Large deviations and limit theorems of law of large numbers' type for the processes related to the interface models

（界面モデルに関連した確率過程に対する大偏差原理と大数の法則型極限定理）

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Chapter 1

Introduction

Under various physical situations especially at low temperatures, more than one distinct pure phases coexist in space and different phases are separated by fairly sharp hypersurfaces called interfaces. Typical examples are snow crystals in the vapor or alloys consisting of two types of metals. It is known that, at the macroscopic level, the most probable shape of a crystal surrounded by an interface having a definite total volume is characterized as a minimizer of the total surface tension and such shape is called Wulff shape. Mathematically, this can be shown as a consequence of large deviation principle. Precisely saying, if the rate functional of the large deviation principle has a unique minimizer, then the law of large number holds. However, non-trivial is the case where minimizers are not unique. In non-trivial case, the probability estimate at the level of large deviation is not sufficient, but its precise version is required.

In this thesis, we study the following four topics related to the interface models.

(1) Concentration under scaling limits for weakly pinned Gaussian random walks.

(2) Scaling limits for weakly pinned random walks with two large deviation minimizers.

(3) Law of large numbers for Wiener measure with density having two large deviation minimizers.

(4) Large deviations for the $\nabla\varphi$ interface model with self potentials.

In Chapter 2, we study scaling limits for d-dimensional mean-zero Gaussian random walks perturbed by an attractive force toward a certain subspace of $\mathbb{R}^d$, especially under the critical situation that the rate functional of the corresponding large deviation principle admits two minimizers. We obtain different type of limits, in a positive recurrent regime, depending on the co-dimension of the subspace and the conditions imposed at the final time under the presence or absence of a wall. The scaling limits in this models are related to Winterbottom shape. The motivation comes from the study of polymers or $(1+1)$-dimensional interfaces with δ-pinning.

In Chapter 3, we extend those obtained in Chapter 2 from the mean-zero Gaussian to non-Gaussian setting under the absence of the wall. More precisely, the scaling limits for d-dimensional random walks perturbed by an attractive force toward the origin are studied under the critical situation that the rate functional of the corresponding large deviation principle admits two minimizers.

In Chapter 4, we discusses the situation that the large deviation rate functional has two distinct minimizers, for a model described by Wiener measures with certain densities involving a scaling. The motivation comes from the study of the so-called $\nabla\varphi$ interface model with weak self potentials. The pinned Wiener measures case was discussed by [17].

The aim of Chapter 5 is to study, in the framework of the $\nabla\varphi$ interface model, the macroscopic behavior of microscopic interfaces under the finite volume Gibbs measures with self potentials, especially by establishing the large deviation principle. The large deviation principle for the $\nabla\varphi$ interface model was first studied by Ben Arous and Deuschel [1] in 0-boundary and quadratic potential case. And the large deviation principle for general potential with 0-boundary conditions without the self potential was discussed by Deuschel et al. [9]. As its extention, the large deviation principle for general potential with the weak self potentials was discussed by Funaki and Sakagawa [22]. We consider the case where the self potential depends on the position and on both macroscopic and microscopic heights of the interfaces as well. The assumption on the upper bound for the self potential required by [22] is relaxed. The notation will be different in each chapter.

Chapter 2

Concentration under scaling limits for weakly pinned Gaussian random walks

2.1 Introduction and main results

This chapter deals with Gaussian random walks on $\mathbb{R}^d$ perturbed by an attractive force toward a subspace M of $\mathbb{R}^d$, especially under the critical situation that the rate functional of the corresponding large deviation principle admits exactly two minimizers. The macroscopic time, observed after scaling, runs over the interval $D = [0, 1]$. The starting point of the (macroscopically scaled) walks at $t = 0$ is always specified, while we will or will not specify the arriving point at $t = 1$. We thus consider four different cases, in addition to the conditions at $t = 1$, depending whether a wall is located at the boundary of the upper half space of $\mathbb{R}^d$ or not, and study how the macroscopic scaling limits differ in these four cases. This chapter is based on a joint work with Professor E. Bolthausen and Professor T. Funaki.

2.1.1 Weakly pinned Gaussian random walks

In this subsection, we introduce (temporally inhomogeneous) Markov chains called the weakly pinned Gaussian random walks. Let $D_N = ND \cap \mathbb{Z} \equiv \{0, 1, 2, \dots, N\}$ be the range of (microscopic) time for the Markov chains corresponding to the macroscopic one D. The state spaces of the Markov chains are $\mathbb{R}^d$ or the upper half space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ according as we do not or do put a wall at $\partial \mathbb{R}^d_+$, where $\mathbb{R}_+ = [0, \infty)$. Let M be an m-dimensional subspace of $\mathbb{R}^d$ for $0 \le m \le d-1$ and let $M^\perp$ be its orthogonal complement.

We consider the measure $\nu(dy) = dy^{(1)}\delta_0(dy^{(2)})$ on $\mathbb{R}^d$ obtained by extending the surface measure $dy^{(1)}$ on M under the decomposition $y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^d \cong M \times M^\perp$; in particular, if $M = \{0\}, y = y^{(2)}$ and $\nu(dy) = \delta_0(dy)$. The co-dimension of M will be denoted by $r \equiv \operatorname{codim} M = d - m$. We assume $M \subset \partial\mathbb{R}^d_+$ when the state space of the Markov chains is $\mathbb{R}^d_+$.

Given $a, b \in \mathbb{R}^d$ (or $\in \mathbb{R}^d_+$), the starting point of the Markov chains $\phi = (\phi_i)_{i\in D_N}$ is always $aN \in \mathbb{R}^d$ (or $\in \mathbb{R}^d_+$), while, for the arriving point at $i = N$, we consider two cases: $\phi_N = bN$ (we call Dirichlet case) or without giving any condition on ϕ_N (we call free case). The distributions of the Markov chains ϕ on $(\mathbb{R}^d)^{N+1}$ or $(\mathbb{R}^d_+)^{N+1}$ with a strength $\varepsilon \geq 0$ of the pinning force toward M, imposing the Dirichlet or free conditions at N and putting or without putting a wall at $\partial\mathbb{R}^d_+$, are described by the following four probability measures $\mu_N^{D,\varepsilon}, \mu_N^{D,\varepsilon,+}, \mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$, respectively:

$$(2.1.1)\quad \mu_N^{D,\varepsilon,(+)}(d\phi) = \frac{1}{Z_N^{D,\varepsilon,(+)}} e^{-H_N(\phi)}\delta_{aN}(d\phi_0) \prod_{i\in D_N\setminus\{0,N\}} \big(\varepsilon\,\nu(d\phi_i) + d\phi_i^{(+)}\big)\delta_{bN}(d\phi_N),$$

$$(2.1.2)\quad \mu_N^{F,\varepsilon,(+)}(d\phi) = \frac{1}{Z_N^{F,\varepsilon,(+)}} e^{-H_N(\phi)}\delta_{aN}(d\phi_0) \prod_{i\in D_N\setminus\{0\}} \big(\varepsilon\,\nu(d\phi_i) + d\phi_i^{(+)}\big),$$

where $d\phi_i^{(+)}$ denotes the Lebesgue measure on $\mathbb{R}^d$ (or on $\mathbb{R}^d_+$), and $Z_N^{D,\varepsilon,(+)}$ and $Z_N^{F,\varepsilon,(+)}$ are the normalizing constants, respectively. The function $H_N(\phi)$ called the Hamiltonian is given by

$$H_N(\phi) = \frac{1}{2}\sum_{i=0}^{N-1} |\phi_{i+1} - \phi_i|^2,$$

in which $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^d$. Note that, if $\varepsilon = 0$ (i.e., without pinning), ϕ under $\mu_N^{F,0}$ is a d-dimensional Brownian motion viewed at integer times.

We sometimes denote the partition functions as $Z_N^{D,\varepsilon,(+)} = Z_N^{a,b,\varepsilon,(+)}$ and $Z_N^{F,\varepsilon,(+)} = Z_N^{a,F,\varepsilon,(+)}$ to clarify the specific conditions at $i = 0$ and N. The Markov chain ϕ satisfies the condition $\phi_0 = aN$ (a.s.) at $i = 0$ under these four measures. At $i = N$, the Dirichlet condition $\phi_N = bN$ is satisfied under $\mu_N^{D,\varepsilon}$ and $\mu_N^{D,\varepsilon,+}$, while the free condition (i.e., no specific condition) is fulfilled under $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$. The superscripts D and F are put to indicate the conditions at $i = N$. Both $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ are probability measures on $(\mathbb{R}^d)^{N+1}$ defined under the absence of wall, while $\mu_N^{D,\varepsilon,+}$ and $\mu_N^{F,\varepsilon,+}$ are those on $(\mathbb{R}^d_+)^{N+1}$ defined under the presence of a wall at $\partial\mathbb{R}^d_+$. The following table exhibits the difference of these four measures in short:

at $i = N$	No wall	Wall at $\partial\mathbb{R}^d_+$
Dirichlet condition	$\mu_N^{D,\varepsilon}$	$\mu_N^{D,\varepsilon,+}$
Free condition	$\mu_N^{F,\varepsilon}$	$\mu_N^{F,\varepsilon,+}$

When $d = 1$ and $m = 0$, the Markov chain $\big(\phi_i \in \mathbb{R}\,(\text{or} \ \in \mathbb{R}_+)\big)_{i\in D_N}$ may be interpreted as the heights of interfaces located in a plane measured from the position i on a reference line (x-axis), so that the system is called $(1+1)$-dimensional interface model with δ-pinning at 0, see [7], [10], [16], [26]. See [24] for a relation to the polymer models.

2.1.2 Scaling limits and large deviation rate functionals

We will sometimes drop the superscripts ε if there is no confusion.

Let $h^N = \{h^N(t), t \in D\}$ be the macroscopic path of the Markov chain determined from the microscopic one ϕ under a proper scaling, namely, it is defined through a polygonal approximation of $\big(h^N(i/N) = \phi_i/N\big)_{i\in D_N}$ so that

$$h^N(t) = \frac{[Nt] - Nt + 1}{N}\phi_{[Nt]} + \frac{Nt - [Nt]}{N}\phi_{[Nt]+1}, \quad t \in D.$$

Then, the sample path large deviation principle holds for h^N under $\mu_N^D, \mu_N^{D,+}, \mu_N^F$ and $\mu_N^{F,+}$, respectively, on the space $\mathcal{C} = C([0,1],\mathbb{R}^d)$ equipped with the uniform topology as $N \to \infty$, see Theorem 2.4.1 in Section 2.4 (or Theorem 2.2 of [22] for μ_N^D when $d = 1$ and $m = 0$, and [8], [28] when $\varepsilon = 0$). The speeds are always N and the unnormalized rate functionals are given by $\Sigma^D, \Sigma^{D,+}, \Sigma^F$ and $\Sigma^{F,+}$, respectively, all of which are of the form:

$$\Sigma(h) = \frac{1}{2}\int_D |\dot{h}(t)|^2\,dt - \xi|\{t \in D; h(t) \in M\}|, \tag{2.1.3}$$

for $h \in H^1_{a,b}(D) = \{h \in H^1(D); h(0) = a, h(1) = b\}$ in the Dirichlet case respectively $h \in H^1_{a,F}(D) = \{h \in H^1(D); h(0) = a\}$ in the free case with certain non-negative constants ξ, where $|\cdot|$ stands for the Lebesgue measure on D and $H^1(D) = H^1(D,\mathbb{R}^d)$ is the usual Sobolev space. We define $\Sigma(h) = +\infty$ for h's outside of these spaces, and also for h such that $h(t) \notin \mathbb{R}^d_+$ for some $t \in D$ under the presence of a wall. The constants ξ differ depending on the absence or presence of a wall as explained below.

We determine two non-negative constants $\xi^\varepsilon = \xi^\varepsilon_r$ and $\xi^{\varepsilon,+} = \xi^{\varepsilon,+}_r$ by the thermodynamic limits:

$$(2.1.4)\qquad \xi^{\varepsilon}=\lim_{N\to\infty}\frac{1}{N}\log\frac{Z_{N,r}^{0,0,\varepsilon}}{Z_{N,r}^{0,0}},\qquad \xi^{\varepsilon,+}=\lim_{N\to\infty}\frac{1}{N}\log\frac{Z_{N,r}^{0,0,\varepsilon,+}}{Z_{N,r}^{0,0,+}},$$

and another two constants $\xi^{F,\varepsilon}$ and $\xi^{F,\varepsilon,+}$ by

$$(2.1.5)\qquad \xi^{F,\varepsilon}=\lim_{N\to\infty}\frac{1}{N}\log\frac{Z_{N,r}^{0,F,\varepsilon}}{Z_{N,r}^{0,F}},\qquad \xi^{F,\varepsilon,+}=\lim_{N\to\infty}\frac{1}{N}\log\frac{Z_{N,r}^{0,F,\varepsilon,+}}{Z_{N,r}^{0,F,+}},$$

where the partition functions in the numerators are associated with the random walks in $\mathbb{R}^r$ with pinning at $M'=\{0\}\subset\mathbb{R}^r$ (i.e., $m=0$) taking $a=b=0\in\mathbb{R}^r$ in the Dirichlet case and $a=0\in\mathbb{R}^r$ in the free case, while the denominators $Z_{N,r}^{0,0}, Z_{N,r}^{0,0,+}, Z_{N,r}^{0,F}$ and $Z_{N,r}^{0,F,+}$ are defined without pinning effect and equal to their corresponding numerators with $\varepsilon=0$. See (2.2.1) for $Z_{N,r}^{0,0,\varepsilon}$, (2.2.13) for $Z_{N,r}^{0,F,\varepsilon}$ and others. As we will state in Theorem 2.1.1, the constants ξ defined for two different cases actually coincide with each other, i.e., $\xi^{\varepsilon}=\xi^{F,\varepsilon}$ and $\xi^{\varepsilon,+}=\xi^{F,\varepsilon,+}$ hold.

The constants ξ in (2.1.3) are defined by $\xi=\xi^{\varepsilon}_{\mathrm{codim}\,M}$ for the functionals $\Sigma=\Sigma^D,\Sigma^F$ and $\xi=\xi^{\varepsilon,+}_{\mathrm{codim}\,M}$ for $\Sigma=\Sigma^{D,+},\Sigma^{F,+}$, respectively, with the choice of $r=\mathrm{codim}\,M$.

The non-positive constants $\tau^{\varepsilon}=-\xi^{\varepsilon}$ and $\tau^{\varepsilon,+}=-\xi^{\varepsilon,+}$ are sometimes called the pinning free energy and the wall (more precisely, wall+pinning or wetting) free energy, respectively. Explicit formulae determining ξ^{ε} and $\xi^{\varepsilon,+}$ are found in (2.2.4) and (2.2.12). In particular, we will see that $\xi^{\varepsilon}>\xi^{\varepsilon,+}\ge 0$ for all $\varepsilon\ge 0$ unless $\xi^{\varepsilon}=0$, see Remark 2.2.1-(1). Furthermore, we have the following result on the phase transition (localization/delocalization transition) in ε, which is called pinning or wetting transitions in the framework of the interface model under the absence or presence of a wall, respectively.

Theorem 2.1.1. (1) *The limits in* (2.1.4) *and* (2.1.5) *exist for every* $\varepsilon\ge 0$, *and we have that* $\xi^{\varepsilon}=\xi^{F,\varepsilon}$ *and* $\xi^{\varepsilon,+}=\xi^{F,\varepsilon,+}$.
(2) (Absence of wall) *If* $r\ge 3$, *there exists* $\varepsilon_c>0$ *determined by* (2.2.3) *such that* $\xi^{\varepsilon}>0$ *if and only if* $\varepsilon>\varepsilon_c$ *and* $\xi^{\varepsilon}=0$ *if and only if* $0\le\varepsilon\le\varepsilon_c$. *If* $r=1$ *and* 2, *the above statement holds with* $\varepsilon_c=0$.
(3) (Presence of wall) *For all* $r\ge 1$, *there exists* $\varepsilon_c^+>0$ (*in fact,* $\varepsilon_c^+>\varepsilon_c$) *determined by* (2.2.11) *such that* $\xi^{\varepsilon,+}>0$ *if and only if* $\varepsilon>\varepsilon_c^+$ *and* $\xi^{\varepsilon,+}=0$ *if and only if* $0\le\varepsilon\le\varepsilon_c^+$.

In short, the pinning transition occurs if $r\ge 3$, while the wetting transition occurs for all dimensions. The Markov chain, being transient at $\varepsilon=0$, turns to be recurrent when the strength ε of the attractive force toward 0 increases and exceeds the critical value ε_c or ε_c^+; see [24] for random walks with discrete values. The asymptotic behavior of the free energies ξ^ε and $\xi^{\varepsilon,+}$ for ε close to their critical values is studied in Appendix A. This gives, in particular, the critical exponents for the free energies.

The large deviation principle (Theorem 2.4.1) immediately implies the concentration properties for $\mu_N = \mu_N^D$, $\mu_N^{D,+}$, μ_N^F and $\mu_N^{F,+}$:

$$\lim_{N\to\infty} \mu_N(\mathrm{dist}_\infty(h^N, \mathcal{H}) \le \delta) = 1, \tag{2.1.6}$$

for every $\delta > 0$, where $\mathcal{H} = \{h^*; \text{minimizers of } \Sigma\}$ with $\Sigma = \Sigma^D, \Sigma^{D,+}, \Sigma^F, \Sigma^{F,+}$, respectively, and dist_∞ denotes the distance on $\mathcal{C}$ under the uniform norm $\|\cdot\|_\infty$. More precisely, for any $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\mu_N(\mathrm{dist}_\infty(h^N, \mathcal{H}) > \delta) \le e^{-c(\delta)N}$$

for large enough N.

2.1.3 Minimizers of the rate functionals

By rotation, we may assume without loss of generality

$$M = \{x = (x^{(1)}, 0) \in \mathbb{R}^m \times \mathbb{R}^r\} \subset \mathbb{R}^d. \tag{2.1.7}$$

Under such coordinate of $\mathbb{R}^d$, we decompose $a = (a^{(1)}, a^{(2)})$ and $b = (b^{(1)}, b^{(2)}) \in M \times M^\perp$.

There are at most two possible minimizers of Σ^D. One is $\bar{h}^D$ defined by linearly interpolating between a and b: $\bar{h}^D(t) = (1-t)a + tb$, $t \in D$. If $|a^{(2)}| + |b^{(2)}| < \sqrt{2\xi^\varepsilon}$ (i.e., $t_1 + t_2 < 1$ for t_1 and t_2 defined below), we define $\hat{h}^D (\equiv \hat{h}^{D,\varepsilon})$ by $\hat{h}^D(t) = \big(\hat{h}^{D,(1)}(t), \hat{h}^{D,(2)}(t)\big)$, where $\hat{h}^{D,(1)}(t) = (1-t)a^{(1)} + tb^{(1)}$,

$$\hat{h}^{D,(2)}(t) = \begin{cases} (t_1 - t)a^{(2)}/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1-t_2], \\ (t + t_2 - 1)b^{(2)}/t_2, & t \in [1-t_2, 1], \end{cases} \tag{2.1.8}$$

and $t_1 = |a^{(2)}|/\sqrt{2\xi^\varepsilon}$ and $t_2 = |b^{(2)}|/\sqrt{2\xi^\varepsilon}$. The last relation is sometimes called Young's relation. Those for $\Sigma^{D,+}$ are two functions $\bar{h}^D$ and $\hat{h}^{D,+} (\equiv \hat{h}^{D,\varepsilon,+})$ defined similarly to $\hat{h}^D$ with ξ^ε replaced by $\xi^{\varepsilon,+}$.

In the free case, first for Σ^F, we set $\bar{h}^F(t) = a$, $t \in D$, and if $|a^{(2)}| < \sqrt{2\xi^\varepsilon}$ (i.e., $t_1 < 1$), $\hat{h}^F(t) = (\hat{h}^{F,(1)}(t), \hat{h}^{F,(2)}(t))$ with $\hat{h}^{F,(1)}(t) = a^{(1)}$,

$$\hat{h}^{F,(2)}(t) = \begin{cases} (t_1 - t)a^{(2)}/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1], \end{cases}$$

where $t_1 = |a^{(2)}|/\sqrt{2\xi^\varepsilon}$. Those for $\Sigma^{F,+}$ are $\bar{h}^F$ and $\hat{h}^{F,+}(\equiv \hat{h}^{F,\varepsilon,+})$ which is defined similarly to $\hat{h}^F$ with ξ^ε replaced by $\xi^{\varepsilon,+}$.

Then, the following lemma can be shown similarly to the case where $d = 1$ and $m = 0$, cf. Sections 6.3 and 6.4 of [16].

Lemma 2.1.2. *The set of minimizers of Σ^D is contained in $\{\bar{h}^D, \hat{h}^D\}$. Similarly, the sets of minimizers of $\Sigma^{D,+}$, Σ^F and $\Sigma^{F,+}$ are contained in $\{\bar{h}^D, \hat{h}^{D,+}\}$, $\{\bar{h}^F, \hat{h}^F\}$ and $\{\bar{h}^F, \hat{h}^{F,+}\}$, respectively.*

The structure of the sets of minimizers is clarified in terms of a and b in Appendix B especially when $d = 1$ and $m = 0$.

2.1.4 Main results

We are concerned with the critical case where $\bar{h}$ and $\hat{h}$ are different and both are simultaneously the minimizers of Σ^D (or $\Sigma^{D,+}$), and similar situations for Σ^F (or $\Sigma^{F,+}$); especially when $d = 1$ and $m = 0$, this is equivalent to $\xi = \xi^\varepsilon$ (or $\xi^{\varepsilon,+}$) > 0 and $(a, b) \in \mathcal{C}_1$ (see Proposition B.1) in the Dirichlet case and $|a| = \sqrt{\xi/2}$ in the free case. Otherwise, h^N converges to the unique minimizer of Σ as $N \to \infty$ in probability, recall (2.1.6). We therefore assume the following conditions in each situation:

$(C)_D \qquad \varepsilon > \varepsilon_c \quad \text{and} \quad \Sigma^D(\bar{h}^D) = \Sigma^D(\hat{h}^D),$

$(C)_{D,+} \qquad \varepsilon > \varepsilon_c^+ \quad \text{and} \quad \Sigma^{D,+}(\bar{h}^D) = \Sigma^{D,+}(\hat{h}^{D,+}),$

$(C)_F \qquad \varepsilon > \varepsilon_c \quad \text{and} \quad \Sigma^F(\bar{h}^F) = \Sigma^F(\hat{h}^F),$

$(C)_{F,+} \qquad \varepsilon > \varepsilon_c^+ \quad \text{and} \quad \Sigma^{F,+}(\bar{h}^F) = \Sigma^{F,+}(\hat{h}^{F,+}).$

Note that the second condition in $(C)_D$ (or $(C)_{D,+}$) is equivalent to

$$\sqrt{2\xi}(|a^{(2)}| + |b^{(2)}|) - \xi = \frac{1}{2}|a^{(2)} - b^{(2)}|^2,$$

while that in $(C)_F$ (or $(C)_{F,+}$) is equivalent to $|a^{(2)}| = \sqrt{\xi/2}$.

We are now in a position to state our main results. We say that the limit under μ_N is h^* if

$$\lim_{N\to\infty} \mu_N(\|h^N - h^*\|_\infty \le \delta) = 1$$

holds for every $\delta > 0$. We say that two functions $\bar{h}$ and $\hat{h}$ coexist in the limit under μ_N with probabilities $\bar{\lambda}$ and $\hat{\lambda}$ if $\bar{\lambda}, \hat{\lambda} > 0$, $\bar{\lambda} + \hat{\lambda} = 1$ and

$$\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = \bar{\lambda},$$
$$\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = \hat{\lambda}$$

hold for every $0 < \delta < |a^{(2)}| \wedge |b^{(2)}|$; it is evident from Lemma 2.1.2 and (2.1.6) that one has to check these properties only for arbitrary small $\delta > 0$.

Theorem 2.1.3. (Dirichlet case) (1) (No wall) *Under the condition* $(C)_D$, *the limit under* $\mu_N^{D,\varepsilon}$ *is* $\hat{h}^D$ *if* $\operatorname{codim} M = 1$ *and* $\bar{h}^D$ *if* $\operatorname{codim} M \ge 3$. *If* $\operatorname{codim} M = 2$, $\bar{h}^D$ *and* $\hat{h}^D$ *coexist in the limit under* $\mu_N^{D,\varepsilon}$ *with probabilities* $\bar{\lambda}^{D,\varepsilon}$ *and* $\hat{\lambda}^{D,\varepsilon}$, *respectively, given by* (2.3.16).
(2) (Wall at $\partial\mathbb{R}^d_+$) *Under the condition* $(C)_{D,+}$, *the limit under* $\mu_N^{D,\varepsilon,+}$ *is* $\hat{h}^{D,+}$ *if* $\operatorname{codim} M = 1$ *and* $\bar{h}^D$ *if* $\operatorname{codim} M \ge 3$. *If* $\operatorname{codim} M = 2$, $\bar{h}^D$ *and* $\hat{h}^{D,+}$ *coexist in the limit under* $\mu_N^{D,\varepsilon,+}$ *with probabilities* $\bar{\lambda}^{D,\varepsilon,+}$ *and* $\hat{\lambda}^{D,\varepsilon,+}$, *respectively, given by* (2.3.21).

Theorem 2.1.4. (Free case) (1) (No wall) *Under the condition* $(C)_F$, *if* $\operatorname{codim} M = 1$, $\bar{h}^F$ *and* $\hat{h}^F$ *coexist in the limit under* $\mu_N^{F,\varepsilon}$ *with probabilities* $\bar{\lambda}^{F,\varepsilon}$ *and* $\hat{\lambda}^{F,\varepsilon}$, *respectively, given by* (2.3.27). *If* $\operatorname{codim} M \ge 2$, *the limit under* $\mu_N^{F,\varepsilon}$ *is* $\bar{h}^F$.
(2) (Wall at $\partial\mathbb{R}^d_+$) *Under the condition* $(C)_{F,+}$, *if* $\operatorname{codim} M = 1$, $\bar{h}^F$ *and* $\hat{h}^{F,+}$ *coexist in the limit under* $\mu_N^{F,\varepsilon,+}$ *with probabilities* $\bar{\lambda}^{F,\varepsilon,+}$ *and* $\hat{\lambda}^{F,\varepsilon,+}$, *respectively, given by* (2.3.28). *If* $\operatorname{codim} M \ge 2$, *the limit under* $\mu_N^{F,\varepsilon,+}$ *is* $\bar{h}^F$.

The central limit theorem holds for the times when the Markov chains first or last hit M. Set

$$i_\ell = \min\{i \in D_N; \phi_i \in M\},$$
$$i_r = \max\{i \in D_N; \phi_i \in M\},$$

and consider them under a proper scaling:

$$X = \frac{1}{\sqrt{N}}\big(i_\ell - t_1 N\big) \quad \text{and} \quad Y = \frac{1}{\sqrt{N}}\big(i_r - (1-t_2)N\big),$$

where we set $\min\emptyset = N$ (in the Dirichlet case), $= N+1$ (in the free case), $\max\emptyset = 0$, and Y is considered only for the Dirichlet case.

Theorem 2.1.5. (1) (Dirichlet case) *Under* $\mu_N^{D,\varepsilon}$ *or* $\mu_N^{D,\varepsilon,+}$, *conditioned on the event* $\{i_\ell \le N-1\}$ *if* $\operatorname{codim} M \ge 2$, *the pair of random variables* (X, Y) *weakly converges to*

(U_1, U_2) *as* $N \to \infty$*, where* $U_1 = N(0, |a^{(2)}|/(2\xi)^{3/2})$ *and* $U_2 = N(0, |b^{(2)}|/(2\xi)^{3/2})$ *(with* $\xi = \xi^\varepsilon$ *or* $\xi^{\varepsilon,+}$*) are mutually independent centered Gaussian random variables.*
(2) (Free case) *Under* $\mu_N^{F,\varepsilon}$ *or* $\mu_N^{F,\varepsilon,+}$ *conditioned on the event* $\{i_\ell \le N\}$*,* X *weakly converges to* $U = N(0, |a^{(2)}|/(2\xi)^{3/2})$ *as* $N \to \infty$ *(with* $\xi = \xi^\varepsilon$ *or* $\xi^{\varepsilon,+}$*).*

The conditioning on the event $\{i_\ell \le N-1\}$ is unnecessary when $\operatorname{codim} M = 1$, since the probability of such event converges to one as $N \to \infty$ in this case.

The proof of Theorems 2.1.3 and 2.1.4, together with Theorem 2.1.5, will be given in Section 2.3. The conditions $(C)_D$–$(C)_{F,+}$ guarantee that the leading exponential decay rates of the probabilities of the neighborhoods of the two different minimizers balance with each other. This enforces us to study their precise asymptotics, which can be obtained as an application of the renewal theory and discussed in Section 2.2. The proof of Theorem 2.1.1 is also given in Section 2.2. Section 2.4 is for the sample path large deviation principles. In Appendix A we study the critical exponents for the free energies, while in Appendix B we clarify the structure of the set of minimizers of Σ when $d = 1$ and $m = 0$. It is straightforward to generalize our results to $H_N(\phi)$ of the form

$$H_N(\phi) = \frac{1}{2} \sum_{i=0}^{N-1} (\phi_{i+1} - \phi_i) \cdot A(\phi_{i+1} - \phi_i)$$

with a positive symmetric $d \times d$ matrix A if M is an eigensubspace of A.

A dichotomy in concentrations on $\bar{h}$ or $\hat{h}$ is shown in the Dirichlet case for a model with the Hamiltonians perturbed by weak self potentials, see [17]. The scaling limits for the two-dimensional model (more precisely, a model with two-dimensional time parameters) under the volume conservation law are studied by [4]. Some related results are obtained by [31] and [32] for the one-dimensional discrete SOS model and the two-dimensional Ising model, respectively. The corresponding fluctuation limits are studied by [10] for general interaction potential and by [26] and [7] for a discrete model under the Dirichlet condition at $t = 0$ with $a = 0$.

2.2 Precise asymptotics for the partition functions

In this section, we will prove a number of results on the precise asymptotic behavior of the ratios of partition functions associated with the Gaussian random walks in $\mathbb{R}^r$ with pinning at $0 \in \mathbb{R}^r$ and starting at $0 \in \mathbb{R}^r$ (and reaching 0 in the Dirichlet case), which were mentioned in Section 2.1.2 to determine ξ's. In particular, these will imply the statements in Theorem 2.1.1. A similar method is used in [7] and [24]. We will omit the subscript r of the partition functions, for example, $Z_{N,r}^{0,0,\varepsilon}$ is simply denoted by $Z_N^{0,0,\varepsilon}$ in this section.

2.2.1 Dirichlet case without wall

We denote $D_N^\circ := D_N \setminus \{0, N\} (= \{1, 2, \dots, N-1\})$. The partition function $Z_N^{0,0,\varepsilon}$ is given by

$$(2.2.1) \qquad Z_N^{0,0,\varepsilon} = \int_{(\mathbb{R}^r)^{N+1}} e^{-H_N(\phi)} \delta_0(d\phi_0) \prod_{i \in D_N^\circ} \big(\varepsilon \delta_0(d\phi_i) + d\phi_i\big) \delta_0(d\phi_N),$$

and $Z_N^{0,0} = Z_N^{0,0,0}$, i.e., $\varepsilon = 0$. An explicit calculation shows that

$$Z_N^{0,0} = \frac{(2\pi)^{rN/2}}{(2\pi N)^{r/2}}.$$

Lemma 2.2.1. *The renewal equation holds for $Z_N^{0,0,\varepsilon}$, $N \geq 2$ with $Z_1^{0,0,\varepsilon} = Z_1^{0,0} = 1$:*

$$Z_N^{0,0,\varepsilon} = Z_N^{0,0} + \varepsilon \sum_{i \in D_N^\circ} Z_i^{0,0} Z_{N-i}^{0,0,\varepsilon}.$$

Proof. Expand the product measure in (2.2.1) by specifying i_ℓ as

$$\begin{aligned} &\prod_{i \in D_N^\circ} \big(\varepsilon \delta_0(d\phi_i) + d\phi_i\big) \\ &\quad = \prod_{j \in D_N^\circ} d\phi_j + \sum_{i \in D_N^\circ} \prod_{j \in D_{N,i,-}^\circ} d\phi_j \cdot \varepsilon \delta_0(d\phi_i) \cdot \prod_{j \in D_{N,i,+}^\circ} \big(\varepsilon \delta_0(d\phi_j) + d\phi_j\big), \end{aligned}$$

where $D_{N,i,-}^\circ = \{1, 2, \dots, i-1\}$ and $D_{N,i,+}^\circ = \{i+1, i+2, \dots, N-1\}$. The i on the right hand side represents the first i such that the factor $\varepsilon \delta_0(d\phi_i)$ appears in the expansion, i.e., $i = i_\ell$. If such i does not exist, we have the measure $\prod_{j \in D_N^\circ} d\phi_j$. This expansion immediately leads to the conclusion. □

Let us define the function

$$(2.2.2) \qquad g(x) = \sum_{n=1}^\infty \frac{x^n}{(2\pi n)^{r/2}}, \quad 0 \leq x < 1.$$

Note that g is increasing, $g(0) = 0$, $g(1) \, (:= g(1-)) < \infty$ if $r \geq 3$ and $g(1-) = \infty$ if $r = 1, 2$. Set

$$(2.2.3) \qquad \varepsilon_c = \begin{cases} 1/g(1) > 0, & r \geq 3, \\ 0, & r = 1, 2. \end{cases}$$

For each $\varepsilon > \varepsilon_c$, we determine $x = x^\varepsilon \in (0,1)$ as the unique solution of $g(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^\varepsilon = -\log x^\varepsilon \quad \text{and} \quad C^{D,\varepsilon} = \frac{(2\pi)^{r/2}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}. \tag{2.2.4}$$

Proposition 2.2.2. *For each $\varepsilon > \varepsilon_c$, we have the precise asymptotics for the ratio of two partition functions:*

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim C^{D,\varepsilon} N^{r/2} e^{N\xi^\varepsilon},$$

as $N \to \infty$, where $\sim$ means that the ratio of both sides tends to 1.

Proof. We set $u_0 = a_0 = b_0 = 0$ and, for $n = 1, 2, \ldots$, $u_n = x^n Z_n^{0,0,\varepsilon}/(2\pi)^{rn/2}$, $a_n = \varepsilon x^n Z_n^{0,0}/(2\pi)^{rn/2}$ and $b_n = x^n Z_n^{0,0}/(2\pi)^{rn/2} = x^n/(2\pi n)^{r/2}$, where $x = x^\varepsilon$. Then, Lemma 2.2.1 shows that

$$u_n = b_n + \sum_{i=0}^{n} a_i u_{n-i} \tag{2.2.5}$$

for every $n \geq 0$. However, the definition of $x = x^\varepsilon$ implies that

$$\sum_{n=0}^{\infty} a_n = \varepsilon \sum_{n=1}^{\infty} \frac{x^n}{(2\pi n)^{r/2}} = 1.$$

Thus, an application of the renewal theory shows that $\lim_{n\to\infty} u_n = B/A$ (cf. Chapter XIII of [14]), where

$$B = \sum_{n=0}^{\infty} b_n = g(x) = 1/\varepsilon,$$

and

$$A = \sum_{n=0}^{\infty} n a_n = \varepsilon \sum_{n=1}^{\infty} \frac{n x^n}{(2\pi n)^{r/2}} = \varepsilon x g'(x).$$

We therefore obtain

$$\lim_{n\to\infty} \frac{x^n}{(2\pi)^{rn/2}} Z_n^{0,0,\varepsilon} = \frac{1}{\varepsilon^2 x g'(x)}.$$

Finally, using $Z_N^{0,0} = (2\pi)^{rN/2}/(2\pi N)^{r/2}$ again, the conclusion is shown by

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim \frac{(2\pi N)^{r/2}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)} (x^\varepsilon)^{-N} = C^{D,\varepsilon} N^{r/2} e^{N\xi^\varepsilon}.$$

□

2.2.2 Dirichlet case with wall

We recall that

$$Z_N^{0,0,+} = \int_{(\mathbb{R}_+^r)^{N+1}} e^{-H_N(\phi)} \delta_0(d\phi_0) \prod_{i\in D_N^\circ} d\phi_i^+ \delta_0(d\phi_N), \tag{2.2.6}$$

where $d\phi_i^+$ is the Lebesgue measure on $\mathbb{R}_+^r = \mathbb{R}^{r-1}\times\mathbb{R}_+$, and this leads to a representation of the ratio of two partition functions:

$$q_N := \frac{Z_N^{0,0,+}}{Z_N^{0,0}} = P_{0,0}^{0,1}\big(B(i/N) \geq 0 \text{ for all } 1 \leq i \leq N-1\big), \tag{2.2.7}$$

by means of a one-dimensional Brownian bridge $\{B(t), t \in [0,1]\}$ satisfying $B(0) = B(1) = 0$ under $P_{0,0}^{0,1}$. It is known (see (20) in [10]) that q_N is given by

$$q_N = \frac{1}{N}. \tag{2.2.8}$$

The partition function $Z_N^{0,0,\varepsilon,+}$, given by (2.2.1) with $(\mathbb{R}^r)^{N+1}$ replaced by $(\mathbb{R}_+^r)^{N+1}$, satisfies the renewal equation:

$$Z_N^{0,0,\varepsilon,+} = Z_N^{0,0,+} + \varepsilon \sum_{i\in D_N^\circ} Z_i^{0,0,+} Z_{N-i}^{0,0,\varepsilon,+}, \tag{2.2.9}$$

for $N \geq 2$ with $Z_1^{0,0,\varepsilon,+} = Z_1^{0,0,+} = 1$. The proof of (2.2.9) is similar to Lemma 2.2.1. With the function

$$g^+(x) = \sum_{n=1}^\infty \frac{x^n}{n(2\pi n)^{r/2}}, \quad 0 \leq x \leq 1, \tag{2.2.10}$$

noting that $g^+(1) < \infty$ for all $r \geq 1$, we define

$$\varepsilon_c^+ = 1/g^+(1) > 0. \tag{2.2.11}$$

We then determine, for each $\varepsilon > \varepsilon_c^+$, $x = x^{\varepsilon,+} \in (0,1)$ as the unique solution of $g^+(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^{\varepsilon,+} = -\log x^{\varepsilon,+} \quad \text{and} \quad C^{D,\varepsilon,+} = \frac{(2\pi)^{r/2}}{\varepsilon^2 g(x^{\varepsilon,+})}. \tag{2.2.12}$$

Proposition 2.2.3. *We have the precise asymptotics*

$$\frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,+}} \sim C^{D,\varepsilon,+} N^{1+r/2} e^{N\xi^{\varepsilon,+}}$$

as $N \to \infty$ *for each* $\varepsilon > \varepsilon_c^+$.

Proof. Define three sequences u_n, a_n and b_n as in the proof of Proposition 2.2.2 with x, $Z_n^{0,0,\varepsilon}$ and $Z_n^{0,0}$ replaced by $x^{\varepsilon,+}$, $Z_n^{0,0,\varepsilon,+}$ and $Z_n^{0,0,+}$, respectively. Then, we have the relation (2.2.5) from (2.2.9) and also $\sum_{n=0}^{\infty} a_n = 1$ from $Z_n^{0,0,+} = Z_n^{0,0}/n$, recall (2.2.7) and (2.2.8). Thus, relying on the renewal theory again (and noting $x(g^+)'(x) = g(x)$), one obtains

$$Z_N^{0,0,\varepsilon,+} \sim \frac{(2\pi)^{rN/2}}{x^N} \frac{1}{\varepsilon^2 g(x)},$$

as $N \to \infty$. Since $Z_N^{0,0,+} = (2\pi)^{rN/2}/N(2\pi N)^{r/2}$, the conclusion is shown as

$$\frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,+}} \sim \frac{N(2\pi N)^{r/2}}{\varepsilon^2 g(x^{\varepsilon,+})} \left(x^{\varepsilon,+}\right)^{-N} = C^{D,\varepsilon,+} N^{1+r/2} e^{N\xi^{\varepsilon,+}}.$$

□

Remark 2.2.1. (1) *Comparing* (2.2.3), (2.2.11) *with* $g(1) > g^+(1)$, *we see* $0 \le \varepsilon_c < \varepsilon_c^+$. *Set* $x^{\varepsilon,(+)} = 1$ *and* $\xi^{\varepsilon,(+)} = 0$ *for* $0 \le \varepsilon \le \varepsilon_c^{(+)}$. *Then, since* $g(x) > g^+(x)$ *for* $0 < x < 1$, *we have* $x^\varepsilon < x^{\varepsilon,+}$ *and therefore* $\xi^{\varepsilon,+} < \xi^\varepsilon$ *for every* $\varepsilon > \varepsilon_c$. *Indeed,* $\xi^{\varepsilon,(+)}$ *defined through the thermodynamic limit in Section* 2.1.2 *is equal to* 0 *for every* $0 \le \varepsilon \le \varepsilon_c^{(+)}$.
(2) *Propositions* 2.2.2 *and* 2.2.3 *combined with* (2.2.7), (2.2.8) *imply that*

$$\mu_N^{0,0,\varepsilon}(\phi_i \in \mathbb{R}_+^r \ \text{ for all } \ i \in D_N) = \frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,\varepsilon}} \sim \frac{C^{D,\varepsilon,+}}{C^{D,\varepsilon}} e^{-N(\xi^\varepsilon - \xi^{\varepsilon,+})}$$

as $N \to \infty$, *if* $\varepsilon > \varepsilon_c^+$.

2.2.3 Free case without wall

We now move to the case with the free condition at $t = 1$ (or microscopically at $i = N$), and denote $D_N^{\circ,F} := D_N \setminus \{0\} (= \{1, 2, \dots, N\})$. The partition function $Z_N^{0,F,\varepsilon}$ is given by

$$Z_N^{0,F,\varepsilon} = \int_{(\mathbb{R}^r)^{N+1}} e^{-H_N(\phi)} \delta_0(d\phi_0) \prod_{i \in D_N^{\circ,F}} \left(\varepsilon\delta_0(d\phi_i) + d\phi_i\right), \tag{2.2.13}$$

and we have $Z_N^{0,F} (= Z_N^{0,F,0}) = (2\pi)^{rN/2}$.

Lemma 2.2.4. *The renewal equation holds for* $Z_N^{0,F,\varepsilon}$, $N \ge 1$ *with* $Z_0^{0,F,\varepsilon} = 1$:

$$Z_N^{0,F,\varepsilon} = Z_N^{0,F} + \varepsilon \sum_{i \in D_N^{\circ,F}} Z_i^{0,0} Z_{N-i}^{0,F,\varepsilon}.$$

Proof. The proof is concluded, similarly to Lemma 2.2.1, by expanding the product measure in (2.2.13) as

$$\prod_{i\in D_N^{\circ,F}} \big(\varepsilon\delta_0(d\phi_i)+d\phi_i\big)$$
$$= \prod_{j\in D_N^{\circ,F}} d\phi_j + \sum_{i\in D_N^{\circ,F}} \prod_{j\in D_{N,i,-}^{\circ}} d\phi_j\cdot\varepsilon\delta_0(d\phi_i)\cdot \prod_{j\in D_{N,i,+}^{\circ,F}} \big(\varepsilon\delta_0(d\phi_j)+d\phi_j\big),$$

where $D_{N,i,+}^{\circ,F}=\{i+1,i+2,\dots,N\}$. □

Recall the function g defined by (2.2.2), the unique solution $x=x^\varepsilon\in(0,1)$ of $g(x)=1/\varepsilon$ and $\xi^\varepsilon=-\log x^\varepsilon>0$ in (2.2.4) for each $\varepsilon>\varepsilon_c$. We then define a positive constant:

$$C^{F,\varepsilon}=\frac{1}{\varepsilon x^\varepsilon(1-x^\varepsilon)g'(x^\varepsilon)}. \tag{2.2.14}$$

Proposition 2.2.5. *We have the precise asymptotics*

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}\sim C^{F,\varepsilon}e^{N\xi^\varepsilon}$$

as $N\to\infty$ for each $\varepsilon>\varepsilon_c$.

Proof. We set $u_0=b_0=1, a_0=0$ and, for $n=1,2,\dots$, $u_n=x^nZ_n^{0,F,\varepsilon}/(2\pi)^{rn/2}$, $a_n=\varepsilon x^nZ_n^{0,0}/(2\pi)^{rn/2}$ and $b_n=x^nZ_n^{0,F}/(2\pi)^{rn/2}=x^n$, where $x=x^\varepsilon$. Then, Lemma 2.2.4 shows that (2.2.5) holds for every $n\ge 0$. However, the definition of $x=x^\varepsilon$ implies that $\sum_{n=0}^\infty a_n=1$. Thus, an application of the renewal theory shows that $\lim_{n\to\infty}u_n=B/A$, where

$$B=\sum_{n=0}^\infty b_n=\sum_{n=0}^\infty x^n=\frac{1}{1-x},$$

and

$$A=\sum_{n=0}^\infty na_n=\varepsilon xg'(x).$$

We therefore obtain

$$\lim_{n\to\infty}\frac{x^n}{(2\pi)^{rn/2}}Z_n^{0,F,\varepsilon}=\frac{1}{\varepsilon x(1-x)g'(x)}.$$

Finally, using $Z_N^{0,F}=(2\pi)^{rN/2}$ again, the conclusion is shown by

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}\sim\frac{1}{\varepsilon x^\varepsilon(1-x^\varepsilon)g'(x^\varepsilon)}(x^\varepsilon)^{-N}=C^{F,\varepsilon}e^{N\xi^\varepsilon}.$$

□

2.2.4 Free case with wall

The partition function $Z_N^{0,F,\varepsilon,+}$, given by (2.2.13) with $(\mathbb{R}^r)^{N+1}$ replaced by $(\mathbb{R}_+^r)^{N+1}$, satisfies the renewal equation:

$$Z_N^{0,F,\varepsilon,+} = Z_N^{0,F,+} + \varepsilon \sum_{i \in D_N^{\circ,F}} Z_i^{0,0,+} Z_{N-i}^{0,F,\varepsilon,+}, \tag{2.2.15}$$

for $N \geq 1$ with $Z_0^{0,F,\varepsilon,+} = 1$. The proof of (2.2.15) is similar to Lemma 2.2.4. Recall the function $g^+(x)$ defined by (2.2.10), the unique solution $x = x^{\varepsilon,+} \in (0,1)$ of $g^+(x) = 1/\varepsilon$ and $\xi^{\varepsilon,+} = -\log x^{\varepsilon,+} > 0$ in (2.2.12) for $\varepsilon > \varepsilon_c^+$. We then define a positive constant:

$$C^{F,\varepsilon,+} = \frac{y^{\varepsilon,+}}{\varepsilon C^{F,+} g(x^{\varepsilon,+})}, \tag{2.2.16}$$

where $y^{\varepsilon,+} = \sum_{N=0}^{\infty} q_N^F (x^{\varepsilon,+})^N$, and q_N^F and $C^{F,+}$ are determined by $q_0^F = 1$ and

$$q_N^F := \frac{Z_N^{0,F,+}}{Z_N^{0,F}} = P_0\big(B(i/N) \geq 0 \text{ for all } 1 \leq i \leq N\big) \sim C^{F,+}/\sqrt{N}, \tag{2.2.17}$$

for $N \geq 1$ with a one-dimensional standard Brownian motion $\{B(t), t \in [0,1]\}$ satisfying $B(0) = 0$ under P_0. See (16) in [10] for the asymptotic behavior of q_N^F in (2.2.17) as $N \to \infty$.

Proposition 2.2.6. *We have the precise asymptotics*

$$\frac{Z_N^{0,F,\varepsilon,+}}{Z_N^{0,F,+}} \sim C^{F,\varepsilon,+} N^{1/2} e^{N\xi^{\varepsilon,+}}$$

as $N \to \infty$ for each $\varepsilon > \varepsilon_c^+$.

Proof. We set $u_0 = b_0 = 1, a_0 = 0$ and, for $n = 1, 2, \dots$, $u_n = x^n Z_n^{0,F,\varepsilon,+}/(2\pi)^{rn/2}$, $a_n = \varepsilon x^n Z_n^{0,0,+}/(2\pi)^{rn/2}$ and $b_n = x^n Z_n^{0,F,+}/(2\pi)^{rn/2}$, where $x = x^{\varepsilon,+}$. Then, (2.2.15) shows that (2.2.5) holds for every $n \geq 0$. However, the definition of $x = x^{\varepsilon,+}$ implies that $\sum_{n=0}^{\infty} a_n = 1$. Thus, relying on the renewal theory, one obtains that

$$Z_N^{0,F,\varepsilon,+} \sim \frac{(2\pi)^{rN/2}}{x^N} \frac{y^{\varepsilon,+}}{\varepsilon g(x)}.$$

Since $Z_N^{0,F,+} \sim C^{F,+} N^{-1/2} Z_N^{0,F} = C^{F,+} N^{-1/2} (2\pi)^{rN/2}$, we have

$$\frac{Z_N^{0,F,\varepsilon,+}}{Z_N^{0,F,+}} \sim \frac{y^{\varepsilon,+}}{\varepsilon C^{F,+} g(x^{\varepsilon,+})} N^{1/2} (x^{\varepsilon,+})^{-N} = C^{F,\varepsilon,+} N^{1/2} e^{N\xi^{\varepsilon,+}}.$$

□

2.3 Proof of Theorems 2.1.3, 2.1.4 and 2.1.5

We assume the conditions $(C)_D$–$(C)_{F,+}$ in this section and give the proof of Theorems 2.1.3 and 2.1.4 together with Theorem 2.1.5. Our first immediate observation is that, under the coordinates introduced in (2.1.7), the two components $\phi^{(1)} = (\phi_i^{(1)})_{i\in D_N} \in M^{N+1}$ and $\phi^{(2)} = (\phi_i^{(2)})_{i\in D_N} \in (M^\perp)^{N+1}$ of the Markov chain $\phi = \big(\phi_i = (\phi_i^{(1)}, \phi_i^{(2)})\big)_{i\in D_N} \in (\mathbb{R}^d)^{N+1}$ are independent. In fact, for instance in the Dirichlet case without wall, the distribution of ϕ and its normalizing constant are decomposed into the products:

$$\mu_N^{a,b,\varepsilon} = \mu_{N,m}^{a^{(1)},b^{(1)},0} \times \mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon} \quad \text{and} \quad Z_N^{a,b,\varepsilon} = Z_{N,m}^{a^{(1)},b^{(1)},0} \times Z_{N,r}^{a^{(2)},b^{(2)},\varepsilon}. \tag{2.3.1}$$

The subscripts m and r indicate that the objects are defined for $\mathbb{R}^m$ and $\mathbb{R}^r$, respectively.

We may assume without loss of generality $d = r$ and $m = 0$. To see this, we choose the norm $\|h\|_\infty = \max_{t\in D}|h(t)|$ for $h \in \mathcal{C}\,(= C([0,1],\mathbb{R}^d))$ with $|h(t)| = \max\{|h^{(1)}(t)|, |h^{(2)}(t)|\}$ for $h(t) = (h^{(1)}(t), h^{(2)}(t)) \in \mathbb{R}^m \times \mathbb{R}^r$, which is equivalent to the Euclidean norm of $h(t)$ in $\mathbb{R}^d$. As we are only concerned with the ratio of probabilities of neighborhoods of $\hat{h}$ and $\bar{h}$, the factor coming from the first component $\phi^{(1)}$ cancels. Thus, the proof can be reduced to the Markov chains on $\mathbb{R}^r$ (or $\mathbb{R}^r_+$) with pinning at $M' = \{0\}$. We will omit the subscript r: $\mu_{N,r}^{D,\varepsilon}$ and $Z_{N,r}^{D,\varepsilon}$ are simply denoted by $\mu_N^{D,\varepsilon}$ and $Z_N^{D,\varepsilon}$, respectively, and $a^{(2)}, b^{(2)}, \bar{h}^{(2)}, \hat{h}^{(2)}$ are denoted by $a, b, \bar{h}, \hat{h}$ and others.

2.3.1 Proof of Theorems 2.1.3-(1) and 2.1.5 for $\mu_N^{D,\varepsilon}$

If $0 \le j < k \le N$, we write $\mu_{j,k}^{a,b}$ for the measure on $(\mathbb{R}^r)^{\{j,\dots,k\}} = \big\{\phi = (\phi_i)_{j\le i\le k}; \phi_i \in \mathbb{R}^r\big\}$ without pinning, under the Dirichlet conditions $\phi_j = aN$ and $\phi_k = bN$:

$$\mu_{j,k}^{a,b}(d\phi) = \frac{1}{Z_{j,k}^{a,b}} e^{-H_{j,k}(\phi)} \delta_{aN}(d\phi_j) \prod_{i=j+1}^{k-1} d\phi_i \, \delta_{bN}(d\phi_k), \tag{2.3.2}$$

where $Z_{j,k}^{a,b} = Z_{k-j}^{a,b}$ is the normalizing constant and $H_{j,k}(\phi) := \frac{1}{2}\sum_{i=j}^{k-1}|\phi_{i+1}-\phi_i|^2$. The corresponding measure with pinning is denoted by $\mu_{j,k}^{a,b,\varepsilon}(d\phi)$. Clearly

$$\begin{aligned} Z_n^{a,b} &= e^{-N^2|a-b|^2/2n} Z_n^{0,0}, \\ Z_n^{0,0} &= \frac{(2\pi)^{rn/2}}{(2\pi n)^{r/2}}. \end{aligned} \tag{2.3.3}$$

Under the measure $\mu_{j,k}^{a,b}$, the macroscopic path determined from $(\phi_i)_{j\le i\le k}$ concentrates on the straight line between $(j/N, a)$ and $(k/N, b)$:

$$g_{[j/N,k/N]}^{a,b}(t) := \left(1 - \frac{Nt-j}{k-j}\right)a + \frac{Nt-j}{k-j}b, \quad \frac{j}{N} \le t \le \frac{k}{N},$$

in particular, $g^{a,b}_{[0,1]} = \overline{h}$. More precisely

Lemma 2.3.1. *For any $\delta' > 0$, there exists $c(\delta') > 0$ and $N_0(\delta') \in \mathbb{N}$ such that for any $a, b \in \mathbb{R}^r$, $0 \le j < k \le N$:*

$$\mu^{a,b}_{j,k}\left(\left\{\phi; \max_{i:j\le i\le k}\left|\frac{\phi_i}{N} - g^{a,b}_{[j/N,k/N]}\left(\frac{i}{N}\right)\right| \ge \delta'\right\}\right) \le e^{-c(\delta')N}$$

for $N \ge N_0(\delta')$.

Proof. This is straightforward from the fact that for any i with $j \le i \le k$, ϕ_i is normally distributed under $\mu^{a,b}_{j,k}$ with mean $(1 - (i-j)/(k-j))\,Na + ((i-j)/(k-j))\,Nb$, and standard deviation bounded by const $\times\sqrt{N}$. □

We write

$$\gamma^{a,b}_{j,k}(\delta) := \mu^{a,b}_{j,k}\left(\left\|h^N_{[j/N,k/N]} - \hat{h}_{[j/N,k/N]}\right\|_\infty \le \delta\right)$$

where $\hat{h} = \hat{h}^{D,(2)}$ in this subsection, and $f_{[u,v]}$ is the restriction of a function $f : [0,1] \to \mathbb{R}^d$ to the subinterval $[u,v]$ of $[0,1]$. Also $\gamma^{a,b,\varepsilon}_{j,k}(\delta)$ is the similarly defined quantity with pinning. We sometimes also write $U_\delta\big(\hat{h}_{[u,v]}\big)$ for the δ-neighborhood with respect to $\|\cdot\|_\infty$ in the space of functions on $[u,v]$ of $\hat{h}_{[u,v]}$; when the subscript $[u,v]$ is dropped, it is considered on $[0,1]$. We similarly write $U_\delta\big(\bar{h}\big)$ for $\bar{h} = \bar{h}^{D,(2)}$.

We remind the reader that it suffices to evaluate

$$\lim_{N\to\infty} \frac{\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\big(\hat{h}\big)\right)}{\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\big(\overline{h}\big)\right)}$$

for arbitrarily *small* $\delta > 0$.

An expansion of the product measure $\prod_{i\in D^\circ_N}(\varepsilon\delta_0(d\phi_i) + d\phi_i)$ in (2.1.1) by specifying $0 < i_\ell \le i_r < N$ gives rise to

$$\begin{aligned} p_N &:= \frac{Z^{D,\varepsilon}_N}{Z^{a,b}_N}\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\big(\hat{h}\big)\right) \\ &= \gamma^{a,b}_{0,N}(\delta) + \sum_{j=1}^{N-1} \varepsilon\,\Xi^\varepsilon_{N,j,j}\gamma^{a,0}_{0,j}(\delta)\,\gamma^{0,b}_{j,N}(\delta) \\ &\quad + \sum_{0<j<k<N} \varepsilon^2\,\Xi^\varepsilon_{N,j,k}\gamma^{a,0}_{0,j}(\delta)\,\gamma^{0,0,\varepsilon}_{j,k}(\delta)\,\gamma^{0,b}_{k,N}(\delta) \\ &=: I^1_N + I^2_N + I^3_N, \end{aligned} \tag{2.3.4}$$

where

$$\Xi^{\varepsilon}_{N,j,k} = \frac{Z^{a,0}_{j} Z^{0,0,\varepsilon}_{k-j} Z^{0,b}_{N-k}}{Z^{a,b}_{N}} \tag{2.3.5}$$

for $0 < j \le k < N$. We set $Z^{0,0,\varepsilon}_{0} = 1$ to define $\Xi^{\varepsilon}_{N,j,j}$.

The first term I^1_N covers all paths without touching 0: $i_\ell = N, i_r = 0$ and I^2_N is for those touching 0 once: $0 < i_\ell = i_r (= j) < N$, while I^3_N is for those touching 0 at least twice: $0 < i_\ell (= j) < i_r (= k) < N$.

If δ is chosen small enough, then $U_\delta\big(\hat{h}\big) \cap U_\delta\big(g^{a,b}_{[0,1]}\big) = \emptyset$. Using Lemma 2.3.1, it follows that I^1_N is exponentially small in N. Similarly, noting that $\Xi^{\varepsilon}_{N,j,j}$ is bounded in N, for I^2_N one has that either $U_\delta\big(\hat{h}_{[0,j/N]}\big) \cap U_\delta\big(g^{a,0}_{[0,j/N]}\big) = \emptyset$ or $U_\delta\big(\hat{h}_{[j/N,1]}\big) \cap U_\delta\big(g^{0,b}_{[j/N,1]}\big) = \emptyset$ and it follows that I^2_N is exponentially small, i.e., we have

$$I^1_N + I^2_N \le e^{-cN} \tag{2.3.6}$$

for N sufficiently large, where $c > 0$.

By (2.3.3), the ratio of the partition functions in (2.3.5) can be rewritten for $j < k$ as

$$\Xi^{\varepsilon}_{N,j,k} = \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} \frac{Z^{0,0,\varepsilon}_{k-j}}{Z^{0,0}_{k-j}} \tag{2.3.7}$$

where $s_1 = j/N,\ s_2 = (N-k)/N$,

$$\tilde{f}(s_1,s_2) := \frac{1}{2}\left(\frac{|a|^2}{s_1} + \frac{|b|^2}{s_2} - |a-b|^2\right), \tag{2.3.8}$$

and

$$\alpha_{N,j,k} = \left[\frac{N}{(2\pi)^2 j(k-j)(N-k)}\right]^{r/2}.$$

In the part I^3_N, we decompose the j-k-summation into the part over

$$A := \left\{(j,k)\,;|j - Nt_1| \le N^{3/5},\ |k - N(1-t_2)| \le N^{3/5}\right\}, \tag{2.3.9}$$

and over its complement. We always assume that N is large enough so that $Nt_1 + N^{3/5} < N(1-t_2) - N^{3/5}$. Using Proposition 2.2.2, we get

$$\begin{aligned}\sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\, \gamma^{0,0,\varepsilon}_{j,k}(\delta)\, \gamma^{0,b}_{k,N}(\delta) &\le \sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \\ &\le C \sum_{(j,k)\notin A} \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} (k-j)^{r/2} e^{(k-j)\xi}\end{aligned}$$

$$= C \sum_{(j,k)\notin A} \alpha_{N,j,k} (k-j)^{r/2} e^{-Nf(s_1,s_2)},$$

for some $C > 0$, where $\xi = \xi^\varepsilon$ and

$$\begin{aligned} f(s_1,s_2) &= \tilde{f}(s_1,s_2) - \xi(1-s_1-s_2) \\ &= \frac{|a|^2}{2t_1^2 s_1}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^2 s_2}(s_2-t_2)^2. \end{aligned} \tag{2.3.10}$$

In the second equation, we have used $|a-b|^2/2 = (|a|^2/t_1 + |b|^2/t_2)/2 - \xi(1-t_1-t_2)$ and $|a|/t_1 = |b|/t_2 = \sqrt{2\xi}$ from Condition $(C)_D$. On the complement A^c, we have

$$Nf(s_1,s_2) \geq CN^{1/5},$$

with some $C > 0$ and therefore

$$\sum_{(j,k)\notin A} \Xi^\varepsilon_{N,j,k} \leq e^{-cN^{1/5}} \tag{2.3.11}$$

for some $c > 0$, and large enough N.

For $(j,k) \in A$, we can expand $f(s_1,s_2)$:

$$f(s_1,s_2) = \frac{|a|^2}{2t_1^3}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^3}(s_2-t_2)^2 + O\left(N^{-6/5}\right).$$

Furthermore, the straight lines $g^{a,0}_{[0,s_1]}$ and $g^{0,b}_{[1-s_2,1]}$ are within distance $\delta/2$ to the restrictions of $\hat{h}_{[0,s_1]}$ and $\hat{h}_{[1-s_2,1]}$, respectively, if N is large enough, and therefore, using Lemma 2.3.1 and Theorem 2.4.1 below (in fact, Proposition 2.4.3 is sufficient), we get

$$\begin{aligned} \sum_{(j,k)\in A} \Xi^\varepsilon_{N,j,k}\left(1-e^{-cN}\right) &\leq \sum_{(j,k)\in A} \Xi^\varepsilon_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\, \gamma^{0,0,\varepsilon}_{j,k}(\delta)\, \gamma^{0,b}_{k,N}(\delta) \\ &\leq \sum_{(j,k)\in A} \Xi^\varepsilon_{N,j,k}, \end{aligned} \tag{2.3.12}$$

for some $c > 0$. It therefore suffices to estimate $\sum_{(j,k)\in A} \Xi^\varepsilon_{N,j,k}$. By using Proposition 2.2.2 and substituting $j - [Nt_1]$ and $k - [N(1-t_2)]$ into j and k, we have by a Riemann sum approximation

$$\begin{aligned} \varepsilon^2 \sum_{(j,k)\in A} \Xi^\varepsilon_{N,j,k} &\sim C_1 N^{-r/2} \sum_{|j|\leq N^{3/5}} e^{-c_1(j/\sqrt{N})^2} \sum_{|k|\leq N^{3/5}} e^{-c_2(k/\sqrt{N})^2} \\ &\sim C_1 N^{1-r/2} \int_{-\infty}^{\infty} e^{-c_1x^2}\,dx \int_{-\infty}^{\infty} e^{-c_2x^2}\,dx \\ &= \frac{C_1\pi}{\sqrt{c_1c_2}} N^{1-r/2}, \end{aligned} \tag{2.3.13}$$

as $N \to \infty$, with $C_1 = \varepsilon^2 C^{D,\varepsilon}/(2\pi)^r (t_1 t_2)^{r/2}$ and $c_1 = |a|^2/2t_1^3 = (2\xi)^{3/2}/2|a|$, $c_2 = |b|^2/2t_2^3 = (2\xi)^{3/2}/2|b|$.

Summarizing, we get from (2.3.4), (2.3.6), (2.3.11), (2.3.13), and for sufficiently large N

$$p_N = \frac{C_1 \pi}{\sqrt{c_1 c_2}} N^{1-r/2} \left(1 - O\left(e^{-cN}\right)\right) + O\left(e^{-cN^{1/5}}\right) + O\left(e^{-cN}\right) \sim \frac{C_1 \pi}{\sqrt{c_1 c_2}} N^{1-r/2}. \tag{2.3.14}$$

On the other hand, the definition (2.1.1) of $\mu_N^{D,\varepsilon}$ implies for every $0 < \delta < |a| \wedge |b|$ that

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon} \left(h^N \in U_\delta(\bar{h})\right) = \mu_N^{D,0} \left(h^N \in U_\delta(\bar{h})\right) \sim 1,$$

where $\bar{h} = \bar{h}^{D,(2)}$. Comparing with (2.3.14), we have the conclusion of Theorem 2.1.3-(1) by recalling that (2.1.6) implies

$$\lim_{N\to\infty} \left\{\mu_N^{D,\varepsilon} \left(h^N \in U_\delta(\hat{h})\right) + \mu_N^{D,\varepsilon} \left(h^N \in U_\delta(\bar{h})\right)\right\} = 1. \tag{2.3.15}$$

In particular, if $r = 2$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{D,\varepsilon}, \hat{\lambda}^{D,\varepsilon}) := \left(\frac{1}{1 + C_2}, \frac{C_2}{1 + C_2}\right), \tag{2.3.16}$$

where $C_2 = \varepsilon^2 C^{D,\varepsilon}/\{2\pi(2|a^{(2)}||b^{(2)}|\xi^\varepsilon)^{1/2}\}(= C_1\pi/\sqrt{c_1 c_2}) > 0$, and ξ^ε and $C^{D,\varepsilon}$ are the constants given in (2.2.4).

Proof of Theorem 2.1.5 for $\mu_N^{D,\varepsilon}$: For $x_1 < x_2$ and $y_1 < y_2$, let

$$A(x_1, x_2; y_1, y_2) := \left\{(j, k) \in A; \sqrt{N}x_1 \le j - t_1 N \le \sqrt{N}x_2,\ \sqrt{N}y_1 \le k - (1 - t_2)N \le \sqrt{N}y_2\right\}.$$

By the same computation as that leading to (2.3.13), (2.3.14), we obtain

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon} \left((i_\ell, i_r) \in A(x_1, x_2; y_1, y_2),\ h^N \in U_\delta(\hat{h})\right) \sim C_1 N^{1-r/2} \int_{x_1}^{x_2} e^{-c_1 x^2}\, dx \int_{y_1}^{y_2} e^{-c_2 x^2}\, dx.$$

Combining with (2.3.14), we obtain

$$\lim_{N\to\infty} \mu_N^{D,\varepsilon} \left((i_\ell, i_r) \in A(x_1, x_2; y_1, y_2) \middle| h^N \in U_\delta(\hat{h})\right) = \frac{\sqrt{c_1 c_2}}{\pi} \int_{x_1}^{x_2} e^{-c_1 x^2}\, dx \int_{y_1}^{y_2} e^{-c_2 x^2}\, dx.$$

On the other hand, by the estimates leading to (2.3.14), we also have

$$\begin{aligned}
&\mu_N^{D,\varepsilon}\left(\left\{h^N \in U_\delta(\hat{h})\right\} \Delta \{i_\ell \le N-1\}\right) \\
&\le \frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}} I_N^1 + \mu_N^{D,\varepsilon}\left(\{i_\ell \le N-1\} \setminus \left\{h^N \in U_\delta(\hat{h})\right\}\right) \\
&\le e^{-cN},
\end{aligned}$$

for some $c > 0$, where Δ denotes the symmetric difference, and where the estimate of the second summand comes from the fact that if h^N touches 0, but is not in $U_\delta(\hat{h})$, then it is outside $U_\delta(\hat{h}) \cup U_\delta(\bar{h})$. So by the large deviation estimate (cf. Theorem 2.4.1 below), the probability of the event that this happens is exponentially small. Therefore, we can replace the conditioning on $\left\{h^N \in U_\delta(\hat{h})\right\}$ by that on $\{i_\ell \le N-1\}$, and obtain

$$\lim_{N\to\infty} \mu_N^{D,\varepsilon}\left((i_\ell, i_r) \in A(x_1,x_2;y_1,y_2) \Big| i_\ell \le N-1\right) = \frac{\sqrt{c_1c_2}}{\pi}\int_{x_1}^{x_2} e^{-c_1x^2}\,dx \int_{y_1}^{y_2} e^{-c_2x^2}\,dx,$$

which proves the claim. Remark that the conditioning on $\{i_\ell \le N-1\}$ is not needed for $r = 1$, as $\mu_N^{D,\varepsilon}(i_\ell \le N-1) \to 1$.

2.3.2 Proof of Theorems 2.1.3-(2) and 2.1.5 for $\mu_N^{D,\varepsilon,+}$

For $a, b \in \mathbb{R}_+^r$ and $0 \le j < k \le N$, let $\mu_{j,k}^{a,b,+}$ be the measure on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$, defined similarly to $\mu_{j,k}^{a,b}$, with the normalizing constant $Z_{j,k}^{a,b,+} = Z_{k-j}^{a,b,+}$, i.e., the measure defined by the formula (2.3.2) with $Z_{j,k}^{a,b}$ and $d\phi_i$ replaced by $Z_{j,k}^{a,b,+}$ and $d\phi_i^+$, respectively. One can define the measure $\mu_{j,k}^{0,0,\varepsilon,+}$ on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$ with pinning and the Dirichlet conditions $\phi_j = \phi_k = 0$ having the normalizing constant $Z_{k-j}^{0,0,\varepsilon,+}$. Taking $\hat{h} = \hat{h}^{D,+,(2)}$ in this subsection, an expansion similar to (2.3.4) gives rise to

$$p_N^+ := \frac{Z_N^{D,\varepsilon,+}}{Z_N^{a,b,+}} \mu_N^{D,\varepsilon,+}\left(h^N \in U_\delta(\hat{h})\right) = I_N^{1,+} + I_N^{2,+} + I_N^{3,+},$$

where $I_N^{\alpha,+}$ are the terms corresponding to I_N^α in (2.3.4) for $\alpha = 1,2,3$, in which we replace the measures $\mu_{j,k}^{a,b}$ by $\mu_{j,k}^{a,b,+}$, $\mu_{j,k}^{0,0,\varepsilon}$ by $\mu_{j,k}^{0,0,\varepsilon,+}$, and $\Xi_{N,j,k}^\varepsilon$ by $\Xi_{N,j,k}^{\varepsilon,+}$ defined as

$$\Xi_{N,j,k}^{\varepsilon,+} = \frac{Z_j^{a,0,+} Z_{k-j}^{0,0,\varepsilon,+} Z_{N-k}^{0,b,+}}{Z_N^{a,b,+}} \tag{2.3.17}$$

for $0 < j \le k < N$, where $Z_0^{0,0,\varepsilon,+} = 1$ as before. We prepare a lemma to find the asymptotic behavior of $\Xi_{N,j,k}^{\varepsilon,+}$. We will denote the rth coordinates of a and $b \in \mathbb{R}_+^r = \mathbb{R}^{r-1} \times \mathbb{R}_+$ by a^r and $b^r \in \mathbb{R}_+$, respectively.

Lemma 2.3.2. (1) *If* $a^r, b^r > 0$ *(i.e.,* $a, b \in (\mathbb{R}^r_+)^\circ$*), we have as* $N \to \infty$

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} \sim 1.$$

(2) *If* $a^r = 0$ *(i.e.,* $a \in \partial\mathbb{R}^r_+$*) and* $b^r > 0$*, we have*

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} \sim \beta\left(b^r\right) = \exp\left\{-\sum_{n=1}^{\infty}\frac{p(b^r\sqrt{n})}{n}\right\},$$

where $p(x) = \int_x^\infty e^{-y^2/2}\,dy/\sqrt{2\pi}$.

Proof. (1) As we have observed in (2.2.7), the ratio of two partition functions has a representation and a bound:

$$\begin{aligned} 1 \geq \frac{Z_N^{a,b,+}}{Z_N^{a,b}} &= P^{0,1}_{a^r\sqrt{N},b^r\sqrt{N}}\big(B(i/N) \geq 0 \text{ for all } 1 \leq i \leq N-1\big) \\ &\geq P^{0,1}_{0,0}\big(B(t) \geq -\big(a^r(1-t)+b^r t\big)\sqrt{N} \text{ for all } t \in [0,1]\big) \to 1 \end{aligned}$$

as $N \to \infty$. The equality in the first line is by the scaling law of a (one-dimensional) Brownian bridge. The second line follows by noting that a Brownian bridge $B(t)$ satisfying $B(0) = a^r\sqrt{N}$ and $B(1) = b^r\sqrt{N}$ can be represented as $B(t) = \bar{B}(t) + \big(a^r(1-t)+b^r t\big)\sqrt{N}$ with another Brownian bridge $\bar{B}$ such that $\bar{B}(0) = \bar{B}(1) = 0$.

(2) If $a^r = 0$, we have

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} = P^{0,N}_{0,0}\big(\bar{B}(i) + b^r i \geq 0 \text{ for all } 1 \leq i \leq N-1\big) =: c^0_N(b^r),$$

with a Brownian bridge $\bar{B}$ such that $\bar{B}(0) = \bar{B}(N) = 0$. Replacing $\bar{B}$ with the standard Brownian motion B, one can prove that

$$c_N(b^r) := P^{0,N}_0\big(B(i) + b^r i \geq 0 \text{ for all } 1 \leq i \leq N-1\big) \longrightarrow \beta\left(b^r\right). \tag{2.3.18}$$

In fact, Theorem 1 (p. 413) of [15] shows that

$$\log\frac{1}{1-\tau\left(s\right)} = \sum_{n=1}^{\infty}\frac{s^n}{n}P_0(B(n) > b^r n),$$

for $\tau\left(s\right) = \sum_{n=1}^{\infty}\big(c_n(b^r) - c_{n+1}(b^r)\big)s^n, 0 \leq s \leq 1$. This identity by taking $s = 1$ implies (2.3.18), since $1 - \tau(1) = \lim_{N\to\infty} c_N(b^r)$ noting that the limit exists by monotonicity. To complete the proof of (2), rewriting $c^0_N(b^r)$ into

$$c^0_N(b^r) = P^{0,N}_0\left(B(i) - \frac{i}{N}B(N) + b^r i \geq 0 \text{ for all } 1 \leq i \leq N-1\right),$$

one can compare it with $c_N(b^r)$ as

$$c_N(b^r-\theta) - P_0^{0,N}\left(B(N) > \theta N\right) \le c_N^0(b^r) \le c_N(b^r+\theta) + P_0^{0,N}\left(B(N) < -\theta N\right),$$

for every $\theta > 0$. The conclusion is shown by letting $N \to \infty$ and then $\theta \downarrow 0$. $\square$

The proof of Theorem 2.1.3-(2) can be given along the same line as Theorem 2.1.3-(1). Indeed, by Lemma 2.3.2 and then by (2.2.7), (2.2.8) and Proposition 2.2.3, if $\varepsilon > \varepsilon_c^+$, we have

$$\begin{aligned}\Xi_{N,j,k}^{\varepsilon,+} &\sim \beta\left(a^r\right)\beta\left(b^r\right)\frac{Z_j^{a,0}Z_{k-j}^{0,0,\varepsilon,+}Z_{N-k}^{0,b}}{Z_N^{a,b}} \\ &= \beta\left(a^r\right)\beta\left(b^r\right)\alpha_{N,j,k}e^{-N\tilde{f}(s_1,s_2)}\cdot\frac{Z_{k-j}^{0,0,\varepsilon,+}}{Z_{k-j}^{0,0}} \\ &\sim \beta\left(a^r\right)\beta\left(b^r\right)\alpha_{N,j,k}C^{D,\varepsilon,+}(k-j)^{r/2}e^{-Nf^+(s_1,s_2)},\end{aligned}$$

as $j, N-k, N$ and $k-j \to \infty$, where $f^+(s_1,s_2)$ is the function $f(s_1,s_2)$ in (2.3.10) with $\xi = \xi^{\varepsilon,+}$, so that

$$f^+(s_1,s_2) = \frac{|a|^2}{2t_1^2 s_1}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^2 s_2}(s_2-t_2)^2,$$

by the condition $(C)_{D,+}$. If we define A as in (2.3.9), we get

$$\varepsilon^2 \sum_{(j,k)\in A} \Xi_{N,j,k}^{\varepsilon,+} \sim C_3 N^{1-r/2},$$

with $C_3 = \varepsilon^2\beta\left(a^r\right)\beta\left(b^r\right)C^{D,\varepsilon,+}\pi/(2\pi)^r(t_1t_2)^{r/2}\sqrt{c_1c_2} > 0$, which is shown similarly to (2.3.13) (just replace $C^{D,\varepsilon}$ with $\beta\left(a^r\right)\beta\left(b^r\right)C^{D,\varepsilon,+}$), and this proves that

$$p_N^+ \sim C_3 N^{1-r/2}. \tag{2.3.19}$$

On the other hand, we have

$$\frac{Z_N^{D,\varepsilon,+}}{Z_N^{a,b,+}}\mu_N^{D,\varepsilon,+}\left(h^N \in U_\delta(\bar{h})\right) = \mu_N^{D,0,+}\left(h^N \in U_\delta(\bar{h})\right) \sim 1, \tag{2.3.20}$$

for $0 < \delta < |a| \wedge |b|$. The conclusion of Theorem 2.1.3-(2) follows from the combination of (2.3.19) and (2.3.20). In particular, if $r = 2$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{D,\varepsilon,+}, \hat{\lambda}^{D,\varepsilon,+}) := \left(\frac{1}{1+C_3}, \frac{C_3}{1+C_3}\right), \tag{2.3.21}$$

where $C_3 = \varepsilon^2\beta\left(a^r\right)\beta\left(b^r\right)C^{D,\varepsilon,+}/\left\{2\pi(2|a^{(2)}||b^{(2)}|\xi^{\varepsilon,+})^{1/2}\right\} > 0$, $\beta\left(a^r\right)$ is in Lemma 2.3.2-(2), and $\xi^{\varepsilon,+}$ and $C^{D,\varepsilon,+}$ are the constants given in (2.2.12).

The proof of Theorem 2.1.5 under $\mu_N^{D,\varepsilon,+}$ is parallel to that for $\mu_N^{D,\varepsilon}$ and omitted.

2.3.3 Proof of Theorems 2.1.4-(1) and 2.1.5 for $\mu_N^{F,\varepsilon}$

Let $\mu_N^{a,F} (= \mu_N^{F,0})$ be the measure defined on $(\mathbb{R}^r)^{D_N}$ without pinning and having the normalizing constant $Z_N^{a,F} (= Z_N^{a,F,0})$:

$$\mu_N^{a,F}(d\phi) = \frac{1}{Z_N^{a,F}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i \in D_N^{\circ,F}} d\phi_i. \tag{2.3.22}$$

For $0 \le j < k \le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon}$ on $(\mathbb{R}^r)^{\{j,\dots,k\}}$ with pinning, the condition $\phi_j = 0$ at j, and the free condition (no specific condition) at k, having the normalizing constant $Z_{k-j}^{0,F,\varepsilon}$. The expansion of the product measure $\prod_{i \in D_N^{\circ,F}} (\varepsilon\delta_0(d\phi_i) + d\phi_i)$ in (2.1.2) by specifying $0 < i_\ell \le N+1$ leads to

$$\begin{aligned} p_N^F &:= \frac{Z_N^{F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\left(h^N \in U_\delta(\hat{h})\right) \\ &= \mu_N^{a,F}\left(h^N \in U_\delta(\hat{h})\right) \\ &\quad + \sum_{j \in D_N^{\circ,F}} \varepsilon \Xi_{N,j}^{F,\varepsilon} \mu_{0,j}^{a,0}\left(h_{[0,j/N]}^N \in U_\delta(\hat{h}_{[0,j/N]})\right) \mu_{j,N}^{0,F,\varepsilon}\left(h_{[j/N,1]}^N \in U_\delta(\hat{h}_{[j/N,1]})\right) \\ &=: I_N^{1,F} + I_N^{2,F}, \end{aligned} \tag{2.3.23}$$

where $\hat{h} = \hat{h}^{F,(2)}$ in this subsection and

$$\Xi_{N,j}^{F,\varepsilon} = \frac{Z_j^{a,0} Z_{N-j}^{0,F,\varepsilon}}{Z_N^{a,F}}$$

for $j \in D_N^{\circ,F}$. Noting that $Z_n^{a,F} = Z_n^{0,F} = (2\pi)^{rn/2}$ and recalling (2.3.3) for $Z_j^{a,0}$, we see that

$$\Xi_{N,j}^{F,\varepsilon} = (2\pi j)^{-r/2} e^{-N\tilde{f}(s_1)} \cdot \frac{Z_{N-j}^{0,F,\varepsilon}}{Z_{N-j}^{0,F}},$$

where $s_1 = j/N$ and $\tilde{f}(s_1) = |a|^2/2s_1$.

We put here

$$A := \left\{ j \in D_N^{\circ,F}; |j - Nt_1| \le N^{3/5} \right\}$$

and arrive in the same way as in Section 2.3.1, using the large deviation estimate for $\mu_{0,j}^{a,0}$ and $\mu_{j,N}^{0,F,\varepsilon}$ (cf. Theorem 2.4.1 below), to

$$p_N^F = \varepsilon \sum_{j \in A} \Xi_{N,j}^{F,\varepsilon} \left(1 - O\left(e^{-cN}\right)\right) + O\left(e^{-cN^{1/5}}\right) + O\left(e^{-cN}\right), \tag{2.3.24}$$

for some $c > 0$. Furthermore, we get by Proposition 2.2.5,

$$\varepsilon \sum_{j \in A} \Xi_{N,j}^{F,\varepsilon} \sim \varepsilon C^{F,\varepsilon} (2\pi)^{-r/2} \sum_{j \in A} (Ns_1)^{-r/2} e^{-Nf^F(s_1)},$$

where $f^F(s) = \tilde{f}(s) - \xi(1-s)$ with $\xi = \xi^\varepsilon$. By the second condition of $(C)_F$, being equivalent to $\xi^\varepsilon = 2|a|^2$, one can rewrite f^F as

$$f^F(s) = \frac{2|a|^2}{s}(s-1/2)^2. \tag{2.3.25}$$

This finally proves, recalling $t_1 = 1/2$ and (2.3.24), that

$$p_N^F \sim \varepsilon C^{F,\varepsilon} \pi^{-r/2} N^{(1-r)/2} \int_{-\infty}^{\infty} e^{-4|a|^2 x^2}\, dx = \frac{\varepsilon C^{F,\varepsilon} \pi^{(1-r)/2}}{2|a|} N^{(1-r)/2}. \tag{2.3.26}$$

On the other hand, for every $0 < \delta < |a|$, we have that

$$\frac{Z_N^{F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\left(h^N \in U_\delta(\bar{h})\right) = \mu_N^{F,0}\left(h^N \in U_\delta(\bar{h})\right) \sim 1,$$

where $\bar{h} = \bar{h}^{F,(2)}$. Comparing this with (2.3.26), and recalling (2.1.6), the conclusion of Theorem 2.1.4-(1) is proved. In particular, if $r = 1$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{F,\varepsilon}, \hat{\lambda}^{F,\varepsilon}) := \left(\frac{2|a^{(2)}|}{\varepsilon C^{F,\varepsilon} + 2|a^{(2)}|}, \frac{\varepsilon C^{F,\varepsilon}}{\varepsilon C^{F,\varepsilon} + 2|a^{(2)}|} \right), \tag{2.3.27}$$

where $C^{F,\varepsilon}$ is the constant given in (2.2.14).

The proof of Theorem 2.1.5 under $\mu_N^{F,\varepsilon}$ conditioned on the event $\{i_\ell \le N\}$ is similar based on the computation like in (2.3.26), note that the variance of the limiting Gaussian distribution is $1/8|a^{(2)}|^2$ which is equal to $|a^{(2)}|/(2\xi)^{3/2}$.

2.3.4 Proof of Theorems 2.1.4-(2) and 2.1.5 for $\mu_N^{F,\varepsilon,+}$

For $a \in \mathbb{R}_+^r$, let $\mu_N^{a,F,+} (= \mu_N^{F,0,+})$ be the measure defined on $(\mathbb{R}_+^r)^{D_N}$ similarly to $\mu_N^{a,F}$ without pinning and having the normalizing constant $Z_N^{a,F,+} (= Z_N^{a,F,0,+})$, i.e., the measure defined by (2.3.22) with $Z_N^{a,F}$ and $d\phi_i$ replaced by $Z_N^{a,F,+}$ and $d\phi_i^+$, respectively. For $0 \le j < k \le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon,+}$ on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$ with pinning and the normalizing constant $Z_{k-j}^{0,F,\varepsilon,+}$. Taking $\hat{h} = \hat{h}^{F,+,(2)}$ and $\bar{h} = \bar{h}^{F,(2)}$ in this subsection, a similar expansion to (2.3.23) leads to

$$p_N^{F,+} := \frac{Z_N^{F,\varepsilon,+}}{Z_N^{a,F,+}} \mu_N^{F,\varepsilon,+}\left(h^N \in U_\delta(\hat{h})\right) = I_N^{1,F,+} + I_N^{2,F,+},$$

where $I_N^{\alpha,F,+}$ are the terms corresponding to $I_N^{\alpha,F}$ for $\alpha = 1, 2$, in which we replace the measures $\mu_N^{a,F}, \mu_{0,j}^{a,0}$ and $\mu_{j,N}^{0,F,\varepsilon}$ with $\mu_N^{a,F,+}, \mu_{0,j}^{a,0,+}$ and $\mu_{j,N}^{0,F,\varepsilon,+}$, and the constant $\Xi_{N,j}^{F,\varepsilon}$ with

$$\Xi_{N,j}^{F,\varepsilon,+} = \frac{Z_j^{a,0,+} Z_{N-j}^{0,F,\varepsilon,+}}{Z_N^{a,F,+}},$$

respectively. The next lemma can be shown similarly to Lemma 2.3.2-(1).

Lemma 2.3.3. *If $a^r > 0$, we have*

$$\frac{Z_N^{a,F,+}}{Z_N^{a,F}} \sim 1$$

as $N \to \infty$.

Using Lemmas 2.3.2-(2), 2.3.3, and then (2.2.17) and Proposition 2.2.6, if $\varepsilon > \varepsilon_c^+$, we have

$$\begin{aligned}
\Xi_{N,j}^{F,\varepsilon,+} &\sim \beta\left(a^r\right) \frac{Z_j^{a,0} Z_{N-j}^{0,F,\varepsilon,+}}{Z_N^{a,F}} \\
&= \beta(a^r)(2\pi j)^{-r/2} e^{-N\tilde{f}(s_1)} \cdot q_{N-j}^F \frac{Z_{N-j}^{0,F,\varepsilon,+}}{Z_{N-j}^{0,F,+}} \\
&\sim \beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} (2\pi j)^{-r/2} e^{-N f^{F,+}(s_1)},
\end{aligned}$$

where $f^{F,+}$ is the function f^F with $\xi = \xi^{\varepsilon,+}$, which can be rewritten as (2.3.25) by the second condition of $(C)_{F,+}$. Therefore, we obtain in the same way as in Section 2.3.3

$$\begin{aligned}
p_N^{F,+} &\sim \varepsilon\beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} (2\pi)^{-r/2} \sum_{|j-Nt_1|\le N^{3/5}} (Ns_1)^{-r/2} e^{-N f^{F,+}(s_1)} \\
&\sim \frac{\varepsilon\beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} \pi^{(1-r)/2}}{2|a|} N^{(1-r)/2}.
\end{aligned}$$

In particular, if $r = 1$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{F,\varepsilon,+}, \hat{\lambda}^{F,\varepsilon,+}) := \left(\frac{2|a^{(2)}|}{\varepsilon\beta(a^r)C^{F,+}C^{F,\varepsilon,+} + 2|a^{(2)}|}, \frac{\varepsilon\beta(a^r)C^{F,+}C^{F,\varepsilon,+}}{\varepsilon\beta(a^r)C^{F,+}C^{F,\varepsilon,+} + 2|a^{(2)}|} \right), \tag{2.3.28}$$

where $\beta\left(a^r\right)$ is in Lemma 2.3.2-(2), $C^{F,\varepsilon,+}$ is in (2.2.16) and $C^{F,+}$ is in (2.2.17), respectively; in fact, $a^{(2)} = a^r$ if $r = 1$.

The rest of the proof is essentially the same as Section 2.3.3.

2.4 Large deviation principle

This section is devoted to the sample path large deviation principle. Note that we do not require the conditions $(C)_D$–$(C)_{F,+}$.

2.4.1 Formulation of results

Theorem 2.4.1. *The large deviation principle* (LDP) *holds for* $h^N = \{h^N(t), t \in D\}$ *distributed under* $\mu_N = \mu_N^{D,\varepsilon}, \mu_N^{D,\varepsilon,+}, \mu_N^{F,\varepsilon}$ *and* $\mu_N^{F,\varepsilon,+}$ *on the spaces* $\mathcal{C}$ *or* $\mathcal{C}^+ = C([0,1], \mathbb{R}_+^d)$

as $N \to \infty$ with the speed N and the good rate functionals $I = I^{D,\varepsilon}, I^{D,\varepsilon,+}, I^{F,\varepsilon}$ and $I^{F,\varepsilon,+}$ of the form:

$$I(h) = \begin{cases} \Sigma(h) - \inf_H \Sigma, & \text{if } h \in H, \\ +\infty \quad , & \text{otherwise,} \end{cases} \tag{2.4.1}$$

with $\Sigma = \Sigma^{D,\varepsilon}, \Sigma^{D,\varepsilon,+}, \Sigma^{F,\varepsilon}$ and $\Sigma^{F,\varepsilon,+}$ given by (2.1.3)*, where $H = H^1_{a,b}(D)$, $H^{1,+}_{a,b}(D) = H^1_{a,b}(D) \cap \mathcal{C}^+$, $H^1_{a,F}(D)$ and $H^{1,+}_{a,F}(D) = H^1_{a,F}(D) \cap \mathcal{C}^+$, respectively. Namely, for every open set $\mathfrak{O}$ and closed set $\mathfrak{C}$ of $\mathcal{C}$ or $\mathcal{C}^+$ equipped with the uniform topology, we have that*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{O}) \geq -\inf_{h\in\mathfrak{O}} I(h),$$
$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{C}) \leq -\inf_{h\in\mathfrak{C}} I(h),$$

in each of four situations.

The LDP for $\mu_N^{D,\varepsilon}$ is shown in [22], Theorem 2.2, when $d = 1$. Indeed, one can give the proof of Theorem 2.4.1 essentially just by copying the proof stated in [22] line by line. But, for completeness, we give another proof with slightly different flavor, which might be simpler in some aspect.

2.4.2 Preliminaries

The case without pinning

We start with the LDP for the case without pinning, which is actually standard.

Proposition 2.4.2. *The LDP holds for h^N under $\mu_N = \mu_N^{a,b}, \mu_N^{a,b,+}, \mu_N^{a,F}$ and $\mu_N^{a,F,+}$ on the spaces $\mathcal{C}$ or $\mathcal{C}^+$ as $N \to \infty$ with the speed N and the unnormalized rate functional*

$$\Sigma_0(h) = \frac{1}{2} \int_D |\dot{h}(t)|^2 \, dt.$$

Proof. We first discuss the situation without a wall. The assertion for $\mu_N^{a,F}$ follows by Schilder's theorem (or Mogul'skii's theorem [28], [8]), while for $\mu_N^{a,b}$, we may employ the contraction principle for the LDP in addition as in the proof of Lemma 6.1 of [22].

We now put a wall at $\partial\mathbb{R}^d_+$. Assuming $a, b \in \mathbb{R}^d_+$, let us denote $\mu_N^{a,b}$ or $\mu_N^{a,F}$ by μ_N and $\mu_N^{a,b,+}$ or $\mu_N^{a,F,+}$ by μ_N^+, correspondingly. Then, μ_N^+ is a conditional distribution of μ_N on the event $A_N^+ = \{\phi_i \in \mathbb{R}^d_+ \text{ for all } i \in D_N\}$. First, we consider the case where $a, b \in (\mathbb{R}^d_+)^\circ$. Then, the LDP for μ_N shown above proves that

$$\lim_{N\to\infty} \frac{1}{N} \log \mu_N(A_N^+) = 0. \tag{2.4.2}$$

Since a closed set $\mathfrak{C}$ of $\mathcal{C}^+$ is closed in $\mathcal{C}$, combined with (2.4.2), the LD upper bound for μ_N implies that for μ_N^+. The LD lower bound for μ_N^+ is also easy, since $\tilde{\mathfrak{O}} = \mathfrak{O} \cap \{h(t) \in (\mathbb{R}_+^d)^\circ$ for all $t \in D\}$ is open in $\mathcal{C}$ for every open set $\mathfrak{O}$ in $\mathcal{C}^+$ and $\mu_N(\mathfrak{O} \cap \mathcal{C}^+) = \mu_N(\tilde{\mathfrak{O}})$.

The case where a or/and $b \in \partial\mathbb{R}_+^d$ is more involved. The idea is to reduce the proof of the LDP for such case to the case where $a, b \in (\mathbb{R}_+^d)^\circ$. For $\mu_N^{a,F,+}$, we have a nice coupling $(h^{N,a}, h^{N,a'})$ for every pair of a and $a' \in \mathbb{R}_+^d$ realized on a common probability space distributed under $\mu_N^{a,F,+}$ and $\mu_N^{a',F,+}$ ($h^{N,a} \sim \mu_N^{a,F,+}$, $h^{N,a'} \sim \mu_N^{a',F,+}$), respectively, such that $\|h^{N,a} - h^{N,a'}\|_\infty \le |a - a'|$ a.s. (which is uniform in N). In fact, we may apply Lemma 2.2 of [20] in one dimension componentwisely noting that components $\{\phi^\alpha = (\phi_i^\alpha)_{i \in D_N}\}_{\alpha=1}^d$ are mutually independent under $\mu_N^{a,F,+}$. This coupling implies

$$\mu_N^{a,F,+}(\mathfrak{C}) \le \mu_N^{a^\gamma,F,+}(\mathfrak{C}^\gamma) \quad \text{and} \quad \mu_N^{a,F,+}(\mathfrak{O}) \ge \mu_N^{a^\gamma,F,+}(\mathfrak{O}^\gamma)$$

for every closed $\mathfrak{C}$ and open $\mathfrak{O}$ in $\mathcal{C}^+$ and $\gamma > 0$, where $a^\gamma = a + \gamma e^d \in (\mathbb{R}_+^d)^\circ$, $e^d = (0, \dots, 0, 1)$ is the dth unit vector, $\mathfrak{C}^\gamma = \{h \in \mathcal{C}^+; B(h,\gamma) \cap \mathfrak{C} \neq \emptyset\}$, $\mathfrak{O}^\gamma = \{h; B(h,\gamma) \subset \mathfrak{O}\}$ and $B(h,\gamma) = \{g; \|g - h\|_\infty \le \gamma\}$. Since $\mathfrak{C}^\gamma$ and $\mathfrak{O}^\gamma$ are closed and open in $\mathcal{C}^+$, respectively, we have the LDP for $\mu_N^{a,F,+}$ with $a \in \partial\mathbb{R}_+^d$ from that for $\mu_N^{a^\gamma,F,+}$ with $a^\gamma \in (\mathbb{R}_+^d)^\circ$ by noting that

$$\lim_{\gamma \downarrow 0} \inf_{h \in \mathfrak{C}^\gamma} \Sigma_0(h) = \inf_{h \in \mathfrak{C}} \Sigma_0(h) \quad \text{and} \quad \lim_{\gamma \downarrow 0} \inf_{h \in \mathfrak{O}^\gamma} \Sigma_0(h) = \inf_{h \in \mathfrak{O}} \Sigma_0(h).$$

Indeed, the first one is shown by the closedness of $\mathfrak{C}$ and the lower semicontinuity of Σ_0, while the second is from the openness of $\mathfrak{O}$. The proof of the LDP for $\mu_N^{a,b,+}$ with a or/and $b \in \partial\mathbb{R}_+^d$ is similar. □

Reduction to the case of $m = 0$

As we have seen in (2.3.1), the probability measure $\mu_N^{a,b,\varepsilon}$ is decomposed into the product:

$$\mu_N^{a,b,\varepsilon} = \mu_{N,m}^{a^{(1)},b^{(1)},0} \times \mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon}.$$

Once Theorem 2.4.1 is shown for the second component $\mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon}$, combining with Proposition 2.4.2 for the first component, Theorem 2.4.1 for $\mu_N^{a,b,\varepsilon}$ is shown. In fact, the LDP lower and upper bounds are shown first for products $\mathfrak{O} = \mathfrak{O}_1 \times \mathfrak{O}_2$ and $\mathfrak{C} = \mathfrak{C}_1 \times \mathfrak{C}_2$ of open and closed sets $\mathfrak{O}_1, \mathfrak{C}_1$ in $C(D, \mathbb{R}^m)$ and $\mathfrak{O}_2, \mathfrak{C}_2$ in $C(D, \mathbb{R}^r)$, respectively; note that $\{t \in D; h^{(2)}(t) = 0\} = \{t \in D; h(t) \in M\}$. Then, these estimates can be extended easily to general open set $\mathfrak{O}$ in $C(D, \mathbb{R}^d)$ and closet set $\mathfrak{C}$ in $C(D, \mathbb{R}^d)$. Other three measures $\mu_N^{a,b,\varepsilon,+}$, $\mu_N^{a,F,\varepsilon}$ and $\mu_N^{a,F,\varepsilon,+}$ can be treated similarly. We may thus assume that $d = r$ and

$m = 0$. In particular, $M = \{0\}$ and, therefore, the unnormalized rate functional should have the form

$$\Sigma(h) = \frac{1}{2}\int_D |\dot h(t)|^2\, dt - \xi|\{t \in D; h(t) = 0\}|. \tag{2.4.3}$$

Estimates via stochastic domination

The proof of the lower bound in Theorem 2.4.1 will be reduced to the following estimates for the measures with pinning starting at 0, see the **Lower bound** in Section 2.4.3 below.

Proposition 2.4.3. *For every $\delta > 0$, there exist $C, c > 0$ such that*

$$\mu_N^\varepsilon(\|h^N\|_\infty \geq \delta) \leq Ce^{-cN}$$

for $\mu_N^\varepsilon = \mu_N^{0,0,\varepsilon}, \mu_N^{0,0,\varepsilon,+}, \mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$.

The idea of the proof of Proposition 2.4.3 is simple. We will apply a coupling argument. For instance, under $\mu_N^{0,F,\varepsilon}$, the Markov chains $\phi^\varepsilon = (\phi_i^\varepsilon)_{i\in D_N}$ occasionally jump to the origin $0 \in \mathbb{R}^d$. It is therefore natural to expect to have a coupling, compared with the Markov chains $\phi^0 = (\phi_i^0)_{i\in D_N}$ without pinning distributed under $\mu_N^{0,F,0}$ (i.e., $\varepsilon = 0$), such that $|\phi_i^\varepsilon| \leq |\phi_i^0|, i \in D_N$ for Euclidean norms. This can be shown based on the FKG inequality, see Remark 2.4.1-(1) below. Once such coupling is established, the estimates stated in Proposition 2.4.3 are immediate from Proposition 2.4.2 for measures without pinning. We will actually establish the coupling not for the Euclidean norms of the Markov chains but for one-dimensional chains obtained by conditioning the original ones, in particular, to deal with the case with a wall.

Let $\mathcal{X}_N^\alpha$ and $\mathcal{X}_N^{\alpha,+}$, $1 \leq \alpha \leq d$, be the sets of all $\psi = \big(\psi^\beta = (\psi_i^\beta)_{i\in D_N}\big)_{\beta\neq\alpha} \in (\mathbb{R}^{d-1})^{D_N}$ respectively $\in (\mathbb{R}_+^{d-1})^{D_N}$ such that $\psi_i^\beta = 0$ for all $\beta \neq \alpha$ if $\psi_i^\gamma = 0$ for some $\gamma \neq \alpha$ and i. Note that $\mu_N^{0,0,\varepsilon}(\mathcal{X}_N^\alpha) = \mu_N^{0,F,\varepsilon}(\mathcal{X}_N^\alpha) = 1$ for $1 \leq \alpha \leq d$, $\mu_N^{0,0,\varepsilon,+}(\mathcal{X}_N^{\alpha,+}) = \mu_N^{0,F,\varepsilon,+}(\mathcal{X}_N^{\alpha,+}) = 1$ for $1 \leq \alpha \leq d-1$ and $\mu_N^{0,0,\varepsilon,+}(\mathcal{X}_N^d) = \mu_N^{0,F,\varepsilon,+}(\mathcal{X}_N^d) = 1$.

For $1 \leq \alpha \leq d$ and $\psi \in \mathcal{X}_N^{\alpha,(+)}$ satisfying $\psi_0 = \psi_N = 0$ in the Dirichlet case and $\psi_0 = 0$ in the free case, let $\nu_{N,\psi}^{\varepsilon,\alpha}$ (more precisely, $\nu_{N,\psi}^{0,0,\varepsilon,\alpha}, \nu_{N,\psi}^{0,0,\varepsilon,\alpha,+}, \nu_{N,\psi}^{0,F,\varepsilon,\alpha}$ and $\nu_{N,\psi}^{0,F,\varepsilon,\alpha,+}$) be the conditional distribution on the space $\mathcal{Y}_N = \mathbb{R}^{D_N}$ (or $\mathcal{Y}_N^+ = \mathbb{R}_+^{D_N}$) of the αth coordinate $\phi^\alpha = (\phi_i^\alpha)_{i\in D_N}$ under μ_N^ε $(= \mu_N^{0,0,\varepsilon}, \mu_N^{0,0,\varepsilon,+}, \mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$, respectively) under the condition that the other coordinates $\big(\phi^\beta = (\phi_i^\beta)_{i\in D_N}\big)_{\beta\neq\alpha}$ satisfy $\phi^\beta = \psi^\beta$ for $1 \leq \beta \neq \alpha \leq d$. For instance, we set $\nu_{N,\psi}^{0,0,\varepsilon,\alpha}(dx) = \mu_N^{0,0,\varepsilon}\big(\phi^\alpha \in dx \big| (\phi^\beta)_{\beta\neq\alpha} = \psi\big)$ for $x \in \mathcal{Y}_N$. For $\psi \in \mathcal{X}_N^{\alpha,(+)}$ satisfying the above conditions, we define a probability measure

$\bar{\nu}_{N,\psi}$ on $\mathcal{Y}_N$, which describes a Markov chain in a random environment ψ, by

$$(2.4.4) \qquad \bar{\nu}_{N,\psi}(dx) = \frac{1}{Z_{N,\psi}} e^{-\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2/2} \prod_{i\in\mathbf{i}(\psi)} \delta_0(dx_i) \prod_{i\in D_N\setminus\mathbf{i}(\psi)} dx_i,$$

where $\mathbf{i}(\psi) = \{i \in D_N; \psi_i = 0\}$. We will write $\bar{\nu}_{N,\psi}$ in two ways: $\bar{\nu}_{N,\psi}^{0,0}$ and $\bar{\nu}_{N,\psi}^{0,F}$ to clarify which case we are discussing; in particular, $\mathbf{i}(\psi) \supset \{0, N\}$ or $\mathbf{i}(\psi) \supset \{0\}$ in the Dirichlet or free cases, respectively. These measures are independent of ε and α. These probability measures restricted on $\mathcal{Y}_N^+$ and renormalized properly are denoted by $\bar{\nu}_{N,\psi}^{0,0,+}$ and $\bar{\nu}_{N,\psi}^{0,F,+}$, respectively. We also define the probability measures $\tilde{\nu}_N^{0,0}, \tilde{\nu}_N^{0,F}, \tilde{\nu}_N^{0,0,+}$ and $\tilde{\nu}_N^{0,F,+}$ by replacing $\mathbf{i}(\psi)$ on the right hand side of (2.4.4) with $\{0, N\}$ in the Dirichlet case and $\{0\}$ in the free case, respectively; note that these measures are independent of ψ. In fact, these are the same as $\mu_N^{0,0,0}, \mu_N^{0,F,0}, \mu_N^{0,0,0,+}$ and $\mu_N^{0,F,0,+}$ (i.e., μ_N^{ε} with $\varepsilon = 0$) in $d = 1$, respectively.

The following lemma gives the conditional distributions $\nu_{N,\psi}^{\varepsilon,\alpha}$ of μ_N^{ε}:

Lemma 2.4.4. *For $\varepsilon > 0$, we have that*

(1) $\nu_{N,\psi}^{0,0,\varepsilon,\alpha} = \bar{\nu}_{N,\psi}^{0,0}$ ($\mu_N^{0,0,\varepsilon}$-*a.s.*ψ), $\nu_{N,\psi}^{0,F,\varepsilon,\alpha} = \bar{\nu}_{N,\psi}^{0,F}$ ($\mu_N^{0,F,\varepsilon}$-*a.s.*ψ), $1 \le \alpha \le d$,

(2) $\nu_{N,\psi}^{0,0,\varepsilon,\alpha,+} = \bar{\nu}_{N,\psi}^{0,0}$ ($\mu_N^{0,0,\varepsilon,+}$-*a.s.*$\psi$), $\nu_{N,\psi}^{0,F,\varepsilon,\alpha,+} = \bar{\nu}_{N,\psi}^{0,F}$ ($\mu_N^{0,F,\varepsilon,+}$-*a.s.*$\psi$), $1 \le \alpha \le d-1$,

(3) $\nu_{N,\psi}^{0,0,\varepsilon,d,+} = \bar{\nu}_{N,\psi}^{0,0,+}$ ($\mu_N^{0,0,\varepsilon,+}$-*a.s.*$\psi$), $\nu_{N,\psi}^{0,F,\varepsilon,d,+} = \bar{\nu}_{N,\psi}^{0,F,+}$ ($\mu_N^{0,F,\varepsilon,+}$-*a.s.*$\psi$).

Proof. Conditioned by the σ-field $\mathcal{F}_0 = \sigma\{\phi_i = 0, i \in D_N\}$ of $\mathcal{X}_N = (\mathbb{R}^d)^{D_N}$, random variables ϕ^α and $(\phi^\beta)_{\beta\neq\alpha}$ are mutually independent under $\mu_N^{0,0,\varepsilon}$. Thus, for every $F = F(\phi^\alpha)$ and $G = G((\phi^\beta)_{\beta\neq\alpha})$, we have

$$\begin{aligned} E^{\mu_N^{0,0,\varepsilon}}[FG] &= E^{\mu_N^{0,0,\varepsilon}}\left[E^{\mu_N^{0,0,\varepsilon}}[F|\mathcal{F}_0]E^{\mu_N^{0,0,\varepsilon}}[G|\mathcal{F}_0]\right] \\ &= E^{\mu_N^{0,0,\varepsilon}}\left[E^{\bar{\nu}^{0,0}_{N,(\phi^\beta)_{\beta\neq\alpha}}}[F]G\right]. \end{aligned}$$

This completes the proof of the first identity in (1). The rest is similar. □

The space $\mathcal{Y}_N^+$ is equipped with a natural partial order $x \le y$ for $x = (x_i)_{i\in D_N}, y = (y_i)_{i\in D_N} \in \mathcal{Y}_N^+$ defined by $x_i \le y_i$ for every $i \in D_N$. For two probability measures ν_1 and ν_2 on $\mathcal{Y}_N^+$, we say that ν_2 stochastically dominates ν_1 and write $\nu_1 \le \nu_2$ if $E^{\nu_1}[F] \le E^{\nu_2}[F]$ holds for all bounded non-decreasing (in the above partial order) functions F on $\mathcal{Y}_N^+$. Note that $\nu_1 \le \nu_2$ is equivalent to the existence of two $\mathcal{Y}_N^+$-valued random variables X and Y, realized on a common probability space and distributed under ν_1 and ν_2 ($X \sim \nu_1, Y \sim \nu_2$), respectively, in such a manner that $X \le Y$ a.s., see [33], [23]. Let $R : \mathcal{Y}_N \to \mathcal{Y}_N^+$ be the mapping defined by $Rx = (|x_i|)_{i\in D_N} \in \mathcal{Y}_N^+$ for $x = (x_i)_{i\in D_N} \in \mathcal{Y}_N$.

Lemma 2.4.5. (*Stochastic domination*) *For all* $\varepsilon > 0$ *and* ψ, *we have that*

$$\begin{aligned} &\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} \le \tilde{\nu}_N^{0,0} \circ R^{-1}, && \bar{\nu}_{N,\psi}^{0,F} \circ R^{-1} \le \tilde{\nu}_N^{0,F} \circ R^{-1}, \\ &\bar{\nu}_{N,\psi}^{0,0,+} \le \tilde{\nu}_N^{0,0,+}, && \bar{\nu}_{N,\psi}^{0,F,+} \le \tilde{\nu}_N^{0,F,+}, \end{aligned}$$

where $\nu \circ R^{-1}$ *stands for the image measure of* ν *under the mapping* R.

Proof. All four probability measures on the right hand side satisfy the FKG inequality. In fact, since $x = (x_i)_{i \in D_N}$ is a reflecting Brownian motion (i.e., one-dimensional Bessel process) viewed at integer times under $\tilde{\nu}_N^{0,F} \circ R^{-1}$ and a pinned reflecting Brownian motion under $\tilde{\nu}_N^{0,0} \circ R^{-1}$, the measures $\tilde{\nu}_N^{0,0} \circ R^{-1}$ and $\tilde{\nu}_N^{0,F} \circ R^{-1}$ satisfy the FKG inequality; see Section 5.3 of [19] for the FKG inequality for Bessel processes. On the other hand, the densities of $\tilde{\nu}_N^{0,0}$ and $\tilde{\nu}_N^{0,F}$ fulfill the Holley's condition on $\mathcal{Y}_N$ (since x is a Brownian motion or a pinned Brownian motion under these measures, see [19], [23]), and therefore their restrictions $\tilde{\nu}_N^{0,0,+}$ and $\tilde{\nu}_N^{0,F,+}$ satisfy the same condition on $\mathcal{Y}_N^+$. This implies the FKG inequality for $\tilde{\nu}_N^{0,0,+}$ and $\tilde{\nu}_N^{0,F,+}$.

The four probability measures on the left hand side are given by the weak limits of probability measures having non-increasing densities with respect to the corresponding measures on the right hand side. For instance, we have

$$\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} = \lim_{\theta \downarrow 0} \nu_{\theta;N,\psi},$$

where $\nu_{\theta;N,\psi}(dx) = \prod_{i \in \mathbf{i}(\psi)} f_\theta(x_i)\, \tilde{\nu}_N^{0,0} \circ R^{-1}(dx)/Z_{\theta;N,\psi}$ with a suitable normalizing constant $Z_{\theta;N,\psi}$ and a non-negative non-increasing function f_θ on $\mathbb{R}_+$ such that $f_\theta(x)dx$ weakly converges to $\delta_0(dx)$ as $\theta \downarrow 0$. Since the FKG inequality for $\tilde{\nu}_N^{0,0} \circ R^{-1}$ implies the stochastic domination $\nu_{\theta;N,\psi} \le \tilde{\nu}_N^{0,0} \circ R^{-1}$, by taking the limit $\theta \downarrow 0$, we have that $\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} \le \tilde{\nu}_N^{0,0} \circ R^{-1}$. The other three stochastic dominations can be shown similarly. □

Proof of Proposition 2.4.3. The conclusion follows by

$$\begin{aligned} \mu_N^\varepsilon(\|h^N\|_\infty \ge \delta) &\le \sum_{\alpha=1}^d \mu_N^\varepsilon(\|h^{N,\alpha}\|_\infty \ge \delta/\sqrt{d}) \\ &= \sum_{\alpha=1}^d E^{\mu_N^\varepsilon}\left[\mu_N^\varepsilon\big(\|h^{N,\alpha}\|_\infty \ge \delta/\sqrt{d}\big|(\phi^\beta)_{\beta\ne\alpha}\big)\right] \\ &\le \sum_{\alpha=1}^d \tilde{\nu}_N(\|h^N\|_\infty \ge \delta/\sqrt{d}) \le Ce^{-cN}, \end{aligned}$$

where $h^{N,\alpha}$ is the αth coordinate of $h^N \in \mathcal{C}$, and $\tilde{\nu}_N = \tilde{\nu}_N^{0,0}, \tilde{\nu}_N^{0,0,+}, \tilde{\nu}_N^{0,F}$ and $\tilde{\nu}_N^{0,F,+}$ according as $\mu_N^{\varepsilon} = \mu_N^{0,0,\varepsilon}, \mu_N^{0,0,\varepsilon,+}, \mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$, respectively. In the third line, we have first used Lemmas 2.4.4 and 2.4.5, and then applied Proposition 2.4.2 with $a, b = 0$ and $d = 1$. □

Remark 2.4.1. (1) *At least under the absence of a wall, one can show the stochastic domination for the Euclidean norms of Markov chains:*

$$\text{(2.4.5)} \qquad \mu_N^{0,0,\varepsilon} \circ \bar{R}^{-1} \le \mu_N^{0,0,0} \circ \bar{R}^{-1} \quad \textit{and} \quad \mu_N^{0,F,\varepsilon} \circ \bar{R}^{-1} \le \mu_N^{0,F,0} \circ \bar{R}^{-1},$$

where $\bar{R} : (\mathbb{R}^d)^{D_N} \to \mathcal{Y}_N^+$ *is defined by* $R\phi = (|\phi_i|)_{i \in D_N} \in \mathcal{Y}_N^+$ *for* $\phi = (\phi_i)_{i \in D_N} \in (\mathbb{R}^d)^{D_N}$*. In fact,* $(|\phi_i|)_{i \in D_N}$ *is a* d*-dimensional Bessel process viewed at integer times under* $\mu_N^{0,F,0}$*, and therefore* $\mu_N^{0,F,0} \circ \bar{R}^{-1}$ *and* $\mu_N^{0,0,0} \circ \bar{R}^{-1}$ *(i.e.,* $\varepsilon = 0$*) satisfy the FKG inequality, see Section* 5.3 *of* [19]*. Then,* (2.4.5) *is shown by expressing* $\mu_N^{0,0,\varepsilon} \circ \bar{R}^{-1}$ *as a weak limit of a sequence of probability measures having non-increasing densities with respect to* $\mu_N^{0,0,0} \circ \bar{R}^{-1}$ *as in the proof of Lemma* 2.4.5*. The free case is similar.*
(2) *Proposition* 2.4.3 *can be shown due to the renewal theory. This method is applicable to the situation that the FKG inequality does not work.*
(3) *What we needed in Section* 2.3 *are, in fact except* (2.3.15)*, only the estimates given in Proposition* 2.4.3 *rather than the full large deviation principle.*

2.4.3 Proof of Theorem 2.4.1 for $\mu_N^{D,\varepsilon}$

We first note that, for the proof of Theorem 2.4.1 for $\mu_N^{D,\varepsilon}$, it is enough to show the following two estimates for every $g \in H_{a,b}^1(D)$:

$$\text{(2.4.6)} \qquad \liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \ge -I^{D,\varepsilon}(g),$$

for every $\delta > 0$, and

$$\text{(2.4.7)} \qquad \limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \le -I^{D,\varepsilon}(g) + \theta,$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), where $I^{D,\varepsilon}$ is defined by (2.4.1) with $\Sigma = \Sigma^{D,\varepsilon}$ and $H = H_{a,b}^1(D)$. This step of reduction is standard, for instance, see (6.6) and the estimate just above (6.11) in [22].

Lower bound

Let $\mathcal{J}_K, K \geq 1$ be the family of all $\mathbf{j} = \{j_1^p, j_2^p \in \mathbb{N}\}_{p=1}^K$ such that $0 < j_1^1 \leq j_2^1 < j_1^2 \leq j_2^2 < \cdots < j_1^K \leq j_2^K < N$. For $\mathbf{j} \in \mathcal{J}_K, K \geq 1$, we set

$$\Xi_{N,\mathbf{j}}^{\varepsilon} = \frac{1}{Z_N^{D,\varepsilon}} Z_{j_1^1}^{a,0} \cdot \prod_{p=1}^{K} Z_{j_2^p - j_1^p}^{0,0,\varepsilon} \cdot \prod_{p=1}^{K-1} Z_{j_1^{p+1} - j_2^p}^{0,0} \cdot Z_{N-j_2^K}^{0,b}$$

and

$$\Psi_{N,\mathbf{j}}^{\varepsilon}(g;\delta) = \mu_{j_1^1}^{a,0}(\|h^N - g\|_{\infty,[0,j_1^1/N]} < \delta) \cdot \prod_{p=1}^{K} \mu_{j_2^p - j_1^p}^{0,0,\varepsilon}(\|h^N\|_{\infty,[j_1^p/N, j_2^p/N]} < \delta)$$
$$\times \prod_{p=1}^{K-1} \mu_{j_1^{p+1} - j_2^p}^{0,0}(\|h^N - g\|_{\infty,[j_2^p/N, j_1^{p+1}/N]} < \delta) \cdot \mu_{N-j_2^K}^{0,b}(\|h^N - g\|_{\infty,[j_2^K/N,1]} < \delta),$$

where $g \in H_{a,b}^1(D)$; note that g is not appearing in the second term of $\Psi_{N,\mathbf{j}}^{\varepsilon}(g;\delta)$. We say that a sequence $\mathbf{j}_N = \{j_1^{p,N}, j_2^{p,N}\}_{p=1}^K \in \mathcal{J}_K$ is macroscopically $\mathbf{t} = \{t_1^p, t_2^p\}_{p=1}^K \in \mathcal{T}_K$ if $\lim_{N\to\infty} j_\ell^{p,N}/N = t_\ell^p$ hold for every $1 \leq p \leq K$ and $\ell = 1,2$, where $\mathcal{T}_K$ is a family of all $\mathbf{t}$ such that $0 < t_1^1 < t_2^1 < t_1^2 < t_2^2 < \cdots < t_1^K < t_2^K < 1$.

We now assume that $g \in H_{a,b}^1(D)$ satisfies the condition:

$$\{t \in D; g(t) = 0\} = \bigcup_{p=1}^{K} [t_1^p, t_2^p] \quad \text{with} \quad \mathbf{t} \in \mathcal{T}_K. \tag{2.4.8}$$

Lemma 2.4.6. *If a sequence* $\mathbf{j}_N$ *is macroscopically* $\mathbf{t}$ *and if* $g \in H_{a,b}^1(D)$ *satisfies* (2.4.8), *we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \Xi_{N,\mathbf{j}_N}^{\varepsilon} = \xi \sum_{p=1}^{K} (t_2^p - t_1^p) - \Sigma_0(a,b;t_1^1,t_2^K) + \inf_{H_{a,b}^1(D)} \Sigma(h), \tag{1}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log \Psi_{N,\mathbf{j}_N}^{\varepsilon}(g;\delta) \geq -\Sigma_0(g) + \Sigma_0(a,b;t_1^1,t_2^K), \tag{2}$$

for every $\delta > 0$*, where* $\xi = \xi^\varepsilon$ *and* $\Sigma_0(a,b;t_1^1,t_2^K) = \{|a|^2/t_1^1 + |b|^2/(1-t_2^K)\}/2$.

Proof. The first task for (1) is to calculate the limit as $N \to \infty$ of ratio of two partition functions $Z_N^{a,b}$ and $Z_N^{D,\varepsilon}$ up to an exponential order. To this end, we recall the expansion (2.3.4) which implies by letting $\delta \to \infty$

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} = 1 + \sum_{j \in D_N^\circ} \varepsilon \Xi_{N,j,j}^{\varepsilon} + \sum_{0<j<k<N} \varepsilon^2 \Xi_{N,j,k}^{\varepsilon}.$$

However, from (2.3.7) and Proposition 2.2.2, $\Xi^\varepsilon_{N,j,j}$ and $\Xi^\varepsilon_{N,j,k}$ behave as

$$e^{-N\tilde f(s_1,s_2)+N\xi(1-s_1-s_2)}$$

except algebraic factors as $N\to\infty$, where $s_1=j/N$, $s_2=(N-k)/N$ (we regard $k=j$ for $\Xi^\varepsilon_{N,j,j}$) and $\tilde f(s_1,s_2)$ is given by (2.3.8). Let $\hat h_{s_1,s_2}\in\mathcal{C}, 0<s_1\le 1-s_2<1$ be the function $\hat h^{D,(2)}$ defined by (2.1.8) with $t_1,t_2,a^{(2)},b^{(2)}$ replaced by s_1,s_2,a,b, respectively. Then, since

$$\tilde f(s_1,s_2)-\xi(1-s_1-s_2)=\Sigma(\hat h_{s_1,s_2})-\Sigma_0(\bar h^D),$$

and also by Lemma 2.1.2, we obtain that

$$\begin{aligned}\text{(2.4.9)}\qquad \lim_{N\to\infty}\frac1N\log\frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}} &= -\left[0\vee\sup_{0<s_1\le 1-s_2<1}\left\{-\Sigma(\hat h_{s_1,s_2})+\Sigma_0(\bar h^D)\right\}\right]\\ &= -\Sigma_0(\bar h^D)+\inf_{H^1_{a,b}(D)}\Sigma(h).\end{aligned}$$

The equality (1) follows from (2.4.9) and Proposition 2.2.2 recalling (2.3.3) (with $r=d$). The inequality (2) is a consequence of Propositions 2.4.2 and 2.4.3 noting the condition (2.4.8). □

We are now in a position to conclude the proof of the LD lower bound (2.4.6) for $\mu_N^{D,\varepsilon}$. We may assume the condition (2.4.8) for $g\in H^1_{a,b}(D)$, cf. [22]; if $K=0$ (i.e., $g(t)\neq 0$ for all $t\in D$), (2.4.6) follows from Proposition 2.4.2 and (2.4.9). Determine $\mathbf{j}_N$ from $\mathbf{t}$ in (2.4.8) by $j_\ell^{p,N}=[Nt_\ell^p], 1\le p\le K, \ell=1,2$, which is macroscopically $\mathbf{t}$. Then, we have

$$\text{(2.4.10)}\qquad \mu_N^{D,\varepsilon}(\|h^N-g\|_\infty<\delta)\ge\varepsilon^{2K}\Xi^\varepsilon_{N,\mathbf{j}_N}\Psi^\varepsilon_{N,\mathbf{j}_N}(g;\delta),$$

by noting that $g=0$ on $[t_1^p,t_2^p]$. In fact, this follows by restricting the probability on the left hand side on the event $\{\phi_{j_\ell^{p,N}}=0$ for all $1\le p\le K,\ell=1,2\}\cap\{\phi_i\neq 0$ for all $i\in\cup_{p=0}^K[j_2^{p,N},j_1^{p+1,N}]\cap\mathbb{Z}\}$, where $j_2^{0,N}=0$ and $j_1^{K+1,N}=N$. The LD lower bound (2.4.6) is shown from Lemma 2.4.6 and (2.4.10) for g satisfying (2.4.8). The rest of the proof is similar to [22], Proof of Theorem 2.2, Step 1.

Remark 2.4.2. *We have implicitly assumed that $a\neq 0$ and $b\neq 0$. The proof can be easily modified when $a=0$ or $b=0$. Indeed, if $a=0$, we take $j_1^1=0$ for $\mathbf{j}\in\mathcal{J}_K$ and remove the first factors $Z^{a,0}_{j_1^1}$ from $\Xi^\varepsilon_{N,\mathbf{j}}$ and $\mu^{a,0}_{j_1^1}(\|h^N-g\|_{\infty,[0,j_1^1/N]}<\delta)$ from $\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta)$, respectively. The factor $|a|^2/t_1^1$ does not appear in $\Sigma_0(a,b;t_1^1,t_2^K)$ in Lemma* 2.4.6. *Similar modification is possible when $b=0$, and the LD lower bound can be shown when $a=0$ or $b=0$.*

Upper bound

Let $g \in H^1_{a,b}(D)$ be a function satisfying the condition:

$$\text{for every } \gamma > 0 \text{ small enough,}$$

$$\{t \in D; |g(t)| \le \gamma\} = \bigcup_{p=1}^{K} [t_1^{p,\gamma}, t_2^{p,\gamma}] \big(=: I^\gamma \big) \quad \text{with} \quad \mathbf{t}^\gamma = \{t_1^{p,\gamma}, t_2^{p,\gamma}\}_{p=1}^K \in \mathcal{T}_K. \tag{2.4.11}$$

Then, if $0 < \delta < \gamma$, since $|g(t)| > \gamma$ implies on the event $\|h^N - g\|_\infty < \delta$ that $|h^N(t)| > \gamma - \delta > 0$ and therefore $\phi_i \ne 0$ for $i \in N(I^\gamma)^c \cap \mathbb{Z}$, we have

$$\begin{aligned} \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \le & \frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}} \mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \\ & + \sum_{k=1}^K \varepsilon^{2k} \sum_{\mathbf{j} \in \mathcal{J}_k(\mathbf{t}^\gamma)} \Xi_{N,\mathbf{j}}^\varepsilon \Psi_{N,\mathbf{j}}^\varepsilon(g; \delta + \gamma). \end{aligned} \tag{2.4.12}$$

Here, for $\mathbf{t} \in \mathcal{T}_K$ and $1 \le k \le K$, $\mathbf{j} = \{j_1^p, j_2^p\}_{p=1}^k \in \mathcal{J}_k(\mathbf{t})$ means that there exists $\mathbf{s} = \{s_1^p, s_2^p\}_{p=1}^k \in \mathcal{T}_k$ such that $\mathbf{s} \subset \mathbf{t}$ and $s_1^p \le j_1^p/N \le j_2^p/N \le s_2^p$ for every $1 \le p \le k$.

We elaborate the results in Lemma 2.4.6 to some extent, i.e., we need uniform upper bounds for $\Xi_{N,\mathbf{j}}^\varepsilon$ and $\Psi_{N,\mathbf{j}}^\varepsilon(g;\delta)$. For $\tilde\gamma > 0$, let $\mathcal{T}_{k,\tilde\gamma}$ be the set of all $\mathbf{t} \in \mathcal{T}_k$ such that $t_2^p - t_1^p \ge \tilde\gamma$ $(1 \le p \le k)$ and $t_1^p - t_2^{p-1} \ge \tilde\gamma$ $(1 \le p \le k+1)$, where $t_2^0 = 0$ and $t_1^{k+1} = 1$. The function $g \in H^1_{a,b}(D)$ satisfying the condition (2.4.11) is fixed in the next lemma.

Lemma 2.4.7. *For every $\tilde\gamma > 0$ and $\theta > 0$, there exist $\delta > 0$, $N_0 \ge 1$ and $\eta > 0$ such that*

$$\Xi_{N,\mathbf{j}}^\varepsilon \le \exp\left\{ N \left(\xi \sum_{p=1}^k (t_2^p - t_1^p) - \Sigma_0(a,b;t_1^1,t_2^k) + \inf_{H^1_{a,b}(D)} \Sigma(h) + \theta \right) \right\}, \tag{1}$$

$$\Psi_{N,\mathbf{j}}^\varepsilon(g;\delta) \le \exp\left\{ N \left(-\frac{1}{2} \int_{D\setminus I} |\dot g(t)|^2 \, dt + \Sigma_0(a,b;t_1^1,t_2^k) + \theta \right) \right\}, \tag{2}$$

hold for all $N \ge N_0$ and $\mathbf{j} \in \mathcal{J}_k$, $\mathbf{t} \in \mathcal{T}_{k,\tilde\gamma}, k \ge 1$, satisfying $|j_\ell^p/N - t_\ell^p| \le \eta$ for each $1 \le p \le k$ and $\ell = 1, 2$, where $I = \cup_{p=1}^k [t_1^p, t_2^p]$.

Proof. The bound (1) is shown by looking over each step of the proof of Lemma 2.4.6-(1) attentively; we omit the details. To show the bound (2), since the second term (i.e., the product of probabilities under $\mu^{0,0,\varepsilon}_{j_2^p - j_1^p}$) of $\Psi_{N,\mathbf{j}}^\varepsilon(g;\delta)$ is estimated by 1 from above, we may deal with other terms. Since those terms can be treated essentially in a same way, we discuss only the first term denoting j_1^1 simply by j. Choose a sufficiently small $\eta > 0$

(in particular, $\eta < \delta \wedge 1$) in such a manner that $|g(s) - g(t)| \le \delta$ holds for $|s-t| \le \eta$. Then, we have

$$I_N^j(\delta) := \mu_j^{a,0}(\|h^N - g\|_{\infty,[0,j/N]} < \delta) = \mu_j^{a,0}(\|\frac{j}{N}h^j(\cdot) - g(\frac{j}{N}\cdot)\|_\infty < \delta),$$

where $h^N(t)$ on the left hand side is defined for $t \in [0, j/N]$ while $h^j(t)$ on the right is for $t \in D = [0,1]$, and this implies the following uniform estimate in j satisfying $|j/N - s| \le \eta$ with $s = t_1^1$:

$$\mu_j^{a,0}(\|h^N - g\|_{\infty,[0,j/N]} < \delta) \le \mu_j^{a,0}(h^j \in \mathcal{A}_\delta),$$

where $\mathcal{A}_\delta = \{h \in \mathcal{C}; \|h(\cdot) - g(s\cdot)/s\|_\infty < c\delta\}$ for some $c > 0$. Indeed, one can take $c = \big(2s + \|g\|_\infty\big)/s(s-\eta)$ by estimating

$$\|h(\cdot) - g(s\cdot)/s\|_\infty \le \|h(\cdot) - g(u\cdot)/u\|_\infty + \|g(u\cdot) - g(s\cdot)\|_\infty/u + |u^{-1} - s^{-1}|\, \|g\|_\infty,$$

for $u = j/N$. Since the event $\mathcal{A}_\delta$ is independent of j, from Proposition 2.4.2, this leads to the uniform upper bound for $I_N^j(\delta)$:

$$I_N^j(\delta) \le \exp\left\{N\left(-\frac{1}{2}\int_0^s |\dot g(t)|^2\,dt + \frac{|a|^2}{2s} + \theta\right)\right\}$$

for N large enough, $\delta > 0$ small enough and all j satisfying $|j/N - s| \le \eta$. Thus, repeating this procedure for other terms, we obtain the upper bound (2). □

The LD upper bound (2.4.7) follows from (2.4.12) and Lemma 2.4.7 for $g \in H^1_{a,b}(D)$ satisfying the condition (2.4.11) by choosing $\gamma > 0$ sufficiently small. The rest of the proof is similar to [22], Proof of Theorem 2.2, Step 2.

2.4.4 Proof of Theorem 2.4.1 for $\mu_N^{D,\varepsilon,+}, \mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$

For $\mu_N^{F,\varepsilon}$, we modify the definition of $\Xi^\varepsilon_{N,\mathbf{j}}$ and $\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta)$ by replacing their first/last terms with $1/Z_N^{F,\varepsilon}$, $Z^{0,F}_{N-j_2^K}$ and $\mu^{0,F}_{N-j_2^K}(\|h^N - g\|_{\infty,[j_2^K/N,1]} < \delta)$, respectively. The modification is also clear under the presence of a wall. Then, one can follow the steps presented in Section 2.4.3 and obtain Theorem 2.4.1 for $\mu_N^{D,\varepsilon,+}$, $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$.

A Critical exponents for the free energies

Here we study the asymptotic behavior of the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$ near the critical values $\varepsilon_c = \varepsilon_{c,r}$ and $\varepsilon_c^+ = \varepsilon_{c,r}^+$, respectively. In general, when the physical order parameter exhibits a power law behavior in ε close to its critical value, the power is called the critical exponent. Our results give such critical exponents associated with the free energies.

Recall that the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$ are defined by the thermodynamic limits (2.1.4) and characterized by the equations: $g_r(e^{-\xi_r^\varepsilon}) = g_r^+(e^{-\xi_r^{\varepsilon,+}}) = 1/\varepsilon$. We put the subscripts r for g_r and g_r^+ to indicate their dependence on r. These functions are defined by (2.2.2) and (2.2.10), respectively, i.e., $g_r(x) = f_r(x)/(2\pi)^{r/2}$ and $g_r^+(x) = f_{r+2}(x)/(2\pi)^{r/2}$, where f_r is the so-called polylogarithm given by the power series:

$$f_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{r/2}} \tag{A.1}$$

for $0 \le x < 1$ (or $0 \le x \le 1$). The critical values $\varepsilon_{c,r}$ and $\varepsilon_{c,r}^+$ are determined by $\varepsilon_{c,r} = 1/g_r(1)$ and $\varepsilon_{c,r}^+ = 1/g_r^+(1)$ as in (2.2.3) and (2.2.11), respectively. Recall that $\varepsilon_{c,r} = 0$ for $r = 1, 2$, $\varepsilon_{c,r} = (2\pi)^{r/2}/\zeta(r/2) > 0$ for $r \ge 3$ and $\varepsilon_{c,r}^+ = (2\pi)^{r/2}/\zeta(r/2+1) > 0$ for all $r \ge 1$, where ζ is the Riemann's ζ-function.

The results of this section are summarized in the following proposition.

Proposition A.1. (1) (Absence of wall) *As $\varepsilon \downarrow \varepsilon_{c,r}$, we have that*

$$\xi_r^\varepsilon \sim \begin{cases} C_r(\varepsilon - \varepsilon_{c,r})^2, & r = 1, 3, \\ e^{-2\pi/\varepsilon}, & r = 2, \\ C_4\, \varphi(\varepsilon - \varepsilon_{c,4}), & r = 4, \\ C_r(\varepsilon - \varepsilon_{c,r}), & r \ge 5, \end{cases}$$

where $C_1 = 1/2, C_3 = 2\pi^2/\varepsilon_{c,3}^4$, $C_4 = 4\pi^2/\varepsilon_{c,4}^2$, $C_r = 2\pi\varepsilon_{c,r-2}/\varepsilon_{c,r}^2$ for $r \ge 5$ and $\varphi(x) = -x/\log x$ for sufficiently small $x > 0$.
(2) (Presence of wall) *As $\varepsilon \downarrow \varepsilon_{c,r}^+$, we have that*

$$\xi_r^{\varepsilon,+} \sim \begin{cases} C_1^+(\varepsilon - \varepsilon_{c,1}^+)^2, & r = 1, \\ C_2^+\, \varphi(\varepsilon - \varepsilon_{c,2}^+), & r = 2, \\ C_r^+(\varepsilon - \varepsilon_{c,r}^+), & r \ge 3, \end{cases}$$

where $C_1^+ = (2\pi)^2 C_3 \big(= 1/2(\varepsilon_{c,1}^+)^4\big)$, $C_2^+ = 2\pi C_4 \big(= 2\pi/(\varepsilon_{c,2}^+)^2\big)$ and $C_r^+ = 2\pi C_{r+2} \big(= 2\pi\varepsilon_{c,r-2}^+/(\varepsilon_{c,r}^+)^2\big)$ for $r \ge 3$.

Remark A.1. *Proposition* A.1 *indicates that the critical exponents κ_r and κ_r^+ associated with the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$, respectively, are given by $\kappa_1 = 2, \kappa_2 = \infty$, $\kappa_3 = 2$, $\kappa_4 = 1+$ and $\kappa_r = 1$ for $r \ge 5$, while $\kappa_1^+ = 2, \kappa_2^+ = 1+$ and $\kappa_r^+ = 1$ for $r \ge 3$. Here $\kappa = \infty$ means that the free energy vanishes faster than any power of ε, and $\kappa = 1+$*

means that the exponent is 1 *with a logarithmic correction. The asymptotic behavior of* ξ_1^ε *is studied in* [6].

Lemma A.2. *We have that* $2\pi\varepsilon_{c,r}^+ = \varepsilon_{c,r+2}$ *and* $\xi_r^{\varepsilon,+} = \xi_{r+2}^{2\pi\varepsilon}$ *for all* $\varepsilon \geq 0$ *and* $r \geq 1$.

Proof. The conclusion is immediate from the relation $g_r^+(x) = 2\pi g_{r+2}(x)$. □

The following asymptotics for the functions f_r as $x \uparrow 1$ may be well-known, but we give the proof for the completeness.

Lemma A.3. *As* $x \uparrow 1$*, we have that*

$$\begin{aligned} f_1(x) &\sim \sqrt{\pi}(1-x)^{-1/2}, \\ f_2(x) &= -\log(1-x), \\ f_3(1) - f_3(x) &\sim 2\sqrt{\pi}(1-x)^{1/2}, \\ f_4(1) - f_4(x) &\sim -(1-x)\log(1-x), \\ f_r(1) - f_r(x) &\sim \zeta(r/2-1)(1-x), \quad r \geq 5. \end{aligned}$$

Proof. The result for f_1 is a consequence of the Tauberian theorem, see [15], Theorem 5, p. 447. When $r = 2$, (A.1) is nothing but the Taylor expansion of $-\log(1-x)$ at $x = 0$. For $r = 3$ and 4, the relation $xf_r'(x) = f_{r-2}(x)$ for $0 < x < 1$ implies

$$f_r(1) - f_r(x) = \int_x^1 \frac{f_{r-2}(y)}{y}\, dy,$$

and this combined with the results for f_1 and f_2 shows the asymptotics for f_3 and f_4. If $r \geq 5$, f_r is differentiable at $x = 1$ from the left and $f_r'(1-) = f_{r-2}(1) = \zeta((r-2)/2)$. This shows the last asymptotic formula. □

Proof of Proposition A.1. The assertion (1) follows from Lemma A.3 recalling that $f_r(e^{-\xi_r^\varepsilon}) = (2\pi)^{r/2}/\varepsilon$. Note that $1 - e^{-\xi_r^\varepsilon} \sim \xi_r^\varepsilon$ as $\varepsilon \downarrow \varepsilon_{c,r}$ and, if $r \geq 3$ in addition, $f_r(1) = (2\pi)^{r/2}/\varepsilon_{c,r}$. Also note that $\psi^{-1}(x) \sim \varphi(x)$ holds as $x \downarrow 0$ for the inverse function ψ^{-1} of $\psi(\xi) = -\xi\log\xi$ defined for small enough $\xi > 0$. The assertion (2) follows from (1) combined with Lemma A.2. □

B Structure of minimizers in $d = 1$

In this section, we consider the case where $d = 1$ and $m = 0$ so that $a, b \in \mathbb{R}$, and clarify the structure of the set of minimizers of $\Sigma = \Sigma^D, \Sigma^{D,+}, \Sigma^F$ and $\Sigma^{F,+}$. Indeed, for each $\xi > 0$, the minimizers of Σ^D (or $\Sigma^{D,+}$) are completely characterizable in terms of $(a, b) \in \mathbb{R}^2$ (or $(a, b) \in \mathbb{R}^2_+$), and those of Σ^F (or $\Sigma^{F,+}$) in $a \in \mathbb{R}$ (or $a \in \mathbb{R}_+$) as well. The result is summarized in the following proposition. In particular, if a and b have different signs, Σ^D (or $\Sigma^{D,+}$) admits a unique minimizer h^*. We simply write $\xi, \bar{h}$ and $\hat{h}$ omitting the superscripts D, F, ε and $+$.

Proposition B.1. *Assume that $\xi > 0$, namely, $\varepsilon > 0$ is arbitrary under the absence of wall and $\varepsilon > \varepsilon_c^+$ under the presence of wall.*
(1) (Dirichlet case) *Let $\mathcal{O}$ be the bounded open region of $\mathbb{R}^2$ surrounded by its boundary $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1 = \{\sqrt{|a|} + \sqrt{|b|} = (2\xi)^{1/4}, ab > 0\}$ and $\mathcal{C}_2 = \{|a| + |b| = (2\xi)^{1/2}, ab \le 0\}$, which consists of four curves (see Figure A). Then, the set $\mathcal{H}^D$ of all minimizers of Σ^D (or $\Sigma^{D,+}$) is given as follows: $\mathcal{H}^D = \{\hat{h}\}$ on $\mathcal{O}$, $\mathcal{H}^D = \{\bar{h}\}$ on $\mathbb{R}^2 \setminus \bar{\mathcal{O}}$, $\mathcal{H}^D = \{\hat{h}, \bar{h}\}$ on $\mathcal{C}_1$ and $\mathcal{H}^D = \{\bar{h}\}$ on $\mathcal{C}_2$. Note that $\hat{h} = \bar{h}$ on $\mathcal{C}_2 \cup \{(0, 0)\}$ and $\hat{h} \neq \bar{h}$ on $\mathcal{C}_1$.*
(2) (Free case) *Let $\mathcal{H}^F$ be the set of all minimizers of Σ^F (or $\Sigma^{F,+}$). Then, $\mathcal{H}^F = \{\hat{h}\}$ on $\{|a| < \sqrt{\xi/2}\}$, $\mathcal{H}^F = \{\bar{h}\}$ on $\{|a| > \sqrt{\xi/2}\}$ and $\mathcal{H}^F = \{\hat{h}, \bar{h}\}$ on $\{|a| = \sqrt{\xi/2}\}$. Note that $\hat{h} = \bar{h}$ at $a = 0$ and $\hat{h} \neq \bar{h}$ at $|a| = \sqrt{\xi/2}$.*

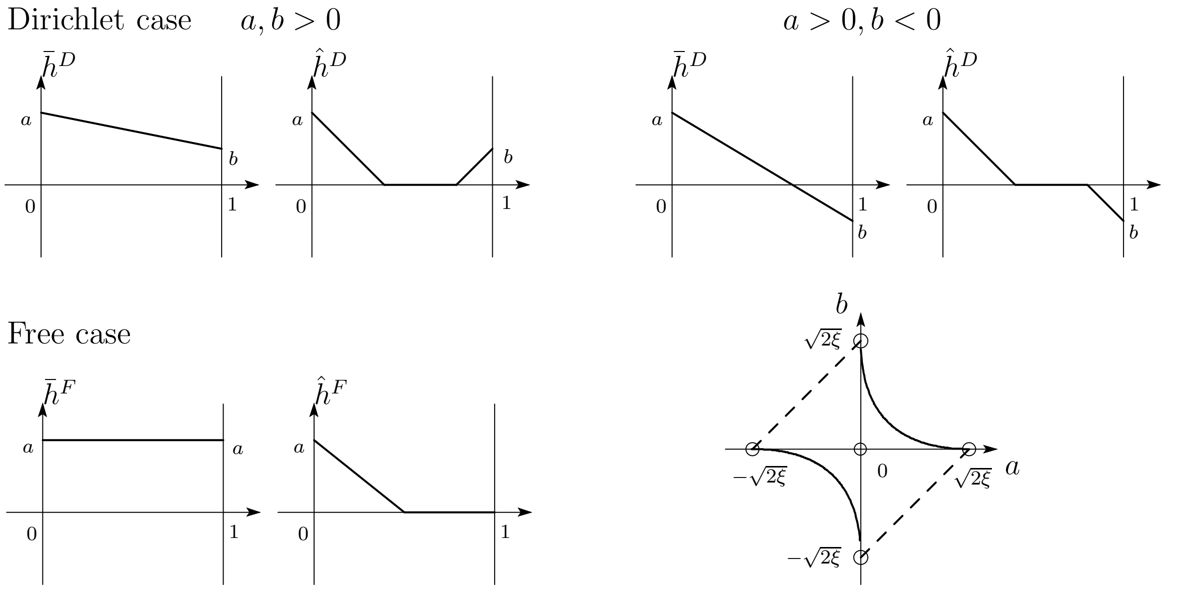


Figure A

Proof. We first give the proof of (1) assuming $a, b > 0$. If $a + b \ge \sqrt{2\xi}$ in addition, then $\bar{h}$ is the minimizer since it is the unique candidate in this case. If $a + b < \sqrt{2\xi}$, noting that

$$\Sigma(\hat{h}) = \frac{a^2}{2t_1} + \frac{b^2}{2t_2} - \xi(1 - t_1 - t_2) = \sqrt{2\xi}(a + b) - \xi$$

by Young's relation, we have

$$2\{\Sigma(\bar{h}) - \Sigma(\hat{h})\} = a^2 - 2a(b + 2\xi) + (b - \sqrt{2\xi})^2.$$

Therefore, we easily see that $\Sigma(\bar{h}) = \Sigma(\hat{h})$ is equivalent to $\sqrt{a} + \sqrt{b} = (2\xi)^{1/4}$ (noting that $a + b < \sqrt{2\xi}$) and the conclusion of (1) follows when $a, b > 0$. The case where $a, b < 0$ can be reduced to this case by symmetry. The case where $a > 0$ and $b < 0$ is also a simple computation. The minimizers of Σ^F (or $\Sigma^{F,+}$) are easily studied so that the proof of (2) is immediate. $\square$

Chapter 3

Scaling limits for weakly pinned random walks with two large deviation minimizers

3.1 Introduction and main results

This chapter deals with random walks on $\mathbb{R}^d$ perturbed by an attractive force toward the origin 0, especially under the critical situation that the rate functional of the corresponding large deviation principle admits exactly two minimizers. The macroscopic time, observed after scaling, runs over the interval $D = [0, 1]$. The starting point of the (macroscopically scaled) walks at $t = 0$ is always specified, while we will or will not specify the arriving point at $t = 1$. We consider such two different cases.

The mean-zero Gaussian random walks, perturbed by an attractive force toward a subspace M of $\mathbb{R}^d$, are studied in Chapter 2 under the presence or absence of a wall located at the boundary of the upper half space of $\mathbb{R}^d$. This chapter investigates the situation that $M = \{0\}$ and the wall is absent. We extend the class of transition probability densities $p(x)$ of the random walks from mean-zero Gaussian (i.e. $p(x) = e^{-|x|^2/2}/(2\pi)^{d/2}$) to general functions satisfying Assumption 3.1.1 stated below. Otherwise, the problems discussed here are exactly the same as in Chapter 2. This chapter is based on a joint work with Professor T. Funaki.

3.1.1 Weakly pinned random walks

This subsection introduces (temporally inhomogeneous) Markov chains called the weakly pinned random walks. Let $D_N = ND \cap \mathbb{Z} \equiv \{0, 1, 2, \ldots, N\}$ be the range of (microscopic) time for the Markov chains corresponding to the macroscopic one D. The state space of the Markov chains is $\mathbb{R}^d$.

Given $a, b \in \mathbb{R}^d$, the starting point of the Markov chains $\phi = (\phi_i)_{i \in D_N}$ is always $aN \in \mathbb{R}^d$ (i.e. $\phi_0 = aN$), while, for the arriving point at $i = N$, we consider two cases: under conditioning ϕ as $\phi_N = bN$ (we call Dirichlet case) or without giving any condition on ϕ_N (we call free case). The distributions of the Markov chains ϕ on $(\mathbb{R}^d)^{N+1}$ with the strength $\varepsilon \geq 0$ of the pinning force toward the origin 0, imposing the Dirichlet or free conditions at N, are described by the following two probability measures $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ on $(\mathbb{R}^d)^{N+1}$, respectively:

$$\mu_N^{D,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,b,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N-1} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big) \delta_{bN}(d\phi_N), \tag{3.1.1}$$

$$\mu_N^{F,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big), \tag{3.1.2}$$

where

$$\mathfrak{p}_N(\phi) = \prod_{i=1}^{N} p(\phi_i - \phi_{i-1}),$$

with a measurable function $p : \mathbb{R}^d \to [0, \infty)$ satisfying $\int_{\mathbb{R}^d} p(x)\, dx = 1$, $d\phi_i$ denotes the Lebesgue measure on $\mathbb{R}^d$, and $Z_N^{a,b,\varepsilon}$ and $Z_N^{a,F,\varepsilon}$ are the normalizing constants, respectively. Note that, if $\varepsilon = 0$, ϕ under $\mu_N^{F,0}$ is the random walk with the transition probability $p(y-x)dy, x, y \in \mathbb{R}^d$ and its conditioning as $\phi_N = bN$ under $\mu_N^{D,0}$. We always assume the following conditions on the transition probability density p:

Assumption 3.1.1. (1) *The function p satisfies $\sup_{x \in \mathbb{R}^d} e^{\lambda \cdot x} p(x) < \infty$ for all $\lambda \in \mathbb{R}^d$, where $\lambda \cdot x = \sum_{\alpha=1}^d \lambda^\alpha x^\alpha$ denotes the inner product of $\lambda = (\lambda^\alpha)_{\alpha=1}^d$ and $x = (x^\alpha)_{\alpha=1}^d$ in $\mathbb{R}^d$. In particular, the Cramér's condition is satisfied:*

$$\Lambda(\lambda) \equiv \log \int_{\mathbb{R}^d} e^{\lambda \cdot x} p(x)\, dx < \infty. \tag{3.1.3}$$

(2) *The Legendre transform of Λ defined by*

$$\Lambda^*(v) = \sup_{\lambda \in \mathbb{R}^d} \{\lambda \cdot v - \Lambda(\lambda)\}, \quad v \in \mathbb{R}^d, \tag{3.1.4}$$

is finite for all $v \in \mathbb{R}^d$, and satisfies $\Lambda^* \in C^3(\mathbb{R}^d)$.

Remark 3.1.1. *Assumption* 3.1.1-(2) *is imposed to simplify the arguments. For instance, if p has a compact support, then so does Λ^* and the macroscopically scaled Markov chain h^N defined below can move only within the distance being inside the support of Λ^*.*

When $d = 1$, the Markov chain ϕ may be interpreted as the heights of interfaces located in a plane measured from the position i on the reference line (x-axis), so that the system is called $(1+1)$-dimensional interface model with δ-pinning at 0, see [16]. For general $d \geq 1$, ϕ can be interpreted as the $(1+d)$-dimensional directed polymers, see [24].

We will assume that $a, b \neq 0$, since the case $a = 0$ or $b = 0$ is similar or even simpler.

3.1.2 Scaling limits and large deviation rate functionals

Let $h^N = \{h^N(t); t \in D\}$ be the macroscopic path of the Markov chain determined from the microscopic one ϕ under a proper scaling, namely, it is defined through a polygonal approximation of $\big(h^N(i/N) = \phi_i/N\big)_{i \in D_N}$ so that

$$h^N(t) = \frac{[Nt] - Nt + 1}{N}\phi_{[Nt]} + \frac{Nt - [Nt]}{N}\phi_{[Nt]+1}, \quad t \in D.$$

Then, the sample path large deviation principle holds for h^N under $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$, respectively, on the space $\mathcal{C} = C(D, \mathbb{R}^d)$ equipped with the uniform topology as $N \to \infty$, see Theorem 3.5.1 below (or [8], [28] for $\mu_N^{F,0}$ when $\varepsilon = 0$). The speeds are always N and the unnormalized rate functionals are given by Σ^D and Σ^F, respectively, both of which are of the form:

$$\Sigma(h) = \int_D \Lambda^*(\dot{h}(t))\, dt - \xi^\varepsilon |\{t \in D; h(t) = 0\}|, \tag{3.1.5}$$

for $h \in \mathcal{AC}_{a,b} = \{h \in \mathcal{AC}; h(0) = a, h(1) = b\}$ in the Dirichlet case respectively $h \in \mathcal{AC}_{a,F} = \{h \in \mathcal{AC}; h(0) = a\}$ in the free case with certain non-negative constants $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon}$, where Λ^* is the Legendre transform of Λ defined by (3.1.4) and $|\cdot|$ stands for the Lebesgue measure on D and $\mathcal{AC} = \mathcal{AC}(D, \mathbb{R}^d)$ is the family of all absolutely continuous functions $h \in \mathcal{C}$. We define $\Sigma(h) = +\infty$ for h's outside of these spaces. The Cramér's condition (3.1.3) implies that $\Lambda \in C^\infty(\mathbb{R}^d)$ and strictly convex, and Λ^* is also strictly convex on the relative interior $\mathrm{ri}(\mathrm{dom}\Lambda^*)$ of $\mathrm{dom}\Lambda^* \equiv \{v \in \mathbb{R}^d : \Lambda^*(v) < \infty\}$, see Theorem VII.5.5 in [13]; note that $\mathrm{ri}(\mathrm{dom}\Lambda^*) = \mathbb{R}^d$ by Assumption 3.1.1-(2).

We determine a non-negative constant $\xi^{D,\varepsilon}$ by the thermodynamic limit:

$$\xi^{D,\varepsilon} = \lim_{N \to \infty} \frac{1}{N} \log \frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}}, \tag{3.1.6}$$

and another non-negative constant $\xi^{F,\varepsilon}$ by

$$\xi^{F,\varepsilon} = \lim_{N \to \infty} \frac{1}{N} \log \frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}, \tag{3.1.7}$$

where the partition functions are given by taking $a = b = 0$ in the Dirichlet case and $a = 0$ in the free case, and the denominators $Z_N^{0,0}$ and $Z_N^{0,F}$ are defined without pinning effect and equal to their corresponding numerators with $\varepsilon = 0$. See (3.3.6) below for the explicit formula of $Z_N^{0,0,\varepsilon}$ and (3.3.11) for $Z_N^{0,F,\varepsilon}$. The constants ξ^ε in (3.1.5) are defined by $\xi^\varepsilon = \xi^{D,\varepsilon}$ for the functional $\Sigma = \Sigma^D$ and $\xi^\varepsilon = \xi^{F,\varepsilon}$ for Σ^F, respectively.

Explicit formulae determining the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$ are found in (3.3.9) and (3.3.13) below, respectively. Furthermore, we have the following result which extends Theorem 2.1.1 to our setting. We denote the mean drift of p by $m = \int_{\mathbb{R}^d} xp(x)\,dx \in \mathbb{R}^d$.

Theorem 3.1.1. (1) *The limits $\xi^{D,\varepsilon}$ in* (3.1.6) *and $\xi^{F,\varepsilon}$ in* (3.1.7) *exist for every $\varepsilon \geq 0$.*
(2) *There exist two critical values $0 \leq \varepsilon_c^D \leq \varepsilon_c^F$ determined by* (3.3.8) *and* (3.3.12) *below, respectively, such that $\xi^{D,\varepsilon} > 0$ iff $\varepsilon > \varepsilon_c^D$ (therefore $\xi^{D,\varepsilon} = 0$ iff $0 \leq \varepsilon \leq \varepsilon_c^D$) and $\xi^{F,\varepsilon} > 0$ iff $\varepsilon > \varepsilon_c^F$ (therefore $\xi^{F,\varepsilon} = 0$ iff $0 \leq \varepsilon \leq \varepsilon_c^F$).*
(3) *If $d \geq 3$, $\varepsilon_c^D > 0$, while if $d = 1$ and 2, $\varepsilon_c^D = 0$.*
(4) *We have $\varepsilon_c^D = \varepsilon_c^F$ iff $m = 0$, and in this case $\xi^{D,\varepsilon} = \xi^{F,\varepsilon}$ holds for all $\varepsilon \geq 0$. If $m \neq 0$, we have $\varepsilon_c^D < \varepsilon_c^F$ and $\xi^{F,\varepsilon} < \xi^{D,\varepsilon}$ holds for every $\varepsilon > \varepsilon_c^D$.*

The last assertion of Theorem 3.1.1 can be interpreted as follows. In such a case that the original unperturbed random walk has non-zero drift $m \neq 0$, if the strength ε of the pinning belongs to the range $\varepsilon \in (\varepsilon_c^D, \varepsilon_c^F)$, the weakly pinned Markov chain is transient (or delocalized) in the free case while it is recurrent (or localized) in the Dirichlet case. This happens because the Dirichlet condition has an effect to make the drift of the random walk vanish.

The large deviation principle (Theorem 3.5.1) immediately implies the concentration properties for $\mu_N = \mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$: for any $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\mu_N(\mathrm{dist}_\infty(h^N, \mathcal{H}) > \delta) \leq e^{-c(\delta)N} \tag{3.1.8}$$

for large enough N, where $\mathcal{H} = \{h^*; \text{minimizers of } \Sigma\}$ with $\Sigma = \Sigma^D$ and Σ^F, respectively, and dist_∞ denotes the distance on $\mathcal{C}$ under the uniform norm $\|\cdot\|_\infty$.

Let us now study the minimizers or their candidates of the rate functionals Σ. Define two functions $\bar{h}_{a,b}$ and $\hat{h}_{a,b;\theta_1,\theta_2}$ on D for $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < 1$ by

$$\bar{h}_{a,b}(t) = (1-t)a + tb, \quad t \in D,$$

and

$$\hat{h}_{a,b;\theta_1,\theta_2}(t) = \begin{cases} (\theta_1 - t)a/\theta_1, & t \in [0, \theta_1), \\ 0, & t \in [\theta_1, 1-\theta_2], \\ (t + \theta_2 - 1)b/\theta_2, & t \in (1-\theta_2, 1], \end{cases}$$

respectively. For each $v \in \mathbb{R}^d \setminus \{0\}$ and $c \geq -\Lambda^*(0)$, let $s = s(v,c) \geq 0$ be the unique solution of the equation

$$sv \cdot \nabla\Lambda^*(sv) - \Lambda^*(sv)\big(\equiv \Lambda\big(\nabla\Lambda^*(sv)\big)\big) = c, \tag{3.1.9}$$

where $v \cdot \nabla\Lambda^* = \sum_{\alpha=1}^d v^\alpha \partial\Lambda^*/\partial v^\alpha$. We define $t_1^D, t_2^D > 0$ by $t_1^D = 1/s(-a, \xi^{D,\varepsilon} - \Lambda^*(0)), t_2^D = 1/s(b, \xi^{D,\varepsilon} - \Lambda^*(0))$ and $t_1^F > 0$ by $t_1^F = 1/s(-a, \xi^{F,\varepsilon} - \Lambda^*(0))$, respectively; if such s does not exist, we set $t_1^D = \infty$ etc. Denote the sets of the minimizers of Σ^D and Σ^F by $\mathcal{M}^D$ and $\mathcal{M}^F$, respectively.

If $\xi^{D,\varepsilon} = 0$ or $\xi^{F,\varepsilon} = 0$, the minimizer of Σ^D or Σ^F is unique and given by $\bar{h}^D := \bar{h}_{a,b}$ or $\bar{h}^F := \bar{h}_{a,a+m}$ for each functional. We therefore consider under the condition $\varepsilon > \varepsilon_c^D$ or $\varepsilon > \varepsilon_c^F$. The following lemma extends Lemma 2.1.2 or the result shown in Sections 6.3 and 6.4 of [16], which discuss the special case: $d = 1$ and $\Lambda^*(v) = v^2/2$.

Lemma 3.1.2. (1) *The solution $s = s(v,c)$ of the equation* (3.1.9) *is unique.*
(2) (Dirichlet case) *If $t_1^D + t_2^D < 1$, $\mathcal{M}^D$ is contained in $\{\bar{h}^D, \hat{h}^D\}$, where $\hat{h}^D := \hat{h}_{a,b;t_1^D,t_2^D}$ (i.e., $\theta_1 = t_1^D, \theta_2 = t_2^D$). If $t_1^D + t_2^D \geq 1$, then $\mathcal{M}^D = \{\bar{h}^D\}$.*
(3) (Free case)

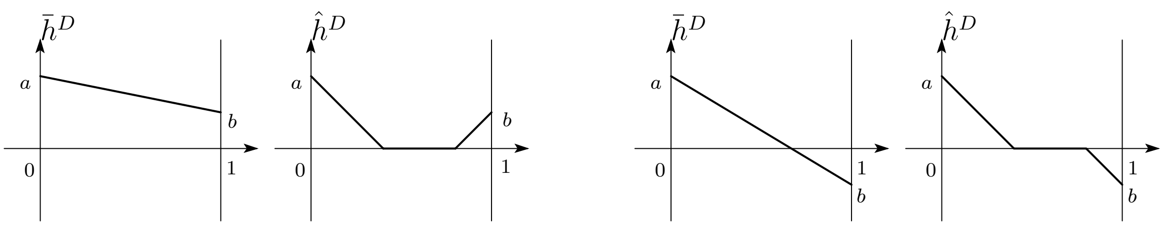
(i) *If $\xi^{F,\varepsilon} > \Lambda^*(0)$ and $t_1^F < 1$, then $\mathcal{M}^F$ is contained in $\{\bar{h}^F, \hat{h}^F\}$, where $\hat{h}^F := \hat{h}_{a,0;t_1^F,0}$ (i.e., $b = 0, \theta_1 = t_1^F, \theta_2 = 0$).*

(ii) *If $\xi^{F,\varepsilon} = \Lambda^*(0)$, $t_1^F < 1$ and $a = -t_1^F m$, then $\mathcal{M}^F$ coincides with the set $\big\{\hat{h}_{a,\theta m;t_1^F,\theta};\ \theta \in [0, 1 - t_1^F]\big\}$; note that $\bar{h}^F = \hat{h}_{a,(1-t_1^F)m;t_1^F,1-t_1^F}$ in this case.*

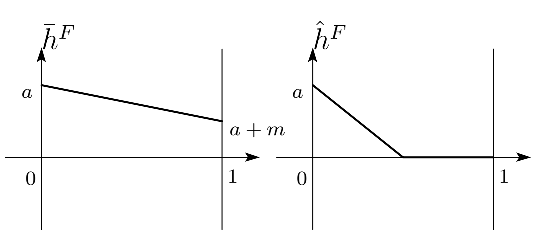
(iii) *In all other cases (i.e., if "$\xi^{F,\varepsilon} \geq \Lambda^*(0)$ and $t_1^F \geq 1$" or "$\xi^{F,\varepsilon} = \Lambda^*(0), t_1^F < 1$ and $a \neq -t_1^F m$" or "$0 < \xi^{F,\varepsilon} < \Lambda^*(0)$"), then $\mathcal{M}^F = \{\bar{h}^F\}$.*

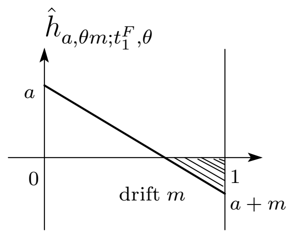
The graphs of the functions $\bar{h}^D, \hat{h}^D, \bar{h}^F, \hat{h}^F$ and $\hat{h}_{a,\theta m;t_1^F,\theta}$ in $d = 1$ are shown below.

Dirichlet case $\quad a, b > 0 \qquad\qquad\qquad\qquad a > 0, b < 0$



Free case $\quad \xi^{F,\varepsilon} > \Lambda^*(0) \qquad\qquad\qquad\qquad \xi^{F,\varepsilon} = \Lambda^*(0)$





In the free case, under the condition $\xi^{F,\varepsilon} = \Lambda^*(0)$, the minimizers $\hat{h}_{a,\theta m;t_1^F,\theta}$ starting at a are floated by the drift m without any cost and, once they hit 0, the price $\Lambda^*(0)$ to stay there balances with the gain $\xi^{F,\varepsilon}$ staying there so that they can leave 0 at any time.

3.1.3 Main results

We are concerned with the critical case where $\bar{h}$ and $\hat{h}$ are different and both are simultaneously the minimizers of Σ^D, and similar situations for Σ^F. We will exclude the special case appeared in Lemma 3.1.2-(3)-(ii), for which the set of the minimizers of Σ^F is continuously parameterized by θ. Otherwise, h^N converges to the unique minimizer of Σ as $N \to \infty$ in probability, recall (3.1.8). We therefore assume the following conditions in each situation:

$(C)_D$ $\quad \varepsilon > \varepsilon_c^D,\ t_1^D + t_2^D < 1 \quad$ and $\quad \Sigma^D(\bar{h}^D) = \Sigma^D(\hat{h}^D)$,

$(C)_F$ $\quad \varepsilon > \varepsilon_c^F,\ \xi^{F,\varepsilon} > \Lambda^*(0),\ t_1^F < 1 \quad$ and $\quad \Sigma^F(\bar{h}^F) = \Sigma^F(\hat{h}^F)$.

We are now in a position to state our main results. We say that the limit under μ_N is h^* if

$$\lim_{N\to\infty} \mu_N(\|h^N - h^*\|_\infty \le \delta) = 1$$

holds for every $\delta > 0$. We say that two functions $\bar{h}$ and $\hat{h}$ coexist in the limit under μ_N with probabilities $\bar{\lambda}$ and $\hat{\lambda}$ if $\bar{\lambda}, \hat{\lambda} > 0$, $\bar{\lambda} + \hat{\lambda} = 1$ and

$$\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = \bar{\lambda},$$
$$\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = \hat{\lambda}$$

hold for every $0 < \delta < |a| \wedge |b|$.

Theorem 3.1.3. (1) (Dirichlet case) *Under the condition $(C)_D$, the limit under $\mu_N^{D,\varepsilon}$ is $\hat{h}^D$ if $d = 1$ and $\bar{h}^D$ if $d \ge 3$. If $d = 2$, $\bar{h}^D$ and $\hat{h}^D$ coexist in the limit under $\mu_N^{D,\varepsilon}$ with probabilities $\bar{\lambda}^{D,\varepsilon}$ and $\hat{\lambda}^{D,\varepsilon}$, respectively, given by* (3.4.15).

(2) (Free case) *Under the condition* $(C)_F$*, if* $d = 1$*,* $\bar{h}^F$ *and* $\hat{h}^F$ *coexist in the limit under* $\mu_N^{F,\varepsilon}$ *with probabilities* $\bar{\lambda}^{F,\varepsilon}$ *and* $\hat{\lambda}^{F,\varepsilon}$*, respectively, given by* (3.4.22)*. If* $d \geq 2$*, the limit under* $\mu_N^{F,\varepsilon}$ *is* $\bar{h}^F$*.*

The central limit theorem holds for the times when the Markov chains first or last touch the origin 0. Set

$$i_\ell = \min\{i \in D_N; \phi_i = 0\},$$
$$i_r = \max\{i \in D_N; \phi_i = 0\},$$

and consider them under a proper scaling:

$$X = \frac{1}{\sqrt{N}}\big(i_\ell - t_1 N\big) \quad \text{and} \quad Y = \frac{1}{\sqrt{N}}\big(i_r - (1-t_2)N\big),$$

where we set $\min\emptyset = N$ (in the Dirichlet case), $= N+1$ (in the free case), $\max\emptyset = 0$, and Y is considered only for the Dirichlet case.

Theorem 3.1.4. (1) (Dirichlet case) *Under* $\mu_N^{D,\varepsilon}$*, conditioned on the event* $\{i_\ell \leq N-1\}$ *if* $d \geq 2$*, the pair of random variables* (X, Y) *weakly converges to* (U_1, U_2) *as* $N \to \infty$*, where* $U_1 = N(0, 1/2c_1)$ *and* $U_2 = N(0, 1/2c_2)$ *are mutually independent centered Gaussian random variables, and* c_1 *and* c_2 *are given by* (3.4.10)*.*
(2) (Free case) *Under* $\mu_N^{F,\varepsilon}$ *conditioned on the event* $\{i_\ell \leq N\}$*,* X *weakly converges to* $U = N(0, 1/2c_3)$ *as* $N \to \infty$*, where* c_3 *is given by* (3.4.21)*.*

Section 3.2 proves Lemma 3.1.2. The proof of Theorems 3.1.3 and 3.1.4 will be given in Section 3.4. The conditions $(C)_D$ and $(C)_F$ guarantee that the leading exponential decay rates of the probabilities of the neighborhoods of the two different minimizers balance with each other. This enforces us to study their precise asymptotics, which are discussed in Section 3.3. The proof of Theorem 3.1.1 is also given in Section 3.3. Section 3.5 is for the sample path large deviation principles. Mogul'skii's result [28] for the free case without pinning is extended to the Dirichlet case. In Section 3.6, we study the critical exponents for the free energies ξ^ε by establishing their asymptotic behavior in ε close to their critical values.

3.2 Proof of Lemma 3.1.2

For each $v \in \mathbb{R}^d \setminus \{0\}$, set $f(s) = sv \cdot \nabla\Lambda^*(sv) - \Lambda^*(sv)$ for $s \geq 0$. Then, we see that $f'(s) = s\sum_{\alpha,\beta=1}^d v^\alpha v^\beta \partial^2\Lambda^*/\partial v^\alpha \partial v^\beta(sv) > 0$ for $s > 0$ and $f(0) = -\Lambda^*(0)$, and this proves the assertion (1).

To show (2) and (3), we first notice the following: For $0 \le s_1 < s_2 \le 1$ and $h \in \mathcal{AC}([s_1, s_2])$ such that $h(s_1) = a$ and $h(s_2) = b$, Jensen's inequality implies that

$$\frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \Lambda^*(\dot{h}(t))\, dt \ge \Lambda^* \left(\frac{1}{s_2 - s_1} \int_{s_1}^{s_2} \dot{h}(t)\, dt \right) = \Lambda^* \left(\frac{b-a}{s_2 - s_1} \right),$$

in which the equality holds only when $\dot{h}(t) = (b-a)/(s_2 - s_1)$ because of the strict convexity of Λ^* on $\mathbb{R}^d$. Thus the minimizer h of the functional $\int_{s_1}^{s_2} \Lambda^*(\dot{h}(t))\, dt$ is linearly interpolating between a and b: $h(t) = (b-a)(t - s_1)/(s_2 - s_1) + a$, $t \in [s_1, s_2]$. This means that the graph of any minimizer of Σ must be a line as long as it does not touch 0, therefore, the minimizers of Σ are in the class of functions $\{\bar{h}_{a,b}, \hat{h}_{a,b;\theta_1,\theta_2}; \theta_1, \theta_2 > 0, \theta_1 + \theta_2 < 1\}$.

To find the minimizers of Σ in the class of $\{\hat{h}_{a,b;\theta_1,\theta_2}\}$, we set

$$\text{(3.2.1)} \quad F_{a,b}(\theta_1, \theta_2) := \Sigma\big(\hat{h}_{a,b;\theta_1,\theta_2}\big) = \theta_1 \Lambda^*(-a/\theta_1) + \theta_2 \Lambda^*(b/\theta_2) + (1 - \theta_1 - \theta_2)\big(\Lambda^*(0) - \xi^\varepsilon\big).$$

Then, we have that

$$\begin{aligned} \frac{\partial F_{a,b}}{\partial \theta_1}(\theta_1, \theta_2) &= \Lambda^*(-a/\theta_1) + (a/\theta_1) \cdot \nabla \Lambda^*(-a/\theta_1) - \big(\Lambda^*(0) - \xi^\varepsilon\big), \\ \frac{\partial F_{a,b}}{\partial \theta_2}(\theta_1, \theta_2) &= \Lambda^*(b/\theta_2) - (b/\theta_2) \cdot \nabla \Lambda^*(b/\theta_2) - \big(\Lambda^*(0) - \xi^\varepsilon\big). \end{aligned}$$

If the minimizer of Σ^D is in the class of $\{\hat{h}_{a,b;\theta_1,\theta_2}\}$, then it satisfies $\partial F_{a,b}/\partial \theta_1 = \partial F_{a,b}/\partial \theta_2 = 0$, which is equivalent to $\theta_1 = t_1^D$ and $\theta_2 = t_2^D$; note that $\hat{h}_{a,b;\theta_1,\theta_2}$ can not be a minimizer if $\theta_1 + \theta_2 = 1$ from the reason mentioned above. This proves the assertion (2).

Let us consider the minimizer of Σ^F in the class of $\{\hat{h}_{a,b;\theta_1,\theta_2}\}$. Now, $b \in \mathbb{R}^d$ also moves as a parameter. The function $F_{a,b}(\theta_1, \theta_2)$, as a function of b, is minimized at $b/\theta_2 = m$ (recall $\Lambda^*(m) = 0$), and it becomes $F_a(\theta_1, \theta_2) \equiv F_{a,\theta_2 m}(\theta_1, \theta_2) = \theta_1 \Lambda^*(-a/\theta_1) + (1 - \theta_1 - \theta_2)\big(\Lambda^*(0) - \xi^\varepsilon\big)$. The function $F_a(\theta_1, \theta_2)$, as a function of θ_2, is minimized at $\theta_2 = 0$ if $\xi^\varepsilon > \Lambda^*(0)$ (which proves the assertion (3)-(i)), at $\theta_2 = 1 - \theta_1$ if $\xi^\varepsilon < \Lambda^*(0)$ and at all $\theta_2 \in [0, 1 - \theta_1]$ if $\xi^\varepsilon = \Lambda^*(0)$. In case $\xi^\varepsilon < \Lambda^*(0)$, $\theta_2 = 1 - \theta_1$ means that $\hat{h}_{a,b;\theta_1,\theta_2}$ actually touch 0 only at $t = \theta_1$ (note that we are concerned with the case $m \ne 0$, since $m = 0$ implies $\Lambda^*(0) = 0$ so that $\xi^\varepsilon < \Lambda^*(0)$ can not happen), and therefore the minimizer of Σ^F must be $\bar{h}^F$. In case $\xi^\varepsilon = \Lambda^*(0)$, for all $\theta_2 \in [0, 1 - \theta_1]$, we have $F_a(\theta_1, \theta_2) = \theta_1 \Lambda^*(-a/\theta_1)$ which is minimized at $\theta_1 = t_1^F$, so that the candidates of the minimizers are of the form $\hat{h}_{a,\theta_2 m; t_1^F, \theta_2}$, $\theta_2 \in [0, 1 - t_1^F]$. Comparing its energy with that of the another candidate $\bar{h}^F$: $\Sigma^F(\bar{h}^F) = 0$, it must hold $F_a(t_1^F, \theta_2) = 0$, which is satisfied only when $-a/t_1^F = m$. This proves the assertion (3)-(ii). The proof of Lemma 3.1.2 is thus concluded.

Remark 3.2.1. *The condition* (3.1.9) *is known as the Young's relation, which prescribes the free boundary points* t_1^D, t_2^D *and* t_1^F*.*

3.3 Precise asymptotics for the partition functions

This section establishes the precise asymptotic behavior of the ratios of partition functions associated with the random walks in $\mathbb{R}^d$ with pinning at $0 \in \mathbb{R}^d$ and starting at $0 \in \mathbb{R}^d$ (and reaching 0 in the Dirichlet case), which were mentioned in Section 3.1.2 to determine $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$. In particular, these will imply the statements in Theorem 3.1.1. A similar result is obtained by Chapter 2.

We introduce several notation; see Section 5.5 of [16] when $d = 1$. For $\lambda \in \mathbb{R}^d$, we define the Cramér transform p_λ of p by

$$p_\lambda(x) = e^{\lambda \cdot x - \Lambda(\lambda)} p(x), \quad x \in \mathbb{R}^d.$$

Note that, under Assumption 3.1.1, the function Λ is in $C^\infty(\mathbb{R}^d)$ and strictly convex, since its Hesse matrix $(\partial^2 \Lambda(\lambda)/\partial\lambda^\alpha \partial\lambda^\beta)_{1 \le \alpha,\beta \le d}$ is equal to the covariance matrix $\mathcal{Q}(\lambda) = (q^{\alpha\beta}(\lambda))_{1 \le \alpha,\beta \le d}$ of p_λ, which is strictly positive definite. Here, $q^{\alpha\beta}(\lambda) = \int_{\mathbb{R}^d} (x^\alpha - v^\alpha(\lambda))(x^\beta - v^\beta(\lambda)) p_\lambda(x)\, dx$ and $v^\alpha(\lambda) = \int_{\mathbb{R}^d} x^\alpha p_\lambda(x)\, dx$; in particular, $m = v(0)$. Two functions $v = v(\lambda) : \mathbb{R}^d \to \mathbb{R}^d$ and $\lambda = \lambda(v) : \mathbb{R}^d \to \mathbb{R}^d$ are defined by

$$\begin{aligned} v = v(\lambda) &:= \nabla\Lambda(\lambda) \left(= \int_{\mathbb{R}^d} x p_\lambda(x)\, dx \right), \quad \lambda \in \mathbb{R}^d, \\ \lambda = \lambda(v) &:= \nabla\Lambda^*(v), \quad v \in \mathbb{R}^d. \end{aligned} \tag{3.3.1}$$

Note that $\lambda = \lambda(v)$ is the inverse function of $v = v(\lambda)$: $\lambda = \lambda(v) \Leftrightarrow v = v(\lambda)$ and the supremum in the right hand side of (3.1.4) for $\Lambda^*(v)$ is attained at $\lambda = \lambda(v)$:

$$\Lambda^*(v) = \lambda(v) \cdot v - \Lambda(\lambda(v)), \tag{3.3.2}$$

cf. Lemma 2.2.31-(b) of [8].

3.3.1 Dirichlet case

For $0 \le j < k \le N$, we denote by $\mu_{j,k}^{a,b}$ the probability measure on $(\mathbb{R}^d)^{\{j,\dots,k\}} = \{\phi = (\phi_i)_{j \le i \le k}; \phi_i \in \mathbb{R}^d\}$ without pinning under the Dirichlet conditions $\phi_j = aN$ and

$\phi_k = bN$:

$$\mu_{j,k}^{a,b}(d\phi) = \frac{\mathfrak{p}_{j,k}(\phi)}{Z_{j,k}^{a,b}} \delta_{aN}(d\phi_j) \prod_{i=j+1}^{k-1} d\phi_i \, \delta_{bN}(d\phi_k), \tag{3.3.3}$$

where $\mathfrak{p}_{j,k}(\phi) = \prod_{i=j+1}^{k} p(\phi_i - \phi_{i-1})$ and $Z_{j,k}^{a,b} = Z_{k-j}^{a,b} \left(= Z_{k-j}^{a,b,N}\right)$ is the normalizing constant. Then, we have the following lemma. A similar result for random walks on $\mathbb{Z}^d$ can be found in Proposition B.2 of [6]. Recall that the matrices $\mathcal{Q}(\lambda)$ are strictly positive definite for all $\lambda \in \mathbb{R}^d$ from the definition.

Lemma 3.3.1. *As $n \to \infty$ by keeping $n/N \to r \in (0, 1]$, we have*

$$Z_n^{a,b} \sim \frac{1}{(2\pi n)^{d/2} \sqrt{\det Q((b-a)/r)}} \exp\left\{ -n\Lambda^* \left(\frac{(b-a)N}{n} \right) \right\},$$

where $\sim$ means that the ratio of both sides tends to 1 and $Q(v) := \mathcal{Q}(\lambda(v))$ is the covariance matrix of $p_{\lambda(v)}$ for $v \in \mathbb{R}^d$; recall that p_λ is the Cramér transform of p and the function $\lambda(v)$ is defined by (3.3.1). *In particular, we have*

$$Z_n^{0,0} \sim \frac{1}{(2\pi n)^{d/2} \sqrt{\det Q}} e^{-n\Lambda^*(0)}, \tag{3.3.4}$$

as $n \to \infty$, where $Q := Q(0)$ is the covariance matrix of $p_{\lambda(0)}$.

Proof. From its definition, the normalizing constant $Z_n^{a,b}$ can be rewritten as $Z_n^{a,b} = p^{n*}((b-a)N)$ in terms of the n-fold convolution p^{n*} of p. However, by a simple computation recalling (3.3.2), we can rewrite $p^{n*}(x)$ as

$$p^{n*}(x) = e^{-n\Lambda^*(x/n)} \big(p_{\lambda(x/n)}\big)^{n*}(x). \tag{3.3.5}$$

Define the probability densities $\tilde{p}_v$ and $q_{n,v}$ for $v \in \mathbb{R}^d$ by $\tilde{p}_v(x) = p_{\lambda(v)}(x+v)$ and $q_{n,v}(x) = n^{d/2} \big(\tilde{p}_v\big)^{n*}(\sqrt{n}x)$, respectively. Note that the mean of $\tilde{p}_v$ is 0 and its covariance matrix is $Q(v)$ (i.e., same as that of $p_{\lambda(v)}$) and $q_{n,v}$ is the distribution density of $n^{-1/2} \sum_{i=1}^n X_i^{(v)}$, where $\{X_i^{(v)}\}_{i=1}^n$ is an i.i.d. sequence with distribution densities $\tilde{p}_v$. Since Assumption 3.1.1-(1) implies $\sup_{|v| \le K} \sup_{x \in \mathbb{R}^d} \tilde{p}_v(x) < \infty$ for every $K > 0$, the local limit theorem, which holds uniformly in v and formulated in Lemma 3.3.2 below applied for $p^{(v)} = \tilde{p}_v$, proves

$$\lim_{n\to\infty} \sup_{|v| \le K} \left| q_{n,v}(0) - \frac{1}{(2\pi)^{d/2} \sqrt{\det Q(v)}} \right| = 0.$$

This shows

$$\sup_{|v|\le K}\left|\left(p_{\lambda(v)}\right)^{n*}(nv)-\frac{1}{(2\pi n)^{d/2}\sqrt{\det Q(v)}}\right| = o\left(n^{-d/2}\right),$$

as $n\to\infty$, since $\left(\tilde p_v\right)^{n*}(x) = \left(p_{\lambda(v)}\right)^{n*}(x+nv)$. Therefore, in particular by taking $v=(b-a)N/n$ which runs over a certain bounded set of $\mathbb{R}^d$ as long as $n/N\to r>0$, the identity (3.3.5) with $x=(b-a)N$ shows that

$$\begin{aligned} &p^{n*}((b-a)N)\\ &\quad = \left(\frac{1}{(2\pi n)^{d/2}\sqrt{\det Q((b-a)/r)}} + o\left(n^{-d/2}\right)\right)\exp\left\{-n\Lambda^*\left(\frac{(b-a)N}{n}\right)\right\}. \end{aligned}$$

Thus, the proof of the lemma is concluded. □

We need to extend Theorem 19.1 of [3] in the following form, in which the random variables depend on an extra parameter v running over a certain set Θ and the local limit theorem is established uniformly in v. The proof is essentially the same so that it is omitted.

Lemma 3.3.2. *For each $v\in\Theta$, let an $\mathbb{R}^d$-valued i.i.d. sequence $\{X_n^{(v)}\}_{n=1}^\infty$ be given. We assume that $E[X_n^{(v)}]=0$, $\mathrm{Cov}(X_n^{(v)})=V^{(v)}$, which is a symmetric positive definite matrix, and the distribution of $X_n^{(v)}$ has a density $p^{(v)}(x)$. Then, if $\sup_{v\in\Theta}\sup_{x\in\mathbb{R}^d}p^{(v)}(x)<\infty$ and if $c_1I\le V^{(v)}\le c_2I$ hold for all $v\in\Theta$ with some constants $0<c_1<c_2<\infty$ and the $d\times d$ identity matrix I, the distribution of $n^{-1/2}\sum_{i=1}^n X_i^{(v)}$ has a density $q_n^{(v)}(x)$ and it holds that*

$$\lim_{n\to\infty}\sup_{v\in\Theta}\sup_{x\in\mathbb{R}^d}\left|q_n^{(v)}(x)-\phi_{0,V^{(v)}}(x)\right| = 0,$$

where $\phi_{0,V}(x)$ stands for the density of the Gaussian distribution on $\mathbb{R}^d$ with mean 0 and covariance V.

The partition function $Z_N^{0,0,\varepsilon}$ is given by

$$Z_N^{0,0,\varepsilon} = \int_{(\mathbb{R}^d)^{N+1}}\mathfrak{p}_N(\phi)\delta_0(d\phi_0)\prod_{i=1}^{N-1}\left(\varepsilon\delta_0(d\phi_i)+d\phi_i\right)\delta_0(d\phi_N), \tag{3.3.6}$$

and $Z_N^{0,0}=Z_N^{0,0,0}$, i.e. $\varepsilon=0$.

Let us define the function $g:[0,\infty)\to[0,\infty]$ by

$$g(x)=\sum_{n=1}^\infty x^n Z_n^{0,0}. \tag{3.3.7}$$

Note that g is increasing, $g(0) = 0$, $g(x) < \infty$ iff $x \in [0, e^{\Lambda^*(0)}]$ when $d \geq 3$ and $g(x) < \infty$ iff $x \in [0, e^{\Lambda^*(0)})$ when $d = 1, 2$ by (3.3.4). Set

$$\varepsilon_c^D = 1/g(e^{\Lambda^*(0)}). \tag{3.3.8}$$

In particular, $\varepsilon_c^D > 0$ if $d \geq 3$ and $\varepsilon_c^D = 0$ if $d = 1, 2$. For each $\varepsilon > \varepsilon_c^D$, we determine $x = x^\varepsilon \in (0, e^{\Lambda^*(0)})$ as the unique solution of $g(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^{D,\varepsilon} = \Lambda^*(0) - \log x^\varepsilon \quad \text{and} \quad C^{D,\varepsilon} = \frac{(2\pi)^{d/2}\sqrt{\det Q}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}. \tag{3.3.9}$$

Proposition 3.3.3. *For each $\varepsilon > \varepsilon_c^D$, we have the precise asymptotics for the ratio of two partition functions:*

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim C^{D,\varepsilon} N^{d/2} e^{N\xi^{D,\varepsilon}},$$

as $N \to \infty$.

Proof. We first note the renewal equation for $Z_N^{0,0,\varepsilon}$, $N \geq 2$ with $Z_1^{0,0,\varepsilon} = Z_1^{0,0} = 1$:

$$Z_N^{0,0,\varepsilon} = Z_N^{0,0} + \varepsilon \sum_{i=1}^{N-1} Z_i^{0,0} Z_{N-i}^{0,0,\varepsilon}, \tag{3.3.10}$$

see Lemma 2.2.1. Then, in a very similar manner to the proof of Proposition 2.2.2 (remind that the partition functions in Chapter 2 in the Gaussian case have an extra factor $(2\pi)^{dn/2}$ because p is unnormalized there), taking $u_0 = a_0 = b_0 = 0$ and $u_n = (x^\varepsilon)^n Z_n^{0,0,\varepsilon}$, $a_n = \varepsilon (x^\varepsilon)^n Z_n^{0,0}$, $b_n = (x^\varepsilon)^n Z_n^{0,0}$ for $n \geq 1$ in the present setting and noting that $\sum_{n=0}^\infty a_n = 1$, the renewal theory applied for the equation for $\{u_n\}$ obtained from (3.3.10) shows that

$$\lim_{n\to\infty} (x^\varepsilon)^n Z_n^{0,0,\varepsilon} = \frac{\sum_{n=0}^\infty b_n}{\sum_{n=0}^\infty n a_n} = \frac{1}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}.$$

The conclusion is shown by combining this with (3.3.4). □

The free energy $\xi^{D,\varepsilon}$ defined by (3.1.6) is, if exists, non-negative and non-decreasing in ε, since $Z_n^{0,0,\varepsilon}$ is increasing in ε. Therefore, since (3.3.9) implies $\lim_{\varepsilon \downarrow \varepsilon_c^D} \xi^{D,\varepsilon} = 0$, we see that $\xi^{D,\varepsilon} = 0$ for $0 \leq \varepsilon \leq \varepsilon_c^D$.

3.3.2 Free case

We now move to the case with the free condition at $t = 1$ (or microscopically at $i = N$). The partition function $Z_N^{0,F,\varepsilon}$ is given by

$$Z_N^{0,F,\varepsilon} = \int_{(\mathbb{R}^d)^{N+1}} \mathfrak{p}_N(\phi)\delta_0(d\phi_0) \prod_{i=1}^{N} \big(\varepsilon\delta_0(d\phi_i) + d\phi_i\big), \tag{3.3.11}$$

and we have $Z_N^{0,F} (= Z_N^{0,F,0}) = 1$.

Recall the function g defined by (3.3.7) and set

$$\varepsilon_c^F = 1/g(1). \tag{3.3.12}$$

We see that $\varepsilon_c^F \geq \varepsilon_c^D$ from $\Lambda^*(0) \geq 0$ and $\varepsilon_c^F = \varepsilon_c^D$ is equivalent to $\Lambda^*(0) = 0$, namely, $m \equiv \int_{\mathbb{R}^d} xp(x)\, dx = 0$. For each $\varepsilon > \varepsilon_c^F$, we define two positive constants:

$$\xi^{F,\varepsilon} = -\log x^\varepsilon \quad \text{and} \quad C^{F,\varepsilon} = \frac{1}{\varepsilon x^\varepsilon (1 - x^\varepsilon) g'(x^\varepsilon)}. \tag{3.3.13}$$

Proposition 3.3.4. *We have the precise asymptotics*

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}} \sim C^{F,\varepsilon} e^{N\xi^{F,\varepsilon}}$$

as $N \to \infty$ for each $\varepsilon > \varepsilon_c^F$.

Proof. We first note the renewal equation for $Z_N^{0,F,\varepsilon}$, $N \geq 1$ with $Z_0^{0,F,\varepsilon} = 1$:

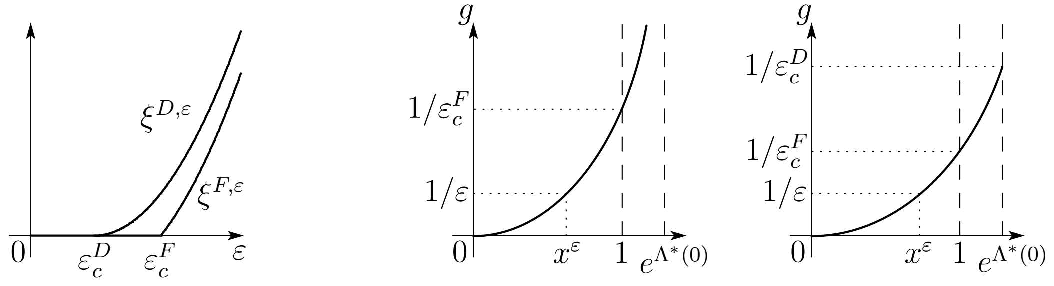
$$Z_N^{0,F,\varepsilon} = Z_N^{0,F} + \varepsilon \sum_{i=1}^{N} Z_i^{0,0} Z_{N-i}^{0,F,\varepsilon},$$

see Lemma 2.2.4. Then, in a similar manner to Proposition 2.2.5, taking $u_0 = b_0 = 1, a_0 = 0$ and $u_n = (x^\varepsilon)^n Z_n^{0,F,\varepsilon}$, $a_n = \varepsilon (x^\varepsilon)^n Z_n^{0,0}$, $b_n = (x^\varepsilon)^n Z_n^{0,F} = (x^\varepsilon)^n$ for $n \geq 1$ in the present setting, an application of the renewal theory shows that

$$\lim_{n\to\infty} (x^\varepsilon)^n Z_n^{0,F,\varepsilon} = \frac{1}{\varepsilon x^\varepsilon (1 - x^\varepsilon) g'(x^\varepsilon)}.$$

Note that the limit is finite only if $\sum_{n=0}^\infty b_n = \sum_{n=0}^\infty (x^\varepsilon)^n < \infty$, that is $x^\varepsilon < 1$, i.e., $\varepsilon > \varepsilon_c^F$. The conclusion is now shown by recalling $Z_N^{0,F} = 1$. □

Since (3.3.13) implies $\lim_{\varepsilon \downarrow \varepsilon_c^F} \xi^{F,\varepsilon} = 0$, we see that $\xi^{F,\varepsilon} = 0$ for $0 \leq \varepsilon \leq \varepsilon_c^F$ for the free energy $\xi^{F,\varepsilon}$ defined by (3.1.7).



$d \geq 3,\ m \neq 0$ $\qquad$ $d = 1, 2,\ m \neq 0$ $\qquad$ $d \geq 3,\ m \neq 0$

3.4 Proof of Theorems 3.1.3 and 3.1.4

We assume the conditions $(C)_D$ and $(C)_F$ in this section and give the proof of Theorems 3.1.3 and 3.1.4. Recall the definition (3.3.3) of the probability measure $\mu_{j,k}^{a,b}$ on $(\mathbb{R}^d)^{\{j,\ldots,k\}}$ for $0 \leq j < k \leq N$. The corresponding measure with pinning is denoted by $\mu_{j,k}^{a,b,\varepsilon}$.

3.4.1 Proof of Theorems 3.1.3-(1) and 3.1.4-(1)

Under the measure $\mu_{j,k}^{a,b}$, the macroscopic path determined from $(\phi_i)_{j \leq i \leq k}$ concentrates on the straight line $g_{[j/N,k/N]}^{a,b}(t)$ between $(j/N, a)$ and $(k/N, b)$, in particular, $g_{[0,1]}^{a,b} = \bar{h}^D$. More precisely, by the large deviation principle (cf. Proposition 3.5.2 below), we have the following lemma.

Lemma 3.4.1. *For any $\delta' > 0$, there exists $c(\delta') > 0$ and $N_0(\delta') \in \mathbb{N}$ such that for any $a, b \in \mathbb{R}^d$, $0 \leq j < k \leq N$:*

$$\mu_{j,k}^{a,b}\left(\max_{i: j \leq i \leq k} \left| \frac{\phi_i}{N} - g_{[j/N,k/N]}^{a,b}\left(\frac{i}{N}\right) \right| \geq \delta' \right) \leq e^{-c(\delta')N}$$

for $N \geq N_0(\delta')$.

We write

$$\gamma_{j,k}^{a,b}(\delta) := \mu_{j,k}^{a,b}\left(\left\| h_{[j/N,k/N]}^N - \hat{h}_{[j/N,k/N]} \right\|_\infty \leq \delta \right),$$

where $\hat{h} = \hat{h}^D$ in this subsection, and $f_{[u,v]}$ is the restriction of a function $f : [0,1] \to \mathbb{R}^d$ to the subinterval $[u, v]$ of $[0, 1]$. The probability $\gamma_{j,k}^{a,b,\varepsilon}(\delta)$ is similarly defined with pinning, i.e., under $\mu_{j,k}^{a,b,\varepsilon}$. We sometimes write $U_\delta(\hat{h}_{[u,v]})$ for the δ-neighborhood with respect to $\|\cdot\|_\infty$ in the space of functions on $[u, v]$ of $\hat{h}_{[u,v]}$; when the subscript $[u, v]$ is dropped, it is considered on $[0, 1]$. We similarly write $U_\delta(\bar{h})$ for $\bar{h} = \bar{h}^D$.

To complete the proof of Theorem 3.1.3-(1), it suffices to evaluate the limit

$$\lim_{N\to\infty} \frac{\mu_N^{D,\varepsilon}\left(h^N \in U_\delta(\hat{h})\right)}{\mu_N^{D,\varepsilon}\left(h^N \in U_\delta(\bar{h})\right)}$$

for arbitrarily small $\delta > 0$; recall the concentration property (3.1.8) or (3.4.14) below.

An expansion of the product measure $\prod_{i=1}^{N-1}(\varepsilon\delta_0(d\phi_i) + d\phi_i)$ in (3.1.1) by specifying $0 < i_\ell \le i_r < N$ gives rise to

$$\begin{aligned} R_N^D &:= \frac{Z_N^{a,b,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon}\left(h^N \in U_\delta(\hat{h})\right) \\ &= \gamma_{0,N}^{a,b}(\delta) + \sum_{j=1}^{N-1} \varepsilon \Xi_{N,j,j}^{\varepsilon} \gamma_{0,j}^{a,0}(\delta)\, \gamma_{j,N}^{0,b}(\delta) \\ &\quad + \sum_{0<j<k<N} \varepsilon^2 \Xi_{N,j,k}^{\varepsilon} \gamma_{0,j}^{a,0}(\delta)\, \gamma_{j,k}^{0,0,\varepsilon}(\delta)\, \gamma_{k,N}^{0,b}(\delta) \\ &=: I_N^1 + I_N^2 + I_N^3, \end{aligned} \tag{3.4.1}$$

where

$$\Xi_{N,j,k}^{\varepsilon} = \frac{Z_j^{a,0} Z_{k-j}^{0,0,\varepsilon} Z_{N-k}^{0,b}}{Z_N^{a,b}} \tag{3.4.2}$$

for $0 < j \le k < N$. We set $Z_0^{0,0,\varepsilon} = 1$ to define $\Xi_{N,j,j}^{\varepsilon}$. The meaning of this expansion is explained below (2.3.4). If δ is chosen small enough, we have from Lemma 3.4.1

$$I_N^1 + I_N^2 \le e^{-cN} \tag{3.4.3}$$

for N sufficiently large, with $c > 0$.

By Lemma 3.3.1, the ratio of the partition functions in (3.4.2) has the asymptotics for $j < k$ as $N \to \infty$:

$$\Xi_{N,j,k}^{\varepsilon} \sim \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} \frac{Z_{k-j}^{0,0,\varepsilon}}{Z_{k-j}^{0,0}}, \tag{3.4.4}$$

where $s_1 = j/N$, $s_2 = (N-k)/N$,

$$\tilde{f}(s_1, s_2) := s_1 \Lambda^*\left(-\frac{a}{s_1}\right) + s_2 \Lambda^*\left(\frac{b}{s_2}\right) + (1 - s_1 - s_2)\Lambda^*(0) - \Lambda^*(b-a), \tag{3.4.5}$$

and

$$\alpha_{N,j,k} = \frac{1}{(2\pi)^d} \left[\frac{N}{j(k-j)(N-k)}\right]^{d/2} \left[\frac{\det Q(b-a)}{\det Q(-a/s_1)\, \det Q\, \det Q(b/s_2)}\right]^{1/2}.$$

In the part I_N^3, we decompose the summation in j and k into the part over

$$A := \left\{ (j,k)\,;|j - Nt_1| \le N^{3/5},\ |k - N(1-t_2)| \le N^{3/5} \right\}, \tag{3.4.6}$$

and over its complement, where $t_1 = t_1^D$ and $t_2 = t_2^D$ are determined by the Young's relation (3.1.9). We always assume that N is large enough so that $Nt_1 + N^{3/5} < N(1-t_2) - N^{3/5}$. Using Proposition 3.3.3, we get

$$\begin{aligned} &\sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\, \gamma^{0,0,\varepsilon}_{j,k}(\delta)\, \gamma^{0,b}_{k,N}(\delta) \\ &\qquad \le \sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \le C \sum_{(j,k)\notin A} \alpha_{N,j,k}(k-j)^{d/2} e^{-Nf(s_1,s_2)}, \end{aligned}$$

for some $C > 0$, where

$$f(s_1,s_2) = \tilde{f}(s_1,s_2) - \xi^{D,\varepsilon}(1 - s_1 - s_2). \tag{3.4.7}$$

However, since the third condition in $(C)_D$ is equivalent to $f(t_1,t_2) = 0$ and the Young's relation (3.1.9) implies $\partial f/\partial s_1(t_1,t_2) = \partial f/\partial s_2(t_1,t_2) = 0$, the Taylor's theorem gives the expansion of $f(s_1,s_2)$:

$$\begin{aligned} f(s_1,s_2) =& \frac{1}{2t_1^3}(a\cdot\nabla)^2 \Lambda^*\left(-\frac{a}{t_1}\right)(s_1-t_1)^2 \\ &+ \frac{1}{2t_2^3}(b\cdot\nabla)^2 \Lambda^*\left(\frac{b}{t_2}\right)(s_2-t_2)^2 + O(|s_1-t_1|^3 + |s_2-t_2|^3), \end{aligned} \tag{3.4.8}$$

for s_1 and s_2 close to t_1 and t_2, respectively. Therefore, since $f(s_1,s_2) > 0$ except $(s_1,s_2) = (t_1,t_2)$, we have

$$Nf(s_1,s_2) \ge CN^{1/5},$$

on the complement A^c with some $C > 0$ and thus

$$\sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \le e^{-cN^{1/5}} \tag{3.4.9}$$

for some $c > 0$, and large enough N.

For $(j,k) \in A$, the expansion (3.4.8) shows

$$f(s_1,s_2) = c_1(s_1-t_1)^2 + c_2(s_2-t_2)^2 + O\left(N^{-6/5}\right),$$

where

$$c_1 = \frac{1}{2t_1^3}(a\cdot\nabla)^2\Lambda^*\left(-\frac{a}{t_1}\right), \quad c_2 = \frac{1}{2t_2^3}(b\cdot\nabla)^2\Lambda^*\left(\frac{b}{t_2}\right). \tag{3.4.10}$$

Furthermore, the straight lines $g^{a,0}_{[0,s_1]}$ and $g^{0,b}_{[1-s_2,1]}$ are within distance $\delta/2$ to the restrictions of $\hat{h}_{[0,s_1]}$ and $\hat{h}_{[1-s_2,1]}$, respectively, if N is large enough, and therefore, using Lemma 3.4.1 and Theorem 3.5.1 below (in fact, Proposition 3.5.7 is sufficient), we get

$$\begin{aligned} \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}\left(1-e^{-cN}\right) &\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}\gamma^{a,0}_{0,j}(\delta)\,\gamma^{0,0,\varepsilon}_{j,k}(\delta)\,\gamma^{0,b}_{k,N}(\delta) \\ &\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}, \end{aligned} \tag{3.4.11}$$

for some $c > 0$. It therefore suffices to estimate $\sum_{(j,k)\in A}\Xi^{\varepsilon}_{N,j,k}$. By using (3.4.4), Proposition 3.3.3 and substituting $j-[Nt_1]$ and $k-[N(1-t_2)]$ into j and k, we have by a Riemann sum approximation

$$\begin{aligned} \varepsilon^2\sum_{(j,k)\in A}\Xi^{\varepsilon}_{N,j,k} &\sim C_1N^{-d/2}\sum_{|j|\le N^{3/5}} e^{-c_1(j/\sqrt{N})^2}\sum_{|k|\le N^{3/5}} e^{-c_2(k/\sqrt{N})^2} \\ &\sim C_1N^{1-d/2}\int_{-\infty}^{\infty} e^{-c_1x^2}\,dx\int_{-\infty}^{\infty} e^{-c_2x^2}\,dx \\ &= \frac{C_1\pi}{\sqrt{c_1c_2}}N^{1-d/2}, \end{aligned} \tag{3.4.12}$$

as $N\to\infty$, with

$$C_1 = \frac{\varepsilon^2C^{D,\varepsilon}}{(4\pi^2t_1t_2)^{d/2}}\left[\frac{\det Q(b-a)}{\det Q(-a/t_1)\,\det Q\,\det Q(b/t_2)}\right]^{1/2},$$

where $C^{D,\varepsilon}$ is the constant given in (3.3.9).

Summarizing, we get from (3.4.1), (3.4.3), (3.4.9) and (3.4.12), for sufficiently large N

$$\begin{aligned} R^D_N &= \frac{C_1\pi}{\sqrt{c_1c_2}}N^{1-d/2}\left(1-O\left(e^{-cN}\right)\right)+O\left(e^{-cN^{1/5}}\right)+O\left(e^{-cN}\right) \\ &\sim \frac{C_1\pi}{\sqrt{c_1c_2}}N^{1-d/2}. \end{aligned} \tag{3.4.13}$$

On the other hand, the definition (3.1.1) of $\mu^{D,\varepsilon}_N$ implies for every $0<\delta<|a|\wedge|b|$ that

$$\frac{Z^{a,b,\varepsilon}_N}{Z^{a,b}_N}\mu^{D,\varepsilon}_N\left(h^N\in U_\delta(\bar{h})\right) = \mu^{D,0}_N\left(h^N\in U_\delta(\bar{h})\right)\sim 1,$$

where $\bar{h}=\bar{h}^D$. Comparing with (3.4.13), we have the conclusion of Theorem 3.1.3-(1) by recalling that (3.1.8) implies

$$\lim_{N\to\infty}\left\{\mu^{D,\varepsilon}_N\left(h^N\in U_\delta(\hat{h})\right)+\mu^{D,\varepsilon}_N\left(h^N\in U_\delta(\bar{h})\right)\right\}=1. \tag{3.4.14}$$

In particular, if $d=2$, the coexistence of $\bar h$ and $\hat h$ occurs in the limit with probabilities

$$(\bar\lambda^{D,\varepsilon},\hat\lambda^{D,\varepsilon}) := \left(\frac{1}{1+C_2},\frac{C_2}{1+C_2}\right), \tag{3.4.15}$$

where $C_2 = C_1\pi/\sqrt{c_1c_2} > 0$.

Proof of Theorem 3.1.4-(1): The proof goes in a similar manner to that of Theorem 2.1.5 for $\mu_N^{D,\varepsilon}$. Indeed, for $x_1 < x_2$ and $y_1 < y_2$, let

$$\begin{aligned} &A(x_1,x_2;y_1,y_2) \\ &:= \left\{(j,k)\in A; \sqrt{N}x_1 \le j - t_1N \le \sqrt{N}x_2,\ \sqrt{N}y_1 \le k-(1-t_2)N \le \sqrt{N}y_2\right\}. \end{aligned}$$

Then, by the same computation as that leading to (3.4.12), (3.4.13) and by the large deviation estimate (cf. Theorem 3.5.1 below), we obtain

$$\lim_{N\to\infty}\mu_N^{D,\varepsilon}\left((i_\ell,i_r)\in A(x_1,x_2;y_1,y_2)\Big| i_\ell\le N-1\right) = \frac{\sqrt{c_1c_2}}{\pi}\int_{x_1}^{x_2} e^{-c_1x^2}\,dx\int_{y_1}^{y_2} e^{-c_2x^2}\,dx,$$

which proves the claim.

3.4.2 Proof of Theorems 3.1.3-(2) and 3.1.4-(2)

Let $\mu_N^{a,F} (=\mu_N^{F,0})$ be the measure defined on $(\mathbb{R}^d)^{D_N}$ without pinning and having the normalizing constant $Z_N^{a,F} (=Z_N^{a,F,0})$:

$$\mu_N^{a,F}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F}}\delta_{aN}(d\phi_0)\prod_{i=1}^N d\phi_i. \tag{3.4.16}$$

For $0\le j<k\le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon}$ on $(\mathbb{R}^d)^{\{j,\dots,k\}}$ with pinning, the condition $\phi_j=0$ at j, and the free condition (no specific condition) at k, having the normalizing constant $Z_{k-j}^{0,F,\varepsilon}$. The expansion of the product measure $\prod_{i=1}^N(\varepsilon\delta_0(d\phi_i)+d\phi_i)$ in (3.1.2) by specifying $0<i_\ell\le N+1$ leads to

$$\begin{aligned} R_N^F &:= \frac{Z_N^{a,F,\varepsilon}}{Z_N^{a,F}}\mu_N^{F,\varepsilon}\left(h^N\in U_\delta(\hat h)\right) \\ &= \mu_N^{a,F}\left(h^N\in U_\delta(\hat h)\right) \\ &\quad + \sum_{j=1}^N \varepsilon\Xi_{N,j}^{F,\varepsilon}\mu_{0,j}^{a,0}\left(h_{[0,j/N]}^N\in U_\delta(\hat h_{[0,j/N]})\right)\mu_{j,N}^{0,F,\varepsilon}\left(h_{[j/N,1]}^N\in U_\delta(\hat h_{[j/N,1]})\right) \\ &=: I_N^{1,F}+I_N^{2,F}, \end{aligned} \tag{3.4.17}$$

where $\hat{h} = \hat{h}^F$ in this subsection and

$$\Xi^{F,\varepsilon}_{N,j} = \frac{Z^{a,0}_j Z^{0,F,\varepsilon}_{N-j}}{Z^{a,F}_N}$$

for $1 \le j \le N$. Noting that $Z^{a,F}_n = Z^{0,F}_n = 1$ and recalling Lemma 3.3.1 for $Z^{a,0}_j$, we see that

$$\Xi^{F,\varepsilon}_{N,j} \sim (2\pi j)^{-d/2}\big(\det Q(-a/s_1)\big)^{-1/2} e^{-N\tilde{f}(s_1)} \cdot \frac{Z^{0,F,\varepsilon}_{N-j}}{Z^{0,F}_{N-j}},$$

where $s_1 = j/N$ and $\tilde{f}(s_1) = s_1\Lambda^*(-a/s_1)$.

We put here

$$A := \left\{j; |j - Nt_1| \le N^{3/5}\right\},$$

where $t_1 = t^F_1$, and arrive in the same way as in Section 3.4.1, using the large deviation estimate for $\mu^{a,0}_{0,j}$ and $\mu^{0,F,\varepsilon}_{j,N}$ (cf. Theorem 3.5.1 below), to

$$R^F_N = \varepsilon \sum_{j\in A} \Xi^{F,\varepsilon}_{N,j}\left(1 - O\left(e^{-cN}\right)\right) + O\left(e^{-cN^{1/5}}\right) + O\left(e^{-cN}\right), \tag{3.4.18}$$

for some $c > 0$. Furthermore, we get by Proposition 3.3.4,

$$\varepsilon \sum_{j\in A} \Xi^{F,\varepsilon}_{N,j} \sim \varepsilon C^{F,\varepsilon}(2\pi)^{-d/2}\big(\det Q(-a/t_1)\big)^{-1/2} \sum_{j\in A}(Ns_1)^{-d/2} e^{-Nf^F(s_1)},$$

where $C^{F,\varepsilon}$ is the constant given in (3.3.13) and $f^F(s) = \tilde{f}(s) - \xi^{F,\varepsilon}(1-s)$. By the final condition in $(C)_F$, the Young's relation (3.1.9) and the Taylor's theorem, we have the expansion of f^F:

$$f^F(s_1) = \frac{1}{2t_1^3}(a\cdot\nabla)^2\Lambda^*\left(-\frac{a}{t_1}\right)(s_1 - t_1)^2 + O(|s_1 - t_1|^3), \tag{3.4.19}$$

for s_1 close to t_1. This finally proves, recalling (3.4.18), that

$$\begin{aligned} R^F_N &\sim C_3 N^{-d/2} \sum_{|j|\le N^{3/5}} e^{-c_3(j/\sqrt{N})^2} \\ &\sim C_3 N^{(1-d)/2} \int_{-\infty}^{\infty} e^{-c_3x^2}\,dx = C_3\sqrt{\frac{\pi}{c_3}} N^{(1-d)/2}, \end{aligned} \tag{3.4.20}$$

as $N \to \infty$, with

$$C_3 = \frac{\varepsilon C^{F,\varepsilon}}{(2\pi t_1)^{d/2}\sqrt{\det Q(-a/t_1)}} \quad \text{and} \quad c_3 = \frac{1}{2t_1^3}(a\cdot\nabla)^2\Lambda^*\left(-\frac{a}{t_1}\right). \tag{3.4.21}$$

On the other hand, for every $0 < \delta < |a|$, we have that

$$\frac{Z_N^{a,F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\left(h^N \in U_\delta(\bar{h})\right) = \mu_N^{F,0}\left(h^N \in U_\delta(\bar{h})\right) \sim 1,$$

where $\bar{h} = \bar{h}^F$. Comparing this with (3.4.20), and recalling (3.1.8), the conclusion of Theorem 3.1.3-(2) is proved. In particular, if $d = 1$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{F,\varepsilon}, \hat{\lambda}^{F,\varepsilon}) := \left(\frac{1}{1+C_4}, \frac{C_4}{1+C_4}\right), \tag{3.4.22}$$

where $C_4 = C_3\sqrt{\pi/c_3} > 0$.

The proof of Theorem 3.1.4-(2) is similar based on the computation like in (3.4.20), note that the variance of the limiting Gaussian distribution is $1/2c_3$.

3.5 Large deviation principle

The goal of this section is to show the sample path large deviation principle (LDP). Here we do not require the conditions $(C)_D$ nor $(C)_F$.

3.5.1 Formulation of results

Theorem 3.5.1. *The LDP holds for $h^N = \{h^N(t); t \in D\}$ distributed under $\mu_N = \mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ on the space $\mathcal{C}$ as $N \to \infty$ with the speed N and the good rate functionals $I = I^D$ and I^F of the form:*

$$I(h) = \begin{cases} \Sigma(h) - \inf_{\mathcal{AC}} \Sigma, & \text{if } h \in \mathcal{AC}, \\ \quad +\infty \quad, & \text{otherwise}, \end{cases} \tag{3.5.1}$$

with $\Sigma = \Sigma^D$ and Σ^F given by (3.1.5)*, where $\mathcal{AC} = \mathcal{AC}_{a,b}$ and $\mathcal{AC}_{a,F}$, respectively, and $\inf_{\mathcal{AC}} \Sigma$ is taken over the space $\mathcal{AC}$. Namely, for every open set $\mathfrak{O}$ and closed set $\mathfrak{C}$ of $\mathcal{C}$ equipped with the uniform topology, we have that*

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{O}) \geq - \inf_{h \in \mathfrak{O}} I(h), \\ &\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{C}) \leq - \inf_{h \in \mathfrak{C}} I(h), \end{aligned} \tag{3.5.2}$$

in each of two situations.

3.5.2 The LDP without pinning

We will show the LDP for $\{h^N\}_N$ distributed under $\mu_N^{a,b} = \mu_N^{D,0}$. The LDP for $\mu_N^{a,F}$, i.e., the case with the free condition at the right end point, was established by Mogul'skii [28]; see also Section 5.1 of [8].

Results

Let $\mathcal{C}_{a,b}$ be the family of all $h \in \mathcal{C}$ such that $h(0) = a$ and $h(1) = b$. We set

$$\Sigma_0(h) = \begin{cases} \displaystyle\int_D \Lambda^*(\dot{h}(t))\,dt, & \text{if} \quad h \in \mathcal{AC}_{a,b}, \\ \quad +\infty, & \text{if} \quad h \in \mathcal{C}_{a,b} \setminus \mathcal{AC}_{a,b}, \end{cases}$$

and

$$I_0(h) = \Sigma_0(h) - \inf_{\mathcal{AC}_{a,b}} \Sigma_0.$$

Proposition 3.5.2. *The family of macroscopic paths $\{h^N\}_N$ distributed under $\mu_N^{a,b} = \mu_N^{D,0}$ satisfies the LDP on the space $\mathcal{C}_{a,b}$ with speed N and the good rate functional $I_0(h)$, namely, for every open set $\mathfrak{O}$ and closed set $\mathfrak{C}$ of $\mathcal{C}_{a,b}$, we have the lower and upper bounds* (3.5.2) *for $\mu_N^{a,b}$ and I_0 in place of μ_N and I, respectively.*

Remark 3.5.1. *Deuschel, Giacomin and Ioffe* [9] *proved the LDP for $\mu_N^{a,b}$ in the L^2-topology, even for the Markov fields rather than the Markov chains discussed in this chapter, under the log-concavity condition on p. Such condition was needed to characterize all (infinite-volume) Gibbs measures for the corresponding gradient fields, which are simply the superpositions of certain product measures in our setting. Therefore, their method would work also in our setting. To improve the topology, one may show the exponential tightness which is actually easy; see Corollary* 4.2.6 *of* [8].

We will follow the method used by Guo, Papanicolaou and Varadhan [25] to show the equivalence of ensemble for a sequence of canonical (conditional) probability measures, with an external field depending on t. This will be applied to show the law of large numbers (LLN) for the perturbed measure. Then, we will use the Cramér's trick to prove Proposition 3.5.2.

LLN for a perturbed measure

For $\lambda \in \mathcal{C}$, we introduce the perturbed measure $\mu^{a,b}_{N,\lambda}$ by

$$\mu^{a,b}_{N,\lambda}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z^{a,b}_{N,\lambda}} \prod_{i=1}^{N} e^{\lambda\left(\frac{i}{N}\right)\cdot(\phi_i-\phi_{i-1})} \prod_{i=1}^{N-1} d\phi_i,$$

under the boundary conditions $\phi_0 = aN, \phi_N = bN$.

Let $h \in \mathcal{C}_{a,b}$ be a polygon with corners at $t = k/m, 0 \le k \le m, m \in \mathbb{N}$. We assume that N divides by m for simplicity. We define $\lambda_h \in \mathcal{C}$ by $\lambda_h(t) = \lambda(\dot{h}(t)), t \in D$.

Proposition 3.5.3. *For the polygon h, we have that*

$$\lim_{N\to\infty} \mu^{a,b}_{N,\lambda_h}\left(\|h^N - h\|_\infty \ge \delta\right) = 0$$

for every $\delta > 0$.

Proof. Step 1. The exponential tightness of the distributions on the space $\mathcal{C}_{a,b}$ of $\{h^N\}$ under $\mu^{a,b}_{N,\lambda}$ will be shown later, see Lemma 3.5.6 below. Then, the conclusion follows by showing the convergence of $\langle h^N, J\rangle$ to $\langle h, J\rangle$ in probability as $N \to \infty$ for every test function $J \in C^\infty(D, \mathbb{R}^d)$. To this end, it suffices to show that $\langle \dot{h}^N, J\rangle$ converges to $\langle \dot{h}, J\rangle$ in probability for every test function J.

Step 2. Note that

$$\langle \dot{h}^N, J\rangle = \frac{1}{N}\sum_{i=1}^{N} \eta_i \cdot \tilde{J}_i,$$

where $\eta_i = \phi_i - \phi_{i-1}, 1 \le i \le N$ and $\tilde{J}_i = N\int_{(i-1)/N}^{i/N} J(t)\,dt$.

We define the probability measure $\nu_{N,\lambda}$ on $(\mathbb{R}^d)^N$ by

$$\nu_{N,\lambda}(d\eta) = \frac{1}{\tilde{Z}_{N,\lambda}} \prod_{i=1}^{N} p(\eta_i) e^{\lambda\left(\frac{i}{N}\right)\cdot\eta_i} d\eta_i, \quad \eta = (\eta_i)_{i=1}^N \in (\mathbb{R}^d)^N.$$

The conditional probability measure of $\nu_{N,\lambda}$ on the hyperplane $\{\eta | \frac{1}{N}\sum_{i=1}^N \eta_i = b - a\}$ is denoted by $\nu^{b-a}_{N,\lambda}$:

$$\nu^{b-a}_{N,\lambda}(\cdot) := \nu_{N,\lambda}\left(\cdot \middle| \frac{1}{N}\sum_{i=1}^{N} \eta_i = b - a\right).$$

Let $f_{N,\lambda}(x)$ be the probability density of $\frac{1}{N}\sum_{i=1}^N \eta_i$ under the distribution $\nu_{N,\lambda}$,i.e.,

$$f_{N,\lambda}(x)dx = \nu_{N,\lambda}\left(\frac{1}{N}\sum_{i=1}^{N} \eta_i \in dx\right), \quad x \in \mathbb{R}^d.$$

The following lemma is an extension of Theorem 3.4 of [25] to the case with non-constant external field λ:

Lemma 3.5.4. *We have that*

$$\lim_{N\to\infty}\frac{1}{N}\log f_{N,\lambda}(y) = -\min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=y}} \frac{1}{m}\sum_{\ell=1}^{m}\{\Lambda^*(x_\ell)-\lambda_\ell\cdot x_\ell+\Lambda(\lambda_\ell)\},$$

uniformly in y on every compact subset of $\mathbb{R}^d$, where λ_ℓ is the value of the step function $\lambda(t)$ on the ℓth interval $D_\ell = ((\ell-1)/m, \ell/m], 1 \le \ell \le m$.

Proof. Let X_ℓ be the average of η over the domain $\tilde{D}_\ell := ND_\ell \cap \mathbb{Z} \equiv ((\ell-1)N/m, \ell N/m] \cap \mathbb{Z}$:

$$X_\ell := \frac{m}{N}\sum_{i\in\tilde{D}_\ell}\eta_i,$$

and let $f^{(\ell)}_{N/m,\lambda}(x_\ell), x_\ell \in \mathbb{R}^d$ be the probability density of X_ℓ under $\nu_{N,\lambda}$. Then, noting the independence of $\{X_1,\dots,X_m\}$ under $\nu_{N,\lambda}$, we see that $f_{N,\lambda}(x)dx$ is nothing but the distribution of $\frac{1}{m}(x_1+\cdots+x_m)$ under the product probability measure

$$\prod_{\ell=1}^{m} f^{(\ell)}_{N/m,\lambda}(x_\ell)\,dx_\ell.$$

This implies that

(3.5.3)
$$f_{N,\lambda}(y) = m\int_{(\mathbb{R}^d)^{m-1}} f^{(m)}_{N/m,\lambda}\big(my-(x_1+\cdots+x_{m-1})\big)\prod_{\ell=1}^{m-1} f^{(\ell)}_{N/m,\lambda}(x_\ell)\,dx_\ell, \quad y\in\mathbb{R}^d.$$

In fact, taking any test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, one can rewrite the integral $\int_{\mathbb{R}^d}\varphi(y)\times f_{N,\lambda}(y)\,dy$ by change of variables and obtains (3.5.3). However, from Theorem 3.4 in [25] applied for $f^{(\ell)}_{N/m,\lambda}$ (we take $-\log p(x)-\lambda_\ell\cdot x+\Lambda(\lambda_\ell)$ as the potential $\phi(x)$ in [25]), we see that

$$\lim_{N\to\infty}\frac{m}{N}\log f^{(\ell)}_{N/m,\lambda}(x) = -\left(\Lambda^*\right)^{(\ell)}(x),$$

uniformly in x on every compact subset of $\mathbb{R}^d$, where

$$(3.5.4)\qquad \begin{aligned}\left(\Lambda^*\right)^{(\ell)}(v) &= \sup_{\lambda\in\mathbb{R}^d}\{\lambda\cdot v-\Lambda(\lambda+\lambda_\ell)+\Lambda(\lambda_\ell)\}\\ &= \Lambda^*(v)-\lambda_\ell\cdot v+\Lambda(\lambda_\ell).\end{aligned}$$

Now, the combination of (3.5.3) and (3.5.4) proves the conclusion. □

We now return to the proof of Proposition 3.5.3. Our goal is to show that $\frac{1}{N}\sum_{i=1}^N \eta_i \cdot \tilde{J}_i$ converges to $\langle \dot{h}, J\rangle$ in probability under $\nu_{N,\lambda}^{b-a}$ with $\lambda = \lambda_h$. To show this, we estimate by the exponential Chebyshev's inequality

$$\begin{aligned}(3.5.5)\qquad &\frac{1}{N}\log \nu_{N,\lambda}^{b-a}\left(\left|\frac{1}{N}\sum_{i=1}^N \eta_i\cdot\tilde{J}_i - \langle \dot{h}, J\rangle\right| > \delta\right)\\ &\le \frac{1}{N}\log\left[\int e^{N\theta\{\frac{1}{N}\sum_{i=1}^N \eta_i\cdot\tilde{J}_i - \langle\dot{h},J\rangle-\delta\}}\,d\nu_{N,\lambda}^{b-a}\right.\\ &\qquad\left. + \int e^{-N\theta\{\frac{1}{N}\sum_{i=1}^N \eta_i\cdot\tilde{J}_i - \langle\dot{h},J\rangle+\delta\}}\,d\nu_{N,\lambda}^{b-a}\right]\end{aligned}$$

for every $\theta > 0$. For the first integral on the right hand side, we have that

$$\int e^{\theta\sum_{i=1}^N \eta_i\cdot\tilde{J}_i}\,d\nu_{N,\lambda}^{b-a} = \left.\frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^N\eta_i\in dx\}} e^{\theta\sum_{i=1}^N\eta_i\cdot\tilde{J}_i}\,d\nu_{N,\lambda}}{\nu_{N,\lambda}(\frac{1}{N}\sum_{i=1}^N\eta_i\in dx)}\right|_{x=b-a}.$$

The denominator is equal to $f_{N,\lambda}(x)dx$, while the numerator is equal to

$$\frac{\tilde{Z}_{N,\lambda}^\theta}{\tilde{Z}_{N,\lambda}} f_{N,\lambda}^\theta(x)dx.$$

Here $f_{N,\lambda}^\theta$ is the probability density of $\frac{1}{N}\sum_{i=1}^N\eta_i$ under the distribution

$$\nu_{N,\lambda}^\theta(d\eta) = \frac{1}{\tilde{Z}_{N,\lambda}^\theta}\prod_{i=1}^N p(\eta_i)e^{\lambda(\frac{i}{N})\cdot\eta_i}e^{\theta\eta_i\cdot\tilde{J}_i}d\eta_i.$$

If J is a step function on D, which takes constant-value J_ℓ on each subinterval $D_\ell, 1\le \ell\le m$, we can apply Lemma 3.5.4 also for $f_{N,\lambda}^\theta$ by taking $\lambda_\ell+\theta J_\ell$ in place of λ_ℓ and have that

$$\lim_{N\to\infty}\frac{1}{N}\log f_{N,\lambda}^\theta(y) = -\min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\dots+x_m)=y}}\frac{1}{m}\sum_{\ell=1}^m\{\Lambda^*(x_\ell) - (\lambda_\ell+\theta J_\ell)\cdot x_\ell + \Lambda(\lambda_\ell+\theta J_\ell)\},$$

uniformly in y on every compact subset of $\mathbb{R}^d$. On the other hand, we have

$$\begin{aligned}\frac{1}{N}\log\frac{\tilde{Z}_{N,\lambda}^\theta}{\tilde{Z}_{N,\lambda}} &= \frac{1}{N}\log E^{\nu_{N,\lambda}}\left[e^{\theta\sum_{i=1}^N\eta_i\cdot\tilde{J}_i}\right]\\ &= \frac{1}{N}\log\left[\frac{\prod_{\ell=1}^m(e^{\Lambda(\lambda_\ell+\theta J_\ell)})^{N/m}}{\prod_{\ell=1}^m(e^{\Lambda(\lambda_\ell)})^{N/m}}\right] = \frac{1}{m}\sum_{\ell=1}^m\log\frac{e^{\Lambda(\lambda_\ell+\theta J_\ell)}}{e^{\Lambda(\lambda_\ell)}}.\end{aligned}$$

These computations are summarized into

$$(3.5.6)\qquad \lim_{N\to\infty}\frac{1}{N}\log\int e^{\theta\sum_{i=1}^N\eta_i\cdot\tilde{J}_i - N\theta\langle\dot{h},J\rangle - N\theta\delta}\,d\nu_{N,\lambda}^{b-a}$$

$$= - \min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^{m}\{\Lambda^*(x_\ell) - (\lambda_\ell + \theta J_\ell)\cdot x_\ell\}$$

$$+ \min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^{m}\{\Lambda^*(x_\ell) - \lambda_\ell\cdot x_\ell\} - \theta\langle \dot{h}, J\rangle - \theta\delta.$$

We prepare the following lemma to prove that the right hand side of (3.5.6) is negative if $\theta > 0$ is sufficiently small.

Lemma 3.5.5. *For a step function λ satisfying $\int_0^1 v(\lambda(t))\,dt = b - a$, the minimizer of the variational problem*

$$\min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \textit{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^{m}\{\Lambda^*(x_\ell) - \lambda_\ell\cdot x_\ell\}$$

is given by $\bar{x} = \{\bar{x}_\ell = v(\lambda_\ell)\}_{\ell=1}^m$.

Proof. At the minimal point $x = \{x_\ell\}_{\ell=1}^m$, $\nabla\Lambda^*(x_\ell) - \lambda_\ell = c$ should be satisfied with a constant $c \in \mathbb{R}^d$ chosen as $\frac{1}{m}\sum_{\ell=1}^m v(\lambda_\ell + c) = b - a$. But this is fulfilled by $c = 0$. □

Lemma 3.5.5 can be applied for the first variational problem in the right hand side of (3.5.6) as well. In fact, choosing $c(\theta) \in \mathbb{R}^d$ in such a way that $\int_0^1 v(\lambda(t)+\theta J(t)+c(\theta))\,dt = b - a$, we can rewrite the first variational problem into

$$\min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^{m}\{\Lambda^*(x_\ell) - (\lambda_\ell + \theta J_\ell + c(\theta))\cdot x_\ell\} + c(\theta)\cdot(b-a),$$

which is equal to

$$\frac{1}{m}\sum_{\ell=1}^{m}\big\{\Lambda^*\big(v(\lambda_\ell + \theta J_\ell + c(\theta))\big) - (\lambda_\ell + \theta J_\ell + c(\theta))\cdot v(\lambda_\ell + \theta J_\ell + c(\theta))\big\} + c(\theta)\cdot(b-a),$$

by Lemma 3.5.5. We expand this formula in θ. Then, since $c(0) = 0$, the main term (the first order term) coincides with the second term in the right hand side of (3.5.6) by noting Lemma 3.5.5 again. The second order term (the term of order θ in the expansion) is given by

$$\frac{\theta}{m}\sum_{\ell=1}^{m}\big\{\nabla\Lambda^*(v(\lambda_\ell))\cdot\nabla v(\lambda_\ell)(J_\ell + c'(0)) - (J_\ell + c'(0))\cdot v(\lambda_\ell) - \lambda_\ell\cdot\nabla v(\lambda_\ell)(J_\ell + c'(0))\big\}$$
$$+ \theta c'(0)\cdot(b-a)$$

$$= -\frac{\theta}{m}\sum_{\ell=1}^{m} J_\ell \cdot v(\lambda_\ell) = -\theta\langle \dot{h}, J\rangle,$$

recall that $\nabla\Lambda^*(v(\lambda_\ell)) = \lambda_\ell$, $\frac{1}{m}\sum_{\ell=1}^{m} v(\lambda_\ell) = b-a$ and note that $\nabla v(\lambda)$ defines a $d\times d$ matrix. This exactly cancels with the term $-\theta\langle \dot{h}, J\rangle$ appearing in (3.5.6) and we have proved that the right hand side of (3.5.6) is strictly negative if $\theta > 0$ is sufficiently small.

We can treat the second integral in the right hand side of (3.5.5) in a similar manner, and this completes the proof of Proposition 3.5.3. □

The final task of this subsection is to establish the exponential tightness of the distributions on the space $\mathcal{C}_{a,b}$ of $\{h^N\}$ under $\mu_{N,\lambda}^{a,b}$. In fact, once the next lemma is shown, this follows in a similar manner to the proof of Lemma 5.1.7 in [8].

Lemma 3.5.6. *Let λ be a step function on D as in Lemma 3.5.4. Then, for every $\delta < 1$, we have that*

$$\limsup_{N\to\infty} \frac{1}{N}\log E^{\mu_{N,\lambda}^{a,b}}\left[e^{\delta\sum_{i=1}^{N}\Lambda^*(\phi_i-\phi_{i-1})}\right] < \infty.$$

Proof. For $\delta < 1$, let $p^{(\delta)}(x)$ be the probability density defined by

$$p^{(\delta)}(x) = \frac{1}{z^{(\delta)}}p(x)e^{\delta\Lambda^*(x)},$$

where $z^{(\delta)} = \int_{\mathbb{R}^d} p(x)e^{\delta\Lambda^*(x)}\,dx < \infty$ if $\delta < 1$ from Lemma 5.1.14 in [8]. Then, $p^{(\delta)}$ satisfies the Cramér's condition:

$$\Lambda^{(\delta)}(\lambda) = \log\int_{\mathbb{R}^d} e^{\lambda\cdot x}p^{(\delta)}(x)\,dx < \infty. \tag{3.5.7}$$

Indeed, by applying Lemma 5.1.14 in [8] for the Cramér transform $p_{\bar\lambda}$ of p, we see that

$$\int_{\mathbb{R}^d} e^{\delta(\Lambda_{\bar\lambda})^*(x)}p_{\bar\lambda}(x)\,dx < \infty, \tag{3.5.8}$$

for all $\delta < 1$ and $\bar\lambda \in \mathbb{R}$, where

$$\Lambda_{\bar\lambda}(\lambda) \equiv \log\int_{\mathbb{R}^d} e^{\lambda\cdot x}p_{\bar\lambda}(x)\,dx = \Lambda(\lambda+\bar\lambda) - \Lambda(\bar\lambda)$$

and $(\Lambda_{\bar\lambda})^*$ is its Legendre transform

$$(\Lambda_{\bar\lambda})^*(v) \equiv \sup_{\lambda\in\mathbb{R}^d}\{\lambda\cdot v - \Lambda_{\bar\lambda}(\lambda)\} = \Lambda^*(v) - \bar\lambda\cdot v + \Lambda(\bar\lambda).$$

Inserting this into (3.5.8), we see that

$$\int_{\mathbb{R}^d} e^{(1-\delta)\bar{\lambda}\cdot x} p^{(\delta)}(x)\, dx < \infty,$$

which implies (3.5.7) by taking $\bar{\lambda} = \lambda/(1-\delta)$ for each $\lambda \in \mathbb{R}$.

Let $\nu_{N,\lambda}^{(\delta)}$ be the probability measure $\nu_{N,\lambda}$ defined by taking $p^{(\delta)}$ in place of p, that is,

$$\nu_{N,\lambda}^{(\delta)}(d\eta) = \frac{1}{\tilde{Z}_{N,\lambda}^{(\delta)}} \prod_{i=1}^{N} p^{(\delta)}(\eta_i) e^{\lambda(\frac{i}{N})\cdot\eta_i} d\eta_i,$$

with the normalizing constant $\tilde{Z}_{N,\lambda}^{(\delta)}$ and let $f_{N,\lambda}^{(\delta)}(x)$ be the probability density of $\frac{1}{N}\sum_{i=1}^{N} \eta_i$ under the distribution $\nu_{N,\lambda}^{(\delta)}$. Then, since $p^{(\delta)}$ satisfies the Cramér's condition, Lemma 3.5.4 can be applied for $p^{(\delta)}$ and we obtain that

$$\lim_{N\to\infty} \frac{1}{N} \log f_{N,\lambda}^{(\delta)}(y) = - \min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=y}} \frac{1}{m} \sum_{\ell=1}^{m} \{(\Lambda^{(\delta)})^*(x_\ell) - \lambda_\ell \cdot x_\ell + \Lambda^{(\delta)}(\lambda_\ell)\}, \tag{3.5.9}$$

which is finite for each $y \in \mathbb{R}^d$.

We now rewrite the expectation in the statement of the lemma as

$$\begin{aligned} E^{\mu_{N,\lambda}^{a,b}}\left[e^{\delta \sum_{i=1}^{N} \Lambda^*(\phi_i - \phi_{i-1})} \right] &= \left. \frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^{N} \eta_i \in dx\}} e^{\delta \sum_{i=1}^{N} \Lambda^*(\eta_i)}\, d\nu_{N,\lambda}}{\nu_{N,\lambda}(\frac{1}{N}\sum_{i=1}^{N} \eta_i \in dx)} \right|_{x=b-a} \\ &= \frac{(z^{(\delta)})^N \tilde{Z}_{N,\lambda}^{(\delta)}}{\tilde{Z}_{N,\lambda}} \times \frac{f_{N,\lambda}^{(\delta)}(b-a)}{f_{N,\lambda}(b-a)}. \end{aligned}$$

However, it is easy to see that

$$\frac{1}{N} \log \tilde{Z}_{N,\lambda}^{(\delta)} = \frac{1}{N} \sum_{i=1}^{N} \Lambda^{(\delta)}(\lambda(\frac{i}{N})) = \frac{1}{m} \sum_{\ell=1}^{m} \Lambda^{(\delta)}(\lambda_\ell) < \infty.$$

This holds also for $\frac{1}{N} \log \tilde{Z}_{N,\lambda}$; take $\delta = 0$. Thus, (3.5.9) together with this formula taken $\delta = 0$ (for $f_{N,\lambda}(b-a)$) completes the proof of the lemma recalling that $z^{(\delta)} < \infty$. □

Proof of the lower bound in Proposition 3.5.2

Let h be the polygon considered in Proposition 3.5.3 and denote $\lambda = \lambda_h$. Then, for every $\delta > 0$, we have

$$\mu_N^{a,b}(\|h^N - h\|_\infty \le \delta) = \frac{Z_{N,\lambda}^{a,b}}{Z_N^{a,b}} E^{\mu_{N,\lambda}^{a,b}} \left[e^{-\sum_{i=1}^{N} \lambda(i/N)\cdot(\phi_i - \phi_{i-i})}, \|h^N - h\|_\infty \le \delta \right]. \tag{3.5.10}$$

Here,

$$
\begin{aligned}
Z_{N,\lambda}^{a,b} &= \int_{(\mathbb{R}^d)^{N-1}} \mathfrak{p}_N(\phi) \prod_{i=1}^{N} e^{\lambda\left(\frac{i}{N}\right)\cdot(\phi_i-\phi_{i-1})} \prod_{i=1}^{N-1} d\phi_i \bigg|_{\phi_0=aN,\phi_N=bN} \\
&= \int \prod_{i=1}^{N} p(\eta_i) e^{\lambda\left(\frac{i}{N}\right)\cdot\eta_i}\, d\eta_i \bigg|_{\frac{1}{N}\sum_{i=1}^N \eta_i = b-a} \\
&= \frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^N \eta_i \in dx\}} \prod_{i=1}^N p(\eta_i) e^{\lambda\left(\frac{i}{N}\right)\cdot\eta_i}\, d\eta_i}{dx} \Bigg|_{x=b-a} \\
&= \tilde{Z}_{N,\lambda} f_{N,\lambda}(b-a),
\end{aligned}
$$

and $Z_N^{a,b} = \tilde{Z}_{N,0} f_{N,0}(b-a)$. Since it holds that

$$
\left| \sum_{i=1}^{N} \lambda\big(\frac{i}{N}\big) \cdot (\phi_i - \phi_{i-1}) - N \int_0^1 \lambda(t) \cdot \dot{h}(t)\, dt \right| \le 2N\delta \|\lambda\|_{L^1(D)}
$$

on the event $\{\|h^N - h\|_\infty \le \delta\}$, we have from (3.5.10) that

$$
\begin{aligned}
(3.5.11) \quad & \liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - h\|_\infty \le \delta) \\
&\ge \lim_{N\to\infty} \frac{1}{N} \log \frac{\tilde{Z}_{N,\lambda}}{\tilde{Z}_{N,0}} + \lim_{N\to\infty} \frac{1}{N} \log \frac{f_{N,\lambda}(b-a)}{f_{N,0}(b-a)} \\
&\quad - \int_0^1 \lambda(t) \cdot \dot{h}(t)\, dt - 2\delta \|\lambda\|_{L^1(D)} + \lim_{N\to\infty} \frac{1}{N} \log \mu_{N,\lambda}^{a,b}(\|h^N - h\|_\infty \le \delta).
\end{aligned}
$$

However, by the computations made in the last subsection, the first term in the right hand side of (3.5.11) is equal to

$$
\frac{1}{m} \sum_{\ell=1}^{m} \Lambda(\lambda_\ell),
$$

while the second in (3.5.11) is equal to

$$
\begin{aligned}
&- \min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^{m} \{\Lambda^*(x_\ell) - \lambda_\ell \cdot x_\ell + \Lambda(\lambda_\ell)\} \\
&+ \min_{\substack{x_1,\dots,x_m \in \mathbb{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^{m} \Lambda^*(x_\ell).
\end{aligned}
$$

Proposition 3.5.3 implies that the last term in (3.5.11) is 0. Thus, we have that

$$
(3.5.12) \quad \liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - h\|_\infty \le \delta)
$$

$$\begin{aligned}
&\geq - \min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \{\Lambda^*(x_\ell) - \lambda_\ell\cdot x_\ell\} + \min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \Lambda^*(x_\ell) \\
&\quad - \int_0^1 \lambda(t)\cdot\dot{h}(t)\,dt - 2\delta\|\lambda\|_{L^1(D)} \\
&\geq -\int_0^1 \Lambda^*(\dot{h}(t))\,dt + \inf\Sigma_0 - 2\delta\|\lambda\|_{L^1(D)} \\
&= -I_0(h) - 2\delta\|\lambda\|_{L^1(D)}.
\end{aligned}$$

Here, the second inequality follows from

$$\begin{aligned}
&\min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \{\Lambda^*(x_\ell) - \lambda_\ell\cdot x_\ell\} \\
&\qquad = \frac{1}{m}\sum_{\ell=1}^m \{\Lambda^*(v(\lambda_\ell)) - \lambda_\ell\cdot v(\lambda_\ell)\} = -\int_0^1 \Lambda(\lambda(t))\,dt
\end{aligned}$$

by Lemma 3.5.5 and (3.3.2),

$$\begin{aligned}
&\min_{\substack{x_1,\dots,x_m\in\mathbb{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \Lambda^*(x_\ell) \\
&\qquad = \inf_{\substack{g:\ \text{polygons s.t.}\\ g(0)=a,g(1)=b}} \int_0^1 \Lambda^*(\dot{g}(t))\,dt \geq \inf\Sigma_0
\end{aligned}$$

and $\lambda(t)\cdot\dot{h}(t) = \Lambda^*(\dot{h}(t)) + \Lambda(\lambda(t))$ by (3.3.2).

Now take an arbitrary open set $\mathfrak{O}$ of $\mathcal{C}_{a,b}$. Then, since $\{\|h^N - h\|_\infty \leq \delta\} \subset \{h^N \in \mathfrak{O}\}$ for every polygon $h \in \mathfrak{O}$ and every sufficiently small $\delta > 0$, we see from (3.5.12) that

$$\liminf_{N\to\infty} \frac{1}{N}\log\mu_N^{a,b}\big(h^N \in \mathfrak{O}\big) \geq -\inf_{h\in\mathfrak{O}:\text{polygons}} I_0(h).$$

However, the (local Lipschitz) continuity of Λ^* implies that

$$\inf_{h\in\mathfrak{O}} I_0(h) = \inf_{h\in\mathfrak{O}:\text{polygons}} I_0(h)$$

and this completes the proof of the lower bound in the proposition.

Proof of the upper bound in Proposition 3.5.2

For the upper bound, it is enough to show the following estimate for every $g \in \mathcal{AC}_{a,b}$:

$$\limsup_{N\to\infty} \frac{1}{N}\log\mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \leq -I_0(g) + \theta, \tag{3.5.13}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), see the remark below (3.5.15). The exponential tightness for $\mu_N^{a,b}$ follows from Lemma 3.5.6.

For every $g \in \mathcal{AC}_{a,b}$, since Assumption 3.1.1-(1) implies $\sup_{x\in\mathbb{R}^d} p(x) < \infty$, by Lemma 3.3.1, we have

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - g\|_\infty < \delta)\\ \le &\limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta) + \Lambda^*(b-a).\end{aligned}$$

By the relation $\mu_{N-1}^{a,F}(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta) = \mu_{N-1}^{a,F}(\|h^{N-1} - \frac{N}{N-1} g(\frac{N-1}{N}\cdot)\|_\infty < \frac{N}{N-1}\delta)$ and the continuity of g, we can get

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta) \le \limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}(\|h^{N-1} - g\|_\infty < 2\delta).$$

Finally, by the LD upper bound for $\mu_N^{a,F}$, the relation $\Lambda^*(b-a) = \inf_{\mathcal{AC}_{a,b}} \Sigma_0$ and the lower semi-continuity of $\Sigma_0(h)$, we have

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \le - \inf_{h\in\{\|h-g\|_\infty \le 2\delta\}} \Sigma_0(h) + \inf_{\mathcal{AC}_{a,b}} \Sigma_0 \le -I_0(g) + \theta,$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ).

3.5.3 Proof of Theorem 3.5.1

For the proof of Theorem 3.5.1 for $\mu_N^{D,\varepsilon}$, it is enough to show the following two estimates for every $g \in \mathcal{AC}_{a,b}$:

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \ge -I^D(g), \tag{3.5.14}$$

for every $\delta > 0$, and

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \le -I^D(g) + \theta, \tag{3.5.15}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), where I^D is defined by (3.5.1) with $\Sigma = \Sigma^D$ and $\mathcal{AC} = \mathcal{AC}_{a,b}$. This step of reduction is standard, for instance, see (6.6) and the estimate just above (6.11) in [22].

The proof of the lower bound (3.5.14) is similar to the **Lower bound** in Section 2.4.3. The only difference is that we should replace $\Sigma_0(a,b;t_1^1,t_2^K)$ in Lemma 2.4.6 by

$$\Sigma_0(a,b;t_1^1,t_2^K) = t_1^1 \Lambda^* \left(-\frac{a}{t_1^1}\right) + t_2^K \Lambda^* \left(\frac{b}{t_2^K}\right).$$

In fact, from (3.4.4), Proposition 3.3.3 and the formula (3.4.5) for $\tilde{f}(s_1, s_2)$, one can show that

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{a,b}}{Z_N^{a,b,\varepsilon}} = -\Sigma_0(\bar{h}^D) + \inf_{\mathcal{AC}_{a,b}} \Sigma(h). \tag{3.5.16}$$

The equality (1) in Lemma 2.4.6 follows from (3.5.16) and Proposition 3.3.3 recalling Lemma 3.3.1. Another inequality (2) in that lemma is a consequence of Propositions 3.5.2 and 3.5.7 stated below. All other arguments are exactly the same.

Proposition 3.5.7. *For every $\delta > 0$, there exist $C, c > 0$ such that*

$$\mu_N^\varepsilon(\|h^N\|_\infty \geq \delta) \leq Ce^{-cN}$$

for $\mu_N^\varepsilon = \mu_N^{0,0,\varepsilon}$ and $\mu_N^{0,F,\varepsilon}$.

This proposition is shown in Proposition 2.4.3 of Chapter 2 or Proposition 2.1 of [18] for the Gaussian case. The general case can be proved from Proposition 3.5.2 by tracing the method used in Section 2.2 of [18], which is based on a renewal theory.

The proof of the upper bound (3.5.15) is also similar to the **Upper bound** in Section 2.4.3. We should replace $\frac{1}{2}\int_{D\setminus I} |\dot{g}(t)|^2\, dt$ with $\int_{D\setminus I} \Lambda^*(\dot{g}(t))\, dt$ in the statement of Lemma 2.4.7 and the estimate on $I_N^j(\delta)$ in its proof with

$$I_N^j(\delta) \leq \exp\left\{ N\left(-\int_0^s \Lambda^*(\dot{g}(t))\, dt + s\Lambda^*\left(-\frac{a}{s}\right) + \theta \right)\right\}.$$

Otherwise, all arguments are the same.

For the proof of Theorem 3.5.1 for $\mu_N^{F,\varepsilon}$, we may modify some arguments in the proof for $\mu_N^{D,\varepsilon}$ as indicated in Section 2.4.4.

3.6 Critical exponents for the free energies

This section studies the asymptotic behavior of the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$ near the critical values ε_c^D and ε_c^F, respectively; recall (3.3.8), (3.3.9), (3.3.12) and (3.3.13) for the definition of these quantities. The results are summarized in the following proposition.

Proposition 3.6.1. (1) (Dirichlet case) *As $\varepsilon \downarrow \varepsilon_c^D$, we have that*

$$\xi^{D,\varepsilon} \sim \begin{cases} C_d(\varepsilon - \varepsilon_c^D)^2, & d = 1, 3, \\ e^{-2\pi\sqrt{\det Q}/\varepsilon}, & d = 2, \\ -C_4(\varepsilon - \varepsilon_c^D)/\log(\varepsilon - \varepsilon_c^D), & d = 4, \\ C_d(\varepsilon - \varepsilon_c^D), & d \geq 5, \end{cases}$$

where $C_1 = 1/(2\det Q)$, $C_3 = 2\pi^2 \det Q/(\varepsilon_c^D)^4$, $C_4 = 4\pi^2\sqrt{\det Q}/(\varepsilon_c^D)^2$ *and* $C_d = 1/\big((\varepsilon_c^D)^2 \sum_{n=1}^\infty n e^{n\Lambda^*(0)} Z_n^{0,0}\big)$ *for* $d \geq 5$.

(2) (Free case)

(i) *If* $m = 0$, $\xi^{F,\varepsilon}$ *behaves exactly in the same way as* $\xi^{D,\varepsilon}$.

(ii) *If* $m \neq 0$, *as* $\varepsilon \downarrow \varepsilon_c^F$, *we have that*

$$\xi^{F,\varepsilon} \sim C_d^F(\varepsilon - \varepsilon_c^F),$$

for every $d \geq 1$, *where* $C_d^F = 1/\big((\varepsilon_c^F)^2 \sum_{n=1}^\infty n Z_n^{0,0}\big)$.

For the proof of the proposition, we prepare a lemma which establishes the asymptotic behavior of the function:

$$q_d(x) = (2\pi)^{d/2}\sqrt{\det Q}\, g\big(e^{\Lambda^*(0)}x\big), \quad 0 \leq x \leq 1,$$

as $x \uparrow 1$, where $g(x) \equiv g_d(x)$ is the function defined by (3.3.7). We only consider the case $1 \leq d \leq 4$, since the case $d \geq 5$ is easy.

Lemma 3.6.2. *As* $x \uparrow 1$, *we have that*

$$q_d(x) \sim \begin{cases} \sqrt{\pi}(1-x)^{-1/2}, & d = 1, \\ -\log(1-x), & d = 2, \end{cases}$$

and

$$q_d(1) - q_d(x) \sim \begin{cases} 2\sqrt{\pi}(1-x)^{1/2}, & d = 3, \\ -(1-x)\log(1-x), & d = 4. \end{cases}$$

Proof. Let $f_d(x) = \sum_{n=1}^\infty x^n/n^{d/2}$, $0 \leq x \leq 1$, be the function defined by (A.1) of Chapter 2, whose asymptotics as $x \uparrow 1$ can be found in Lemma A.3 there. Then, we have that

$$q_d(x) - f_d(x) = \sum_{n=1}^\infty \frac{x^n}{n^{d/2}} \Big\{(2\pi n)^{d/2}\sqrt{\det Q}\, e^{n\Lambda^*(0)} Z_n^{0,0} - 1\Big\}.$$

However, since (3.3.4) in Lemma 3.3.1 shows that the difference in the braces in the right hand side tends to 0 as $n \to \infty$, one can show that, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$|q_d(x) - f_d(x)| \leq \delta f_d(x) + C_\delta.$$

If $d = 1, 2$, since $f_d(x) \to \infty$ as $x \uparrow 1$, this implies that the asymptotics of q_d are the same as f_d. To show the asymptotics of $q_d(1) - q_d(x)$ for $d = 3, 4$, we see that, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$|xq_d'(x) - f_{d-2}(x)| \le \delta f_{d-2}(x) + C_\delta.$$

This can be proved similarly as above. Since $f_{d-2}(x) \to \infty$ as $x \uparrow 1$, this shows the asymptotics for $d = 3, 4$, cf. the proof of Lemma A.3 of Chapter 2. □

Proof of Proposition 3.6.1. The assertion (1) for $1 \le d \le 4$ follows from Lemma 3.6.2 in a similar manner to the proof of Proposition A.1 of Chapter 2 recalling that $q_d\big(e^{-\xi^{D,\varepsilon}}\big) = (2\pi)^{d/2}\sqrt{\det Q}/\varepsilon$. The proof of the assertion (1) for $d \ge 5$ is easy from

$$g\big(e^{\Lambda^*(0)}\big) - g\big(e^{-\xi^{D,\varepsilon}+\Lambda^*(0)}\big) = \frac{1}{\varepsilon_c^D} - \frac{1}{\varepsilon}.$$

Indeed, the left hand side is asymptotically equivalent to $\xi^{D,\varepsilon} e^{\Lambda^*(0)} g'\big(e^{\Lambda^*(0)} - \big) = \xi^{D,\varepsilon} \sum_{n=1}^\infty n e^{n\Lambda^*(0)} Z_n^{0,0}$, while the right hand side behaves as $(\varepsilon - \varepsilon_c^D)/(e_c^D)^2$; note that the series appeared above converges. The proof of the assertion (2) is immediate, since we have $\xi^{F,\varepsilon} = \xi^{D,\varepsilon}$ and $\varepsilon_c^F = \varepsilon_c^D$ if $m = 0$. The proof of the assertion (3) is similar as above by noting that

$$g(1) - g\big(e^{-\xi^{F,\varepsilon}}\big) = \frac{1}{\varepsilon_c^F} - \frac{1}{\varepsilon}.$$

□

Chapter 4

Law of large numbers for Wiener measure with density having two large deviation minimizers

4.1 Introduction and results

In this chapter, we are interested in the law of large numbers for a sequence of probability measures $\{\mu_N\}_{N=1,2,\dots}$ on the space $\mathcal{C} = C(I, \mathbb{R}), I = [0, 1]$, under the critical situation that the rate functional of the corresponding large deviation principle admits two minimizers. The sequence of probability measures $\{\mu_N\}_{N=1,2,\dots}$ is defined from the Wiener measures involving a proper scaling with densities determined by a class of potentials W. Such measures naturally arise as a continuous analog of the $\nabla\varphi$ interface model with weak self potentials in one dimension. The relation to the $\nabla\varphi$ interface model was stated in Section 3 in [17]. The large deviation principle (LDP) is easily established for $\{\mu_N\}$ and the (unnormalized) rate functional is given by Σ^W, see (4.1.3) below. The purpose of this chapter is to prove the law of large numbers (LLN) for $\{\mu_N\}$ under the situation that Σ^W admits two minimizers $\bar{h}$ and $\hat{h}$. We shall specify the conditions for the potentials W, under which the limit points under μ_N are either $\bar{h}$ or $\hat{h}$ as $N \to \infty$.

We now formulate our problem more precisely. Let ν_0 be the law on the space $\mathcal{C}$ of the Brownian motion such that $x(0) = 0$. The canonical coordinate of $x \in \mathcal{C}$ is described by $x = \{x(t); t \in I\}$. For $a \in \mathbb{R}$, $x \in \mathcal{C}$ and $N = 1, 2, \dots$, we set

$$h^N(t) = \frac{1}{\sqrt{N}} x(t) + \bar{h}(t), \quad t \in I, \tag{4.1.1}$$

where $\bar{h}(t) \equiv a$. The law on $\mathcal{C}$ of h^N with x distributed under ν_0 is denoted by ν_N. Let

$W = W(r)$ be a (measurable) function on $\mathbb{R}$ satisfying the condition:

(W.1) There exists $A > 0$ such that $\lim_{r\to\infty} W(r) = 0$, $\lim_{r\to-\infty} W(r) = -A$ and $-A \le W(r) \le 0$ for every $r \in \mathbb{R}$.

We consider the distribution, indeed a finite volume Gibbs measure, μ_N on $\mathcal{C}$ defined by

$$\mu_N(dh) = Z_N^{-1} \exp\left\{-N \int_I W(Nh(t))\, dt\right\} \nu_N(dh), \tag{4.1.2}$$

where Z_N is the normalizing constant. Under μ_N, as $N \to \infty$, negative h has an advantage since the density factor becomes larger if it takes negative values. This causes a competition, especially when $a > 0$, between the effect of the potential W pushing h to the negative side and the boundary condition $a > 0$ keeping h at the positive side.

The large deviation principle (LDP) holds for μ_N on $\mathcal{C}$ as $N \to \infty$ under the uniform topology. The speed is N and its (unnormalized) rate functional is given by

$$\Sigma^W(h) = \frac{1}{2}\int_I \dot{h}^2(t)\, dt - A\big|\{t \in I; h(t) \le 0\}\big|, \tag{4.1.3}$$

for $h \in H^1_{a,F}(I)$, i.e., for absolutely continuous h with derivatives $\dot{h}(t) = dh/dt \in L^2(I)$ satisfying $h(0) = a$, where $|\cdot|$ stands for the Lebesgue measure. For more precise formulation, cf. [22], [30] and Theorem 6.4 in [16] for a discrete model. The LDP immediately implies the concentration property for μ_N:

$$\lim_{N\to\infty} \mu_N\left(\mathrm{dist}_\infty(h, \mathcal{H}^W) \le \delta\right) = 1$$

for every $\delta > 0$, where $\mathcal{H}^W = \{h^*;\ \text{minimizers of } \Sigma^W\}$ and dist_∞ denotes the distance under the uniform norm $\|\cdot\|_\infty$. In particular, if Σ^W has a unique minimizer h^*, then the law of large numbers (LLN) holds under μ_N:

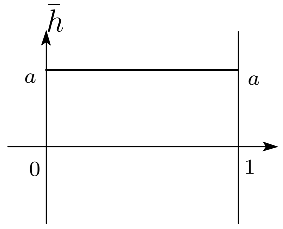
$$\lim_{N\to\infty} \mu_N(\|h - h^*\|_\infty \le \delta) = 1 \tag{4.1.4}$$

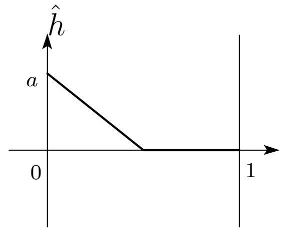
for every $\delta > 0$.

We consider the structure of $\mathcal{H}^W$. It is easy to see that $\mathcal{H}^W = \{\bar{h}\}$ when $a \le 0$. We now assume that $a > 0$. Let $\hat{h}$ be the curve composed of two straight line segments connecting three points $(0, a), P(T, 0)$ and $(1, 0)$ in this order. The angles at the corner P is equal to $\theta \in [0, \pi/2]$, which is determined by the Young's relation (free boundary condition): $\tan\theta = \sqrt{2A}$. More precisely saying, if $0 < a \le \sqrt{2A}$ we have $T = a/\sqrt{2A}$, and

$$\hat{h}(t) = \begin{cases} a - \sqrt{2A}\, t, & t \in I_1 = [0, T], \\ 0, & t \in I_2 = [T, 1]. \end{cases}$$

Moreover, we can see that $\mathcal{H}^W = \{\bar{h}\}$ when $a > \sqrt{2A}$. Then, $\{\bar{h}, \hat{h}\}$ is the set of all critical points of Σ^W (cf. Section 6.3 in [16]), and this implies that $\mathcal{H}^W \subset \{\bar{h}, \hat{h}\}$.





This chapter is concerned with the case where both $\bar{h}$ and $\hat{h}$ are minimizers of Σ^W, i.e. $\Sigma^W(\bar{h}) = \Sigma^W(\hat{h})$; note that $\Sigma^W(\bar{h}) = 0$ and $\Sigma^W(\hat{h}) = a(1+\sqrt{2A})/2 - A$. In fact, in the following, we always assume the conditions (W.1) and

(W.2) $$a > 0 \quad \text{and} \quad \Sigma^W(\bar{h}) = \Sigma^W(\hat{h}).$$

If the condition (W.2) holds, we have $a = \sqrt{2A}/2$ and $T = 1/2$.

We are now in a position to state our main results.

Theorem 4.1.1. (*Concentration on* $\bar{h}$) *In addition to the conditions* (W.1) *and* (W.2), *if*

(W.3) $$W(r) = 0 \text{ for all } r \geq K$$

is fulfilled for some $K \in \mathbb{R}$, *then* (4.1.4) *holds with* $h^* = \bar{h}$.

Theorem 4.1.2. (*Concentration on* $\hat{h}$) *In addition to* (W.1) *and* (W.2), *if the following three conditions*

(W.4) $\exists\lambda_1, \alpha_1 > 0$ *such that* $W(r) \sim -\lambda_1 r^{-\alpha_1}$ (*i.e. the ratio tends to* 1) *as* $r \to \infty$

(W.5) $\exists\lambda_2, \alpha_2 > 0$ *such that* $W(r) \leq -A + \lambda_2 |r|^{-\alpha_2}$ *as* $r \to -\infty$

(W.6) $0 < \alpha_1 < \min\{\alpha_2/(\alpha_2+1), \alpha_2/2\}$ *and* $\displaystyle\int_{I_1} \hat{h}(t)^{-\alpha_1}\,dt > \int_I \bar{h}(t)^{-\alpha_1}\,dt$

are fulfilled, then (4.1.4) *holds with* $h^* = \hat{h}$.

The rate functional Σ^W of the LDP is determined only from the limit values $W(\pm\infty)$, but for Theorems 4.1.1 and 4.1.2 we need more delicate information on the asymptotic properties of W as $r \to \pm\infty$ to control the next order. Let us try to explain the roles of the above conditions in a rather intuitive way. The condition (W.3) (with $K = 0$) means that W is large at least for $r \geq 0$ so that the force pushing the interface (or the Brownian path) downward is weak and not enough to push it down to the level of $\hat{h}$. On the other hand, since the values of $Nh(t)$ in (4.1.2) are very large for t close to 0, compared with (W.3), the interface is pushed downward because of the condition (W.4) and, once it reaches near the level 0, the condition (W.5) forces it to stay there. This makes the interface reach the level of $\hat{h}$. The second condition in (W.6) is fulfilled if $1/2 < \alpha_1 < 1$, and such α_1, which simultaneously satisfies the first condition in (W.6), exists if $\alpha_2 > 1$.

The same kind of problem is discussed for weakly pinned Gaussian random walks in [5]. In one dimension, they proved the coexistence of $\bar{h}$ and $\hat{h}$ under the free boundary condition at the right edge and the concentration on $\hat{h}$ under the Dirichlet boundary condition at the right edge. The problem for the pinned Wiener measures with our densities is discussed by [17].

Section 4.2 gives the proofs of Theorems 4.1.1 and 4.1.2.

4.2 Proofs of results

We consider the following quantity:

$$(4.2.1) \qquad \lim_{N\to\infty} \frac{\mu_N\left(\|h-\hat{h}\|_\infty \le \delta\right)}{\mu_N\left(\|h-\bar{h}\|_\infty \le \delta\right)}$$

for arbitrary small $\delta > 0$.

4.2.1 Proof of Theorem 4.1.1

If the limit of (4.2.1) is equal to 0, then (4.1.4) holds with $h^* = \bar{h}$. In view of the scaling, we may assume $K = 0$ in the condition (W.3) without loss of generality. Introduce the first hitting time $0 \le \tau \le 1$ of $h^N(t)$ to 0 on the event $\Omega_0 = \{h^N \text{ hits } 0\}$ by $\tau = \inf\{t \in I; h^N(t) = 0\}$. Then, from the condition (W.3) with $K = 0$, the strong Markov property of $h^N(t)$ under ν_N shows that

$$\begin{aligned}
&Z_N\mu_N(\|h-\hat{h}\|_\infty \le \delta) \\
&\quad \le \int_{S\ge T-c} E^{\nu_0^S}\left[\exp\left\{-N\int_S^1 W(\sqrt{N}x(s))\,ds\right\}\right]\nu_N(\tau\in dS) \\
&\qquad + \nu_N(\Omega_0^c, \|h-\hat{h}\|_\infty \le \delta),
\end{aligned}$$

where ν_0^S (more generally ν_α^S) is the law on the space $C([S,1],\mathbb{R})$ of the Brownian motion such that $x(S) = 0$ (or $x(S) = \alpha$) and $c = \delta/\sqrt{2A}$ arises from the condition

$\|h-\hat{h}\|_\infty \le \delta$. However, in the first term, the conditions (W.1) and (W.3) with $K=0$ imply that

$$-N\int_S^1 W(\sqrt{N}x(s))\,ds \le ANX^{S,1},$$

where $X^{S,1} = |\{s \in [S,1]; x(s) < 0\}|$ is the occupation time of x on the negative side. Since $X^{S,1} = (1-S)X^{0,1}$ in law and $\nu_0(X^{0,1} \in ds) = 1/\{\pi\sqrt{s(1-s)}\}ds$ (see Proposition 4.11 in [27], p.273), we obtain that

$$E^{\nu_0^S}\left[\exp\left\{-N\int_S^1 W(\sqrt{N}x(s))\,ds\right\}\right] \le \int_I \frac{e^{AN(1-S)s}}{\pi\sqrt{s(1-s)}}\,ds.$$

Simple calculation yields that

$$\begin{aligned}\int_I \frac{e^{AN(1-S)s}}{\pi\sqrt{s(1-s)}}\,ds &= \frac{2}{\pi}\int_0^{\pi/2} e^{AN(1-S)/2}\cosh\left(\frac{AN(1-S)}{2}\sin\theta\right)\,d\theta \\ &\le \frac{2}{\pi}\int_0^{\pi/2} e^{AN(1-S)(1+\sin\theta)/2}\,d\theta.\end{aligned}$$

Then, by Laplace's method, we have

$$\int_I \frac{e^{AN(1-S)s}}{\pi\sqrt{s(1-s)}}\,ds \le \frac{2}{\sqrt{A(1-S)\pi}}\frac{1}{\sqrt{N}}e^{AN(1-S)},$$

for sufficiently large N, see [34].

On the other hand, the distribution of τ under ν_N is given by

$$\nu_N(\tau \in dS) = \frac{a\sqrt{N}}{\sqrt{2\pi S^3}}e^{-\frac{a^2N}{2S}}\,dS,$$

for $0 < S < 1$, see (6.3) in [27], p.80.

Combining these all facts, for N large enough, we have

$$\begin{aligned}(4.2.2)\qquad & Z_N\mu_N(\|h-\hat{h}\|_\infty \le \delta) \\ &\le \frac{2a}{\sqrt{2A}\pi}\int_{S\ge T-c}\frac{e^{-Nf(S)}}{\sqrt{S^3(1-S)}}\,dS + \nu_N(\|h-\hat{h}\|_\infty \le \delta),\end{aligned}$$

where

$$f(S) = \frac{a^2}{2S} - A(1-S).$$

Since $f(S) = \Sigma^W(\hat{h}_S) - \Sigma^W(\hat{h})$ for the curve $\hat{h}_S$ defined similarly to $\hat{h}$ with T replaced by S, we see that $f(S) \ge 0$ and f attains its minimal value 0 at $S = T(=1/2)$. Furthermore, by the condition (W.2), it behaves near T as

$$f(S) = \frac{2a^2}{S}\left(S-\frac{1}{2}\right)^2 \sim 4a^2\left(S-\frac{1}{2}\right)^2.$$

This proves that the first term in the right hand side of (4.2.2) behaves as $O(1/\sqrt{N})$ as $N \to \infty$. Therefore, for every $0 < \delta < \|\bar{h} - \hat{h}\|_\infty$, by noting that $\nu_N(\|h - \hat{h}\|_\infty \le \delta) \le e^{-CN}$ for some $C > 0$ (since the LDP holds for ν_N with speed N and the rate functional $\Sigma^0(h)$, which is defined by $A \equiv 0$ in (4.1.3)), we have that

$$\lim_{N\to\infty} Z_N \mu_N(\|h - \hat{h}\|_\infty \le \delta) = 0.$$

On the other hand, the condition (W.3) implies for every $0 < \delta < (a \wedge b)$ that

$$\lim_{N\to\infty} Z_N \mu_N(\|h - \bar{h}\|_\infty \le \delta) = \lim_{N\to\infty} \nu_0(\|x\|_\infty \le \sqrt{N}\delta) = 1.$$

Thus, the proof of Theorem 4.1.1 is concluded.

4.2.2 Proof of Theorem 4.1.2

We prove the limit of (4.2.1) is equal to ∞. From the definition (4.1.2) of μ_N and by recalling (4.1.1), we have

$$\begin{aligned} &Z_N \mu_N(\|h - \hat{h}\|_\infty \le \delta) \\ &\quad = E^{\nu_0}\left[\exp\left\{-N\int_I W(\sqrt{N}x(t) + N\bar{h}(t))\,dt\right\}, \|x + \sqrt{N}(\bar{h} - \hat{h})\|_\infty \le \sqrt{N}\delta\right] \\ &\quad = E^{\nu_0}\left[\exp\left\{\hat{F}_N(x)\right\}, \|x\|_\infty \le \sqrt{N}\delta\right], \end{aligned}$$

where

$$\hat{F}_N(x) = -N\int_I W(\sqrt{N}x(t) + N\hat{h}(t))\,dt + \sqrt{N}\int_I (\dot{\bar{h}} - \dot{\hat{h}})(t)\,dx(t) - \frac{N}{2}\int_I (\dot{\bar{h}} - \dot{\hat{h}})^2(t)\,dt.$$

The third line follows by means of the Cameron-Martin formula for ν_0 transforming $x + \sqrt{N}(\bar{h} - \hat{h})$ into x. However, since $\dot{\bar{h}}(t) \equiv 0$ and $\int_I \dot{\hat{h}}(t)\,dt = \hat{h}(1) - \hat{h}(0) = -a$, we have

$$\frac{1}{2}\int_I (\dot{\bar{h}} - \dot{\hat{h}})^2(t)\,dt = AT,$$

by the condition (W.2). Moreover, since $\dot{\hat{h}} = -\sqrt{2A}$ on I_1° and 0 on I_2°,

$$\int_I (\dot{\bar{h}} - \dot{\hat{h}})(t)\,dx(t) = \sqrt{2A}(x(T) - x(0)) = \sqrt{2A}x(T),$$

recall that $x(0) = 0$ under ν_0. Therefore, we can rewrite $\hat{F}_N(x)$ as

$$\hat{F}_N(x) = -N\int_{I_1} W(\sqrt{N}x(t) + N\hat{h}(t))\,dt + \sqrt{2AN}x(T) - N\int_{I_2} \left\{W(\sqrt{N}x(t)) + A\right\}dt$$

$$=: F_N^{(1)}(x) + F_N^{(2)}(x) + F_N^{(3)}(x).$$

To give a lower bound on $F_N^{(1)}$, we consider subintervals $\tilde{I}_1 = [0, T - \sqrt{2/A}\,\delta]$ of I_1. Then, since $\hat{h} \geq 2\delta$ on $\tilde{I}_1$, on the event $\mathcal{A}_1 = \{\|x\|_\infty \leq \sqrt{N}\delta\}$, we have for $t \in \tilde{I}_1$,

$$\sqrt{N}x(t) + N\hat{h}(t) \geq -N\delta + N\hat{h}(t) \geq N\delta \longrightarrow \infty \quad (\text{as } N \to \infty),$$

and also $\sqrt{N}x(t) + N\hat{h}(t) \leq N(\hat{h}(t) + \delta)$. Accordingly, by the condition (W.4), for every sufficiently small $\varepsilon > 0$, the integrand of $F_N^{(1)}$ times $-N$ is bounded from below as

$$-NW(\sqrt{N}x(t) + N\hat{h}(t)) \geq (\lambda_1 - \varepsilon)N^{1-\alpha_1}(\hat{h}(t) + \delta)^{-\alpha_1},$$

which implies, by recalling $-W \geq 0$, that

$$F_N^{(1)} \geq (\lambda_1 - \varepsilon)N^{1-\alpha_1} \int_{\tilde{I}_1} (\hat{h}(t) + \delta)^{-\alpha_1}\, dt =: (\lambda_1 - \varepsilon)C_1(\delta)N^{1-\alpha_1},$$

on $\mathcal{A}_1$ for sufficiently large N.

To give lower bounds on $F_N^{(2)}$ and $F_N^{(3)}$, we introduce two more events

$$\begin{aligned} &\mathcal{A}_2 = \{x(T) \geq 0\}, \\ &\mathcal{A}_3 = \{x(t) \leq -N^{-\kappa} \text{ for all } t \in \tilde{I}_2 := [T + N^{-\frac{1}{2}-\kappa}, 1]\}, \end{aligned}$$

where $0 < \kappa < 1/2$ will be chosen later. Then, obviously $F_N^{(2)} \geq 0$ on $\mathcal{A}_2$. If $x \in \mathcal{A}_3$, noting that $-W(r) - A \geq -A$ for all $r \in \mathbb{R}$, we have from (W.5)

$$\begin{aligned} F_N^{(3)} &\geq -AN^{\frac{1}{2}-\kappa} + N \int_{\tilde{I}_2} \big\{ -W(\sqrt{N}x(t)) - A \big\}\, dt \\ &\geq -AN^{\frac{1}{2}-\kappa} - \lambda_2 N^{1-\alpha_2(\frac{1}{2}-\kappa)}|\tilde{I}_2|, \end{aligned}$$

for sufficiently large N. These estimates on $F_N^{(1)}, F_N^{(2)}$ and $F_N^{(3)}$ are summarized into

$$\hat{F}_N \geq (\lambda_1 - \varepsilon)C_1(\delta)N^{1-\alpha_1} - AN^{\frac{1}{2}-\kappa} - \lambda_2 N^{1-\alpha_2(\frac{1}{2}-\kappa)}|\tilde{I}_2| \tag{4.2.3}$$

on $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$ for sufficiently large N.

The next lemma gives a lower bound on the probability $\nu_0(\mathcal{A}_2 \cap \mathcal{A}_3)$.

Lemma 4.2.1. *There exists $C > 0$ such that*

$$\nu_0(\mathcal{A}_2 \cap \mathcal{A}_3) \geq CN^{-\frac{1}{4}-\frac{3}{2}\kappa} \exp\{-18N^{\frac{1}{2}-\kappa}\}.$$

Proof. Consider an auxiliary event

$$\mathcal{A}_4 = \{-3N^{-\kappa} \le x(T + N^{-\frac{1}{2}-\kappa}) \le -2N^{-\kappa}\}.$$

Then, by the Markov property, we have

$$\begin{aligned}\nu_0(\mathcal{A}_2 \cap \mathcal{A}_3) &\ge \nu_0(\mathcal{A}_2 \cap \mathcal{A}_3 \cap \mathcal{A}_4)\\ &= E^{\nu_0}\left[\nu_{0,\alpha}^{0,T+N^{-\frac{1}{2}-\kappa}}(x(T) \ge 0) \cdot \nu_\alpha^{T+N^{-\frac{1}{2}-\kappa}}(x(t) \le -N^{-\kappa}, {}^\forall t \in \tilde{I}_2), \mathcal{A}_4\right],\end{aligned}$$

where $\alpha = x(T + N^{-\frac{1}{2}-\kappa})$ and $\nu_{0,\alpha}^{0,T+N^{-\frac{1}{2}-\kappa}}$ is the law on the space $C([0, T + N^{-\frac{1}{2}-\kappa}], \mathbb{R})$ of the Brownian bridge such that $x(0) = 0$, $x(T + N^{-\frac{1}{2}-\kappa}) = \alpha$. However,

$$\nu_{0,\alpha}^{0,T+N^{-\frac{1}{2}-\kappa}}(x(T) \ge 0) \ge C_1 N^{\frac{\kappa}{2}-\frac{1}{4}} \exp\{-18N^{\frac{1}{2}-\kappa}\} - C_2 N^{-\frac{1}{2}} \exp\{-2TN\},$$

for sufficiently large N with $C_1, C_2 > 0$, see the proof of Lemma 2.2 in [17]. On $\mathcal{A}_4$, we have

$$\nu_\alpha^{T+N^{-\frac{1}{2}-\kappa}}\big(x(t) \le -N^{-\kappa}, {}^\forall t \in \tilde{I}_2\big) \ge P_0\big(\max_{t\in I} |B(t)| \le \bar{t}^{-1/2} N^{-\kappa}\big) \ge C_3 N^{-\kappa},$$

where $\bar{t} = 1 - T - N^{-\frac{1}{2}-\kappa}$ and $C_3 > 0$. Therefore, we obtain

$$\nu_0(\mathcal{A}_2 \cap \mathcal{A}_3) \ge C_4 N^{\frac{\kappa}{2}-\frac{1}{4}} \cdot N^{-\kappa} \cdot \exp\{-18N^{\frac{1}{2}-\kappa}\} \cdot \nu_0(\mathcal{A}_4),$$

for sufficiently large N with $C_4 > 0$. However, we obtain $\nu_0(\mathcal{A}_4) \ge N^{-\kappa}$, see the proof of Lemma 2.2 in [17]. This completes the proof of the lemma. □

Since Lemma 4.2.1 shows

$$\begin{aligned}\nu_0(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) &\ge \nu_0(\mathcal{A}_2 \cap \mathcal{A}_3) - \nu_0(\mathcal{A}_1^c)\\ &\ge \nu_0(\mathcal{A}_2 \cap \mathcal{A}_3) - e^{-\delta^2 N/4} \ge \exp\{-20N^{\frac{1}{2}-\kappa}\},\end{aligned}$$

for sufficiently large N (recall $\frac{1}{2} - \kappa < 1$), we have from (4.2.3)

$$\begin{aligned}&Z_N \mu_N(\|h - \hat{h}\|_\infty \le \delta)\\ &\quad\ge \exp\big\{(\lambda_1 - \varepsilon) C_1(\delta) N^{1-\alpha_1} - A N^{\frac{1}{2}-\kappa} - \lambda_2 N^{1-\alpha_2(\frac{1}{2}-\kappa)} |\tilde{I}_2| - 20 N^{\frac{1}{2}-\kappa}\big\}\\ &\quad\ge \exp\big\{(\lambda_1 - 2\varepsilon) C_1(\delta) N^{1-\alpha_1}\big\},\end{aligned} \tag{4.2.4}$$

for sufficiently large N if $1 - \alpha_1 > 0$ (i.e. $\alpha_1 < 1$), $\frac{1}{2} - \kappa < 1 - \alpha_1$ (i.e. $\kappa > \alpha_1 - \frac{1}{2}$) and $1 - \alpha_2(\frac{1}{2} - \kappa) < 1 - \alpha_1$ (i.e. $\kappa < \frac{1}{2} - \frac{\alpha_1}{\alpha_2}$). One can choose such $\kappa : \alpha_1 - \frac{1}{2} < \kappa < \frac{1}{2} - \frac{\alpha_1}{\alpha_2}$ under the first condition in (W.6), which implies that $\alpha_1(1 + \frac{1}{\alpha_2}) < 1$ and $\frac{1}{2} - \frac{\alpha_1}{\alpha_2} > 0$.

On the other hand, we have

$$Z_N \mu_N(\|h - \bar{h}\|_\infty \le \delta) = E^{\nu_0}\left[\exp\left\{\bar{F}_N(x)\right\}, \|x\|_\infty \le \sqrt{N}\delta\right], \tag{4.2.5}$$

where

$$\bar{F}_N(x) = -N \int_I W(\sqrt{N}x(t) + N\bar{h}(t))\, dt.$$

However, since $\sqrt{N}x(t) + N\bar{h}(t) \ge N(\bar{h}(t) - \delta)$ on the event $\mathcal{A}_1$, the condition (W.4) shows

$$\bar{F}_N \le (\lambda_1 + \varepsilon)N^{1-\alpha_1} \int_I (\bar{h}(t) - \delta)^{-\alpha_1}\, dt =: (\lambda_1 + \varepsilon)C_2(\delta)N^{1-\alpha_1}. \tag{4.2.6}$$

Comparing (4.2.4) and (4.2.5) with (4.2.6), since $(\lambda_1 - 2\varepsilon)C_1(\delta) > (\lambda_1 + \varepsilon)C_2(\delta)$ for sufficiently small δ and $\varepsilon > 0$ by the second condition in (W.6), the proof of Theorem 4.1.2 is concluded.

Chapter 5

Large deviations for the $\nabla\varphi$ interface model with self potentials

5.1 Introduction and result

In this chapter, we are interested in a macroscopic behavior of microscopic interfaces distributed under the finite volume Gibbs measures. In general, the interfaces are hypersurfaces which separate different phases like vapor and water. It is known that, at the macroscopic level, the most probable shape of a crystal surrounded by an interface having a definite total volume is characterized as a minimizer of the total surface tension and such shape is called Wulff shape. Mathematically, this can be shown as a consequence of large deviation principle. We prove the large deviations for the $\nabla\varphi$ interface model under the scaling limit for microscopic interfaces with self potentials. A survey on the $\nabla\varphi$ interface model is in [16], while a review of the results on the Ising model together with some explanations on the physical background can be found in [2].

The large deviation principle for the $\nabla\varphi$ interface model was first studied by Ben Arous and Deuschel [1]. They considered the Gibbs measure with quadratic potential having 0-boundary conditions. Deuschel, Giacomin and Ioffe [9] generalized the results to the non-Gaussian setting under the 0-boundary conditions. Then, taking an effect of self potentials into account, Funaki and Sakagawa [22] extended them for the Gibbs measure added a weak self potential under general Dirichlet boundary conditions, but they required that the self potentials take values between two limits as the height variables tends to $\pm\infty$, i.e. the condition (W2) below with $\gamma = \alpha \vee \beta$.

In our case, the self potentials may depend on microscopic and macroscopic height variables, and also on the macroscopic position of the interfaces. The values of our self potentials may be larger than two limits of them as the height variable tends to $\pm\infty$. In other words, our self potentials are rather free from the upper bound and therefore admit a wide class of functions.

We now formulate our problem more precisely. Let D be a bounded domain in $\mathbb{R}^d$ with a piecewise Lipschitz boundary and set $D_N = ND \cap \mathbb{Z}^d$. The location of the interface is described by a height variable $\phi = \{\phi(x) \in \mathbb{R}; x \in D_N\}$, which measures the vertical distance between the interface and the reference hyperplane D_N. We denote $\partial^+ D_N = \{x \notin D_N; |x-y| = 1 \text{ for some } y \in D_N\}$ and $\overline{D_N} = D_N \cup \partial^+ D_N$.

The Hamiltonian of ϕ on D_N with a boundary condition $\psi = \{\psi(x); x \in \partial^+ D_N\}$ and a self potential S is given by

$$H_N^{\psi,S}(\phi) = H_N^{\psi}(\phi) + \sum_{x \in D_N} S\left(\frac{x}{N}, \frac{1}{N}\phi(x), \phi(x)\right),$$

where

$$H_N^{\psi}(\phi) = \sum_{x,y \in \overline{D_N}, |x-y|=1} V((\phi \vee \psi)(x) - (\phi \vee \psi)(y))$$

and $\phi \vee \psi$ is the height variable on $\overline{D_N}$ determined by $(\phi \vee \psi)(x) = \phi(x)$ for $x \in D_N$ and $= \psi(x)$ for $x \in \partial^+ D_N$. The interaction potential V satisfies the following three conditions:

(V1) *(smoothness)* $V \in C^2(\mathbb{R})$,

(V2) *(symmetry)* $V(\eta) = V(-\eta)$ for every $\eta \in \mathbb{R}$,

(V3) *(strict convexity)* there exist $c_-, c_+ > 0$ such that $c_- \le V''(\eta) \le c_+$ for every $\eta \in \mathbb{R}$.

The self potential $S : D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the form $S(\theta, s, r) \equiv Q(\theta, s)W(r)$, where $Q : D \times \mathbb{R} \to [0, \infty)$ and $W : \mathbb{R} \to \mathbb{R}$. We assume the following conditions on Q and W, respectively:

(Q1) Q is non-negative, bounded and piecewise continuous,

(Q2) $|Q(\theta, s) - Q(\theta, s')| \le c(\theta)|s - s'|$, with $c : D \to [0, \infty)$ satisfying $\|c\|_{L^2(D)} < \infty$,

(W1) W is measurable,

(W2) the limits $\alpha = \lim_{r\to+\infty} W(r)$, $\beta = \lim_{r\to-\infty} W(r)$ exist, $\gamma = \sup_{r\in\mathbb{R}} W(r) < \infty$, and $W(r) \ge \alpha \wedge \beta$ for every $r \in \mathbb{R}$.

Remark 5.1.1. *Note that Funaki and Sakagawa* [22] *considered the case where S is decomposed into $S(\theta,s,r)=Q(\theta)W(r)$ and $\gamma=\alpha\vee\beta$ in the condition* (W2).

The macroscopic boundary condition will given by $g|_{\partial D}$ for $g\in C^\infty(\mathbb{R}^d)$. We assume the following conditions for the corresponding microscopic boundary condition $\psi\in\mathbb{R}^{\partial^+D_N}$.

(PS1) $\max_{x\in\partial^+D_N}|\psi(x)|\le CN$,

(PS2) $\sum_{x\in\partial^+D_N}\left|\psi(x)-Ng\left(\frac{x}{N}\right)\right|^{p_0}\le CN^d$ for some $C>0$ and $p_0>2$.

The corresponding finite volume Gibbs measure on $\mathbb{R}^{D_N}$ is now defined by

$$\mu_N^{\psi,S}(d\phi)=\frac{1}{Z_N^{\psi,S}}\exp\{-H_N^{\psi,S}(\phi)\}\prod_{x\in D_N}d\phi(x),$$

where $Z_N^{\psi,S}$ is the normalization factor. The finite volume Gibbs measure without self potential is denoted by μ_N^{ψ}.

Our scaled random interface $\{h^N(\theta);\theta\in D\}$ is defined by a polilinear interpolation of the macroscopically scaled height variables, i.e., $h^N(\theta)=\frac{1}{N}\phi(x)$ for $\theta=\frac{x}{N}, x\in\overline{D_N}$ and

$$h^N(\theta)=\sum_{\lambda\in\{0,1\}^d}\left[\prod_{i=1}^d(\lambda_i\{N\theta_i\}+(1-\lambda_i)(1-\{N\theta_i\}))\right]h^N\left(\frac{[N\theta]+\lambda}{N}\right),$$

for general $\theta\in D$, where $[\cdot]$ and $\{\cdot\}$ denote the integral and the fractional parts, respectively (cf. [9]). We define another scaled profile $\{\overline{h}^N(\theta);\theta\in D\}$ by a step function, i.e., $\overline{h}^N(\theta)=\frac{1}{N}\phi([N\theta])$ for $\theta\in D$.

For $h\in H^1(D)$, define a surface free energy by

$$\Sigma(h)=\int_D\sigma(\nabla h(\theta))\,d\theta,$$

where $\sigma(u)$ is the surface tension with the tilt $u\in\mathbb{R}^d$ (cf. [9], [16]).

Now we state our main theorem which establishes the large deviation principle for $\mu_N^{\psi,S}$ with weak self potentials in a wider class than those treated by Funaki and Sakagawa [22].

Theorem 5.1.1. *The family of random surfaces $\{h^N(\theta);\theta\in D\}$ distributed under $\mu_N^{\psi,S}$ satisfies the large deviation principle on $L^2(D)$ with speed N^d and the rate functional $I^S(h)$, that is, for every closed set $\mathcal{C}$ and open set $\mathcal{O}$ of $L^2(D)$ we have that*

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\mu_N^{\psi,S}(h^N\in\mathcal{C})\le-\inf_{h\in\mathcal{C}}I^S(h),$$

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mu_N^{\psi,S}(h^N \in \mathcal{O}) \geq - \inf_{h\in\mathcal{O}} I^S(h).$$

The functional $I^S(h)$ is given by

$$I^S(h) = \begin{cases} \Sigma^S(h) - \inf\{\Sigma^S(h); h \in H_g^1(D)\}, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $H_g^1(D) = \{h \in H^1(D); h - g|_D \in H_0^1(D)\}$ and

$$\Sigma^S(h) = \Sigma(h) + \int_D Q(\theta, h(\theta)) \left(\alpha 1_{\{h(\theta)>0\}} + \beta 1_{\{h(\theta)<0\}} + (\alpha \wedge \beta) 1_{\{h(\theta)=0\}}\right) d\theta. \tag{5.1.1}$$

It is well known that, once the large deviation principle is established, we immediately obtain the law of large numbers for h^N under $\mu_N^{\psi,S}$, that is, if the rate functional Σ^S has a unique minimizer h^* in $H_g^1(D)$, we have

$$\lim_{N\to\infty} \mu_N^{\psi,S}(\|h^N - h^*\|_{L^2(D)} > \delta) = 0$$

for every $\delta > 0$.

The law of large numbers under the situation that the rate functional has two distinct minimizers for the model motivated by the $\nabla\varphi$ interface model with self potentials in one dimension was discussed by [17] and [29]. The law of large number under the Gaussian Gibbs measures with δ-pinning, especially under the rate functional of the corresponding large deviation principle has two minimizers in one dimension was studied by [5]. In particular, they proved that two minimizers coexist in Free boundary case. The δ-pinning potential is defined from a certain limit of the square well pinning potential. The large deviation principle for the square well pinning potential has not been proven yet. Dunlop et al. [12] first proved the localization under the Gaussian Gibbs measures with the square well pinning potential and 0-boundary conditions in two dimension. The result of [12] was extended for general convex potential by Deuschel and Velenik [11].

In the next section, we will give the proof of Theorem 5.1.1.

5.2 Proof of Theorem 5.1.1

In this section, we only consider the case where $\beta \leq \alpha$. The case $\alpha < \beta$ can be treated in a similar way. We decompose the self potential S into $S = \gamma Q + Q(W - \gamma)$. Then, $\tilde{W} = W - \gamma$ satisfies the condition (W2'), which is the condition (W2) with $\gamma = 0$.

Remark 5.2.1. *If Q does not depend on $\phi(x)$, since the contribution of the first term γQ in $\exp\{-H_N^{\psi,S}(\phi)\}$ of $\mu_N^{\psi,S}$ cancels with the normalization factor, we can prove Theorem* 5.1.1 *in a similar way to the proof of* [22, *Theorem* 2.1]. *However, since Q of our self potential depends on $\phi(x)$, we cannot prove Theorem* 5.1.1 *by tracing the method used for the proof of* [22, *Theorem* 2.1]. *The following proposition recovers the thread.*

The following proposition is for the finite Gibbs measure $\mu_N^{\psi,\gamma Q}$ with $S = \gamma Q$.

Proposition 5.2.1. *The family of random surfaces $\{h^N(\theta); \theta \in D\}$ distributed under $\mu_N^{\psi,\gamma Q}$ satisfies the large deviation principle on $L^2(D)$ with speed N^d and the rate functional given by*

$$I^{\gamma Q}(h) = \begin{cases} \Sigma^{\gamma Q}(h) - \inf\{\Sigma^{\gamma Q}(h); h \in H_g^1(D)\}, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$\Sigma^{\gamma Q}(h) = \Sigma(h) + \gamma \int_D Q(\theta, h(\theta))\, d\theta.$$

To prove Proposition 5.2.1, we prepare the following lemma.

Lemma 5.2.2. *Assume the conditions* (Q1) *and* (Q2) *on $Q(\theta, s)$. Let $g \in L^2(D)$ and $0 < \delta < 1$ be fixed. If $h^N \in B_2(g, \delta) = \{h \in L^2(D); \|h - g\|_{L^2(D)} < \delta\}$ for N large enough, then there exists some constant $C > 0$ such that*

$$\left| \gamma \frac{1}{N^d} \sum_{x \in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right) - \gamma \int_D Q(\theta, g(\theta))\, d\theta \right| < C\delta,$$

for every N sufficiently large.

Proof. If $h^N \in B_2(g, \delta)$, then $\|\overline{h}^N - g\|_{L^2(D)} < C_1\delta + a_{N,k}$, where C_1 is a positive constant and $a_{N,k}$ tends to 0 as $N \to \infty$ and $k \to \infty$, see (3.2) in [22]. Therefore, if $h^N \in B_2(g, \delta)$, then the left hand side of the desired inequality can be bounded by

$$\begin{aligned} \le& |\gamma| \int_D \left| Q\left(\frac{[N\theta]}{N}, \overline{h}^N\left(\frac{[N\theta]}{N}\right)\right) - Q\left(\frac{[N\theta]}{N}, g(\theta)\right) \right| d\theta \\ &+ |\gamma| \int_D \left| Q\left(\frac{[N\theta]}{N}, g(\theta)\right) - Q(\theta, g(\theta)) \right| d\theta \\ \le& |\gamma|(C_1\delta + a_{N,k})\|c\|_{L^2(D)}|D| + |\gamma| C_2\delta \le C\delta, \end{aligned}$$

for every N and k large enough, where C_2 and C are positive constants. □

Proof of Proposition 5.2.1. Step1 (lower bound). Let $g \in L^2(D)$ and $\delta > 0$. Then, by Lemma 5.2.2 and the large deviation principle lower bound for μ_N^ψ (cf. [22, Proposition 3.1]), we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in B_2(g,\delta)) \\
&\geq - \inf_{h \in B_2(g,\delta)} I(h) - \gamma \int_D Q(\theta, g(\theta))\, d\theta - C\delta \\
&\geq - \left\{ I(g) + \gamma \int_D Q(\theta, g(\theta))\, d\theta \right\} - C\delta,
\end{aligned}$$

where

$$I(h) = \begin{cases} \Sigma(h) - \inf\{\Sigma(h); h \in H_g^1(D)\}, & \text{if } h \in H_g^1(D), \\ +\infty, & \text{otherwise.} \end{cases}$$

is the rate functional of the large deviation principle for μ_N^ψ.

Now let us take an arbitrary open set $\mathcal{O}$ of $L^2(D)$. Then, for every $h \in \mathcal{O}$ and $\delta > 0$ such that $B_2(h,\delta) \subset \mathcal{O}$,

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in \mathcal{O}) \geq - \left\{ I(h) + \gamma \int_D Q(\theta, h(\theta))\, d\theta \right\} - C\delta.$$

Letting $\delta \downarrow 0$, since $h \in \mathcal{O}$ is arbitrary, we have the lower bound

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in \mathcal{O}) \geq - \inf_{h \in \mathcal{O}} \left\{ I(h) + \gamma \int_D Q(\theta, h(\theta))\, d\theta \right\}. \tag{5.2.1}$$

Step2 (upper bound). Let $g \in L^2(D)$ and $\delta > 0$ be fixed. Then, by Lemma 5.2.2 and the large deviation principle upper bound for μ_N^ψ (cf. [22, Proposition 3.1]), we have

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in B_2(g,\delta)) \leq - \inf_{h \in \bar{B}_2(g,\delta)} I(h) - \gamma \int_D Q(\theta, g(\theta))\, d\theta + C\delta.$$

Then, by using the lower semi-continuity of $I(h)$, we see that for every $g \in L^2(D)$, there exists $\delta > 0$ small enough such that

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in B_2(g,\delta)) \leq - \left\{ I(g) + \gamma \int_D Q(\theta, g(\theta))\, d\theta \right\}.$$

The standard argument in the theory of large deviation principle (cf. [8]) yields the upper bound

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,\gamma Q}}{Z_N^\psi} \mu_N^{\psi,\gamma Q}(h^N \in \mathcal{C}) \leq - \inf_{h \in \mathcal{C}} \left\{ I(h) + \gamma \int_D Q(\theta, h(\theta))\, d\theta \right\} \tag{5.2.2}$$

for every compact set $\mathcal{C}$ of $L^2(D)$. However, the exponential tightness for $\mu_N^{\psi,\gamma Q}$ can be proved in a similar way to those for μ_N^{ψ} (see Remark 4.1 of [22]). Thus, (5.2.2) holds for every closed set $\mathcal{C}$ of $L^2(D)$.

Taking $\mathcal{O} = \mathcal{C} = L^2(D)$ in (5.2.1) and (5.2.2), we have the conclusion. □

To prove Theorem 5.1.1, we also prepare the following lemmas.

Lemma 5.2.3. *Assume the conditions* (Q1), (Q2), (W1) *and* (W2') *on* $S(\theta, s, r) = Q(\theta, s)W(r)$. *Let* $g \in L^2(D)$ *and* $0 < \delta < 1$ *be fixed. If* $h^N \in B_2(g, \delta) = \{h \in L^2(D); \|h - g\|_{L^2(D)} < \delta\}$ *with* N *large enough, then there exists some constant* $C > 0$ *such that*

$$\sum_{x \in D_N} S\left(\frac{x}{N}, \frac{1}{N}\phi(x), \phi(x)\right) - N^d \int_D Q(\theta, g(\theta)) \left(\alpha 1_{\{g(\theta) \geq \sqrt{\delta}\}} + \beta 1_{\{g(\theta) \leq -\sqrt{\delta}\}}\right) d\theta < C N^d \delta,$$

for every N *sufficiently large.*

Lemma 5.2.4. *Assume the conditions* (Q1), (Q2), (W1) *and* (W2') *on* $S(\theta, s, r) = Q(\theta, s)W(r)$.

(1) *The functional* $\Sigma^S(h)$ *is lower semi-continuous on* $L^2(D)$.

(2) *Let* $\Sigma_-^S(h)$ *be the functional defined by* (5.1.1) *with* $1_{\{h(\theta) \leq 0\}}$ *replaced by* $1_{\{h(\theta) < 0\}}$. *Then, for every open set* $\mathcal{O}$ *of* $L^2(D)$, *we have that*

$$\inf_{h \in \mathcal{O}} \Sigma^S(h) = \inf_{h \in \mathcal{O}} \Sigma_-^S(h).$$

Lemmas 5.2.3 and 5.2.4 are very similar to Lemmas 3.1 and 3.2 of [22], respectively. Therefore, we only give some remarks instead of completely proving the lemmas.

Remark 5.2.2. *In the proof of Lemma* 5.2.4-(2), *by replacing* $h^n(\theta)$ *which was defined in the proof of* [22, *Lemma* 3.2-(2)] *with*

$$h^n(\theta) = \begin{cases} h(\theta) - f^n(\theta), & \text{if } h(\theta) \leq 0, \\ h(\theta), & \text{if } h(\theta) > 0, \end{cases} \tag{5.2.3}$$

where $f^n \in C_0^\infty(D)$ *are functions such that* $f^n(\theta) \equiv \frac{1}{n}$ *on* $D_n = \{\theta \in D; \mathrm{dist}(\theta, \partial D) \geq \frac{1}{n}\}$ *and* $|\nabla f^n(\theta)| \leq C$ *with* $C > 0$, *we can get the conclusion in a similar way to the proof of* [22, *Lemma* 3.2-(2)]. *Moreover, the case* $\alpha < \beta$ *can be proved in a similar way replacing* (5.2.3) *by*

$$h^n(\theta) = \begin{cases} h(\theta), & \text{if } h(\theta) < 0, \\ h(\theta) + f^n(\theta), & \text{if } h(\theta) \geq 0. \end{cases}$$

Proof of Theorem 5.1.1. Step1 (lower bound). Let $g \in L^2(D)$ and $\delta > 0$. Then, by Lemma 5.2.3 and the large deviation principle lower bound for $\mu_N^{\psi,\gamma Q}$ (Proposition 5.2.1), we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in B_2(g,\delta)) \\
&\geq - \inf_{h\in B_2(g,\delta)} I^{\gamma Q}(h) - \int_D Q(\theta, g(\theta)) \left((\alpha-\gamma)1_{\{g(\theta)\geq\sqrt{\delta}\}} + (\beta-\gamma)1_{\{g(\theta)\leq-\sqrt{\delta}\}}\right) d\theta - C\delta \\
&\geq - \left\{ I^{\gamma Q}(g) + \int_D Q(\theta, g(\theta)) \left((\alpha-\gamma)1_{\{g(\theta)\geq\sqrt{\delta}\}} + (\beta-\gamma)1_{\{g(\theta)\leq-\sqrt{\delta}\}}\right) d\theta \right\} - C\delta.
\end{aligned}$$

Now let us take an arbitrary open set $\mathcal{O}$ of $L^2(D)$. Then, we have

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{O}) \\
&\geq - \inf_{h\in\mathcal{O}} \left\{ I^{\gamma Q}(h) + \int_D Q(\theta, h(\theta)) \left((\alpha-\gamma)1_{\{h(\theta)>0\}} + (\beta-\gamma)1_{\{h(\theta)<0\}}\right) d\theta \right\} \qquad (5.2.4)
\end{aligned}$$

in a similar way to the proof of the lower bound of Proposition 5.2.1.

However, by Lemma 5.2.4-(2), one can replace $1_{\{h(\theta)<0\}}$ with $1_{\{h(\theta)\leq 0\}}$ on the right hand side of (5.2.4). Therefore, we get

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{O}) \geq - \inf_{h\in\mathcal{O}} \Sigma^S(h) + \inf_{h\in H_g^1(D)} \Sigma^{\gamma Q}(h). \qquad (5.2.5)$$

Step2 (upper bound). Let $g \in L^2(D)$ and $\delta > 0$ be fixed. We define

$$\begin{aligned}
L_N^+ &= N\{\theta \in D; g(\theta) > \sqrt{\delta}\} \cap \mathbb{Z}^d, \\
L_N^- &= N\{\theta \in D; g(\theta) < -\sqrt{\delta}\} \cap \mathbb{Z}^d, \\
I_N &= N\{\theta \in D; |g(\theta)| \leq \sqrt{\delta}\} \cap \mathbb{Z}^d.
\end{aligned}$$

By the assumption (W2) on W, for every $\varepsilon > 0$ there exists $K = K_\varepsilon > 0$ such that $W(r) - \gamma \geq -(\alpha-\beta-\varepsilon)1_{\{r\leq K\}} + \alpha - \gamma - \varepsilon$ for any $r \in \mathbb{R}$. Therefore, we have

$$\begin{aligned}
&\exp\left\{ -\sum_{x\in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right) (W(\phi(x)) - \gamma) \right\} \\
&\leq \exp\left\{ (-\alpha+\gamma+\varepsilon) \sum_{x\in D_N} Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right) \right\} \\
&\quad \times \exp\left\{ \sum_{x\in D_N} \left\{ Q\left(\frac{x}{N}, \frac{1}{N}\phi(x)\right) (\alpha-\beta-\varepsilon) 1_{\{\phi(x)\leq K\}} \right\} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{(-\alpha+\gamma+\varepsilon)\sum_{x\in D_N} Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)\right\}\\
&\quad\times\sum_{\Lambda\subset D_N}\prod_{x\in\Lambda}\left(e^{(\alpha-\beta-\varepsilon)Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)}-1\right)1_{\{\phi(x)\le K\}}.
\end{aligned}
$$

Now, if $\phi(x)\le K$ for $x\in L_N^+$, then $\frac{1}{N}\phi(x)-g\left(\frac{x}{N}\right)<-\frac{1}{2}\sqrt{\delta}$ for N large enough. Thus, since $\|\overline{h}^N-\overline{g}^N\|_{L^2(D)}<\frac{1}{C_0}(\delta+\|g-g^N\|_{L^2(D)})$, if $\phi(x)\le K$ for every $x\in\Lambda\subset L_N^+$ on $\{h^N\in B_2(g,\delta)\}$, then we have for N large enough

$$
\frac{2\delta^2}{C_0^2}>\frac{1}{N^d}\sum_{x\in D_N}\left(\frac{1}{N}\phi(x)-g\left(\frac{x}{N}\right)\right)^2>\frac{|\Lambda|\delta}{4N^d},
$$

namely, $|\Lambda|<\frac{8N^d\delta}{C_0^2}$, where $C_0=C_0(d,p)>0$ is the constant, see in [22, p.188]. Combining these all facts and Lemma 5.2.2

$$
\begin{aligned}
&\frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}}\mu_N^{\psi,S}(h^N\in B_2(g,\delta))\\
\le&\frac{1}{Z_N^{\psi,\gamma Q}}\int_{\mathbb{R}^{D_N}}1_{\{h^N\in B_2(g,\delta)\}}e^{-H_N^{\psi}(\phi)-\sum_{x\in D_N}\gamma Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)}e^{(-\alpha+\gamma+\varepsilon)\sum_{x\in D_N}Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)}\\
&\times\sum_{\Lambda\subset L_N^+;|\Lambda|<\frac{8N^d\delta}{C_0^2}}\prod_{x\in\Lambda}\left(e^{(\alpha-\beta-\varepsilon)Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)}-1\right)\\
&\times\sum_{\Lambda'\subset I_N\cup L_N^-}\prod_{x\in\Lambda'}\left(e^{(\alpha-\beta-\varepsilon)Q\left(\frac{x}{N},\frac{1}{N}\phi(x)\right)}-1\right)1_{\{\phi(x)\le K\text{ for every }x\in\Lambda\cup\Lambda'\}}\prod_{x\in D_N}d\phi(x)\\
\le&\exp\left\{2CN^d\delta+N^d\int_D Q(\theta,g(\theta))\left((-\alpha+\gamma+\varepsilon)+(\alpha-\beta-\varepsilon)1_{\{g(\theta)\le\sqrt{\delta}\}}\right)d\theta\right\}\\
&\times\left(e^{(\alpha-\beta-\varepsilon)\|Q\|_\infty}-1\right)^{\frac{8N^d\delta}{C_0^2}}|\{\Lambda\subset L_N^+:|\Lambda|<8N^d\delta C_0^{-2}\}|\mu_N^{\psi,\gamma Q}(h^N\in B_2(g,\delta)).
\end{aligned}
$$

On the other hand, by using Stirling's formula, we see that

$$
|\{\Lambda\subset L_N^+:|\Lambda|<8N^d\delta C_0^{-2}\}|\le\left(\frac{C}{\delta}\right)^{CN^d\delta}N^d(1+o(1))
$$

as $N\to\infty$, for some constant $C>0$ independent of N and δ (cf. [22, p.189]). Hence, by the large deviation principle upper bound for $\mu_N^{\psi,\gamma Q}$ (Proposition 5.2.1), we obtain

$$
\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N^d}\log\frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}}\mu_N^{\psi,S}(h^N\in B_2(g,\delta))\\
\le&-\inf_{h\in\bar{B}_2(g,\delta)}I^{\gamma Q}(h)+\int_D Q(\theta,g(\theta))\left((-\alpha+\gamma+\varepsilon)+(\alpha-\beta-\varepsilon)1_{\{g(\theta)\le\sqrt{\delta}\}}\right)d\theta+C(\delta),
\end{aligned}
$$

where $C(\delta)$ is a constant independent of N and converges to 0 as $\delta \to 0$. Then, by using the lower semi-continuity of $I^{\gamma Q}(h)$, we see that for every $g \in L^2(D)$ and $\varepsilon > 0$, there exists $\delta > 0$ small enough such that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in B_2(g,\delta)) \\
&\quad \leq - \left\{ I^{\gamma Q}(g) + \int_D Q(\theta, g(\theta)) \left((\alpha - \gamma) + (\beta - \alpha) 1_{\{g(\theta) \leq 0\}} \right) d\theta \right\} + \varepsilon |D| \|Q\|_\infty.
\end{aligned}$$

Therefore, the standard argument in the theory of large deviation principle yields

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{C}) \\
&\quad \leq - \inf_{h \in \mathcal{C}} \left\{ I^{\gamma Q}(h) + \int_D Q(\theta, h(\theta)) \left((\alpha - \gamma) + (\beta - \alpha) 1_{\{h(\theta) \leq 0\}} \right) d\theta \right\},
\end{aligned}$$

for every compact set $\mathcal{C}$ of $L^2(D)$. The exponential tightness for $\mu_N^{\psi,S}$ can be proved in a similar way to those for μ_N^{ψ} (cf. Remark 4.1 of [22]). Thus, for every closed set $\mathcal{C}$ of $L^2(D)$, we get

$$\limsup_{N\to\infty} \frac{1}{N^d} \log \frac{Z_N^{\psi,S}}{Z_N^{\psi,\gamma Q}} \mu_N^{\psi,S}(h^N \in \mathcal{C}) \leq - \inf_{h \in \mathcal{C}} \Sigma^S(h) + \inf_{h \in H_g^1(D)} \Sigma^{\gamma Q}(h). \tag{5.2.6}$$

Taking $\mathcal{O} = \mathcal{C} = L^2(D)$ in (5.2.5) and (5.2.6), we have the conclusion. $\square$

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