Lambda}_X^{i+j+1} \overset{(-1)^i\theta_n\wedge}{\longrightarrow} \cdots)_{j\geq 0},$$

and let us define a *preweight filtration* $P_\bullet$ on $W_nA_X^{i\bullet}$ as follows:

$$(4.1.4)\quad P_kW_nA_X^{i\bullet} = (\cdots \overset{(-1)^i\theta_n\wedge}{\longrightarrow} (P_{2j+k+1}+P_j)(W_n\widetilde{\Lambda}_X^{i+j+1})/P_jW_n\widetilde{\Lambda}_X^{i+j+1} \overset{(-1)^i\theta_n\wedge}{\longrightarrow} \cdots)_{j\geq 0}.$$

First, we ignore the compatibility of the spectral sequence (4.1.1;n) with the Frobenius. Then, by [M1, 3.7], we have

$$(4.1.5)\quad \begin{aligned}\mathrm{gr}_k^PW_nA_X^{i\bullet} &= \bigoplus_{j\geq \max\{-k,0\}} \mathrm{gr}^P_{2j+k+1}W_n\widetilde{\Lambda}_X^{i+j+1}\{-j\}\\ &= \bigoplus_{j\geq \max\{-k,0\}} W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-j\}.\end{aligned}$$

The compatibility of (4.1.5) with the Frobenius is considerably delicate. Because the proof for this fact is quite long, we prove this in §9 below (the proof of (9.9) below).

Thus we obtain (4.1.1;n).

By (8.6) (2) below, we obtain the complex $WA_X^{i\bullet}$ by taking the projective limit with respect to the projection $\pi\colon W_{n+1}A_X^{i\bullet} \longrightarrow W_nA_X^{i\bullet}$ ($n\in\mathbb{Z}_{>0}$). By (8.6) (5) and by the proof of (9.9) (cf. (9.11)), we obtain (4.1.1). □

Definition 4.2. We call (4.1.1;n) the *preweight spectral sequence* of $W_n\Lambda_X^i$ and call (4.1.1) the *weight spectral sequence* of $W\Lambda_X^i$. We define the *preweight filtration* P on $H^{h-i}(X, W_n\Lambda_X^i)$ and the *weight filtration* P on $H^{h-i}(X, W\Lambda_X^i)$, respectively, as follows:

$$\mathrm{gr}_k^PH^{h-i}(X, W_\star\Lambda_X^i) = E_\infty^{h-k,k}(W_\star\Lambda_X^i)\quad (\star = n \text{ or nothing}).$$

(4.1.1) tells us that the graded pieces of the slope filtration on $H^h_{\text{log-crys}}(X/W)\otimes_W K_0$ have weight filtrations (cf. (4.7.3) below).

REMARK 4.3. In this paper, as in (4.1.4), we call the filtrations P on the (sheaves of) W_n-modules in [M1] the *preweight filtrations*. The name "weight filtration" in e,g., [M1, 3.6] is not appropriate; to take the projective limit is essential for making the yoga of weight work as in the l-adic and p-adic Galois representations.

REMARK 4.4. (1) The spectral sequence (4.1.1;n) is a generalization of a spectral sequence in [RZS, §2], where i is assumed to be zero. (4.1.1;n) has an interesting geometric application: using (4.1.1;n), we have

$$\text{(4.4.1)}\qquad \begin{aligned}&\operatorname{length}_{W_n} H^0(X, (W_n\Lambda^d_X)^{\otimes r})\\ &\quad\geq \operatorname{length}_{W_n} H^0(\overset{\circ}{X}{}^{(1)}, (W_n\Omega^d_{\overset{\circ}{X}{}^{(1)}})^{\otimes r}) \quad (\forall r\in\mathbb{Z}_{\geq 1}),\end{aligned}$$

where $d := \dim \overset{\circ}{X}$; hence we have the following inequality which we call the "lower semicontinuity of (log) Kodaira dimensions" (cf. "The κ-invariant conjecture" of Persson [Per, Introduction p. ix]):

$$\kappa(X, n) \geq \max\{\kappa(X_j, n) | X_j : \text{ an irreducible component of } \overset{\circ}{X}\},$$

where

$$\kappa(X, n) := \varlimsup_{r\longrightarrow\infty} \frac{\log(\operatorname{length}_{W_n} H^0(X, (W_n\Lambda^d_X)^{\otimes r}))}{\log r}$$

and

$$\kappa(X_j, n) := \varlimsup_{r\longrightarrow\infty} \frac{\log(\operatorname{length}_{W_n} H^0(X_j, (W_n\Omega^d_{X_j})^{\otimes r}))}{\log r}).$$

See [Nakk3] for the proof of (4.4.1).

(2) In (10.3) below, we shall describe the boundary morphisms of the E_1-terms of (4.1.1;n).

In order to prove the E_2-degeneration of (4.1.1) modulo torsion, we need *Gysin morphisms in Hodge-Witt cohomologies.*

Let i be an integer. Let Y be a proper smooth scheme over κ and let D be a smooth divisor on Y. By (9.6) (2) below (cf. [M1, 4.6]), we have the following two exact sequences:

$$\text{(4.4.2;}+\text{)}\qquad 0 \longrightarrow W\Omega^\bullet_Y \longrightarrow W\Omega^\bullet_Y(\log D) \xrightarrow{\text{Res}} W\Omega^\bullet_D(-1)\{-1\} \longrightarrow 0,$$

$$0 \longrightarrow W\Omega^i_Y \longrightarrow W\Omega^i_Y(\log D) \xrightarrow{\text{Res}} W\Omega^{i-1}_D(-1) \longrightarrow 0. \tag{4.4.3;+}$$

Let

$$d_+ \colon W\Omega^\bullet_D(-1)\{-1\} \longrightarrow W\Omega^\bullet_Y[1] \tag{4.4.4;+}$$

and

$$d_+^{(i)} \colon W\Omega^{i-1}_D(-1) \longrightarrow W\Omega^i_Y[1] \tag{4.4.5;+}$$

be the boundary morphisms of (4.4.2; +) and (4.4.3; +), respectively. We call the induced boundary morphism $H^j(D, W\Omega^{i-1}_D)(-1) \longrightarrow H^{j+1}(Y, W\Omega^i_Y)$ by $-d_+^{(i)}$ the *Gysin morphism in Hodge-Witt cohomologies.* Recall that the boundary morphism $d_+^{(i)}$ of cohomologies is a classical boundary morphism (Convention (5)) and note the minus sign in $-d_+^{(i)}$.

By [I2, II (3.1.1)], we have the following slope spectral sequences

$$\begin{aligned} E_1^{ij}(Y/W)_+ &:= H^j(Y, W\Omega^i_Y) \\ &\Longrightarrow E^{i+j}(Y/W)_+ := H^{i+j}_{\text{crys}}(Y/W), \end{aligned} \tag{4.4.6;+}$$

$$\begin{aligned} E_1^{ij}(D/W)_+ &:= H^j(D, W\Omega^i_D) \\ &\Longrightarrow E^{i+j}(D/W)_+ := H^{i+j}_{\text{crys}}(D/W). \end{aligned} \tag{4.4.7;+}$$

The spectral sequences (4.4.6; +) and (4.4.7; +) degenerates at E_1 modulo torsion by [I2, II (3.2)].

The following easy lemma is necessary for the proof of (4.7) below.

Lemma 4.5. *The morphisms d_+ and $\{d_+^{(i)}\}_i$ are compatible modulo torsion in the following sense: the induced morphism*

$$E_\infty^{ij}(D/W)_+(-1) \otimes_W K_0 \longrightarrow E_\infty^{i+1,j+1}(Y/W)_+ \otimes_W K_0$$

by

$$d_+ \colon E^{i+j}(D/W)_+(-1) \otimes_W K_0 \longrightarrow E^{i+j+2}(Y/W)_+ \otimes_W K_0$$

is equal to the induced morphism by

$$d_+^{(i)} \colon E_1^{ij}(D/W)_+(-1) \otimes_W K_0 \longrightarrow E_1^{i+1,j+1}(Y/W)_+ \otimes_W K_0.$$

PROOF. The restriction of d_+ obtained by the following spectral sequence

$$0 \longrightarrow W\Omega_Y^{\bullet \geq i} \longrightarrow W\Omega_Y^{\bullet \geq i}(\log D) \xrightarrow{\mathrm{Res}} (W\Omega_D^{\bullet}(-1)\{-1\})^{\geq i} \longrightarrow 0$$

induces $d_+^{(i)}$ on (4.4.3; +). Thus (4.5) is clear. $\square$

REMARK 4.6. For the proof of (10.3) which will be used in (4.7) below, we have to pay the following attention to signs. The reader shall know that, in this place of this paper, this remark is indispensable for the determination of signs before the Gysin morphism G in (10.3) after he reads the proofs of (10.1) and (4.7).

Let l and i be two non-negative integers. Let $(I^{\bullet i}, \delta)$, $(J^{\bullet i}, \delta)$, $(K^{\bullet i}, \delta)$ be the Godement resolutions of $W\Omega_Y^i$, $W\Omega_Y^i(\log D)$ and $W\Omega_D^{i-1}(-1)$, respectively. Let us make a convention on signs of the boundary morphisms of $I^{\bullet\bullet}$ as follows: Let d be the induced morphism $I^{li} \longrightarrow I^{l,i+1}$ by the morphism $d\colon W\Omega_Y^i \longrightarrow W\Omega_Y^{i+1}$. Then we fix the boundary morphisms as follows:

$$\delta\colon I^{li} \longrightarrow I^{l+1,i},$$

$$(-1)^l d\colon I^{li} \longrightarrow I^{l,i+1}.$$

We make the same convention for $J^{\bullet\bullet}$ and $K^{\bullet\bullet}$. Then we have double complexes $I^{\bullet\bullet}$, $J^{\bullet\bullet}$ and $K^{\bullet\bullet}$, and single complexes $(I^{\bullet}, d_I)$, $(J^{\bullet}, d_J)$ and $(K^{\bullet}, d_K)$.

Now we change all signs in the boundary morphisms for $I^{\bullet\bullet}$, $J^{\bullet\bullet}$ and $K^{\bullet\bullet}$. For example, we have a double complex $(I^{\bullet\bullet}, \{(-1)^{l+1}d\}_{l\in\mathbb{N}}, -\delta)$ and a single complex $(I^{\bullet}, -d_I)$. Furthermore we have the following two exact sequences of complexes:

$$0 \longrightarrow (I^{\bullet}, -d_I) \longrightarrow (J^{\bullet}, -d_J) \longrightarrow (K^{\bullet}, -d_K) \longrightarrow 0,$$

$$0 \longrightarrow (I^{\bullet i}, -\delta) \longrightarrow (J^{\bullet i}, -\delta) \longrightarrow (K^{\bullet i}, -\delta) \longrightarrow 0.$$

Hence we have two boundary morphisms

$$d_-\colon (K^{\bullet}, -d_K) \longrightarrow (I^{\bullet}, -d_I)[1]$$

and

$$d_-^{(i)}\colon (K^{\bullet i}, -\delta) \longrightarrow (I^{\bullet i}, -\delta)[1].$$

They are equal to

$$(4.4.4;-) \qquad d_-\colon (W\Omega^\bullet_D(-1)\{-1\}, -d) \longrightarrow (W\Omega^\bullet_Y, -d)[1],$$

and

$$(4.4.5;-) \qquad d_-^{(i)}\colon W\Omega^{i-1}_D(-1) \longrightarrow W\Omega^i_Y[1]$$

respectively.

On the other hand, by using the Convention (6), we have

$$H^{i+j}(\Gamma(D,(K^\bullet,-d_K))) = H^{i+j}(\Gamma(D,(K^\bullet,d_K))) = H^{i+j}(D, W\Omega^\bullet_D)$$

and

$$H^{i+j}(\Gamma(Y,(I^\bullet,-d_I))) = H^{i+j}(\Gamma(Y,(I^\bullet,d_I))) = H^{i+j}(Y, W\Omega^\bullet_Y).$$

Hence we have the analogue of $(4.4.6;+)$:

$$(4.4.6;-) \qquad \begin{aligned} E_1^{ij}(Y/W)_- &:= H^j(Y, W\Omega^i_Y) \\ &\Longrightarrow E^{i+j}(Y/W)_- := H^{i+j}_{\mathrm{crys}}(Y/W). \end{aligned}$$

By the same way, we have the analogue of $(4.4.7;+)$:

$$(4.4.7;-) \qquad \begin{aligned} E_1^{ij}(D/W)_- &:= H^j(D, W\Omega^i_D) \\ &\Longrightarrow E^{i+j}(D/W)_- := H^{i+j}_{\mathrm{crys}}(D/W). \end{aligned}$$

Then, by the same proof of (4.5), we have the compatibility of $\{d_-^{(i)}\}$ with d_- modulo torsion. Moreover, it is easy to see that, by the constructions of d_- and $d_-^{(i)}$, the induced morphisms

$$d_-\colon H^h_{\mathrm{crys}}(D/W)(-1) \longrightarrow H^{h+2}_{\mathrm{crys}}(Y/W)$$

and

$$d_-^{(i)}\colon H^j(D, W\Omega^i_D)(-1) \longrightarrow H^{j+1}(Y, W\Omega^{i+1}_Y)$$

are equal to the induced morphisms by $-d_+$ and $-d_+^{(i)}$, respectively.

THEOREM 4.7. *The spectral sequence* (4.1.1) *degenerates at* E_2 *modulo torsion.*

PROOF. Let Y be a proper smooth scheme over κ. By [I2, II (3.5)], we have the following slope decomposition

$$\bigoplus_{i=0}^{h} H^{h-i}(Y, W\Omega_Y^i)\otimes_W K_0 = H^h_{\rm crys}(Y/W)\otimes_W K_0. \tag{4.7.1}$$

The direct sum decomposition (4.7.1) is functorial; moreover (4.7.1) is also compatible with Gysin morphisms in crystalline cohomologies and those in Hodge-Witt cohomologies by (4.5). By (4.1.1) and (10.3) below (cf. [M1, 4.9, 4.12]), we have an analogous diagram of [loc. cit., 4.14] for $\{H^{h-i}(X, W\Lambda_X^i)\}_{h,i\in\mathbb{Z}_{\geq 0}}$. Hence we obtain

$$\begin{aligned}\dim_{K_0}&(E_2^{-k,h+k}(X/W)\otimes_W K_0)\\ &= \sum_{i=0}^{h}\dim_{K_0}(E_2^{-k,h+k}(W\Lambda_X^i)\otimes_W K_0).\end{aligned} \tag{4.7.2}$$

By the formalism of $W\Lambda_X^{\bullet}$ [HK, §4], we also have the following slope decomposition as in [I2, II (3.5)]:

$$\bigoplus_{i=0}^{h} H^{h-i}(X, W\Lambda_X^i)\otimes_W K_0 = H^h_{\text{log-crys}}(X/W)\otimes_W K_0. \tag{4.7.3}$$

By (4.7.3) and (3.6), we have the following equality:

$$\begin{aligned}\sum_{i=0}^{h}&\dim_{K_0}(H^{h-i}(X, W\Lambda_X^i)\otimes_W K_0)\\ &= \sum_{k\in\mathbb{Z}}\dim_{K_0}(E_2^{-k,h+k}(X/W)\otimes_W K_0).\end{aligned} \tag{4.7.4}$$

By (4.7.2) and (4.7.4), we obtain

$$\begin{aligned}\sum_{i=0}^{h}&\dim_{K_0}(H^{h-i}(X, W\Lambda_X^i)\otimes_W K_0)\\ &= \sum_{i=0}^{h}\sum_{k\in\mathbb{Z}}\dim_{K_0}(E_2^{-k,h+k}(W\Lambda_X^i)\otimes_W K_0).\end{aligned}$$

Since

$$\dim_{K_0}(H^{h-i}(X, W\Lambda^i_X)\otimes_W K_0) \leq \sum_{k\in\mathbb{Z}} \dim_{K_0}(E_2^{-k,h+k}(W\Lambda^i_X)\otimes_W K_0)$$

for each i, we obtain

$$\dim_{K_0}(H^{h-i}(X, W\Lambda^i_X)\otimes_W K_0) = \sum_{k\in\mathbb{Z}} \dim_{K_0}(E_2^{-k,h+k}(W\Lambda^i_X)\otimes_W K_0).$$

Thus we can finish the proof of (4.7). □

Remark 4.8. Assume that the base field is the complex number field. Let X be a proper analytic SNCL variety over a log point $(\mathrm{Spec}(\mathbb{C})^{\mathrm{an}}, \mathbb{N}\oplus\mathbb{C}^*)$. Then we have an analogue of (4.1.1)

(4.8.1)
$$E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-i-j}(\overset{\circ}{X}{}^{(2j+k+1)}, \Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}/\mathbb{C}})(-j-k) \Longrightarrow H^{h-i}(X, \Lambda^i_{X/\mathbb{C}})$$

by [St2, (5.5)] and by the same proof as that of (4.1). ((4.8.1) has geometric applications as in (4.4) (1).)

Let $\epsilon\colon X^{\log} \longrightarrow X$ be the real blow up of X ([KN, (1.2)]) which is denoted by τ in [loc. cit.]; there is a natural map $X^{\log} \longrightarrow \mathbb{S}^1$ of topological spaces, where $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Let $\mathbb{R} \longrightarrow \mathbb{S}^1$ be the universal covering of $\mathbb{S}^1$. Take a fiber product $X_\infty := X^{\log} \times_{\mathbb{S}^1} \mathbb{R}$ ([U]). In [FN], Fujisawa and Nakayama have constructed the following spectral sequence

(4.8.2)
$$E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}, \mathbb{Q})(-j-k) \Longrightarrow H^h(X_\infty, \mathbb{Q}).$$

The spectral sequence (4.8.2) degenerates at E_2 by the theory of the weight in Hodge theory (cf. [D4, (8.1.9) (iv)]) if the irreducible components of $\overset{\circ}{X}$ are Kähler manifolds or the analytifications of algebraic varieties (cf. [D1, (5.3)]). Moreover, they have proved that there exists an isomorphism $H^h(X_\infty, \mathbb{C}) \overset{\sim}{\longrightarrow} H^h(X, A^\bullet_{\mathbb{C}})$. Here we fix the horizontal boundary morphism and the vertical one of the Steenbrink double complex $A^{\bullet\bullet}_{\mathbb{C}}$ (cf. [St2, (5.3)]) as in (2.2.1;$\star$), and $A^{\bullet}_{\mathbb{C}}$ is the single complex of this $A^{\bullet\bullet}_{\mathbb{C}}$. By [St2, (5.5)], we have an isomorphism $H^h(X, A^{\bullet}_{\mathbb{C}}) \overset{\sim}{\longleftarrow} H^h(X, \Lambda^{\bullet}_{X/\mathbb{C}})$. If the irreducible components of $\overset{\circ}{X}$ are Kähler manifolds or the analytifications of algebraic varieties, then the log Hodge-de Rham spectral sequence

$$(4.8.3) \qquad E_1^{i,j} = H^j(X, \Lambda^i_{X/\mathbb{C}}) \Longrightarrow H^{i+j}(X, \Lambda^{\bullet}_{X/\mathbb{C}})$$

degenerates at E_1 by mixed Hodge theory (cf. [D4, (8.1.9) (v)]). (If X is algebraic, we can also prove the degeneration at E_1 of (4.8.3) by the method of Deligne-Illusie [DI, (2.7)] and by [Ka2, (4.12) (3)].) Hence we have $\dim_{\mathbb{C}} H^h(X_\infty, \mathbb{C}) = \sum_{i+j=h} \dim_{\mathbb{C}} H^j(X, \Lambda^i_{X/\mathbb{C}})$. In [Nakk4], we have proved that the boundary morphisms between the E_1-terms of (4.8.2) are expressed by Gysin morphisms and the induced morphisms of closed immersions. Hence (4.8.1) degenerates at E_2 by the same proof as that of (4.7).

5. p-adic weight spectral sequences of open smooth varieties

Let κ, W and K_0 be as in §2. Let $f\colon X \longrightarrow \mathrm{Spec}(\kappa)$ be a proper smooth scheme and D an SNCD on X. Set $U := X \setminus D$. Consider the following log structure M on X: $M := \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\}$. We denote by $H^h((X,D)/W)$ the log crystalline cohomology $H^h((X,M)/W)$. By Shiho's comparison theorem ([Sh, Corollary 2.4.13, Theorem 3.1.1]), $H^h((X,D)/W) \otimes_W K_0 = H^h_{\mathrm{rig}}(U/K_0)$. In particular, $H^h((X,D)/W) \otimes_W K_0$ is independent of the choice of the compactification (X,D) of U.

For the completeness of this paper, let us state some facts which will be needed in (5.2) and (5.9) below. We can apply the general theory of log de Rham-Witt complexes in [HK, §4] (cf. [Hy1], [Hy2]) to our situation above because the morphism $(X,M) \longrightarrow (\mathrm{Spec}(\kappa), \kappa^*)$ of log schemes is log smooth and of Cartier type. In particular, we have a log de Rham-Witt complex $W_\star\Omega^{\bullet}_X(\log D)$ on X ($\star = n \in \mathbb{Z}_{>0}$ or nothing) and a canonical isomorphism

$$(5.0.1) \qquad H^h((X,D)/W) \overset{\sim}{\longrightarrow} H^h(X, W\Omega^{\bullet}_X(\log D))$$

by (7.19) below.

By [M1, 1.4.5] and by (9.3) (1) below, there exists an isomorphism

$$(5.0.2;n) \qquad \mathrm{Res}\colon \mathrm{gr}_k^P W_n\Omega^{\bullet}_X(\log D) \overset{\sim}{\longrightarrow} W_n\Omega^{\bullet}_{D^{(k)}}(-k)\{-k\} \quad (n \in \mathbb{Z}_{>0}).$$

Here $D^{(0)} := X$, and $D^{(k)}$ ($k \in \mathbb{Z}_{>0}$) is the disjoint union of all k-fold intersections of the distinct irreducible components of D, and P is the *preweight filtration* on $W_n\Omega^\bullet_X(\log D)$ defined in [M1, (1.4.1), 1.4.4] (cf. (4.3)). By (9.6) (2) below, there exists an isomorphism

$$\text{(5.0.2)} \qquad \mathrm{Res}\colon \mathrm{gr}_k^P W\Omega^\bullet_X(\log D) \overset{\sim}{\longrightarrow} W\Omega^\bullet_{D^{(k)}}(-k)\{-k\}.$$

By (5.0.2), we obtain the following spectral sequence (cf. [M2, (3.1)]; see (9.3) (1) below.):

$$\text{(5.0.3)} \qquad E_{1,\mathrm{dW}}^{-k,h+k} = H^{h-k}_{\mathrm{crys}}(D^{(k)}/W)(-k) \Longrightarrow H^h((X,D)/W).$$

By [M1, 4.9], the boundary morphisms $\{d_1^{\bullet\bullet}\}$ of (5.0.3) are described by Gysin morphisms.

Recall the complex $W_\star\Omega^\bullet_X(-\log D)$ ($\star = n \in \mathbb{Z}_{>0}$ or nothing) in [Hy1, 1] and set $H^h_{\mathrm{c}}((X,D)/W_\star) := H^h(X, W_\star\Omega^\bullet_X(-\log D))$. By [M1, 3.15.1] (cf. (6.29) (1) below), $W_n\Omega^\bullet_X(-\log D)$ has a resolution

$$\text{(5.0.4;}n\text{)} \qquad \begin{aligned} 0 \longrightarrow W_n\Omega^\bullet_X(-\log D) &\longrightarrow W_n\Omega^\bullet_X \\ &\longrightarrow W_n\Omega^\bullet_{D^{(1)}} \longrightarrow W_n\Omega^\bullet_{D^{(2)}} \longrightarrow \cdots . \end{aligned}$$

Because the projections

$$\pi\colon W_{n+1}\Omega^\bullet_X(-\log D) \longrightarrow W_n\Omega^\bullet_X(-\log D)$$

and

$$\pi\colon W_{n+1}\Omega^\bullet_{D^{(k)}} \longrightarrow W_n\Omega^\bullet_{D^{(k)}} \quad (k \in \mathbb{N})$$

are surjective ([Hy1, p. 301]), we have the following exact sequence by taking the projective limit of (5.0.4;n):

$$\text{(5.0.4)} \qquad \begin{aligned} 0 \longrightarrow W\Omega^\bullet_X(-\log D) &\longrightarrow W\Omega^\bullet_X \\ &\longrightarrow W\Omega^\bullet_{D^{(1)}} \longrightarrow W\Omega^\bullet_{D^{(2)}} \longrightarrow \cdots . \end{aligned}$$

Let $\{D_n\}_{n=1}^m$ be the irreducible components of D. For positive integers $n_0, \ldots, n_k \leq m$, set $D_{n_0\cdots n_k} := D_{n_0} \cap \cdots \cap D_{n_k}$. Let $\iota^{n_0\cdots\hat{n}_j\cdots n_k}_{n_0\cdots n_k}\colon D_{n_0\cdots n_k} \overset{\subset}{\longrightarrow} D_{n_0\cdots\hat{n}_j\cdots n_k}$ be the natural closed immersion. Set

$$\text{(5.0.5)} \qquad \begin{aligned} \iota^{(k)*} := \sum_{1\leq n_0<\cdots<n_k\leq m} \sum_{j=0}^{k} (-1)^j \iota^{n_0\cdots\hat{n}_j\cdots n_k *}_{n_0\cdots n_k}\colon \\ W\Omega^\bullet_{D^{(k)}} \longrightarrow W\Omega^\bullet_{D^{(k+1)}}. \end{aligned}$$

Then the following holds:

PROPOSITION 5.1. *The cohomology* $H^h_{\mathrm{c}}((X,D)/W)$ *with compact support has the following spectral sequence*

$$E^{k,h-k}_{1,\mathrm{dW,c}} = H^{h-k}_{\mathrm{crys}}(D^{(k)}/W) \Longrightarrow H^h_{\mathrm{c}}((X,D)/W) \tag{5.1.1}$$

such that the boundary morphism $d_1^{k,h-k}\colon E^{k,h-k}_{1,\mathrm{dW,c}} \longrightarrow E^{k+1,h-k}_{1,\mathrm{dW,c}}$ *is equal to the morphism* $\iota^{(k)*}\colon H^{h-k}_{\mathrm{crys}}(D^{(k)}/W) \longrightarrow H^{h-k}_{\mathrm{crys}}(D^{(k+1)}/W)$ *induced by* (5.0.5).

PROOF. By (5.0.4), $W\Omega^\bullet_X(-\log D)$ is quasi-isomorphic to the single complex of the following double complex

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
d\big\uparrow & & -d\big\uparrow & & d\big\uparrow & & \\
W\Omega^2_X & \xrightarrow{\iota^{(0)*}} & W\Omega^2_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W\Omega^2_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots \\
d\big\uparrow & & -d\big\uparrow & & d\big\uparrow & & \\
W\Omega^1_X & \xrightarrow{\iota^{(0)*}} & W\Omega^1_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W\Omega^1_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots \\
d\big\uparrow & & -d\big\uparrow & & d\big\uparrow & & \\
W\Omega^0_X & \xrightarrow{\iota^{(0)*}} & W\Omega^0_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W\Omega^0_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots .
\end{array} \tag{5.1.2}$$

Hence $H^h_{\mathrm{c}}((X,D)/W)$ has the following spectral sequence by considering the stupid filtration σ of (5.1.2) with respect to the columns:

$$E^{k,h-k}_{1,\mathrm{dW,c}} = H^h(D^{(k)}, W\Omega^\bullet_{D^{(k)}}[-k]) \Longrightarrow H^h_{\mathrm{c}}((X,D)/W). \tag{5.1.3}$$

If k is even,

$$H^h(D^{(k)}, W\Omega^\bullet_{D^{(k)}}[-k]) = H^h(D^{(k)}, W\Omega^\bullet_{D^{(k)}}\{-k\}) = H^{h-k}(D^{(k)}, W\Omega^\bullet_{D^{(k)}}).$$

If k is odd, we have

$$H^h(D^{(k)}, W\Omega^\bullet_{D^{(k)}}[-k]) = H^h(D^{(k)}, W\Omega^\bullet_{D^{(k)}}\{-k\}) = H^{h-k}(D^{(k)}, W\Omega^\bullet_{D^{(k)}})$$

by using the Convention (6). Therefore we can identify (5.1.3) with the following spectral sequence:

$$E_{1,\mathrm{dW,c}}^{k,h-k} = H_{\mathrm{crys}}^{h-k}(D^{(k)}/W) \Longrightarrow H_{\mathrm{c}}^{h}((X,D)/W). \tag{5.1.4}$$

We claim that the boundary morphism $d_1^{k,h-k}$ is equal to the morphism

$$\iota^{(k)*} \colon H_{\mathrm{crys}}^{h-k}(D^{(k)}/W) \longrightarrow H_{\mathrm{crys}}^{h-k}(D^{(k+1)}/W).$$

Indeed, we have the following exact sequence

$$\begin{aligned} 0 \longrightarrow W\Omega^{\bullet}_{D^{(k+1)}}[-(k+1)] &\longrightarrow \sigma^{k}/\sigma^{k+2}((5.1.2)) \\ &\longrightarrow W\Omega^{\bullet}_{D^{(k)}}[-k] \longrightarrow 0. \end{aligned} \tag{5.1.5}$$

Hence $d_1^{k,h-k}$ is equal to the induced morphism by the boundary morphism

$$W\Omega^{\bullet}_{D^{(k)}}[-k] \longrightarrow W\Omega^{\bullet}_{D^{(k+1)}}[-(k+1)][1] = W\Omega^{\bullet}_{D^{(k+1)}}[-k]$$

by the Convention (4) and (5). By the Convention (3), (4) and by taking the Godement resolution of three complexes in (5.1.5) (cf. the proof of (10.1) below), we can easily check that this is equal to $\iota^{(k)*}$. □

We call (5.0.3) (resp. (5.1.1)) the *p-adic weight spectral sequence* of $H^h((X,D)/W)$ (resp. $H_{\mathrm{c}}^h((X,D)/W)$).

When κ is a finite field, Mokrane has proved the E_2-degeneration of (5.0.3) modulo torsion in [M2, (3.2)] (under the assumption of the projectivity of X; we need not assume this projectivity by [CL1, (1.2)] or (2.2) (4)). The following (1) is a generalization of this result.

THEOREM 5.2. (1) *The spectral sequence* (5.0.3) *degenerates at* E_2 *modulo torsion.*

(2) *The spectral sequence* (5.1.1) *degenerates at* E_2 *modulo torsion.*

PROOF. (1): The proof is the same as that of (3.6): all we need are the existence of a model of (X,D) over a smooth affine scheme over a finite field ([EGA IV-3, (8.9.1) (iii)]), the log base change theorem ([Ka2, (6.10)]), the strict perfectness of the complex which produces the log crystalline cohomologies of (X,M) (cf. the proof of (3.2)), the existence of the Gysin morphism over a PD-scheme, Deligne's remark and the purity of the weight in [CL1, (1.2)] or (2.2) (4).

(2): (2) follows from (1), (5.3) (2) below and the duality between the E_2-terms of (5.0.3)$\otimes_W K_0$ and (5.1.1)$\otimes_W K_0$. (Note that W_n is an injective module over W_n itself.) □

The following (1) is a log version of Ekedahl's duality (cf. [Ek1, II (2.2.23)]), and (2) is a Poincaré duality which has been proved in [M2, (4.2)] (cf. [Hy2]), though the proof in [loc. cit.] is quite sketchy.

THEOREM 5.3. *Assume that X is of pure dimension d. Let n be a positive integer. Then the following hold*:

(1) *There exists a canonical perfect pairing of W_n-modules*

$$H^j(X, W_n\Omega^i_X(\log D))\otimes_{W_n} H^{d-j}(X, W_n\Omega^{d-i}_X(-\log D)) \longrightarrow W_n. \tag{5.3.1}$$

(2) *There exists a canonical perfect pairing of W_n-modules*

$$H^j(X, W_n\Omega^\bullet_X(\log D))\otimes_{W_n} H^{2d-j}(X, W_n\Omega^\bullet_X(-\log D)) \longrightarrow W_n. \tag{5.3.2}$$

PROOF. Let $f_n\colon W_n(X) \longrightarrow \mathrm{Spec}(W_n)$ be the projection. Then, by [Ek1, I] (here we have to use [Co, (2.2.7)]), $f_n^! W_n = W_n\Omega^d_X[d]$. Hence (1) follows from [Ha1, VII Corollary 3.4 (c)] (cf. [Co, Theorem 3.4.4]) and [Hy2, (3.3.1)]. (2) follows from (1) as in [B1, VII 2.1.5]. □

REMARK 5.4. (1) By the finite length version of (5.0.1), (5.3) (2) and Tsuji's Poincaré duality ([T]), we obtain

$$H^h_{\mathrm{c}}((X, D)/W_n) = H^h(((X, M)/W_n)^{\log}_{\mathrm{crys}}, K_{X/W_n}).$$

(See [loc. cit., §5] for the definition of K_{X/W_n}. See also [NS, §19] for another proof of the equality above.) Hence, by [T], [Sh, Corollary 2.4.13, Theorem 3.1.1] and [B4, (2.4)], we obtain $H^h_{\mathrm{c}}((X, D)/W)\otimes_W K_0 = H^h_{\mathrm{rig,c}}(U/K_0)$. (Note that the duality in [B4, (2.4)] holds for a separated smooth scheme of finite type over κ by the same proof of [loc. cit.] once the definitions of the rigid cohomology $H^h_{\mathrm{rig}}(U/K_0)$ and the rigid cohomology $H^h_{\mathrm{rig,c}}(U/K_0)$ with compact support are given for U/κ ([B3, p. 335, Remarque], [B2, (3.1)].) In particular, $H^h_{\mathrm{c}}((X,D)/W)\otimes_W K_0$ is independent of the choice of the compactification (X,D) of U.

(2) Though the compatibility of the pairing in (5.3) (2) with the Frobenius is claimed in [M2, (4.2)], it is not proved in [loc. cit.]; the Frobenius on $W_n\Omega^i_X(\pm\log D)$ is not defined in [loc. cit.]; in (9.1.2) and before (9.4) below, we define the Frobenius on $W_n\Omega^i_X(\pm\log D)$, and in (9.4) (1), we prove the compatibility.

Next, we prove the E_2-degenerations of the weight spectral sequences constructed by Chiarellotto and Le Stum in [CL2] by the method of rigid cohomologies.

Let V be a complete discrete valuation ring of mixed characteristics. Let κ (resp. K) be the (not necessarily perfect) residue (resp. fraction) field of V. Let p be the characteristic of κ. Let $\sigma\in\mathrm{Aut}(V)$ be a fixed lift of the p-th power endomorphism of κ. For simplicity, assume that there exists a closed immersion $X \overset{\subset}{\longrightarrow} \mathcal{P}$ into a formal V-scheme such that $\mathcal{P}/\mathrm{Spf}(V)$ is formally smooth around X. Then there exist the following weight spectral sequences ([CL2, (3.8), (3.5)]):

$$(5.4.1)\qquad E^{-k,h+k}_{1,\mathrm{rig}} = H^{h-k}_{\mathrm{rig}}(D^{(k)}/K)(-k) \Longrightarrow H^h_{\mathrm{rig}}(U/K),$$

$$(5.4.2)\qquad E^{k,h-k}_{1,\mathrm{rig,c}} = H^{h-k}_{\mathrm{rig}}(D^{(k)}/K) \Longrightarrow H^h_{\mathrm{rig,c}}(U/K),$$

though the Tate twist in (5.4.1) has been forgotten in [loc. cit.]. The boundary morphisms $\{d_1^{\bullet\bullet}\}$ of (5.4.1) (resp. (5.4.2)) are described by Gysin morphisms of divisors (resp. the induced morphisms of closed immersions). They have proved the E_2-degenerations of (5.4.1) and (5.4.2) when κ is a finite field.

Remark 5.5. As in the case (5.1.2) for (5.1.4), we have made the analogous convention on signs of the Mayer-Vietoris exact sequence in [CL2, p. 165] for (5.4.2) to rid ambiguous conventions on signs.

Theorem 5.6. *The spectral sequences* (5.4.1) *and* (5.4.2) *degenerate at* E_2.

Proof. By [B4, Remarque on p. 498], rigid cohomologies commute with the extension of fields. Hence we may assume that κ is perfect.

By [B3, (1.9)], $E_{2,\mathrm{rig,c}}^{k,h-k} = E_{2,\mathrm{dW,c}}^{k,h-k}\otimes_W K$. By (5.4) (1), $H^h_{\mathrm{rig,c}}(U/K) = H^h_{\mathrm{c}}((X,D)/W)\otimes_W K$. Hence the E_2-degeneration of (5.4.2) follows from (5.2) (2); the duality [B4, (2.4)] shows the E_2-degeneration of (5.4.1). □

Remark 5.7. By the comparison of the chern class of an invertible sheaf on a proper smooth variety in rigid cohomologies and in crystalline cohomologies ([Pet]), we can prove the E_2-degeneration of (5.4.1) directly by [Sh, Corollary 2.4.13, Theorem 3.1.1] and (5.2) (1) without using the E_2-degeneration of (5.4.2).

Finally, we prove the E_2-degenerations of the weight spectral sequences of the log Hodge-Witt sheaves on X. Let us assume again that the base field κ is a perfect field of characteristic $p > 0$. Let i be a non-negative integer. By (5.0.2;n) and (5.0.2), we obtain the following spectral sequences

$$(5.7.1;n)\qquad E_1^{-k,h+k} = H^{h-i}(D^{(k)}, W_n\Omega^{i-k}_{D^{(k)}})(-k) \Longrightarrow H^{h-i}(X, W_n\Omega^i_X(\log D)),$$

$$(5.7.1)\qquad E_1^{-k,h+k} = H^{h-i}(D^{(k)}, W\Omega^{i-k}_{D^{(k)}})(-k) \Longrightarrow H^{h-i}(X, W\Omega^i_X(\log D)).$$

By (5.0.4;n) and (5.0.4), we obtain the following spectral sequences

$$(5.7.2;n)\qquad E_1^{k,h-k} = H^{h-i-k}(D^{(k)}, W_n\Omega^{i}_{D^{(k)}}) \Longrightarrow H^{h-i}(X, W_n\Omega^i_X(-\log D)),$$

$$(5.7.2)\qquad E_1^{k,h-k} = H^{h-i-k}(D^{(k)}, W\Omega^{i}_{D^{(k)}}) \Longrightarrow H^{h-i}(X, W\Omega^i_X(-\log D)).$$

Remark 5.8. As remarked in (4.4) (1), (5.7.1;n) and (5.7.2;n) have some geometric applications.

Theorem 5.9. *The spectral sequences* (5.7.1) *and* (5.7.2) *degenerate at* E_2 *modulo torsion.*

Proof. By [M2, (3.3)], there exists the following slope decomposition

$$\bigoplus_{i=0}^{h} H^{h-i}(X, W\Omega^i_X(\log D))\otimes_W K_0 = H^h((X,D)/W)\otimes_W K_0.$$

By the formalism of $W\Omega^{\bullet}_X(-\log D)$ ([Hy1]), there also exists the following slope decomposition

$$\bigoplus_{i=0}^{h} H^{h-i}(X, W\Omega^i_X(-\log D))\otimes_W K_0 = H^h_{\mathrm{c}}((X,D)/W)\otimes_W K_0.$$

The rest of the proof is the same as that of (4.7) by using (5.2) (1), (2). □

Remark 5.10. Assume that the base field is the complex number field $\mathbb{C}$. Let (X, D) be a proper smooth analytic variety over $\mathrm{Spec}(\mathbb{C})^{\mathrm{an}}$ with an SNCD. Then, by the obvious analogues of (5.0.2;n) and (5.0.4;n), there exist the following two spectral sequences:

$$(5.10.1)\quad E_1^{-k,h+k} = H^{h-i}(D^{(k)}, \Omega^{i-k}_{D^{(k)}/\mathbb{C}})(-k) \Longrightarrow H^{h-i}(X, \Omega^i_{X/\mathbb{C}}(\log D)),$$

$$(5.10.2)\quad E_1^{k,h-k} = H^{h-i-k}(D^{(k)}, \Omega^i_{D^{(k)}/\mathbb{C}}) \Longrightarrow H^{h-i}(X, \Omega^i_{X/\mathbb{C}}(-\log D)).$$

If X is Kähler or algebraic, then the E_2-degeneration of the spectral sequence (5.10.1) has been proved in [D3, (3.2.13) (iii)], though the upper indices of the E_1-terms and those of the convergent terms in [loc. cit.] should be replaced as in (5.10.1). The E_2-degeneration of (5.10.2) follows from that of (5.10.1) and from the duality between (5.10.1) and (5.10.2).

Part II. Fundamental Properties of (Idealized) Log de Rham-Witt Complexes and Those of p-Adic Weight Spectral Sequences of Log Varieties

Because methods in the theory of log de Rham-Witt complexes are delicate and because it is sometimes difficult to find mistakes in the proofs for some facts in the theory in published papers and to realize that no one has given correct proofs for some facts in the theory in them, there are many non-minor mistakes and unproved facts in the theory in them. In the Part II, we complete many of unproved facts, make the theory perfect by using some ideas and results in published papers; results in [Nakk4] clarify some mistakes in published papers as counter-examples (cf. (10.6) (1), (11.15) (1) below).

We divide the Part II into six sections. In §6 we give theory of formal de Rham-Witt complexes and as a corollary of this theory, we give a precise proof of the fact that the log Hodge-Witt sheaf with compact support of a smooth scheme with an NCD is compatible with the canonical filtration as modules over the Cartier-Dieudonné-Raynaud algebra. In §7 we complete some fundamental results in [HK]. In §8 we study some fundamental properties of projections in the theory of log de Rham-Witt complexes. In §9 we prove the compatibility of the (pre)weight spectral sequences with the Frobenii. In §10 we give a right proof of the description of the boundary morphisms between the E_1-terms of (2.0.1); we also describe the boundary morphisms between the E_1-terms of (4.1.1). In §11 we give a right proof of the coincidence of the p-adic monodromy operators in [HK] and in [Hy2]. In §9, §10 and §11, we pay attention to signs; even one mistake in signs may make many statements and proofs wrong (see (11.9) (1), (2) below), and consequently someone has to correct the mistakes.

I hope that the reader shall read all in the Part II and references quoted in the Part II closely and that he does not believe that there are only careless mistakes in references.

6. Formal de Rham-Witt complexes

In this section we give theory of formal de Rham-Witt complexes (see (6.9) below for the definition) associated to certain complexes, following [Nakk1]. As a corollary, we give a detailed proof of [M1, 1.3.3] because the proof in [loc. cit.] is sketchy (see (6.29) below for details) and [loc. cit.] has been used in [M1, 3.15.1] and [M1, 3.15.1] is one of key ingredients for the construction of the spectral sequence (2.0.1).

Let $(\mathcal{T}, \mathcal{A})$ be a ringed topos. Let $\Omega'^{\bullet}$ and $\Omega^{\bullet}$ are complexes of $\mathcal{A}$-modules and let $\phi\colon \Omega'^{\bullet} \longrightarrow \Omega^{\bullet}$ be an $\mathcal{A}$-linear morphism of complexes of $\mathcal{A}$-modules. Let p be a prime number. Set $\Omega'^{\bullet}_1 := \Omega'^{\bullet}/p(\Omega'^{\bullet})$, $\Omega^{\bullet}_1 := \Omega^{\bullet}/p\Omega^{\bullet}$, and $\mathcal{A}_1 := \mathcal{A}/p\mathcal{A}$. We assume that the following conditions (6.0.1) $\sim$ (6.0.5) hold:

(6.0.1) $\Omega'^i = 0 = \Omega^i$ for $i < 0$.

(6.0.2) Ω'^i and Ω^i $(\forall i \in \mathbb{N})$ are p-torsion-free, p-adically complete $\mathcal{A}$-modules.

(6.0.3) $\phi(\Omega'^i) \subset \{\omega \in p^i\Omega^i \mid d\omega \in p^{i+1}\Omega^{i+1}\}$ $(\forall i \in \mathbb{N})$.

(6.0.4) There exists an $\mathcal{A}_1$-linear isomorphism

$$C^{-1}\colon \Omega'^i_1 \overset{\sim}{\longrightarrow} \mathcal{H}^i(\Omega^\bullet_1) \quad (\forall i \in \mathbb{N}).$$

(6.0.5) A composite morphism (mod p) $\circ\, p^{-i}\phi\colon \Omega'^i \longrightarrow \Omega^i \longrightarrow \Omega^i_1$ factors through $\mathrm{Ker}(d\colon \Omega^i_1 \longrightarrow \Omega^{i+1}_1)$, and the following diagram is commutative:

$$\begin{array}{ccc} \Omega'^i & \xrightarrow{\text{mod } p} & \Omega'^i_1 \\ {\scriptstyle p^{-i}\phi}\big\downarrow & & \big\downarrow{\scriptstyle C^{-1}} \\ \Omega^i & \xrightarrow{\text{mod } p} & \mathcal{H}^i(\Omega^\bullet_1). \end{array}$$

Definition 6.1. Let $\mathrm{C}^+_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$ be the category of objects $(\Omega'^\bullet, \Omega^\bullet, \phi, C^{-1})$ satisfying (6.0.1) $\sim$ (6.0.5); a morphism in $\mathrm{C}^+_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$ is defined in an obvious way. Let $\mathrm{C}^+_{\mathrm{F}}(\mathcal{T},\mathcal{A})$ be the full subcategory of $\mathrm{C}^+_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$ whose objects satisfy an equality $\Omega'^\bullet = \Omega^\bullet$. Let $\mathrm{C}^{\mathrm{b}}_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$ be a full-subcategory of $\mathrm{C}^+_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$ of objects $(\Omega'^\bullet, \Omega^\bullet, \phi, C^{-1})$ such that $\Omega'^\bullet$ and $\Omega^\bullet$ are bounded. Set $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T},\mathcal{A}) := \mathrm{C}^+_{\mathrm{F}}(\mathcal{T},\mathcal{A}) \cap \mathrm{C}^{\mathrm{b}}_{\mathrm{F,rel}}(\mathcal{T},\mathcal{A})$. For simplicity of notation, we denote $(\Omega^\bullet, \Omega^\bullet, \phi, C^{-1}) \in \mathrm{C}^+_{\mathrm{F}}(\mathcal{T},\mathcal{A})$ by $(\Omega^\bullet, \phi, C^{-1})$.

For a gauge $\epsilon\colon \mathbb{Z} \longrightarrow \mathbb{N}$ ([BO, 8.7 Definition]), let η be the associated cogauge to ϵ defined by

$$\eta(i) := \begin{cases} \epsilon(i) + i & (i \geq 0), \\ \epsilon(0) & (i \leq 0). \end{cases}$$

Let $\Omega'^\bullet_\epsilon$ (resp. $\Omega^\bullet_\eta$) be the largest complex of $\Omega'^\bullet$ (resp. $\Omega^\bullet$) whose i-th degree is contained in $p^{\epsilon(i)}\Omega'^i$ (resp. $p^{\eta(i)}\Omega^i$) ([BO, 8.6 Definition]). Then the following holds:

Theorem 6.2. ([Nakk1]) *Let $\epsilon\colon \mathbb{Z} \longrightarrow \mathbb{N}$ be a gauge with associated cogauge η. Then the morphism $\phi\colon \Omega'^\bullet \longrightarrow \Omega^\bullet$ induces a quasi-isomorphism*

$$\phi_\epsilon\colon \Omega'^\bullet_\epsilon \longrightarrow \Omega^\bullet_\eta.$$

PROOF. The proof is the same as that in [BO, 8.8 Theorem]; the conditions (6.0.1) $\sim$ (6.0.5) are nothing but ones which enable the proof in [loc. cit.] to work in (6.2). □

COROLLARY 6.3. ([Nakk1]) *Assume that* $\Omega'^{\bullet}_{\epsilon}$ *and* $\Omega^{\bullet}_{\eta}$ *are bounded above and that they consist of flat* $\mathcal{A}$*-modules. Let* $\mathcal{M}$ *be an* $\mathcal{A}$*-module. Then the morphism*

(6.3.1)
$$\phi_{\epsilon} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{M}} \colon \Omega'^{\bullet}_{\epsilon} \otimes_{\mathcal{A}} \mathcal{M} \longrightarrow \Omega^{\bullet}_{\eta} \otimes_{\mathcal{A}} \mathcal{M}$$

is a quasi-isomorphism.

PROOF. Let $\mathrm{MC}(\phi_{\epsilon})$ be the mapping cone of ϕ_{ϵ}. Then $\mathrm{MC}(\phi_{\epsilon}) \otimes_{\mathcal{A}} \mathcal{M}$ is acyclic. Hence $\phi_{\epsilon} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{M}}$ is a quasi-isomorphism. □

COROLLARY 6.4. *Let* i (*resp.* n) *be a non-negative* (*resp. positive*) *integer. Then the following hold*:

(1) (cf. [HK, (2.24)])

(6.4.1)
$$\frac{p^i\{\omega \in \Omega^i \mid d\omega \in p^{n+1}\Omega^{i+1}\}}{p^{i+n}\{\omega \in \Omega^i \mid d\omega \in p\Omega^{i+1}\} + p^{i-1}d\{\omega \in \Omega^{i-1} \mid d\omega \in p\Omega^i\}} \overset{\phi}{\underset{\sim}{\longleftarrow}} \frac{\{\omega \in \Omega'^i \mid d\omega \in p^n\Omega'^{i+1}\}}{p^n\Omega'^i + d\Omega'^{i-1}}.$$

(2) ([Nakk1], cf. [IR, III (1.5)])

(6.4.2)
$$\frac{p^i\{\omega \in \Omega^i \mid d\omega \in p^{n+1}\Omega^{i+1}\}}{p^{i+n}\{\omega \in \Omega^i \mid d\omega \in p\Omega^{i+1}\} + p^i d\Omega^{i-1}} \overset{\phi}{\underset{\sim}{\longleftarrow}} \frac{\{\omega \in \Omega'^i \mid d\omega \in p^n\Omega'^{i+1}\}}{p^n\Omega'^i + pd\Omega'^{i-1}}.$$

PROOF. (1): Set $\mathcal{A} := \mathbb{Z}_p$ and $\mathcal{M} := \mathbb{Z}/p^n$. Let ϵ be any gauge such that $\epsilon(i-1) = 0$. Then (6.3.1) at the degree i is equal to (6.4.1); we finish the proof.

(2): Set $\mathcal{A} := \mathbb{Z}_p$ and $\mathcal{M} := \mathbb{Z}/p^n$. Let ϵ be any gauge such that $\epsilon(i-1) = 1$ and $\epsilon(i) = 0$. Then (6.3.1) at the degree i is equal to (6.4.2). □

Henceforth, assume that $(\Omega^\bullet, \phi, C^{-1}) \in \mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p)$, where, by abuse of notation, $\mathbb{Z}_p$ in the notation $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p)$ is the constant sheaf in $\mathcal{T}$ defined by $\mathbb{Z}_p$. For a positive integer n and a non-negative integer i, set

$$(6.4.3) \qquad \begin{aligned} &Z^i_n := \{\omega \in \Omega^i |\ d\omega \in p^n\Omega^{i+1}\}, \quad B^i_n := p^n\Omega^i + d\Omega^{i-1}, \\ &\mathfrak{W}_n\Omega^i := Z^i_n/B^i_n. \end{aligned}$$

Then we have an isomorphism

$$(6.4.4) \qquad \mathfrak{W}_1\Omega^i = \mathcal{H}^i(\Omega^\bullet_1) \overset{C^{-1}}{\underset{\sim}{\longleftarrow}} \Omega^i_1$$

of $\mathbb{F}_p$-modules by (6.0.4).

Let $\epsilon\colon \mathbb{Z} \longrightarrow \mathbb{N}$ be a gauge with associated cogauge η. Since $\Omega^\bullet$ is bounded and consists of torsion-free $\mathbb{Z}_p$-modules, so are $\Omega^\bullet_\epsilon$ and $\Omega^\bullet_\eta$.

Let us define morphisms

$$F\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i, \quad V\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i, \quad d\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_n\Omega^{i+1},$$

$$\mathbf{p}\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i \quad \text{and} \quad \pi\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i$$

of sheaves of W-modules in $\mathcal{T}$ as follows: F (resp. V) is a morphism induced by $\mathrm{id}\colon \Omega^i \longrightarrow \Omega^i$ (resp. $p \times \mathrm{id}\colon \Omega^i \longrightarrow \Omega^i$); d is a morphism induced by $p^{-n}d\colon \{\omega \in \Omega^i |\ d\omega \in p^n\Omega^{i+1}\} \longrightarrow \Omega^{i+1}$; $\mathbf{p}$ is a morphism induced by $p^{-(i-1)}\phi\colon \Omega^i \longrightarrow \Omega^i$ (note that $-(i-1)$ is positive if $i = 0$); π is the following composite morphism (cf. [HK, (4.2)]):

$$(6.4.5) \qquad \begin{aligned} \mathfrak{W}_{n+1}\Omega^i = Z^i_{n+1}/B^i_{n+1} &\overset{p^i}{\underset{\sim}{\longrightarrow}} p^iZ^i_{n+1}/p^iB^i_{n+1} \\ &\overset{\text{proj.}}{\longrightarrow} p^iZ^i_{n+1}/(p^{i+n}Z^i_1 + p^{i-1}dZ^{i-1}_1) \\ &\overset{\phi}{\underset{\sim}{\longleftarrow}} Z^i_n/B^i_n = \mathfrak{W}_n\Omega^i_Y. \end{aligned}$$

Then, the following formulas hold:

$$(6.4.6) \qquad \begin{aligned} &d^2 = 0,\ FdV = d,\ FV = VF = p\ , F\mathbf{p} = \mathbf{p}F,\ V\mathbf{p} = \mathbf{p}V, \\ &d\mathbf{p} = \mathbf{p}d,\ \mathbf{p}\pi = \pi\mathbf{p} = p. \end{aligned}$$

The morphism π is surjective. Hence

$$\mathrm{Im}(p\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i) = \mathrm{Im}(\mathbf{p}\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i)$$

(cf. [Hy2, p. 245]).

PROPOSITION 6.5. *The morphism* π *is equal to the following composite morphism*:

(6.5.1)
$$\mathfrak{W}_{n+1}\Omega^i = Z^i_{n+1}/B^i_{n+1} \overset{\text{proj.}}{\longrightarrow} Z^i_{n+1}/(p^n Z^i_1 + d\Omega^{i-1}) \underset{\sim}{\overset{(p^{-i}\phi)^{-1}}{\longrightarrow}} \frac{Z^i_n}{p^n\Omega^i + pd\Omega^{i-1}} \overset{\text{proj.}}{\longrightarrow} Z^i_n/B^i_n = \mathfrak{W}_n\Omega^i.$$

PROOF. (6.5) immediately follows from the following commutative diagram:

$$\begin{array}{ccc} p^iZ^i_{n+1}/(p^{i+n}Z^i_1 + p^i d\Omega^{i-1}) & \underset{(6.4.2)}{\overset{\phi}{\underset{\sim}{\longleftarrow}}} & Z^i_n/(p^n\Omega^i + pd\Omega^{i-1}) \\ {\scriptstyle\text{proj.}}\downarrow & & \downarrow{\scriptstyle\text{proj.}} \\ p^iZ^i_{n+1}/(p^{i+n}Z^i_1 + p^{i-1}dZ^{i-1}_1) & \underset{(6.4.1)}{\overset{\phi}{\underset{\sim}{\longleftarrow}}} & Z^i_n/B^i_n. \end{array}$$
□

REMARK 6.6. Let $(W_n\Omega^{\bullet}_U)''$ be the de Rham-Witt complex defined in [I2, I (1.3)] for a scheme U over κ. As suggested in [IR, III (1.5)], we see that π in (6.5.1) for U over κ fits into the following commutative diagram

(6.6.1)
$$\begin{array}{ccc} (W_{n+1}\Omega^i_U)'' & \underset{\sim}{\overset{C^{-(n+1)}}{\longrightarrow}} & W_{n+1}\Omega^i_U \\ {\scriptstyle\text{proj.}}\downarrow & & \downarrow{\scriptstyle\pi} \\ (W_n\Omega^i_U)'' & \underset{\sim}{\overset{C^{-n}}{\longrightarrow}} & W_n\Omega^i_U. \end{array}$$

The commutativity of (6.6.1) will be used in (7.21) below.

Let κ be a perfect field of characteristic $p > 0$. Let W (resp. W_n) be the Witt ring of κ (resp. the Witt ring of κ of length n). Let σ be the Frobenius automorphism of W. Let $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W)$ be a subcategory of

$\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p)$ consisting of $(\Omega^\bullet, \phi, C^{-1})$ such that $\Omega^\bullet$ is a complex of W-modules (W is the constant sheaf defined by the Witt ring W) which is compatible with the $\mathbb{Z}_p$-module structure and such that ϕ is σ-linear; a morphism in $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W)$ is assumed to be W-linear. Henceforth we consider only objects of $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W)$.

LEMMA 6.7. *Let $\star$ be a positive integer n or nothing. Set*

$$\mathfrak{W}\Omega^\bullet = \varprojlim_{\pi} \mathfrak{W}_n\Omega^\bullet.$$

Then there exists a natural $W_\star$-module structure on $\mathfrak{W}_\star\Omega^i$.

PROOF. Let $c := (c_0, \dots, c_{n-1})$ ($c_i \in \kappa$ $(0 \leq i \leq n-1)$) be an element of W_n. Let ω be a section of Z^i_n. Then we define $c \cdot [\omega]$ as follows: $c \cdot [\omega] = [(\sum_{j=0}^{n-1} p^j \widetilde{c}_j^{\,p^{n-j}}) \cdot \omega]$, where $\widetilde{c}_j \in W_n$ is a lift of c_j. It is a routine work to check that this action is well-defined. By the same calculation as that in the proof of the commutativity of (7.1.3) below and by taking the projective limit, we see that there exists a natural W-module structure on $\mathfrak{W}\Omega^i$. $\square$

By the proof of (6.7), we see that F (resp. V) is a σ-linear (resp. σ^{-1}-linear) morphism and that π and $\mathbf{p}$ are W-linear morphisms.

LEMMA 6.8. (cf. [Hy2, (1.3.2)], [HK, (4.5)])
(1)

$$\text{(6.8.1)} \qquad \operatorname{Ker}(p\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i) = \operatorname{Ker}(\pi\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i).$$

(2) *The morphism $\mathbf{p}\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i$ is injective.*

(3) *$\mathfrak{W}\Omega^i$ is a sheaf of torsion-free W-modules.*

(4) *$d\pi = \pi d$, $F\pi = \pi F$, $V\pi = \pi V$; the morphisms d, F and V induce morphisms $d\colon \mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}\Omega^{i+1}$, $F\colon \mathfrak{W}\Omega^\bullet \longrightarrow \mathfrak{W}\Omega^\bullet$ and $V\colon \mathfrak{W}\Omega^\bullet \longrightarrow \mathfrak{W}\Omega^\bullet$, respectively.*

PROOF. (1): Since $\mathbf{p}\pi = p$, it suffices to show the inclusion $\subset$. Let ω be a local section of Z^i_{n+1} such that there exists a local section $\eta \in \Omega^i$ and $\zeta \in \Omega^{i-1}$ satisfying an equality $p\omega = p^{n+1}\eta + d\zeta$. Then we see that $\eta \in Z^i_1$ and that $\zeta \in Z^{i-1}_1$. Hence $p^i\omega$ is the zero in the third sheaf of (6.4.5). This shows (6.8.1).

(2): (2) immediately follows from (1).

(3): Let $(\omega_n)_{n=1}^{\infty}$ $(\omega_n \in \mathfrak{W}_n\Omega^i)$ be a local section of $\mathfrak{W}\Omega^i$. Assume that $p\omega_{n+1} = 0$. Then $\mathbf{p}(\omega_n) = 0$. By (2), we have $\omega_n = 0$.

(4): Since $\mathbf{p}d\pi = d\mathbf{p}\pi = pd = \mathbf{p}\pi d$ and since $\mathbf{p}$ is injective, we have $d\pi = \pi d$. The rest of the proof is the same by using two relations $F\mathbf{p} = \mathbf{p}F$ and $V\mathbf{p} = \mathbf{p}V$ in (6.4.6). □

Let R be the Cartier-Dieudonné-Raynaud algebra over κ ([IR, I (1.1)]). Set $R_n := R/(V^nR + dV^nR)$. Then $\mathfrak{W}\Omega^\bullet$ (resp. $\mathfrak{W}_\star\Omega^\bullet$) is naturally an R-module (resp. $R_\star$-module). Let $\mathrm{C}^{\mathrm{b}}(\mathcal{T}, R_\star)$ be the category of bounded complexes of left $R_\star$-modules. We have a functor

$$\mathfrak{W}_\star \colon \mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W) \longrightarrow \mathrm{C}^{\mathrm{b}}(\mathcal{T}, R_\star), \tag{6.8.2}$$

where, by abuse of notation, $R_\star$ in (6.8.2) is the constant sheaf in $\mathcal{T}$ defined by $R_\star$.

Definition 6.9. We call $\mathfrak{W}\Omega^\bullet$ (resp. $\mathfrak{W}_n\Omega^\bullet$) the *formal de Rham-Witt complex* (resp. *formal de Rham-Witt complex of length* n) of $(\Omega^\bullet, \phi, C^{-1}) \in \mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W)$.

Definition 6.10. Let n be a positive integer. Set

$$\mathrm{Fil}^r\mathfrak{W}_n\Omega^i := \begin{cases} 0 & (r > n), \\ \mathrm{Ker}(\pi^{n-r} \colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_r\Omega^i) & (0 < r \leq n), \\ \mathfrak{W}_n\Omega^i & (r \leq 0). \end{cases} \tag{6.10.1}$$

$$\mathrm{Fil}^r\mathfrak{W}\Omega^i := \begin{cases} \mathrm{Ker}(\mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}_r\Omega^i) & (r > 0), \\ \mathfrak{W}\Omega^i & (r \leq 0). \end{cases} \tag{6.10.2}$$

Let $\star$ be a positive integer or nothing. For an integer r, set

$$\mathrm{gr}^r\mathfrak{W}_\star\Omega^i := \mathrm{Fil}^r\mathfrak{W}_\star\Omega^i/\mathrm{Fil}^{r+1}\mathfrak{W}_\star\Omega^i. \tag{6.10.3}$$

By the formulas $\mathbf{p}\pi = \pi\mathbf{p}$ and $\mathbf{p}d = d\mathbf{p}$ in (6.4.6), the injective morphism $\mathbf{p} \colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i$ induces injective morphisms

$$\mathbf{p} \colon \mathfrak{W}\Omega^\bullet \longrightarrow \mathfrak{W}\Omega^\bullet \tag{6.10.4}$$

and

$$\mathbf{p}\colon \mathrm{gr}^n\mathfrak{W}\Omega^\bullet \longrightarrow \mathrm{gr}^{n+1}\mathfrak{W}\Omega^\bullet. \tag{6.10.5}$$

Proposition 6.11. ([Nakk1], cf. [I2, I (3.4)]) (1) *The morphism of a multiplication* $p^j\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i$ $(0 \le j \le n+1)$ *induces the morphism* $\mathbf{p}^j\colon \mathfrak{W}_{n+1-j}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i$.

(2)

$$\begin{aligned}\mathrm{Ker}(p^j\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i) \\ = \mathrm{Fil}^{n+1-j}\mathfrak{W}_{n+1}\Omega^i \quad (0 \le j \le n+1).\end{aligned} \tag{6.11.1}$$

Proof. (1): The equality $p^j = \mathbf{p}^j\pi^j$ shows (1).

(2): By the following commutative diagram

$$\begin{array}{ccc} \mathfrak{W}_{n+1}\Omega^i & \xrightarrow{p^j} & \mathfrak{W}_{n+1}\Omega^i \\ {\scriptstyle \pi^j}\big\downarrow & & \big\uparrow{\scriptstyle \mathbf{p}^j}\ \cup \\ \mathfrak{W}_{n+1-j}\Omega^i & = & \mathfrak{W}_{n+1-j}\Omega^i \end{array}$$

and by the injectivity of $\mathbf{p}^j$ ((6.8) (2)), we have (2). $\square$

Now, as suggested by the referee, we assume that $\mathcal{T}$ is the Zariski topos $\widetilde{X}_{\mathrm{zar}}$ of a scheme X over κ. Let $\mathcal{B}$ be a p-torsion free quasi-coherent sheaf of commutative rings with unit elements in $\widetilde{X}_{\mathrm{zar}}$ with a surjective morphism $\mathcal{B} \longrightarrow \mathcal{O}_X$ of sheaves of rings in $\widetilde{X}_{\mathrm{zar}}$. Assume that $\mathrm{Ker}(\mathcal{B} \longrightarrow \mathcal{O}_X) = p\mathcal{B}$ and that each Ω^i $(i \in \mathbb{N})$ is a quasi-coherent $\mathcal{B}$-module. Then, as in the proof of (6.7), we can endow $\mathfrak{W}_n\Omega^i$ with a natural $W_n(\mathcal{O}_X)$-module structure (cf. [IR, III (1.5)]). For a projective system of exact sequences

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{G}_n \longrightarrow \mathcal{H}_n \longrightarrow 0 \quad (n \in \mathbb{N})$$

of quasi-coherent $W_n(\mathcal{O}_X)$-modules, if $\mathcal{F}_n$ satisfies the Mittag-Leffler condition, then we have an exact sequence

$$0 \longrightarrow \varprojlim_n \mathcal{F}_n \longrightarrow \varprojlim_n \mathcal{G}_n \longrightarrow \varprojlim_n \mathcal{H}_n \longrightarrow 0$$

by considering the sections of $\varprojlim_n \mathcal{H}_n$ over a small affine open subscheme of X. We can easily check that $\mathfrak{W}_n\Omega^i$ is a quasi-coherent $W_n(\mathcal{O}_X)$-module and that the morphisms $\pi\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i$ and $\mathbf{p}\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i$ are morphisms of $W(\mathcal{O}_X)$-modules (cf. the proof of (7.1) below). In this situation, we prove the following in turn:
(A): $\mathrm{Fil}^n\mathfrak{W}\Omega^i = V^n\mathfrak{W}\Omega^i + dV^n\mathfrak{W}\Omega^{i-1} \quad (n \in \mathbb{N})$.
(B): $d^{-1}(p^n\mathfrak{W}\Omega^{i+1}) = F^n\mathfrak{W}\Omega^i$.
(C): $R_n \otimes^L_R \mathfrak{W}\Omega^\bullet = \mathfrak{W}_n\Omega^\bullet \quad (n \in \mathbb{Z}_{>0})$.

(A)

Proposition 6.12. (1) ([Nakk1], cf. [I2, I (3.2)]) *Let n be a positive integer and r a non-negative integer such that $r \leq n$. Then*

$$\mathrm{Fil}^r\mathfrak{W}_{n+1}\Omega^i = V^r\mathfrak{W}_{n+1-r}\Omega^i + dV^r\mathfrak{W}_{n+1-r}\Omega^{i-1}. \tag{6.12.1}$$

(2) *Let $F_X\colon X \longrightarrow X$ be the Frobenius endomorphism of X. Assume that*

$$C^{-1}\colon \Omega^i_1 \xrightarrow{\sim} \mathfrak{W}_1\Omega^i = \mathcal{H}^i(F_{X*}(\Omega^\bullet_1))$$

is an isomorphism of $\mathcal{O}_X$-modules. If Ω^j_1 $(j = i-1, i)$ is an $\mathcal{O}_X$-module of finite type and if F_X is finite, then $\mathfrak{W}_n\Omega^i$ is a $W_n(\mathcal{O}_X)$-module of finite type.

Proof. (1): Since $\mathbf{p}\pi = p$, $\mathbf{p}^{n+1-r}\pi^{n+1-r}V^r\mathfrak{W}_{n+1-r}\Omega^i = 0$. Hence $V^r\mathfrak{W}_{n+1-r}\Omega^i \subset \mathrm{Fil}^r\mathfrak{W}_{n+1}\Omega^i$ by (6.8) (2). Since $\pi d = d\pi$ by (6.8) (4), the inclusion $\supset$ is obvious.

We prove the inclusion $\subset$ by descending induction on r. We may assume that $r \geq 1$.

If $r = n$, we have

$$\begin{aligned}\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i &= p^{-i}p^i(p^nZ^i_1 + p^{-1}dZ^{i-1}_1)/p^{-i}p^iB^i_{n+1}\\ &= (p^nZ^i_1 + p^{-1}dZ^{i-1}_1)/B^i_{n+1}\end{aligned} \tag{6.12.2}$$

by (6.4.5). On the other hand,

$$\begin{aligned}V^n\mathfrak{W}_1\Omega^i + dV^n\mathfrak{W}_1\Omega^{i-1} &= (p^nZ^i_1 + p^{-n-1}dp^nZ^{i-1}_1)/B^i_{n+1}\\ &= (p^nZ^i_1 + p^{-1}dZ^{i-1}_1)/B^i_{n+1}.\end{aligned} \tag{6.12.3}$$

Hence, by (6.12.2) and (6.12.3), we obtain (6.12.1) for the case $r = n$.

Next, assume that (6.12.1) holds for a fixed $r \leq n$. Since π is surjective, we have the following exact sequence:

$$(6.12.4)\quad 0 \longrightarrow \mathrm{Fil}^r\mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathrm{Fil}^{r-1}\mathfrak{W}_{n+1}\Omega^i \overset{\pi^{n+1-r}}{\longrightarrow} \mathrm{Fil}^{r-1}\mathfrak{W}_r\Omega^i \longrightarrow 0.$$

Let x be a local section of $\mathrm{Fil}^{r-1}\mathfrak{W}_{n+1}\Omega^i$ such that there exist local sections $y' \in \mathfrak{W}_1\Omega^i$ and $z' \in \mathfrak{W}_1\Omega^{i-1}$ satisfying an equation $\pi^{n+1-r}x = V^{r-1}y' + dV^{r-1}z'$ in $\mathfrak{W}_r\Omega^i$. Since π is surjective, we may assume that there exist local sections $y \in \mathfrak{W}_{n+2-r}\Omega^i$ and $z \in \mathfrak{W}_{n+2-r}\Omega^{i-1}$ such that $y' = \pi^{n+1-r}y$ and $z' = \pi^{n+1-r}z$. Then we have $\pi^{n+1-r}(x - (V^{r-1}y + dV^{r-1}z)) = 0$. Hence $x - (V^{r-1}y + dV^{r-1}z) \in \mathrm{Fil}^r\mathfrak{W}_{n+1}\Omega^i$. Induction on r shows that $x \in V^{r-1}\mathfrak{W}_{n+2-r}\Omega^i + dV^{r-1}\mathfrak{W}_{n+2-r}\Omega^{i-1}$.

(2): By the pull-back of the Frobenius endomorphism $W_{n+1}(\mathcal{O}_X) \longrightarrow F_{X*}W_{n+1}(\mathcal{O}_X)$ and the $W_{n+1}(\mathcal{O}_X)$-module structure of $\mathfrak{W}_{n+1}\Omega^i$, $F_{X*}\mathfrak{W}_{n+1}\Omega^i$ is naturally endowed with a $W_{n+1}(\mathcal{O}_X)$-module structure. By (6.12.1) we have a surjective morphism

$$(6.12.5)\qquad V^n + dV^n\colon \mathfrak{W}_1\Omega^i \oplus \mathfrak{W}_1\Omega^{i-1} \longrightarrow \mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i$$

of abelian sheaves on X_{zar}. By the surjectivity of the morphism (6.12.5) and by using the following relation

$$p^{j-1}b^{p^{n+2-j}}d\omega \equiv p^{j-1}d(b^{p^{n+2-j}}\omega) \mod p^{n+1} \qquad (1 \leq j \leq n) \quad (b \in \mathcal{B}, \omega \in \Omega^i),$$

we see that $F_{X*}\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i$ is an $\mathcal{O}_X = W_{n+1}(\mathcal{O}_X)/VW_{n+1}(\mathcal{O}_X)$-module (cf. [I2, I (3.9)]). It is easy to see that the morphism (6.12.5) is the following morphism of $\mathcal{O}_X$-modules:

$$(6.12.6)\qquad V^n + dV^n\colon F^{n+1}_{X*}\mathfrak{W}_1\Omega^i \oplus F^{n+1}_{X*}\mathfrak{W}_1\Omega^{i-1} \longrightarrow F_{X*}\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i.$$

Because the source of the morphism above is an $\mathcal{O}_X$-module of finite type by the assumptions of (2), so is the target. Hence $\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i$ is a $W_{n+1}(\mathcal{O}_X)$-module of finite type. Now (2) follows from the induction on n and the following exact sequence

$$0 \longrightarrow \mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i \overset{\pi}{\longrightarrow} \mathfrak{W}_n\Omega^i \longrightarrow 0$$

of $W_{n+1}(\mathcal{O}_X)$-modules.

We finish the proof of (6.12). □

Proposition 6.13. *Let r be a positive integer. The following three canonical morphisms*

$$\mathfrak{W}\Omega^i/p^r\mathfrak{W}\Omega^i \longrightarrow \varprojlim_n \mathfrak{W}_n\Omega^i/p^r\mathfrak{W}_n\Omega^i, \tag{6.13.1}$$

$$\mathfrak{W}\Omega^i/V^r\mathfrak{W}\Omega^i \longrightarrow \varprojlim_{n>r} \mathfrak{W}_n\Omega^i/V^r\mathfrak{W}_{n-r}\Omega^i \tag{6.13.2}$$

and

$$\mathfrak{W}\Omega^i/F^r\mathfrak{W}\Omega^i \longrightarrow \varprojlim_n \mathfrak{W}_n\Omega^i/F^r\mathfrak{W}_{n+r}\Omega^i \tag{6.13.3}$$

are isomorphisms.

Proof. We prove only that (6.13.2) is an isomorphism; the proof for (6.13.1) and (6.13.3) is similar.

Assume that $n > 2r$. Set $K_n := \mathrm{Ker}(V^r\colon \mathfrak{W}_{n-r}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i)$. Then, $K_n \subset K'_n := \mathrm{Ker}(p^r\colon \mathfrak{W}_{n-r}\Omega^i \longrightarrow \mathfrak{W}_{n-r}\Omega^i)$. By (6.11.1), $K'_n \subset \mathrm{Fil}^{n-2r}\mathfrak{W}_{n-r}\Omega^i$, that is, $\pi^r(K'_n) = 0$; hence $\pi^r(K_n) = 0$. In particular, $\{K_n\}_{n>r}$ satisfies the Mittag-Leffler condition. Note also that transition morphism $\pi\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i$ is surjective. Hence, by taking the projective limit of the following exact sequence

$$0 \longrightarrow K_n \xrightarrow{\subset} \mathfrak{W}_{n-r}\Omega^i \xrightarrow{V^r} \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i/V^r\mathfrak{W}_{n-r}\Omega^i \longrightarrow 0,$$

we have the following exact sequence

$$0 \longrightarrow \mathfrak{W}\Omega^i \xrightarrow{V^r} \mathfrak{W}\Omega^i \longrightarrow \varprojlim_{n>r} \mathfrak{W}_n\Omega^i/V^r\mathfrak{W}_{n-r}\Omega^i \longrightarrow 0. \tag{6.13.5}$$

The exactness of (6.13.5) implies that (6.13.2) is an isomorphism. □

Proposition 6.14. (A generalization of [I2, I (3.19.2.1)] (cf. [Lo, p. 258])) *Let $n > r$ be two positive integers. Then the following sequence is exact:*

$$\begin{aligned} 0 \longrightarrow \mathfrak{W}_{n-r}\Omega^{i-1}/F^r\mathfrak{W}_n\Omega^{i-1} &\xrightarrow{dV^r} \mathfrak{W}_n\Omega^i/V^r\mathfrak{W}_{n-r}\Omega^i \\ &\longrightarrow \mathfrak{W}_r\Omega^i \longrightarrow 0. \end{aligned} \tag{6.14.1;r,n}$$

Consequently the following sequence is exact:

$$(6.14.1;r) \quad 0 \longrightarrow \mathfrak{W}\Omega^{i-1}/F^r\mathfrak{W}\Omega^{i-1} \xrightarrow{dV^r} \mathfrak{W}\Omega^i/V^r\mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}_r\Omega^i \longrightarrow 0.$$

PROOF. By (6.12), we have only to prove that dV^r in (6.14.1;r,n) is injective. Let $[\omega] \in \mathfrak{W}_{n-r}\Omega^{i-1}$ ($\omega \in Z^{i-1}_{n-r}$) be a local section such that $dV^r[\omega] \in V^r\mathfrak{W}_{n-r}\Omega^i$. Then $d[\omega] = F^r dV^r[\omega] \in p^r\mathfrak{W}_{n-r}\Omega^i$. Hence there exists a local section $\eta \in \Omega^i$ such that $p^{-(n-r)}d\omega = p^r\eta$, that is, $\omega \in Z^{i-1}_n$. Therefore $[\omega] \in F^r\mathfrak{W}_n\Omega^{i-1}$. Thus the sequence (6.14.1;r,n) is exact; by taking the projective limit of (6.14.1;r,n) with respect to π, we obtain (6.14.1;r) by (6.13). □

THEOREM 6.15. (cf. [I2, I (3.31)]) *Let r be a non-negative integer. Then the following formula holds*:

$$(6.15.1) \qquad \mathrm{Fil}^r\mathfrak{W}\Omega^i = V^r\mathfrak{W}\Omega^i + dV^r\mathfrak{W}\Omega^{i-1}.$$

PROOF. By the definition (6.10.2), the equality (6.15.1) immediately follows from the exactness of (6.14.1;r). □

COROLLARY 6.16. (cf. [Lo, (2.16)]) *Let R be the Cartier-Dieudonné-Raynaud algebra over κ. Let n be a positive integer. Set $R_n := R/(V^nR + dV^nR)$. The canonical morphism*

$$(6.16.1) \qquad R_n \otimes_R \mathfrak{W}\Omega^\bullet \longrightarrow \mathfrak{W}_n\Omega^\bullet$$

is an isomorphism.

Let us define the following abelian subsheaves of Ω^i_1 inductively for $n \in \mathbb{N}$:

$$(6.16.2) \qquad \begin{aligned} &Z_0\Omega^i_1 := \Omega^i_1, \quad Z_1\Omega^i_1 := Z\Omega^i_1 := \mathrm{Ker}(d\colon \Omega^i_1 \longrightarrow \Omega^{i+1}_1), \\ &Z_{n+1}\Omega^i_1/B\Omega^i_1 \overset{C^{-1}}{\underset{\sim}{\longleftarrow}} Z_n\Omega^i_1. \end{aligned}$$

$$(6.16.3) \qquad \begin{aligned} &B_0\Omega^i_1 := 0, \quad B_1\Omega^i_1 := B\Omega^i_1 := \mathrm{Im}(d\colon \Omega^{i-1}_1 \longrightarrow \Omega^i_1), \\ &B_{n+1}\Omega^i_1/B\Omega^i_1 \overset{C^{-1}}{\underset{\sim}{\longleftarrow}} B_n\Omega^i_1. \end{aligned}$$

Usually, $Z_n\Omega^i_1$ and $B_n\Omega^i_1$ are endowed with κ-module structures induced by that of Ω^i_1; they can also be inductively endowed with κ-module structures such that C^{-1}'s in (6.16.2) and (6.16.3) are κ-linear isomorphisms. However, in the all parts of this paper, we do not need either κ-module structures of $Z_n\Omega^i_1$ and $B_n\Omega^i_1$.

Lemma 6.17. (1) $Z_n\Omega^i_1 = (Z^i_n + p\Omega^i)/p\Omega^i$ $(n \in \mathbb{Z}_{>0})$.
(2) $B_n\Omega^i_1 = (p^{-(n-1)}dZ^{i-1}_{n-1} + p\Omega^i)/p\Omega^i$ $(n \in \mathbb{Z}_{>0})$.

Proof. (1): It is easy to check that $C^{-1} = p^{-i}\phi$ induces an injective morphism

$$(6.17.1) \quad (Z^i_{n+1} + p\Omega^i)/(p\Omega^i + d\Omega^{i-1}) \overset{C^{-1}}{\longleftarrow} (Z^i_n + p\Omega^i)/p\Omega^i \quad (n \in \mathbb{Z}_{>0}).$$

The morphism (6.17.1) is surjective by (6.4.2). Then induction on $n \in \mathbb{Z}_{>0}$ shows (1).

(2): The proof of (2) is equivalent to proving that C^{-1} induces an isomorphism

$$(6.17.2) \quad (p^{-n}dZ^{i-1}_n + p\Omega^i)/(p\Omega^i + d\Omega^{i-1}) \overset{C^{-1}}{\underset{\sim}{\longleftarrow}} (p^{-(n-1)}dZ^{i-1}_{n-1} + p\Omega^i)/p\Omega^i \quad (n \in \mathbb{Z}_{>0}).$$

As in (1), we have only to prove that the morphism (6.17.2) is surjective.

Let η be a local section of Z^{i-1}_n. Then, by (6.4.2), there exists a local section $\omega \in Z^{i-1}_{n-1}$ such that $p^{i-1}\eta - \phi(\omega) \in p^{i+n-1}Z^{i-1}_1 + p^{i-1}d\Omega^{i-2}$. (Strictly speaking, we have to consider a more local section of η.) Hence $p^{-n}d\eta - p^{-i}\phi(p^{-(n-1)}d\omega) \in dZ^{i-1}_1$. Therefore the morphism (6.17.2) is surjective. Induction on $n \in \mathbb{Z}_{>0}$ shows (2). □

Let us consider the following composite morphisms:

$$(6.17.3) \quad C\colon Z_{n+1}\Omega^i_1/B\Omega^i_1 \overset{\text{proj.}}{\longrightarrow} Z_{n+1}\Omega^i_1/B_2\Omega^i_1 \overset{C}{\underset{\sim}{\longrightarrow}} Z_n\Omega^i_1/B\Omega^i_1 \quad (n \geq 0),$$

$$(6.17.4) \quad C\colon B_{n+1}\Omega^i_1/B\Omega^i_1 \overset{\text{proj.}}{\longrightarrow} B_{n+1}\Omega^i_1/B_2\Omega^i_1 \overset{C}{\underset{\sim}{\longrightarrow}} B_n\Omega^i_1/B\Omega^i_1 \quad (n \geq 1).$$

The morphisms (6.17.3) and (6.17.4) induce the following morphisms, respectively:

$$C^n\colon Z_{n+1}\Omega^i_1/B\Omega^i_1 \longrightarrow Z\Omega^i_1/B\Omega^i_1 = \mathfrak{W}_1\Omega^i, \tag{6.17.5}$$

$$C^n\colon B_{n+2}\Omega^i_1/B\Omega^i_1 \longrightarrow B_2\Omega^i_1/B\Omega^i_1. \tag{6.17.6}$$

Let us define an abelian subsheaf $\mathfrak{W}_1K^i_n$ in $\mathcal{T} = \widetilde{X}_{\mathrm{zar}}$ of $\mathfrak{W}_1\Omega^i_1 \oplus \mathfrak{W}_1\Omega^{i-1}_1$ fitting into the following exact sequence (cf. [Hy2, (2.3)], [HK, (4.4)]):

$$\begin{aligned} 0 \longrightarrow \mathfrak{W}_1K^i_n \longrightarrow (B_{n+2}\Omega^i_1/B\Omega^i_1) \oplus (Z_{n+1}\Omega^{i-1}_1/B\Omega^{i-1}_1)& \\ \overset{(C^n, dC^n)}{\longrightarrow} B_2\Omega^i_1/B\Omega^i_1 \longrightarrow 0.& \end{aligned} \tag{6.17.7}$$

Note that $d\colon Z_1\Omega^{i-1}_1/B\Omega^{i-1}_1 = \mathfrak{W}_1\Omega^{i-1} \overset{d}{\longrightarrow} B_2\Omega^i_1/B\Omega^i_1$ is well-defined by (6.17) (2).

LEMMA 6.18. *Let n be a positive integer. The following diagram is commutative*:

$$\begin{array}{ccc} \mathfrak{W}_{n+1}\Omega^i & = & \mathfrak{W}_{n+1}\Omega^i \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \pi^n} \\ (Z^i_{n+1} + p\Omega^i)/(p\Omega^i + d\Omega^{i-1}) = Z_{n+1}\Omega^i_1/B\Omega^i_1 & \overset{C^n}{\longrightarrow} & \mathfrak{W}_1\Omega^i \end{array} \tag{6.18.1}$$

PROOF. We proceed by induction on n.

Consider the following diagram:

$$\begin{array}{ccccc} \mathfrak{W}_{n+1}\Omega^i & \overset{\pi}{\longrightarrow} & \mathfrak{W}_n\Omega^i & \overset{\mathrm{proj.}}{\longrightarrow} & Z_n\Omega^i_1/B\Omega^i_1 \\ {\scriptstyle \mathrm{proj.}}\downarrow & & & & \| \\ Z_{n+1}\Omega^i_1/B\Omega^i_1 & \overset{\mathrm{proj.}}{\longrightarrow} & Z_{n+1}\Omega^i_1/B_2\Omega^i_1 & \underset{\sim}{\overset{C^{-1}}{\longleftarrow}} & Z_n\Omega^i_1/B\Omega^i_1 \end{array} \tag{6.18.2}$$

Consider sections $[\omega] \in \mathfrak{W}_{n+1}\Omega^i$ ($\omega \in Z^i_{n+1}$) and $[\eta] \in \mathfrak{W}_n\Omega^i$ ($\eta \in Z^i_n$) such that $p^i\omega - \phi(\eta) \in p^{i+n}Z^i_1 + p^{i-1}dZ^{i-1}_1$ ((6.4.1)). Since $p^nZ^i_1 + p^{-1}dZ^{i-1}_1 \subset p\Omega^i + p^{-1}dZ^{i-1}_1$, the image of $\omega - p^{-i}\phi(\eta)$ in $Z_{n+1}\Omega^i_1/B_2\Omega^i_1$ is the zero by (6.17) (2). By the definition of π in (6.4.5) and by the commutative diagram

(6.0.5), (6.18.2) is commutative. We prove the commutativity of (6.18.1) by induction on n.

(6.18.1) for $n = 1$ is equal to (6.18.2) for $n = 1$.

By the inductive hypothesis, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{W}_n\Omega^i & \xrightarrow{\pi^{n-1}} & \mathfrak{W}_1\Omega^i \\
{\scriptstyle \text{proj.}}\downarrow & & \| \\
Z_n\Omega_1^i/B\Omega_1^i & \xrightarrow{C^{n-1}} & \mathfrak{W}_1\Omega^i
\end{array}
\tag{6.18.3}
$$

The commutativity of (6.18.3) and that of (6.18.2) show the commutativity of (6.18.1). □

Proposition 6.19. (cf. [Hy2, (2.3.1)], [HK, (4.4)]) *The surjective morphism*

$$(V^n, dV^n)\colon \mathfrak{W}_1\Omega^i \oplus \mathfrak{W}_1\Omega^{i-1} \longrightarrow \mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i \tag{6.19.1}$$

((6.12)) *induces an isomorphism*

$$(V^n, dV^n)\colon (\mathfrak{W}_1\Omega^i \oplus \mathfrak{W}_1\Omega^{i-1})/\mathfrak{W}_1K_n^i \xrightarrow{\sim} \mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i. \tag{6.19.2}$$

Proof. Let $([\omega],[\eta])$ ($\omega \in Z_1^i$, $\eta \in Z_1^{i-1}$) be a local section of $\mathfrak{W}_1\Omega^i \oplus \mathfrak{W}_1\Omega^{i-1}$. Assume that $V^n([\omega]) + dV^n([\eta]) = 0$. Then there exist local sections $\omega' \in \Omega^i$ and $\eta' \in d\Omega^{i-1}$ such that $p^n\omega + p^{-1}d\eta = p^{n+1}\omega' + d\eta'$. Then $d(\eta - p\eta') = -p^{n+1}(\omega - p\omega')$. Since $[\eta] = [\eta - p\eta']$ and $[\omega] = [\omega - p\omega']$, we may assume that $d\eta = -p^{n+1}\omega$. Hence $[\eta] \in Z_{n+1}\Omega_1^{i-1}/B\Omega_1^i$ and $[\omega] \in B_{n+2}\Omega_1^i/B\Omega_1^i$ by (6.17). Moreover, η (resp. ω) defines a section $[\eta]_{n+1} \in \mathfrak{W}_{n+1}\Omega^{i-1}$ (resp. $[\omega]_{n+1} \in \mathfrak{W}_{n+1}\Omega^i$), and we have a formula $d[\eta]_{n+1} = -[\omega]_{n+1}$ and hence $\pi^n d[\eta]_{n+1} = -\pi^n[\omega]_{n+1}$. Since $\pi d = d\pi$, we have $d\pi^n[\eta]_{n+1} = -\pi^n[\omega]_{n+1}$. By (6.18), $dC^n[\eta] = -C^n[\omega]$. Therefore $([\omega],[\eta]) \in \mathfrak{W}_1K_n^i$.

Conversely, let $([\omega],[\eta])$ ($\omega \in p^{-(n+1)}dZ_{n+1}^{i-1}$, $\eta \in Z_{n+1}^{i-1}$) be a local section of $\mathfrak{W}_1K_n^i$. Let N_{n+1}^i be the kernel of the morphism (6.19.1). First consider the case $[\omega] \in B_{n+1}\Omega_1^i/B\Omega_1^i$ and $[\eta] = 0$. Then $V^n[\omega] \in [p^np^{-(n+1-1)}dZ_n^{i-1}]_{n+1} \subset [d\Omega^{i-1}]_{n+1} = 0$. Hence

$$V^n((B_{n+1}\Omega_1^i/B\Omega_1^i, 0)) = 0.$$

Next, let us consider the general case. First note that C^n induces an isomorphism $C^n\colon B_{n+2}\Omega^i_1/B_{n+1}\Omega^i_1 \xrightarrow{\sim} B_2\Omega^i_1/B\Omega^i_1$. Therefore, by the definition of $\mathfrak{W}_1K^i_n$, we have $-C^{-n}dC^n[\eta]=[\omega]$ in $B_{n+2}\Omega^i_1/B_{n+1}\Omega^i_1$. Because

$$\begin{aligned} C^{-n}dC^n[\eta] &= C^{-n}d\pi^n[\eta]_{n+1} = C^{-n}\pi^n d[\eta]_{n+1} \\ &= C^{-n}\pi^n[p^{-(n+1)}d\eta]_{n+1} = \mathrm{proj.}([p^{-(n+1)}d\eta]_{n+1}), \end{aligned}$$

we have $-[p^{-(n+1)}d\eta]=[\omega]$ in $B_{n+2}\Omega^i_1/B_{n+1}\Omega^i_1$. Because $V^n((B_{n+1}\Omega^i_1/B\Omega^i_1,0))=0$, we have $[p^n\omega+p^np^{-(n+1)}d\eta]=0$ in $\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i$. This shows that $\mathfrak{W}_1K^i_n\subset N^i_{n+1}$. □

Remark 6.20. We immediately obtain [HK, (4.4)] by the equality (6.19.2); our proof is different from the proof of [loc. cit.].

(B)

Proposition 6.21. (cf. [I2, I (3.21)], [IR, II (1.3)], [M1, 1.3.4])

(6.21.1) $$\mathrm{Ker}(d\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_n\Omega^{i+1}) = F^n(\mathfrak{W}_{2n}\Omega^i).$$

Proof. The proof is the same as that in [M1, 1.3.4]: we immediately obtain (6.21) by taking the long exact sequence of the following exact sequence

$$0\longrightarrow \Omega^\bullet/p^n\Omega^\bullet \xrightarrow{p^n} \Omega^\bullet/p^{2n}\Omega^\bullet \longrightarrow \Omega^\bullet/p^n\Omega^\bullet \longrightarrow 0. \ \square$$

Proposition 6.22. (cf. [I2, I (3.13) $\sim$ (3.17)], [Hy2, (2.4.1), (2.4.2)])
(1) *Let n be a non-negative integer. The injective morphism $\mathbf{p}^m\colon \mathrm{gr}^n\mathfrak{W}\Omega^\bullet \longrightarrow \mathrm{gr}^{n+m}\mathfrak{W}\Omega^\bullet$ is a quasi-isomorphism.*

(2) *Let n be a non-negative integer. Let m be a positive integer. Then the following canonical projection*

(6.22.1) $$\pi\colon \mathfrak{W}_{n+m+1}\Omega^\bullet/p^m\mathfrak{W}_{n+m+1}\Omega^\bullet \longrightarrow \mathfrak{W}_{n+m}\Omega^\bullet/p^m\mathfrak{W}_{n+m}\Omega^\bullet$$

is a quasi-isomorphism.

(3) *Let n be a positive integer. The following canonical projection*

$$\mathfrak{W}\Omega^\bullet/p^n\mathfrak{W}\Omega^\bullet \longrightarrow \mathfrak{W}_n\Omega^\bullet \tag{6.22.2}$$

is a quasi-isomorphism.

Proof. (1): We may assume that $m = 1$. Set $\mathfrak{W}_0\Omega^\bullet = 0$. Since $\mathbf{p}$ is injective, we have only to prove, as in the proof of [I2, I (3.13)], that $\mathrm{gr}^{n+1}\mathfrak{W}\Omega^\bullet/\mathbf{p}\mathrm{gr}^n\mathfrak{W}\Omega^\bullet$ is acyclic. By the commutative diagram

$$\begin{array}{ccc}
\mathrm{gr}^n\mathfrak{W}\Omega^\bullet & \xrightarrow{\mathbf{p}} & \mathrm{gr}^{n+1}\mathfrak{W}\Omega^\bullet \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{Ker}(\mathfrak{W}_{n+1}\Omega^\bullet \longrightarrow \mathfrak{W}_n\Omega^\bullet) & \xrightarrow{\mathbf{p}} & \mathrm{Ker}(\mathfrak{W}_{n+2}\Omega^\bullet \longrightarrow \mathfrak{W}_{n+1}\Omega^\bullet),
\end{array}$$

we have only to prove that $\mathrm{Fil}^{n+1}\mathfrak{W}_{n+2}\Omega^\bullet/\mathbf{p}\mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^\bullet$ is acyclic. By (6.12), we have $\mathrm{Fil}^{n+1}\mathfrak{W}_{n+2}\Omega^i = V^{n+1}\mathfrak{W}_1\Omega^i + dV^{n+1}\mathfrak{W}_1\Omega^{i-1}$. Let ω be a section of $\mathfrak{W}_1\Omega^i$ such that $dV^{n+1}\omega \in p\mathfrak{W}_{n+2}\Omega^{i+1}$. Then $d\omega = F^{n+1}dV^{n+1}\omega \in pF^{n+1}\mathfrak{W}_{n+2}\Omega^{i+1} = 0$. Hence, by (6.21), $\omega \in F\mathfrak{W}_2\Omega^i$. Hence $V^{n+1}\omega \in p\mathfrak{W}_{n+2}\Omega^i = \mathbf{p}\mathfrak{W}_{n+1}\Omega^i$.

(2): By the proof of (1), we have only to prove that the following sequence

$$\begin{aligned}0 \longrightarrow \mathrm{gr}^{n+m}\mathfrak{W}\Omega^\bullet/\mathbf{p}^m\mathrm{gr}^n\mathfrak{W}\Omega^\bullet &\longrightarrow \mathfrak{W}_{n+m+1}\Omega^\bullet/p^m\mathfrak{W}_{n+m+1}\Omega^\bullet \\ &\xrightarrow{\pi} \mathfrak{W}_{n+m}\Omega^\bullet/p^m\mathfrak{W}_{n+m}\Omega^\bullet \longrightarrow 0\end{aligned} \tag{6.22.3}$$

is exact.

First let us prove the exactness at $\mathrm{gr}^{n+m}\mathfrak{W}\Omega^\bullet/\mathbf{p}^m\mathrm{gr}^n\mathfrak{W}\Omega^\bullet$. Let ω be a local section of $\mathrm{Fil}^{n+m}\mathfrak{W}_{n+m+1}\Omega^i \cap p^m\mathfrak{W}_{n+m+1}\Omega^i$. Then there exists a local section $\eta \in \mathfrak{W}_{n+1}\Omega^i$ such that $\omega = \mathbf{p}^m\eta$. Since $\pi(\omega) = 0$ and $\pi\mathbf{p} = \mathbf{p}\pi$, $0 = \mathbf{p}^m\pi(\eta)$. Since $\mathbf{p}$ is injective, $\pi(\eta) = 0$; thus $\eta \in \mathrm{Fil}^n\mathfrak{W}_{n+1}\Omega^i$. Therefore the morphism $\mathrm{gr}^{n+m}\mathfrak{W}\Omega^i/\mathbf{p}^m\mathrm{gr}^n\mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}_{n+m+1}\Omega^i/p^m\mathfrak{W}_{n+m+1}\Omega^i$ is injective. Thus we have only to prove the exactness at $\mathfrak{W}_{n+m+1}\Omega^i/p^m\mathfrak{W}_{n+m+1}\Omega^i$. Let ω be a local section of $\mathfrak{W}_{n+m+1}\Omega^i$ such that $\pi(\omega) \in p^m\mathfrak{W}_{n+m}\Omega^i$. Since π is surjective, we may assume that there exists a local section $\eta \in \mathfrak{W}_{n+m+1}\Omega^i$ such that $\pi(\omega) = p^m\pi(\eta)$. Thus $\omega \in p^m\mathfrak{W}_{n+m+1}\Omega^i + \mathrm{Fil}^{n+m}\mathfrak{W}_{n+m+1}\Omega^i$. Now we have proved the exactness of (6.22.3).

(3): The following sequence

$$0 \longrightarrow \mathfrak{W}_{m-n}\Omega^\bullet \overset{\mathbf{p}^n}{\longrightarrow} \mathfrak{W}_m\Omega^\bullet \longrightarrow \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet \longrightarrow 0 \tag{6.22.4}$$

is exact for $m \geq n$. Since the transition morphisms π's are surjective, we have the following exact sequence

$$0 \longrightarrow \mathfrak{W}\Omega^\bullet \overset{p^n}{\longrightarrow} \mathfrak{W}\Omega^\bullet \longrightarrow \varprojlim_m \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet \longrightarrow 0. \tag{6.22.5}$$

by taking the projective limit of (6.22.4). Hence

$$\begin{aligned}\mathcal{H}^i(\mathfrak{W}\Omega^\bullet/p^n\mathfrak{W}\Omega^\bullet) &= \mathcal{H}^i(\varprojlim_m \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet)\\ &= \mathcal{H}^i(R\varprojlim_m \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet).\end{aligned}$$

We have the following Leray spectral sequence

$$E_2^{ij} = R^i\varprojlim_m(\mathcal{H}^j(\mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet)) \Longrightarrow \mathcal{H}^{i+j}(R\varprojlim_m \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet).$$

By (2), $E_2^{ij} = 0$ if $i > 0$. Hence

$$\begin{aligned}\mathcal{H}^j(R\varprojlim_m \mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet) = E_2^{0j} &= \varprojlim_m \mathcal{H}^j(\mathfrak{W}_m\Omega^\bullet/p^n\mathfrak{W}_m\Omega^\bullet)\\ &= \mathcal{H}^j(\mathfrak{W}_n\Omega^\bullet). \ \square\end{aligned}$$

PROPOSITION 6.23. (cf. [I2, I (3.21.1.5)], [Lo, (1.20)]) *Let n be a non-negative integer. Then $d^{-1}(p^n\mathfrak{W}\Omega^{i+1}) = F^n\mathfrak{W}\Omega^i$.*

PROOF. We have only to prove the inclusion $\subset$. Let $(\omega_m)_{m=1}^\infty$ ($\omega_m \in \mathfrak{W}_m\Omega^i$) be a local section of $d^{-1}(p^n\mathfrak{W}\Omega^{i+1})$. Then $[\omega] \in \mathcal{H}^i(\mathfrak{W}\Omega^\bullet/p^n\mathfrak{W}\Omega^\bullet) = \mathcal{H}^i(\mathfrak{W}_n\Omega^\bullet)$. By (6.21.1), there exists a local section $\eta_{2n} \in \mathfrak{W}_{2n}\Omega^i$ such that $\omega_n = F^n(\eta_{2n})$. By (6.14.1;$r$), we may assume that there exists a local section η of $\mathfrak{W}\Omega^i$ whose image in $\mathfrak{W}_{2n}\Omega^i$ is η_{2n}. Then $[\omega - F^n\eta] = 0$ in $\mathcal{H}^i(\mathfrak{W}\Omega^\bullet/p^n\mathfrak{W}\Omega^\bullet)$. Hence $\omega - F^n\eta \in d\mathfrak{W}\Omega^{i-1} + p^n\mathfrak{W}\Omega^i$. Since $d = FdV$ and $FV = p$, $\omega \in F^n\mathfrak{W}\Omega^i$. $\square$

(C)

THEOREM 6.24. (cf. [IR, II (1.2)], [Lo, (2.17)]) *The isomorphism* (6.16.1) *induces the following isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathcal{T}, W_n[d])$:

$$R_n \otimes^L_R \mathfrak{W}\Omega^\bullet \xrightarrow{\sim} \mathfrak{W}_n\Omega^\bullet \tag{6.24.1}$$

PROOF. The proof is the same as that of [IR, II (1.2)] by using (6.15) and (6.23): the following sequence

$$0 \longrightarrow \mathfrak{W}\Omega^{i-1} \overset{(F^n,-F^nd)}{\longrightarrow} \mathfrak{W}\Omega^{i-1} \oplus \mathfrak{W}\Omega^i \overset{(dV^n,V^n)}{\longrightarrow} \mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i \longrightarrow 0$$

is exact. □

COROLLARY 6.25. *Let M be a positive integer. Let $(\Omega^\bullet_m, \phi_m, C_m^{-1})$ $(0 \leq m \leq M)$ be objects of $\mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W)$. Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & \mathfrak{W}_n\Omega^2_0 & \longrightarrow & \mathfrak{W}_n\Omega^2_1 & \longrightarrow & \mathfrak{W}_n\Omega^2_2 & \longrightarrow & \cdots \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & \mathfrak{W}_n\Omega^1_0 & \longrightarrow & \mathfrak{W}_n\Omega^1_1 & \longrightarrow & \mathfrak{W}_n\Omega^1_2 & \longrightarrow & \cdots \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & \mathfrak{W}_n\Omega^0_0 & \longrightarrow & \mathfrak{W}_n\Omega^0_1 & \longrightarrow & \mathfrak{W}_n\Omega^0_2 & \longrightarrow & \cdots
\end{array} \tag{6.25.1;n}$$

be a commutative diagram of R_n-modules $(n \in \mathbb{Z}_{>0})$. If all horizontal lines of $(6.25.1; 1)$ *are exact, then $\mathfrak{W}_n\Omega^\bullet_0$ $(n \in \mathbb{Z}_{>0})$ is quasi-isomorphic to the single complex of the following double complex:*

$$
(6.25.2)\qquad
\begin{array}{ccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
{\scriptstyle \pm d}\uparrow & & {\scriptstyle \mp d}\uparrow & & {\scriptstyle \pm d}\uparrow & & \\
\mathfrak{W}_n\Omega^2_1 & \longrightarrow & \mathfrak{W}_n\Omega^2_2 & \longrightarrow & \mathfrak{W}_n\Omega^2_3 & \longrightarrow & \cdots \\
{\scriptstyle \pm d}\uparrow & & {\scriptstyle \mp d}\uparrow & & {\scriptstyle \pm d}\uparrow & & \\
\mathfrak{W}_n\Omega^1_1 & \longrightarrow & \mathfrak{W}_n\Omega^1_2 & \longrightarrow & \mathfrak{W}_n\Omega^1_3 & \longrightarrow & \cdots \\
{\scriptstyle \pm d}\uparrow & & {\scriptstyle \mp d}\uparrow & & {\scriptstyle \pm d}\uparrow & & \\
\mathfrak{W}_n\Omega^0_1 & \longrightarrow & \mathfrak{W}_n\Omega^0_2 & \longrightarrow & \mathfrak{W}_n\Omega^0_3 & \longrightarrow & \cdots .
\end{array}
$$

Proof. Using (6.24.1) and Ekedahl's Nakayama lemma [Ek2, I (1.1.3)], we can check (6.25) without difficulty. □

Remark 6.26. Let $\star$ be a positive integer or nothing. Since $\mathfrak{W}_\star\colon \mathrm{C}^{\mathrm{b}}_{\mathrm{F}}(\mathcal{T}, \mathbb{Z}_p; W) \longrightarrow \mathrm{D}^{\mathrm{b}}(R_\star)$ is functorial, the obvious generalizations of the results (6.1) $\sim$ (6.25) for copsimplicial sheaves hold.

Now we apply the theory above to (idealized) log de Rham-Witt complexes.

Let s be a log point whose underlying scheme $\mathrm{Spec}(\kappa)$ is the spectrum of the perfect field of characteristic $p > 0$. Let X be an SNCL variety over s. In order to express the projection $\pi\colon W_{n+1}\widetilde{\Lambda}^i_X \longrightarrow W_n\widetilde{\Lambda}^i_X$ in [Hy2, (1.3.2)] locally, we need to recall the definition of the admissible lift of X (cf. [M1, 2.4.3], [Hy2, (1.1)]); the definition here is slightly different from that in [M1, 2.4.3], and it is the same as that in [GK] for an affine SNCL variety.

Let $W\{x_0, \dots, x_d\}$ be the p-adic completion of the polynomial ring $W[x_0, \dots, x_d]$ over W. Assume that $\overset{\circ}{X}$ is affine (for simplicity). A pair $(\mathcal{Y}, \mathcal{X})$ of log formal schemes over $\mathrm{Spf}(W)$ is called an *admissible lift* of X if $\overset{\circ}{\mathcal{Y}}$ is formally etale over $\mathrm{Spf}(W\{x_0, \dots, x_d\})$ with structural morphism $\mathrm{Spf}(W\{x_0, \dots, x_d\}) \longrightarrow \mathrm{Spf}(W\{t\})$ given by $t \longmapsto x_0 \cdots x_r$ $(0 \leq r \leq d)$, if the log structure of $\mathcal{Y}$ is associated to a morphism $\mathbb{N}^{r+1} \ni (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \longmapsto x_{i-1} \in \mathcal{O}_{\mathcal{Y}}$, if $\mathcal{X} = \mathcal{Y}\hat{\otimes}_{W\{t\}, t\to 0}W$ with the pullback of the log structure of $\mathcal{Y}$ and if there exists the following cartesian diagram of (formal) fine log schemes:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ s & \longrightarrow & (\mathrm{Spf}(W), \mathbb{N} \oplus W^*). \end{array}$$

Here the morphism $s \longrightarrow (\mathrm{Spf}(W), \mathbb{N} \oplus W^*)$ is the natural exact closed immersion. The existence of the admissible lift of X has been shown in [GK, 1.1] by using [SGA 1, I Proposition 8.1].

Let $\Phi\colon (\mathcal{Y}, \mathcal{X}) \longrightarrow (\mathcal{Y}, \mathcal{X})$ be a lift of the Frobenius endomorphism of the log scheme $(\mathcal{Y} \otimes_W \kappa, \overset{\circ}{X})$ such that $\Phi^*(W\{t\}) \subset W\{t\}$ (cf. [Hy2, (1.1)]). We say that the triple $(\mathcal{Y}, \mathcal{X}, \Phi)$ is *admissible*. Note that $\Phi^*(\overset{\circ}{\mathcal{X}}) = p\overset{\circ}{\mathcal{X}}$, and it is easy to check that there exists an element of $u \in W\{t\}$ such that $\Phi^*(t) = t^p(1 + pu)$.

Proposition 6.27. (1) *Let L be a fine log structure of* $\mathrm{Spec}(\kappa)$. *Let Y be a log smooth scheme of Cartier type over* $(\mathrm{Spec}(\kappa), L)$. *Let $\mathcal{Y}/(\mathrm{Spf}(W), W(L))$ be a formally log smooth lift of $Y/(\mathrm{Spec}(\kappa), L)$ with a lift $\Phi\colon \mathcal{Y} \longrightarrow \mathcal{Y}$ of the Frobenius of Y. Set $\mathcal{Y}_n := \mathcal{Y} \otimes_W W_n$ $(n \in \mathbb{Z}_{>0})$. Let $\Lambda_n^\bullet$ be the log de Rham complex of $\mathcal{Y}_n/(\mathrm{Spec}(W_n), W_n(L))$. Set $\Lambda^\bullet := \varprojlim_n \Lambda_n^\bullet$. Let $C^{-1}\colon \Lambda_1^i \overset{\sim}{\longrightarrow} \mathcal{H}^i(\Lambda_1^\bullet)$ be the log Cartier inverse isomorphism* ([Ka2, (4.12) (1)]). *Then $(\Lambda^\bullet, \Phi^*, C^{-1})$ satisfies the conditions* (6.0.1) $\sim$ (6.0.5) *for $(\mathcal{T}, \mathcal{A}) = (\widetilde{Y}_{\mathrm{zar}}, \mathbb{Z}_p)$.*

(2) *Let X be an SNCL variety over s. Let $(\mathcal{X}, \mathcal{Y}, \Phi)$ be an admissible triple of X over $W\{t\}$. Set $(\mathcal{Y}_n, \mathcal{X}_n) := (\mathcal{Y}, \mathcal{X}) \otimes_W W_n$, and set $\widetilde{\Lambda}_n^\bullet := \mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)$ and $\widetilde{\Lambda}^\bullet := \varprojlim_n \widetilde{\Lambda}_n^\bullet$. Let $C^{-1}\colon \widetilde{\Lambda}_1^i \overset{\sim}{\longrightarrow} \mathcal{H}^i(\widetilde{\Lambda}_1^\bullet)$ be the log Cartier inverse isomorphism* ([Hy2, (2.1.1)]). *Then $(\widetilde{\Lambda}^\bullet, \Phi^*, C^{-1})$ satisfies the conditions* (6.0.1) $\sim$ (6.0.5) *for $(\mathcal{T}, \mathcal{A}) = (\widetilde{X}_{\mathrm{zar}}, \mathbb{Z}_p)$.*

(3) *Let (X, D) be a smooth scheme with an SNCD over* $\mathrm{Spec}(\kappa)$. *Let $(\mathcal{X}, \mathcal{D})$ be a lift of (X, D) over* $\mathrm{Spf}(W)$ *with a lift $\Phi\colon (\mathcal{X}, \mathcal{D}) \longrightarrow (\mathcal{X}, \mathcal{D})$ of the Frobenius endomorphism of (X, D). Set $(\mathcal{X}_n, \mathcal{D}_n) := (\mathcal{X}, \mathcal{D}) \otimes_W W_n$ $(n \in \mathbb{Z}_{>0})$. Set $\Omega^\bullet_{\mathcal{X}/W}(-\log \mathcal{D}) := \varprojlim_n \Omega^\bullet_{\mathcal{X}_n/W_n}(-\log \mathcal{D}_n)$. Let $C^{-1}\colon \Omega^i_{X/\kappa}(-\log D) \overset{\sim}{\longrightarrow} \mathcal{H}^i(\Omega^\bullet_{X/\kappa}(-\log D))$ be the log Cartier inverse isomorphism* ([DI, (4.2.1.3)]). *Then $(\Omega^\bullet_{\mathcal{X}/W}(-\log \mathcal{D}), \Phi^*, C^{-1})$ satisfies the conditions* (6.0.1) $\sim$ (6.0.5) *for $(\mathcal{T}, \mathcal{A}) = (\widetilde{X}_{\mathrm{zar}}, \mathbb{Z}_p)$.*

Proof. (1): The conditions (6.0.1), (6.0.2) and (6.0.3) are obviously satisfied. The condition (6.0.4) immediately follows from [Ka2, (4.12) (1)].

We show that (6.0.5) holds for the object $(\Lambda^\bullet, \Phi^*, C^{-1})$ in (1). Let m_j $(j = 1, \dots, i)$ be a local section of the log structure of $\mathcal{Y}$. Then there exists a section $a_j \in \mathcal{O}_{\mathcal{Y}}$ such that $\Phi^*(m_j) = m_j^p(1 + pa_j)$. Let f be a section of $\mathcal{O}_{\mathcal{Y}}$. Then there exists a section $g \in \mathcal{O}_{\mathcal{Y}}$ such that $\Phi^*(f) = f^p + pg$. Hence we have

$$\begin{aligned}&(p^{-i}\Phi^*)(f d\log m_1 \cdots d\log m_i)\\ &\qquad = (f^p + pg)(d\log m_1 + \frac{da_1}{1 + pa_1}) \cdots (d\log m_i + \frac{da_i}{1 + pa_i}).\end{aligned}$$

The reduction mod p of this section is equal to $f^p d(\log m_1 + da_1) \cdots (d\log m_i + da_i) \mod p$; furthermore, we have $d(f^p(d\log m_1 + da_1) \cdots (d\log m_i + da_i)) \equiv 0 \mod p$ and, for a positive integer $k < i$,

$$f^p \prod_{l=1}^{k} d\log m_{j_l} \prod_{l=k+1}^{i} da_{j_l} \equiv (-1)^k d(f^p a_{j_{k+1}} \prod_{l=1}^{k} d\log m_{j_l} \prod_{l=k+2}^{i} da_{j_l}) \mod p.$$

Therefore we have the commutativity of (6.0.5) by [Ka2, (4.12) (1)].

(2): The conditions (6.0.1), (6.0.2) and (6.0.3) are obviously satisfied. The conditions (6.0.4) and (6.0.5) are nothing but the first isomorphism in [Hy2, (2.1.1)].

(3): By using the Cartier inverse isomorphism in [DI, (4.2.1.3)], the proof is the same as that of (1). □

Corollary 6.28. *Let L be a fine log structure of* $\mathrm{Spec}(\kappa)$. *Let Y be a log smooth scheme of Cartier type over* $(\mathrm{Spec}(\kappa), L)$. *Let X be an SNCL variety over s. Let (Z, D) be a smooth scheme with an SNCD over* $\mathrm{Spec}(\kappa)$. *Let $\star$ be a positive integer n or nothing. Let $W_\star\Lambda^\bullet$ be an* (*idealized*) *log de Rham-Witt complex $W_\star\Lambda^\bullet_Y$, $W_\star\widetilde{\Lambda}^\bullet_X$ or $W_\star\Omega^\bullet_Z(-\log D)$. Then the following hold*:

(1) ([Hy2, p. 245], [HK, §4], [Lo, §1]) *Let F, V, π, d be standard operators on $W_\star\Lambda^\bullet$. Then, $F\mathbf{p} = \mathbf{p}F$, $V\mathbf{p} = \mathbf{p}V$, $d\mathbf{p} = \mathbf{p}d$, $\mathbf{p}\pi = \pi\mathbf{p} = p$, $F\pi = \pi F$, $V\pi = \pi V$ and $d\pi = \pi d$.*

(2) (cf. [Hy2, (2.2.2), (2.2.3)], [HK, (4.5)]) *The morphism $\mathbf{p}\colon W_n\Lambda^i \longrightarrow W_{n+1}\Lambda^i$ is injective*; *the morphism $\pi\colon W_{n+1}\Lambda^i \longrightarrow W_n\Lambda^i$ is surjective. The sheaf $W\Lambda^i$ $(i \in \mathbb{N})$ is torsion-free.*

(3) (cf. [Lo, (1.16), p. 258]) $\mathrm{Fil}^r W_n\Lambda^i = V^r W_{n-r}\Lambda^i + dV^r W_{n-r}\Lambda^{i-1}$ $(i, r \in \mathbb{Z})$ *and* $\mathrm{Fil}^r W\Lambda^i = V^r W\Lambda^i + dV^r W\Lambda^{i-1}$ $(i, r \in \mathbb{N})$.

(4) *Let* $\mathcal{O}$ *be* $\mathcal{O}_Y$, $\mathcal{O}_X$ *or* $\mathcal{O}_Z$. *Then* $W_n\Lambda^i$ $(i \in \mathbb{N})$ *is a coherent sheaf of* $W_n(\mathcal{O})$*-modules.*

(5) (cf. [Lo, (1.20.3)]) *The canonical projection* $W\Lambda^\bullet/p^n W\Lambda^\bullet \longrightarrow W_n\Lambda^\bullet$ *is a quasi-isomorphism.*

(6) (cf. [Lo, (1.20.1)]) $d^{-1}(p^n W\Lambda^\bullet) = F^n W\Lambda^\bullet$.

(7) (cf. [Lo, (2.17)]) $R_n \otimes^L_R W\Lambda^\bullet = W_n\Lambda^\bullet$ $(n \in \mathbb{Z}_{>0})$.

(8) (cf. [HK, (4.2)]) *The projection* $\pi\colon W_{n+1}\Lambda^i \longrightarrow W_n\Lambda^i$ *has the local expression in* (6.4.5) *and* (6.5.1).

(9) (cf. [M1, 3.15.1]) *The complex* $W_n\Omega^\bullet_Z(-\log D)$ $(n \in \mathbb{Z}_{>0})$ *is quasi-isomorphic to the single complex of the following double complex:*

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
W_n\Omega^2_Z & \xrightarrow{\iota^{(0)*}} & W_n\Omega^2_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W_n\Omega^2_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
W_n\Omega^1_Z & \xrightarrow{\iota^{(0)*}} & W_n\Omega^1_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W_n\Omega^1_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow & & {\scriptstyle d}\uparrow & & \\
W_n\Omega^0_Z & \xrightarrow{\iota^{(0)*}} & W_n\Omega^0_{D^{(1)}} & \xrightarrow{\iota^{(1)*}} & W_n\Omega^0_{D^{(2)}} & \xrightarrow{\iota^{(2)*}} & \cdots .
\end{array}
\tag{6.28.1}
$$

Here $\iota^{(k)}$ $(k \in \mathbb{N})$ *is the morphism defined in* (5.0.5).

Remark 6.29. (1) I think that the proof of [M1, 1.3.3] is sketchy: the sentence “Le lemme résulte alors, par un calcul formel (cf. [I2, I. 3.31] et [IR, II. 1.2]), du lemme suivant.” in [M1, p. 307, l. 8–9] and [M1, 1.3.4] are not enough for the proof of [M1, 1.3.3]: as in the proof of [IR, II (1.2)], (6.28) (3) (cf. **(A)**) and (6.28) (6) (cf. **(B)**) are necessary for the proof of [M1, 1.3.3]; (6.12) for $W_n\Omega^\bullet_X(-\log D)$, **(A)** and **(B)** for $W\Omega^\bullet_X(-\log D)$ have not been proved in [M1]; the proof of (6.28) (7) (cf. **(C)**) is a detailed proof of [M1, 1.3.3]. Though the first isomorphism in [M1, 1.3.3] is a special case of [Lo, (2.17)], [Lo, (2.17)] is not useful for the second isomorphism in [M1, 1.3.3]; as far as I know, any result in any published paper does not imply the second isomorphism in [M1, 1.3.3].

The second isomorphism in [M1, 1.3.3] has been used in the proof of [loc. cit., 3.15.1](=(5.0.4;n) (cf. (6.28.1))) and [loc. cit., 3.15.1] is crucial for proving a fundamental fact which tells us that the morphism

$$\theta\wedge\colon W_n\Lambda^i_X \xrightarrow{\sim} (W_nA^{i0}_X \overset{(-1)^i\theta\wedge}{\longrightarrow} W_nA^{i1}_X \overset{(-1)^i\theta\wedge}{\longrightarrow} \cdots) \quad (n\in\mathbb{Z}_{>0})$$

is a quasi-isomorphism (cf. [M1, 3.15]). See also (9.5) (2) below.

(2) Let Z be a smooth scheme over a perfect field κ of characteristic $p>0$. Let $D_1\cup D_2$ be a union of two NCD's over κ such that $D_1\cup D_2$ is also an NCD over κ. Then we can define an idealized log de Rham-Witt complex $W_\star\Omega^\bullet_Z(\log(D_1-D_2))$ ($\star$ =a positive integer or nothing).

7. Obverse and reverse log de Rham-Witt complexes

In this section we define an abelian sheaf $(W_n\Lambda^i_Y)''$ which is a correction of $(W_n\Lambda^i_Y)'$ in [HK, (4.6)] and which is shown to be a logarithmic version of [Ka1, §2 Corollary 3]. However, for the proof of the correction of [HK, (4.6)], we have to use an expression of $(W_n\Lambda^i_Y)''$ which is different from the logarithmic version of [Ka1, §2 Corollary 3]. Though the proof for the coincidence between our expression of $(W_n\Lambda^i_Y)''$ and the logarithmic version of [Ka1, §2 Corollary 3] is easy, our expression of $(W_n\Lambda^i_Y)''$ is crucial for the proof of the correction of [HK, (4.6)]. We also prove the compatibility of two projections of two log de Rham-Witt complexes. As applications, we complete [HK, (4.6), (4.8)]; [HK, (4.8)] has been used in the proof of Hyodo-Kato's isomorphism ([HK, (5.1)]). Using an analogous compatibility (and the log version of a lemma of Dwork-Dieudonné-Cartier), we also give a right proof of [HK, (4.19)]; though it is claimed in [loc. cit.] that the proof of the [HK, (4.19)] is the same as that of the proof of [I2, II (1.4)], it is not so because the log de Rham-Witt complex in [HK, (4.19)] is defined by the method of Katz-Illusie-Raynaud and the log de Rham-Witt complex in [I2, II (1.4)] is defined by the V-procomplex; as to some properties of the projection of the log de Rham-Witt complex by the method of Katz-Illusie-Raynaud is nontrivial, while some properties of the projection of the de Rham-Witt complex by the method of V-procomplex is evident. [HK, (4.19)] has been used in the construction of the spectral sequence (2.0.1) (and in [Nakk2, (4.10), (4.11)]).

Let κ be a perfect field of characteristic $p>0$ and L a fine log structure on Spec(κ). Let $f\colon Y = (\overset{\circ}{Y}, M) \longrightarrow (\mathrm{Spec}(\kappa), L)$ be a log smooth morphism of Cartier type of fine log schemes. Let $(\mathrm{Spf}(W_\star), W_\star(L))$ ($\star$ is a positive integer or nothing) be the canonical lift of $(\mathrm{Spec}(\kappa), L)$ (cf. [HK, (3.1)]). Let $\alpha\colon M \longrightarrow \mathcal{O}_Y$ be the structural morphism.

First, we define an abelian sheaf $(W_n\Lambda^i_Y)''$. The sheaf $(W_n\Lambda^i_Y)'$ on $\overset{\circ}{Y}$ is a quotient of

$$\begin{aligned}(7.0.1)\qquad &\{W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^{i}(M^{\mathrm{gp}}/f^{-1}(L^{\mathrm{gp}}))\} \\ &\oplus \{W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^{i-1}(M^{\mathrm{gp}}/f^{-1}(L^{\mathrm{gp}}))\}\end{aligned}$$

divided by a $\mathbb{Z}$-submodule $\mathcal{F}_n$ generated by the images of the local sections of the following type

$$(7.0.2)\quad (v_j(\alpha(a_1)) \otimes (a_1 \wedge \cdots \wedge a_i), 0) - p^j(0, v_j(\alpha(a_1)) \otimes (a_2 \wedge \cdots \wedge a_i))$$

$$(a_1, \ldots, a_i \in M,\ 0 \leq j < n),$$

where $v_j(b) := (\underbrace{0, \cdots, 0}_{j \text{ times}}, b, 0, \ldots, 0)$ for a local section $b \in \mathcal{O}_Y$ ([HK, (4.6)]). The sheaf $(W_n\Lambda^i_Y)'$ is wrong and I think that [HK, (4.6)] does not hold. Indeed, consider a case $n \geq 2$ and $i \geq 2$ in [HK, (4.6)], and consider a section $(0, v_j(\alpha(a_2)^e) \otimes (a_2 \wedge \cdots \wedge a_i)) \in (W_n\Lambda^i_Y)'$ for $j \geq 1$ and for $e \geq 1$. Then the image of this section by a morphism $s_n\colon (W_n\Lambda^i_Y)' \longrightarrow \mathcal{H}^i(C_{Y/(W_n, W_n(L))})$ in [HK, (4.9)] which is denoted by s in [loc. cit.] is equal to the class of

$$\begin{aligned}&\widetilde{\alpha(a_2)}^{e(p^{n-j}-1)} d\alpha(\widetilde{a}_2^e) d\log \widetilde{a}_2 \cdots d\log \widetilde{a}_i \\ &= e\widetilde{\alpha(a_2)}^{ep^{n-j}} d\log \widetilde{a}_2 d\log \widetilde{a}_2 \cdots d\log \widetilde{a}_i,\end{aligned}$$

which is the zero. However I think that this section does not belong to $\mathcal{F}_n$ in general. Thus, for the case $n \geq 2$ and $i \geq 2$, I define a log Hodge-Witt sheaf $(W_n\Lambda^i_Y)''$ which is a quotient of (7.0.1) divided by $\mathcal{F}_n$ and a $\mathbb{Z}$-submodule $\mathcal{G}'_n$, where $\mathcal{G}'_n$ is generated by the images of the local sections of the following type

$$(7.0.3)\qquad (0, v_j(\alpha(a_2)^e) \otimes (a_2 \wedge \cdots \wedge a_i))$$

$$(a_2,\dots,a_i\in M,\ 1\leq j<n,\ e\in \mathbb{Z}_{\geq 1})$$

as an abelian sheaf on $Y_{\rm zar}$. K. Kato has given a remark that the $\mathbb{Z}$-submodule $\mathcal{F}_n+\mathcal{G}'_n$ is equal to $\mathcal{F}_n+\mathcal{G}_n$, where $\mathcal{G}_n$ is a $\mathbb{Z}$-submodule generated by the images of the local sections of the following type

$$(7.0.4)\qquad (0,v_j(\alpha(a_2))\otimes(a_2\wedge\cdots\wedge a_i))\quad (a_2,\dots,a_i\in M,\ 1\leq j<n).$$

Indeed, let us consider the section (7.0.3). We may assume that e is a power p^r ($r\in\mathbb{N}$). In this case, $(0,v_j(\alpha(a_2)^{p^r})\otimes(a_2\wedge\cdots\wedge a_i))=p^r(0,v_{j-r}(\alpha(a_2))\otimes(a_2\wedge\cdots\wedge a_i))$ if $r\leq j$ by the relation $VF=p$ on $W_n(\mathcal{O}_Y)$, and $p^j(0,v_0(\alpha(a_2)^{p^{r-j}})\otimes(a_2\wedge\cdots\wedge a_i))$ if $r\geq j$. The former is equivalent to the zero modulo $\mathcal{G}_n$; as for the latter, we have $p^j(0,v_0(\alpha(a_2)^{p^{r-j}})\otimes(a_2\wedge\cdots\wedge a_i))\equiv p^j(v_0(\alpha(a_2)^{p^{r-j}})\otimes(a_2^{p^{r-j}}\wedge a_2\wedge\cdots\wedge a_i),0)$ modulo $\mathcal{F}_n$, which is the zero. Hence $\mathcal{F}_n+\mathcal{G}'_n=\mathcal{F}_n+\mathcal{G}_n$. The economic expression $\mathcal{F}_n+\mathcal{G}_n$ is a logarithmic version of the expression in [Ka1, §2 Corollary 3]. The reader may jump to the conclusion that the expression $\mathcal{F}_n+\mathcal{G}'_n$ is unnecessary. However he must not do so because we shall use the expression $\mathcal{F}_n+\mathcal{G}'_n$ crucially in the proof of (7.5): more concretely we shall use the expression $\mathcal{F}_n+\mathcal{G}'_n$ in a key fact that the isomorphism $(V^n,dV^n)C^{-1}\colon(\Lambda^i_Y\oplus\Lambda^{i-1}_Y)/K^i_n\overset{\sim}{\longrightarrow}{\rm Fil}^n(W_{n+1}\Lambda^i_Y)$ in [HK, (4.4)] factors through the surjective morphism $(\Lambda^i_Y\oplus\Lambda^{i-1}_Y)/K^i_n\longrightarrow{\rm Fil}^n((W_{n+1}\Lambda^i_Y)'')$ (see the proof of (7.5) below for details). I think that [HK, (4.6)] itself and the proof of [loc. cit.] are quite incomplete because the indispensable expression $\mathcal{F}_n+\mathcal{G}'_n$ for the proof of [HK, (4.6)] has not appeared in [HK] and [Ka1].

If $n=1$ or if $i=0$ or $i=1$, then we set $(W_n\Lambda^i_Y)'':=(W_n\Lambda^i_Y)'$ for the unification of notation. For the time being, we consider the sheaf $(W_n\Lambda^i_Y)''$ only as an abelian sheaf on $Y_{\rm zar}$. Later in (7.7) (2), we shall endow $(W_n\Lambda^i_Y)''$ with a natural $W_n(\mathcal{O}_Y)$-module structure.

As explained in the paragraph before the previous one, the morphism s_n in [HK, (4.9)] factors through a morphism

$$(7.0.5)\qquad s_n\colon (W_n\Lambda^i_Y)''\longrightarrow W_n\Lambda^i_Y\quad(n\in\mathbb{Z}_{>0}).$$

In this section we prove that the following diagram is commutative:

$$(7.0.6)\qquad \begin{array}{ccc}(W_{n+1}\Lambda^i_Y)'' & \xrightarrow{s_{n+1}} & W_{n+1}\Lambda^i_Y\\ {\scriptstyle\rm proj.}\big\downarrow & & \big\downarrow{\scriptstyle\pi}\\ (W_n\Lambda^i_Y)'' & \xrightarrow{s_n} & W_n\Lambda^i_Y.\end{array}$$

No one has proved the commutativity of (7.0.6) in a literature even if one replaces $(W_m\Lambda^i_Y)''$ with $(W_m\Lambda^i_Y)'$ $(m = n, n+1)$. Using the commutativity of (7.0.6), we also complete the correction of [HK, (4.6)], that is, we prove that the morphism s_n in (7.0.5) is an isomorphism. In (7.6) below, this correction and the commutativity of (7.0.6) will complete [HK, (4.8)]. In order to prove the commutativity of (7.0.6), we use a local expression of π in (6.28) (8).

Proposition 7.1. *The diagram* (7.0.6) *is commutative.*

Proof. The problem is local. Assume that there exists a formally log smooth lift $\mathcal{Y}$ of Y over $(\mathrm{Spf}(W), W(L))$. Assume, moreover, that there exists a lift $\Phi\colon \mathcal{Y} \longrightarrow \mathcal{Y}$ of the Frobenius of Y. Set $\mathcal{Y}_n := \mathcal{Y} \otimes_W W_n$. Let $\Lambda^\bullet_n$ be the log de Rham complex of $\mathcal{Y}_n/(\mathrm{Spec}(W_n), W_n(L))$. Then $\Lambda^\bullet_n$ is a crystalline complex $C_{Y/(W_n,W_n(L))}$ of $Y/(W_n, W_n(L))$ ([HK, (2.19)]). Set $\Lambda^\bullet := \varprojlim_n \Lambda^\bullet_n$.

Let a_i $(i = 0, \dots, n-1)$ be a local section of $\mathcal{O}_Y$. Let $\widetilde{a}_i \in \mathcal{O}_{\mathcal{Y}_n}$ be a lift of a_i. Let

$$s_n(0,0)\colon (W_n\Lambda^0_Y)'' = W_n(\mathcal{O}_Y) \ni (a_0, \dots, a_{n-1}) \longmapsto \sum_{i=0}^{n-1} p^i\widetilde{a}_i^{p^{n-i}} \in \mathcal{H}^0(\Lambda^\bullet_n) = W_n\Lambda^0_Y \tag{7.1.1}$$

be a morphism defined in [HK, (4.9)] and denoted by τ in [loc. cit.]. Let

$$s_n(1,0)\colon W_n(\mathcal{O}_Y) \ni (a_0, \dots, a_{n-1}) \longmapsto \sum_{i=0}^{n-1} \widetilde{a}_i^{p^{n-i}-1}d\widetilde{a}_i \in \mathcal{H}^1(\Lambda^\bullet_n) \tag{7.1.2}$$

be a morphism defined in [HK, (4.9)] and denoted by δ in [loc. cit.].

First, set $i = 0$ in (7.0.6). Then we can check the commutativity of the following diagram:

$$\begin{array}{ccc} (W_{n+1}\Lambda^0_Y)'' & \xrightarrow{s_{n+1}(0,0)} & W_{n+1}\Lambda^0_Y \\ {\scriptstyle\text{proj.}}\downarrow & & \downarrow{\scriptstyle\pi} \\ (W_n\Lambda^0_Y)'' & \xrightarrow{s_n(0,0)} & W_n\Lambda^0_Y. \end{array} \tag{7.1.3}$$

Indeed, the last term $p^n\widetilde{a}_n^p$ in $s_{n+1}(0,0)((a_0,\dots,a_n))$ is the zero in the third sheaf in (6.4.5) in the case $i=0$. Furthermore, $\Phi^*(\widetilde{a}_i^{p^{n-i}}) \equiv \widetilde{a}_i^{p^{n+1-i}}$ mod p^{n+1-i}. Hence the commutativity of (7.1.3) follows from the local expression of π in (6.4.5).

Next we check the commutativity of the following diagram:

$$
\begin{array}{ccc}
(W_{n+1}\Lambda_Y^0)'' & \xrightarrow{s_{n+1}(1,0)} & W_{n+1}\Lambda_Y^1 \\
{\scriptstyle \text{proj.}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
(W_n\Lambda_Y^0)'' & \xrightarrow{s_n(1,0)} & W_n\Lambda_Y^1.
\end{array}
\tag{7.1.4}
$$

The section $s_{n+1}(1,0)((a_0,\dots,a_n))$ is represented by $\sum_{i=0}^n \widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i$. Let $\widetilde{a}\in\Lambda^0$ be a local lift of a local section $a\in\mathcal{O}_Y$. Then there exists a local section $\widetilde{b}\in p\Lambda^0$ such that $\Phi^*(\widetilde{a})=\widetilde{a}^p+\widetilde{b}$. In the second sheaf in (6.4.5) in the case $i=1$, we have an equality $p\widetilde{a}^{p-1}d\widetilde{a}=\Phi^*(d\widetilde{a})$ since $\Phi^*(d\widetilde{a})-p\widetilde{a}^{p-1}d\widetilde{a}=d\widetilde{b}$; the image of $\widetilde{a}^{p-1}d\widetilde{a}\in W_{n+1}\Lambda_Y^1$ by the composite morphism (6.4.5) is $d\widetilde{a}=0$. Thus we may assume that $a_n=0$ and $\widetilde{a}_n=0$.

By the definition of $s_{n+1}(1,0)$, $s_{n+1}(1,0)((a_0,\dots,a_{n-1},0))$ is represented by a section $\sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i$. We claim that a formula

$$\Phi^*\Big(\sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n-i}-1}d\widetilde{a}_i\Big)=p\sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i$$

holds in the second sheaf in (6.4.5). Indeed, by a formula in [HK, p. 251], we have

$$\{\Phi^*(\widetilde{a}^{p^{n-i}-1})d\Phi^*(\widetilde{a})-(\widetilde{a}^{p^{n-i}-1})^p d\widetilde{a}^p\}=pd\Big\{\sum_{j=1}^{p^{n-i}}c_j(\widetilde{a}^p)^{p^{n-i}-j}(p^{-1}\widetilde{b}^{[j]})\Big\},$$

where $c_j=(p^{n-i}-1)!/(p^{n-i}-j)!$ $(1\le j\le p^{n-i})$ is an integer, and the desired formula follows. Hence we have

$$
\begin{aligned}
\pi\circ s_{n+1}(1,0)((a_0,\dots,a_{n-1},0)) &= \text{the class of } \sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n-i}-1}d\widetilde{a}_i \\
&= s_n(1,0)((a_0,\dots,a_{n-1})).
\end{aligned}
$$

Therefore the commutativity of (7.1.4) follows.

Let $\mathcal{M}$ be the log structure of $\mathcal{Y}$. Let b be a local section of the log structure M of Y and let $\widetilde{b} \in \mathcal{M}$ be a local lift of b. Then there exists a local section $c \in \mathcal{O}_{\mathcal{Y}}$ such that $\Phi^*(\widetilde{b}) = \widetilde{b}^p(1+pc)$. Then we have $\Phi^*(d\log \widetilde{b}) = pd\log \widetilde{b} + d\log(1+pc) = pd\log \widetilde{b} + d(\sum_{j=1}^{\infty}(-1)^{j-1}(pc)^j/j)$; $\Phi^*(d\log \widetilde{b})$ is equal to $pd\log \widetilde{b}$ in the second sheaf of (6.4.5) in the case $i=1$. Hence (7.1) follows from (6.28) (8) and from the following formulas:

$$p^i\Phi^*(\widetilde{a}_i^{p^{n-i}})df = d(p^i\Phi^*(\widetilde{a}_i^{p^{n-i}})f) \quad (f \in \mathcal{O}_{\mathcal{Y}_{n+1}})$$

and

$$\Phi^*(\widetilde{a}_i^{p^{n-i}-1}d\widetilde{a}_i)df = d(\Phi^*(\widetilde{a}_i^{p^{n-i}-1}d\widetilde{a}_i)f) \quad (f \in \mathcal{O}_{\mathcal{Y}_{n+1}}). \ \square$$

Next, we prove the corrected statement of [HK, (4.6)]: we prove that s_n in (7.0.5) is an isomorphism of abelian sheaves. To do so, we need the following three lemmas:

LEMMA 7.2. *Let n be a positive integer. Then the following hold*:

(1) *The abelian subsheaf $B_n\Lambda^i_Y$ of Λ^i_Y* ([Lo, (1.12)]) *is generated by the following local sections*

$$\alpha(a_1^{p^r})d\log a_1 \cdots d\log a_i \quad (a_1, \ldots , a_i \in M,\ 0 \leq r \leq n-1)$$

as an abelian sheaf on Y.

(2) *The abelian subsheaf $Z_n\Lambda^i_Y$ of Λ^i_Y* ([Lo, (1.12)]) *is generated by $B_n\Lambda^i_Y$ and by the following local sections*

$$\alpha(b^{p^n})d\log a_1 \cdots d\log a_i \quad (a_1, \ldots , a_i, b \in M)$$

as an abelian sheaf on Y.

PROOF. The proof is the same as that of [I2, 0, (2.2.8)].

Let $C^{-1}\colon \Lambda^i_Y \xrightarrow{\sim} \mathcal{H}^i(\Lambda^\bullet_Y)$ be the log Cartier inverse isomorphism ([Ka2, (4.12) (1)]).

(1): If $n=1$, then the assertion is obvious since $\mathcal{O}_Y$ is additively generated by $\mathcal{O}^*_Y$. The following formula

$$C^{-1}(\alpha(a_1)^{p^r}d\log a_1 \cdots d\log a_i) = \alpha(a_1)^{p^{r+1}}d\log a_1 \cdots d\log a_i$$

and induction on n show (1).

(2): If $n=1$, then (2) holds by [Ka2, (4.12) (1)]. The rest of the proof is the same as that of (1) by using the isomorphism $C^{-1}\colon Z_n\Lambda^i_Y/B_n\Lambda^i_Y \overset{\sim}{\longrightarrow} Z_{n+1}\Lambda^i_Y/B_{n+1}\Lambda^i_Y$. □

LEMMA 7.3. *Let K^i_n be an abelian subsheaf $B_{n+1}\Lambda^i_Y \oplus Z_n\Lambda^{i-1}_Y$ defined by the following exact sequence* ([Hy2, (2.3)], [HK, (4.4)])

$$(7.3.1)\qquad 0 \longrightarrow K^i_n \longrightarrow B_{n+1}\Lambda^i_Y \oplus Z_n\Lambda^{i-1}_Y \overset{(C^n, dC^n)}{\longrightarrow} B_1\Lambda^i_Y \longrightarrow 0.$$

Then K^i_n is generated by the local sections of the forms of the following three types

$$(\omega, 0) \quad (\omega \in B_n\Lambda^i_Y),$$

$$(0, \eta) \quad (\eta \in B_n\Lambda^{i-1}_Y)$$

and

$$(\alpha(a_1^{p^n})d\log a_1 \cdots d\log a_i, -\alpha(a_1^{p^n})d\log a_2 \cdots d\log a_i) \quad (a_1, \ldots, a_i \in M)$$

as an abelian sheaf on Y.

PROOF. By the definition of $B_n\Lambda^j_Y$ ($j \in \mathbb{N}$), we have $C^n(B_n\Lambda^j_Y) = 0$. We also have the isomorphism $C^n\colon B_{n+1}\Lambda^i_Y/B_n\Lambda^i_Y \overset{\sim}{\longrightarrow} B_1\Lambda^i_Y$.

Take a local section $\alpha(a_1^{p^n})d\log a_2 \cdots d\log a_i$ of $Z_n\Lambda^{i-1}_Y$ ($a_1, \ldots, a_i \in M$). Then the image of this section by the morphism dC^n is equal to $d\alpha(a_1) \wedge d\log a_2 \cdots d\log a_i$. The image of this section by C^{-n} is equal to $\alpha(a_1^{p^n})d\log a_1 d\log a_2 \cdots d\log a_i$. Hence K^i_n is generated by local sections of three types in (7.3) as an abelian sheaf. □

The following statement is due to K. Kato, which will be used in the proof of (7.5):

LEMMA 7.4. *Let $g\colon Z = (\overset{\circ}{Z}, M_Z) \longrightarrow S = (\overset{\circ}{S}, M_S)$ be a morphism of log schemes. Let $\alpha\colon M_Z \longrightarrow \mathcal{O}_Z$ be the structural morphism. Set*

$$\Omega^i = (\mathcal{O}_Z \otimes_{\mathbb{Z}} \bigwedge^i M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}})) \oplus (\mathcal{O}_Z \otimes_{\mathbb{Z}} \bigwedge^{i-1} M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}}))$$

and let N^i *be an abelian subsheaf of* Ω^i *generated by*

$$(\alpha(a_1) \otimes (a_1 \wedge \cdots \wedge a_i), -\alpha(a_1) \otimes (a_2 \wedge \cdots \wedge a_i)) \quad (a_1, \ldots, a_i \in M_Z).$$

Let $\Lambda^i_{Z/S}$ *be an* $\mathcal{O}_Z$*-module defined in* [HK, (2.5)]. *Then, for* $i \geq 1$, *the morphism*

$$\Omega^i \ni (b \otimes (a_1 \wedge \cdots \wedge a_i), c \otimes (a'_2 \wedge \cdots \wedge a'_i)) \longmapsto \tag{7.4.1}$$

$$bd\log a_1 \cdots d\log a_i + dc \wedge d\log a'_2 \cdots d\log a'_i \in \Lambda^i_{Z/S}.$$

induces an isomorphism

$$\Omega^i/N^i \xrightarrow{\sim} \Lambda^i_{Z/S} \tag{7.4.2}$$

of abelian sheaves. (*Note that, for* $i = 0$, $\Omega^i/N^i = \mathcal{O}_Z$.) *In fact, there exists an* $\mathcal{O}_Z$*-module structure on* Ω^i/N^i *such that the isomorphism* (7.4.2) *is an isomorphism of* $\mathcal{O}_Z$*-modules.*

Proof. Some steps are necessary for the proof. We identify $\alpha^{-1}(\mathcal{O}_Z^*)$ with $\mathcal{O}_Z^*$ via the isomorphism $\alpha\colon \alpha^{-1}(\mathcal{O}_Z^*) \xrightarrow{\sim} \mathcal{O}_Z^*$.

Step 1. We define an $\mathcal{O}_Z$-module structure of Ω^1/N^1 as follows:

$$u(b \otimes a, c) = (ub \otimes a - cu \otimes u, uc) \tag{7.4.3}$$

$$(u \in \mathcal{O}_Z^*,\ a \in M_Z^{\text{gp}}/g^{-1}(M_S^{\text{gp}}),\ b, c \in \mathcal{O}_Z).$$

Since $\mathcal{O}_Z$ is additively generated by $\mathcal{O}_Z^*$, Ω^1/N^1 has an $\mathcal{O}_Z$-module structure if this action is well-defined (it is easy to check the axioms of the $\mathcal{O}_Z$-module structure for Ω^1/N^1); we have to show

$$\sum_i (cu_i \otimes u_i, 0) = \sum_j (cv_j \otimes v_j, 0) \quad \text{if} \quad \sum_i u_i = \sum_j v_j \quad (u_i, v_j \in \mathcal{O}_Z^*). \tag{7.4.4}$$

To prove (7.4.4), we need the following formulas:

$$(wu \otimes u, 0) = (-wu \otimes w, wu) \quad (w, u \in \mathcal{O}_Z^*), \tag{7.4.5}$$

$$(7.4.6) \qquad \sum_i (wu_i \otimes u_i, 0) = \sum_j (wv_j \otimes v_j, 0) \quad (w \in \mathcal{O}_Z^*).$$

We can immediately check (7.4.5), (7.4.6) and (7.4.4) in turn. Consequently, Ω^1/N^1 is an $\mathcal{O}_Z$-module because N^1 is stable under the action of $\mathcal{O}_Z^*$. It is easy to check that the morphism (7.4.2) for the case $i = 1$ is an $\mathcal{O}_Z$-linear morphism.

Step 2. We prove that (7.4.2) is an isomorphism for the case where $i = 1$ and where Z and S are schemes with trivial log structures.

We have the following formula

$$(7.4.7) \qquad (0, xy) = y \cdot (0, x) + x \cdot (0, y) \quad (x, y \in \mathcal{O}_X) \quad \text{in} \quad \Omega^1/N^1.$$

Indeed, we may assume that $x, y \in \mathcal{O}_X^*$ by linearity. In this case, we immediately obtain (7.4.7). Therefore, by the universality of $\Omega^1_{Z/S}$, we have the following $\mathcal{O}_Z$-linear morphism

$$(7.4.8) \qquad \Omega^1_{Z/S} \ni dx \longmapsto (0, x) \in \Omega^1/N^1.$$

We can easily check that the morphism (7.4.8) is the inverse of the morphism (7.4.2).

Step 3. We prove that (7.4.2) is an isomorphism for the case $i = 1$.

Recall that $\Lambda^1_{Z/S}$ is a quotient sheaf of $\mathcal{O}_Z$-module

$$\mathcal{O}_Z \otimes_{\mathbb{Z}} (M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}})) \oplus \Omega^1_{Z/S}$$

divided by an $\mathcal{O}_Z$-module generated by local sections $(\alpha(a) \otimes a, -d\alpha(a))$ $(a \in M_Z)$. We can construct the inverse morphism of (7.4.2) for the case $i = 1$ as follows. By the Step 1 and the Step 2, we have an $\mathcal{O}_Z$-linear morphism $\overset{\circ}{G} \colon \Omega^1_{Z/S} \longrightarrow \Omega^1/N^1$ characterized by the following: $\overset{\circ}{G}(da) = (0, a)$ $(a \in \mathcal{O}_Z)$. We define a morphism $G \colon \Lambda^1_{Z/S} \longrightarrow \Omega^1/N^1$ defined by the following: $\Lambda^1_{Z/S} \ni (b \otimes a, \omega) \longrightarrow (b \otimes a, 0) + \overset{\circ}{G}(\omega) \in \Omega^1/N^1$ $(b \in \mathcal{O}_Z,$ $a \in M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}})$, $\omega \in \Omega^1_{Z/S})$. It is easy to check that this morphism is well-defined. Noting that $\Lambda^1_{Z/S}$ and Ω^1/N^1 are quotients of $\mathcal{O}_Z \otimes_{\mathbb{Z}}$ $(M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}}))$, we see that the morphism is the inverse of the morphism (7.4.2) for the case $i = 1$.

Step 4. We prove that (7.4.2) for the general case $i \geq 1$ is an isomorphism.

As in (7.4.3), we can define an $\mathcal{O}_Z$-module structure on Ω^i/N^i $(i \geq 1)$ characterized by the following:

(7.4.9) $$u \cdot (b \otimes (a_1 \wedge \cdots \wedge a_i), c \otimes (a'_2 \wedge \cdots \wedge a'_i)) =$$

$$(ub \otimes (a_1 \wedge \cdots \wedge a_i) - cu \otimes (u \wedge a'_2 \wedge \cdots \wedge a'_i), cu \otimes (a'_2 \wedge \cdots \wedge a'_i))$$

$$(u \in \mathcal{O}_Z^*,\ b, c \in \mathcal{O}_Z,\ a_j, a'_k \in M_Z^{\mathrm{gp}}/g^{-1}(M_S^{\mathrm{gp}}),\ 1 \leq j \leq i,\ 2 \leq k \leq i).$$

It is not difficult to check that there exists a morphism

$$\underbrace{\Lambda^1_{Z/S} \otimes_{\mathcal{O}_Z} \cdots \otimes_{\mathcal{O}_Z} \Lambda^1_{Z/S}}_{i \text{ times}} \longrightarrow \Omega^i/N^i$$

of $\mathcal{O}_Z$-modules characterized by the following

$$(b_1 \otimes a_1, 0) \otimes \cdots \otimes (b_i \otimes a_i, 0) \longmapsto (b_1 \cdots b_i \otimes (a_1 \wedge \cdots \wedge a_i), 0).$$

This morphism induces a morphism $\Lambda^i_{Z/S} \longrightarrow \Omega^i/N^i$, which is the inverse of the morphism (7.4.2). $\square$

Theorem 7.5. *The induced morphism* $s_n \colon (W_n\Lambda^i_Y)'' \longrightarrow W_n\Lambda^i_Y$ *by a morphism defined in* [HK, (4.9)] *is an isomorphism.*

Proof. We proceed by induction on n.

Let $C^{-1} \colon \Lambda^i_Y \xrightarrow{\sim} W_1\Lambda^i_Y$ be the log Cartier inverse isomorphism ([Ka2, (4.12) (1)]). By (7.4) and by the definition of s_1, we have the following commutative diagram:

$$\begin{array}{ccc}
(W_1\Lambda^i_Y)'' & \xrightarrow{s_1} & W_1\Lambda^i_Y \\
\simeq\downarrow & & \| \\
\Lambda^i_Y & \xrightarrow[\sim]{C^{-1}} & W_1\Lambda^i_Y.
\end{array}$$

Hence we obtain (7.5) for the case $n = 1$.

Assume that $n \geq 1$ and that s_n is an isomorphism. Set

$$\Lambda^i_n(Y) := \{W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^i (M^{\rm gp}/f^{-1}(L^{\rm gp}))\} \oplus \{W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^{i-1} (M^{\rm gp}/f^{-1}(L^{\rm gp}))\}$$

and

$$J^i_n(Y) := \{V^n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^i (M^{\rm gp}/f^{-1}(L^{\rm gp}))\} \oplus \{V^n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} \bigwedge^{i-1} (M^{\rm gp}/f^{-1}(L^{\rm gp}))\}.$$

Let ${\rm Fil}^n((W_{n+1}\Lambda^i_Y)'')$ be the image of $J^i_n(Y)$ in $(W_{n+1}\Lambda^i_Y)''$. First we claim that the following sequence

(7.5.1) $$0 \longrightarrow {\rm Fil}^n((W_{n+1}\Lambda^i_Y)'') \longrightarrow (W_{n+1}\Lambda^i_Y)'' \overset{\rm proj.}{\longrightarrow} (W_n\Lambda^i_Y)'' \longrightarrow 0$$

is exact. Indeed, set

$$(K^i_n)'' := {\rm Ker}((W_{n+1}\Lambda^i_Y)'' \overset{\rm proj.}{\longrightarrow} (W_n\Lambda^i_Y)'').$$

Since the sequence

$$0 \longrightarrow \mathcal{O}_Y \overset{V^n}{\longrightarrow} W_{n+1}(\mathcal{O}_Y) \longrightarrow W_n(\mathcal{O}_Y) \longrightarrow 0$$

is exact, we have the following commutative diagram with exact rows:

(7.5.2) $$\begin{array}{ccccccccc} & & J^i_n(Y) & \longrightarrow & \Lambda^i_{n+1}(Y) & \overset{\rm proj.}{\longrightarrow} & \Lambda^i_n(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (K^i_n)'' & \longrightarrow & (W_{n+1}\Lambda^i_Y)'' & \overset{\rm proj.}{\longrightarrow} & (W_n\Lambda^i_Y)'' & \longrightarrow & 0 \end{array}$$

Since proj.: $\mathcal{F}_{n+1} + \mathcal{G}_{n+1} \longrightarrow \mathcal{F}_n + \mathcal{G}_n$ is surjective, the snake lemma for (7.5.2) shows that the morphism $J^i_n(Y) \longrightarrow (K^i_n)''$ is surjective. In other words, (7.5.1) is exact.

As in [HK, p. 252], we define a morphism

(7.5.3) $$\Lambda^i_Y \oplus \Lambda^{i-1}_Y \longrightarrow {\rm Fil}^n((W_{n+1}\Lambda^i_Y)'')$$

of abelian sheaves by the following

$$(bd\log a_1\cdots d\log a_i, 0)\longmapsto (v_n(b)\otimes(a_1\wedge\cdots\wedge a_i),0), \tag{7.5.4}$$

$$(0, bd\log a_2\cdots d\log a_i)\longmapsto (0, v_n(b)\otimes(a_2\wedge\cdots\wedge a_i)) \tag{7.5.5}$$

$$(b\in\mathcal{O}_Y,\ a_1,\dots,a_i\in M).$$

By using the isomorphism $(W_1\Lambda_Y^j)''\overset{\sim}{\longrightarrow}\Lambda_Y^j$ ($j\in\mathbb{N}$) in (7.4) and by noting that $\mathcal{O}_Y$ is generated by $\mathcal{O}_Y^*$ as abelian sheaves, we can check that the morphisms (7.5.4) and (7.5.5) are well-defined (cf. the proof of (7.4)). We claim that the surjective morphism $\Lambda_Y^i\oplus\Lambda_Y^{i-1}\longrightarrow \mathrm{Fil}^n((W_{n+1}\Lambda_Y^i)'')$ defined by (7.5.4) and (7.5.5) factors through $(\Lambda_Y^i\oplus\Lambda_Y^{i-1})/K_n^i\longrightarrow\mathrm{Fil}^n((W_{n+1}\Lambda_Y^i)'')$. (Recall the sheaf K_n^i in (7.3.1).) Indeed, let $(\omega,0)$ ($\omega\in B_n\Lambda_Y^i$) be a section of K_n^i. Then, by (7.2) (1), we may assume that $\omega=\alpha(a_1^{p^r})d\log a_1\cdots d\log a_i$ ($a_1,\dots,a_i\in M$, $0\le r\le n-1$). Then the image of $(\omega,0)$ by the morphism (7.5.4) is

$$(v_n(\alpha(a_1^{p^r}))\otimes(a_1\wedge\cdots\wedge a_i),0)=p^r(v_{n-r}(\alpha(a_1))\otimes(a_1\wedge a_2\wedge\cdots\wedge a_i),0).$$

This is equivalent to $p^n(0,v_{n-r}(\alpha(a_1))\otimes(a_2\wedge\cdots\wedge a_i))$ modulo $\mathcal{F}_{n+1}$. Since $p=FV$, the last form is in the image of $V^{n+1}(W_1(\mathcal{O}_Y))\otimes_{\mathbb{Z}}\bigwedge^{i-1}(M^{\mathrm{gp}}/f^{-1}(L^{\mathrm{gp}}))=0$. Next, let $(0,\eta)$ ($\eta\in B_n\Lambda_Y^{i-1}$) be a section of K_n^i. As above, we may assume that $\eta=\alpha(a_2^{p^r})d\log a_2\cdots d\log a_i$ ($a_2,\dots,a_i\in M$, $0\le r\le n-1$). Then the image of this section by the morphism (7.5.5) is $(0,v_n(\alpha(a_2^{p^r}))\otimes(a_2\wedge\cdots\wedge a_i))$. This section belongs to $\mathcal{F}_{n+1}+\mathcal{G}'_{n+1}=\mathcal{F}_{n+1}+\mathcal{G}_{n+1}$ (cf. (7.0.3) and (7.0.4)). Finally, consider a section

$$(\alpha(a_1^{p^n})d\log a_1\cdots d\log a_i, -\alpha(a_1^{p^n})d\log a_2\cdots d\log a_i)\quad (a_1,\dots,a_i\in M)$$

in (7.3). Then the image of this section by the morphism (7.5.3) is equal to

$$(v_n(\alpha(a_1^{p^n}))\otimes(a_1\wedge\cdots\wedge a_i), -v_n(\alpha(a_1^{p^n}))\otimes(a_2\wedge\cdots\wedge a_i)).$$

This image is equal to $p^n(v_0(\alpha(a_1))\otimes(a_1\wedge\cdots\wedge a_i), -v_0(\alpha(a_1))\otimes(a_2\wedge\cdots\wedge a_i))$, which is a section of $\mathcal{F}_{n+1}$.

Thus we see that the isomorphism

$$(V^n, dV^n)C^{-1}\colon(\Lambda_Y^i\oplus\Lambda_Y^{i-1})/K_n^i\overset{\sim}{\longrightarrow}\mathrm{Fil}^n(W_{n+1}\Lambda_Y^i)$$

in [HK, (4.4)] (cf. (6.19), (6.20)) factors through the surjective morphism $(\Lambda^i_Y \oplus \Lambda^{i-1}_Y)/K^i_n \longrightarrow \mathrm{Fil}^n((W_{n+1}\Lambda^i_Y)'')$ (cf. [HK, p. 252]; "the isomorphism (4.4)

$$(V^i, dV^i)C^{-1}\colon \omega^q_Y \oplus \omega^{q-1}_Y \xrightarrow{\sim} \mathrm{Ker}(\pi_i)\text{''}$$

in [loc. cit.] is mistaken). Hence the morphism s_{n+1} induces an isomorphism

$$s_{n+1}\colon \mathrm{Fil}^n((W_{n+1}\Lambda^i_Y)'') \xrightarrow{\sim} \mathrm{Fil}^n(W_{n+1}\Lambda^i_Y).$$

On the other hand, by the compatibility of two projections ((7.1)), we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^n((W_{n+1}\Lambda^i_Y)'') & \longrightarrow & (W_{n+1}\Lambda^i_Y)'' & \xrightarrow{\text{proj.}} & (W_n\Lambda^i_Y)'' & \longrightarrow & 0 \\ & & {\scriptstyle s_{n+1},\simeq}\big\downarrow & & {\scriptstyle s_{n+1}}\big\downarrow & & {\scriptstyle s_n,\simeq}\big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^n W_{n+1}\Lambda^i_Y & \longrightarrow & W_{n+1}\Lambda^i_Y & \xrightarrow{\pi} & W_n\Lambda^i_Y & \longrightarrow & 0 \end{array} \tag{7.5.6}$$

with two horizontal exact sequences. Therefore the middle vertical morphism s_{n+1} in (7.5.6) is an isomorphism of abelian sheaves. □

REMARK 7.6. Let the notations be as in [HK, (4.8)]. We correct the proof of [loc. cit.]: [HK, p. 253, l. -5] has to be replaced by the following

$$W_n\Lambda^i_Y \underset{(7.5)}{\xleftarrow{\sim}} (W_n\Lambda^i_Y)'' \xrightarrow{s_n} \mathcal{H}^i(C_{Y/T}).$$

Here the number (7.5) is a number in this paper, and the morphism s_n above is similarly defined as in [HK, (4.9)].

REMARK 7.7. (1) As is well-known, $W_n\Lambda^i_Y$ is a $W_n(\mathcal{O}_Y)$-module: let $Y \xrightarrow{\subset} \mathcal{Y}_n$ be a closed immersion into a log smooth scheme over $(\mathrm{Spec}(W_n), W_n(L))$ and let $C_{Y/(W_n,W_n(L))}$ be the crystalline complex associated to the closed immersion above. Then $W_n\Lambda^i_Y = \mathcal{H}^i(C_{Y/(W_n,W_n(L))})$. The action of $c = (c_0, \dots, c_{n-1}) \in W_n(\mathcal{O}_Y)$ on $[\omega] \in \mathcal{H}^i(C_{Y/(W_n,W_n(L))})$ is given by the following formula: $c \cdot [\omega] = [(\sum_{j=0}^{n-1} p^j \tilde{c}_j^{p^{n-j}}) \cdot \omega]$.

(2) We endow $(W_n\Lambda^i_Y)''$ with a $W_n(\mathcal{O}_Y)$-module structure, and we define operators $R\colon (W_n\Lambda^i_Y)'' \longrightarrow (W_{n-1}\Lambda^i_Y)''$, $V\colon (W_n\Lambda^i_Y)'' \longrightarrow (W_{n+1}\Lambda^i_Y)''$, $F\colon (W_n\Lambda^i_Y)'' \longrightarrow (W_{n-1}\Lambda^i_Y)''$, and the boundary morphism $d\colon (W_n\Lambda^i_Y)'' \longrightarrow (W_n\Lambda^{i+1}_Y)''$ as follows: let b, c be local sections of $W_n(\mathcal{O}_Y)$, and let a_j, a'_k be local sections of M $(1 \le j \le i, 2 \le k \le i)$.

(a) Since $W_n\Lambda^i_Y$ is a $W_n(\mathcal{O}_Y)$-module, we can endow $(W_n\Lambda^i_Y)''$ with a $W_n(\mathcal{O}_Y)$-module structure by using the isomorphism s_n in (7.5).

(b) We define R by the following formula

$$R(b\otimes(\wedge^i_{j=1}a_j), c\otimes(\wedge^i_{j=2}a'_j)) = (R(b)\otimes(\wedge^i_{j=1}a_j), R(c)\otimes(\wedge^i_{j=2}a'_j)), \tag{7.7.1}$$

where $R\colon W_n(\mathcal{O}_Y)\longrightarrow W_{n-1}(\mathcal{O}_Y)$ is the usual projection.

(c) We define the morphism V by the following formula

$$V(b\otimes(\wedge^i_{j=1}a_j), c\otimes(\wedge^i_{j=2}a'_j)) = (V(b)\otimes(\wedge^i_{j=1}a_j), pV(c)\otimes(\wedge^i_{j=2}a'_j)), \tag{7.7.2}$$

where $V\colon W_n(\mathcal{O}_Y)\longrightarrow W_{n+1}(\mathcal{O}_Y)$ is the usual Verschiebung.

(d) Assume that $c:=(c_0,\dots,c_{n-1})$ is a unit of $W_n(\mathcal{O}_Y)$. Then $c_0\in\mathcal{O}^*_Y$. We define the morphism $F\colon (W_n\Lambda^i_Y)''\longrightarrow(W_{n-1}\Lambda^i_Y)''$ by the following formula

$$F(b\otimes(\wedge^i_{j=1}a_j), c\otimes(\wedge^i_{j=2}a'_j)) = (F(b)\otimes(\wedge^i_{j=1}a_j), 0) + ((c_0^p,0,\dots,0)\otimes(c_0\wedge\wedge^i_{j=2}a'_j), (c_1,\cdots,c_{n-2},0)\otimes(\wedge^i_{j=2}a'_j)), \tag{7.7.3}$$

where $F\colon W_n(\mathcal{O}_Y)\longrightarrow W_{n-1}(\mathcal{O}_Y)$ is the usual Frobenius morphism. The operator F is well-defined because the following diagram is commutative:

$$\begin{array}{ccc} (W_n\Lambda^i_Y)'' & \xrightarrow[\sim]{s_n} & W_n\Lambda^i_Y \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ (W_{n-1}\Lambda^i_Y)'' & \xrightarrow[\sim]{s_{n-1}} & W_{n-1}\Lambda^i_Y. \end{array} \tag{7.7.4}$$

By (7.5) and by the commutativity of (7.7.4) and (9.2.1) below, the morphism

$$p^iF\colon (W_n\Lambda^i_Y)''\longrightarrow(W_{n-1}\Lambda^i_Y)''$$

is lifted to the Frobenius Φ''_n, that is, the following diagram is commutative:

$$\begin{array}{ccc} (W_n\Lambda^i_Y)'' & \xrightarrow{\Phi''_n} & (W_n\Lambda^i_Y)'' \\ \big\| & & \big\downarrow{\scriptstyle \text{proj.}} \\ (W_n\Lambda^i_Y)'' & \xrightarrow{p^iF} & (W_{n-1}\Lambda^i_Y)''. \end{array} \tag{7.7.5}$$

(e) We define the boundary morphism d by the following formula

$$d(b\otimes(a_1\wedge\cdots\wedge a_i), c\otimes(a'_2\wedge\cdots\wedge a'_i)) = (0, b\otimes(a_1\wedge\cdots\wedge a_i)). \tag{7.7.6}$$

Then we can check that the isomorphism $s_n\colon \bigoplus_{i\geq 0}(W_n\Lambda^i_Y)'' \overset{\sim}{\longrightarrow} \bigoplus_{i\geq 0} W_n\Lambda^i_Y$ is an isomorphism which is compatible with $W_n(\mathcal{O}_Y)$-module structures and with the operators R, V, F and d. In other words, s_n is an isomorphism of $R(\kappa)$-modules, where $R(\kappa)$ is the Cartier-Dieudonné-Raynaud algebra of κ ([IR, I (1.1)]).

We give names to $(W_n\Lambda^i_Y)''$ and $W_n\Lambda^i_Y$ because some mistakes arise by confusing them.

Definition 7.8. We call $(W_n\Lambda^i_Y)''$ (resp. $W_n\Lambda^i_Y$) the *obverse* (resp. *reverse*) *log Hodge Witt sheaf* of $Y/(\mathrm{Spec}(\kappa), L)$, and $(W_n\Lambda^\bullet_Y)''$ (resp. $W_n\Lambda^\bullet_Y$) the *obverse* (resp. *reverse*) *log de Rham-Witt complex* of $Y/(\mathrm{Spec}(\kappa), L)$.

We obviously have the following commutative diagram:

$$\begin{array}{ccc} (W_{n+1}\Lambda^i_Y)'' & \xrightarrow[\sim]{s_{n+1}} & W_{n+1}\Lambda^i_Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (W_{n+1}\Lambda^i_Y)'' & \xrightarrow[\sim]{s_{n+1}} & W_{n+1}\Lambda^i_Y. \end{array} \tag{7.8.1}$$

By (7.1), (7.5) and the commutative diagram (7.8.1), the left vertical morphism in (7.8.1) induces a unique morphism

$$\mathbf{p}''\colon (W_n\Lambda^i_Y)'' \longrightarrow (W_{n+1}\Lambda^i_Y)'' \tag{7.8.2}$$

fitting into the following commutative diagram:

$$\begin{array}{ccc} (W_n\Lambda^i_Y)'' & \xrightarrow[\sim]{s_n} & W_n\Lambda^i_Y \\ {\scriptstyle \mathbf{p}''}\downarrow & & \downarrow{\scriptstyle \mathbf{p}} \\ (W_{n+1}\Lambda^i_Y)'' & \xrightarrow[\sim]{s_{n+1}} & W_{n+1}\Lambda^i_Y. \end{array} \tag{7.8.3}$$

Proposition 7.9. *The morphism* $\mathbf{p}''$ *in* (7.8.2) *is injective.*

Proof. (7.9) follows from (7.8.3) and [HK, (4.5) (1)] or (6.8) (2). □

Next, we give a right proof of [HK, (4.19)].

Because projections and other operators are not clear in some places in [HK] (e.g., [HK, the proofs in (4.15) and (4.16)]), we have to clarify the transition morphism

$$\text{proj.}\colon Ru_{Y/W_{n+1}*}(\mathcal{O}_{Y/W_{n+1}}) \longrightarrow Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}), \tag{7.9.1}$$

though it seems clear in this paper. The morphism (7.9.1) is, by definition, the morphism induced by a natural exact closed immersion

$$(\mathrm{Spec}(W_n), pW_n, W_n(L)) \xrightarrow{\subset} (\mathrm{Spec}(W_{n+1}), pW_{n+1}, W_{n+1}(L))$$

of base PD log schemes. The morphism (7.9.1) induces a morphism of cohomologies:

$$\text{proj.}\colon R^iu_{Y/W_{n+1}*}(\mathcal{O}_{Y/W_{n+1}}) \longrightarrow R^iu_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \quad (i \in \mathbb{N}).$$

Let $\pi\colon W_{n+1}\Lambda^\bullet_Y \longrightarrow W_n\Lambda^\bullet_Y$ be the projection defined in [HK, (4.2)]. We have, by definition, $W_n\Lambda^i_Y := R^iu_{Y/W_n*}(\mathcal{O}_{Y/W_n})$ ([HK, (4.1)]).

Before giving a right proof of [HK, (4.19)], we point out the incomplete part of [HK, (4.19)]. Hyodo and Kato have claimed that a canonical morphism

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \longrightarrow W_n\Lambda^\bullet_Y \tag{7.9.2}$$

in [loc. cit.] is an isomorphism and is compatible with the transition morphisms. However the commutativity of the following diagram(=(7.19.4) below) has not been proved in [loc. cit.]:

$$\begin{array}{ccc} Ru_{Y/W_{n+1}*}(\mathcal{O}_{Y/W_{n+1}}) & \longrightarrow & W_{n+1}\Lambda^\bullet_Y \\ {\scriptstyle\text{proj.}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) & \longrightarrow & W_n\Lambda^\bullet_Y. \end{array} \tag{7.9.3}$$

The proof of the following isomorphism

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \xrightarrow{\sim} W_n\Lambda^\bullet_Y \tag{7.9.4}$$

in [HK, (4.19)] is also incomplete: the reduction of the isomorphism of (7.9.4) to the isomorphism (7.9.4) in the case $n = 1$ has a gap because the commutativity of (7.18.1) below has not been proved. In [loc. cit.], only a canonical morphism

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \longrightarrow W_n\Lambda^\bullet_Y$$

has been constructed for each positive integer n. For the perfect proof of [HK, (4.19)], we need the lemma (7.18) below whose proof is the same as that of (7.1).

Though we do not need a log version of a lemma of Dwork-Dieudnonné-Cartier (see (7.10) below) only for the construction of the morphism (7.9.2) as in the classical case ([I2, pp. 602–603]), we need an explicit description of the morphism (7.9.2) in a local case for the proof of a fundamental theorem in [NS, §19] (=comparison theorem between the preweight-filtered zariskian complex defined in [NS] and the preweight-filtered log de Rham-Witt complex $(W_n\Omega^\bullet_X(\log D), \{P_kW_n\Omega^\bullet_X(\log D)\}_{k\in\mathbb{Z}})$ of a smooth scheme X with an SNCD D over κ); see [loc. cit.] for details. (If the reader wishes to know only a right proof of [HK, (4.19)], he can skip (7.10)–(7.17).)

Thus we first prove this log version:

Lemma 7.10. *Let A be a p-torsion free commutative ring with unit element. Let $(A, P) := (A, P, \alpha)$ be a* prelog ring*, that is, $(P, \cdot)$ is a commutative monoid with unit element and $\alpha\colon (P, \cdot) \longrightarrow (A, \cdot)$ is a morphism of monoids. Set*

$$W(P) := P \oplus (1 + VW(A))$$

with natural morphism $W(\alpha)\colon W(P) \longrightarrow W(A)$ of monoids. Assume that P is integral, and that α induces an isomorphism $\alpha^{-1}(1+pA) \xrightarrow{\sim} 1+pA$, which enables us to identify an element of $1+pA$ with that of $\alpha^{-1}(1+pA)$. Assume, moreover, that there exist an endomorphism $\varphi\colon P \longrightarrow P$ of monoids and an endomorphism $\overset{\circ}{\varphi}\colon A \longrightarrow A$ of rings such that

(7.10.1) $\alpha\varphi = \overset{\circ}{\varphi}\alpha$,

(7.10.2) $\forall x \in P,\ \exists y \in A,\ \varphi(x) = x^p(1 + py)$

and such that

(7.10.3) *$\overset{\circ}{\varphi}$ is a lift of the Frobenius endomorphism of A/p.*

Let $s_{\overset{\circ}{\varphi}}\colon A \longrightarrow W(A)$ *be the morphism defined in* [La, VII (4.12)] (cf. [I2, 0 (1.3.16)]). *Then there exists a natural morphism*

$$s_{\varphi}\colon P \longrightarrow W(P) \tag{7.10.4}$$

of monoids which is a section of the natural projection $W(P) \longrightarrow P$ *and which makes the following diagram commutative*:

$$\begin{array}{ccc} P & \xrightarrow{s_{\varphi}} & W(P) \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle W(\alpha)} \\ A & \xrightarrow{s_{\overset{\circ}{\varphi}}} & W(A). \end{array} \tag{7.10.5}$$

In other words, the morphisms $s_{\overset{\circ}{\varphi}}$ *and* s_{φ} *give a morphism*

$$(s_{\overset{\circ}{\varphi}}, s_{\varphi})\colon (A, P, \alpha) \longrightarrow (W(A), W(P), W(\alpha)) \tag{7.10.6}$$

of prelog rings.

Proof. We use the argument in [La, VII 4].

Let n be a positive integer. Let x be an element of P. Let us define a unique element $y_n \in A$ characterized by the following formula

$$\varphi^n(x) = x^{p^n}(1 + py_n). \tag{7.10.7}$$

Indeed, y_n is uniquely determined since P is integral and A is p-torsion free. Then we have

$$1 + py_n = (1 + py_1)^{p^{n-1}}(1 + p\overset{\circ}{\varphi}(y_{n-1})). \tag{7.10.8}$$

We claim that there exists a unique sequence $(1, s_1, \dots, s_n, \dots)$ of elements in A satisfying the following family of equations:

$$1 + ps_1^{p^{n-1}} + \cdots + p^{n-1}s_{n-1}^{p} + p^n s_n = 1 + py_n \quad (n \in \mathbb{Z}_{>0}). \tag{7.10.9}$$

Since A is p-torsion free, we have only to prove the existence of $s_1, \dots, s_n, \dots$. We proceed by induction on n.

There is nothing to prove in the case $n = 1$. Assume that $s_1, \dots, s_{n-1}$ exist. Then, by (7.10.8), we have

$$(7.10.10)\qquad (1 + py_n) - (1 + \sum_{i=1}^{n-1} p^i s_i^{p^{n-i}})$$
$$= (1 + py_1)^{p^{n-1}}(1 + p\mathring{\varphi}(y_{n-1})) - (1 + \sum_{i=1}^{n-1} p^i s_i^{p^{n-i}})$$
$$\equiv \sum_{i=1}^{n-1} p^i(\mathring{\varphi}(s_i^{p^{n-1-i}}) - s_i^{p^{n-i}}) \mod p^n.$$

Since $\mathring{\varphi}(s_i^{p^{n-i-1}}) \equiv s_i^{p^{n-i}} \mod p^{n-i}$, the right hand side of (7.10.10) is divisible by p^n. Hence we know the existence of $s_n \in A$.

Let x be an element of P and y_n the element in (7.10.7).

We define a map $s_\varphi \colon P \longrightarrow W(P)$ as follows:

$$(7.10.11)\qquad P \ni x \longmapsto (x, (1, s_1, \dots, s_n, \dots,)) \in W(P).$$

Obviously the map (7.10.11) preserves the unit element. Let x_i $(i = 1, 2)$ be an element of P and let $y_n^{(i)}$ be the element of A in (7.10.7) for $x = x_i$. Then $\varphi^n(x_1x_2) = (x_1x_2)^{p^n}\{(1 + py_n^{(1)})(1 + py_n^{(2)})\}$. Hence, by the definition of the multiplicative structure of $W(A)$, the map (7.10.11) preserves the multiplicative structure.

Now we prove the commutativity of the diagram (7.10.5).

Let x be an element of P and $\{s_n\}_{n=1}^{\infty}$ the family of elements of A in (7.10.9). Then we have

$$W(\alpha)s_\varphi(x) = (\alpha(x), 0, \dots, 0, \dots) \cdot (1, s_1, \dots, s_n, \dots)$$
$$= (\alpha(x), \alpha(x)^p s_1, \dots, \alpha(x)^{p^n} s_n, \dots).$$

Furthermore, we have

$$(7.10.12)\qquad \alpha(x)^{p^n} + \sum_{i=1}^{n} p^i(\alpha(x)^{p^i} s_i)^{p^{n-i}} = \alpha(x)^{p^n}(1 + \sum_{i=1}^{n} p^i s_i^{p^{n-i}})$$
$$= \alpha(x)^{p^n}(1 + py_n)$$
$$= \mathring{\varphi}^n(\alpha(x)).$$

Hence, by the definition of $s_{\overset{\circ}{\varphi}}$, $s_{\overset{\circ}{\varphi}}(\alpha(x))=(\alpha(x),\dots,\alpha(x)^{p^n}s_n,\dots)$. Therefore we obtain the commutativity of the diagram (7.10.5). □

Corollary 7.11. *Let $F\colon W(A)\longrightarrow W(A)$ be the Frobenius endomorphism of $W(A)$. Then the following formulas hold*:

$$s_\varphi\varphi=(\varphi,W(\overset{\circ}{\varphi}))s_\varphi, \tag{7.11.1}$$

$$W(\alpha)s_\varphi\varphi=FW(\alpha)s_\varphi. \tag{7.11.2}$$

Proof. Let the notations be as in the proof of (7.10).

(7.11.1): We have the following formula:

$$(\varphi,W(\overset{\circ}{\varphi}))s_\varphi(x)=(\varphi(x),(1,\overset{\circ}{\varphi}(s_1),\dots,\overset{\circ}{\varphi}(s_n),\dots)).$$

The right hand of the formula above is equal to $s_\varphi\varphi(x)$ since $\overset{\circ}{\varphi}(1+\sum_{i=1}^n p^is_i^{p^{n-i}})=1+p\overset{\circ}{\varphi}(y_n)$.

(7.11.2): By the commutative diagram (7.10.5) and the formula $Fs_{\overset{\circ}{\varphi}}=s_{\overset{\circ}{\varphi}}\overset{\circ}{\varphi}$, the right hand side of (7.11.2) is equal to $s_{\overset{\circ}{\varphi}}\overset{\circ}{\varphi}\alpha$. Let $q\colon W(P)\longrightarrow 1+VW(A)$ be the second projection. By (7.11.1), the left hand side of (7.11.2) is equal to $W(\alpha)(\varphi,W(\overset{\circ}{\varphi}))s_\varphi$. This is equal to $([\quad]\alpha\varphi)\times(W(\overset{\circ}{\varphi})qs_\varphi)=([\quad]\overset{\circ}{\varphi}\alpha)\times W(\overset{\circ}{\varphi})qs_\varphi$, where $[\quad]$ is the morphism $A\ni x\longmapsto(x,0,\dots,0,\dots)\in W(A)$. By the calculation (7.10.12), the right hand side of the last formula is equal to $s_{\overset{\circ}{\varphi}}\overset{\circ}{\varphi}\alpha$. Therefore we obtain (7.11.2). □

Remark 7.12. (1) If $P=A^*$ and if α is the natural inclusion $A^*\overset{\subset}{\longrightarrow}A$, then (7.10) is equivalent to the lemma of Dwork-Dieudonné-Cartier in [La, VII 4].

(2) We can also prove the existence of $\{s_n\}_{n=1}^\infty$ in (7.10.9) by using the lemma of Dwork-Dieudonné-Cartier directly as follows.

Let B be the polynomial ring $\mathbb{Z}[X,Z_n|n\in\mathbb{N}]$. Let Φ be a lift of the Frobenius endomorphism of B/p defined by the following formula:

$$\Phi(X)=X^p(1+pZ_1),\quad \Phi(Z_n)=Z_n^p+pZ_{n+1}\quad(n\in\mathbb{Z}_{>0})$$

Then, by the lemma of Dwork-Dieudonné-Cartier, there exists a family of elements $\{T_n\}_{n=0}^{\infty}$ in B with $T_0 := X$ satisfying the following equation for all $n \in \mathbb{N}$:

$$T_0^{p^n} + pT_1^{p^{n-1}} + \cdots + p^n T_n = \Phi^n(X).$$

Then, by induction on n, we see that $T_n \in p^{-n}X^{p^n}B$. Since $T_n \in B$, we see that $T_n \in X^{p^n}B$. Set $S_n := X^{-p^n}T_n \in B$. Then we have

$$1 + pS_1^{p^{n-1}} + \cdots + p^n S_n = X^{-p^n}\Phi^n(X).$$

Since $p^{-n}\mathbb{Z}[Z_m|0 \leq m \leq n] \cap B = \mathbb{Z}[Z_m|0 \leq m \leq n]$ in $p^{-n}B$, we inductively see that $S_n \in \mathbb{Z}[Z_m|0 \leq m \leq n]$.

Now the existence of $\{s_n\}_{n=1}^{\infty}$ in (7.10.9) is clear. Indeed, let $\{z_n\}_{n=1}^{\infty}$ be a family of elements defined by the following formula

$$z_1 = y_1, \quad \mathring{\varphi}(z_n) = z_n^p + pz_{n+1}.$$

in A. Let C be the subring of A generated by $\{z_n\}_{n=1}^{\infty}$ over $\mathbb{Z}$. Since C is a quotient ring of $\mathbb{Z}[Z_n|n \in \mathbb{N}]$, we see that there exists a family $\{s_n\}_{n=1}^{\infty}$ of elements of C in (7.10.9).

Let the notations and the assumptions be as in (7.10). Let n be a positive integer. Set

$$W_n(P) := P \oplus (1 + VW_n(A)). \tag{7.12.1}$$

Then the morphism (7.10.6) induces a natural morphism

$$(A, P) \longrightarrow (W_n(A), W_n(P)) \tag{7.12.2}$$

of prelog rings. Let $\star$ be a positive integer or nothing. The reduction mod p: $A \longrightarrow A/p$ induces a morphism $(W_\star(A), W_\star(P)) \longrightarrow (W_\star(A/p), W_\star(P))$, where $W_\star(P)$ on the right hand side is the canonical lift of P with respect to the morphism $P \longrightarrow W_\star(A/p)$: $W_\star(P) = P \oplus (1 + VW_\star(A/p))$ (cf. [HK, (3.1)]). Hence we have a natural morphism

$$(A/p^\star, P) \longrightarrow (W_\star(A/p), W_\star(P)) \tag{7.12.3}$$

by (7.12.2).

PROPOSITION 7.13. *The constructions* $(A, P) \longmapsto (W_\star(A), W_\star(P))$ *and* $(A, P) \longmapsto (W_\star(A/p), W_\star(P))$ *are functorial in the obvious sense.*

PROOF. The proof is obvious. □

PROPOSITION 7.14. *The morphism* (7.12.3) *for* $n = 1$ *is equal to the following morphism*

$$(A/p, P) \ni (a, x) \longmapsto (a, (x, 1)) \in (A/p, P \oplus 1).$$

PROOF. (7.14) immediately follows from the definitions of $s_{\overset{\circ}{\varphi}}$ and s_φ ((7.10.11)). □

DEFINITION 7.15. We call a quadruplet $(\mathcal{T}, \mathcal{O}_\mathcal{T}, \mathcal{M}, \alpha)$ a *prelog ringed topos* if $(\mathcal{T}, \mathcal{O}_\mathcal{T})$ is a ringed topos and $\alpha\colon \mathcal{M} \longrightarrow \mathcal{O}_\mathcal{T}$ is a morphism of sheaves of monoids in $\mathcal{T}$. We call a prelog ringed topos $(\mathcal{T}, \mathcal{O}_\mathcal{T}, \mathcal{M}, \alpha)$ a *log ringed topos* if α induces an isomorphism $\alpha^{-1}(\mathcal{O}_\mathcal{T}^*) \overset{\sim}{\longrightarrow} \mathcal{O}_\mathcal{T}^*$ as sheaves of monoids in $\mathcal{T}$. We define a *morphism* of (pre)log ringed topoi in an obvious way.

We leave the definition of an *integral* log ringed topos, a *fine* log ringed topos, an *fs* log ringed topos and so on to the reader. We also leave the definition of the pull-back and the direct image of the log structure on a ringed topos by a morphism of ringed topoi to him.

Let $\mathbf{T}$ be the 2-category of log ringed topoi. For an object $\mathcal{T} = (\mathcal{T}, \mathcal{O}_\mathcal{T}, \mathcal{M}_\mathcal{T}, \alpha)$ of $\mathbf{T}$ and for a positive integer n, set $\mathcal{T}_n := (\mathcal{T}, \mathcal{O}_n, \mathcal{M}_n, \alpha_n) := (\mathcal{T}, \mathcal{O}_\mathcal{T}/p^n, \mathcal{M}_n, \alpha_n)$, where $(\mathcal{M}_n, \alpha_n)$ is the associated log structure to the composite morphism $\mathcal{M} \longrightarrow \mathcal{O}_\mathcal{T} \longrightarrow \mathcal{O}_n$. Let $\mathbf{DDC}_p^{\mathbf{top}}$ be a 2-subcategory of $\mathbf{T}$ whose objects are quintuplets $(\mathcal{T}, \mathcal{O}_\mathcal{T}, \mathcal{M}_\mathcal{T}, \alpha; \varphi)$'s satisfying the corresponding conditions in (7.10):

(7.15.1): $\mathcal{O}_\mathcal{T}$ is p-torsion free.

(7.15.2): $\mathcal{M}_\mathcal{T}$ is integral.

(7.15.3): φ is a lift of the Frobenius endomorphism of $\mathcal{T}_1$.

(7.15.4): α induces an isomorphism $\alpha^{-1}(1 + p\mathcal{O}_\mathcal{T}) \overset{\sim}{\longrightarrow} 1 + p\mathcal{O}_\mathcal{T}$.

We define a morphism in $\mathbf{DDC}_p^{\mathbf{top}}$ in an obvious way.

Let $\star$ be a positive integer n or nothing. Then there exists a natural functor

$$W_\star \colon \mathbf{DDC}_p^{\mathbf{top}} \longrightarrow \mathbf{T}$$

whose underlying functor of 2-subcategories of ringed topoi is the restriction of the functor in [I2, 0 1.5].

As in [HK, (3.1)], we define the canonical lift of a log ringed topos of characteristic p and for a positive integer n.

We obtain the following by the morphism (7.12.3) without difficulty:

Corollary 7.16. *Let $\mathcal{T}$ be an object of $\mathbf{DDC}_p^{\mathbf{top}}$. Let $\iota\colon \mathcal{S} \longrightarrow \mathcal{T}_1$ be a morphism of log ringed topoi. Let $W_n(\mathcal{S})$ $(n \in \mathbb{Z}_{>0})$ be the canonical lift of $\mathcal{S}$. Then the composite morphism $\mathcal{S} \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T}_n$ factors through a natural morphism $W_n(\mathcal{S}) \longrightarrow \mathcal{T}_n$.*

Let p be a fixed prime number. Let $\mathbf{FLSch}_p$ be the category of formal log schemes with p-adic topology. Let n be a positive integer. Let $\mathbf{DDC}_p^{\mathbf{fsch}}$ be the full subcategory of $\mathbf{FLSch}_p$ whose objects satisfy the similar conditions to (7.15.1), (7.15.2) and (7.15.3) (the condition (7.15.4) is automatically satisfied). Let $\mathbf{LSch}$ be the category of log schemes. Then there exists a natural functor

$$W_n \colon \mathbf{DDC}_p^{\mathbf{fsch}} \longrightarrow \mathbf{LSch}.$$

We restate a special case of (7.16) (with slight generalization) in order to clarify a relationship with [HK, (4.19)]:

Corollary 7.17. *Let L be a fine log structure on* $\mathrm{Spec}(\kappa)$. *Let $\mathcal{Z}$ be an integral formal log scheme over* $(\mathrm{Spf}(W), W(L))$ *such that $\mathcal{O}_{\mathcal{Z}}$ is p-torsion free. Set $\mathcal{Z}_n := \mathcal{Z} \otimes_W W_n$. Assume that $\mathcal{Z}$ has a lift of the Frobenius endomorphism of $\mathcal{Z}_1$. Let $\iota\colon Y \overset{\subset}{\longrightarrow} \mathcal{Z}_1$ be a* (*not necessarily closed*) *immersion of fine log schemes over the fine log scheme* $(\mathrm{Spec}(\kappa), L)$. *Let $W_n(Y)$* $(n \in \mathbb{Z}_{>0})$ *be the canonical lift of Y over W_n. Then the composite morphism $Y \overset{\subset}{\longrightarrow} \mathcal{Z}_1 \overset{\subset}{\longrightarrow} \mathcal{Z}_n$ of immersions factors through a natural morphism $W_n(Y) \longrightarrow \mathcal{Z}_n$ over* $(\mathrm{Spec}(W_n), W_n(L))$.

Next we prove a lemma which will be needed for the commutativity of (7.9.3).

Let $Y \overset{\subset}{\longrightarrow} Z$ be a (not necessarily closed) immersion from a log scheme over $(\mathrm{Spec}(\kappa), L)$ into a log smooth scheme over $(\mathrm{Spec}(W_n), W_n(L))$. Let D be the log PD-envelope of the immersion $Y \overset{\subset}{\longrightarrow} Z$. Let $C_{Y/(W_n,W_n(L))}$ be the crystalline complex with respect to this immersion; the crystalline complex can be defined for a (not necessarily closed) immersion. Let $W_n(Y) := (W_n(\overset{\circ}{Y}), W_n(M))$ be the canonical lift of $Y = (\overset{\circ}{Y}, M)$ over $(\mathrm{Spec}(W_n)$, $W_n(L))$. Let $\Lambda^{\bullet}_{W_n(Y)/(W_n,W_n(L))}$ be the log de Rham complex of a log scheme $W_n(Y)$ over $(\mathrm{Spec}(W_n), W_n(L))$. Then, in [HK, p. 251–252], a morphism

$$\psi_n \colon \Lambda^{\bullet}_{W_n(Y)/(W_n,W_n(L))} \longrightarrow W_n\Lambda^{\bullet}_Y = \mathcal{H}^{\bullet}(C_{Y/(W_n,W_n(L))})$$

is defined by the following:

$$(a_0, \dots, a_{n-1}) \longmapsto s_n(0,0)((a_0, \dots, a_{n-1})) \quad (a_i \in \mathcal{O}_Y, 0 \le i \le n-1)$$

$$d(a_0, \dots, a_{n-1}) \longmapsto s_n(1,0)((a_0, \dots, a_{n-1})) \quad (a_i \in \mathcal{O}_Y, 0 \le i \le n-1)$$

and

$$d\log b \longmapsto d\log \widetilde{b} \quad (b \in M(\subset W_n(M))),$$

where $\widetilde{b}$ is a lift of b to the log structure of an open log subscheme of Z which contains Y as a closed log subscheme and $s_n(0,0)$ (resp. $s_n(1,0)$) is a morphism defined in (7.1.1) (resp. (7.1.2)). The morphism ψ_n is $W_n(\mathcal{O}_Y)$-linear.

LEMMA 7.18. *The following two diagrams are commutative*:

$$\begin{array}{ccc} \Lambda^{\bullet}_{W_{n+1}(Y)/(W_{n+1},W_{n+1}(L))} & \xrightarrow{\psi_{n+1}} & W_{n+1}\Lambda^{\bullet}_Y \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Lambda^{\bullet}_{W_n(Y)/(W_n,W_n(L))} & \xrightarrow{\psi_n} & W_n\Lambda^{\bullet}_Y, \end{array} \tag{7.18.1}$$

$$\begin{array}{ccc} \Lambda^{i}_{W_n(Y)/(W_n,W_n(L))} & \xrightarrow{\psi_n} & W_n\Lambda^{i}_Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \Lambda^{i+1}_{W_n(Y)/(W_n,W_n(L))} & \xrightarrow{\psi_n} & W_n\Lambda^{i+1}_Y. \end{array} \tag{7.18.2}$$

Proof. We can check the commutativity of (7.18.1) by the same proof of (7.1).

We can check the commutativity of (7.18.2) as follows: The problem is local. Let $\mathcal{Y}$ be a log smooth lift of Y over $(\mathrm{Spf}(W), W(L))$. Set $\mathcal{Y}_n := \mathcal{Y}\otimes_W W_n$. Then we have only to check the following diagram is commutative:

$$\begin{array}{ccc}\Lambda^i_{W_n(Y)/(W_n,W_n(L))} & \xrightarrow{\psi_n} & \mathcal{H}^i(\Lambda^\bullet_{\mathcal{Y}_n/(W_n,W_n(L))}) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ \Lambda^{i+1}_{W_n(Y)/(W_n,W_n(L))} & \xrightarrow{\psi_n} & \mathcal{H}^{i+1}(\Lambda^\bullet_{\mathcal{Y}_n/(W_n,W_n(L))}).\end{array} \tag{7.18.3}$$

This commutativity immediately follows because the right vertical boundary operator in (7.18.3) is the boundary morphism of the following exact sequence

$$0 \longrightarrow \Lambda^\bullet_{\mathcal{Y}_n/(W_n,W_n(L))} \xrightarrow{p^n} \Lambda^\bullet_{\mathcal{Y}_{2n}/(W_{2n},W_{2n}(L))} \xrightarrow{\mathrm{proj.}} \Lambda^\bullet_{\mathcal{Y}_n/(W_n,W_n(L))} \longrightarrow 0. \square$$

Now we give a right proof of [HK, (4.19)]:

Theorem 7.19. *There exists a canonical morphism*

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \longrightarrow W_n\Lambda^\bullet_Y. \tag{7.19.1}$$

The morphism (7.19.1) *is an isomorphism and is compatible with the transition morphisms.*

Proof. (The proof is not the same as that of the proof of [I2, II (1.4)].) Let $Y_\bullet \longrightarrow Y$ be the Čech hypercovering of an open covering of Y for the zariski topology. Take an embedding system $(Y_\bullet, \mathcal{Z}_\bullet)_{\bullet\in\mathbb{N}}$ over $(\mathrm{Spec}(W_n), W_n(L))$ of $Y \longrightarrow (\mathrm{Spec}(\kappa), L)$ such that the immersion $Y_\bullet \longrightarrow \mathcal{Z}_\bullet$ of simplicial log schemes factors through a morphism $W_n(Y_\bullet) \longrightarrow \mathcal{Z}_\bullet$ over $(\mathrm{Spec}(W_n), W_n(L))$. The embedding system above indeed exists by (7.17) and the standard construction of the Čech diagram associated to lifts of open log affine subschemes of Y which covers Y (cf. [I2, II (1.1)]; see also (7.20) below). (As we said before, we can avoid using (7.17) in this paper, but we cannot in [NS].) Let

$$\eta\colon \widetilde{Y_{\bullet}}_{\mathrm{zar}} \longrightarrow \widetilde{Y}_{\mathrm{zar}}$$

be the natural morphism of topoi.

Let $C_{Y/(W_n,W_n(L))}$ be the crystalline complex associated to the embedding system $(Y_\bullet, \mathcal{Z}_\bullet)$. By the proof of [HK, (4.19)], we have the following composite morphism

$$C_{Y/(W_n,W_n(L))} \longrightarrow \Lambda^\bullet_{W_n(Y_\bullet)/(W_n,W_n(L)),[\ ]} \xrightarrow{\psi_n} \eta^{-1}(W_n\Lambda^\bullet_Y).$$

Here $\bigoplus_{i\geq 0} \Lambda^i_{W_n(Y_\bullet)/(W_n,W_n(L)),[\ ]}$ is a sheaf of differential graded algebras over W_n which is a quotient of $\bigoplus_{i\geq 0} \Lambda^i_{W_n(Y_\bullet)/(W_n,W_n(L))}$ divided by a W_n-submodule generated by the local sections of the following form $da^{[j]} - a^{[j-1]}da$ $(a \in \mathrm{Ker}(W_n(\mathcal{O}_Y) \longrightarrow \mathcal{O}_Y),\ j \geq 1)$. By the cohomological descent for a bounded below complex, we have the morphism (7.19.1):

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \longrightarrow W_n\Lambda^\bullet_Y.$$

As in [I2, pp. 602–603], this morphism is independent of the choice of the embedding system above.

Since $Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) = R\eta_*(C_{Y/(W_n,W_n(L))})$, we have the following commutative diagram:

$$(7.19.2)\qquad \begin{array}{ccc} Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) & \longrightarrow & R\eta_*\Lambda^\bullet_{W_n(Y_\bullet)/(W_n,W_n(L)),[\ ]} \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \mathrm{proj.}} \\ Ru_{Y/W_{n-1}*}(\mathcal{O}_{Y/W_{n-1}}) & \longrightarrow & R\eta_*\Lambda^\bullet_{W_{n-1}(Y_\bullet)/(W_{n-1},W_{n-1}(L)),[\ ]}. \end{array}$$

On the other hand, by (7.18) and the cohomological descent for a bounded below complex, we have the following commutative diagram

$$(7.19.3)\qquad \begin{array}{ccc} R\eta_*\Lambda^\bullet_{W_n(Y_\bullet)/(W_n,W_n(L)),[\ ]} & \longrightarrow & W_n\Lambda^\bullet_Y \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \pi} \\ R\eta_*\Lambda^\bullet_{W_{n-1}(Y_\bullet)/(W_{n-1},W_{n-1}(L)),[\ ]} & \longrightarrow & W_{n-1}\Lambda^\bullet_Y. \end{array}$$

By (7.19.2) and (7.19.3), we have the following commutative diagram:

$$(7.19.4)\qquad \begin{array}{ccc} Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) & \longrightarrow & W_n\Lambda^\bullet_Y \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \pi} \\ Ru_{Y/W_{n-1}*}(\mathcal{O}_{Y/W_{n-1}}) & \longrightarrow & W_{n-1}\Lambda^\bullet_Y. \end{array}$$

Let $C_{Y/(W_m,W_m(L))} \xrightarrow{\mathbf{p}} C_{Y/(W_{m+1},W_{m+1}(L))}$ $(1 \leq m \leq n-1)$ be the induced morphism by the multiplication by p: $C_{Y/(W_{m+1},W_{m+1}(L))} \longrightarrow C_{Y/(W_{m+1},W_{m+1}(L))}$. Let $\mathbf{p^{n-1}}\colon C_{Y/(W_1,L)} \longrightarrow C_{Y/(W_n,W_n(L))}$ be the following composite morphism

$$C_{Y/(W_1,L)} \xrightarrow{\mathbf{p}} C_{Y/(W_2,W_2(L))} \xrightarrow{\mathbf{p}} \cdots \xrightarrow{\mathbf{p}} C_{Y/(W_n,W_n(L))}. \tag{7.19.5}$$

Let $\mathbf{p^{n-1}}\colon \Lambda^{\bullet}_{Y_\bullet/(W_1,L),[\ ]} \longrightarrow \Lambda^{\bullet}_{W_n(Y_\bullet)/(W_n,W_n(L)),[\ ]}$ be an analogue of (7.19.5). Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{Y/(W_1,L)} & \xrightarrow{\mathbf{p^{n-1}}} & C_{Y/(W_n,W_n(L))} & \xrightarrow{\text{proj.}} & C_{Y/(W_{n-1},W_{n-1}(L))} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \eta^{-1}(\Lambda^{\bullet}_{Y/(W_1,L),[\ ]}) & \xrightarrow{\mathbf{p^{n-1}}} & \eta^{-1}(\Lambda^{\bullet}_{W_n(Y)/(W_n,W_n(L)),[\ ]}) & \xrightarrow{\text{proj.}} & \eta^{-1}(\Lambda^{\bullet}_{W_{n-1}(Y)/(W_{n-1},W_{n-1}(L)),[\ ]}) & & .
\end{array} \tag{7.19.6}$$

Moreover, using (7.18), we have the following commutative diagram as in (7.8.3):

$$\begin{array}{ccc}
\eta^{-1}(\Lambda^{\bullet}_{Y/(W_1,W_1(L)),[\ ]}) & \xrightarrow{\mathbf{p^{n-1}}} & \eta^{-1}(\Lambda^{\bullet}_{W_n(Y)/(W_n,W_n(L)),[\ ]}) \\
\psi_1 \downarrow & & \downarrow \psi_n \\
\eta^{-1}(W_1\Lambda^{\bullet}_Y) & \xrightarrow{\mathbf{p^{n-1}}} & \eta^{-1}(W_n\Lambda^{\bullet}_Y).
\end{array} \tag{7.19.7}$$

Hence, by (7.19.6), (7.19.7), (7.18) and by [HK, (4.5) (1)] or by (6.22) (2), we have the following commutative diagram of triangles

$$\begin{array}{ccccccccc}
\longrightarrow & C_{Y/(W_1,L)} & \xrightarrow{\mathbf{p^{n-1}}} & C_{Y/(W_n,W_n(L))} & \xrightarrow{\text{proj.}} & C_{Y/(W_{n-1},W_{n-1}(L))} & \xrightarrow{+1} & \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & \eta^{-1}(W_1\Lambda^{\bullet}_Y) & \xrightarrow{\mathbf{p^{n-1}}} & \eta^{-1}(W_n\Lambda^{\bullet}_Y) & \xrightarrow{\pi} & \eta^{-1}(W_{n-1}\Lambda^{\bullet}_Y) & \xrightarrow{+1} & .
\end{array} \tag{7.19.8}$$

By the cohomological descent, we have the following commutative diagram of triangles:

$$
(7.19.9)\quad
\begin{array}{ccccc}
\longrightarrow & Ru_{Y/W_1*}(\mathcal{O}_{Y/W_1}) & \xrightarrow{\mathbf{p^{n-1}}} & Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \\
 & \downarrow & & \downarrow \\
\longrightarrow & W_1\Lambda^{\bullet}_Y & \xrightarrow{\mathbf{p^{n-1}}} & W_n\Lambda^{\bullet}_Y
\end{array}
$$

$$
\begin{array}{ccc}
\xrightarrow{\text{proj.}} & Ru_{Y/W_{n-1}*}(\mathcal{O}_{Y/W_{n-1}}) & \xrightarrow{+1} \\
 & \downarrow & \\
\xrightarrow{\pi} & W_{n-1}\Lambda^{\bullet}_Y & \xrightarrow{+1} .
\end{array}
$$

By the proof [HK, (4.9)] and (7.14), the left vertical morphism in (7.19.9) is induced by a Cartier inverse isomorphism $C^{-1}\colon \Lambda^i_Y \xrightarrow{\sim} \mathcal{H}^i(\Lambda^{\bullet}_Y)$ $(i \in \mathbb{N})$. In particular, it is an isomorphism. Hence induction on n shows that the middle arrow in (7.19.9) is an isomorphism. We finish the proof. □

REMARK 7.20. (cf. [Sh, Proposition 2.2.11]) The claim of the existence of the embedding system in [HK, p. 237] is not perfect and the argument in [I2, p. 602, p. 604] is not perfect since $\mathrm{cosq}(U_0/X) \longrightarrow \mathrm{cosq}(Y_0/W)$ in [loc. cit.] is not necessarily a closed immersion; in general, $\mathrm{cosq}(U_0/X) \longrightarrow \mathrm{cosq}(Y_0/W)$ is only an immersion. We have only to change the definition of the embedding system of [HK, p. 237] as in [Sh, (2.2.10)]: we allow the (not necessarily closed) immersion in the definition of the embedding system.

REMARK 7.21. Let Y be the special fiber with canonical log structure of a semistable family over a complete discrete valuation ring of mixed characteristics. In [HK, (1.1)], we can find a claim that the complex $W_n\omega^{\bullet}_Y$ in [HK, §1] is equal to $W_n\omega^{\bullet}_Y$ in [Hy2]. Though I do not use this fact in this paper, I give a proof of it as follows because there is no proof for it in literatures.

Let $\iota\colon U \longrightarrow Y$ be an open immersion from a dense open smooth subscheme of Y over κ. Let $(W_n\Omega^{\bullet}_U)''$ (resp. $W_n\Omega^{\bullet}_U$) be the de Rham-Witt complex defined in [I2, I (1.3)] (resp. [IR, III (1.5)]). Let $(W_n\omega^{\bullet}_Y)''$ be the complex defined in [HK, (1.1)]. By [IR, III (1.4), (1.5)] we have an isomorphism

$$\iota_*(W_n\Omega^i_U)'' \overset{C^{-n}}{\underset{\sim}{\longrightarrow}} \iota_*(\mathcal{H}^i((W_n\Omega^\bullet_U)'')) = \iota_*(\mathcal{H}^i(Ru_{U/W_n*}(\mathcal{O}_{U/W_n}))) = \iota_*(W_n\Omega^i_U),$$

which we denote by C^{-n} by abuse of notation. Consider the following diagram:

(7.21.1)
$$\begin{array}{ccc} (W_n\omega^i_Y)'' & \xrightarrow{\subset} & \iota_*(W_n\Omega^i_U)'' \\ & & \downarrow{\scriptstyle C^{-n}}\,\simeq \\ W_n\Lambda^i_Y & \xrightarrow{\subset} & \iota_*(W_n\Omega^i_U). \end{array}$$

By [I2, I (3.27)] and [IR, III (1.4)] and by the definition of $(W_n\omega^i_Y)''$, the image of a local section

$$\omega d\log u_1\cdots d\log u_j \in (W_n\omega^i_Y)''$$

$$(j\le i,\ \omega\in \mathrm{Im}(W_n(\mathcal{O}_Y)(dW_n(\mathcal{O}_Y))^{\otimes(i-j)}$$

$$\longrightarrow \iota_*(W_n\Omega^i_U)''),\ u_1,\dots u_j\in\iota_*(\mathcal{O}^*_U))$$

in $\iota_*(W_n\Omega^i_U)$ is $C^{-n}(\omega)d\log u_1\cdots d\log u_j$. This section is contained in $W_n\Lambda^i_Y$ by the local description of the log scheme Y and by the definition of $W_n\Lambda^i_Y$. Therefore we have a natural injective morphism $C^{-n}\colon (W_n\omega^i_Y)'' \overset{\subset}{\longrightarrow} W_n\Lambda^i_Y$.

The pro-sheaf $(W_\bullet\omega^i_Y)''$ is stable by the operator V on $\iota_*(W_\bullet\Omega^i_U)''$. Indeed, we have $V(d\log u) = pd\log u$ $(u\in\iota_*(\mathcal{O}^*_U))$ by [I2, I (1.15.4)]. Furthermore, since $d(d\log\mathcal{O}^*_U)=0$, $(W_n\omega^\bullet_Y)''$ is stable by the operator d on $\iota_*(W_n\Omega^\bullet_U)''$. Hence we can consider $\mathrm{Fil}^{n-1}((W_n\omega^i_Y)'') := V^{n-1}(W_1\omega^i_Y)'' + dV^{n-1}(W_1\omega^{i-1}_Y)''$ in $(W_n\omega^i_Y)''$. By the commutativity of (6.6.1), we have the following commutative diagram:

(7.21.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^{n-1}((W_n\omega^i_Y)'') & \longrightarrow & (W_n\omega^i_Y)'' & \xrightarrow{\text{proj.}} & (W_{n-1}\omega^i_Y)'' & \longrightarrow & 0 \\ & & {\scriptstyle C^{-n}}\downarrow & & {\scriptstyle C^{-n}}\downarrow & & {\scriptstyle C^{-(n-1)}}\downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^{n-1}(W_n\Lambda^i_Y) & \longrightarrow & W_n\Lambda^i_Y & \xrightarrow{\pi} & W_{n-1}\Lambda^i_Y & \longrightarrow & 0. \end{array}$$

Here the lower horizontal sequence is exact; however we have not yet claimed that the middle term of the upper one is exact. As to the middle term, we claim only that $\mathrm{Fil}^{n-1}((W_n\omega_Y^i)'') \subset \mathrm{Ker}((W_n\omega_Y^i)'' \longrightarrow (W_{n-1}\omega_Y^i)'')$. By [IR, III (1.4.9)], we have the following commutative diagram

$$\begin{array}{ccc} (W_1\omega_Y^i)'' & \xrightarrow{V^{n-1}} & (W_n\omega_Y^i)'' \\ {\scriptstyle C^{-1}}\downarrow & & \downarrow{\scriptstyle C^{-n}} \\ W_1\Lambda_Y^i & \xrightarrow{V^{n-1}} & W_n\Lambda_Y^i. \end{array} \tag{7.21.3}$$

Furthermore, we also have the following commutative diagram by a formula $dF = pFd$ in $(W_n\omega_Y^i)''$, by [IR, III (1.4)], and by the definition of the boundary operator $d\colon W_n\Lambda_Y^{i-1} \longrightarrow W_n\Lambda_Y^i$:

$$\begin{array}{ccc} (W_n\omega_Y^{i-1})'' & \xrightarrow{d} & (W_n\omega_Y^i)'' \\ {\scriptstyle C^{-n}}\downarrow & & \downarrow{\scriptstyle C^{-n}} \\ W_n\Lambda_Y^{i-1} & \xrightarrow{d} & W_n\Lambda_Y^i. \end{array} \tag{7.21.4}$$

By [HK, (4.4)] or by (6.19), the morphism

$$(V^{n-1}, dV^{n-1}) \circ C^{-1}\colon (W_1\omega_Y^i)'' \oplus (W_1\omega_Y^{i-1})'' \longrightarrow \mathrm{Fil}^{n-1}(W_n\Lambda_Y^i)$$

is surjective. Here, note that $(W_1\omega_Y^i)'' = \omega_Y^i$ by the definition of $(W_n\omega_Y^i)''$ in [HK, (1.1)]. Hence, by the commutativity of (7.21.3) and (7.21.4), the morphism $C^{-n}\colon \mathrm{Fil}^{n-1}((W_n\omega_Y^i)'') \longrightarrow \mathrm{Fil}^{n-1}(W_n\Lambda_Y^i)$ is surjective. By induction on n and by (7.21.2), we see that the morphism $C^{-n}\colon (W_n\omega_Y^i)'' \longrightarrow W_n\Lambda_Y^i$ is surjective. Putting all this together, the morphism $C^{-n}\colon (W_n\omega_Y^i)'' \longrightarrow W_n\Lambda_Y^i$ is an isomorphism. Now we have proved the claim on the coincidence.

Except in (7.21), I do not use $W_n\omega_Y^{\bullet}$ in [HK, §1]. In particular, I do not use the lower exact sequence of the diagram in [HK, p. 261], though I shall use a symbolically similar exact sequence in (11.1).

8. Projections

The proposition (8.4) (2) below and the corollary (8.6) below are important because they, the commutative diagram (8.4.3) and (7.19) are necessary for the construction of the weight spectral sequence (2.0.1).

Let n be a positive integer. Let κ, s and W be as in §2 and let X be a (not necessarily proper) SNCL variety over s. Set $W_n := W_n(\kappa)$. Let $W_n\Lambda^\bullet_X(= W_n\omega^\bullet_X$ in [Hy2], [M1, §2]) be the log de Rham-Witt complex of X/s. In [Hy2], X is assumed to be the special fiber with canonical log structure of a semistable scheme over a complete discrete valuation ring of mixed characteristics; however we need not assume this for the results in [Hy2]. Let $W_n\widetilde{\Lambda}^\bullet_X$ be a complex which has been denoted by $W_n\widetilde{\omega}^\bullet_X$ in [loc. cit.]. Following [M1, 3.8], let us set $W_nA^{ij}_X := W_n\widetilde{\Lambda}^{i+j+1}_X/P_jW_n\widetilde{\Lambda}^{i+j+1}_X$ $(i, j \in \mathbb{N})$. Let θ_n be a section of $W_n\widetilde{\Lambda}^1_X$ which has been constructed in [Hy2, (1.2.2)] and [M1, 3.4 (3)].

Let i be a non-negative integer. In his article [M1, 3.8, 3.11], Mokrane has constructed a double pro-complex $W_\bullet A^{\bullet\bullet}_X$ of $W_\bullet$-modules which contains the following complex as a sub-pro-complex:

$$W_\bullet A^{i0}_X \xrightarrow{\wedge\theta_\bullet} W_\bullet A^{i1}_X \xrightarrow{\wedge\theta_\bullet} W_\bullet A^{i2}_X \longrightarrow \cdots . \tag{8.0.1}$$

Furthermore he has defined a morphism of pro-complexes [M1, 3.14]:

$$W_\bullet\Lambda^\bullet_X \xrightarrow{\wedge\theta_\bullet} W_\bullet A^{\bullet 0}_X. \tag{8.0.2}$$

However, in order to construct the pro-complex $W_\bullet A^{i\bullet}_X$ in (8.0.1) and the morphism of pro-complexes in (8.0.2), we have to check that the projection $\pi\colon W_{n+1}\widetilde{\Lambda}^{i+j+1}_X \longrightarrow W_n\widetilde{\Lambda}^{i+j+1}_X$ $(j \in \mathbb{N})$ preserves the preweight filtration P on $W_m\widetilde{\Lambda}^{i+j+1}_X$ $(m = n+1, n)$ (recall our terminology in (4.3)) and that the following two diagrams are commutative:

$$\begin{array}{ccc} W_{n+1}A^{i,j+1}_X & \xrightarrow{\pi} & W_nA^{i,j+1}_X \\ {\scriptstyle\wedge\theta_{n+1}}\uparrow & & \uparrow{\scriptstyle\wedge\theta_n} \\ W_{n+1}A^{ij}_X & \xrightarrow{\pi} & W_nA^{ij}_X, \end{array} \tag{8.0.3}$$

$$\begin{array}{ccc} W_{n+1}A^{i0}_X & \xrightarrow{\pi} & W_nA^{i0}_X \\ {\scriptstyle\wedge\theta_{n+1}}\uparrow & & \uparrow{\scriptstyle\wedge\theta_n} \\ W_{n+1}\Lambda^{i}_X & \xrightarrow{\pi} & W_n\Lambda^{i}_X. \end{array} \tag{8.0.4}$$

These have been claimed in [M1, 3.8]. However no proof for these facts has been given.

Hyodo has also claimed that the following exact sequence

$$0 \longrightarrow W_n\Lambda_X^{\bullet-1} \xrightarrow{\wedge\theta_n} W_n\widetilde{\Lambda}_X^{\bullet} \longrightarrow W_n\Lambda_X^{\bullet} \longrightarrow 0 \tag{8.0.5}$$

is compatible with projections ([Hy2, (1.4.3)]). However the proof of the commutativity of the following diagram has not been given in any literature:

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}_X^{\bullet} & \xrightarrow{\pi} & W_n\widetilde{\Lambda}_X^{\bullet} \\ {\scriptstyle \wedge\theta_{n+1}}\big\uparrow & & \big\uparrow{\scriptstyle \wedge\theta_n} \\ W_{n+1}\Lambda_X^{\bullet-1} & \xrightarrow{\pi} & W_n\Lambda_X^{\bullet-1}. \end{array} \tag{8.0.6}$$

We can find a similar statement in [HK, (1.5)] for a semistable family over a complete discrete valuation ring of mixed characteristics with perfect residue field. However, strictly speaking, even for the semistable family above, [Hy2, (1.4.3)] and [HK, (1.5)] are not the same statements (cf. (11.1)).

In (11.1) below, we shall consider the compatibility of (8.0.5) with the Frobenius. Note that this compatibility has not been considered in any literature except analogous compatibility in [HK, (1.6)] in the case of a semistable family over a complete discrete valuation ring of mixed characteristics.

In §8, we prove that the projection $\pi\colon W_{n+1}\widetilde{\Lambda}_X^i \longrightarrow W_n\widetilde{\Lambda}_X^i$ ($i \in \mathbb{N}$) preserves the preweight filtration P on $W_m\widetilde{\Lambda}_X^i$ ($m = n+1, n$), and we show the commutativity of (8.0.3), (8.0.4) and (8.0.6).

Let the notations be as before (6.27).

Set $\mathcal{Y}_n := \mathcal{Y} \otimes_W W_n$, $\mathcal{X}_n := \mathcal{X} \otimes_W W_n$, $\widetilde{\Lambda}_n^{\bullet} := \mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^{\bullet}_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)$ and $\widetilde{\Lambda}^{\bullet} := \varprojlim_n \widetilde{\Lambda}_n^{\bullet}$. Let i be a non-negative integer. We have $W_n\Lambda_X^i = W_n\widetilde{\Lambda}_X^i/(W_n\widetilde{\Lambda}_X^{i-1} \wedge \theta_n)$ by the definition of $W_n\Lambda_X^i$ ([Hy2, (1.2)]). An equality $\Phi^*(\theta_n) = p\theta_n$ in $W_n\widetilde{\Lambda}_X^1$ holds ([Hy2, p. 245], cf. the proof of (8.1) below). The following diagram is commutative by the characterizations of π's ([Hy2, (1.3.2)]):

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}_X^{i} & \xrightarrow{\pi} & W_n\widetilde{\Lambda}_X^{i} \\ \big\downarrow & & \big\downarrow \\ W_{n+1}\Lambda_X^{i} & \xrightarrow{\pi} & W_n\Lambda_X^{i}. \end{array} \tag{8.0.7}$$

Proposition 8.1. *Let i be a non-negative integer. Then the following two diagrams are commutative*:

(8.1.1)
$$\begin{array}{ccc} W_n\widetilde{\Lambda}^{i+1}_X & \xrightarrow{\mathbf{p}} & W_{n+1}\widetilde{\Lambda}^{i+1}_X \\ {\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_{n+1}\wedge} \\ W_n\widetilde{\Lambda}^{i}_X & \xrightarrow{\mathbf{p}} & W_{n+1}\widetilde{\Lambda}^{i}_X, \end{array}$$

(8.1.2)
$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}^{i+1}_X & \xrightarrow{\pi} & W_n\widetilde{\Lambda}^{i+1}_X \\ {\scriptstyle \theta_{n+1}\wedge}\uparrow & & \uparrow{\scriptstyle \theta_n\wedge} \\ W_{n+1}\widetilde{\Lambda}^{i}_X & \xrightarrow{\pi} & W_n\widetilde{\Lambda}^{i}_X. \end{array}$$

In particular, there exists a morphism

(8.1.3)
$$\begin{aligned} \theta\wedge := \varprojlim_n(\theta_n\wedge)\colon W\widetilde{\Lambda}^i_X := \varprojlim_{\pi}{}_n W_n\widetilde{\Lambda}^i_X \longrightarrow \varprojlim_{\pi}{}_n W_n\widetilde{\Lambda}^{i+1}_X \\ = W\widetilde{\Lambda}^{i+1}_X. \end{aligned}$$

Proof. The question is local; we may assume that X is affine and that there exists an admissible triple $(\mathcal{Y}, \mathcal{X}, \Phi)$ of X (recall the definition of the admissible triple before (6.27)). Then the commutativity of (8.1.1) is equivalent to that of the following diagram:

(8.1.4)
$$\begin{array}{ccc} W_n\widetilde{\Lambda}^{i+1}_X & \xrightarrow{\mathcal{H}^{i+1}(\Phi^*/p^i)} & W_{n+1}\widetilde{\Lambda}^{i+1}_X \\ {\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_{n+1}\wedge} \\ W_n\widetilde{\Lambda}^{i}_X & \xrightarrow{\mathcal{H}^{i}(\Phi^*/p^{i-1})} & W_{n+1}\widetilde{\Lambda}^{i}_X. \end{array}$$

Because there exists an element $u \in W\{t\}$ such that $\Phi^*(t) = t^p(1+pu)$, $\Phi^*(d\log t) = pd\log t + d\log(1+pu) = pd\log t + d(\sum_{j=1}^{\infty}(-1)^{j-1}(pu)^j/j)$; $\Phi^*(d\log t)$ is equivalent to $pd\log t$ modulo an exact form in $\mathcal{H}^1(\widetilde{\Lambda}^\bullet_n)$ ([Hy2, p. 245]). Now the commutativity of (8.1.4) is obvious.

We can easily check the commutativity of (8.1.2) by the following obvious commutative diagram

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}^{i+1}_X & \xrightarrow{p} & W_{n+1}\widetilde{\Lambda}^{i+1}_X \\ {\scriptstyle \theta_{n+1}\wedge}\uparrow & & \uparrow{\scriptstyle \theta_{n+1}\wedge} \\ W_{n+1}\widetilde{\Lambda}^{i}_X & \xrightarrow{p} & W_{n+1}\widetilde{\Lambda}^{i}_X, \end{array}$$

by the commutativity of (8.1.1), by the injectivity of $\mathbf{p}\colon W_n\widetilde{\Lambda}^{i+1}_X \longrightarrow W_{n+1}\widetilde{\Lambda}^{i+1}_X$ ([Hy2, (2.2.2)], cf. (6.8) (2), (6.28) (2)) and by an easy diagram-chasing. □

Corollary 8.2. *The diagram* (8.0.6) *is commutative.*

Proof. We immediately obtain (8.2) by the commutativity of (8.0.7) and (8.1.2). □

Remark 8.3. (11.1) (2) below will give another proof of (8.2).

Proposition 8.4. *Let i be a non-negative integer. Then the following hold*:

(1) *The morphism $\mathbf{p}\colon W_n\widetilde{\Lambda}^i_X \longrightarrow W_{n+1}\widetilde{\Lambda}^i_X$ preserves the preweight filtration P on $W_m\widetilde{\Lambda}^i_X$ $(m = n, n+1)$.*

(2) *The projection $\pi\colon W_{n+1}\widetilde{\Lambda}^i_X \longrightarrow W_n\widetilde{\Lambda}^i_X$ preserves the preweight filtration P on $W_m\widetilde{\Lambda}^i_X$ $(m = n+1, n)$.*

Proof. We may assume that $i > 0$. The question is local; we may assume that X is affine and that there exists an admissible triple $(\mathcal{Y}, \mathcal{X}, \Phi)$ of X; especially, $\overset{\circ}{\mathcal{Y}}$ (resp. $\overset{\circ}{\mathcal{X}}$) is formally etale over $\mathrm{Spf}(W\{x_0, \ldots, x_d\})$ (resp. $\mathrm{Spf}(W\{x_0, \ldots, x_d\}/(x_0 \cdots x_r))$) with a structural morphism $\mathrm{Spf}(W\{x_0, \ldots, x_d\}) \longrightarrow \mathrm{Spf}(W\{t\})$ defined by $t \longmapsto x_0 \cdots x_r$ $(0 \leq r \leq d)$. Set $\mathcal{Y}_n := \mathcal{Y} \otimes_W W_n$, $\mathcal{X}_n := \mathcal{X} \otimes_W W_n$, $\widetilde{\Lambda}^\bullet_n := \mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)$ and $\widetilde{\Lambda}^\bullet := \varprojlim_n \widetilde{\Lambda}^\bullet_n$.

(1): Because Φ is a lift of the Frobenius of the log scheme $(\mathcal{Y}_1, \mathcal{X}_1)$, there exists a section $y_j \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ such that $\Phi^*(x_j) = x_j^p(1 + py_j)$ $(0 \leq j \leq r)$. Then, as in the proof of (8.1), $\Phi^*(d\log x_j)$ is equivalent to $pd\log x_j$ modulo an exact form in $\varprojlim_n \Omega^1_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)$.

Furthermore, the morphism

$$\Phi^*/p^{j-1}\colon \widetilde{\Lambda}^j_n \longrightarrow \widetilde{\Lambda}^j_{n+1} \quad (j \in \mathbb{N})$$

obviously induces a morphism

$$\Omega^j_{\mathcal{Y}_n/W_n}/\Omega^j_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n) \longrightarrow \Omega^j_{\mathcal{Y}_{n+1}/W_{n+1}}/\Omega^j_{\mathcal{Y}_{n+1}/W_{n+1}}(-\log \mathcal{X}_{n+1}).$$

Therefore the morphism $\mathbf{p}\colon W_n\widetilde{\Lambda}^i_X \longrightarrow W_{n+1}\widetilde{\Lambda}^i_X$ preserves the preweight filtration P.

(2): ([M1, 3.8] is incomplete.) We prove (2) by descending induction on the numbers of the preweight filtration P on $W_n\widetilde{\Lambda}^i_X$.

Let k be a positive integer less than or equal to $\min\{i, r+1\}$. If $k = \min\{i, r+1\}$, then $P_kW_n\widetilde{\Lambda}^i_X = W_n\widetilde{\Lambda}^i_X$, and there is nothing to prove. Assume that the projection $\pi\colon W_{n+1}\widetilde{\Lambda}^i_X \longrightarrow W_n\widetilde{\Lambda}^i_X$ induces a morphism $P_kW_{n+1}\widetilde{\Lambda}^i_X \longrightarrow P_kW_n\widetilde{\Lambda}^i_X$. By [M1, 3.7] and by the proof of (1), there exists the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{k-1}W_n\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_n\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_n\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0 \\ & & {\scriptstyle \mathcal{H}^i(\Phi^*/p^{i-1})}\big\downarrow & & {\scriptstyle \mathcal{H}^i(\Phi^*/p^{i-1})}\big\downarrow & & {\scriptstyle \mathcal{H}^{i-k}(\Phi^*/p^{i-k-1})}\big\downarrow & & \\ 0 & \longrightarrow & P_{k-1}W_{n+1}\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0. \end{array} \tag{8.4.1}$$

Obviously the following diagram is commutative:

$$\begin{array}{ccc} P_kW_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{i-k}_{X^{(k)}} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P_kW_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{i-k}_{X^{(k)}}. \end{array} \tag{8.4.2}$$

By the commutativity of (8.4.1) and (8.4.2), by the injectivity of $\mathbf{p}\colon W_n\Omega^{i-k}_{X^{(k)}} \longrightarrow W_{n+1}\Omega^{i-k}_{X^{(k)}}$ in the trivial log case of [Hy2, (2.2.2)] (cf. (6.8) (2), (6.28) (2)) and by the definition of $\pi\colon W_{n+1}\Omega^{i-k}_{X^{(k)}} \longrightarrow W_n\Omega^{i-k}_{X^{(k)}}$ in the case above of [Hy2, (1.3.2)], the following diagram

$$\begin{array}{ccc} P_kW_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{i-k}_{X^{(k)}} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ P_kW_n\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_n\Omega^{i-k}_{X^{(k)}} \end{array} \tag{8.4.3}$$

is commutative. Therefore we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_{k-1}W_{n+1}\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0 \\
 & & & & \pi\big\downarrow & & \pi\big\downarrow & & \\
0 & \longrightarrow & P_{k-1}W_n\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_n\widetilde{\Lambda}^i_X & \xrightarrow{\text{Res}} & W_n\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0.
\end{array}
\tag{8.4.4}
$$

Consequently the composite morphism $P_{k-1}W_{n+1}\widetilde{\Lambda}^i_X \overset{\subset}{\longrightarrow} P_kW_{n+1}\widetilde{\Lambda}^i_X \overset{\pi}{\longrightarrow} P_kW_n\widetilde{\Lambda}^i_X$ induces a morphism $\pi\colon P_{k-1}W_{n+1}\widetilde{\Lambda}^i_X \longrightarrow P_{k-1}W_n\widetilde{\Lambda}^i_X$. Thus we can finish the proof of (2). □

Remark 8.5. It is easy to see that $P_kW_n\widetilde{\Lambda}^i_X$ is a quasi-coherent sheaf of $W_n(\mathcal{O}_X)$-modules. By (6.28) (4) and by the upper exact sequence of (8.4.1) and by the descending induction on k, we see that $P_kW_n\widetilde{\Lambda}^i_X$ ($k \in \mathbb{N}$) is a coherent sheaf of $W_n(\mathcal{O}_X)$-modules.

Corollary 8.6. (1) *The diagrams* (8.0.3) *and* (8.0.4) *are commutative.*

(2) *Let i be a non-negative integer. Then the projection $\pi\colon W_{n+1}\widetilde{\Lambda}^\bullet_X \longrightarrow W_n\widetilde{\Lambda}^\bullet_X$ induces a morphism $\pi\colon W_{n+1}A^{i\bullet}_X \longrightarrow W_nA^{i\bullet}_X$ of complexes with boundary morphisms in* (4.1.3).

(3) *Let $W_nA^{\bullet\bullet}_X$ be the p-adic double Steenbrink complex in* (2.2.1; n). *Then the projection $\pi\colon W_{n+1}\widetilde{\Lambda}^\bullet_X \longrightarrow W_n\widetilde{\Lambda}^\bullet_X$ induces a morphism $\pi\colon W_{n+1}A^{\bullet\bullet}_X \longrightarrow W_nA^{\bullet\bullet}_X$ of double complexes.*

(4) *Let k be a non-negative integer. Then the morphism $\pi\colon P_kW_{n+1}\widetilde{\Lambda}^\bullet_X \longrightarrow P_kW_n\widetilde{\Lambda}^\bullet_X$ is surjective.*

(5) *Set $P_kW\widetilde{\Lambda}^\bullet_X := \varprojlim_n P_kW_n\widetilde{\Lambda}^\bullet_X$. Then the following sequence*

$$
0 \longrightarrow P_kW\widetilde{\Lambda}^\bullet_X \longrightarrow P_{k+1}W\widetilde{\Lambda}^\bullet_X \overset{\text{Res}}{\longrightarrow} W\Omega^\bullet_{X^{(k+1)}}\{-k-1\} \longrightarrow 0 \tag{8.6.1}
$$

is exact.

Proof. (1): (1) follows from the commutativity of (8.1.2) and from (8.4) (2).

(2): (2) immediately follows from a part of (1).

(3): The morphism π commutes with the boundary morphism $d\colon W_n\widetilde{\Lambda}^i_X \longrightarrow W_n\widetilde{\Lambda}^{i+1}_X$ ([Hy2, p. 245], cf. (6.8) (4)). Hence (3) follows from (2).

(4): We proceed by induction on k. The problem is local. Let $(\mathcal{Y},\mathcal{X})$ be an admissible lift of X. Let notations be as in the proof of (8.4), and set $Y := \mathcal{Y}_1$. Consider the following exact sequence

$$0 \longrightarrow \Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n) \longrightarrow \Omega^{\bullet}_{\mathcal{Y}_n/W_n} \longrightarrow \Omega^{\bullet}_{\mathcal{Y}_n/W_n}/\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n) \longrightarrow 0.$$

By [Hy2, Editorial comments (6)], the natural morphism

$$\begin{aligned} W_n\Omega^i_Y(-\log X) &= \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n)) \\ &\longrightarrow \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)) = W_n\Omega^i_Y(\log X) \end{aligned}$$

is injective. Since this morphism factors through the following morphism

$$\mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n)) \longrightarrow \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}), \tag{8.6.2}$$

the morphism (8.6.2) is also injective. Hence we have the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n)) &\longrightarrow \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}) \\ &\longrightarrow \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{Y}_n/W_n}/\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(-\log \mathcal{X}_n)) \longrightarrow 0. \end{aligned}$$

The exact sequence above is nothing but the following one:

$$0 \longrightarrow W_n\Omega^i_Y(-\log X) \longrightarrow W_n\Omega^i_Y \longrightarrow P_0W_n\widetilde{\Lambda}^i_X \longrightarrow 0. \tag{8.6.3}$$

Because the morphism $\pi\colon W_{n+1}\Omega^i \longrightarrow W_n\Omega^i_Y$ is surjective, so is the transition morphism $\pi\colon P_0W_{n+1}\widetilde{\Lambda}^i_X \longrightarrow P_0W_n\widetilde{\Lambda}^i_X$.

Let k be a positive integer. Because $\pi\colon W_{n+1}\Omega^{\bullet}_{X^{(k)}} \longrightarrow W_n\Omega^{\bullet}_{X^{(k)}}$ is surjective, the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{k-1}W_{n+1}\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_{n+1}\widetilde{\Lambda}^i_X & \longrightarrow & W_{n+1}\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0 \\ & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \\ 0 & \longrightarrow & P_{k-1}W_n\widetilde{\Lambda}^i_X & \longrightarrow & P_kW_n\widetilde{\Lambda}^i_X & \longrightarrow & W_n\Omega^{i-k}_{X^{(k)}} & \longrightarrow & 0 \end{array}$$

with exact rows and induction on k show (4).

(5): By (4), the projective system $\{P_kW_n\widetilde{\Lambda}^{\bullet}_X\}_n$ satisfies the Mittag-Leffler condition. Hence (8.6.1) is exact by (8.5). (The compatibility of (8.6.1) with the Frobenius will be obtained in (9.12) below.) □

Next let us consider the case of an open variety. Let (X, D) be a smooth (not necessarily proper) variety with an SNCD over κ. Then [Hy1, p. 301] (cf. (6.28) (2)) tells us that there exists a projection

$$\pi\colon W_{n+1}\Omega^i_X(\pm\log D) \longrightarrow W_n\Omega^i_X(\pm\log D) \quad (i \in \mathbb{N}). \tag{8.6.4}$$

The morphism π in [HK, (4.2)] in the case of the open variety above is equal to the morphism $\pi\colon W_{n+1}\Omega^i_X(\log D) \longrightarrow W_n\Omega^i_X(\log D)$ in (8.6.4) because π in [HK, (4.2)] satisfies a relation $\mathbf{p}\pi = p$ (cf. (6.28) (1), (6.4.6)), which is the characterization of π in [Hy1].

Proposition 8.7. *Let $\pi\colon W_{n+1}\Omega^i_X(\log D) \longrightarrow W_n\Omega^i_X(\log D)$ be the projection. Then the following hold*:

(1) *The morphism π preserves the preweight filtration P.*

(2) *The following diagram*

$$\begin{array}{ccc} P_kW_{n+1}\Omega^\bullet_X(\log D) & \xrightarrow{\text{Res}} & W_{n+1}\Omega^\bullet_{D^{(k)}}\{-k\} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ P_kW_n\Omega^\bullet_X(\log D) & \xrightarrow{\text{Res}} & W_n\Omega^\bullet_{D^{(k)}}\{-k\} \end{array} \tag{8.7.1}$$

is commutative.

(3) *The natural projection $\pi\colon P_kW_{n+1}\Omega^\bullet_X(\log D) \longrightarrow P_kW_n\Omega^\bullet_X(\log D)$ is surjective.*

(4) *Set $P_kW\Omega^\bullet_X(\log D) := \varprojlim_n P_kW_n\Omega^\bullet_X(\log D)$. Then the following sequence*

$$0 \longrightarrow P_{k-1}W\Omega^\bullet_X(\log D) \longrightarrow P_kW\Omega^\bullet_X(\log D) \xrightarrow{\text{Res}} W\Omega^\bullet_{D^{(k)}}\{-k\} \longrightarrow 0 \tag{8.7.2}$$

is exact.

Proof. (1): The proof is the same as that of (8.4) (2).

(2): Using [M1, 1.4.5] (cf. (9.0.1) below) and a relation $\mathbf{p}\pi = p$ in [Hy1, p. 301], we can give the same proof of (2) as that of the commutativity of (8.4.3).

(3): The proof of (3) is similar to that of (8.6) (4) and easier than it.

(4): (4) follows from (8.7.1), [M1, 1.4.5] and (3). (The compatibility of (8.7.2) with the Frobenius will be obtained in (9.6.2) below.) □

REMARK 8.8. By (6.28) (4) and by [M1, 1.4.5] and by the induction on k, we see that $P_kW_n\Omega^i_X(\log D)$ $(k \in \mathbb{N})$ is a coherent sheaf of $W_n(\mathcal{O}_X)$-modules.

9. Frobenius compatibility

In this section we prove the compatibility of the Frobenius in the finite length version of (2.0.1), in (4.1.1;n), in (5.0.2;n), and in (5.4) (2); the proofs of the former two are the same; the proofs of the latter two are much easier than those of the former two. Unexpectedly, the proofs of the former two is not easy because the easily defined operators F's on (log) de-Rham-Witt complexes due to the method of Katz-Illusie-Raynaud [IR, III (1.5)] are morphisms from W_n-modules to W_{n-1}-modules; we show the compatibility with the Frobenius from W_n-modules to themselves (cf. [I2, I (2.18.7)]).

For the time being, let the notations be as in §5. First, we clarify what has been proved and what has not. As to (5.0.2;n), Mokrane has shown in [M1, 1.4.5] that there exists an isomorphism

$$\mathrm{gr}_k^P W_n\Omega^\bullet_X(\log D) \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} W_n\Omega^\bullet_{D^{(k)}}\{-k\} \tag{9.0.1}$$

which makes the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{gr}_k^P W_{n+1}\Omega^\bullet_X(\log D) & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & W_{n+1}\Omega^\bullet_{D^{(k)}}\{-k\} \\
{\scriptstyle p^\bullet F}\big\downarrow & & \big\downarrow{\scriptstyle p^\bullet F} \\
\mathrm{gr}_k^P W_n\Omega^\bullet_X(\log D) & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & W_n\Omega^\bullet_{D^{(k)}}\{-k\}.
\end{array} \tag{9.0.2}$$

We prove (9.3.1) and (9.4) (2) below, which are stronger than the above.

We can find an analogous compatibility in [M1, 3.22 (2)]. In the proof of it, we find a Tate twist $(j+1)$ $(j \in \mathbb{N})$ in a pro-complex; this is misleading. For example, a sentence in [Ch2, p. 159, l. 5∼7] "The double complex $(W_nA^{\bullet\bullet}, d', d'')$ is endowed with a Frobenius endomorphism Φ_n defined on each W_nA^{ij} by the usual Frobenius twisted by p^{-j-1}" has non-sense since W_nA^{ij} is a torsion W-module; to give the definition of Φ_n in the sentence above is non-trivial. In (9.9) below we shall give the precise meaning of this sentence. Moreover, we have to check the compatibility of Φ_n with various operators.

Now let us prove the compatibility of Frobenius in the finite length version of (2.0.1), in (4.1.1;n), in (5.0.2;n), and in (5.4) (2). We first prove (5.0.2;n) and (5.4) (2).

The following is necessary for the proof of (5.0.2;n).

LEMMA 9.1. *Let $Z \longrightarrow (\mathrm{Spec}(\kappa), L)$ be a log smooth morphism of Cartier type of fine log schemes. Let $(W_n\Lambda^i_Z)''$ be the obverse log Hodge-Witt sheaf on $\overset{\circ}{Z}$ defined in §7 and let $s_n\colon (W_n\Lambda^i_Z)'' \overset{\sim}{\longrightarrow} W_n\Lambda^i_Z$ ($n \in \mathbb{Z}_{>0}$) be a canonical isomorphism defined in* (7.0.5) ((7.5)). *Then the following hold*:

(1) *The morphism s_n is functorial, that is, for a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \big\downarrow & & \big\downarrow \\ (\mathrm{Spec}(\kappa'), L') & \longrightarrow & (\mathrm{Spec}(\kappa), L) \end{array}$$

of fine log schemes, where κ' is a perfect field of characteristic $p > 0$ and where the left vertical morphism is a log smooth morphism of Cartier type, the following diagram is commutative:

$$\begin{array}{ccc} (W_n\Lambda^i_Z)'' & \underset{\sim}{\xrightarrow{s_n}} & W_n\Lambda^i_Z = R^i u_{Z/W_n *}(\mathcal{O}_{Z/W_n}) \\ {\scriptstyle g^*}\big\downarrow & & \big\downarrow{\scriptstyle g^*} \\ g_*(W'_n\Lambda^i_Y)'' & \underset{\sim}{\xrightarrow{g_*(s_n)}} & g_*W'_n\Lambda^i_Y = g_*R^i u_{Y/W'_n *}(\mathcal{O}_{Y/W'_n}). \end{array} \tag{9.1.1}$$

Here W'_n is the Witt ring of κ' of length n.

(2) (*Only for our memory*) *Let $\Phi''_n\colon (W_n\Lambda^i_Z)'' \longrightarrow (W_n\Lambda^i_Z)''$ and $\Phi_n\colon W_n\Lambda^i_Z \longrightarrow W_n\Lambda^i_Z$ be two morphisms induced by the absolute Frobenius endomorphism of Z. Then the following diagram is commutative*:

$$\begin{array}{ccc} (W_n\Lambda^i_Z)'' & \underset{\sim}{\xrightarrow{s_n}} & W_n\Lambda^i_Z \\ {\scriptstyle \Phi''_n}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_n} \\ (W_n\Lambda^i_Z)'' & \underset{\sim}{\xrightarrow{s_n}} & W_n\Lambda^i_Z. \end{array} \tag{9.1.2}$$

Proof. (1): By the general nonsense, the morphism g indeed induces a morphism

$$R^iu_{Z/W_n*}(\mathcal{O}_{Z/W_n}) \longrightarrow g_*R^iu_{Y/W'_n*}(\mathcal{O}_{Y/W'_n}).$$

Indeed, g^* induces a morphism

$$Ru_{Z/W_n*}(\mathcal{O}_{Z/W_n}) \longrightarrow Ru_{Z/W_n*}Rg^{\log}_{\mathrm{crys}*}(\mathcal{O}_{Y/W'_n}) = Rg_*Ru_{Y/W'_n*}(\mathcal{O}_{Y/W'_n}).$$

One can easily check that there exists a natural morphism

$$\mathcal{H}^i(Rg_*Ru_{Y/W'_n*}(\mathcal{O}_{Y/W'_n})) \longrightarrow g_*\mathcal{H}^i(Ru_{Y/W'_n*}(\mathcal{O}_{Y/W'_n})).$$

Thus we have a morphism $W_n\Lambda^i_Z \longrightarrow g_*W'_n\Lambda^i_Y$.

The commutativity of (9.1.1) is a local question. It is easy to check this commutativity by the local expression of s_n (cf. [HK, p. 251]) and by the existence of the local lift of $g\colon Y \longrightarrow Z$.

(2): (2) is a special case of (1). □

Lemma 9.2. (cf. [I2, I (2.18.7)]) *Let the notations be as in* (9.1). *Let* $\Phi''_{n,n+1}$ *be the following composite morphism*

$$(W_{n+1}\Lambda^i_Z)'' \overset{\Phi''_{n+1}}{\longrightarrow} (W_{n+1}\Lambda^i_Z)'' \overset{\mathrm{proj.}}{\longrightarrow} (W_n\Lambda^i_Z)''.$$

Then the following hold:

(1) *The following diagram is commutative*:

(9.2.1)
$$\begin{array}{ccc}
(W_{n+1}\Lambda^i_Z)'' & \xrightarrow[\sim]{s_{n+1}} & W_{n+1}\Lambda^i_Z \\
{\scriptstyle \Phi''_{n,n+1}}\downarrow & & \downarrow{\scriptstyle p^iF} \\
(W_n\Lambda^i_Z)'' & \xrightarrow[\sim]{s_n} & W_n\Lambda^i_Z.
\end{array}$$

(2) *The morphism* Φ_n *in* (9.1.2) *fits into the following diagram commutative*:

(9.2.2)
$$\begin{array}{ccc}
W_{n+1}\Lambda^i_Z & \xrightarrow{\pi} & W_n\Lambda^i_Z \\
{\scriptstyle p^iF}\downarrow & & \downarrow{\scriptstyle \Phi_n} \\
W_n\Lambda^i_Z & = & W_n\Lambda^i_Z.
\end{array}$$

(3) *The morphisms* Φ_{n+1} *and* Φ_n *fit into the following commutative diagram*:

$$
\begin{array}{ccc}
W_{n+1}\Lambda^i_Z & \xrightarrow{\Phi_{n+1}} & W_{n+1}\Lambda^i_Z \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
W_n\Lambda^i_Z & \xrightarrow{\Phi_n} & W_n\Lambda^i_Z .
\end{array}
\tag{9.2.3}
$$

Proof. The proof is an analogue of (7.1):

(1): The question is local. As in (7.1), we may assume that a formally log smooth lift $\mathcal{Z}$ of Z over $(\mathrm{Spf}(W), W(L))$ exists. Set $\mathcal{Z}_n := \mathcal{Z} \otimes_W W_n$. Let the notations be as in (7.1). Since the morphism $F\colon W_{n+1}\Lambda^0_Z \longrightarrow W_n\Lambda^0_Z$ is induced by the projection $\Lambda^\bullet_{n+1} \longrightarrow \Lambda^\bullet_n$, we obtain the following commutative diagram by a simple calculation:

$$
\begin{array}{ccc}
W_{n+1}(\mathcal{O}_Z) & \xrightarrow{s_{n+1}(0,0)} & W_{n+1}\Lambda^0_Z \\
{\scriptstyle \Phi''_{n,n+1}}\big\downarrow & & \big\downarrow{\scriptstyle F} \\
W_n(\mathcal{O}_Z) & \xrightarrow{s_n(0,0)} & W_n\Lambda^0_Z .
\end{array}
\tag{9.2.4}
$$

Because $p\widetilde{a}_n^{p-1}d\widetilde{a}_n$ is an exact form, we have

$$
\begin{aligned}
pF\circ s_{n+1}(1,0)(a_0,\ldots,a_n) &= p\mathcal{H}^1(\mathrm{proj.})\Big(\sum_{i=0}^{n}\widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i\Big) \\
&= p\sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i + p\widetilde{a}_n^{p-1}d\widetilde{a}_n \\
&= p\sum_{i=0}^{n-1}\widetilde{a}_i^{p^{n+1-i}-1}d\widetilde{a}_i
\end{aligned}
$$

in $\mathcal{H}^1(\Lambda^\bullet_{n+1})$. By this formula, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
W_{n+1}(\mathcal{O}_Z) & \xrightarrow{s_{n+1}(1,0)} & W_{n+1}\Lambda^1_Z \\
{\scriptstyle \Phi''_{n,n+1}}\big\downarrow & & \big\downarrow{\scriptstyle pF} \\
W_n(\mathcal{O}_Z) & \xrightarrow{s_n(1,0)} & W_n\Lambda^1_Z .
\end{array}
\tag{9.2.5}
$$

By (9.2.4), (9.2.5) and by the definition of the morphism s_n ((7.0.5), cf. [HK, (4.9)]), we obtain (1) as in (7.1).

(2): The commutativity of (9.2.2) follows from that of (7.1), (9.1.2) and (9.2.1).

(3): The commutativity of (9.2.3) immediately follows from the commutativity of (7.1) and (9.1.2). □

Let the notations be as in §5. The following (1) is the precise content of (5.0.2;n):

PROPOSITION 9.3. (1) *Let* (X, D) *be a smooth scheme with an SNCD over* κ. *The Frobenius endomorphism* $\Phi_n\colon W_n\Omega^\bullet_X(\log D) \longrightarrow W_n\Omega^\bullet_X(\log D)$ *defined in* (9.1) (2) *preserves the preweight filtration* P *and makes the following diagram commutative*:

$$
\begin{array}{ccc}
\mathrm{gr}^P_k W_n\Omega^\bullet_X(\log D) & \xrightarrow[\sim]{\mathrm{Res}} & W_n\Omega^\bullet_{D^{(k)}}\{-k\} \\
{\scriptstyle \mathrm{gr}^P_k(\Phi_n)}\downarrow & & \downarrow{\scriptstyle p^k\Phi_n} \\
\mathrm{gr}^P_k W_n\Omega^\bullet_X(\log D) & \xrightarrow[\sim]{\mathrm{Res}} & W_n\Omega^\bullet_{D^{(k)}}\{-k\}.
\end{array}
\tag{9.3.1}
$$

Consequently there exists the following spectral sequence (*cf.* [M2, (3.1)]: *the convergent term* $H^{2i+j}_{\mathrm{cris}}(U/W)$ *in* [loc. cit.] *has to be replaced by* $H^{i+j}_{\mathrm{cris}}(U/W)$.) :

$$
\begin{aligned}
E_1^{-k,h+k} &= H^{h-k}_{\mathrm{crys}}(D^{(k)}/W_n)(-k) \\
&\Longrightarrow H^h((X,D)/W_n) \quad (n \in \mathbb{Z}_{>0}).
\end{aligned}
\tag{9.3.2}
$$

As to the preservation, the following holds more generally:

(2) *The morphism* g^* *in* (9.1.1) *for the log Hodge-Witt sheaves of smooth schemes with NCD's over* κ *preserves the preweight filtration.*

PROOF. (1): By the same proof as that of (8.4) (1), it follows that the endomorphism $\Phi_n\colon W_n\Omega^\bullet_X(\log D) \longrightarrow W_n\Omega^\bullet_X(\log D)$ preserves the preweight filtration P.

Next, we prove the commutativity of (9.3.1). This is a local question. Let $(\mathcal{X}, \mathcal{D})/\mathrm{Spf}(W)$ be a formally log smooth lift of $(X, D)/\mathrm{Spec}(\kappa)$. Set $(\mathcal{X}_n, \mathcal{D}_n) := (\mathcal{X}, \mathcal{D}) \otimes_W W_n$ ($n \in \mathbb{N}$). Assume that $\mathcal{X}$ is formally etale over

$\mathrm{Spf}(W\{x_1,\ldots,x_d\})$ and $\mathcal{D}$ is defined by an equation $x_1\cdots x_r=0$ $(1\leq r\leq d)$. Let $\Phi\colon(\mathcal{X},\mathcal{D})\longrightarrow(\mathcal{X},\mathcal{D})$ be a lift of the Frobenius endomorphism of (X,D). Then, as in the proof of (8.1), $\Phi^*(d\log x_i)$ $(1\leq i\leq r)$ is equivalent to $p d\log x_i$ in $W_n\Omega^1_X(\log D)=\mathcal{H}^1(\Omega^\bullet_{\mathcal{X}_n/W_n}(\log\mathcal{D}_n))$. By the definition of Φ, Φ induces a morphism $\mathcal{D}^{(k)}\longrightarrow\mathcal{D}^{(k)}$ which is a lift of the Frobenius of $D^{(k)}$. Hence we have the commutative diagram (9.3.1) by (9.0.1) due to Mokrane.

(2): We can prove (2) in a similar way: the number of log poles in a logarithmic differential form does not increase by the pull-back of a local lift of a morphism of smooth schemes with NCD's over κ. □

Let us also prove that the two pairings in (5.3) are compatible with the Frobenius. First, we have to define the Frobenius endomorphism of $W_n\Omega^i_X(-\log D)$. This is essentially given in [Hy1] and [Hy2, p. 245]: if we are given a triple $(\mathcal{X},\mathcal{D},\Phi)$ as in the proof of (9.3) (1), we define the Frobenius endomorphism to be the induced morphism $\Phi_n=\mathcal{H}^i(\Phi^*)$ on $W_n\Omega^i_X(-\log D)=\mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_n/W_n}(-\log\mathcal{D}_n))$ by Φ; this morphism is independent of the choice of the lift $(\mathcal{X},\mathcal{D},\Phi)$ by the product construction as explained in [Hy1, 2] (see also (9.5) (1) below for another method to define the Frobenius endomorphism).

Proposition 9.4. (1) *The two pairings in* (5.3) *are compatible with the Frobenius.*

(2) *An equality* $\Phi_n\circ\pi=p^iF\colon W_{n+1}\Omega^i_X(\pm\log D)\longrightarrow W_n\Omega^i_X(\pm\log D)$ *holds.*

(3) *The morphisms* Φ_{n+1} *and* Φ_n *fit into the following commutative diagram:*

$$\begin{array}{ccc} W_{n+1}\Omega^i_X(\pm\log D) & \xrightarrow{\Phi_{n+1}} & W_{n+1}\Omega^i_X(\pm\log D)\\ \pi\downarrow & & \downarrow\pi\\ W_n\Omega^i_X(\pm\log D) & \xrightarrow{\Phi_n} & W_n\Omega^i_X(\pm\log D). \end{array} \tag{9.4.1;±}$$

Proof. The questions are local. Let the notations be as in (9.3) (1).
(1): The pairing

$$W_n\Omega^i_X(\log D)\otimes_{W_n}W_n\Omega^{d-i}_X(-\log D)\longrightarrow W_n\Omega^d_X(-\log D)=W_n\Omega^d_X$$

is compatible with the Frobenius. Here the last equality $W_n\Omega^d_X(-\log D) = W_n\Omega^d_X$ follows from (6.28) (9). Since the isomorphism $s_n^{-1}\colon W_n\Omega^d_X \xrightarrow{\sim} (W_n\Omega^d_X)''$ is compatible with the Frobenius ((9.1.2)), the cup product

$$H^j(X, W_n\Omega^i_X(\log D)) \otimes_{W_n} H^{d-j}(X, W_n\Omega^{d-i}_X(-\log D)) \longrightarrow W_n(-d)$$

is compatible with the Frobenius. Thus the pairing (5.3.1) is compatible with the Frobenius. Similarly, the natural wedge product

$$W_n\Omega^\bullet_X(\log D) \otimes_{W_n} W_n\Omega^\bullet_X(-\log D) \longrightarrow W_n\Omega^\bullet_X(-\log D)$$

is compatible with the Frobenius. Hence the compatibility of the pairing (5.3.2) with the Frobenius follows as above.

(2): Though the case for $W_m\Omega^i_X(\log D)$ $(m = n, n+1)$ is a special case of the commutativity of (9.2.2), we prove (2) for $W_m\Omega^i_X(\pm\log D)$ at the same time for notational reason. First, assume that $i = 0$. Then (2) is a special case of the commutativity of (9.2.2). Next, assume that $i > 0$. By the definition of the morphism $\mathbf{p}\colon W_n\Omega^i_X(\pm\log D) \longrightarrow W_{n+1}\Omega^i_X(\pm\log D)$ ([Hy1, p. 301]), we have an equality

$$\begin{aligned}\Phi_n = p^{i-1}\mathbf{p}\circ\mathcal{H}^i(\text{proj.})\colon &\mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_n/W_n}(\pm\log\mathcal{D}_n)) \\ &\longrightarrow \mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_{n-1}/W_{n-1}}(\pm\log\mathcal{D}_{n-1})) \\ &\longrightarrow \mathcal{H}^i(\Omega^\bullet_{\mathcal{X}_n/W_n}(\pm\log\mathcal{D}_n)).\end{aligned}$$

Since $\mathbf{p}\circ\mathcal{H}^i(\text{proj.}) = \mathcal{H}^i(\text{proj.})\circ\mathbf{p}$, (2) follows from an equality $\mathbf{p}\circ\pi = p$ ([Hy1, p. 301], cf. (6.28) (1)).

(3): The problem is local. By (2) and by the surjectivity of the morphism $\pi\colon W_{n+2}\Omega^i_X(\pm\log D) \longrightarrow W_{n+1}\Omega^i_X(\pm\log D)$, we have only to prove that the following diagram is commutative:

(9.4.2;±)
$$\begin{array}{ccc} W_{n+2}\Omega^i_X(\pm\log D) & \xrightarrow{\mathcal{H}^i(\text{proj.})} & W_{n+1}\Omega^i_X(\pm\log D) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W_{n+1}\Omega^i_X(\pm\log D) & \xrightarrow{\mathcal{H}^i(\text{proj.})} & W_n\Omega^i_X(\pm\log D). \end{array}$$

By the same proof as that of (6.8) (4), we obtain the commutativity of (9.4.2;±). □

REMARK 9.5. (1) Let M be the log structure defined by an NCD D on X:

$$M := \{f \in \mathcal{O}_X | f \text{ is invertible outside } D\}.$$

Let us consider the log crystalline site $((X,M)/(W_n, W_n^*))^{\log}_{\text{crys}}$. Let $(U, T, \iota, M_T, \delta)$ be an object of $((X,M)/(W_n, W_n^*))^{\log}_{\text{crys}}$. Because $\iota\colon (U, M|_U) \longrightarrow (T, M_T)$ is an exact closed immersion, $M_T/\mathcal{O}_T^* = M_U/\mathcal{O}_U^*$ on $U_{\text{zar}} = T_{\text{zar}}$; hence the defining equation of the SNCD divisor $D \cap U$ in U lifts locally to a section of M_T. We define the ideal sheaf $\mathcal{I}_{X/W_n} \subset \mathcal{O}_{X/W_n}$ by the following: $\mathcal{I}_{X/W_n}(T)=$ ((the ideal generated by the image of the local section above by the structural morphism $M_T \longrightarrow \mathcal{O}_T$)). The sheaf $\mathcal{I}_{X/W_n}$ is a special case of a sheaf defined in [T, §5] ($\mathcal{I}_{X/W_n}$ is denoted by $\mathcal{K}_{X/W_n}$ in [loc. cit.]). By this remark, we see that $\mathcal{I}_{X/W_n}$ is a crystal on $((X,M)/(W_n, W_n^*))^{\log}_{\text{crys}}$ (cf. [T, (5.3)]). Let $u_{X/W_n}\colon \widetilde{((X,M)/(W_n, W_n^*))}^{\log}_{\text{crys}} \longrightarrow \widetilde{X}_{\text{zar}}$ be the projection. If (X, D) lifts to a smooth scheme $\mathcal{X}_n/W_n$ with a relative SNCD $\mathcal{D}_n$ over W_n, then $R^i u_{X/W_n*}(\mathcal{I}_{X/W_n}) = \mathcal{H}^i(\Omega^{\bullet}_{\mathcal{X}_n/W_n}(-\log \mathcal{D}_n))$ by [Ka2, (6.9)]. Hence, by [Hy1, p. 301],

$$R^i u_{X/W_n*}(\mathcal{I}_{X/W_n}) = W_n\Omega^i_X(-\log D). \tag{9.5.1}$$

In particular, (9.5.1) also tells us that $W_n\Omega^i_X(-\log D)$ is independent of the choice of $(\mathcal{X}_n, \mathcal{D}_n)$. Using (9.5.1), we also see that Φ_n on $W_n\Omega^i_X(-\log D)$ is independent of the triple $(\mathcal{X}, \mathcal{D}, \Phi)$ in the proof of (9.3) (1).

(2) Let (X, D) be a smooth scheme with SNCD over κ with structural morphism $f\colon X \longrightarrow W_n$. Let $(\widetilde{X/W_n})_{\text{crys}}$ be the classical crystalline topos of $X/(W_n, pW_n, [\ ])$ and $\overset{\circ}{u}_{X/W_n}\colon (\widetilde{X/W_n})_{\text{crys}} \longrightarrow \widetilde{X}_{\text{zar}}$ the classical projection of topoi. Let $\overset{\circ}{\mathcal{O}}_{X/W_n}$ be the structure sheaf in $(\widetilde{X/W_n})_{\text{crys}}$. Let $a^{(k)}\colon D^{(k)} \longrightarrow X$ be the natural morphism. In [NS] and [Nakk6], as a special case, we have obtained the following commutative diagram:

$$\begin{array}{ccc} Ru_{X/W_n*}(\mathcal{I}_{X/W_n}) & \xrightarrow{\sim} & \\ \simeq\downarrow & & \\ W_n\Omega^{\bullet}_X(-\log D) & \longrightarrow & \end{array} \tag{9.5.2}$$

$$\begin{array}{c}[R\overset{\circ}{u}_{X/W_n*}(\overset{\circ}{\mathcal{O}}_{X/W_n}) \longrightarrow (a^{(1)}_{*}Ru_{D^{(1)}/W_n*}(\mathcal{O}_{D^{(1)}/W_n}), -d) \longrightarrow \cdots] \\ \big\downarrow \simeq \\ [W_n\Omega^{\bullet}_{X} \longrightarrow (a^{(1)}_{*}W_n\Omega^{\bullet}_{D^{(1)}}, -d) \longrightarrow \cdots].\end{array}$$

Here the upper right hand side of (9.5.2) is, by definition, the single complex of $\overset{\circ}{u}_{X/W_n*}(I^{\bullet\bullet})$ in $\mathrm{D}^+(f^{-1}(W_n))$, where $I^{\bullet\bullet}$ is a double complex of $\overset{\circ}{\mathcal{O}}_{X/W_n}$-modules such that, for each nonnegative integer k, $I^{k\bullet}$ is a $\overset{\circ}{u}_{X/W_n*}$-acyclic resolution of $(a^{(k)}_{\mathrm{crys}*}(\mathcal{O}_{D^{(k)}/W_n}), (-1)^k d)$. In particular, the canonical morphism

$$W_n\Omega^{\bullet}_{X}(-\log D) \longrightarrow [W_n\Omega^{\bullet}_{X} \longrightarrow (a^{(1)}_{*}W_n\Omega^{\bullet}_{D^{(1)}}, -d) \longrightarrow \cdots].$$

is a quasi-isomorphism. In the arguments above, we do not use Ekedahl's Nakayama lemma ([Ek2, I (1.1.3)]).

Let Y be an SNCL variety over the log point s. By the above and by the same argument as that of [M1, 3.15], we can give another proof of the fact that the following canonical morphism

$$\theta_n\wedge\colon W_n\Lambda^{\bullet}_{Y} \longrightarrow W_nA^{\bullet}_{Y}$$

is a quasi-isomorphism (cf. (6.29) (1)).

Corollary 9.6. (1) *The following diagram*

$$(9.6.1)\qquad \begin{array}{ccc}\mathrm{gr}^P_k W_{n+1}\Omega^{\bullet}_{X}(\log D) & \xrightarrow[\sim]{\mathrm{Res}} & W_{n+1}\Omega^{\bullet}_{D^{(k)}}(-k)\{-k\} \\ \pi\big\downarrow & & \big\downarrow\pi \\ \mathrm{gr}^P_k W_n\Omega^{\bullet}_{X}(\log D) & \xrightarrow[\sim]{\mathrm{Res}} & W_n\Omega^{\bullet}_{D^{(k)}}(-k)\{-k\}\end{array}$$

is commutative.

(2) *The exact sequence* (8.7.2) *is compatible with the Frobenius in the following sense*: *the following sequence*

$$(9.6.2)\qquad 0 \longrightarrow P_{k-1}W\Omega^{\bullet}_{X}(\log D) \longrightarrow P_kW\Omega^{\bullet}_{X}(\log D) \xrightarrow{\mathrm{Res}} W\Omega^{\bullet}_{D^{(k)}}(-k)\{-k\} \longrightarrow 0$$

is exact.

PROOF. (1) follows from (8.7.1), the commutativity of (9.4.1;+) and that of (9.3.1). (2) follows from (1) and (8.7.2). □

Now we come back to the SNCL case.

Let the notations be as in the case of the SNCL variety in §8. We can avoid the obscure point in the proof in [M1, 3.22 (2)] by (9.7.3) below: it is not necessary to give the meaning of the Tate twist $(j+1)$ in [loc. cit.] if one considers only the pro-system $W_\bullet A_X^{\bullet\bullet}$; later, in (9.9) below, we shall consider the compatibility of $W_n A_X^{\bullet\bullet}$ with the Frobenius for a positive integer n. We shall use (9.7.3) in the proof of (9.9).

PROPOSITION 9.7. *Let i be a non-negative integer. Then the following hold*:

(1) *The following diagram is commutative*:

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}_X^{i+1} & \xrightarrow{p^iF} & W_n\widetilde{\Lambda}_X^{i+1} \\ {\scriptstyle\theta_{n+1}\wedge}\uparrow & & \uparrow{\scriptstyle\theta_n\wedge} \\ W_{n+1}\widetilde{\Lambda}_X^{i} & \xrightarrow{p^iF} & W_n\widetilde{\Lambda}_X^{i}. \end{array} \tag{9.7.1}$$

(2) *Let $W_nA_X^{i\bullet}$ be the complex defined in* (4.1.3). *For the positive integers n's, the canonical quasi-isomorphisms $W_n\Lambda_X^i \xrightarrow{\theta_n\wedge} W_nA_X^{i\bullet}$* ([M1, 3.15], (6.28) (9), (6.29) (1)) *make the following diagram commutative*:

$$\begin{array}{ccc} W_{n+1}A_X^{i\bullet} & \xrightarrow{p^iF} & W_nA_X^{i\bullet} \\ {\scriptstyle\theta_{n+1}\wedge}\uparrow & & \uparrow{\scriptstyle\theta_n\wedge} \\ W_{n+1}\Lambda_X^{i} & \xrightarrow{p^iF} & W_n\Lambda_X^{i}. \end{array} \tag{9.7.2}$$

(3) (*cf.* [M1, (3.22) (2)]) *Let k be an integer. The Poincaré residue isomorphisms*

$$\mathrm{Res}\colon \mathrm{gr}_k^P W_nA_X^{i\bullet} \xrightarrow{\sim} \bigoplus_{j\geq\max\{-k,0\}} W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-j\} \quad (n\in\mathbb{Z}_{>0})$$

make the following diagram commutative:

$$
(9.7.3) \qquad \begin{array}{ccc}
\mathrm{gr}_k^P W_{n+1}A_X^{i\bullet} & \xrightarrow[\sim]{\mathrm{Res}} & \bigoplus_{j\geq \max\{-k,0\}} W_{n+1}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-j\} \\
{\scriptstyle p^iF}\big\downarrow & & \big\downarrow{\scriptstyle p^{j+k}(p^{i-j-k}F)} \\
\mathrm{gr}_k^P W_nA_X^{i\bullet} & \xrightarrow[\sim]{\mathrm{Res}} & \bigoplus_{j\geq \max\{-k,0\}} W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-j\}.
\end{array}
$$

Proof. The questions are local. Let the notations be as in the proof of (8.4).

(1): Because the morphism $F\colon W_{n+1}\widetilde{\Lambda}^i_X = \mathcal{H}^i(\widetilde{\Lambda}^\bullet_{n+1}) \longrightarrow W_n\widetilde{\Lambda}^i_X = \mathcal{H}^i(\widetilde{\Lambda}^\bullet_n)$ is induced by the projection proj.: $\widetilde{\Lambda}^\bullet_{n+1} \longrightarrow \widetilde{\Lambda}^\bullet_n$ ([Hy2, (1.3)]), (1) is obvious.

(2): By the definitions of F's in [Hy2, (1.3)] and in [M1, 3.8], (2) immediately follows as in (1).

(3): For a positive integer l, let $\mathcal{X}^{(l)}$ be the disjoint union of all l-fold intersections of the distinct irreducible components of the scheme $\mathcal{X}$. Let j be a non-negative integer such that $j \geq -k$. By [M1, 3.7], we have an isomorphism

$$
\mathrm{Res}\colon \mathrm{gr}^P_{2j+k+1}\mathcal{H}^{i+j+1}(\widetilde{\Lambda}^\bullet_n) \xrightarrow{\sim} \mathcal{H}^{i-j-k}(\Omega^\bullet_{\mathcal{X}_n^{(2j+k+1)}/W_n}).
$$

(3) follows from the following obvious commutative diagram

$$
\begin{array}{ccc}
\mathrm{gr}^P_{2j+k+1}\mathcal{H}^{i+j+1}(\widetilde{\Lambda}^\bullet_{n+1}) & \xrightarrow[\sim]{\mathrm{Res}} & \mathcal{H}^{i-j-k}(\Omega^\bullet_{\mathcal{X}_{n+1}^{(2j+k+1)}/W_{n+1}}) \\
{\scriptstyle \mathcal{H}^{i+j+1}(p^i\mathrm{proj.})}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{H}^{i-j-k}(p^i\mathrm{proj.})} \\
\mathrm{gr}^P_{2j+k+1}\mathcal{H}^{i+j+1}(\widetilde{\Lambda}^\bullet_n) & \xrightarrow[\sim]{\mathrm{Res}} & \mathcal{H}^{i-j-k}(\Omega^\bullet_{\mathcal{X}_n^{(2j+k+1)}/W_n}).
\end{array}
$$

□

The following tells us that the Frobenius on torsion p-adic Steenbrink complexes can be constructed by the method of Katz-Illusie-Raynaud ([IR, III (1.5)]).

THEOREM 9.8. *Let n, k be two positive integers and j a non-negative integer. Then the following hold*:

(1) *There exists a unique morphism $\Psi_n^{(k;j)}\colon W_n\Lambda^j_X \longrightarrow W_n\Lambda^j_X$ which makes the following diagram commutative*:

$$
\begin{array}{ccc}
W_{n+1}\Lambda^j_X & \xrightarrow{\pi} & W_n\Lambda^j_X \\
{\scriptstyle p^kF}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_n^{(k;j)}} \\
W_n\Lambda^j_X & = & W_n\Lambda^j_X.
\end{array} \tag{9.8.1}
$$

The morphisms $\Psi_n^{(k;j)}$ and $\Psi_n^{(k+1;j+1)}$ fit into the following commutative diagram:

$$
\begin{array}{ccc}
W_n\Lambda^j_X & \xrightarrow{d} & W_n\Lambda^{j+1}_X \\
{\scriptstyle \Psi_n^{(k;j)}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_n^{(k+1;j+1)}} \\
W_n\Lambda^j_X & \xrightarrow{d} & W_n\Lambda^{j+1}_X.
\end{array} \tag{9.8.2}
$$

The morphism $\Psi_n^{(k;k)}\colon W_n\Lambda^k_X \longrightarrow W_n\Lambda^k_X$ is equal to $\Phi_n\colon W_n\Lambda^k_X \longrightarrow W_n\Lambda^k_X$ in (9.1.2). *The morphisms $\Psi_{n+1}^{(k;j)}$ and $\Psi_n^{(k;j)}$ fit into the following commutative diagram*:

$$
\begin{array}{ccc}
W_{n+1}\Lambda^j_X & \xrightarrow{\Psi_{n+1}^{(k;j)}} & W_{n+1}\Lambda^j_X \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
W_n\Lambda^j_X & \xrightarrow{\Psi_n^{(k;j)}} & W_n\Lambda^j_X.
\end{array} \tag{9.8.3}
$$

(2) *There exists a unique morphism $\widetilde{\Phi}_n^{(0j)}\colon W_nA^{0j}_X \longrightarrow W_nA^{0j}_X$ which makes the following diagram commutative*:

$$
\begin{array}{ccc}
W_{n+1}A^{0j}_X & \xrightarrow{\pi} & W_nA^{0j}_X \\
{\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\Phi}_n^{(0j)}} \\
W_nA^{0j}_X & = & W_nA^{0j}_X.
\end{array} \tag{9.8.4}
$$

The morphisms $\widetilde{\Phi}_n^{(0\bullet)}$ $(\bullet \geq 0)$ *make the following two diagrams commutative*:

$$
\begin{array}{ccc}
W_nA_X^{0,j+1} & \xrightarrow{\widetilde{\Phi}_n^{(0,j+1)}} & W_nA_X^{0,j+1} \\
{\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_n\wedge} \\
W_nA_X^{0j} & \xrightarrow{\widetilde{\Phi}_n^{(0j)}} & W_nA_X^{0j},
\end{array}
\tag{9.8.5}
$$

$$
\begin{array}{ccc}
W_{n+1}A_X^{0j} & \xrightarrow{\widetilde{\Phi}_{n+1}^{(0j)}} & W_{n+1}A_X^{0j} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
W_nA_X^{0j} & \xrightarrow{\widetilde{\Phi}_n^{(0j)}} & W_nA_X^{0j}.
\end{array}
\tag{9.8.6}
$$

(3) *There exists a unique morphism* $\widetilde{\Psi}_n^{(k;j)} \colon W_n\widetilde{\Lambda}_X^j \longrightarrow W_n\widetilde{\Lambda}_X^j$ *which makes the following diagram commutative*:

$$
\begin{array}{ccc}
W_{n+1}\widetilde{\Lambda}_X^j & \xrightarrow{\pi} & W_n\widetilde{\Lambda}_X^j \\
{\scriptstyle p^kF}\downarrow & & \downarrow{\scriptstyle \widetilde{\Psi}_n^{(k;j)}} \\
W_n\widetilde{\Lambda}_X^j & = & W_n\widetilde{\Lambda}_X^j.
\end{array}
\tag{9.8.7}
$$

The family $\{\widetilde{\Psi}_n^{(k;\bullet)}\}$ $(\bullet \geq 0, k \geq 1)$ *makes the following three diagrams commutative*:

$$
\begin{array}{ccc}
W_n\widetilde{\Lambda}_X^j & \xrightarrow{d} & W_n\widetilde{\Lambda}_X^{j+1} \\
{\scriptstyle \widetilde{\Psi}_n^{(k;j)}}\downarrow & & \downarrow{\scriptstyle \widetilde{\Psi}_n^{(k+1;j+1)}} \\
W_n\widetilde{\Lambda}_X^j & \xrightarrow{d} & W_n\widetilde{\Lambda}_X^{j+1},
\end{array}
\tag{9.8.8}
$$

$$
\begin{array}{ccc}
W_n\widetilde{\Lambda}_X^{j+1} & \xrightarrow{\widetilde{\Psi}_n^{(k;j+1)}} & W_n\widetilde{\Lambda}_X^{j+1} \\
{\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_n\wedge} \\
W_n\widetilde{\Lambda}_X^j & \xrightarrow{\widetilde{\Psi}_n^{(k;j)}} & W_n\widetilde{\Lambda}_X^j,
\end{array}
\tag{9.8.9}
$$

$$\begin{array}{ccc} W_n\widetilde{\Lambda}^{j+1}_X & \xrightarrow{\widetilde{\Psi}^{(k;j+1)}_n} & W_n\widetilde{\Lambda}^{j+1}_X \\ {\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_n\wedge} \\ W_n\Lambda^j_X & \xrightarrow{\Psi^{(k;j)}_n} & W_n\Lambda^j_X. \end{array} \tag{9.8.10}$$

The morphisms $\widetilde{\Psi}^{(k;j)}_{n+1}$ *and* $\widetilde{\Psi}^{(k;j)}_n$ *fit into the following commutative diagram*:

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}^{j}_X & \xrightarrow{\widetilde{\Psi}^{(k;j)}_{n+1}} & W_{n+1}\widetilde{\Lambda}^{j}_X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ W_n\widetilde{\Lambda}^j_X & \xrightarrow{\widetilde{\Psi}^{(k;j)}_n} & W_n\widetilde{\Lambda}^j_X. \end{array} \tag{9.8.11}$$

(4) *The morphism* $\widetilde{\Psi}^{(k;j)}_n$ *preserves the preweight filtration* P *on* $W_n\widetilde{\Lambda}^j_X$; *for an integer* $i \geq 1$, $\widetilde{\Psi}^{(i;i+\bullet+1)}_n \colon W_n\widetilde{\Lambda}^{i+\bullet+1}_X \longrightarrow W_n\widetilde{\Lambda}^{i+\bullet+1}_X$ *induces an endomorphism*

$$\widetilde{\Phi}^{(i\bullet)}_n \colon W_nA^{i\bullet}_X \longrightarrow W_nA^{i\bullet}_X$$

of complexes.

(5) *Let* i *be a positive integer. The family* $\{\widetilde{\Phi}^{(ij)}_n\}_{i\geq 1, j\geq 0}$ *makes the following four diagrams commutative*:

$$\begin{array}{ccc} W_{n+1}A^{ij}_X & \xrightarrow{\pi} & W_nA^{ij}_X \\ {\scriptstyle p^iF}\downarrow & & \downarrow{\scriptstyle \widetilde{\Phi}^{(ij)}_n} \\ W_nA^{ij}_X & = & W_nA^{ij}_X, \end{array} \tag{9.8.12}$$

$$\begin{array}{ccc} W_nA^{ij}_X & \xrightarrow{(-1)^{j+1}d} & W_nA^{i+1,j}_X \\ {\scriptstyle \widetilde{\Phi}^{(ij)}_n}\downarrow & & \downarrow{\scriptstyle \widetilde{\Phi}^{(i+1,j)}_n} \\ W_nA^{ij}_X & \xrightarrow{(-1)^{j+1}d} & W_nA^{i+1,j}_X, \end{array} \tag{9.8.13}$$

$$\begin{array}{ccc} W_nA^{i,j+1}_X & \xrightarrow{\widetilde{\Phi}^{(i,j+1)}_n} & W_nA^{i,j+1}_X \\ {\scriptstyle (-1)^i\theta_n\wedge}\uparrow & & \uparrow{\scriptstyle (-1)^i\theta_n\wedge} \\ W_nA^{ij}_X & \xrightarrow{\widetilde{\Phi}^{(ij)}_n} & W_nA^{ij}_X, \end{array} \tag{9.8.14}$$

$$\begin{array}{ccc} W_{n+1}A_X^{ij} & \xrightarrow{\widetilde{\Phi}{}^{(ij)}_{n+1}} & W_{n+1}A_X^{ij} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ W_nA_X^{ij} & \xrightarrow{\widetilde{\Phi}{}^{(ij)}_{n}} & W_nA_X^{ij}. \end{array} \tag{9.8.15}$$

(6) *The following diagram is commutative*:

$$\begin{array}{ccc} W_nA_X^{0j} & \xrightarrow{(-1)^{j+1}d} & W_nA_X^{1j} \\ {\scriptstyle \widetilde{\Phi}{}^{(0j)}_{n}}\downarrow & & \downarrow{\scriptstyle \widetilde{\Phi}{}^{(1j)}_{n}} \\ W_nA_X^{0j} & \xrightarrow{(-1)^{j+1}d} & W_nA_X^{1j}. \end{array} \tag{9.8.16}$$

(7) *The following diagram is commutative*:

$$\begin{array}{ccc} W_n\widetilde{\Lambda}_X^j & \xrightarrow{\widetilde{\Psi}{}^{(k;j)}_{n}} & W_n\widetilde{\Lambda}_X^j \\ \downarrow & & \downarrow \\ W_n\Lambda_X^j = W_n\widetilde{\Lambda}_X^j/(\theta_n \wedge W_n\widetilde{\Lambda}_X^{j-1}) & \xrightarrow{\Psi^{(k;j)}_{n}} & W_n\Lambda_X^j = W_n\widetilde{\Lambda}_X^j/(\theta_n \wedge W_n\widetilde{\Lambda}_X^{j-1}). \end{array} \tag{9.8.17}$$

Proof. Let Z/κ be a smooth scheme. In this proof we do not use the obverse de Rham-Witt complex constructed in [I2, I]; we use only the reverse de Rham-Witt complex $W_n\Omega_Z^{\bullet}$ constructed in [IR, III (1.5)](=[HK, (4.1)] in the trivial log case). In particular, $W_n(\mathcal{O}_Z)$ is, by definition, $R^0u_{Z/W_n*}(\mathcal{O}_{Z/W_n})$.

(1): The following composite morphism

$$W_n\Lambda_X^j \xrightarrow{\mathbf{p}} W_{n+1}\Lambda_X^j \xrightarrow{p^{k-1}} W_{n+1}\Lambda_X^j \xrightarrow{F} W_n\Lambda_X^j \tag{9.8.18}$$

is a desired morphism because $\mathbf{p}\pi = p$ ([Hy2, (1.3.2)]). The uniqueness of $\Psi_n^{(k;j)}$ follows from the surjectivity of π ([Hy2, (2.2.3)]).

Next, let us prove the commutativity of (9.8.2). Because $\mathbf{p}\colon W_n\Lambda_X^{\bullet} \longrightarrow W_{n+1}\Lambda_X^{\bullet}$ is a morphism of complexes ([Hy2, (1.3.1)]) and because the following diagram

$$\begin{array}{ccc} W_{n+1}\Lambda_X^j & \xrightarrow{d} & W_{n+1}\Lambda_X^{j+1} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle pF} \\ W_n\Lambda_X^j & \xrightarrow{d} & W_n\Lambda_X^{j+1} \end{array} \tag{9.8.19}$$

is commutative, we obtain the commutativity of (9.8.2) by (9.8.18).

The equality $\Psi_n^{(k;k)} = \Phi_n$ follows from the commutativity of (9.2.2) and (9.8.1), and from the surjectivity of π.

Finally, let us prove the commutativity of (9.8.3). This follows from (9.8.18) and from relations $\mathbf{p}\pi = \pi\mathbf{p}$ and $F\pi = \pi F$ in (6.28) (1).

(2): By [M1, 3.7], the Poincaré residue morphism

$$W_nA_X^{0j} = W_n\widetilde{\Lambda}_X^{j+1}/P_jW_n\widetilde{\Lambda}_X^{j+1} \overset{\text{Res}}{\longrightarrow} W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}}) \tag{9.8.20}$$

is an isomorphism. Let $\Phi_n\colon W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}}) \longrightarrow W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}})$ be the Frobenius endomorphism defined in (9.1.2). Set $\widetilde{\Phi}_n^{(0j)} := \text{Res}^{-1} \circ \Phi_n \circ \text{Res}$. Because the operators F's are induced by the projections, there exists the following commutative diagram

$$\begin{array}{ccc} W_{n+1}A_X^{0j} & \xrightarrow[\sim]{\text{Res}} & W_{n+1}(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}}) \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ W_nA_X^{0j} & \xrightarrow[\sim]{\text{Res}} & W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}}). \end{array} \tag{9.8.21}$$

Then the commutativity of (9.8.4) follows from that of (9.8.21), (8.4.3) and (9.2.2). The uniqueness of $\widetilde{\Phi}_n^{(0j)}$ follows from the surjectivity of π.

By [M1, 4.12], the left wedge product $\theta_n\wedge$ induces a morphism of the sum (with signs) of the induced morphism $W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+1)}}) \longrightarrow W_n(\mathcal{O}_{\overset{\circ}{X}{}^{(j+2)}})$ by restriction morphisms. The morphism Φ_n defined in (9.1.2) is functorial. Hence the commutativity of (9.8.5) is obvious.

Let $\pi\colon W_{n+2}\widetilde{\Lambda}_X^{j+1} \longrightarrow W_{n+1}\widetilde{\Lambda}_X^{j+1}$ be the projection. Consider the composite diagram (9.8.6) $\circ\, \pi$. The commutativity of (9.8.6) follows from that of (9.8.4), from the relation $F\pi = \pi F$ in [Hy2, p. 245] ((6.28) (1), (6.8) (4)), and from the surjectivity of $\pi\colon W_{n+2}\widetilde{\Lambda}_X^{j+1} \longrightarrow W_{n+1}\widetilde{\Lambda}_X^{j+1}$.

(3): As in (1), the following composite

$$W_n\widetilde{\Lambda}_X^j \overset{\mathbf{p}}{\longrightarrow} W_{n+1}\widetilde{\Lambda}_X^j \overset{p^{k-1}}{\longrightarrow} W_{n+1}\widetilde{\Lambda}_X^j \overset{F}{\longrightarrow} W_n\widetilde{\Lambda}_X^j \tag{9.8.22}$$

is a desired morphism. The uniqueness of $\widetilde{\Psi}_n^{(k;j)}$ follows from the surjectivity of $\pi\colon W_{n+1}\widetilde{\Lambda}_X^j \longrightarrow W_n\widetilde{\Lambda}_X^j$ ([Hy2, (2.2.3)]).

The proof of the commutativity of (9.8.8) is the same as that of the commutativity of (9.8.2) by using the relation $\mathbf{p}d = d\mathbf{p}$ in [Hy2, (1.3.1)] and the commutativity of the following diagram:

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}^{j}_{X} & \xrightarrow{d} & W_{n+1}\widetilde{\Lambda}^{j+1}_{X} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle pF} \\ W_{n}\widetilde{\Lambda}^{j}_{X} & \xrightarrow{d} & W_{n}\widetilde{\Lambda}^{j+1}_{X}. \end{array} \tag{9.8.23}$$

The commutativity of (9.8.9) follows from (9.8.22) and (8.1.1). The commutativity of (9.8.10) follows from that of (9.8.18), (9.8.22), (8.1.1) and from the commutativity of the following diagram:

$$\begin{array}{ccc} W_{n}\widetilde{\Lambda}^{j}_{X} & \xrightarrow{\mathbf{p}} & W_{n+1}\widetilde{\Lambda}^{j}_{X} \\ \big\downarrow & & \big\downarrow \\ W_{n}\Lambda^{j}_{X} & \xrightarrow{\mathbf{p}} & W_{n+1}\Lambda^{j}_{X}. \end{array} \tag{9.8.24}$$

By using relations in (6.28) (1), the proof of the commutativity of (9.8.11) is the same as that of the commutativity of (9.8.3).

(4): Because $\mathbf{p}\colon W_n\widetilde{\Lambda}^j_X \longrightarrow W_{n+1}\widetilde{\Lambda}^j_X$ preserves the preweight filtration P ((8.4) (1)), so does $\widetilde{\Psi}^{(k;j)}_n$ by (9.8.22). Hence (4) follows from the commutativity of (9.8.9).

(5): By (4), the commutativity of (9.8.12), (9.8.13), (9.8.14) and (9.8.15) immediately follows from that of (9.8.7), (9.8.8), (9.8.9) and (9.8.11), respectively.

(6): By the surjectivity of $\pi\colon W_{n+1}A^{0j}_X \longrightarrow W_nA^{0j}_X$, by the relation $\pi d = d\pi$ in [Hy2, p. 245], and by the commutativity of (9.8.4) and (9.8.7), it suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc} W_{n+1}A^{0j}_{X} & \xrightarrow{d} & W_{n+1}A^{1j}_{X} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle pF} \\ W_{n}A^{0j}_{X} & \xrightarrow{d} & W_{n}A^{1j}_{X}. \end{array} \tag{9.8.25}$$

This follows from the commutativity of (9.8.23).

(7): (7) follows from (9.8.18) and (9.8.22), and (9.8.24). $\square$

By (9.8), we have the commutativity of (9.9.1) and (9.9.2) below:

THEOREM 9.9. *Let* $W_nA_X^{\bullet *}$ *be the double complex in* (2.2.1; n). *Then there exists a unique endomorphism* $\widetilde{\Phi}_n^{(\bullet *)}\colon W_nA_X^{\bullet *} \longrightarrow W_nA_X^{\bullet *}$ ($n \in \mathbb{Z}_{>0}$) *of double complexes which makes the following diagram commutative*:

(9.9.1)
$$\begin{array}{ccc} W_{n+1}A_X^{\bullet *} & \xrightarrow{\pi} & W_nA_X^{\bullet *} \\ {\scriptstyle p^{\bullet}F}\downarrow & & \downarrow{\scriptstyle \tilde{\Phi}_n^{(\bullet *)}} \\ W_nA_X^{\bullet *} & = & W_nA_X^{\bullet *}. \end{array}$$

The endomorphism $\widetilde{\Phi}_n^{(\bullet *)}$ *induces an endomorphism*

$$\widetilde{\Phi}_n\colon W_nA_X^{\bullet} \longrightarrow W_nA_X^{\bullet}$$

of complexes; $\widetilde{\Phi}_n$ *fits into the following commutative diagram*:

(9.9.2)
$$\begin{array}{ccc} W_nA_X^{\bullet} & \xrightarrow{\tilde{\Phi}_n} & W_nA_X^{\bullet} \\ {\scriptstyle \theta_n\wedge}\uparrow & & \uparrow{\scriptstyle \theta_n\wedge} \\ W_n\Lambda_X^{\bullet} & \xrightarrow{\Phi_n} & W_n\Lambda_X^{\bullet}. \end{array}$$

The Poincaré residue isomorphism Res *induces an isomorphism* $\mathrm{Res}\colon \mathrm{gr}_k^PW_nA_X^{\bullet} \xrightarrow{\sim} \bigoplus_{j\geq \max\{-k,0\}} (W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d)\{-2j-k\}$ *which makes the following diagram commutative*:

(9.9.3)
$$\begin{array}{ccc} \mathrm{gr}_k^PW_nA_X^{\bullet} & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & \bigoplus_{j\geq \max\{-k,0\}} (W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d)\{-2j-k\} \\ {\scriptstyle \tilde{\Phi}_n}\downarrow & & \downarrow{\scriptstyle p^{j+k}\Phi_n} \\ \mathrm{gr}_k^PW_nA_X^{\bullet} & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & \bigoplus_{j\geq \max\{-k,0\}} (W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d)\{-2j-k\}, \end{array}$$

Consequently there exists the following spectral sequence (*cf.* [M1, 3.23]):

(9.9.4)
$$E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}, (W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d))(-j-k)$$
$$\Longrightarrow H^h_{\text{log-crys}}(X/W_n) \quad (n \in \mathbb{Z}_{>0}).$$

PROOF. We have only to prove the commutativity of (9.9.3).

Let i, j be fixed integers. Then we have the following commutative diagram:

(9.9.5)
$$\begin{array}{ccc}
\bigoplus_{j\geq\max\{-k,0\}} \mathrm{gr}^P_{2j+k+1}W_n\widetilde{\Lambda}^{i+j+1}_X & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & \bigoplus_{j\geq\max\{-k,0\}} W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}} \\
\tilde{\Phi}_n\big\downarrow & & \big\downarrow p^{j+k}\Phi_n \\
\bigoplus_{j\geq\max\{-k,0\}} \mathrm{gr}^P_{2j+k+1}W_n\widetilde{\Lambda}^{i+j+1}_X & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & \bigoplus_{j\geq\max\{-k,0\}} W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}.
\end{array}$$

Indeed, by [M1, 3.7] and (8.4.3), the morphism $\pi\colon \mathrm{gr}^P_kW_{n+1}A^{ij}_X \longrightarrow \mathrm{gr}^P_kW_nA^{ij}_X$ is surjective. Consider the diagram (9.9.5) $\circ\, \pi$. Then the commutativity of the diagram (9.9.5) $\circ\, \pi$ follows from that of (9.8.7), (8.4.3), (9.2.2) in the trivial log case, and (9.7.3). Since the horizontal boundary morphisms of $\mathrm{gr}^P_kW_nA^{\bullet\bullet}_X$ are *twisted by signs* in this paper (and also in [M1, 3.8]; [M1, (3.22) (2)] is mistaken), the Poincaré residue isomorphism induces an isomorphism

$$\mathrm{Res}\colon (\mathrm{gr}^P_{2j+k+1}W_n\widetilde{\Lambda}^{\bullet+j+1}_X, (-1)^{j+1}d) \overset{\sim}{\longrightarrow} (W_n\Omega^{\bullet-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d)$$

of complexes. Hence we can finish the proof of (9.9); at the same time, we have proved the Frobenius compatibility in (4.1.5) and hence in (4.1.1;n). □

REMARK 9.10. We have to make the following identification clearly: because identification changes many things, it is important. Using the Convention (6), we obtain

(9.10.1)
$$\begin{aligned}
H^*(\overset{\circ}{X}{}^{(e)}, (W_n\Omega^\bullet_{\overset{\circ}{X}{}^{(e)}}, -d)) &= H^*(\overset{\circ}{X}{}^{(e)}, W_n\Omega^\bullet_{\overset{\circ}{X}{}^{(e)}}) \\
&= H^*_{\mathrm{crys}}(\overset{\circ}{X}{}^{(e)}/W_n) \quad (e\in\mathbb{Z}_{>0}).
\end{aligned}$$

Using this identification, we obtain the following spectral sequence

(9.10.2)
$$\begin{aligned}
E_1^{-k,h+k} = &\bigoplus_{j\geq\max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j-k) \\
&\Longrightarrow H^h_{\text{log-crys}}(X/W_n)
\end{aligned}$$

by (9.9.4). The Frobenius compatibility in (9.10.2) is not obtained in [M1]. We call the spectral sequence (9.10.2) the *p-adic preweight spectral sequence* of X/s.

Definition 9.11. (1) Set $W\Lambda^\bullet_X = \varprojlim_n W_n\Lambda^\bullet_X$ and $WA^\bullet_X = \varprojlim_n W_nA^\bullet_X$. Set $\Phi := \varprojlim_n \Phi_n\colon W\Lambda^\bullet_X \longrightarrow W\Lambda^\bullet_X$ ((9.2.3)), $\widetilde{\Phi}^{\bullet *} := \varprojlim_n \widetilde{\Phi}^{\bullet *}_n\colon WA^{\bullet *}_X \longrightarrow WA^{\bullet *}_X$ ((9.8.15)) and $\widetilde{\Phi} := \varprojlim_n \widetilde{\Phi}_n\colon WA^\bullet_X \longrightarrow WA^\bullet_X$ ((9.8.15)). Because $\pi\colon W_{n+1}\Lambda^\bullet_X \longrightarrow W_n\Lambda^\bullet_X$ and $\pi\colon W_{n+1}A^\bullet_X \longrightarrow W_nA^\bullet_X$ are surjective, we have an obvious analogue of (9.9) for $W\Lambda^\bullet_X$, $WA^\bullet_X$, Φ and $\widetilde{\Phi}$.

(2) We call the following spectral sequence

$$E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W)(-j-k) \Longrightarrow H^h_{\text{log-crys}}(X/W). \tag{9.11.1}$$

the *p-adic weight spectral sequence* of X/s. (The spectral sequence (9.11.1) is obtained by the same argument as that of (9.9) and (9.10) for $W\Lambda^\bullet_X$, $WA^\bullet_X$, Φ, $\widetilde{\Phi}^{\bullet *}$ and $\widetilde{\Phi}$.)

Finally, we define the Frobenius endomorphism $\widetilde{\Phi}_n\colon W_n\widetilde{\Lambda}^i_X \longrightarrow W_n\widetilde{\Lambda}^i_X$ ($i \geq 0$) by the following formula:

$$\widetilde{\Phi}_n = \begin{cases} \widetilde{\Psi}^{(i;i)}_n & (i \geq 1), \\ \Phi_n \text{ in (9.1.2)} & (i = 0). \end{cases} \tag{9.11.2}$$

Assume that we are given an admissible triple $(\mathcal{Y}, \mathcal{X}, \Phi)$ of X (§6). Then, $\widetilde{\Phi}_n$ is equal to

$$\mathcal{H}^i(\Phi^*\mathrm{mod}\ p^n)\colon \mathcal{H}^i(\mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)) \longrightarrow \mathcal{H}^i(\mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n))$$

by (9.8.22). By (9.8.11), we obtain the following commutative diagram:

$$\begin{array}{ccc} W_{n+1}\widetilde{\Lambda}^i_X & \xrightarrow{\widetilde{\Phi}_{n+1}} & W_{n+1}\widetilde{\Lambda}^i_X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W_n\widetilde{\Lambda}^i_X & \xrightarrow{\widetilde{\Phi}_n} & W_n\widetilde{\Lambda}^i_X. \end{array} \tag{9.11.3}$$

By (9.11.3), we can set $\widetilde{\Phi} := \varprojlim_n \widetilde{\Phi}_n$.

Let k be an integer. Because the Frobenius on $P_kW_n\widetilde{\Lambda}^{\bullet}_X$ is not defined in [M1], the proof of [M1, 3.7] is not complete; we complete it by the following:

Proposition 9.12. *For a non-negative integer k, $P_kW_n\widetilde{\Lambda}^{\bullet}_X$ is stable by the Frobenius $\widetilde{\Phi}_n$ on $W_n\widetilde{\Lambda}^{\bullet}_X$. The following two exact sequences*

$$0 \longrightarrow P_kW_n\widetilde{\Lambda}^{\bullet}_X \longrightarrow P_{k+1}W_n\widetilde{\Lambda}^{\bullet}_X \overset{\text{Res}}{\longrightarrow} W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(k+1)}}(-k-1)\{-k-1\} \longrightarrow 0 \quad (n \in \mathbb{N}) \tag{9.12.1;n}$$

and

$$0 \longrightarrow P_kW\widetilde{\Lambda}^{\bullet}_X \longrightarrow P_{k+1}W\widetilde{\Lambda}^{\bullet}_X \overset{\text{Res}}{\longrightarrow} W\Omega^{\bullet}_{\overset{\circ}{X}{}^{(k+1)}}(-k-1)\{-k-1\} \longrightarrow 0 \tag{9.12.1}$$

are compatible with the Frobenius. Here the Frobenius on $W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(k+1)}}$ $(n \in \mathbb{N})$ is, by definition, given in (9.1.2)*, and the Frobenius on $W\Omega^{\bullet}_{\overset{\circ}{X}{}^{(k+1)}}$ is the projective limit of those on $W_n\Omega^{\bullet}_{\overset{\circ}{X}{}^{(k+1)}}$ $(n \in \mathbb{N})$.*

Proof. By the local description of $\widetilde{\Phi}_n$ and by the same proof as that of (8.4) (1), we see that $P_kW_n\widetilde{\Lambda}^{\bullet}_X$ is stable by $\widetilde{\Phi}_n$. Now (9.12.1;n) follows from this stability, from [M1, 3.7] and from the same proof as that of (9.3) (1).

(9.12.1) follows from (9.12.1;n) and (8.6.1). □

10. The boundary morphism of the p-adic weight spectral sequence of an SNCL variety

In this section we give the description of the boundary morphisms of the E_1-terms (of the finite length version) of (2.0.1) with boundary morphisms in (2.2.1;$\star$); the proof of (10.1) below completes the proof of [M1, 4.14]; in addition, in (10.2) (4) below, we correct signs of boundary morphisms in [M1, 4.14].

Theorem 10.1. *Let k be an integer. Let G and ρ be the Gysin morphism and the induced morphism by closed immersions defined in* [M1, 4.10] *and* [M1, 4.12], *respectively. Let* $W_\star A_X^\bullet$ ($\star$ = *a positive integer* n *or nothing*) *be the single complex of the double complex in* (2.2.1;$\star$). *Let* $d_1\colon H^h(X, \mathrm{gr}_k^P W_\star A_X^\bullet) \longrightarrow H^{h+1}(X, \mathrm{gr}_{k-1}^P W_\star A_X^\bullet)$ *be the boundary morphism obtained from the following exact sequence*

$$(10.1.1;\star)\quad 0 \longrightarrow \mathrm{gr}_{k-1}^P W_\star A_X^\bullet \longrightarrow (P_k/P_{k-2}) W_\star A_X^\bullet \longrightarrow \mathrm{gr}_k^P W_\star A_X^\bullet \longrightarrow 0.$$

Then, under the identification in (9.10), *the morphism* d_1 *is identified with the following morphism*:

$$(10.1.2;\star)\qquad \sum_{j\geq \max\{-k,0\}} [(-1)^j G\{-2j-k+1\} + (-1)^{j+k}\rho\{-2j-k\}]\colon$$

$$\bigoplus_{j\geq \max\{-k,0\}} H_{\mathrm{crys}}^{h-2j-k}(\overset{\circ}{X}{}^{(2j+k+1)}/W_\star)(-j-k) \longrightarrow$$

$$\bigoplus_{j\geq \max\{-k+1,0\}} H_{\mathrm{crys}}^{h-2j-k+2}(\overset{\circ}{X}{}^{(2j+k)}/W_\star)(-j-k+1).$$

Proof. Because signs are considerably delicate, I give the proof in full detail.

As in (9.10), we have $\mathrm{gr}_l^P W_\star A_X^\bullet = \bigoplus_{j\geq 0} \mathrm{gr}_l^P W_\star A_X^{\bullet j}\{-j\}$ ($l \in \mathbb{Z}$). Let i, j be two non-negative integers. Let $(I^{\bullet ij}, \delta)$, $(J^{\bullet ij}, \delta)$, $(K^{\bullet ij}, \delta)$ be the Godement resolutions of $\mathrm{gr}_{k-1}^P W_\star A_X^{ij}$, $(P_k/P_{k-2}) W_\star A_X^{ij}$ and $\mathrm{gr}_k^P W_\star A_X^{ij}$, respectively. Let us make a convention on signs of the boundary morphisms of $J^{\bullet\bullet\bullet}$ as follows: Let d_1 (resp. d_2) be the naturally induced morphism $J^{lij} \longrightarrow J^{l,i+1,j}$ (resp. $J^{lij} \longrightarrow J^{l,i,j+1}$) by the horizontal (resp. vertical) morphism $(-1)^{j+1} d\colon (P_k/P_{k-2}) W_\star A_X^{ij} \longrightarrow (P_k/P_{k-2}) W_\star A_X^{i+1,j}$ (resp. $(-1)^i \theta_\star \wedge\colon (P_k/P_{k-2}) W_\star A_X^{ij} \longrightarrow (P_k/P_{k-2}) W_\star A_X^{i,j+1}$). Then we fix the boundary morphisms as follows:

$$(10.1.3;\mathrm{G})\qquad \delta\colon J^{lij} \longrightarrow J^{l+1,i,j},$$

$$(10.1.3;\mathrm{h})\qquad (-1)^l d_1\colon J^{lij} \longrightarrow J^{l,i+1,j},$$

$$(10.1.3;\mathrm{v})\qquad (-1)^l d_2\colon J^{lij} \longrightarrow J^{l,i,j+1}.$$

We make the same convention for $I^{\bullet\bullet\bullet}$ and $K^{\bullet\bullet\bullet}$. Then we have triple complexes $I^{\bullet\bullet\bullet}$, $J^{\bullet\bullet\bullet}$ and $K^{\bullet\bullet\bullet}$. Note that the boundary morphisms of $I^{\bullet\bullet\bullet} \longrightarrow I^{\bullet,\bullet,\bullet+1}$ and $K^{\bullet\bullet\bullet} \longrightarrow K^{\bullet,\bullet,\bullet+1}$ are the zeros. Let $J^{\bullet\bullet}$ be a double complex defined by $J^{ij} := \bigoplus_{i'+i''=i} J^{i'i''j}$. Similarly we have double complexes $I^{\bullet\bullet}$ and $K^{\bullet\bullet}$. Then, for each j, $I^{\bullet j}$, $J^{\bullet j}$ and $K^{\bullet j}$ are flasque resolutions of $\mathrm{gr}^P_{k-1} W_\star A_X^{\bullet j}$, $(P_k/P_{k-2})W_\star A_X^{\bullet j}$ and $\mathrm{gr}^P_k W_\star A_X^{\bullet j}$, respectively, and we have the following commutative diagram

(10.1.4)
$$\begin{array}{ccccc} 0 \longrightarrow & I^{\bullet j}\{-j\} & \longrightarrow & J^{\bullet j}\{-j\} \\ & \uparrow & & \uparrow \\ 0 \longrightarrow & \mathrm{gr}^P_{k-1} W_\star A_X^{\bullet j}\{-j\} & \longrightarrow & (P_k/P_{k-2})W_\star A_X^{\bullet j}\{-j\} \end{array}$$
$$\begin{array}{ccc} \longrightarrow & K^{\bullet j}\{-j\} & \longrightarrow 0 \\ & \uparrow & \\ \longrightarrow & \mathrm{gr}^P_k W_\star A_X^{\bullet j}\{-j\} & \longrightarrow 0. \end{array}$$

Let $I^\bullet$, $J^\bullet$ and $K^\bullet$ be the single complexes of $I^{\bullet\bullet}$, $J^{\bullet\bullet}$ and $K^{\bullet\bullet}$, respectively. Then we also have the following exact sequence

(10.1.5)
$$0 \longrightarrow \Gamma(X, I^\bullet) \longrightarrow \Gamma(X, J^\bullet) \longrightarrow \Gamma(X, K^\bullet) \longrightarrow 0.$$

Let d_h (resp. d_v) be the horizontal (resp. vertical) boundary morphism of $J^{\bullet\bullet}$.

Let $\omega = (\omega^{ij})_{i+j=q} \in \Gamma(X, K^q) = \bigoplus_{i+j=q} \Gamma(X, K^{ij})$ ($q \in \mathbb{N}$) be a cocycle. Let $\widetilde{\omega}^{ij} \in \Gamma(X, J^{ij})$ be a lift of ω^{ij}. Then the image of ω by the boundary morphism of (10.1.5) is $(\eta^{i+1,j})_{i+j=q}$, where

(10.1.6;$i, j-1$)
$$\eta^{i+1,j} := d_h(\widetilde{\omega}^{ij}) + d_v(\widetilde{\omega}^{i+1,j-1}).$$

First, we consider the horizontal morphism d_h in (10.1.6;$i, j-1$). The section $d_h(\widetilde{\omega}^{ij})$ is obtained by the following exact sequence

$$0 \longrightarrow I^{\bullet j}\{-j\} \longrightarrow J^{\bullet j}\{-j\} \longrightarrow K^{\bullet j}\{-j\} \longrightarrow 0.$$

By (10.1.4), this is isomorphic to

(10.1.7)
$$\begin{aligned} 0 &\longrightarrow (\mathrm{gr}^P_{k-1} W_\star A_X^{\bullet j}\{-j\}, (-1)^{j+1}d) \\ &\longrightarrow ((P_k/P_{k-2})W_\star A_X^{\bullet j}\{-j\}, (-1)^{j+1}d) \\ &\longrightarrow (\mathrm{gr}^P_k W_\star A_X^{\bullet j}\{-j\}, (-1)^{j+1}d) \longrightarrow 0. \end{aligned}$$

Moreover, by [M1, 4.11] and (9.9.3), the following diagram is commutative:

(10.1.8)
$$\begin{array}{ccc} (\mathrm{gr}_k^P W_\star A_X^{\bullet j}\{-j\}, d) & \xrightarrow{d} & (\mathrm{gr}_{k-1}^P W_\star A_X^{\bullet j}\{-j\}, d)[1] \\ {\scriptstyle \mathrm{Res}}\downarrow\simeq & & {\scriptstyle \mathrm{Res}}\downarrow\simeq \\ \bigoplus_{j\geq\max\{-k,0\}} W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k)\{-2j-k\} & \xrightarrow{-G\{-2j-k+1\}} & \bigoplus_{j\geq\max\{-k+1,0\}} W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k)}}(-j-k+1)\{-2j-k+1\}[1]. \end{array}$$

As in the diagram [RZ, p. 31], we use the Convention (6) for the following cohomologies:

(10.1.9)
$$\begin{aligned} &H^h(X, (\mathrm{gr}_l^P W_\star A_X^{\bullet j}\{-j\}, (-1)^{j+1}d)) \\ &\quad = H^h(X, (\mathrm{gr}_l^P W_\star A_X^{\bullet j}\{-j\}, d)) \quad (l = k, k-1). \end{aligned}$$

Hence the part of the Gysin morphism in (10.1.2;$\star$) is obtained by noting the signs of the horizontal boundary morphism in (2.2.1;$\star$) and by the Convention (5).

Secondly, we consider the vertical morphism d_v in (10.1.6;$i, j-1$). However we consider it in (10.1.6;i, j) because we wish to give the formula (10.1.2;$\star$). The morphism $d_v'\colon K^{i+1,j} \ni \omega^{i+1,j} \longmapsto d_v(\widetilde{\omega}^{i+1,j}) \in I^{i+1,j+1}$ is well-defined. Indeed, we can easily check this by noting that the vertical morphism of $K^{\bullet j} \longrightarrow K^{\bullet j+1}$ is the zero and the injectivity of the morphism $I^{\bullet j+1} \longrightarrow J^{\bullet j+1}$. Then we have the following four commutative diagrams:

(10.1.10)
$$\begin{array}{ccc} \mathrm{gr}_{k-1}^P W_\star A_X^{\bullet j+1}\{-j-1\} & \longrightarrow & (P_k/P_{k-2}) W_\star A_X^{\bullet j+1}\{-j-1\} \\ {\scriptstyle (-1)^\bullet\theta_\star\wedge}\uparrow & & \uparrow{\scriptstyle (-1)^\bullet\theta_\star\wedge} \\ \mathrm{gr}_k^P W_\star A_X^{\bullet j}\{-j\} & \longleftarrow & (P_k/P_{k-2}) W_\star A_X^{\bullet j}\{-j\}, \end{array}$$

(10.1.11)
$$\begin{array}{ccc} (P_k/P_{k-2}) W_\star A_X^{\bullet j+1}\{-j-1\} & \longrightarrow & J^{\bullet j+1}\{-j-1\} \\ {\scriptstyle (-1)^\bullet\theta_\star\wedge}\uparrow & & \uparrow{\scriptstyle d_v} \\ (P_k/P_{k-2}) W_\star A_X^{\bullet j}\{-j\} & \longrightarrow & J^{\bullet j}\{-j\}, \end{array}$$

$$
(10.1.12)\qquad
\begin{array}{ccc}
I^{\bullet j+1}\{-j-1\} & \longrightarrow & J^{\bullet j+1}\{-j-1\} \\
{\scriptstyle d'_v}\uparrow & & \uparrow{\scriptstyle d_v} \\
K^{\bullet j}\{-j\} & \longleftarrow & J^{\bullet j}\{-j\},
\end{array}
$$

$$
(10.1.13)\qquad
\begin{array}{ccc}
\mathrm{gr}^P_{k-1}W_{\star}A^{\bullet j+1}_X\{-j-1\} & \longrightarrow & I^{\bullet j+1}\{-j-1\} \\
{\scriptstyle (-1)^{\bullet}\theta_{\star}\wedge}\uparrow & & \uparrow{\scriptstyle d'_v} \\
\mathrm{gr}^P_k W_{\star}A^{\bullet j}_X\{-j\} & \longrightarrow & K^{\bullet j}\{-j\}.
\end{array}
$$

The complex $\mathrm{gr}^P_k W_{\star}A^{\bullet}_X$ is isomorphic to the complex $\bigoplus_{j\geq\max\{-k,0\}}$ (10.1.14;j), where (10.1.14;j) is the following complex

$$
(10.1.14;j)\qquad \cdots \overset{(-1)^{j+1}d}{\longrightarrow} \underset{(i,j)}{W_n\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k)} \overset{(-1)^{j+1}d}{\longrightarrow} \cdots,
$$

where $(*,*)$ below the sheaf above means the bi-degree. On the other hand, $\mathrm{gr}^P_{k-1}W_{\star}A^{\bullet}_X$ is isomorphic to $\bigoplus_{j+1\geq\max\{-k+1,0\}}$ (10.1.15;$j+1$), where (10.1.15;$j+1$) is

$$
(10.1.15;j+1)\qquad \cdots \overset{(-1)^{j+2}d}{\longrightarrow} \underset{(i,j+1)}{W_{\star}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k)} \overset{(-1)^{j+2}d}{\longrightarrow} \cdots.
$$

By the same proof as that of [M1, 4.12], we have the following commutative diagram for $l\leq m$:

$$
(10.1.16)\qquad
\begin{array}{ccc}
\mathrm{gr}^P_{l+1}W_{\star}\widetilde{\Lambda}^{m+1}_X & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & W_{\star}\Omega^{m-l}_{\overset{\circ}{X}{}^{(l+1)}} \\
{\scriptstyle \theta_{\star}\wedge}\uparrow & & \uparrow{\scriptstyle (-1)^{m-l}\rho} \\
\mathrm{gr}^P_l W_{\star}\widetilde{\Lambda}^m_X & \overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} & W_{\star}\Omega^{m-l}_{\overset{\circ}{X}{}^{(l)}}.
\end{array}
$$

Hence the morphism $(-1)^i\theta_{\star}\wedge\colon \mathrm{gr}^P_k W_{\star}A^{ij}_X \longrightarrow \mathrm{gr}^P_{k-1}W_{\star}A^{i,j+1}_X[1]$ is identified with a morphism $(-1)^i(-1)^{(i+j+1)-(2j+k+1)}\rho = (-1)^{j+k}\rho$. Hence, by the commutative diagram (10.1.13), the morphism $d'_v\colon K^{\bullet j}\{-j\} \longrightarrow$

$I^{\bullet j+1}\{-j-1\}$ is induced by $(-1)^{j+k}\rho$. Now, as in (10.1.9), let us make the following identifications:

(10.1.17) $$H^{\bullet}(\overset{\circ}{X}{}^{(2j+k+1)}, (W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-2j-k\}, (-1)^{j+1}d))(-j-k) = H^{\bullet}(\overset{\circ}{X}{}^{(2j+k+1)}, W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-2j-k\})(-j-k),$$

(10.1.18) $$H^{\bullet}(\overset{\circ}{X}{}^{(2j+k+2)}, (W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+2)}}\{-2j-k\}, (-1)^{j+2}d))(-j-k) = H^{\bullet}(\overset{\circ}{X}{}^{(2j+k+2)}, W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+2)}}\{-2j-k\})(-j-k).$$

Under these identifications, the desired vertical morphism

$$H^{h}(\overset{\circ}{X}{}^{(2j+k+1)}, W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-2j-k\})(-j-k) \longrightarrow H^{h}(\overset{\circ}{X}{}^{(2j+k+2)}, W_{\star}\Omega^{\bullet}_{\overset{\circ}{X}{}^{(2j+k+2)}}\{-2j-k\})(-j-k)$$

is $(-1)^{j+k}\rho$.

Finally, we consider the Frobenius compatibility. By the commutativity of (9.9.5) and (10.1.16), the following diagram

(10.1.19) $$\begin{array}{ccc} \mathrm{gr}^{P}_{2j+k+2}W_{\star}\widetilde{\Lambda}^{i+j+2}_{X} & \xrightarrow{\widetilde{\Psi}^{(i;i+j+2)}_{\star}} & \mathrm{gr}^{P}_{2j+k+2}W_{\star}\widetilde{\Lambda}^{i+j+2}_{X} \\ {\scriptstyle (-1)^{i}\theta_{\star}\wedge}\uparrow & & \uparrow{\scriptstyle (-1)^{i}\theta_{\star}\wedge} \\ \mathrm{gr}^{P}_{2j+k+1}W_{\star}\widetilde{\Lambda}^{i+j+1}_{X} & \xrightarrow{\widetilde{\Psi}^{(i;i+j+1)}_{\star}} & \mathrm{gr}^{P}_{2j+k+1}W_{\star}\widetilde{\Lambda}^{i+j+1}_{X} \end{array}$$

is identified with the following commutative diagram

(10.1.20) $$\begin{array}{ccc} W_{\star}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k) & \xrightarrow{\Phi_{\star}} & W_{\star}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k) \\ {\scriptstyle (-1)^{j+k}\rho}\uparrow & & \uparrow{\scriptstyle (-1)^{j+k}\rho} \\ W_{\star}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k) & \xrightarrow{\Phi_{\star}} & W_{\star}\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j-k). \end{array}$$

Therefore we have proved (10.1). □

REMARK 10.2. (1) The description (10.1.2;$\star$) is *not* the same as the p-adic analogue of Rapoport-Zink's description of the boundary morphism of the E_1-terms of the l-adic weight spectral sequence in [RZ, (2.10)] because their Čech-Gysin morphism [RZ, p. 39] corresponds to $-G$ in [M1, p. 324]. If we wish to have the same description in the p-adic case as that in [RZ, (2.10)], we need to make the following identification for example:

(10.2.1)
$$\begin{aligned}
&H^h(X, \mathrm{gr}_k^P W_\star A_X^\bullet) \\
&\overset{\mathrm{Res}}{\underset{\sim}{\longrightarrow}} \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, (W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}}, (-1)^{j+1}d))(-j-k) \\
&= \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, (W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}}, d))(-j-k) \\
&= \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_\star)(-j-k) \\
&\xrightarrow{\bigoplus_{j\geq\max\{-k,0\}}(-1)^{j+k}\times} \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_\star)(-j-k)
\end{aligned}$$

for all h, j and k such that $j \geq \max\{-k, 0\}$. Here we have used the Convention (6) for the second equality.

However the description of the boundary morphism in [RZ, (2.10)] is mistaken in signs if we use the resolution [RZ, (2.6)], the formula

$$a_{r*}Rb_r^!\Lambda = a_{r*}b_r^!I^\bullet = a_{r*}\Lambda(-r)[-2r] \quad (r \in \mathbb{Z}_{>0})$$

in [loc. cit., p. 37] and the double complex $\overline{C}$ in [loc. cit., p. 38]. See [Nakk4] for the details and for the description using the formula above and for another description.

(2) Let the notations be as in [M1, p. 323]. The formula $\mathrm{Res}^J_{I_q}(\omega) = \alpha \wedge d\log x_{i_q}|_{D_{I_q}}$ in [M1, p. 323, l. -9] is miswritten; the correct one is: $\mathrm{Res}^J_{I_q}(\omega) = (-1)^{q-1}\alpha \wedge d\log x_{i_q}|_{D_{I_q}}$.

(3) The morphism $(-1)^{j+k}\rho\colon W_\star\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j - k) \longrightarrow$

$W_\star\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+2)}}(-j-k)$ does *not* induce a morphism of complexes:

$$(-1)^{j+k}\rho\colon (W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}}\{-2j-k\}(-j-k),(-1)^{j+1}d) \\ \longrightarrow (W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+2)}}\{-2j-k-1\}(-j-k),(-1)^{j+2}d).$$

(4) Let us point out six mistakes in [M1, 4.13, 4.14] and correct them. In this remark, as in [M1, 3.8], let us consider a double complex $W_nA_X^{\bullet\bullet}$ with boundary morphisms (2.2.3; n). Let $W_nA_X^\bullet$ be the single complex of $W_nA_X^{\bullet\bullet}$. The mistakes in [loc. cit.] are as follows:

(a): The sign $(-1)^{j-1}$ before the term G in [loc. cit.] is necessary; because the horizontal boundary morphism $W_nA_X^{ij}\longrightarrow W_nA_X^{i+1,j}$ is $(-1)^jd(=(-1)^{j-1}(-d))$, signs of the horizontal boundary morphism $\mathrm{gr}^P_{k+1}W_nA_X^{\bullet\bullet}\longrightarrow \mathrm{gr}^P_kW_nA_X^{\bullet+1,\bullet}[1]$ and that of the usual boundary morphism $d\colon \mathrm{gr}^P_{k'+1}W_n\widetilde{\Lambda}^\bullet_X\longrightarrow \mathrm{gr}^P_{k'}W_n\widetilde{\Lambda}^\bullet_X[1]$ ($k'\in\mathbb{Z}$) are different.

(b): $\mathrm{Gr}_{j+1}W_\bullet A^\bullet$ and $\mathrm{Gr}_jW_\bullet A^\bullet[1]$ in the diagram of [M1, 4.13] have to be replaced by $\mathrm{Gr}_{k+1}W_\bullet A^\bullet$ and $\mathrm{Gr}_kW_\bullet A^\bullet[1]$, respectively.

(c): The sign $(-1)^{k+1}$ before the term ρ in [M1, 4.13] must be replaced by $(-1)^k$; the number of the preweight filtration P on the source of the boundary morphism $\mathrm{gr}^P_{k+1}W_nA_X^\bullet\longrightarrow \mathrm{gr}^P_kW_nA_X^\bullet$ is $k+1$ but not k.

(d): The Poincaré residue isomorphism in the diagram in [M1, 4.13] are not isomorphisms of complexes.

(e): The Frobenius on the pro-system $\mathrm{gr}^P_kW_\bullet A_X^\bullet$ is not considered.

(f): The two shifts $[*]$ and $\{*\}$ are considered as the same operators (we can find this confusion in many references).

By putting the above together and by the proof of (10.1), [M1, 4.14] is corrected in a stronger form than [loc. cit.] as follows: the following diagram is commutative:

(10.2.2)
$$\begin{array}{cc} H^h(X,\mathrm{gr}^P_{k+1}W_nA_X^\bullet) & \xrightarrow{d_1} \\ \Big\| & \\ \displaystyle\bigoplus_{\substack{j\geq -k-1\\ j\geq 0}} H^{h-2j-k-1}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+2)}/W_n)(-j-k-1) & \longrightarrow \end{array}$$

$$\begin{array}{c} H^{h+1}(X, \mathrm{gr}_k^P W_n A_X^{\bullet}) \\ \Big\| \\ \displaystyle\bigoplus_{\substack{j\geq -k \\ j\geq 0}} H^{h+1-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j-k), \end{array}$$

where the lower horizontal morphism is

$$\sum_{j\geq \max\{-k-1,0\}} [(-1)^{j-1}G\{-2j-k\} + (-1)^k\rho\{-2j-k-1\}] \tag{10.2.3}$$

and the two vertical identities are canonical isomorphisms.

(5) I do not understand why [M1, 4.13] is obviously obtained by [M1, 4.11, 4.12]: the notion of double complexes do not exist in a derived category.

Let i be a fixed non-negative integer. Next, we consider the boundary morphism

$$H^h(X, \mathrm{gr}_k^P W_\star A_X^{i\bullet}\{-i\}) \longrightarrow H^{h+1}(X, \mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}\{-i\}) \tag{10.2.4}$$

arising from the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}\{-i\} \longrightarrow (P_k/P_{k-2})W_\star A_X^{i\bullet}\{-i\} \\ \longrightarrow \mathrm{gr}_k^P W_\star A_X^{i\bullet}\{-i\} \longrightarrow 0. \end{aligned} \tag{10.2.5}$$

Let the notations be as in the proof of (10.1). Then we have the resolutions $I^{\bullet i\bullet}$, $J^{\bullet i\bullet}$ and $K^{\bullet i\bullet}$ of $\mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}$, $(P_k/P_{k-2})W_\star A_X^{i\bullet}$ and $\mathrm{gr}_k^P W_\star A_X^{i\bullet}$, respectively. As in the proof of (10.1), we make the same convention on signs of, e.g., the boundary morphisms $J^{lij} \longrightarrow J^{l+1,i,j}$ and $J^{lij} \longrightarrow J^{l,i,j+1}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & s(I^{\bullet i\bullet})\{-i\} & \longrightarrow & s(J^{\bullet i\bullet})\{-i\} \\ & \uparrow & & \uparrow \\ 0 \longrightarrow & \mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}\{-i\} & \longrightarrow & (P_k/P_{k-2})W_\star A_X^{i\bullet}\{-i\} \end{array} \tag{10.2.6}$$

$$\begin{array}{ccc} \longrightarrow & s(K^{\bullet i\bullet})\{-i\} & \longrightarrow 0 \\ & \uparrow & \\ \longrightarrow & \mathrm{gr}_k^P W_\star A_X^{i\bullet}\{-i\} & \longrightarrow 0. \end{array}$$

Here s means the single complex of a double complex. We consider the usual boundary morphism

$$(10.2.7)\qquad d_1\colon H^h(X,\mathrm{gr}_k^P W_\star A_X^{i\bullet}\{-i\})=H^h(\Gamma(X,s(K^{\bullet i\bullet})\{-i\}))\longrightarrow$$
$$H^{h+1}(\Gamma(X,s(I^{\bullet i\bullet})\{-i\}))=H^{h+1}(X,\mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}\{-i\})$$

of the lower exact sequence of (10.2.6).

Let $\{X_n\}_{n=1}^M$ be the smooth irreducible components of $\overset{\circ}{X}$. For positive integers $1\leq n_0<\cdots<n_{k-1}\leq M$, set $X_{(n_0\cdots n_{k-1})}:=X_{n_0}\cap\cdots\cap X_{n_{k-1}}$. Let l be an integer. Let

$$G^{n_0\cdots\hat{n}_j\cdots n_{k-1}}_{n_0\cdots n_{k-1}}\colon H^l(X_{(n_0\cdots n_{k-1})},W_\star\Omega^i_{X_{(n_0\cdots n_{k-1})}})(-1)\longrightarrow$$
$$H^{l+1}(X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})},W_\star\Omega^{i+1}_{X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})}})$$

be the Gysin morphism of the closed immersion $X_{(n_0\cdots n_{k-1})}\overset{\subset}{\longrightarrow}X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})}$ which is defined in §4. Set

$$(10.2.8)\qquad G:=\sum_{1\leq n_0<n_1<\cdots<n_{k-1}\leq M}\sum_{j=0}^{k-1}(-1)^jG^{n_0\cdots\hat{n}_j\cdots n_{k-1}}_{n_0\cdots n_{k-1}}\colon$$
$$H^l(X^{(k)},W_\star\Omega^i_{X^{(k)}})(-1)\longrightarrow H^{l+1}(X^{(k-1)},W_\star\Omega^{i+1}_{X^{(k-1)}}).$$

Let

$$\iota^{n_0\cdots\hat{n}_j\cdots n_{k-1}*}_{n_0\cdots n_{k-1}}\colon H^l(X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})},W_\star\Omega^i_{X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})}})\longrightarrow$$
$$H^l(X_{(n_0\cdots n_{k-1})},W_\star\Omega^i_{X_{(n_0\cdots n_{k-1})}})$$

be the induced morphism by the closed immersion $X_{(n_0\cdots n_{k-1})}\overset{\subset}{\longrightarrow}X_{(n_0\cdots\hat{n}_j\cdots n_{k-1})}$. Set

$$(10.2.9)\qquad \rho:=\sum_{1\leq n_0<n_1<\cdots<n_{k-1}\leq M}\sum_{j=0}^{k-1}(-1)^j\iota^{n_0\cdots\hat{n}_j\cdots n_{k-1}*}_{n_0\cdots n_{k-1}}\colon$$
$$H^l(X^{(k-1)},W_\star\Omega^i_{X^{(k-1)}})\longrightarrow H^l(X^{(k)},W_\star\Omega^i_{X^{(k)}}).$$

Theorem 10.3. *Let the notations be as above. Then the morphism*

$$d_1\colon H^h(X, \mathrm{gr}_k^P W_\star A_X^{i\bullet}\{-i\}) \longrightarrow H^{h+1}(X, \mathrm{gr}_{k-1}^P W_\star A_X^{i\bullet}\{-i\})$$

in (10.2.7) *is identified with the following morphism*:

$$(10.3.1;\star)\qquad \sum_{j\geq \max\{-k,0\}} [(-1)^j G\{-2j-k+1\} + (-1)^{j+k}\rho\{-2j-k\}]\colon$$

$$\bigoplus_{j\geq \max\{-k,0\}} H^{h-i-j}(\overset{\circ}{X}{}^{(2j+k+1)}, W_\star\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \longrightarrow$$

$$\bigoplus_{j\geq \max\{-k+1,0\}} H^{h-i-j+1}(\overset{\circ}{X}{}^{(2j+k)}, W_\star\Omega^{i-j-k+1}_{\overset{\circ}{X}{}^{(2j+k)}})(-j-k+1).$$

Proof. By noting the sign in (10.1.3;G) and the remark (4.6), the proof is the same as that of (10.1). □

Remark 10.4. If we use the boundary morphisms of the p-adic Steenbrink complex in (2.2.2; $\star$), there exists the following spectral sequence

$$(10.4.1;\star)\qquad E_1^{-k,h+k} = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_\star)(-j-k) \Longrightarrow H^h_{\text{log-crys}}(X/W_\star).$$

Set

$$(10.4.2;\star)\qquad \rho' := (-1)^\bullet\rho\colon W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}} \longrightarrow W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+2)}}.$$

Then the boundary morphism $E_1^{-k,h+k} \longrightarrow E_1^{-k+1,h+k}$ in (10.4.1;$\star$) is identified with

$$(10.4.3;\star)\qquad \sum_{j\geq \max\{-k,0\}} [-G\{-2j-k+1\} + \rho'].$$

Though ρ' is not a usual induced morphism of log de Rham-Witt complexes, the convention (2.2.2; $\star$) makes us free from the convention on the work for signs in (10.1). However, because (2.2.1; $\star$) is naturally related to a left cup-product $\theta\wedge$ in the lower exact sequence of a commutative diagram (11.8.1; $\star$) below, we have used (2.2.1; $\star$) in this paper.

We conclude this section by giving the proof of the correction of the duality in [M1, 4.15]:

Proposition 10.5. *Assume that $\overset{\circ}{X}$ is of pure dimension d. Then the following hold*:

(1) *Let n be a positive integer. Let $\{E_{r,n}^{\bullet\bullet}\}_{r\geq 1}$ be the E_r-terms of the preweight spectral sequence* (9.10.2). *Then the Poincaré duality pairing*

$$\langle\ ,\ \rangle\colon E_{1,n}^{-k,2d-h-k}\otimes_{W_n}E_{1,n}^{k,h+k}\longrightarrow W_n(-d) \tag{10.5.1}$$

induces the following perfect pairing

$$\langle\ ,\ \rangle\colon E_{2,n}^{-k,2d-h-k}\otimes_{W_n}E_{2,n}^{k,h+k}\longrightarrow W_n(-d). \tag{10.5.2}$$

(2) *The analogue of* (1) *for the weight spectral sequence* (9.11.1) $\otimes_W K_0$ *of $H^{\bullet}_{\text{log-crys}}(X/W)\otimes_W K_0$ holds.*

Proof. (1) By (9.10.2) we have

$$E_{1,n}^{-k,2d-h-k}=\bigoplus_{j\geq\max\{-k,0\}}H^{2d-h-2j-3k}_{\text{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j-k)$$

and

$$\begin{aligned}E_{1,n}^{k,h+k}&=\bigoplus_{j\geq\max\{k,0\}}H^{h+2k-2j+k}_{\text{crys}}(\overset{\circ}{X}{}^{(2j-k+1)}/W_n)(-j+k)\\&=\bigoplus_{j\geq\max\{-k,0\}}H^{h-2j+k}_{\text{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j).\end{aligned}$$

Since $(2d-h-2j-3k)+(h-2j+k)=2\dim\overset{\circ}{X}{}^{(2j+k+1)}$ and since $-(d-2j-k)-(j+k)-j=-d$, we have indeed the Poincaré duality pairing (10.5.1).

Set $d_{j,k} := \dim \overset{\circ}{X}{}^{(2j+k+1)}$. Let $d_{1,n}^{-k,h+k}\colon E_{1,n}^{-k,h+k} \longrightarrow E_{1,n}^{-k+1,h+k}$ be the boundary morphism of the E_1-terms of the preweight spectral sequence (9.10.2). The morphism $d_{1,n}^{-k,2d-h-k}$ is given by the following diagram

$$\begin{array}{c} H^{2d_{j,k}-(h-2j+k)}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+2)}/W_n)(-j-k) \\ {\scriptstyle (-1)^{j+k}\rho}\big\uparrow \\ H^{2d_{j,k}-(h-2j+k)}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j-k) \\ {\scriptstyle (-1)^{j}G}\big\downarrow \\ H^{2d_{j,k}-(h-2j+k)+2}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k)}/W_n)(-j-k+1). \end{array} \tag{10.5.3}$$

By (10.1.2; n), the morphism $d_{1,n}^{k,h+k}$ is given by

$$[(-1)^{j+k}G+(-1)^{j}\rho]\colon \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j+k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j) \longrightarrow \bigoplus_{j\geq \max\{-(k+1),0\}} H^{h-2j+k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+2)}/W_n)(-j).$$

Hence the morphism $d_{1,n}^{k-1,h+k}$ is given by the following diagram

$$\begin{array}{c} H^{h-2j+k-2}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+2)}/W_n)(-j-1) \\ {\scriptstyle (-1)^{(j+1)+k-1}G}\big\downarrow \\ H^{h-2j+k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k+1)}/W_n)(-j) \\ {\scriptstyle (-1)^{j}\rho}\big\uparrow \\ H^{h-2j+k}_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2j+k)}/W_n)(-j). \end{array} \tag{10.5.4}$$

Using the adjoint property of G and ρ, we have

$$\langle d_{1,n}^{-k,2d-h-k}(*), **\rangle = \langle *, d_{1,n}^{k-1,h+k}(**)\rangle. \tag{10.5.5}$$

Hence we obtain (1).

(2) immediately follows from (1). □

REMARK 10.6. (1) In [M1, (4.15)], we have to kill the torsions of the E_r-terms in [loc. cit.] since the Poincaré duality does not hold over W in general. Furthermore we have to make a restriction $r \leq 2$ in [loc. cit.] since the E_2-degeneration was not proved in [loc. cit.] (we do not need the restriction in this paper thanks to (3.6)). The upper indexes of the E_r-terms of the duality in [loc. cit] are mistaken; there exist proper SNCL surfaces over κ such that $E_2^{10} \otimes_W K_0 \simeq K_0$ but $E_2^{12} \otimes_W K_0 = 0$. See [Nakk4] for the examples.

(2) If we make the identification (10.2.1), we have the formula

$$\langle d_{1,n}^{-k,2d-h-k}(*), **\rangle = -\langle *, d_{1,n}^{k-1,h+k}(**)\rangle. \tag{10.6.1}$$

instead of (10.5.5). The description (10.1.2;$\star$) is better than the description of $d_{1,n}^{-k,h+k}$ by the use (10.2.1).

11. p-adic monodromy operators

In this section, we define a complex $(W_n\widetilde{\Lambda}^\bullet_X)''$ which is a correction of $(W_n\widetilde{\omega}^\bullet_X)'$ ($n \in \mathbb{Z}_{>0}$) in [HK, (4.20)]. Next we establish a relation between $(W_n\widetilde{\Lambda}^\bullet_X)''$ and $W_n\widetilde{\Lambda}^\bullet_X$ which has been denoted by $W_n\widetilde{\omega}^\bullet_X$ in [Hy2] and [M1]. There is a similar relation in [HK, (4.20)] in a case where X is the special fiber of a semistable family over a complete discrete valuation ring of mixed characteristics with residue field κ; but our relation in (11.1) below is different from the relation in [loc. cit.] a priori even in the semistable case. In (11.5) (2) below, our relation will show that the monodromy operator in [HK, §3] coincides with that in [Hy2, Introduction] via a canonical isomorphism; the proof of the coincidence in [M1, 2.3] is mistaken; see (11.12) (2) below for the reason. This coincidence gives a right proof of the interpretation of the p-adic monodromy operator of Hyodo-Kato by a canonical operator ν of the p-adic Steenbrink complex; this claim in [M1, 3.18] is also mistaken since ν in [M1, 3.13] is *not* a morphism of complexes if $\dim X \geq 2$: see (11.9) (1) below for details.

(11.5) (2) below is necessary for the proof of the p-adic monodromy-weight conjecture for a proper SNCL curve over a log point ([M1]; see also (11.13) below), for a proper semistable family of surfaces over a complete discrete valuation ring with simple normal crossing special fiber ([Nakk4]) and for other cases ([Nakk3], [Nakk4]). Of course, if one considers the p-adic monodromy operator in [Hy2, Introduction] as the definition (cf. (11.3.8) below), then (11.5) (2) is not necessary for proving the p-adic monodromy weight conjecture in the cases above; the analogous proof in [M1, 3.18] is enough for proving it in the cases above if one considers the Frobenius action into consideration in [Hy2, Introduction] and [M1, 3.18] (cf. (11.7.1;$\star$) below), and if one makes a correction in (11.9) below (cf. (11.8.1) below). However, the definition in [Hy2, Introduction] is obviously valid only when the base scheme is the spectrum of a perfect field or at most a perfect scheme, and hence it is not very good to consider the p-adic monodromy operator in [loc. cit.] as the definition of it; in a future paper we shall give the definition of a p-adic monodromy operator in the same way as that in [HK, (3.6)] for a more general base scheme by using the crystalline complex.

Let X be an SNCL variety with log structure M over s. Let $\alpha\colon M \longrightarrow \mathcal{O}_X$ be the structural morphism. Consider two abelian subsheaves $\widetilde{\mathcal{F}}_n$ and $\widetilde{\mathcal{G}}_n$ in

$$(11.0.1)\qquad (W_n(\mathcal{O}_X)\otimes_{\mathbb{Z}}\bigwedge^{i}(M^{\mathrm{gp}}/\alpha^{-1}(\kappa^*))) \oplus (W_n(\mathcal{O}_X)\otimes_{\mathbb{Z}}\bigwedge^{i-1}(M^{\mathrm{gp}}/\alpha^{-1}(\kappa^*))):$$

the sheaf $\widetilde{\mathcal{F}}_n$ is, by definition, generated by the forms of the types (7.0.2), and $\widetilde{\mathcal{G}}_n$ is, by definition, generated by the forms of the type (7.0.4). Set

$$(11.0.2)\quad (W_n\widetilde{\Lambda}^i_X)'' := \{(W_n(\mathcal{O}_X)\otimes_{\mathbb{Z}}\bigwedge^{i}(M^{\mathrm{gp}}/\alpha^{-1}(\kappa^*))) \oplus (W_n(\mathcal{O}_X)\otimes_{\mathbb{Z}}\bigwedge^{i-1}(M^{\mathrm{gp}}/\alpha^{-1}(\kappa^*)))\}/(\widetilde{\mathcal{F}}_n+\widetilde{\mathcal{G}}_n).$$

For the time being, we consider $(W_n\widetilde{\Lambda}^i_X)''$ only as an abelian sheaf on X_{zar}. We define a boundary morphism $d\colon (W_n\widetilde{\Lambda}^i_X)'' \longrightarrow (W_n\widetilde{\Lambda}^{i+1}_X)''$ by the same formula as that of (7.7.6). Thus we have a complex $(W_n\widetilde{\Lambda}^{\bullet}_X)''$ of abelian sheaves. Set $(W\widetilde{\Lambda}^{\bullet}_X)'' := \varprojlim_n (W_n\widetilde{\Lambda}^{\bullet}_X)''$. The Frobenius of X induces a morphism

$$\widetilde{\Phi}''_{\star}\colon (W_{\star}\widetilde{\Lambda}^i_X)'' \longrightarrow (W_{\star}\widetilde{\Lambda}^i_X)'' \quad (\star = n\in\mathbb{Z}_{>0} \text{ or nothing }).$$

Let us also recall the Frobenius

$$\widetilde{\Phi}_{\star}\colon W_{\star}\widetilde{\Lambda}^i_X \longrightarrow W_{\star}\widetilde{\Lambda}^i_X \quad (\star = n\in\mathbb{Z}_{>0} \text{ or nothing })$$

defined in (9.11.2).

Let e be a global section of the log structure M_s of s whose image in $\Gamma(s, M_s/\mathcal{O}_s^*)$ is a generator. Let τ be the image of e in $\Gamma(X, M)$. Let $d\log\tau_\star$ be the image of $1 \otimes \tau \in \Gamma(X, W_\star(\mathcal{O}_X) \otimes M^{\mathrm{gp}})$ in $\Gamma(X, (W_\star\widetilde{\Lambda}^1_X)'')$. Then $d\log\tau_\star$ is independent of the choice of e.

Theorem 11.1. (1) *There exists a canonical isomorphism*

$$\widetilde{s}_n \colon (W_n\widetilde{\Lambda}^\bullet_X)'' \xrightarrow{\sim} W_n\widetilde{\Lambda}^\bullet_X$$

which makes the following two diagrams commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (W_n\Lambda^\bullet_X)''(-1)[-1] & \xrightarrow{d\log\tau_n\wedge} & (W_n\widetilde{\Lambda}^\bullet_X)'' & \longrightarrow & (W_n\Lambda^\bullet_X)'' & \longrightarrow & 0 \\ & & {\scriptstyle s_n,\ \simeq}\downarrow & & {\scriptstyle \tilde{s}_n,\ \simeq}\downarrow & & {\scriptstyle s_n,\ \simeq}\downarrow & & \\ 0 & \longrightarrow & W_n\Lambda^\bullet_X(-1)[-1] & \xrightarrow{\theta_n\wedge} & W_n\widetilde{\Lambda}^\bullet_X & \longrightarrow & W_n\Lambda^\bullet_X & \longrightarrow & 0, \end{array} \tag{11.1.1;n}$$

$$\begin{array}{ccc} (W_{n+1}\widetilde{\Lambda}^\bullet_X)'' & \xrightarrow[\sim]{\tilde{s}_{n+1}} & W_{n+1}\widetilde{\Lambda}^\bullet_X \\ {\scriptstyle \mathrm{proj.}}\downarrow & & \downarrow{\scriptstyle \pi} \\ (W_n\widetilde{\Lambda}^\bullet_X)'' & \xrightarrow[\sim]{\tilde{s}_n} & W_n\widetilde{\Lambda}^\bullet_X. \end{array} \tag{11.1.2}$$

(*Note that, in* (11.1.1;n), *we have taken the left wedge products* $d\log\tau_n\wedge$ *and* $\theta_n\wedge$, *and we have shifted the two left complexes in* (11.1.1;n) *by* $[-1]$ *in order that the horizontal sequences in* (11.1.1;n) *are exact sequences of complexes.*)

(2) *The two projections* $\mathrm{proj}\colon (W_{n+1}\widetilde{\Lambda}^\bullet_X)'' \longrightarrow (W_n\widetilde{\Lambda}^\bullet_X)''$, $\mathrm{proj}\colon (W_{n+1}\Lambda^\bullet_X)'' \longrightarrow (W_n\Lambda^\bullet_X)''$ *and the other two projections* $\pi\colon W_{n+1}\widetilde{\Lambda}^\bullet_X \longrightarrow W_n\widetilde{\Lambda}^\bullet_X$, $\pi\colon W_{n+1}\Lambda^\bullet_X \longrightarrow W_n\Lambda^\bullet_X$ *induce a morphism from* (11.1.1;$n+1$) *to* (11.1.1;n).

(3) *There exists the following commutative diagram*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (W\Lambda^\bullet_X)''(-1)[-1] & \xrightarrow{d\log\tau\wedge} & (W\widetilde{\Lambda}^\bullet_X)'' & \longrightarrow & (W\Lambda^\bullet_X)'' & \longrightarrow & 0 \\ & & {\scriptstyle s,\ \simeq}\downarrow & & {\scriptstyle \tilde{s},\ \simeq}\downarrow & & {\scriptstyle s,\ \simeq}\downarrow & & \\ 0 & \longrightarrow & W\Lambda^\bullet_X(-1)[-1] & \xrightarrow{\theta\wedge} & W\widetilde{\Lambda}^\bullet_X & \longrightarrow & W\Lambda^\bullet_X & \longrightarrow & 0, \end{array} \tag{11.1.1}$$

Proof. (1): For the time being, we ignore the Frobenius action. Following [HK, (4.9)], we first construct a morphism $\widetilde{s}_n$ locally. Let M be the log structure of X. Assume that $\overset{\circ}{X}$ is affine. Let $(\mathcal{Y},\mathcal{X})$ be an admissible lift of X. Assume that $\overset{\circ}{\mathcal{Y}}$ is formally etale over $\mathrm{Spf}(W\{x_0,\dots,x_d\})$ with structural morphism $\mathrm{Spf}(W\{x_0,\dots,x_d\}) \longrightarrow \mathrm{Spf}(W\{t\})$ defined by $t \longmapsto x_0\cdots x_r$ $(r \leq d)$. Let $\mathcal{M}_{\mathcal{Y}}$ be the log structures of $\mathcal{Y}$ associated to a morphism $\mathbb{N}^{r+1} \ni (0,\dots,0,\overset{i}{1},0,\dots,0) \longmapsto x_{i-1} \in \mathcal{O}_{\mathcal{Y}}$. Let $\mathcal{M}_{\mathcal{X}}$ be the pull-back of $\mathcal{M}_{\mathcal{Y}}$ to $\overset{\circ}{\mathcal{X}}$. Let $\mathcal{M}_{\mathcal{X}_n}$ be the pull-back of $\mathcal{M}_{\mathcal{X}}$ to $\overset{\circ}{\mathcal{X}}_n := \overset{\circ}{\mathcal{X}} \otimes_W W_n$. Let $\Omega^{\bullet}_{\mathcal{X}_n/W_n}(\log \mathcal{M}_{\mathcal{X}_n})$ be the log de Rham complex of $\mathcal{X}_n/(\mathrm{Spec}(W_n),W_n^*)$. Set $\widetilde{\Lambda}^{\bullet}_n = \mathcal{O}_{\mathcal{X}_n} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Omega^{\bullet}_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n)$. Then the natural surjection $\Omega^{\bullet}_{\mathcal{Y}_n/W_n}(\log \mathcal{X}_n) \longrightarrow \Omega^{\bullet}_{\mathcal{X}_n/W_n}(\log \mathcal{M}_{\mathcal{X}_n})$ induces an isomorphism

$$\widetilde{\Lambda}^{\bullet}_n \xrightarrow{\sim} \Omega^{\bullet}_{\mathcal{X}_n/W_n}(\log \mathcal{M}_{\mathcal{X}_n}). \tag{11.1.3}$$

As in [HK, (4.9)], we define three morphisms

$$\widetilde{s}_n(0,0)\colon W_n(\mathcal{O}_X) \ni (a_0,\dots,a_{n-1}) \longmapsto \sum_{i=0}^{n-1} p^i \widetilde{a}_i^{p^{n-i}} \in \mathcal{H}^0(\widetilde{\Lambda}^{\bullet}_n), \tag{11.1.4}$$

$$\widetilde{s}_n(1,0)\colon W_n(\mathcal{O}_X) \ni (a_0,\dots,a_{n-1}) \longmapsto \sum_{i=0}^{n-1} \widetilde{a}_i^{p^{n-i}-1} d\widetilde{a}_i \in \mathcal{H}^1(\widetilde{\Lambda}^{\bullet}_n), \tag{11.1.5}$$

$$d\log\colon M^{\mathrm{gp}} \ni b \longmapsto d\log \widetilde{b} \in \mathcal{H}^1(\widetilde{\Lambda}^{\bullet}_n), \tag{11.1.6}$$

where $\widetilde{a}_i \in \mathcal{O}_{\mathcal{X}_n}$ and $\widetilde{b} \in \mathcal{M}_{\mathcal{X}_n}$ are lifts of a_i and b, respectively. Then, by the same proof as that of [loc. cit.], $\widetilde{s}_n(0,0)$, $\widetilde{s}_n(1,0)$ and $d\log$ are well-defined. As in [loc. cit.], we have a morphism $\widetilde{s}_n\colon (W_n\widetilde{\Lambda}^i_X)'' \longrightarrow W_n\widetilde{\Lambda}^i_X$ $(i \in \mathbb{N})$ of abelian sheaves. It is a routine work to check that $\widetilde{s}_n$ induces a morphism $\widetilde{s}_n\colon (W_n\widetilde{\Lambda}^{\bullet}_X)'' \longrightarrow W_n\widetilde{\Lambda}^{\bullet}_X$ of complexes.

Next, we claim that $\widetilde{s}_n$ is independent of the choice of the admissible lift of X. Let $(\mathcal{Y}',\mathcal{X}')$ be another admissible lift of X. Let $\mathcal{Y}''_n$ be a scheme over $W_n[t]$ constructed in, e.g., [M1, 3.4]: the scheme $\mathcal{Y}''_n$ is obtained by the reduction mod p^n of an open scheme of a blow up of the product $\mathcal{Y} \times_{\mathrm{Spf}(W)} \mathcal{Y}'$

along certain closed formal subscheme of the product [loc. cit.]. Then $\mathcal{Y}''_n$ is a smooth scheme over W_n with relative SNCD $\mathcal{X}''_n$ defined by $t=0$ and such that there exists a commutative diagram

$$\begin{array}{ccccccc} X & \overset{\subset}{\longrightarrow} & \mathcal{X}_n & \longrightarrow & \mathcal{Y}_n & \longrightarrow & \mathrm{Spec}(W_n[t]) \\ \| & & \uparrow & & \uparrow & & \| \\ X & \overset{\Delta_n,\subset}{\longrightarrow} & \mathcal{X}''_n & \longrightarrow & \mathcal{Y}''_n & \longrightarrow & \mathrm{Spec}(W_n[t]) \\ \| & & \downarrow & & \downarrow & & \| \\ X & \overset{\subset}{\longrightarrow} & \mathcal{X}'_n & \longrightarrow & \mathcal{Y}'_n & \longrightarrow & \mathrm{Spec}(W_n[t]). \end{array} \tag{11.1.7}$$

Here the morphism $\Delta_n\colon X \overset{\subset}{\longrightarrow} \mathcal{X}''_n$ is an exact closed immersion obtained by the reduction mod p^n of the strict transform in $\mathcal{Y}''$ of the image of the diagonal embedding $X \overset{\subset}{\longrightarrow} \mathcal{Y} \times_{\mathrm{Spf}(W)} \mathcal{Y}'$. Let $\mathcal{D}_n$ be the usual divided power of the exact closed immersion Δ_n. Let $\widetilde{\Lambda}''^{\bullet}_n$ be the log de Rham complex of a morphism of log schemes $\mathcal{X}''_n \longrightarrow (\mathrm{Spec}(W_n), W_n^*)$. Let $\widetilde{s}'_n\colon (W_n\widetilde{\Lambda}^{\bullet}_X)'' \longrightarrow \mathcal{H}^{\bullet}(\widetilde{\Lambda}'^{*}_n)$ be an analogue of $\widetilde{s}_n$ for the admissible lift $(\mathcal{Y}', \mathcal{X}')$. Let $p_1\colon (\mathcal{Y}''_n, \mathcal{X}''_n) \longrightarrow (\mathcal{Y}_n, \mathcal{X}_n)$ and $p_2\colon (\mathcal{Y}''_n, \mathcal{X}''_n) \longrightarrow (\mathcal{Y}'_n, \mathcal{X}'_n)$ be the "projections". By [M1, 3.4] (cf. [Hy2, p. 247–248]), p_1^* and p_2^* induces an isomorphism

$$p_1^*\colon \mathcal{H}^{\bullet}(\widetilde{\Lambda}^*_n) \overset{\sim}{\longrightarrow} \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n} \otimes_{\mathcal{O}_{\mathcal{X}''_n}} \widetilde{\Lambda}''^{*}_n),$$

$$p_2^*\colon \mathcal{H}^{\bullet}(\widetilde{\Lambda}'^{*}_n) \overset{\sim}{\longrightarrow} \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n} \otimes_{\mathcal{O}_{\mathcal{X}''_n}} \widetilde{\Lambda}''^{*}_n).$$

It suffices to prove that the following diagram is commutative:

$$\begin{array}{ccc} (W_n\widetilde{\Lambda}^{\bullet}_X)'' & \overset{\widetilde{s}_n}{\longrightarrow} & \mathcal{H}^{\bullet}(\widetilde{\Lambda}^*_n) \\ \| & & \downarrow p_1^* \\ (W_n\widetilde{\Lambda}^{\bullet}_X)'' & & \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n} \otimes_{\mathcal{O}_{\mathcal{X}''_n}} \widetilde{\Lambda}''^{*}_n) \\ \| & & \uparrow p_2^* \\ (W_n\widetilde{\Lambda}^{\bullet}_X)'' & \overset{\widetilde{s}'_n}{\longrightarrow} & \mathcal{H}^{\bullet}(\widetilde{\Lambda}'^{*}_n). \end{array} \tag{11.1.8}$$

Let $\widetilde{s}''_n\colon W_n\widetilde{\Lambda}^{\bullet}_X \longrightarrow \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n} \otimes_{\mathcal{O}_{\mathcal{X}''_n}} \widetilde{\Lambda}''^{\bullet}_n)$ be a morphism defined as in (11.1.4), (11.1.5) and (11.1.6) by using the "diagonal" exact closed immersion

$\Delta_n\colon X \overset{\subset}{\longrightarrow} \mathcal{X}''_n$ ($\widetilde{s}''_n$ is well-defined as in [HK, (4.9)]). By the symmetry, it suffice to prove the commutativity of the following diagram

$$\begin{array}{ccc} (W_n\widetilde{\Lambda}^{\bullet}_X)'' & \xrightarrow{\widetilde{s}_n} & \mathcal{H}^{\bullet}(\widetilde{\Lambda}^*_n) \\ {\scriptstyle \widetilde{s}''_n}\downarrow & & \downarrow{\scriptstyle p_1^*} \\ \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\widetilde{\Lambda}''^*_n) & = & \mathcal{H}^{\bullet}(\mathcal{O}_{\mathcal{D}_n}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\widetilde{\Lambda}''^*_n). \end{array} \tag{11.1.9}$$

Since $p_1\circ\Delta_n$ is the given exact closed immersion $X \overset{\subset}{\longrightarrow} \mathcal{X}_n$ and since $\widetilde{s}_n$ and $\widetilde{s}''_n$ are defined by lifts of sections of $W_n(\mathcal{O}_X)$ and M^{gp}, the commutativity of (11.1.9) follows from the well-definedness of the morphism $\widetilde{s}''_n$, which is proved by the same argument as that of [HK, (4.9)]. Therefore $\widetilde{s}_n$ is independent of the choice of the admissible lift of X.

As in [HK, (4.20)], by [Hy2, (1.4.3)], we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & (W_n\Lambda^{\bullet}_X)''[-1] & \xrightarrow{d\log\tau_n\wedge} & (W_n\widetilde{\Lambda}^{\bullet}_X)'' & \longrightarrow & (W_n\Lambda^{\bullet}_X)'' & \longrightarrow & 0 \\ & & {\scriptstyle s_n}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\widetilde{s}_n}\downarrow & & {\scriptstyle s_n}\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & W_n\Lambda^{\bullet}_X[-1] & \xrightarrow{\theta_n\wedge} & W_n\widetilde{\Lambda}^{\bullet}_X & \longrightarrow & W_n\Lambda^{\bullet}_X & \longrightarrow & 0. \end{array} \tag{11.1.10}$$

Because s_n is an isomorphism, so is $\widetilde{s}_n$.

The compatibility of the upper row of (11.1.1;n) with the Frobenius is obvious. Because s_n is compatible with the Frobenius ((9.1.2)), we have only to prove that $\widetilde{s}_n$ is compatible with the Frobenius. This follows from the local description of $\widetilde{s}_n$ and from the proof of (7.1).

Using (6.28) (8), we can prove the commutativity of (11.1.2) by the same proof as that of (7.1).

(2): (2) follows from the commutativity of (11.1.1;n), that of (11.1.2) and (7.1).

(3): (3) follows from (1) and (2). □

COROLLARY 11.2. *The sheaf* $\bigoplus_{i\geq 0}(W_\star\widetilde{\Lambda}^i_X)''$ ($\star = n$ *or nothing) has a natural module structure over the Cartier-Dieudonné-Raynaud algebra of* κ.

PROOF. Since the sheaf $\bigoplus_{i\geq 0}W_\star\widetilde{\Lambda}^i_X$ has a module structure over the Cartier-Dieudonné-Raynaud algebra of κ, $\bigoplus_{i\geq 0}(W_\star\widetilde{\Lambda}^i_X)''$ also has it by (11.1). □

COROLLARY 11.3. (1) *Let* $\mathbf{p}''\colon (W_n\widetilde{\Lambda}^{\bullet}_X)'' \longrightarrow (W_{n+1}\widetilde{\Lambda}^{\bullet}_X)''$ *be the induced morphism by the multiplication by* $p\colon (W_{n+1}\widetilde{\Lambda}^{\bullet}_X)'' \longrightarrow (W_{n+1}\widetilde{\Lambda}^{\bullet}_X)''$. *Then* $\mathbf{p}''$ *fits into the following commutative diagram*

$$\begin{array}{ccc} (W_n\widetilde{\Lambda}^{\bullet}_X)'' & \xrightarrow[\sim]{\widetilde{s}_n} & W_n\widetilde{\Lambda}^{\bullet}_X \\ {\scriptstyle \mathbf{p}''}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{p}} \\ (W_{n+1}\widetilde{\Lambda}^{\bullet}_X)'' & \xrightarrow[\sim]{\widetilde{s}_{n+1}} & W_{n+1}\widetilde{\Lambda}^{\bullet}_X. \end{array} \tag{11.3.1}$$

(2) *The abelian sheaf* $(W\widetilde{\Lambda}^{i}_X)''$ $(i \in \mathbb{Z})$ *on* X_{zar} *is torsion-free.*

PROOF. (1): (1) follows from (11.1.2) and the obvious analogue of (7.8.1).

(2): By the injectivity of $\mathbf{p}\colon W_n\widetilde{\Lambda}^{i}_X \longrightarrow W_{n+1}\widetilde{\Lambda}^{i}_X$ ([Hy2, (2.2.2)]), $\varprojlim_n W_n\widetilde{\Lambda}^{i}_X$ is torsion-free. Hence (2) follows from the commutative diagram (11.3.1). □

Let h be an integer. We define a monodromy operator as the boundary morphism of the upper exact sequence of (11.1.1;n):

$$\begin{aligned} (N_{\mathrm{dRW},n})''\colon H^h(X,(W_n\Lambda^{\bullet}_X)'') &\longrightarrow H^h(X,(W_n\Lambda^{\bullet}_X)''(-1)[-1][1]) \\ &= H^h(X,(W_n\Lambda^{\bullet}_X)'')(-1). \end{aligned} \tag{11.3.2}$$

As in [Hy2, Introduction], we define a monodromy operator as the boundary morphism of the lower exact sequence of (11.1.1;n):

$$\begin{aligned} N_{\mathrm{dRW},n}\colon H^h(X,W_n\Lambda^{\bullet}_X) &\longrightarrow H^h(X,W_n\Lambda^{\bullet}_X(-1)[-1][1]) \\ &= H^h(X,W_n\Lambda^{\bullet}_X)(-1). \end{aligned} \tag{11.3.3}$$

As in [HK, (3.6)], let $((X_\bullet,M_\bullet),(Y_\bullet,N_\bullet))$ be an embedding system (recall (7.20)) of a composite morphism $(X,M) \longrightarrow (\mathrm{Spec}(\kappa),L) \longrightarrow (\mathrm{Spec}(W_n[t]),\mathcal{L})$, where $\mathcal{L}$ is the log structure associated to a morphism $\mathbb{N} \ni 1 \longmapsto t \in W_n[t]$. Let C_{X/W_n} and $C_{X/(W_n,W_n(M_s))}$ be the crystalline complexes with respect to the embedding system above and $((X_\bullet,M_\bullet),(Y_\bullet\otimes_{W_n[t]} W_n, N_\bullet\otimes_{W_n[t]} W_n))$, respectively. Let

$$\eta\colon \widetilde{X_{\bullet}}_{\mathrm{zar}} \longrightarrow \widetilde{X}_{\mathrm{zar}} \tag{11.3.4}$$

be a natural morphism of topoi. Then we have a triangle

$$(11.3.5)\qquad R\eta_*(C_{X/(W_n,W_n(M_s))})(-1)[-1] \overset{d\log\tau_n\wedge}{\longrightarrow} R\eta_*(W_n\otimes_{W_n\langle t\rangle}C_{X/W_n}) \longrightarrow R\eta_*(C_{X/(W_n,W_n(M_s))}) \overset{+1}{\longrightarrow}$$

by [HK, (3.6)], and let

$$(11.3.6)\qquad N_{\mathrm{crys},n}\colon H^h_{\text{log-crys}}(X/W_n) \longrightarrow H^h_{\text{log-crys}}(X/W_n)(-1)$$

be the boundary morphism of the triangle above(cf. [loc. cit.]). By the upper exact sequence of (11.1.1), we also have the following monodromy operator:

$$(11.3.7)\qquad (N_{\mathrm{dRW}})''\colon H^h(X,(W\Lambda^\bullet_X)'') \longrightarrow H^h(X,(W\Lambda^\bullet_X)'')(-1).$$

Similarly, by the lower exact sequence of (11.1.1), we obtain the following monodromy operator:

$$(11.3.8)\qquad N_{\mathrm{dRW}}\colon H^h(X,W\Lambda^\bullet_X) \longrightarrow H^h(X,W\Lambda^\bullet_X)(-1).$$

Then the following hold:

THEOREM 11.4. (1) *Via the identification* $H^h(X,(W_\star\Lambda^\bullet_X)'') \overset{s_\star}{\underset{\sim}{\longrightarrow}} H^h(X,W_\star\Lambda^\bullet_X)$ *in* (7.5), $(N_{\mathrm{dRW},\star})'' = N_{\mathrm{dRW},\star}$ ($\star = n$ *or nothing*).

(2) *Via the identification* $H^h_{\text{log-crys}}(X/W_n) = H^h(X,W_n\Lambda^\bullet_X)$ *in* (7.19), $N_{\mathrm{dRW},n} = N_{\mathrm{crys},n}$

PROOF. (1): (1) is obvious by (11.1).

(2): Let the notations be as before (11.4). Moreover, the embedding system $(X_\bullet,M_\bullet) \overset{\subset}{\longrightarrow} (Y_\bullet,N_\bullet)$ can be assumed to factor through the exact closed immersion $(X_\bullet,M_\bullet) \overset{\subset}{\longrightarrow} (W_n(X_\bullet),W_n(M_\bullet))$. Let $\bigoplus_{i\geq 0}\Lambda^i_{W_n(X_\bullet)/W_n,[\ ]}$ be a differential graded algebra over W_n which is a quotient of $\bigoplus_{i\geq 0}\Lambda^i_{W_n(X_\bullet)/(W_n,W_n^*)}$ divided by a W_n-submodule (not a $\mathbb{Z}$-submodule) generated by the local sections of the form $da^{[j]} - a^{[j-1]}da$ ($a\in \mathrm{Ker}(W_n(\mathcal{O}_X)\longrightarrow \mathcal{O}_X)$, $j\geq 1$). Then we have a morphism $C_{X/W_n} \longrightarrow \Lambda^\bullet_{W_n(X_\bullet)/W_n,[\ ]}$ of complexes of W_n-modules. This morphism factors through a morphism $W_n\otimes_{W_n\langle t\rangle}C_{X/W_n} \longrightarrow \Lambda^\bullet_{W_n(X_\bullet)/W_n,[\ ]}$. As in [HK, p. 251], by using (7.4), we see that there exists a unique morphism $\Lambda^1_{W_n(X_\bullet)/(W_n,W_n^*)} \longrightarrow$

$\eta^{-1}(W_n\widetilde{\Lambda}^1_X)$ of $W_n(\mathcal{O}_X)$-modules such that da ($a \in W_n(\mathcal{O}_{X_\bullet})$) is mapped to $\widetilde{s}_n(1,0)(a)$ and such that $d\log b$ ($b \in M_\bullet$) is mapped to $d\log \widetilde{b}$. This morphism induces a morphism

$$\widetilde{\psi}_n \colon \bigoplus_{i\geq 0} \Lambda^i_{W_n(X_\bullet)/W_n} \longrightarrow \bigoplus_{i\geq 0} \eta^{-1}(W_n\widetilde{\Lambda}^i_X).$$

By (11.3) (2), by the same proof as that of [HK, (4.19)] and by the obvious analogue of the commutative diagram (7.18.2), $\widetilde{\psi}_n$ induces a morphism of complexes

$$\widetilde{\psi}_n \colon \Lambda^\bullet_{W_n(X_\bullet)/W_n,[\ ]} \longrightarrow \eta^{-1}(W_n\widetilde{\Lambda}^\bullet_X).$$

Hence we have a morphism

$$W_n \otimes_{W_n\langle t\rangle} C_{X/W_n} \longrightarrow \eta^{-1}(W_n\widetilde{\Lambda}^\bullet_X). \tag{11.4.1}$$

By [HK, (3.6)], by (11.1) (1) and by the construction (11.4.1), we have the following commutative diagram of triangles:

$$\begin{array}{ccccc} R\eta_*(C_{X/(W_n,W_n(M_s))})(-1)[-1] & \xrightarrow{d\log\tau_n\wedge} & R\eta_*(W_n\otimes_{W_n\langle t\rangle} C_{X/W_n}) & \longrightarrow \\ \downarrow & & \downarrow & \\ W_n\Lambda^\bullet_X(-1)[-1] & \xrightarrow{\theta_n\wedge} & W_n\widetilde{\Lambda}^\bullet_X & \longrightarrow \\ R\eta_*(C_{X/(W_n,W_n(M_s))}) & \xrightarrow{+1} & \cdots & \\ \downarrow & & & \\ W_n\Lambda^\bullet_X & \xrightarrow{+1} & \cdots . & \end{array} \tag{11.4.2}$$

Hence we have (2). $\square$

Corollary 11.5. *The following hold*:
(1) $N_{\text{crys},n} = N''_{\text{dRW},n}$.
(2) *Let the notations be as before* (11.4). *Set*

$$H^h_{\text{log-crys}}(X/W) := H^h(X, R\varprojlim_n R\eta_*(C_{X/(W_n,W_n(M_s))}))$$

and $d\log\tau\wedge * := R\varprojlim_n(d\log\tau_n \wedge *)$. *Let*

$$N_{\text{crys}} \colon H^h_{\text{log-crys}}(X/W) \longrightarrow H^h_{\text{log-crys}}(X/W)(-1)$$

be the boundary morphism of the following triangle

$$\begin{aligned}(11.5.1)\qquad & R\varprojlim_n R\eta_*(C_{X/(W_n,W_n(M_s))})(-1)[-1] \\ & \overset{d\log\tau\wedge *}{\longrightarrow} R\varprojlim_n R\eta_*(W_n\otimes_{W_n\langle t\rangle} C_{X/W_n}) \\ & \longrightarrow R\varprojlim_n R\eta_*(C_{X/(W_n,W_n(M_s))}) \overset{+1}{\longrightarrow}\cdots\end{aligned}$$

in $\mathrm{D}^+(\overset{\circ}{f}{}^{-1}(W_n))$. *Here* $\overset{\circ}{f}\colon \overset{\circ}{X} \longrightarrow \mathrm{Spec}(W_n)$ *is the structural morphism. Then* $N_{\mathrm{crys}} = N_{\mathrm{dRW}}$.

Proof. (1): (1) immediately follows from (11.4) (1), (2).

(2): Since the transitive morphisms of the projective system $\{(W_n\Lambda^\bullet_X)''\}_n$ are surjective and since $(W_n\Lambda^i_X)''$ $(i\in\mathbb{N})$ is a sheaf of quasi-coherent $W_n(\mathcal{O}_X)$-modules, we have

$$R\varprojlim_n (W_n\Lambda^\bullet_X)'' = \varprojlim_n (W_n\Lambda^\bullet_X)''.$$

Hence (2) follows from the obvious analogue (for $\tilde{\psi}_n$) of the commutative diagram (7.18.1) and from the commutative diagram (11.4.2). □

Definition 11.6. (1) Let $\star$ be a positive integer n or nothing. For a non-negative integer k, the double complex $W_\star A^{\bullet *}_X(-k)$ is, by definition, $W_\star A^{\bullet *}_X(-k) := W_\star A^{\bullet *}_X$ with Frobenius action $p^k\widetilde{\Phi}^{\bullet *}_\star$ ((9.8), (9.9), (9.11)). The complex $W_\star A^\bullet_X(-k)$ is defined in a similar way.

(2) The morphism $\nu_\star\colon W_\star A^\bullet_X \longrightarrow W_\star A^\bullet_X$ is, by definition, the induced morphism by a morphism $(-1)^{i+j+1}\mathrm{proj.}\colon W_\star A^{ij}_X \longrightarrow W_\star A^{i-1,j+1}_X$. (It is easy to check that $\nu_\star$ is indeed a morphism of complexes with the convention (2.2.1;$\star$).) We call ν the *p-adic quasi-monodromy operator* of X.

Proposition 11.7. (1) *The p-adic quasi-monodromy operator* $\nu_\star$ *of* X *is a morphism of complexes which is compatible with the Frobenius*:

$$(11.7.1;\star)\qquad \nu_\star\colon W_\star A^\bullet_X \longrightarrow W_\star A^\bullet_X(-1).$$

(2) *Let* k *be a positive integer. Under the identification* (9.9.3) *and* (9.10.1)

$$H^h(X,\mathrm{gr}^P_k W_\star A^\bullet_X) = \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k),$$

the induced isomorphism

$$\nu_\star^k\colon \bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \overset{\sim}{\longrightarrow}$$

$$\bigoplus_{j\geq \max\{-k,0\}} H^{h-2j-k}(X, W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k)$$

by the isomorphism $\nu_\star^k\colon \mathrm{gr}_k^P W_\star A_X^\bullet \overset{\sim}{\longrightarrow} \mathrm{gr}_{-k}^P W_\star A_X^\bullet(-k)$ *is the identity if* k *is even and* $(-1)^{h+1}$ *if* k *is odd.*

Proof. (1): We have only to prove that the following diagram is commutative:

$$\begin{array}{ccc} W_\star A_X^{i-1,j+1} & \overset{\text{proj.}}{\longleftarrow} & W_\star A_X^{ij} \\ {\scriptstyle p\widetilde{\Phi}_\star^{(i-1,j+1)}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\Phi}_\star^{(ij)}} \\ W_\star A_X^{i-1,j+1} & \overset{\text{proj.}}{\longleftarrow} & W_\star A_X^{ij}. \end{array} \tag{11.7.2}$$

This immediately follows from (9.8.22).

(2): The complex $\mathrm{gr}_{-k}^P W_\star A_X^\bullet$ is equal to

$$\bigoplus_{j\geq k} \mathrm{gr}_{-k}^P W_\star A_X^{\bullet j}\{-j\} = \bigoplus_{j\geq 0} \mathrm{gr}_{-k}^P W_\star A_X^{\bullet j+k}\{-j-k\},$$

and the complex $\mathrm{gr}_{-k}^P W_\star A_X^{\bullet j+k}(-k)\{-j-k\}$ is isomorphic to the following complex:

$$\cdots \overset{(-1)^{j+k+1}d}{\longrightarrow} \underset{(i-k,j+k)}{W_\star\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j)(-k)} \overset{(-1)^{j+k+1}d}{\longrightarrow} \cdots. \tag{11.7.3;$j+k$}$$

The morphism $\nu_\star^k$ induces a morphism $((-1)^{i+j+1})^k\colon$ (10.1.14;j) $\longrightarrow$ (11.7.3;$j+k$). First, assume that k is even. Then the complexes (10.1.14;j) and (11.7.3;$j+k$) are the same and $\nu_\star^k = \mathrm{id}$. Next, assume that k is odd. Then (11.7.3;$j+k$) is equal to

$$\cdots \overset{-(-1)^{j+1}d}{\longrightarrow} \underset{(i-k,j+k)}{W_\star\Omega^{i-j-k}_{\overset{\circ}{X}{}^{(2j+k+1)}}(-j)(-k)} \overset{-(-1)^{j+1}d}{\longrightarrow} \cdots. \tag{11.7.4;$j+k$}$$

and $\nu_\star^k = (-1)^{i+j+1}$. The rest of the proof follows from the lemma below. □

Lemma 11.8. *Let $(E^\bullet, d)$ be a complex of objects in an abelian category $\mathcal{A}$. Let $f^\bullet\colon (E^\bullet, d) \longrightarrow (E^\bullet, -d)$ and $g^\bullet\colon (E^\bullet, -d) \longrightarrow (E^\bullet, d)$ be morphisms of complexes defined by $f^l = g^l := (-1)^{l+1}\colon E^l \longrightarrow E^l$ ($l \in \mathbb{Z}$). Then the composite morphisms*

$$\mathcal{H}^h((E^\bullet, d)) \overset{\mathcal{H}^h(f^\bullet)}{\longrightarrow} \mathcal{H}^h((E^\bullet, -d)) \overset{\text{Convention (6)}}{=} \mathcal{H}^h((E^\bullet, d))$$

and

$$\mathcal{H}^h((E^\bullet, d)) \overset{\text{Convention (6)}}{=} \mathcal{H}^h((E^\bullet, -d)) \overset{\mathcal{H}^h(g^\bullet)}{\longrightarrow} \mathcal{H}^h((E^\bullet, d))$$

are the multiplications by $(-1)^{h+1}$.

Proof. The proof is easy. □

Our conventions (2.2.1; $\star$) and (11.1.1; $\star$) lead us to change the boundary morphisms of the double complex $W_\star B_X^{\bullet\bullet}$ as follows (cf. [St1, p. 246], [M1, p. 318]): the (i,j)-component $W_nB_X^{ij}$ ($i,j \in \mathbb{Z}_{\geq 0}$) is, by definition, $W_nA_X^{i-1,j}(-1) \oplus W_nA_X^{ij}$. The horizontal boundary morphism $d'\colon W_nB_X^{ij} \longrightarrow W_nB_X^{i+1,j}$ is, by definition,

$$d'(\omega_1, \omega_2) = ((-1)^j d\omega_1, (-1)^{j+1} d\omega_2)$$

and the vertical one $d''\colon W_nB_X^{ij} \longrightarrow W_nB_X^{i,j+1}$ is

$$d''(\omega_1, \omega_2) = ((-1)^i \theta_n \wedge \omega_1 + \nu_n(\omega_2), (-1)^i \theta_n \wedge \omega_2).$$

Here we have omitted a notation mod P_{j+1} in the definition of d'' for simplicity. It is easy to check that $W_nB_X^{\bullet\bullet}$ is indeed a double complex. Let $W_nB_X^\bullet$ be the single complex of $W_nB_X^{\bullet\bullet}$.

Let $\lambda_n\colon : W_n\widetilde{\Lambda}_X^\bullet \longrightarrow W_nB_X^\bullet$ be a morphism of complexes defined by $\lambda_n(\omega) := (\omega \mod P_0, \theta_n \wedge \omega \mod P_0)$ ($\omega \in W_n\widetilde{\Lambda}_X^\bullet$). Then, by the lower exact sequence of (11.1.1;n), there exists the following commutative diagram (cf. [loc. cit.]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_nA_X^\bullet(-1)[-1] & \longrightarrow & W_nB_X^\bullet & \longrightarrow & W_nA_X^\bullet & \longrightarrow & 0 \\ & & {\scriptstyle (\theta_n\wedge *)[-1]}\big\uparrow & & {\scriptstyle \lambda_n}\big\uparrow & & {\scriptstyle \theta_n\wedge}\big\uparrow & & \\ 0 & \longrightarrow & W_n\Lambda_X^\bullet(-1)[-1] & \overset{\theta_n\wedge}{\longrightarrow} & W_n\widetilde{\Lambda}_X^\bullet & \longrightarrow & W_n\Lambda_X^\bullet & \longrightarrow & 0. \end{array} \quad (11.8.1;n)$$

Set $WB^\bullet_X := \varprojlim_n W_nB^\bullet_X$. Moreover, by (8.1.2), [Hy2, (2.2.3)] and (8.4) (2), there also exists the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & WA^\bullet_X[-1](-1) & \longrightarrow & WB^\bullet_X & \longrightarrow & WA^\bullet_X & \longrightarrow & 0 \\ & & {\scriptstyle (\theta\wedge *)[-1]}\uparrow & & {\scriptstyle\lambda}\uparrow & & {\scriptstyle\theta\wedge}\uparrow & & \\ 0 & \longrightarrow & W\Lambda^\bullet_X[-1](-1) & \xrightarrow{\theta\wedge} & W\widetilde{\Lambda}^\bullet_X & \longrightarrow & W\Lambda^\bullet_X & \longrightarrow & 0. \end{array} \tag{11.8.1}$$

Note that, in the proof of [M1, 3.18], the Frobenius action is not considered.

REMARK 11.9. As in [M1, 3.8], let us consider the p-adic double Steenbrink complex $(W_nA^{\bullet\bullet}_X, d', d'')$ with boundary morphisms in (2.2.3; n) in this remark.

(1) As in [M1, 3.13], let us consider a morphism

$$\nu^{ij}_n \colon W_nA^{ij}_X \ni \omega \longmapsto (-1)^{i+j+1}\omega \mod P_{j+1} \in W_nA^{i-1,j+1}_X \quad (i, j \in \mathbb{N}).$$

Then

$$(\nu^{i+1,j}_n \oplus \nu^{i,j+1}_n)(d' + d'')(\omega) = ((-1)^i d\omega, (-1)^{i+j}\omega \wedge \theta)$$

and

$$(d' + d'')\nu^{ij}_n(\omega) = ((-1)^i d\omega, (-1)^{i+j+1}\omega \wedge \theta).$$

Hence the family $\{\nu^{ij}_n\}$ does *not* induce a morphism

$$(W_nA^\bullet_X, d' + d'') \longrightarrow (W_nA^\bullet_X(-1), d' + d'')$$

of complexes nor

$$(W_nA^\bullet_X, d' + d'') \longrightarrow (W_nA^\bullet_X(-1), -(d' + d''))$$

if $\dim X \geq 2$ and if $p \neq 2$ or if $n \geq 2$. (If $\dim X = 1$, then $\sum_{ij} \nu^{ij}_n \circ (d' + d'') = 0 = (d' + d'') \circ \sum_{ij} \nu^{ij}_n$ and hence $\sum_{ij} \nu^{ij}_n$ happens to be a morphism $(W_nA^\bullet_X, d' + d'') \longrightarrow (W_nA^\bullet_X(-1), \pm(d' + d''))$ of complexes.) In particular, if $\dim X \geq 2$ and if $p \neq 2$ or if $n \geq 2$, $\{\nu^{ij}_n\}$ does not induce a morphism $H^h(X, W_nA^\bullet_X) \longrightarrow H^h(X, W_nA^\bullet_X)(-1)$ of cohomologies.

(2) Große-Klönne pointed out to me that the sign in the operator in ν_n in [M1, 3.13] is mistaken in the following point.

In the diagram in [M1, p. 319], two $[-1]$'s has been used as $\{-1\}$ in this paper.

Let $(W_nB_X^{\bullet\bullet}, d', d'')$ be the double complex defined in the proof of [M1, 3.18]. The morphism d'' has been defined by the following formula:

$$d''(\omega_1, \omega_2) = (\omega_1 \wedge \theta_n + \nu_n(\omega_2), \omega_2 \wedge \theta_n) \quad ((\omega_1, \omega_2) \in W_nB_X^{\bullet\bullet}).$$

As in [loc. cit.], let $\Psi\colon W_n\widetilde{\Lambda}_X^i \longrightarrow W_nB_X^{i0}$ $(i \in \mathbb{N})$ be a morphism defined by $\Psi(\omega) = (\omega \mod P_0, \omega \wedge \theta_n \mod P_0)$. In the proof of [M1, 3.18], it is claimed that Ψ induces a morphism of complexes. This does not hold: since $d'' \circ \Psi(\omega) = (2\omega \wedge \theta_n \mod P_0, 0) \in W_nB_X^{i1}$ for an odd positive integer i, the composite morphism $d'' \circ \Psi$ is not the zero in general. Hence one has to change signs in ν_n in [M1, 3.13].

Thus, let us consider a map $\epsilon\colon \mathbb{N} \times \mathbb{N} \longrightarrow \{\pm 1\}$, a new morphism $\nu_n^{ij}\colon W_nA_X^{ij} \ni \omega \longmapsto \epsilon(i,j)\omega \mod P_{j+1}W_nA_X^{i-1,j+1}$ $(i, j \in \mathbb{N})$ and a new vertical morphism

$$d'''\colon W_nB_X^{ij} \ni (\omega_1, \omega_2) \longmapsto (\omega_1 \wedge \theta_n + \nu_n^{ij}(\omega_2), \omega_2 \wedge \theta_n) \in W_nB_X^{i,j+1}.$$

Then, two relations $d''' \circ \Psi = 0$ and $(d''')^2 = 0$ implies that $\epsilon(i, 0) = -1$ for $i \leq \dim X - 1$ and $\epsilon(i, j+1) = -\epsilon(i, j)$ for $i + j \leq \dim X - 2$. Hence we have $\epsilon(i, j) = (-1)^{j+1}$ for $i + j \leq \dim X - 1$.

Thus we have to change the sign in ν_n in [M1, 3.13] as follows: we define a new morphism $\mu_n^{ij} := (-1)^{j+1}\mathrm{proj}\colon W_nA_X^{ij} \longrightarrow W_nA_X^{i-1,j+1}$ $(i, j \in \mathbb{N})$. Große-Klönne has proposed this morphism; the elimination of the other possibility for signs explained above is due to me.

Furthermore I would like to give the following remark: the family $\mu_n := \{\mu_n^{ij}\}$ does not induce a morphism $(W_nA_X^\bullet, d'+d'') \longrightarrow (W_nA_X^\bullet(-1), d'+d'')$ of complexes if $\dim X \geq 2$ and if $p \neq 2$ or if $n \geq 2$ as in (1); however μ_n induces a morphism $(W_nA_X^\bullet, d' + d'') \longrightarrow (W_nA_X^\bullet(-1), -(d' + d''))$ of complexes. Therefore, by using the Convention (6), μ_n induces a morphism

$$\begin{aligned}\mu_n\colon H^h(X, (W_nA_X^\bullet, d' + d'')) &\longrightarrow H^h(X, (W_nA_X^\bullet(-1), -(d' + d''))) \\ &= H^h(X, (W_nA_X^\bullet, d' + d''))\end{aligned}$$

of cohomologies unconditionally. By the identification

$$H^h(X, \mathrm{gr}_k^P W_\star A_X^\bullet) = \bigoplus_{j \geq \max\{-k, 0\}} H^{h-2j-k}(X, W_\star \Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k)$$

by the use of the Convention (6), μ_n^k induces an isomorphism

$$\mu_n^k\colon \bigoplus_{j\geq\max\{-k,0\}} H^{h-2j-k}(X, W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k) \overset{\sim}{\longrightarrow} \bigoplus_{j\geq\max\{-k,0\}} H^{h-2j-k}(X, W_\star\Omega^\bullet_{\overset{\circ}{X}{}^{(2j+k+1)}})(-j-k).$$

If k is an odd (resp. even) positive integer, μ_n^k is equal to $\bigoplus_{j\geq\max\{-k,0\}}(-1)^{j+1}$ (resp. the identity).

Let $W_nC_X^{\bullet\bullet}$ be a double complex defined as follows: the (i,j)-component of $W_nC_X^{ij}$ $(i,j\in\mathbb{N})$ is, by definition, $W_nA_X^{i-1,j}(-1)\oplus W_nA_X^{ij}$. The horizontal (resp. vertical) boundary morphism $d'\colon W_nC_X^{ij}\longrightarrow W_nC_X^{i+1,j}$ (resp. $d''\colon W_nC_X^{ij}\longrightarrow W_nC_X^{i,j+1}$) is, by definition, $d'(\omega_1,\omega_2)=((-1)^jd\omega_1, (-1)^jd\omega_2)$ (resp. $d''(\omega_1,\omega_2)=(\omega_1\wedge\theta_n+\mu_n(\omega_2),\omega_2\wedge\theta_n)$. (We have to check that $W_nC_X^{\bullet\bullet}$ is indeed a double complex.) Let $\rho_n\colon W_n\widetilde{\Lambda}^i_X\longrightarrow W_nC_X^{i0}$ be a morphism defined by $\rho_n(\omega)=(\omega \mod P_0, \omega\wedge\theta_n \mod P_0)$. Let $W_nC_X^\bullet$ be the single complex of $W_nC_X^{\bullet\bullet}$. Then $\rho_n\colon W_n\widetilde{\Lambda}^\bullet_X\longrightarrow W_nC_X^\bullet$ is clearly a morphism of complexes. Moreover, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_nA_X^\bullet(-1)\{-1\} & \longrightarrow & W_nC_X^\bullet & \longrightarrow & W_nA_X^\bullet & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle(*\wedge\theta_n)\{-1\}} & & \uparrow{\scriptstyle\rho_n} & & \uparrow{\scriptstyle\wedge\theta_n} & & \\
0 & \longrightarrow & W_n\Lambda_X^\bullet(-1)\{-1\} & \xrightarrow{\wedge\theta_n} & W_n\widetilde{\Lambda}_X^\bullet & \longrightarrow & W_n\Lambda_X^\bullet & \longrightarrow & 0.
\end{array} \tag{11.9.1;n}$$

In conclusion, we have solved the problem of signs in [M1, 3.13] by using μ_n (resp. $W_nC_X^{\bullet\bullet}$) instead of ν (resp. $B_n^{\bullet\bullet}$) in [loc. cit.] (resp. [loc. cit., 3.18]).

(3) We can also solve the problem in (2) in the following way.

We use the shift $[-1]$ as $[-1]$ in the Convention (2).

Consider a morphism

$$\xi_n^{ij}\colon W_nA_X^{ij}\ni\omega\longmapsto(-1)^{i+1}\omega \mod P_{j+1}\in W_nA_X^{i-1,j+1} \quad (i,j\in\mathbb{N}).$$

Then it is easy to check that $\xi_n:=\oplus_{ij}\xi_n^{ij}$ is a morphism $(W_nA_X^\bullet, d'+d'')\longrightarrow(W_nA_X^\bullet(-1), d'+d'')$ of complexes. As in (2), we define the following objects:

(a): a double complex $W_nD_X^{\bullet\bullet}\colon W_nD_X^{ij}:=W_nA_X^{i-1,j}(-1)\oplus W_nA_X^{ij}$ $(i,j\in\mathbb{N})$;

$$d'(\omega_1,\omega_2)=((-1)^jd\omega_1,(-1)^{j+1}d\omega_2),$$

$$d''(\omega_1,\omega_2)=(-\omega_1\wedge\theta_n+\xi_n(\omega_2),\omega_2\wedge\theta_n).$$

(b): $\chi_n\colon W_n\widetilde{\Lambda}^i_X\ni\omega\longmapsto((-1)^{i+1}\omega \mod P_0,\omega\wedge\theta_n \mod P_0)\in W_nD^{i0}_X.$

Then we have the following commutative diagram of complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_nA^\bullet_X(-1)[-1] & \longrightarrow & W_nD^\bullet_X & \longrightarrow & W_nA^\bullet_X & \longrightarrow & 0 \\ & & {\scriptstyle(*\wedge\theta_n)[-1]}\uparrow & & {\scriptstyle\chi_n}\uparrow & & {\scriptstyle\wedge\theta_n}\uparrow & & \\ 0 & \longrightarrow & W_n\Lambda^\bullet_X(-1)[-1] & \xrightarrow{\theta_n\wedge} & W_n\widetilde{\Lambda}^\bullet_X & \longrightarrow & W_n\Lambda^\bullet_X & \longrightarrow & 0. \end{array} \tag{11.9.2;n}$$

Proposition 11.10. *Let $\star$ be a positive integer n or nothing. Then there exists the following commutative diagram*:

$$\begin{array}{ccc} H^h(X,W_\star A^\bullet_X) & \xrightarrow{\nu_\star} & H^h(X,W_\star A^\bullet_X)(-1) \\ {\scriptstyle\theta_\star\wedge,\ \simeq}\uparrow & & \uparrow{\scriptstyle\theta_\star\wedge,\ \simeq} \\ H^h(X,W_\star\Lambda^\bullet_X) & \xrightarrow{N_{\mathrm{dRW},\star}} & H^h(X,W_\star\Lambda^\bullet_X)(-1). \end{array} \tag{11.10.1}$$

In particular, the p-adic quasi-monodromy operator $\nu_\star$ can be identified with the p-adic monodromy operator in (11.3.6).

Proof. (11.10) immediately follows from the commutative diagram (11.8.1;$\star$). □

As in [M1, 3.19], we obtain the following:

Corollary 11.11. *The monodromy operator*

$$N_{\mathrm{crys}}\colon H^h_{\text{log-crys}}(X/W)\longrightarrow H^h_{\text{log-crys}}(X/W)(-1)$$

is nilpotent.

Proof. (11.11) immediately follows from (11.5) (2) and (11.10) since the p-adic quasi-monodromy operator ν is nilpotent. □

Remark 11.12. (1) Let $f\colon Y\longrightarrow\Delta$ be a proper semistable family of analytic varieties over the open unit disk. Let $X:=f^{-1}(0)$ be the special fiber of f. Let Δ^* be the punctured disk. Let h be an integer. The anticlockwise generator of $\pi_1(\Delta^*, t)$ ($t \in \Delta^*$) acts on $H^h(Y_t, \mathbb{C})$. This action extends to an automorphism of $R^h f_* \Omega^{\bullet}_{Y/\Delta}(\log X)$ ([St1, (2.18), (2.20)], [D2, II (5.4)]). Let T_0 be the induced action on $H^h(X, \Lambda^{\bullet}_{X/\mathbb{C}})$. Consider the following exact sequence (cf. [St1, (2.19)])

$$(11.12.1) \qquad 0 \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}}(-1)[-1] \overset{d\log t\wedge}{\longrightarrow} \Omega^{\bullet}_{Y/\mathbb{C}}(\log X) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow 0.$$

Let N_0 be the boundary morphism of (11.12.1) (cf. the Convention (5) in §1):

$$(11.12.2) \qquad N_0 \colon H^h(X, \Lambda^{\bullet}_{X/\mathbb{C}}) \longrightarrow H^h(X, \Lambda^{\bullet}_{X/\mathbb{C}}(-1)[-1][1]) = H^h(X, \Lambda^{\bullet}_{X/\mathbb{C}})(-1).$$

Then, by [D2, II (1.17)] (cf. [St1, (2.21)]), $T_0 = \exp(-2\pi\sqrt{-1}N_0)$.

The exact sequence in [St1, (2.20)](=(11.12.1)) and the commutative diagram [St1, (4.22)] are slightly confusing: we have to consider $[-1]$ in [St1, (2.19)] as $[-1]$ in this paper, and to consider $[-1]$ in [St1, (4.22)] as $\{-1\}$ in this paper, and to consider the following commutative diagram:

$$(11.12.3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A^{\bullet}_{\mathbb{C}}\{-1\} & \longrightarrow & B^{\bullet}_{\mathbb{C}} & \longrightarrow & A^{\bullet}_{\mathbb{C}} & \longrightarrow & 0 \\ & & {\scriptstyle\wedge d\log t}\Big\uparrow & & {\scriptstyle\eta}\Big\uparrow & & {\scriptstyle\wedge d\log t}\Big\uparrow & & \\ 0 & \longrightarrow & \Lambda^{\bullet}_{X/\mathbb{C}}(-1)\{-1\} & \xrightarrow{\wedge d\log t} & \Omega^{\bullet}_{Y/\mathbb{C}}(\log X)\otimes_{\mathcal{O}_Y}\mathcal{O}_X & \longrightarrow & \Lambda^{\bullet}_{X/\mathbb{C}} & \longrightarrow & 0. \end{array}$$

We have to mention a very obvious fact that the two shifts $[-1]$ and $\{-1\}$ are not the same; there are many literatures in which this distinction is not made. Moreover, in [St1, (4.22)], there is no explanation for a claim that the boundary morphism of (11.12.1) and that of the following complex

$$(11.12.4) \qquad 0 \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}}(-1)\{-1\} \overset{\wedge d\log t}{\longrightarrow} \Omega^{\bullet}_{Y/\mathbb{C}}(\log X)\otimes_{\mathcal{O}_Y}\mathcal{O}_X \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow 0$$

are the same. We need the following explanation: by the Convention (5), the boundary morphisms of (11.12.1) and (11.12.4) are the induced morphisms of the following morphisms of derived categories, respectively:

$$(11.12.5) \qquad \Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}}(-1)[-1][1] = (\Lambda^{\bullet}_{X/\mathbb{C}}(-1), d),$$

and

$$\Lambda^{\bullet}_{X/\mathbb{C}} \longrightarrow \Lambda^{\bullet}_{X/\mathbb{C}}(-1)\{-1\}[1] = (\Lambda^{\bullet}_{X/\mathbb{C}}(-1), -d). \tag{11.12.6}$$

The point is that the two shifts $\{1\}$ and $[1]$ are not the same operations. The right hand sides of (11.12.5) and (11.12.6) are identified as *complexes* by the following isomorphism:

$$(-1)^i \times \colon \Lambda^i_{X/\mathbb{C}} \xrightarrow{\sim} \Lambda^i_{X/\mathbb{C}} \quad (i \in \mathbb{N}). \tag{11.12.7}$$

By using this identification, the boundary morphisms of cohomologies induced by the exact sequences (11.12.1) and (11.12.4) are the same.

Since the upper and the lower exact sequences in (11.1.1;n) and (11.1.1) are literal p-adic analogues of (11.12.1), we formulate $N_{\mathrm{dRW},\star}$, $N_{\mathrm{crys},\star}$ and $N''_{\mathrm{dRW},\star}$ as in (11.3.2), (11.3.3), (11.3.6) and (11.3.7); we have not followed the formulations of [Hy2, Introduction] and [HK, (3.6)], though we easily establish the relation between our monodromy operators and those in [loc. cit.] as above.

(2) The explanation in [M1, 2.3] is very incomplete because a morphism

$$C_{(X,M)/(W_n,\mathrm{triv})} \otimes_{W_n\langle t\rangle} W_n \longrightarrow W_n\widetilde{\omega}^{\bullet}_X$$

in the notation of [loc. cit.](= the second vertical morphism in (11.4.2)) has not been constructed. Since $(X,M)/(W_n, W_n^*)$ is not log smooth, [HK, (4.19)] is not useful for the construction of the morphism above. Note that the commutative diagram in [M1, p. 311] is different from that in [HK, p. 262]; the complex $W_n\widetilde{\omega}^{\bullet}_X$ in [M1, 2.3] is more general than that in [HK, p. 262]. Moreover, even in the case of the semistable family, we need a proof for a fact that $W_n\widetilde{\omega}^{\bullet}_X$ in [M1, 2.3] in a local case is identified with that in [HK, p. 262]: the commutative diagram in [HK, p. 261] and (11.1.1;n) enable us to identify two $W_n\widetilde{\omega}^{\bullet}_X$'s. In addition, note that the crystalline complex in [M1, 2.1.2, 2.3] is different from that in [HK, (2.19)]: the former crystalline complex is the higher direct image of the latter by the natural morphism (11.3.4) of topoi.

(3) By (11.5) (2) and (11.10) (cf. [M1, 3.18], (11.9)), we can use the strategy of Steenbrink-Rapoport-Zink (cf. [St1], [RZ], the proof of [M1, 3.33]) in order to investigate the relation between the p-adic monodromy filtration and the p-adic weight filtration.

I am not sure that the proof of the p-adic monodromy-weight conjecture for a proper SNCL curve over a log point in [M1, 5.3] is perfect: I cannot understand the implication "on déduit que $\nu_{\mathbb{Q}}\colon E_{2\mathbb{Q}}^{-12}(1) \longrightarrow E_{2\mathbb{Q}}^{10}$ est un isomorphism" in [loc. cit., p. 329, l. 19–20]. For this reason I give the proof of the following proposition, though the argument in the proof is well-known (cf. [St1, p. 254]).

Proposition 11.13. *Let X be a proper SNCL curve over $s = (\mathrm{Spec}(\kappa), \mathbb{N}\oplus\kappa^*)$. Then the monodromy filtration defined by the monodromy operator*

$$N_{\mathrm{crys}}\colon H^1_{\text{log-crys}}(X/W)\otimes_W K_0 \longrightarrow H^1_{\text{log-crys}}(X/W)(-1)\otimes_W K_0$$

and the weight filtration on $H^1_{\text{log-crys}}(X/W)\otimes_W K_0$ coincide.

Proof. By (11.7) (2), the morphism

$$\begin{aligned}\nu\colon E_1^{-12} &= H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)(-1)\otimes_W K_0\\ &\longrightarrow E_1^{10}(-1) = H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)(-1)\otimes_W K_0\end{aligned}$$

is the identify. By (11.5) (2), by (11.10) and by the definition of the monodromy filtration, it suffices to prove that the induced morphism $\nu\colon E_2^{-12} \longrightarrow E_2^{10}(-1)$ is an isomorphism. By (10.1), the boundary morphisms $d_1^{-12}\colon E_1^{-12} \longrightarrow E_1^{02}$ and $d_1^{00}\colon E_1^{00} \longrightarrow E_1^{10}$ are identified with

$$G\colon H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)(-1)\otimes_W K_0 \longrightarrow H^2_{\mathrm{crys}}(\overset{\circ}{X}{}^{(1)}/W)\otimes_W K_0$$

and

$$\rho\colon H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(1)}/W)\otimes_W K_0 \longrightarrow H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)\otimes_W K_0,$$

respectively. By using the $\mathbb{Q}$-structure as in [M1, 5.3], we see that the Poincaré duality perfect pairing

$$\langle\ ,\ \rangle\colon (H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)\otimes_W K_0)\otimes_{K_0}(H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)\otimes_W K_0) \longrightarrow K_0$$

satisfies the following property: if $\langle v, v\rangle = 0$ ($v \in H^0_{\mathrm{crys}}(\overset{\circ}{X}{}^{(2)}/W)\otimes_W K_0$) and if v is $\mathbb{Q}$-rational, then $v = 0$. Now (11.13) follows from the lemma (11.14) below by setting $A = \mathbb{Q}$ in (11.14). $\square$

Let A be a commutative ring with unit element. Let $\rho\colon L \longrightarrow M$ be a morphism of A-modules. Set $L^* = \mathrm{Hom}_A(L, A)$ and $M^* = \mathrm{Hom}_A(M, A)$. Let $\langle\ ,\ \rangle_L \colon L \otimes_A L^* \longrightarrow A$ be the natural perfect pairing. Let $\langle\ ,\ \rangle_M \colon M \otimes_A M \longrightarrow A$ be an A-linear perfect pairing. Using $\langle\ ,\ \rangle_M$, we have an identification $\iota\colon M \ni y \longmapsto (x \longmapsto \langle x, y\rangle_M) \in M^*$. Then the following holds:

LEMMA 11.14. *Assume that, if $\langle v, v\rangle_M = 0$ $(v \in M)$, then $v = 0$. Then the following composite morphism*

$$\mathrm{Ker}(\rho^*\colon M^* \longrightarrow L^*) \xrightarrow{\subset} M^* \xrightarrow{\iota^{-1}} M \longrightarrow \mathrm{Coker}(\rho\colon L \longrightarrow M) \tag{11.14.1}$$

is injective. In particular, if A is a field and if M is a finite dimensional vector space over A, then the morphism (11.14.1) *is an isomorphism.*

PROOF. Let $\langle\ ,\ \rangle'_M \colon M \otimes_A M^* \longrightarrow A$ be the natural perfect pairing. Then we have $\langle v_1, v_2\rangle_M = \langle v_1, \iota(v_2)\rangle'_M$. Let v be an element of M such that $\iota(v) \in \mathrm{Ker}(\rho^*)$ and such that $v = \rho(w)$ $(\exists w \in L)$. Then we have

$$\langle v, v\rangle_M = \langle v, \iota(v)\rangle'_M = \langle \rho(w), \iota(v)\rangle'_M = \langle w, \rho^*(\iota(v))\rangle_L = 0.$$

Hence $v = 0$ by the assumption. Therefore (11.14.1) is injective. □

We conclude this paper by stating the following:

REMARK 11.15. (1) There exist counter-examples of [M1, 6.2.4]: in [Nakk4], we have constructed proper SNCL surfaces over the log point s such that $\dim_{K_0} E_2^{-12} \neq \dim_{K_0} E_2^{10}$ and $\dim_{K_0} E_2^{-14} \neq \dim_{K_0} E_2^{12}$. Therefore the proof of [M1, 6.2.3] for the first and third log crystalline cohomologies is mistaken. The surfaces above cannot be special fibers of algebraic proper semistable families over any complete discrete valuation ring of neither equal characteristic nor mixed characteristics with residue field κ, though they are special fibers of formal proper semistable families over $\mathrm{Spf}(W)$. These examples are also counter-examples of the conjecture in [Ch2, Introduction]. See [Nakk4] for more details.

(2) In [Nakk4], we have proved the p-adic monodromy-weight conjecture for an algebraic proper semistable family of surfaces over a complete discrete valuation ring of equal characteristic and mixed characteristics.

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Department of Mathematics
Tokyo Denki University
2–2 Kanda Nishiki-cho Chiyoda-ku
Tokyo 101–8457, Japan
E-mail: nakayuki@cck.dendai.ac.jp