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A Stochastic Representation for Fully Nonlinear PDEs and Its Application to Homogenization

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Abstract. We establish a stochastic representation formula for solutions to fully nonlinear second-order partial differential equations of parabolic type. For this purpose, we introduce forward-backward stochastic differential equations with random coefficients. We next apply them to homogenization of fully nonlinear parabolic equations. As a byproduct, we obtain an estimate concerning the convergence rate of solutions. The results partially generalize homogenization of Hamilton-Jacobi-Bellman equations studied by R. Buckdahn and the author.

Introduction

In this paper, we consider the following second-order partial differential equations (PDEs) of parabolic type

$$
(0.1)\qquad \begin{cases} -u_t + H\big(x, u, u_x, u_{xx}\big) = 0\,, & \text{in} \quad [0,T)\times\mathbb{R}^d\,,\\ u(T,x) = h(x)\,, & \text{on} \quad \mathbb{R}^d\,, \end{cases}
$$

where u_t denotes the partial derivative of u in t, and $u_x = (u_{x^i})$ and $u_{xx} = (u_{x^i x^j})$ stand for its first and second derivatives in x, respectively. The function H on $\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ is called *Hamiltonian* of equation (0.1), where $\mathbb{S}^d$ is the totality of all symmetric d-dimensional matrices, which is considered as a subset of $\mathbb{R}^{d\times d}$.

The present paper consists of two principal sections except for this introductory section. Section 1 is concerned with a stochastic representation for solutions to PDE (0.1). It is well known by the theory of forward-backward

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stochastic differential equations (FBSDEs, for short) that if the Hamiltonian of (0.1) is of the form

$$H(x,y,p,X) := -\frac{1}{2}\sum_{i,j,k=1}^{d}(\sigma^{ik}\sigma^{jk})(x,y)X_{ij} - \sum_{i=1}^{d} b^i(x,y)p_i - f(x,y,p) \tag{0.2}$$

with suitable functions σ^{ij}, b^i and f, then we have the following stochastic representation for solutions to quasilinear PDE (0.1) through the so-called nonlinear Feynman-Kac formula:

$$u(s,X_s) = Y_s\,, \qquad u_x(s,X_s) = Z_s\,, \tag{0.3}$$

where $u(t,x)$ is a solution of PDE (0.1)-(0.2) and $(Y_s, Z_s)_{s\in[t,T]}$ is a unique pair of adapted solution to the following FBSDE

$$\begin{cases} dX_s = b(X_s,Y_s)\,ds + \sigma(X_s,Y_s)\,dW_s\,, & X_t = x\,,\\ -dY_s = f(X_s,Y_s,Z_s)\,ds - \sigma^*(X_s,Y_s)Z_s\,dW_s\,, & Y_T = h(X_T)\,. \end{cases} \tag{0.4}$$

Here $W = (W_s)$ denotes a d-dimensional Brownian motion on a probability space and σ^* is the transposed matrix of σ. Note that FBSDE (0.4) must be solved as a triplet $(X_\cdot, Y_\cdot, Z_\cdot)$ of adapted processes. We refer to [17] and [21] for more information about this subject.

It is also known that Hamilton-Jacobi-Bellman equations can be represented in a similar manner; let $u(t,x)$ be a solution of (0.1) with the Hamiltonian defined by

$$H(x,y,p,X) := \sup_{\alpha\in U}\left\{-\frac{1}{2}\sum_{i,k,j=1}^{d}(\sigma^{ik}\sigma^{jk})(x,\alpha)X_{ij} - \sum_{i=1}^{d} b^i(x,\alpha)p_i - f(x,y,p,\alpha)\right\}, \tag{0.5}$$

where the parameter α lies in an index set U. We denote by $(X_s^\alpha, Y_s^\alpha, Z_s^\alpha)$ a unique adapted solution to the following decoupled FBSDEs associated with an adapted control process (α_s) with values in U

$$(0.6)\qquad \begin{cases} dX_s^\alpha = b(X_s^\alpha, \alpha_s)\, ds + \sigma(X_s^\alpha, \alpha_s)\, dW_s\,, & X_t^\alpha = x\,, \\ -dY_s^\alpha = f(X_s^\alpha, Y_s^\alpha, Z_s^\alpha, \alpha_s)\, ds & \\ \qquad\qquad - \sigma^*(X_s^\alpha, \alpha_s) Z_s^\alpha\, dW_s\,, & Y_T^\alpha = h(X_T^\alpha)\,. \end{cases}$$

Then, we have the identity

$$(0.7)\qquad u(t,x) = \inf_\alpha Y_t^\alpha\,,$$

where the infimum is taken over all admissible control processes (see [7] and [22] for details).

The first objective of this paper is to obtain such representation formula for solutions to more general fully nonlinear nondegenerate parabolic PDEs by the BSDE point of view. Roughly speaking, we consider as Hamiltonian any function H such that $H = H(x, y, p, X)$ is of C^2-class, convex in X, and uniformly Lipschitz continuous with respect to (y, p, X) (see (A1)-(A6) of Assumption 1.1). We try to find an appropriate FBSDE of the form (0.6) such that the value $\inf_\alpha Y_t^\alpha$ becomes a solution of the corresponding fully nonlinear PDE. The point is that, if H is convex in X, we can rewrite the Hamiltonian H as that of Bellman type (i.e. "sup" or "inf" type).

Section 2 is concerned with homogenization of fully nonlinear PDEs. The literatures [5], [6], [10], [14] and [20] study homogenization of semilinear and quasilinear PDEs by BSDE approaches (see also [4], [13], [15] and [19] for classical results on homogenization of linear second-order PDEs). The literature [7] treats homogenization of HJB equations, which is a typical example of fully nonlinear equations, by some probabilistic arguments owing to the representation (0.7).

Our purpose is to extend these results, especially that of [7], to more general fully nonlinear PDEs; we consider the following PDEs with small parameter $\varepsilon > 0$

$$(0.8)\qquad \begin{cases} -u_t + H\big(\varepsilon^{-1}x, u, u_x, u_{xx}\big) = 0\,, & \text{in} \quad [0,T) \times \mathbb{R}^d\,, \\ u(T, x) = h(x)\,, & \text{on} \quad \mathbb{R}^d\,, \end{cases}$$

where the Hamiltonian H satisfies Assumption 1.1 below and is supposed to be $\mathbb{Z}^d$-periodic in the first variable, i.e. periodic with period 1 for all components in the first variable. We are interested in the convergence of the family of solutions $\{u^\varepsilon; \varepsilon > 0\}$ as ε tends to zero, as well as specifying the effective Hamiltonian of the limit equation. By virtue of the representation formula obtained in Section 1, it turns out that we can execute the BSDE approach nearly in the same way as in [7]. The point is that we can choose appropriate FBSDEs associated with (0.8) uniformly in $\varepsilon > 0$ in some sense, which makes us possible to take the limit $\varepsilon \downarrow 0$ successfully. We also characterize the effective Hamiltonian precisely in Theorem 2.1. As a byproduct of this approach, we obtain an estimate on the convergence rate of solutions (Corollary 2.7). Such kind of rate of convergence for first-order PDEs has been investigated recently in [8]. However, as far as we know, it has not been well studied for second-order PDEs except some trivial cases (cf. [18]. See also Remark 2.8). In this paper, we investigate it in a straightforward and intuitive way with the aid of probabilistic tools.

Before closing this introductory section, we point out that the investigation of homogenization by analytic approaches is also an interesting subject. Especially, the viscosity solution method might be the most powerful one. In fact, it is by this approach that the homogenization of fully nonlinear second-order PDEs was successfully investigated for the first time; in [11], Evans establishes the so-called perturbed test function method based on the theory of viscosity solution. In the same spirit but in a more refined and unified manner, Alvarez and Bardi prove homogenization of a large class of fully nonlinear possibly degenerate second-order parabolic PDEs with periodic structure. We cite [1], [2] and [12] for the study of homogenization in this direction.

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1. Stochastic Representation

We begin this section with some notation. For elements $\xi = (\xi^1, \ldots, \xi^d)$ and $\eta = (\eta^1, \cdots, \eta^d)$ in $\mathbb{R}^d$, we denote the canonical inner product by $\xi \cdot \eta := \sum_{i=1}^d \xi^i \eta^i$ and its induced norm by $|\xi| := \sqrt{\xi \cdot \xi}$, respectively. We keep the same symbols for Euclidean spaces of different dimensions. We often use the summation convention if the same indices are repeated: $a^{ij} X_{ij} := \sum_{i,j=1}^d a^{ij} X_{ij}$, $a^{ij}\xi^i\xi^j := \sum_{i,j=1}^d a^{ij}\xi^i\xi^j$, etc.

Now, we give the precise conditions of the Hamiltonian in (0.1) that we assume throughout this paper.

ASSUMPTION 1.1. *There exist constants k, K and $\nu > 0$ such that $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$ satisfies the following conditions:*
(A1) *H is twice continuously differentiable with respect to all variables, and all second derivatives are bounded.*
(A2) *H is convex in X.*
(A3) *For every (x, y, p, X) and $\xi \in \mathbb{R}^d$,*

$$\nu|\xi|^2 \leq H(x, y, p, X) - H(x, y, p, X + \xi \otimes \xi) \leq \nu^{-1}|\xi|^2\,,$$

where $\xi \otimes \xi$ stands for the $(d \times d)$-matrix defined by $(\xi \otimes \xi)_{ij} := \xi^i \xi^j$.
(A4) *For every (y, p, X), (y', p', X') and x,*

$$|H(x, y, p, X) - H(x, y', p', X')| \leq K\{|y - y'| + |p - p'| + |X - X'|\}\,.$$

(A5) *$|H(x, 0, 0, 0)| \leq K$.*
(A6) *For every x, x' and (y, p, X),*

$$|H(x, y, p, X) - H(x', y, p, X)| \leq k(1 + |p| + |X|)|x - x'|\,.$$

Let us consider PDE (0.1) with a given terminal condition $h \in C_b^3(\mathbb{R}^d)$ and a Hamiltonian H satisfying Assumption 1.1, where $C_b^3(\mathbb{R}^d)$ stands for the set of all bounded and three times continuously differentiable functions of which all derivatives of order less than or equal to three are bounded. Remark that under these conditions, PDE (0.1) has a unique solution in the Hölder space $C^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{R}^d)$ for some $\delta \in (0, 1)$ (see for example [16], [23] and [24]). Remark also that under (A1), the conditions (A2), (A3) and (A4) are equivalent to the following (A2'), (A3') and (A4'), respectively (cf. [24]):
(A2') For every $Y = (Y_{ij}) \in \mathbb{S}^d$, we have

$$\sum_{i,j,k,l=1}^{d} H_{X_{ij} X_{kl}}(x, y, p, X) Y_{ij} Y_{kl} \geq 0\,,$$

where $H_{X_{ij}X_{kl}}$ denotes the second derivative of H with respect to X_{ij} and X_{kl}.
(A3') $a^{ij}(x,y,p,X) := H_{X_{ij}}(x,y,p,X)$ is symmetric and satisfies

$$\nu|\xi|^2 \leq a^{ij}(x,y,p,X)\xi^i\xi^j \leq \nu^{-1}|\xi|^2 \tag{1.1}$$

with the same ν as in (A3).
(A4') $|H_y|$, $|H_{p^i}|$, $|H_{X_{ij}}| \leq K$ with the same K as in (A4).

This observation leads us easily the following lemma.

Lemma 1.2. *Let us set $E := \mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$. Then, there exist a bounded and continuous function $a : \mathbb{R}^d\times E \longrightarrow \mathbb{S}^d$ satisfying (1.1) and a continuous function $f : \mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times E \longrightarrow \mathbb{R}$ such that for every (x,y,p,X), H is represented as*

$$H(x,y,p,X) = \max_{\zeta\in E}\{-a^{ij}(x,\zeta)X_{ij} - f(x,y,p,\zeta)\}\,, \tag{1.2}$$

and the maximum is attained when $\zeta = (-y,-p,-X)$. Moreover, $a = (a^{ij})$ and f can be taken so that a^{ij} is Lipschitz continuous with respect to x uniformly in ζ, f is Lipschitz continuous with respect to (y,p) uniformly in (x,ζ), and under the notation $\zeta = (\alpha,\beta,\gamma) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$, f satisfies

$$\begin{aligned}-K(\,1+\min\{|y|,|\alpha|\}+\min\{|p|,|\beta|\}\,) &\leq f(x,y,p,\zeta)\\ &\leq \tilde{K}(1+|y|+|p|+|\zeta|)\,,\end{aligned} \tag{1.3}$$

where $\tilde{K}$ is a constant depending only on K.

Proof. We set $\tilde{H}(x,y,p,X) := H(x,-y,-p,-X)$. Clearly, $\tilde{H}$ satisfies (A1)-(A6) with $\tilde{H}$ in place of H. Then, by (A4), we can easily show that

$$\tilde{H}(x,y,p,X) = \max_{(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}^d}\{\tilde{H}(x,\alpha,\beta,X) - K|\alpha-y| - K|\beta-p|\}\,, \tag{1.4}$$

where the right-hand side attains its maximum when $(\alpha,\beta) = (y,p)$. Since $\tilde{H}$ satisfies (A1) and (A2'), we also have the equality

$$\tilde{H}(x,\alpha,\beta,X) = \max_{\gamma\in\mathbb{S}^d}\{\tilde{H}_{X_{ij}}(x,\alpha,\beta,\gamma)(X_{ij}-\gamma_{ij}) + \tilde{H}(x,\alpha,\beta,\gamma)\}\,. \tag{1.5}$$

Note that the maximum of the right-hand side is reached when $\gamma = X$. By plugging (1.5) into (1.4), we have the representation

$$\begin{aligned} &H(x,y,p,X) = \tilde{H}(x,-y,-p,-X) \\ &= \max_{\zeta\in E} \{\tilde{H}_{X_{ij}}(x,\zeta)(-X_{ij}) - \tilde{H}_{X_{ij}}(x,\zeta)\gamma_{ij} + \tilde{H}(x,\zeta) - K|\alpha+y| - K|\beta+p|\}\,, \end{aligned}$$

where $\zeta = (\alpha,\beta,\gamma)$. Thus, in order to get (1.2), it suffices to set $a^{ij}(x,\zeta) := \tilde{H}_{X_{ij}}(x,\zeta)$ and

$$f(x,y,p,\zeta) := \tilde{H}_{X_{ij}}(x,\zeta)\gamma_{ij} - \tilde{H}(x,\zeta) + K|\alpha+y| + K|\beta+p|\,.$$

The continuity of a^{ij}, f and the ellipticity of a^{ij} are obvious by (A1) and (A3'), respectively. Furthermore, from (A1) and (A4'), we can easily check that (a^{ij}) is bounded and Lipschitz continuous with respect to x uniformly in ζ, and f satisfies

$$\begin{aligned} &|f(x,y,p,\zeta) - f(x,y',p',\zeta)| \le K\{|y-y'| + |p-p'|\}\,, \\ &|f(x,y,p,\zeta)| \le \tilde{K}(1+|y|+|p|+|\zeta|) \end{aligned} \tag{1.6}$$

for some $\tilde{K}$ which depends only on K. The first inequality in (1.3) can be verified as follows. From (1.2) with $X = 0$, we can see $f(x,y,p,\zeta) \ge -H(x,y,p,0)$. By using (A4), we get $f(x,y,p,\zeta) \ge -K(1+|y|+|p|)$. On the other hand, (1.5) with $X = 0$ yields $\tilde{H}_{X_{ij}}(x,\zeta)\gamma_{ij} - \tilde{H}(x,\zeta) \ge -\tilde{H}(x,y,p,0)$, which implies, by the definition of f and (A4), that

$$\begin{aligned} f(x,y,p,\zeta) &\ge -\tilde{H}(x,y,p,0) + K\{|\alpha+y| + |\beta+p|\} \\ &\ge -\tilde{H}(x,-\alpha,-\beta,0) \\ &\ge -K(1+|\alpha|+|\beta|)\,. \end{aligned}$$

Hence we have completed the proof. □

Let $\sigma = (\sigma^{ij}) : \mathbb{R}^d \times E \longrightarrow \mathbb{R}^{d\times d}$ be a bounded and continuous function which satisfies $\sum_{k=1}^d (\sigma^{ik}\sigma^{jk})(x,\zeta) = 2a^{ij}(x,\zeta)$. Remark that σ can be chosen so that σ is invertible and Lipschitz continuous with respect to x uniformly in ζ (see for example Section 5.2 of [25]). Let $(\Omega,\mathcal{F},P;W)$ be a probability space with d-dimensional Brownian motion. For $0 \le t \le s \le T$, we set $W_{t,s} := W_s - W_t$ and denote by $\mathcal{F}_{t,s}$ the filtration generated by $(W_{t,r})_{t\le r\le s}$ and augmented by all P-null sets in Ω.

Fix an arbitrary point $(t,x) \in [0,T] \times \mathbb{R}^d$ and consider the following decoupled FBSDE

$$
\begin{cases} dX_s^\zeta = \sigma(X_s^\zeta, \zeta_s)\, dW_{t,s}\,, & X_t^\zeta = x\,, \\ -dY_s^\zeta = f(X_s^\zeta, Y_s^\zeta, Z_s^\zeta, \zeta_s)\, ds & \\ \qquad\qquad - \sigma^*(X_s^\zeta, \zeta_s) Z_s^\zeta\, dW_{t,s}\,, & Y_T^\zeta = h(X_T^\zeta)\,, \end{cases} \tag{1.7}
$$

where $\zeta : \Omega \times [t,T] \longrightarrow E$ is a given $\mathcal{F}_{t,s}$-adapted process such that $E\int_0^T |\zeta_s|^2 ds < \infty$. Then, the classical theory of BSDEs tells us that (1.7) has a unique adapted solution $(X^\zeta, Y^\zeta, Z^\zeta)$ satisfying

$$
E \sup_{t \le s \le T} |X_s^\zeta|^2 + E \sup_{t \le s \le T} |Y_s^\zeta|^2 + E \int_t^T |Z_s^\zeta|^2\, ds < \infty\,.
$$

Now, we define the value function $u(t,x)$ associated with (1.7) by

$$
u(t,x) := \inf_\zeta Y_t^\zeta\,, \tag{1.8}
$$

where the infimum is taken over all $\mathcal{F}_{t,s}$-adapted processes satisfying $E\int_t^T |\zeta_s|^2 ds < \infty$. Remark that the right-hand side of (1.8) is deterministic by definition. We claim here that (1.8) is well-defined for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Indeed, under the notation $y_+ := \max\{y, 0\}$, $y_- := \max\{-y, 0\}$ for $y \in \mathbb{R}$, and $\mathbf{1}_-(y) := 0$ if $y > 0$ and $\mathbf{1}_-(y) := 1$ if $y \le 0$, we can show, by applying Ito's formula to $(Y_s^\zeta)_-^2$, that

$$
\begin{aligned}
(Y_s^\zeta)_-^2 = (Y_T^\zeta)_-^2 &- 2 \int_s^T (Y_r^\zeta)_- f(X_r^\zeta, Y_r^\zeta, Z_r^\zeta, \zeta_r)\, dr \\
&+ 2 \int_s^T (Y_r^\zeta)_- \sigma^*(X_r^\zeta, \zeta_r) Z_r^\zeta\, dW_{t,r} \\
&- \int_s^T \mathbf{1}_-(Y_r^\zeta) |\sigma^*(X_r^\zeta, \zeta_r) Z_r^\zeta|^2\, dr\,.
\end{aligned}
$$

Notice that although the function $(y)_-^2$ does not belong to $C^2(\mathbb{R})$, we can justify the above equality by approximation. Taking account of (1.6), the first inequality in (1.3) and the facts that $y_+ y_- = 0$ and $y_- = y_- \mathbf{1}_-(y)$, we can verify that

$$
\begin{aligned}
-(Y_r^\zeta)_- f(X_r^\zeta, Y_r^\zeta, Z_r^\zeta, \zeta_r) &\le -(Y_r^\zeta)_- f(X_r^\zeta, (Y_r^\zeta)_+, Z_r^\zeta, \zeta_r) + K (Y_r^\zeta)_-^2 \\
&\le (Y_r^\zeta)_- K(1 + (Y_r^\zeta)_+) \\
&\qquad + K (Y_r^\zeta)_- \mathbf{1}_-(Y_r^\zeta) |Z_r^\zeta| + K (Y_r^\zeta)_-^2
\end{aligned}
$$

$$\leq K'(1+(Y_r^\zeta)_-^2)+\frac{\nu}{2}\mathbf{1}_-(Y_r^\zeta)|Z_r^\zeta|^2$$

for some constant K' depending only on K and ν. Since $Y_T^\zeta = h(X_T^\zeta)$ is bounded and $\sigma\sigma^*$ is uniformly elliptic, we obtain

$$E(Y_s^\zeta)_-^2+\nu E\int_s^T \mathbf{1}_-(Y_r^\zeta)|Z_r^\zeta|^2\,dr \leq K''+K''\int_s^T E(Y_r^\zeta)_-^2\,dr$$

for some constant K'' depending only on K, ν, T and the bound of h. The Gronwall lemma implies that $E(Y_t^\zeta)_-^2$ is bounded from above by a constant independent of ζ. In particular, $-(Y_t^\zeta)_-$ is bounded from below uniformly in ζ. Thus, (1.8) is well-defined.

We are in position to state the main result of this section.

THEOREM 1.3. *Let $u(t,x)$ be the function defined by (1.7)-(1.8). Then, u satisfies PDE (0.1) in the classical sense.*

PROOF. We denote by $v(t,x)$ the unique classical solution of PDE (0.1). We shall show $u(t,x)=v(t,x)$ for each fixed (t,x). Let $(X^\zeta,Y^\zeta,Z^\zeta)$ be a solution of FBSDE (1.7) with a given control process ζ, and we set $\overline{Y}_s^\zeta := Y_s^\zeta - v(s,X_s^\zeta)$ and $\overline{Z}_s^\zeta := Z_s^\zeta - v_x(s,X_s^\zeta)$. Then, by applying Ito's formula to $v(s,X_s^\zeta)$, we can easily check that $(\overline{Y}^\zeta,\overline{Z}^\zeta)$ satisfies the following linear BSDE

$$\begin{cases} -d\overline{Y}_s^\zeta = \{\theta(s,X_s^\zeta,\zeta_s)+\phi_s^\zeta\overline{Y}_s^\zeta+\psi_s^\zeta\overline{Z}_s^\zeta\}\,ds-\sigma^*(X_s^\zeta,\zeta_s)\overline{Z}_s^\zeta\,dW_{t,s}\,, \\ \overline{Y}_T^\zeta = 0\,. \end{cases}$$

Here, the function $\theta : [0,T]\times\mathbb{R}^d\times E \longrightarrow \mathbb{R}$ and bounded processes (ϕ_s^ζ) and (ψ_s^ζ) are defined by

$$\begin{aligned} \theta(s,x,\zeta) &:= H(x,v(s,x),v_x(s,x),v_{xx}(s,x)) \\ &\qquad + a^{ij}(x,\zeta)v_{x^ix^j}(s,x)+f(x,v(s,x),v_x(s,x),\zeta)\,, \\ \phi_s^\zeta &:= \int_0^1 f_y(X_s^\zeta,\lambda Y_s^\zeta+(1-\lambda)v(s,X_s^\zeta),v_x(s,X_s^\zeta),\zeta_s)\,d\lambda\,, \\ \psi_s^\zeta &:= \int_0^1 f_p(X_s^\zeta,Y_s^\zeta,\lambda Z_s^\zeta+(1-\lambda)v_x(s,X_s^\zeta),\zeta_s)\,d\lambda\,, \end{aligned}$$

where f_y and $f_p = (f_{p^1}, \dots, f_{p^d})$ are partial derivatives of f with respect to y and p, respectively. From the classical theory of linear BSDEs, $\overline{Y}_t^\zeta$ can be represented as

$$\overline{Y}_t^\zeta = E \int_t^T \Gamma_s^\zeta \, \theta(s, X_s^\zeta, \zeta_s)\, ds \,,$$

$$\Gamma_s^\zeta := \exp\left(\int_t^s \phi_r^\zeta \, dr + \int_t^s \sigma^{-1}(X_r^\zeta, \zeta_r)\psi_r^\zeta \, dW_{t,r} - \frac{1}{2}\int_t^s |\sigma^{-1}(X_r^\zeta, \zeta_r)\psi_r^\zeta|^2 \, dr \right) . \tag{1.9}$$

Since $\theta(s,x,\zeta) \geq 0$ and $\Gamma_s^\zeta > 0$ by definition, we have $\inf_\zeta \overline{Y}_t^\zeta \geq 0$.

On the other hand, we claim that for any small $\rho > 0$, we can construct an adapted control process (ζ_s^*) such that $\theta(s, X_s^{\zeta^*}, \zeta_s^*) < \rho$, where $(X_s^{\zeta^*})$ stands for a solution of forward SDE in (1.7) associated with (ζ_s^*). The idea is as follows. We construct a feed-back control of the form $\zeta_s = (-v(s, X_s^\zeta), -v_x(s, X_s^\zeta), -v_{xx}(s, X_s^\zeta))$ so that $\theta(s, X_s^\zeta, \zeta_s) = 0$ (remind that the maximum in (1.2) is attained when $\zeta = (-y, -p, -X)$). For this purpose, we consider the following SDE

$$dX_s = \tilde{\sigma}(s, X_s)\, dW_{t,s} \,, \qquad X_t = x \,, \tag{1.10}$$

where $\tilde{\sigma}(s,x) := \sigma(x, -v(s,x), -v_x(s,x), -v_{xx}(s,x))$. The continuity of $\tilde{\sigma}$ with respect to (s,x) implies that (1.10) has at least one weak solution. Thus, in order to get the desired control, it suffices to put $\zeta_s := (-v(s,X_s), -v_x(s,X_s), -v_{xx}(s,X_s))$. Nevertheless, since we would like to keep the same Brownian motion and the associated Brownian filtration, we construct a "ρ-optimal" control (ζ_s^*) without changing probability space. Such construction is always possible by choosing an appropriate step control and solving the corresponding SDE step by step (cf. Appendix of [7]). Thus, we have

$$0 \leq \inf_\zeta \overline{Y}_t^\zeta \leq \overline{Y}_t^{\zeta^*} < T\rho \, E \sup_{t \leq s \leq T} \Gamma_s^\zeta \,.$$

Since $E \sup_{t\leq s\leq T} \Gamma_s^\zeta$ is bounded by a constant depending only on K and ν, and since ρ is arbitrary, we finally obtain $u(t,x) - v(t,x) = \inf_\zeta \overline{Y}_t^\zeta = 0$. We have completed the proof. □

2. Application to Homogenization

Let H be a given Hamiltonian satisfying Assumption 1.1, and let $h \in C_b^3(\mathbb{R}^d)$ be a given terminal function. For each $\varepsilon > 0$, we consider PDE (0.8). Throughout this section, we assume that the Hamiltonian $H = H(\eta, y, p, X)$ is $\mathbb{Z}^d$-periodic with respect to η. The aim of this section is to show the following theorem.

THEOREM 2.1. *Let $u^\varepsilon(t,x)$ be a solution of PDE (0.8). Then, for every $(t,x) \in [0,T] \times \mathbb{R}^d$, the family of solutions $\{u^\varepsilon(t,x); \varepsilon > 0\}$ converges to $u^0(t,x)$ as ε goes to zero, where $u^0(t,x)$ is a unique classical solution of the PDE*

$$\tag{2.1} \begin{cases} -u_t + \overline{H}\big(u, u_x, u_{xx}\big) = 0\,, & \textit{in} \quad [0,T) \times \mathbb{R}^d\,, \\ u(T,x) = h(x)\,, & \textit{on} \quad \mathbb{R}^d\,. \end{cases}$$

The effective Hamiltonian $\overline{H} = \overline{H}(y,p,X)$ is defined as a unique constant of the following cell problem

$$\tag{2.2} \overline{H} = H(\eta, y, p, X + v_{\eta\eta}(\eta))\,, \qquad (v(\cdot), \overline{H}) : \textit{unknown.}$$

REMARK 2.2. The cell problem (2.2) is naturally led as follows. By considering, as usual, the formal asymptotic expansion of the form

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon\, v_0(t,x,\varepsilon^{-1}x) + \varepsilon^2 v(t,x,\varepsilon^{-1}x) + \cdots\,,$$

and plugging it into (0.8), it turns out that v_0 must be zero so that the limit as $\varepsilon \to 0$ makes sense and that the equality

$$-u_t^0 + H\big(\varepsilon^{-1}x, u^0(t,x), u_x^0(t,x), u_{xx}^0(t,x) + v_{\eta\eta}(t,x,\varepsilon^{-1}x)\big) + O(\varepsilon) = 0$$

holds. Therefore, if u^0 is a solution of (2.1), $\overline{H}$ and $v(\cdot)$ must satisfy (2.2).

Now, in order to prove Theorem 2.1 rigorously, we must check
(a) Solvability of the cell problem (2.2), that is, well-definedness of $\overline{H}$.
(b) Solvability of the limit equation (2.1).
(c) Convergence of solutions $u^\varepsilon(t,x) \longrightarrow u^0(t,x)$ as $\varepsilon \downarrow 0$.
Concerning (a), it is well-known that for every (y,p,X), there exist a unique

constant $\overline{H}$ such that (2.2) has a unique $\mathbb{Z}^d$-periodic continuous viscosity solution $v(\cdot)$ up to an additive constant (see [2]). Moreover, by classical results on regularity property for fully nonlinear, convex and uniformly elliptic PDEs, $v(\cdot)$ is indeed of $C^{2+\bar{\delta}}$-class in η for some $\bar{\delta} \in (0,1)$ and v satisfies the following estimate:

$$|v(\cdot) - v(0)|_{C^{2,\bar{\delta}}(\mathbb{R}^d)} \leq \hat{K}(1 + |y| + |p| + |X|)\,, \tag{2.3}$$

where $\hat{K}$ is a constant independent of (y,p,X) (remind that the solution $v(\cdot)$ relies on (y,p,X)). See [1] and [3] for details about the estimate (2.3). This fact is one of the keys of our method.

On the other hand, in view of representation formula (1.2), $\overline{H}$ is also written as

$$\overline{H}(y,p,X) = \lim_{\lambda \downarrow 0} \sup_{\zeta} \left\{ -\lambda\, E \int_0^\infty e^{-\lambda s} [a^{ij}(\eta_s^\zeta, \zeta_s) X_{ij} + f(\eta_s^\zeta, y, p, \zeta_s)]\, ds \right\}\,,$$

where (ζ_s) and (η_s^ζ) stand for an E-valued control process and the corresponding controlled process which satisfies, on some probability space with Brownian motion, the following SDE

$$d\eta_s^\zeta = \sigma(\eta_s^\zeta, \zeta_s)\, dW_s\,, \qquad \eta_0^\zeta = 0\,.$$

Note that (ζ_s) is taken so that $E\int_0^\infty |\zeta_s|^2 ds < \infty$. This representation deduces easily that $\overline{H}$ satisfies (A2)-(A4) with $\overline{H}$ in place of H. Thus, by the theory of viscosity solution, the limit equation (2.1) has a unique bounded, continuous viscosity solution (e.g. [9]). Moreover, by the regularity result of Safonov ([23] and [24]), this solution belongs to $C^{1+\delta/2,2+\delta}([0,T] \times \mathbb{R}^d)$ for some $\delta \in (0,1)$, which answers question (b).

To prove (c), let us consider the following FBSDE with parameter $\varepsilon > 0$:

$$\begin{cases} dX_s^{\varepsilon,\zeta} = \sigma(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, dW_{t,s}\,, \\ -dY_s^{\varepsilon,\zeta} = f(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, Y_s^{\varepsilon,\zeta}, Z_s^{\varepsilon,\zeta}, \zeta_s)\, ds \\ \qquad\qquad - \sigma^*(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s) Z_s^{\varepsilon,\zeta}\, dW_{t,s}\,, \\ X_t^{\varepsilon,\zeta} = x\,, \qquad Y_T^{\varepsilon,\zeta} = h(X_T^{\varepsilon,\zeta})\,, \end{cases} \tag{2.4}$$

where σ and f are the functions defined in Section 1. Then, by virtue of Theorem 1.3, the solution of PDE (0.8) can be written as $u^\varepsilon(t,x) = \inf_\zeta Y_t^{\varepsilon,\zeta}$. Thus, Theorem 2.1 is reduced to the following theorem.

THEOREM 2.3. *Let $u^0(t,x)$ be a solution of PDE (2.1) and set* $\overline{Y}_s^{\varepsilon,\zeta} := Y_s^{\varepsilon,\zeta} - u^0(s, X_s^{\varepsilon,\zeta})$. *Then, we have* $\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta} \longrightarrow 0$ *as* $\varepsilon \downarrow 0$.

The proof of this theorem is divided into several parts. We reproduce some arguments used in Section 4.2 of [7]. The main difference between [7] and the present paper is that f is not bounded, as well as the control region E is not compact, and that we would like to get more sharpened estimates than that of [7] for the investigation of convergence rate. The idea is as follows. For each $(s,x) \in [0,T] \times \mathbb{R}^d$, we set

$$v(\eta, s, x) := v(\eta, u^0(s,x), u_x^0(s,x), u_{xx}^0(s,x))\,,$$

where $v(\eta, y, p, X)$ is a solution of the cell problem (2.2) corresponding to (y,p,X). Then, we apply Ito's formula to $\overline{Y}_s^{\varepsilon,\zeta} - \varepsilon^2 v(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, s, X_s^{\varepsilon,\zeta})$ in order to show

$$\lim_{\varepsilon\downarrow 0} \inf_\zeta E[\overline{Y}_s^{\varepsilon,\zeta} - \varepsilon^2 v(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, s, X_s^{\varepsilon,\zeta})] = 0\,.$$

Unfortunately, this procedure is too naive to justify since v is not differentiable in general (even not continuous) with respect to (y,p,X) (cf. Remark 2.8 below). However, we can execute a similar argument locally by freezing the slow variable $X^{\varepsilon,\zeta}$ (see Propositions 2.4 and 2.5 below).

Let us set $\overline{Z}_s^{\varepsilon,\zeta} := Z_s^{\varepsilon,\zeta} - u_x^0(s, X_s^{\varepsilon,\zeta})$. Then $(\overline{Y}_s^{\varepsilon,\zeta}, \overline{Z}_s^{\varepsilon,\zeta})$ satisfies

$$\begin{cases} -d\overline{Y}_s^{\varepsilon,\zeta} = \{\overline{\theta}(s, X_s^{\varepsilon,\zeta}, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s) + \phi_s^{\varepsilon,\zeta} \overline{Y}_s^{\varepsilon,\zeta} + \psi_s^{\varepsilon,\zeta} \overline{Z}_s^{\varepsilon,\zeta}\}\, ds \\ \qquad\qquad - \sigma^*(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s) \overline{Z}_s^{\varepsilon,\zeta}\, dW_{t,s}\,, \\ \overline{Y}_T^{\varepsilon,\zeta} = 0\,, \end{cases}$$

where the function $\overline{\theta} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times E \longrightarrow \mathbb{R}$ and bounded processes $(\phi_s^{\varepsilon,\zeta})$ and $(\psi_s^{\varepsilon,\zeta})$ are defined by

$$\begin{aligned} \overline{\theta}(s,x,\eta,\zeta) &:= \overline{H}(u^0(s,x), u_x^0(s,x), u_{xx}^0(s,x)) \\ &\qquad + a^{ij}(\eta,\zeta) u^0_{x^i x^j}(s,x) + f(\eta, u^0(s,x), u_x^0(s,x), \zeta)\,, \\ \phi_s^{\varepsilon,\zeta} &:= \int_0^1 f_y(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \lambda Y_s^{\varepsilon,\zeta} + (1-\lambda) u^0(s, X_s^{\varepsilon,\zeta}), u_x^0(s, X_s^{\varepsilon,\zeta}), \zeta_s)\, d\lambda\,, \\ \psi_s^{\varepsilon,\zeta} &:= \int_0^1 f_p(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, Y_s^{\varepsilon,\zeta}, \lambda Z_s^{\varepsilon,\zeta} + (1-\lambda) u_x^0(s, X_s^{\varepsilon,\zeta}), \zeta_s)\, d\lambda\,. \end{aligned}$$

Remark that the bounds of $\phi_s^{\varepsilon,\zeta}$ and $\psi_s^{\varepsilon,\zeta}$ are independent of $\varepsilon > 0$. Then, as in the previous section, we obtain the expression

$$\overline{Y}_t^{\varepsilon,\zeta} = E\int_t^T \Gamma_s^{\varepsilon,\zeta}\,\overline{\theta}(s, X_s^{\varepsilon,\zeta}, \varepsilon^{-1}X^{\varepsilon,\zeta}, \zeta_s)\,ds \tag{2.5}$$

with $\Gamma_s^{\varepsilon,\zeta} > 0$ defined similarly as (1.9). Moreover, for any $q \geq 1$, we can show

$$\sup_{\varepsilon>0} E \sup_{t\leq s\leq T} |\Gamma_s^{\varepsilon,\zeta}|^q < \infty\,.$$

Now we set $V(s,x,\eta,\zeta) := a^{ij}(\eta,\zeta)v_{\eta^i\eta^j}(\eta,s,x)$. Remark that V is a bounded function since v satisfies (2.3) and u^0, u_x^0 and u_{xx}^0 are bounded. The following proposition gives us a lower estimate of $\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta}$.

Proposition 2.4. *For any $\rho > 0$, there exists a partition $(t,T] = \bigcup_{j=0}^{N-1}(s_j, s_{j+1}]$ and finite Borel sets $B_1,\ B_2, \ldots, B_{N'} \in \mathcal{B}(\mathbb{R}^d)$ such that for arbitrary $x_k \in B_k$ $(k = 1, \ldots, N')$, we have*

$$\begin{aligned}\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta} + \rho \\ > -\sup_\zeta \left| \sum_{j=0}^{N-1}\sum_{k=1}^{N'} E\int_{s_j}^{s_{j+1}} \Gamma_s^{\varepsilon,\zeta} 1_{\{X_{s_j}^{\varepsilon,\zeta}\in B_k\}} V(s_j, x_k, \varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\,ds \right|.\end{aligned} \tag{2.6}$$

Proof. For $N \in \mathbb{N}$ and $n \in \mathbb{R}_+$, we consider the partition

$$(t,T] = \bigcup_{j=0}^{N-1} \Delta_j := \bigcup_{j=0}^{N-1}(s_j, s_{j+1}]\,, \qquad s_j = t + \frac{j\,(T-t)}{N}\,, \quad j = 0,1,\ldots,N\,,$$

and an open covering of $B(n) := \{\, x \in \mathbb{R}^d\,;\, |x| \leq n \,\}$ consisting of open balls in $\mathbb{R}^d$ with radius $(2n)^{-1}$. From this covering, we can construct a finite and disjoint decomposition $B(n) = \bigcup_{k=1}^{N'} B_k\,,\ B_k \in \mathcal{B}(\mathbb{R}^d)\,,\ k = 1,2,\ldots,N'$.

Now we set

$$A_n = \{\sup_{t\leq s\leq T} |X_s^{\varepsilon,\zeta}| \leq n\,\}\,, \quad B_{n,N} = \{\max_{0\leq j\leq N-1}\sup_{s\in\Delta_j} |X_s^{\varepsilon,\zeta} - X_{s_j}^{\varepsilon,\zeta}| \leq 1/n\,\}\,.$$

Then, for any given $q > 1$, Chebyshev's inequality yields

$$(2.7)\qquad \begin{aligned} P(A_n^c) &\le \frac{C(1+|x|)^{2q}}{n^{2q}}\,, \\ P(B_{n,N}^c) &\le \sum_{j=0}^{N-1} Cn^{2q}\,|s_{j+1}-s_j|^q = \frac{Cn^{2q}(T-t)^q}{N^{q-1}}\,. \end{aligned}$$

Here and in the following, we denote various constants by the same symbol C if they are independent of n, N, ε and control (ζ_s). Since u^0 belongs to $C^{1+\delta/2,2+\delta}([0,T]\times\mathbb{R}^d)$, we have

$$(2.8)\qquad |\overline{\theta}(s,x,\eta,\zeta)-\overline{\theta}(s',x',\eta,\zeta)| \le K'\{|s-s'|^{\delta/2}+|x-x'|^{\delta}\}$$

for some K' depending only on K and the $C^{1+\delta/2,2+\delta}$-norm of u^0, which implies that K' relies only on K, d, ν, δ and the $C^{1+\delta/2,2+\delta}$-norm of h.

Next, for each $k = 1,\dots,N'$, we set $C_{j,k} := \{\,X_{s_j}^{\varepsilon,\zeta} \in B_k\,\}$ and take $x_k \in B_k$ arbitrarily. Note that $A_n \subset \bigcup_{k=1}^{N'} C_{j,k}$ and $C_{jk}\cap C_{jk'} = \emptyset$ if $k \ne k'$. Then, for every $s \in \Delta_j$,

$$\begin{aligned} &\overline{\theta}(s,X_s^{\varepsilon,\zeta},\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) \\ &\quad = \sum_{k=1}^{N'} 1_{A_n\cap B_{n,N}} 1_{C_{j,k}} \{\,\overline{\theta}(s,X_s^{\varepsilon,\zeta},\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) - \overline{\theta}(s_j,x_k,\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s)\,\} \\ &\qquad + \sum_{k=1}^{N'} 1_{A_n\cap B_{n,N}} 1_{C_{j,k}} \overline{\theta}(s_j,x_k,\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) \\ &\qquad + 1_{(A_n\cap B_{n,N})^c}\,\overline{\theta}(s,X_s^{\varepsilon,\zeta},\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s)\,. \end{aligned}$$

Since $\overline{\theta}(s,x,\eta,\zeta) \ge -V(s,x,\eta,\zeta)$ for every (s,x,η,ζ), we have

$$\begin{aligned} &\overline{\theta}(s,X_s^{\varepsilon,\zeta},\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) \\ &\quad \ge \sum_{k=1}^{N'} 1_{A_n\cap B_{n,N}} 1_{C_{j,k}} \{\,\overline{\theta}(s,X_s^{\varepsilon,\zeta},\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) - \overline{\theta}(s_j,x_k,\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s)\,\} \\ &\qquad - \sum_{k=1}^{N'} 1_{C_{j,k}} V(s_j,x_k,\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) \\ &\qquad + \sum_{k=1}^{N'} 1_{(A_n\cap B_{n,N})^c} 1_{C_{j,k}} V(s_j,x_k,\varepsilon^{-1}X_s^{\varepsilon,\zeta},\zeta_s) \end{aligned}$$

$$
\begin{aligned}
&\quad - 1_{(A_n\cap B_{n,N})^c}\, V(s, X^{\varepsilon,\zeta}_s, \varepsilon^{-1}X^{\varepsilon,\zeta}_s, \zeta_s)\\
&=: \Psi^j_1(s) - \Psi^j_2(s) + \Psi^j_3(s) - \Psi^j_4(s)\,.
\end{aligned}
$$

By plugging the right-hand side into (2.5),

$$
\overline{Y}^{\varepsilon,\zeta}_t \geq \sum_{j=0}^{N-1} E\int_{s_j}^{s_{j+1}} \Gamma^{\varepsilon,\zeta}_s\{\Psi^j_1(s) - \Psi^j_2(s) + \Psi^j_3(s) - \Psi^j_4(s)\}\,ds\,.
$$

We estimate the right-hand side one by one. Note first that on the event $A_n \cap B_{n,N} \cap C_{j,k}$, we have

$$
|X^{\varepsilon,\zeta}_s - x_k| \leq |X^{\varepsilon,\zeta}_s - X^{\varepsilon,\zeta}_{s_j}| + |X^{\varepsilon,\zeta}_{s_j} - x_k| \leq 2/n \qquad \text{for all}\ \ s\in\Delta_j\,.
$$

Then, the inequality (2.8) easily yields

$$
\begin{aligned}
&\Big| E\int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s \Psi^j_1(s)\,ds\Big|\\
&\quad\leq K'\,E\Big[\int_{\Delta_j}\Gamma^{\varepsilon,\zeta}_s 1_{A_n\cap B_{n,N}}\sum_{k=1}^{N'} 1_{C_{j,k}}\,\{\,|s-s_j|^{\delta/2} + |X^{\varepsilon,\zeta}_s - x_k|^{\delta}\,\}\,ds\Big]\\
&\quad\leq C\,(s_{j+1}-s_j)\,(\,|s_{j+1}-s_j|^{\delta/2} + n^{-\delta}\,)\,.
\end{aligned}
$$

Furthermore, by using (2.7), we can see that

$$
\begin{aligned}
&\Big| E\int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s \Psi^j_4(s)\,ds\Big|\\
&\quad\leq\ |V|_{L^\infty}(s_{j+i}-s_j)\sqrt{P((A_n\cap B_{n,N})^c)}\,\sqrt{E\sup_{t\leq s\leq T}|\Gamma^{\varepsilon,\zeta}_s|^2}\\
&\quad\leq C\,|V|_{L^\infty}(s_{j+i}-s_j)\,\{n^{-q}(1+|x|)^q + n^q N^{(1-q)/2}\}\,,
\end{aligned}
$$

and in consideration of $\sum_{k=1}^{N'} 1_{C_{j,k}}|V(s_j, x_k, \varepsilon^{-1}X^{\varepsilon,\zeta}_s, \zeta_s)| \leq |V|_{L^\infty} < \infty$, we can show similarly that

$$
\Big| E\int_{\Delta_j} \Gamma^{\varepsilon,\zeta}_s \Psi^j_3(s)\,ds\Big| \leq C\,|V|_{L^\infty}(s_{j+i}-s_j)\,\{n^{-q}(1+|x|)^q + n^q N^{(1-q)/2}\}\,.
$$

Thus, we obtain

$$
\begin{aligned}
\text{(2.9)}\qquad \overline{Y}^{\varepsilon,\zeta}_t \geq &-\sum_{j=0}^{N-1} E\int_{\Delta_j}\Gamma^{\varepsilon,\zeta}_s\Psi^j_2(s)\,ds\\
&- C\,(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta}\,)
\end{aligned}
$$

for some C depending only on $|x|$, δ, K' in (2.8), T and $|V|_{L^\infty}$. Since the above inequality does not depend on the choice of control (ζ_s), we obtain (2.6) by taking n and N so that the second term of the right-hand side in (2.9) is less than $-\rho$. Hence, we have completed the proof. □

Let us now show the reverse inequality.

PROPOSITION 2.5. *For any $\rho > 0$, there exists a partition $(t,T] = \bigcup_{j=0}^{N-1}(s_j, s_{j+1}]$ and finite Borel sets $B_1, B_2, \dots, B_{N'} \in \mathcal{B}(\mathbb{R}^d)$ such that for arbitrary $x_k \in B_k$ $(k = 1, \dots, N')$, we have*

$$\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta} - \rho < \sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{s_j}^{s_{j+1}} \Gamma_s^{\varepsilon,\zeta} 1_{\{X_{s_j}^{\varepsilon,\zeta} \in B_k\}} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| .$$

PROOF. As in the proof of Proposition 2.4, we consider the N-partition $(t,T] := \bigcup_{j=0}^{N-1} \Delta_j$ and the finite and disjoint decomposition $B(n) = \bigcup_{k=1}^{N'} B_k$ for given $N \in \mathbb{N}$ and $n \in \mathbb{R}_+$. Furthermore, let us take $M \in \mathbb{N}$ and $m \in \mathbb{R}_+$, and let us consider the following sub-partition of (Δ_j) and disjoint decomposition of $[0,1)^d$:

$$\Delta_j = \bigcup_{l=0}^{M-1} I_{j,l} := \bigcup_{l=0}^{M-1} (s_j + r_l, s_j + r_{l+1}] , \qquad r_l = \frac{s_{j+1} - s_j}{M}\, l ,$$

$$[0,1)^d = \bigcup_{i=1}^{M'} E_i , \qquad E_i \in \mathcal{B}(\mathbb{R}^d) , \quad \mathrm{diam}(E_i) < 1/m ,$$

where $\mathrm{diam}(E_i) := \sup\{|e - e'|; e,\ e' \in E_i\}$ and the family of Borel sets $\{E_i\}_{i=1}^{M'}$ is constructed, as in the proof of Proposition 2.4, by a covering of $[0,1)^d$ consisting of open balls in $\mathbb{R}^d$ with radius less than $(2m)^{-1}$.

Next, we define $\underline{\zeta} : \mathbb{R}^d \times [0,T] \times \mathbb{R}^d \longrightarrow E$ by

$$\underline{\zeta}(\eta, s, x) := (-u^0(s,x), -u_x^0(s,x), -u_{xx}^0(s,x) - v_{\eta\eta}(\eta, s, x)) .$$

Recall that $v(\eta, s, x)$ is defined by $v(\eta, s, x) = v(\eta, u^0(s,x), u_x^0(s,x), u_{xx}^0(s,x))$ and $v_{\eta\eta} = (v_{\eta^i\eta^j})$ is the matrix of second derivatives with respect to η. Since u^0 is in $C^{1+\delta/2, 2+\delta}([0,T] \times \mathbb{R}^d)$ and v satisfies (2.3), we can check that $\underline{\zeta}$ is bounded with a bound depending only on $\hat{K}$ in (2.3) and the bounds of u^0, u^0_x and u^0_{xx}. Moreover, we have

$$\begin{aligned}\overline{\theta}(s,x,\eta,\underline{\zeta}(\eta,s,x)) &= -a^{ij}(\eta,\underline{\zeta}(\eta,s,x))v_{\eta^i\eta^j}(\eta,s,x)\\ &= -V(s,x,\eta,\underline{\zeta}(\eta,s,x))\,.\end{aligned} \tag{2.10}$$

For each $i = 1,\dots,M'$, we fix arbitrarily $e_i \in E_i$ and construct an $\mathcal{F}_{t,s}$-adapted step control (ζ^*_s) and the corresponding solution $(X^{\varepsilon,\zeta^*}_s)$ of the associated forward SDE in (2.4) such that

$$\begin{aligned}&\zeta^*_s := \underline{\zeta}(e_i,s_j,x_k) \quad \text{if } s \in I_{j,l}\,,\ \ X^{\varepsilon,\zeta^*}_{s_j} \in B_k \quad \text{and}\\ &\varepsilon^{-1}X^{\varepsilon,\zeta^*}_{s_j+r_l} \in E_i \pmod{\mathbb{Z}^d}\,,\end{aligned}$$

and

$$X^{\varepsilon,\zeta^*}_s = x + \int_t^s \sigma(\varepsilon^{-1}X^{\varepsilon,\zeta^*}_r,\zeta^*_r)\,dW_{t,r}\,, \qquad t \le s \le T\,.$$

Such construction is always possible by solving the above SDE step by step. Once we get a solution of forward SDE, the solvability of associated backward SDE in (2.4) is obvious. Note that ζ^*_s takes its values in a bounded region of E and the bound is independent of ε.

Now, let us "freeze" the slow variable X^{ε,ζ^*}. As in Proposition 2.4, we have

$$\begin{aligned}\overline{\theta}(s,&X^{\varepsilon,\zeta^*}_s,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s)\\ &= \sum_{k=1}^{N'} 1_{A_n\cap B_{n,N}}1_{C_{j,k}}\{\,\overline{\theta}(s,X^{\varepsilon,\zeta^*}_s,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s) - \overline{\theta}(s_j,x_k,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s)\,\}\\ &\qquad + 1_{(A_n\cap B_{n,N})^c}\,\overline{\theta}(s,X^{\varepsilon,\zeta^*}_s,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s)\\ &\qquad - \sum_{k=1}^{N'} 1_{(A_n\cap B_{n,N})^c}1_{C_{j,k}}\overline{\theta}(s_j,x_k,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s)\\ &\qquad + \sum_{k=1}^{N'} 1_{C_{j,k}}\overline{\theta}(s_j,x_k,\varepsilon^{-1}X^{\varepsilon,\zeta^*}_s,\zeta^*_s)\\ &=: \Phi^j_1(s) + \Phi^j_2(s) - \Phi^j_3(s) + \Phi^j_4(s)\,.\end{aligned}$$

For each j, l and i, let $D^j_{l,i}$ and $\Lambda^j_{m,M}$ be the events defined by

$$D^j_{l,i} := \{ \varepsilon^{-1} X^{\varepsilon,\zeta^*}_{s_j+r_l} \in E_i \quad (\mathrm{mod}\ \mathbb{Z}^d) \}\,,$$
$$\Lambda^j_{m,M} := \{ \max_{0\le l\le M-1} \sup_{s\in I_{j,i}} \left| \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s - \varepsilon^{-1} X^{\varepsilon,\zeta^*}_{s_j+r_l} \right| \le 1/m \}\,.$$

Remark that similarly to (2.7), we can show

$$\begin{aligned} P((\Lambda^j_{m,M})^c) &\le \sum_{l=0}^{M-1} \left(\frac{m}{\varepsilon}\right)^{2q} C\,|r_{l+1}-r_l|^q \\ &\le \frac{Cm^{2q}}{N^q M^{q-1}\varepsilon^{2q}}\,, \qquad q>1\,. \end{aligned} \tag{2.11}$$

Then, for all $s \in I_{j,l}$, $\Phi^j_4(s)$ can be written as

$$\begin{aligned} \Phi^j_4(s) =& \sum_{k=1}^{N'} 1_{C_{j,k}} 1_{(\Lambda^j_{m,M})^c} \overline{\theta}(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s, \zeta^*_s) \\ &+ \sum_{k=1}^{N'}\sum_{i=1}^{M'} 1_{C_{j,k}} 1_{\Lambda^j_{m,M}\cap D^j_{l,i}} \{\overline{\theta}(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s, \zeta^*_s) - \overline{\theta}(s_j, x_k, e_i, \zeta^*_s)\} \\ &+ \sum_{k=1}^{N'}\sum_{i=1}^{M'} 1_{C_{j,k}} 1_{\Lambda^j_{m,M}\cap D^j_{l,i}} \overline{\theta}(s_j, x_k, e_i, \zeta^*_s) \\ =:& \Phi^{j,l}_{41}(s) + \Phi^{j,l}_{42}(s) + \Phi^{j,l}_{43}(s)\,. \end{aligned}$$

Recall that on the event $C_{j,k} \cap D^j_{l,i}$, the control ζ^*_s is of the form $\zeta^*_s = \underline{\zeta}(e_i, s_j, x_k)$ for all $s \in I_{j,l}$. Therefore, in view of (2.10),

$$\begin{aligned} \Phi^{j,l}_{43}(s) =& \sum_{k=1}^{N'}\sum_{i=1}^{M'} 1_{C_{j,k}} 1_{\Lambda^j_{m,M}\cap D^j_{l,i}} \{V(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s, \zeta^*_s) - V(s_j, x_k, e_i, \zeta^*_s)\} \\ &+ \sum_{k=1}^{N'} 1_{C_{j,k}} 1_{(\Lambda^j_{m,M})^c} V(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s, \zeta^*_s) \\ &- \sum_{k=1}^{N'} 1_{C_{j,k}} V(s_j, x_k, \varepsilon^{-1} X^{\varepsilon,\zeta^*}_s, \zeta^*_s) \\ =:& \Phi^{j,l}_{431}(s) + \Phi^{j}_{432}(s) - \Phi^{j}_{433}(s)\,. \end{aligned}$$

Thus, plugging these equalities into (2.5), we have

$$\begin{aligned}\overline{Y}_t^{\varepsilon,\zeta^*} &= \sum_{j=0}^{N-1} E\int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta^*}\{\Phi_1^j(s)+\Phi_2^j(s)-\Phi_3^j(s)\}\,ds \\ &\quad + \sum_{j=0}^{N-1}\sum_{l=0}^{M-1} E\int_{I_{j,l}} \Gamma_s^{\varepsilon,\zeta^*}\{\Phi_{41}^{j,l}(s)+\Phi_{42}^{j,l}(s) \\ &\qquad\qquad + \Phi_{431}^{j,l}(s)+\Phi_{432}^{j}(s)-\Phi_{433}^{j}(s)\}\,ds\,.\end{aligned}$$

Since $\overline{\theta}(s,x,\eta,\underline{\zeta}(\eta',s',x'))$ is bounded uniformly in (η,s,x) and (η',s',x'), we can show as in Proposition 2.4 that

$$\begin{aligned}&\left|\sum_{j=0}^{N-1} E\int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta^*}\{\Phi_1^j(s)+\Phi_2^j(s)-\Phi_3^j(s)\}\,ds\right| \\ &\qquad \le C(n^{-q}+n^qN^{(1-q)/2}+N^{-\delta/2}+n^{-\delta})\,.\end{aligned}$$

Furthermore, (A1), (A6) and (2.3) yield

$$\begin{aligned}&|\overline{\theta}(s,x,\eta,\zeta)-\overline{\theta}(s,x,\eta',\zeta)| \le C(1+|u_x^0|+|u_{xx}^0|+|\zeta|)|\eta-\eta'|\,, \\ &|V(s,x,\eta,\zeta)-V(s,x,\eta',\zeta)| \\ &\qquad \le C(1+|u^0|+|u_x^0|+|u_{xx}^0|+|\zeta|)(|\eta-\eta'|+|\eta-\eta'|^{\bar{\delta}})\end{aligned}$$

with the same $\bar{\delta}\in(0,1)$ in (2.3). Since V and ζ_s^* are bounded uniformly in ε, we obtain, in view of the estimate (2.11), that

$$\begin{aligned}&\left|\sum_{j=0}^{N-1}\sum_{l=0}^{M-1} E\int_{I_{j,l}} \Gamma_s^{\varepsilon,\zeta^*}\{\Phi_{41}^{j,l}(s)+\Phi_{42}^{j,l}(s)+\Phi_{431}^{j,l}(s)+\Phi_{432}^{j}(s)\}\,ds\right| \\ &\qquad \le C(m^qN^{-q/2}M^{(1-q)/2}\varepsilon^{-q}+m^{-1}+m^{-\bar{\delta}})\,.\end{aligned}$$

Now, let us take $M=([m^{2(q+1)/(q-1)}]+1)([\varepsilon^{-2q/(q-1)}]+1)$, where the symbol $[x]$ stands for the integer part of $x\in\mathbb{R}$. Then, we have

$$m^qN^{-q/2}M^{(1-q)/2}\varepsilon^{-q} \le N^{-q/2}m^qm^{-(q+1)}\varepsilon^q\varepsilon^{-q} \le m^{-1}\,,$$

which implies the following estimate of $\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta}$ from above:

$$\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta} \le \overline{Y}_t^{\varepsilon,\zeta^*} \le \sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta} 1_{C_{j,k}} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| \\ + C(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta} + m^{-1} + m^{-\bar{\delta}})\,.$$

Remark that we can take the limit $m \to +\infty$ independently of n, N and ε. Thus, it remains to take n and N so that the last term is less than ρ. □

By virtue of Propositions 2.4 and 2.5, the proof of Theorem 2.3 is reduced to that of the following lemma.

LEMMA 2.6. *For each fixed N and N', we have*

$$\lim_{\varepsilon \downarrow 0} \sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{\Delta_j} 1_{C_{j,k}} \Gamma_s^{\varepsilon,\zeta} V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| = 0\,.$$

PROOF. Let us set $\overline{v}^{j,k}(\eta) = v(\eta, s_j, x_k) - v(0, s_j, x_k)$. Clearly, $\overline{v}_\eta^{j,k}(\eta) = v_\eta(\eta, s_j, x_k)$ and $\overline{v}_{\eta\eta}^{j,k}(\eta) = v_{\eta\eta}(\eta, s_j, x_k)$. Then, for every $j = 0, 1, \ldots, N-1$, $k = 1, \ldots, N'$ and (ζ_s), Ito's formula yields

$$\Gamma_{s_{j+1}}^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_{s_{j+1}}^{\varepsilon,\zeta}) - \Gamma_{s_j}^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_{s_j}^{\varepsilon,\zeta}) \\ = \frac{1}{\varepsilon^2} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, V(s_j, x_k, \varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \\ + \frac{1}{\varepsilon} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, (\sigma^* \overline{v}_\eta^{j,k})(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, dW_{t,s} \\ + \frac{1}{\varepsilon} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \sigma(\varepsilon^{-1} X_s^{\varepsilon,\zeta}, \zeta_s)\, \psi_s^{\varepsilon,\zeta} \cdot \overline{v}_\eta^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, ds \\ + \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, \psi_s^{\varepsilon,\zeta}\, dW_{t,s} + \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1} X_s^{\varepsilon,\zeta})\, \phi_s^{\varepsilon,\zeta}\, ds\,.$$

Remark that the stochastic integral parts of the right-hand side are $\mathcal{F}_{t,s}$-

martingales. Since $C_{j,k} \in \mathcal{F}_{s_j}$, we have

$$\begin{aligned} E\int_{\Delta_j} 1_{C_{j,k}}\Gamma_s^{\varepsilon,\zeta}\, V(s_j, x_k, \varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \\ = -\varepsilon\, E\Big[\, 1_{C_{j,k}} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \sigma(\varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\, \psi_s^{\varepsilon,\zeta} \cdot \overline{v}_\eta^{j,k}(\varepsilon^{-1}X_s^{\varepsilon,\zeta})\, ds\,\Big] \\ - \varepsilon^2\, E\Big[\, 1_{C_{j,k}} \int_{\Delta_j} \Gamma_s^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1}X_s^{\varepsilon,\zeta})\, \phi_s^{\varepsilon,\zeta}\, ds\,\Big] \\ + \varepsilon^2 E\, 1_{C_{j,k}}\{\, \Gamma_{s_{j+1}}^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1}X_{s_{j+1}}^{\varepsilon,\zeta}) - \Gamma_{s_j}^{\varepsilon,\zeta}\, \overline{v}^{j,k}(\varepsilon^{-1}X_{s_j}^{\varepsilon,\zeta})\,\}\,, \end{aligned}$$

which implies

$$\sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E\int_{\Delta_j} 1_{C_{j,k}}\Gamma_s^{\varepsilon,\zeta} V(s_j, x_k, \varepsilon^{-1}X_s^{\varepsilon,\zeta}, \zeta_s)\, ds \right| \le (\,\varepsilon + \varepsilon^2\,)\, C + \varepsilon^2\, C\, N\,.$$

Thus, we have completed the proof. □

Our proof also leads an estimate on the rate of convergence of solutions.

Corollary 2.7. *The convergence stated in Theorem 2.3 is uniform on compacts. Moreover, let $\delta \in (0,1)$ be the exponent of Hölder continuity for $u^0 \in C^{1+\delta/2,2+\delta}([0,T] \times \mathbb{R}^d)$. Then, for every compact subset Q of $[0,T] \times \mathbb{R}^d$, there exists $C > 0$ independent of $\varepsilon > 0$ such that*

$$\sup_{(t,x)\in Q} |u^\varepsilon(t,x) - u^0(t,x)| \le C\, \varepsilon^{\frac{2\delta}{2+\delta}}\ .$$

Proof. Form the proof of Propositions 2.4, 2.5 and Lemma 2.6, we have

$$\big|\inf_\zeta \overline{Y}_t^{\varepsilon,\zeta}\big| \le C(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta} + \varepsilon + \varepsilon^2 + \varepsilon^2 N)\,,$$

where C may depend on T and $|x|$ but is independent of $q > 1$ and $\varepsilon > 0$.

Let us take $\gamma_1,\ \gamma_2 > 0$ arbitrarily. We define $n \in \mathbb{R}_+$ and $N \in \mathbb{N}$ by

$$n := \varepsilon^{-\gamma_1}\,, \quad N := [\varepsilon^{-\gamma_2}] + 1\,.$$

Then, we have

$$\tag{2.12} \bigl|\inf_{\zeta}\overline{Y}_t^{\varepsilon,\zeta}\bigr| \le C(\varepsilon^{\gamma_1 q}+\varepsilon^{\gamma_2(q-1)/2-\gamma_1 q} + \varepsilon^{\delta\gamma_2/2}+\varepsilon^{\delta\gamma_1}+\varepsilon+\varepsilon^2+\varepsilon^{2-\gamma_2})\,.$$

Remark that estimate (2.12) makes sense only if

$$\tag{2.13} 0<\gamma_1<(q-1)\gamma_2/2q\,,\qquad 0<\gamma_2<2\,.$$

Hereafter, we always assume (2.13). Since $\delta\in(0,1)$ and $q>1$, we can see

$$\bigl|\inf_{\zeta}\overline{Y}_t^{\varepsilon,\zeta}\bigr| \le C\,\varepsilon^{F(\gamma_1,\gamma_2,q)}\,,$$

where $F(\gamma_1,\gamma_2,q) := \min\{\,\gamma_2(q-1)/2-\gamma_1 q\,,\,\delta\gamma_1\,,\,2-\gamma_2\,\}$. By elementary computation, we can calculate the maximum of $F(\gamma_1,\gamma_2,q)$ with constraint (2.13) as

$$\begin{aligned} F_{\max}(q) :=& \max\{F(\gamma_1,\gamma_2,q)\,;\, 0<\gamma_1<(q-1)\gamma_2/2q\,,\quad 0<\gamma_2<2\,\}\\ =&\frac{2\delta(q-1)}{2q+\delta+\delta q}\,, \end{aligned}$$

and the right-hand side is an increasing function of q and converges to $2\delta/(\delta+2)$ as $q\to+\infty$. In particular, we obtain

$$\bigl|\inf_{\zeta}\overline{Y}_t^{\varepsilon,\zeta}\bigr| \le \lim_{q\to+\infty} C\,\varepsilon^{F_{\max}(q)} \le C\,\varepsilon^{\frac{2\delta}{2+\delta}}\,.$$

Hence we have completed the proof. □

REMARK 2.8. If v and u^0 are smooth enough (e.g. $v(\eta,y,p,X)\in C^2(\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S})$ and $u^0(t,x)\in C_b^{2,4}([0,T]\times\mathbb{R}^d)$), we have no need to execute the local argument and can improve the convergence rate in Corollary 2.7. Indeed, let us consider the linear case, i.e. the case where the Hamiltonian of PDE (0.8) is of the form

$$H(\eta,y,p,X) := -\sum_{i,j=1}^d a^{ij}(\eta)X_{ij} - \sum_{i=1}^d b^i(\eta)p_i - c(\eta)y\,.$$

The corresponding FBSDE is given by

$$\begin{cases} dX_s^\varepsilon = b(\varepsilon^{-1}X_s^\varepsilon)\,ds + \sigma(\varepsilon^{-1}X_s^\varepsilon)\,dW_{t,s}\,, & X_t^\varepsilon = x\,, \\ -dY_s^\varepsilon = c(\varepsilon^{-1}X_s^\varepsilon)\,Y_s^\varepsilon\,ds - \sigma^*(\varepsilon^{-1}X_s^\varepsilon)\,Z_s^\varepsilon\,dW_{t,s}\,, & Y_T^\varepsilon = h(X_T^\varepsilon)\,, \end{cases}$$

where $\sigma\sigma^* = 2a$. Then, it is well known that the effective Hamiltonian $\overline{H}$ in (2.2) is written as

$$\overline{H}(\eta, y, p, X) := -\sum_{i,j=1}^{d} \overline{a}^{ij} X_{ij} - \sum_{i=1}^{d} \overline{b}^i p_i - \overline{c}\,y\,,$$

and the coefficients are characterized by

$$\overline{g} = \int_{[0,1)^d} g(\eta) m(\eta)\,d\eta\,, \qquad g = a^{ij},\ b^i,\ c\,,$$

where $m(\eta)$ denotes the invariant measure on $[0,1)^d$ associated with the differential operator $L := a^{ij}(\eta)\partial_{x^i}\partial_{x^j}$.

Now, let $v = v(\eta, y, p, X)$ be a solution of (2.2). To ensure the uniqueness, we impose the condition $v(0, y, p, X) = 0$. Then, we can easily check that v has the following linear structure with respect to (y, p, X):

$$v(\eta, \lambda_1\Theta_1 + \lambda_2\Theta_2) = \lambda_1 v(\eta, \Theta_1) + \lambda_2 v(\eta, \Theta_2)\,,$$

for all $\lambda_i \in \mathbb{R}$ and $\Theta_i = (y_i, p_i, X_i)$, $i = 1, 2$. In particular, v is twice differentiable with respect to (y, p, X) and

$$\begin{aligned} &v_y(\eta, y, p, X) = v(\eta, 1, 0, 0)\,, \quad v_{p^i}(\eta, y, p, X) = v(\eta, 0, e_i, 0)\,, \\ &v_{X_{ij}}(\eta, y, p, X) = v(\eta, 0, 0, E_{ij})\,, \end{aligned}$$

where e_i denotes the i-th unit vector and E_{ij} stands for the matrix whose (k, l)-component is 1 if $(k, l) = (i, j)$ and is zero otherwise.

Let u^0 be a solution of the limit equation (2.1). We assume here that $u^0 \in C_b^{2,4}([0, T] \times \mathbb{R}^d)$. Then, by using Ito's formula, we can easily show that

$$\left| Y_s^\varepsilon - u^0(s, X_s^\varepsilon) - \varepsilon^2 v(\varepsilon^{-1}X_s^\varepsilon, s, X_s^\varepsilon) \right| \le C(\varepsilon + \varepsilon^2)\,.$$

Hence, we obtain the convergence rate of order ε, which coincides formally with the case where $\delta = 2$ in Corollary 2.7.

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