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A Limit Theorem for Solutions of Some Functional Stochastic Difference Equations

By Takashi KATO

Abstract. In this paper, we study a limit theorem for solutions of some functional stochastic difference equations under strong mixing conditions and some dimensional conditions. This work is an extension of the work of Hisao Watanabe.

1. Introduction and Main Results

Diffusion approximations for certain stochastic difference equations or stochastic ordinary differential equations have been discussed in several papers. [9] [15], [16] and [17] treated such problem and derived the weak limit of appropriately scaled and interpolated process, which was given by the solution of a stochastic difference equation as a diffusion process. Concerning this, [5], [6], [10], [11] and many other papers dealt with weak convergence of the solution of a stochastic ordinary differential equation.

In this paper, we study a limit theorem for stochastic processes X^n_t given by the following functional stochastic difference equations

$$X^n_{(k+1)/n} - X^n_{k/n} = \frac{1}{\sqrt{n}} F^n_k(X^n, \omega) + \frac{1}{n} G^n_k(X^n, \omega) \tag{1.1}$$

and by linear interpolation as

$$X^n_t = (1 - nt + k) X^n_{k/n} + (nt - k) X^n_{(k+1)/n} \tag{1.2}$$

for $k/n < t < (k+1)/n$, and

$$X^n_0 = x_0 \in \mathbb{R}^d. \tag{1.3}$$

Here F^n_k and G^n_k are d dimensional random functions on $C([0,\infty);\mathbb{R}^d)$, the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$, such that F^n_k has mean zero.

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Under certain assumptions for F_k^n and G_k^n, we show that the distribution of X^n converges weakly to the solution of a martingale problem corresponding to functional coefficients.

The methods of the proof are based on [5] and [16]. However, we cannot use mixing inequalities in these papers, since the dimension of parameter space $C([0,\infty);\mathbb{R}^d)$ is infinite.

We show another version of mixing inequalities by assuming certain dimensional conditions for the set of random variables $F_k^n(w)$ and $G_k^n(w)$, which may look artificial but we give sufficient conditions for this assumption later.

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Let $(\Omega^n, \mathcal{F}^n, P^n)$, $n \in \mathbb{N} = \{1,2,3,\dots\}$, be complete probability spaces. Let $F_k^n(w,\omega) = (F_k^{n,i}(w,\omega))_{i=1}^d$ and $G_k^n(w,\omega) = (G_k^{n,i}(w,\omega))_{i=1}^d : C([0,\infty); \mathbb{R}^d) \times \Omega^n \longrightarrow \mathbb{R}^d$, $k \in \mathbb{Z}_+ = \{0,1,2,\dots\}$, be random functions. Let $\mathcal{B}_t$ be the σ-algebra of $C([0,\infty);\mathbb{R}^d)$ given by $\mathcal{B}_t = \sigma(w(s)\ ;\ s \le t)$.

We introduce the following conditions.

[$A1$] $F_k^{n,i}$ and $G_k^{n,i}$ are measurable with respect to $\mathcal{B}_{k/n} \otimes \mathcal{F}^n$.

By [A1], we can regard $F_k^{n,i}$ and $G_k^{n,i}$ as random functions defined on the Banach space $C([0,k/n];\mathbb{R}^d)$.

[$A2$] $F_k^{n,i}(w,\omega)$ (respectively, $G_k^{n,i}(w,\omega)$) is twice (respectively, once) continuously Fréchet differentiable in w for P^n-almost surely ω.

We denote by L_T^m the space of real valued continuous m-multilinear operators on $C([0,T];\mathbb{R}^d)$ and denote by $|\cdot|_{L_T^m}$ its norm. Then the m-th Fréchet derivative $\nabla^m F_k^{n,i}(w) : (w_1,\dots,w_m) \longmapsto \nabla^m F_k^{n,i}(w; w_1,\dots,w_m)$ is regarded as the element of $L_{k/n}^m$ for each w (and so is $\nabla^m G_k^{n,i}(w)$). For $m=0$, $L_T^0 = \mathbb{R}$ and $\nabla^0 F_k^{n,i}(w) = F_l^{n,i}(w)$.

Let $p_0 > 3$ and $\gamma_0 > 0$. We assume the moment conditions with respect to p_0 and the dimensional conditions with respect to γ_0 as [$A3$] and [$A4$].

$[A3]$ For each $M > 0$, there exists a constant $C(M) > 0$ such that

$$\sum_{m=0}^{2} \mathrm{E}^n \Big[\sup_{|w|_\infty \le M} \big| \nabla^m F_k^{n,i}(w) \big|_{L_{k/n}^m}^{p_0} \Big] \le C(M) \tag{1.4}$$

and

$$\sum_{m=0}^{1} \mathrm{E}^n \Big[\sup_{|w|_\infty \le M} \big| \nabla^m G_k^{n,i}(w) \big|_{L_{k/n}^m}^{p_0} \Big] \le C(M) \tag{1.5}$$

for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, where $\mathrm{E}^n[\cdot]$ denotes the expectation under the probability measure P^n and $|w|_\infty = \sup_{t \ge 0} |w(t)|$.

Let $\mathcal{C}_M^d$ denote the set of $w \in C([0,\infty);\mathbb{R}^d)$ such that $|w|_\infty \le M$. For a random function $U : C([0,\infty);\mathbb{R}^d) \times \Omega^n \longrightarrow \mathbb{R}$ and $\varepsilon > 0$, let $N_n(\varepsilon, M; U)$ be the smallest integer m such that there exist sets $S_1, \dots, S_m$ which satisfy $\mathcal{C}_M^d = \bigcup_{i=1}^{m} S_i$ and

$$\mathrm{E}^n \Big[\max_{i=1,\dots,m} \sup_{x,y \in S_i} |U(x) - U(y)|^{p_0} \Big]^{1/p_0} < \varepsilon.$$

$[A4]$

$$\sup_{n,k} \sup_{\varepsilon > 0} \varepsilon^{\gamma_0} N_n(\varepsilon, M; F_k^{n,i}) < \infty, \tag{1.6}$$

$$\sup_{n,k} \sup_{l \le k} \sup_{\varepsilon > 0} \varepsilon^{\gamma_0} N_n(\varepsilon, M; \nabla F_k^{n,i}(\cdot\,; I_l^n e_j)) < \infty, \tag{1.7}$$

$$\sup_{n,k} \sup_{l,m \le k} \sup_{\varepsilon > 0} \varepsilon^{\gamma_0} N_n(\varepsilon, M; \nabla^2 F_k^{n,i}(\cdot\,; I_l^n e_j, I_m^n e_\nu)) < \infty, \tag{1.8}$$

$$\sup_{n,k} \sup_{\varepsilon > 0} \varepsilon^{\gamma_0} N_n(\varepsilon, M; G_k^{n,i}) < \infty \tag{1.9}$$

and

$$\sup_{n,k} \sup_{l \le k} \sup_{\varepsilon > 0} \varepsilon^{\gamma_0} N_n(\varepsilon, M; \nabla G_k^{n,i}(\cdot\,; I_l^n e_j)) < \infty \tag{1.10}$$

for each $M > 0$ and $i, j, \nu = 1, \dots d$, where $e_j \in \mathbb{R}^d$ denotes the unit vector along the j-th axis, i.e. $e_j = (0, \dots, 0, \overset{j}{\check{1}}, 0, \dots, 0)$, and the function I_l^n :

$[0,\infty) \longrightarrow \mathbb{R}$ is given by

$$I_l^n(t) = \begin{cases} 0 & \text{if } 0 \le t < \dfrac{l}{n} \\ nt - l & \text{if } \dfrac{l}{n} \le t < \dfrac{l+1}{n} \\ 1 & \text{if } t \ge \dfrac{l+1}{n}. \end{cases}$$

$[A5]$ Let

$$\mathcal{F}_{k,l}^n = \sigma\big(F_m^{n,i}(w), G_m^{n,i}(w) \ ; \ i = 1, \ldots, d, \ k \le m \le l, \ w \in C([0,\infty);\mathbb{R}^d)\big)$$

and

$$\alpha_k = \sup_n \sup_l \sup\{|P^n(A \cap B) - P^n(A)P^n(B)| \ ; \ A \in \mathcal{F}_{0,l}^n, \ B \in \mathcal{F}_{k+l,\infty}^n\}.$$

Then

$$\sum_{k=1}^{\infty} \alpha_k^{\varrho_0} < \infty, \tag{1.11}$$

where $\varrho_0 = \dfrac{1}{2s_0 + 4\gamma_0}$ and $s_0 = \dfrac{p_0}{p_0 - 3}$.

$[A6]$ $\mathrm{E}^n[F_k^{n,i}(w)] = 0$.

We denote by $\mathcal{K}^d$ the family of a compact set K of $C([0,\infty);\mathbb{R}^d)$ such that $\sup_{w \in K} |w|_\infty < \infty$.

$[A7]$ Let

$$\begin{aligned} a_0^{n,ij}(k,w) &= \mathrm{E}^n[F_k^{n,i}(w) F_k^{n,j}(w)], \\ b_0^{n,i}(k,w) &= \mathrm{E}^n[G_k^{n,i}(w)], \\ A^{n,ij}(k,w) &= \sum_{l=1}^{\infty} \mathrm{E}^n\Big[F_{k+l}^{n,i}\Big(w\Big(\,\cdot\, \wedge \frac{k}{n}\Big)\Big) F_k^{n,j}(w)\Big], \\ B^{n,ij}(k,w) &= \sum_{l=1}^{\infty} \mathrm{E}^n\Big[\nabla F_{k+l}^{n,i}\Big(w\Big(\,\cdot\, \wedge \frac{k}{n}\Big); I_k^n e_j\Big) F_k^{n,j}(w)\Big] \end{aligned}$$

for $k \in \mathbb{Z}_+$ and $w \in C([0,\infty);\mathbb{R}^d)$, where $a \wedge b = \min\{a, b\}$. The following limits exist uniformly on any $K \in \mathcal{K}^d$ for each $t \geq 0$:

$$a_0^{ij}(t,w) = \lim_{n\to\infty} a_0^{n,ij}([nt],w), \tag{1.12}$$

$$b_0^{i}(t,w) = \lim_{n\to\infty} b_0^{n,i}([nt],w), \tag{1.13}$$

$$A^{ij}(t,w) = \lim_{n\to\infty} A^{n,ij}([nt],w), \tag{1.14}$$

$$B^{ij}(t,w) = \lim_{n\to\infty} B^{n,ij}([nt],w), \tag{1.15}$$

where $[x]$ denotes the greatest integer less than or equal to x.

[$A8$] Define $a(t,w) = (a^{ij}(t,w))_{i,j=1}^d$ and $b(t,w) = (b^i(t,w))_{i=1}^d$ by

$$a^{ij}(t,w) = a_0^{ij}(t,w) + A^{ij}(t,w) + A^{ji}(t,w)$$

and

$$b^i(t,w) = b_0^i(t,w) + \sum_{j=1}^d B^{ij}(t,w).$$

For each $T > 0$, there exists a positive constant $C(T)$ such that

$$|a^{ij}(t,w)| \leq C(T)\Big(1 + \sup_{0\leq s\leq t} |w(s)|^2\Big) \tag{1.16}$$

and

$$|b^{i}(t,w)| \leq C(T)\Big(1 + \sup_{0\leq s\leq t} |w(s)|\Big) \tag{1.17}$$

for $t \in [0,T]$ and $w \in C([0,\infty);\mathbb{R}^d)$.

[$A9$] Let

$$\mathscr{L}f(t,w) = \frac{1}{2}\sum_{i,j=1}^d a^{ij}(t,w)\frac{\partial^2}{\partial x^i \partial x^j} f(w(t)) + \sum_{i=1}^d b^i(t,w)\frac{\partial}{\partial x^i} f(w(t))$$

for $f \in C^2(\mathbb{R}^d)$. The martingale problem associated with the generator $\mathscr{L}$ and initial value x_0 has a unique solution Q on $C([0,\infty);\mathbb{R}^d)$.

We will introduce the sufficient conditions for [$A4$] and [$A9$] in Section 5.

Define the stochastic process $X_t^n = (X_t^{n,i})_{i=1}^d$ by (1.1), (1.2) and (1.3). Let Q^n be the probability measure induced by X^n on $C([0,\infty);\mathbb{R}^d)$.

THEOREM 1. *Assume* $[A1]-[A9]$*. Then* Q^n *converges weakly to* Q *on* $C([0,\infty);\mathbb{R}^d)$.

Let us give some remarks on Theorem 1.

(i) In fact, using the arguments in [16], we can prove Theorem 1 without assuming the condition (1.10).

(ii) We can replace the assumption $[A5]$ with

$[A5']$ For each $M > 0$

$$\sum_{k=1}^{\infty} \alpha_k(M)^{\varrho_0} < \infty, \tag{1.18}$$

where

$$\mathcal{F}_{k,l}^n(M) = \sigma(F_m^{n,i}(w), G_m^{n,i}(w)\ ;\ i = 1,\dots,d,\ k \le m \le l,\ |w|_\infty \le M)$$

and

$$\alpha_k(M) = \sup_n \sup_l \sup\{|P^n(A\cap B) - P^n(A)P^n(B)|\ ; \\ A \in \mathcal{F}_{0,l}^n(M),\ B \in \mathcal{F}_{k+l,\infty}^n(M)\}.$$

The proof needs no change.

(iii) Assuming the following uniform mixing condition $[A5'']$ instead of $[A5]$, we can remove the dimensional condition $[A4]$:

$[A5'']$ It holds that

$$\sum_{k=1}^{\infty} \phi_k^{\varrho_2} < \infty, \tag{1.19}$$

where $\varrho_2 = \dfrac{p_0 - 2}{2p_0}$ and

$$\phi_k = \sup_n \sup_l \sup\left\{\left|\frac{P^n(A\cap B)}{P^n(A)} - P^n(B)\right|\ ; \\ A \in \mathcal{F}_{0,l}^n, B \in \mathcal{F}_{k+l,\infty}^n, P^n(A) > 0\right\}.$$

Next we provide another version of Theorem 1. We introduce the following conditions.

$[B4]$ For some $\gamma_1 > 0$, (1.6)$-$(1.10) hold with $\log N_n$ instead of N_n.

$[B5]$ Let α_k be as in $[A5]$. Then there exists $\varrho_1 \in \Big(0, \dfrac{1}{2\gamma_1}\Big)$ such that

$$\sum_{k=1}^{\infty} \Big(\frac{1}{\log(1/\alpha_k)}\Big)^{\varrho_1} < \infty. \tag{1.20}$$

THEOREM 2. *Assume* $[A1] - [A3]$, $[B4]$, $[B5]$ *and* $[A6] - [A9]$. *Then* Q^n *converges weakly to* Q *on* $C([0,\infty);\mathbb{R}^d)$.

2. Mixing Inequalities

In this section we prepare some inequalities for strong mixing coefficients. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset \mathcal{F}$ be sub σ-algebras. Define $\alpha(\mathcal{A},\mathcal{B})$ by

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{\ |P(A\cap B) - P(A)P(B)|\ ;\ A \in \mathcal{A},\ B \in \mathcal{B}\ \}.$$

The following lemma is shown in the proof of Theorem 17.2.2 in [4].

LEMMA 1. *Let* $1 \le p, q, r \le \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$, X *be an* $\mathcal{A}$*-measurable random variable and* Y *be a* $\mathcal{B}$*-measurable random variable. Then*

$$\big|\,\mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]\big| \le 8\,\mathrm{E}[|X|^p]^{1/p}\,\mathrm{E}[|Y|^q]^{1/q}\alpha(\mathcal{A},\mathcal{B})^{1/r}. \tag{2.1}$$

Let (S,d) be a metric space, $\varepsilon, p > 0$ and $U : S\times\Omega \longrightarrow \mathbb{R}$ be a continuous random function. We say that a family of sets $(S_i)_{i=1}^m$ is an (ε,p,U)-net of S if $S = \bigcup_{i=1}^m S_i$ and

$$\mathrm{E}\Big[\max_{i=1,\dots,m}\ \sup_{x,y\in S_i} |U(x) - U(y)|^p\Big]^{1/p} < \varepsilon.$$

We denote the minimum of cardinals of (ε, p, U)-nets by $N(\varepsilon, p; U)$.

Lemma 2. *Let* $1 < p, q < \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} < 1$ *and* $U : S \times \Omega \longrightarrow \mathbb{R}$ *be a continuous random function such that* $U(x)$ *is* $\mathcal{A}$*-measurable and* $\mathrm{E}[U(x)] = 0$ *for each* $x \in S$*, and* $X : \Omega \longrightarrow S$*,* $V : \Omega \longrightarrow \mathbb{R}$ *be* $\mathcal{B}$*-measurable random variables. Then for any* $\varepsilon > 0$

$$\begin{aligned}(2.2)\qquad \big|\,\mathrm{E}[U(X)V]\big| \;\le\; & 8\big(\,\mathrm{E}[\sup_{x\in S}|U(x)|^p]^{1/p} + 1\big) \\ & \times \mathrm{E}[|V|^q]^{1/q}\big\{\varepsilon + \varepsilon^{1-r}N(\varepsilon, p; U)\alpha(\mathcal{A}, \mathcal{B})\big\},\end{aligned}$$

where $\dfrac{1}{r} = 1 - \dfrac{1}{p} - \dfrac{1}{q}$.

Proof. We may assume that the right-hand side of (2.2) is finite and $\alpha(\mathcal{A}, \mathcal{B}) > 0$. Set $N_\varepsilon = N(\varepsilon, p; U)$ and $U^* = \sup_{x\in S}|U(x)|$. Let $\delta = p/r$, $\tilde{\delta} = q/r$,

$$I = \mathrm{E}[|U^*|^p]^{1/p}\varepsilon^{-1/\delta}, \quad J = \mathrm{E}[|V|^q]^{1/q}\varepsilon^{-1/\tilde{\delta}}$$

and

$$U_I(x) = U(x)1_{\{|U^*| \le I\}}, \quad V_J = V1_{\{|V| \le J\}}.$$

Then we have

$$\frac{1}{\delta} + \frac{1}{\tilde{\delta}} = r - 1. \tag{2.3}$$

Let $(S_i)_{i=1}^{N_\varepsilon}$ be an (ε, p, U)-net. We may assume that all S_i are disjoint and not empty. Take any $x_i \in S_i$, and define the random variable $\tilde{X} : \Omega \longrightarrow S$ by

$$\tilde{X}(\omega) = \sum_{i=1}^{N_\varepsilon} x_i 1_{\Omega_i}(\omega),$$

where $\Omega_i = \{X \in S_i\}$. Then it follows that

$$\begin{aligned}(2.4)\quad \big|\,\mathrm{E}[U(X)V]\big| \;\le\; & \big|\,\mathrm{E}[(U(X) - U(\tilde{X}))V]\big| + \big|\,\mathrm{E}[(U(\tilde{X}) - U_I(\tilde{X}))V]\big| \\ & + \big|\,\mathrm{E}[U_I(\tilde{X})(V - V_J)]\big| + \big|\,\mathrm{E}[U_I(\tilde{X})V_J]\big|.\end{aligned}$$

By the definition of $\tilde{X}$, we have

$$
\begin{aligned}
(2.5) \qquad & \left|\,\mathrm{E}[(U(X)-U(\tilde{X}))V]\right| \\
\le\ & \mathrm{E}\left[\max_{i=1,\dots,N_\varepsilon}\ \sup_{x,y\in S_i}|U(x)-U(y)|\cdot|V|\right] \\
\le\ & \mathrm{E}\left[\max_{i=1,\dots,N_\varepsilon}\ \sup_{x,y\in S_i}|U(x)-U(y)|^p\right]^{1/p}\mathrm{E}[|V|^q]^{1/q} \\
\le\ & \varepsilon\,\mathrm{E}[|V|^q]^{1/q}.
\end{aligned}
$$

By the Chebyshev inequality and the Hölder inequality, we have

$$
\begin{aligned}
(2.6) \qquad & \left|\,\mathrm{E}[(U(\tilde{X})-U_I(\tilde{X}))V]\right| \le \frac{1}{I^\delta}\,\mathrm{E}\left[|U^*|^{1+\delta}|V|\right] \\
\le\ & \frac{1}{I^\delta}\,\mathrm{E}[|U^*|^p]^{(1+\delta)/p}\,\mathrm{E}[|V|^q]^{1/q} = \mathrm{E}[|U^*|^p]^{1/p}\,\mathrm{E}[|V|^q]^{1/q}\varepsilon.
\end{aligned}
$$

Similarly we obtain

$$
(2.7) \qquad \left|\,\mathrm{E}[U_I(\tilde{X})(V-V_J)]\right| \ \le\ \mathrm{E}[|U^*|^p]^{1/p}\,\mathrm{E}[|V|^q]^{1/q}\varepsilon.
$$

Set $\bar{U}_I(x)=\mathrm{E}[U_I(x)]$ and $\tilde{U}_I(x)=U_I(x)-\bar{U}_I(x)$. Then it follows that

$$
\begin{aligned}
(2.8) \qquad \left|\,\mathrm{E}[U_I(\tilde{X})V_J]\right| \ &\le\ \left|\,\mathrm{E}[\bar{U}_I(\tilde{X})V_J]\right| + \left|\,\mathrm{E}[\tilde{U}_I(\tilde{X})V_J]\right| \\
&\le\ \sup_{x\in S}|\bar{U}_I(x)|\,\mathrm{E}[|V|^q]^{1/q} + \sum_{i=1}^{N_\varepsilon}\left|\,\mathrm{E}[\tilde{U}_I(x_i)V_J 1_{\Omega_i}]\right|.
\end{aligned}
$$

Since $\mathrm{E}[U(x)]=0$, we have

$$
(2.9) \qquad |\bar{U}_I(x)| = \left|\,\mathrm{E}[U_I(x)-U(x)]\right| \le \frac{1}{I^\delta}\,\mathrm{E}[|U^*|^{1+\delta}] = \mathrm{E}[|U^*|^p]^{1/p}\varepsilon.
$$

By Lemma 1 and (2.3), we get

$$
\begin{aligned}
(2.10) \qquad \sum_{i=1}^{N_\varepsilon}\left|\,\mathrm{E}[\tilde{U}_I(x_i)V_J 1_{\Omega_i}]\right| \ &\le\ 8N_\varepsilon IJ\alpha(\mathcal{A},\mathcal{B}) \\
&=\ 8\,\mathrm{E}[|U^*|^p]^{1/p}\,\mathrm{E}[|V|^q]^{1/q}\varepsilon^{1-r}N_\varepsilon\alpha(\mathcal{A},\mathcal{B}).
\end{aligned}
$$

By (2.4)-(2.10), we obtain the assertion. □

Lemma 3. *Let* $1 < p, q < \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} < 1$ *and* $U : S \times \Omega \longrightarrow \mathbb{R}$ *be a continuous random function such that* $U(x)$ *is* $\mathcal{A}$*-measurable and* $\mathrm{E}[U(x)] = 0$ *for each* $x \in S$*, and* $X : \Omega \longrightarrow S$, $V : \Omega \longrightarrow \mathbb{R}$ *be* $\mathcal{B}$*-measurable random variables. Suppose that there exist positive constants* C_0 *and* γ *such that*

$$\sup_{\varepsilon>0} \varepsilon^{\gamma} N(\varepsilon, p; U) \le C_0. \tag{2.11}$$

Then it holds that

$$\begin{aligned} \big|\, \mathrm{E}[U(X)V] \big| \;\le\; & 16(C_0+1)\big(\, \mathrm{E}[\sup_{x\in S} |U(x)|^p]^{1/p} + 1\big) \\ & \times \mathrm{E}[|V|^q]^{1/q} \alpha(\mathcal{A}, \mathcal{B})^{\varrho}, \end{aligned} \tag{2.12}$$

where $\varrho = \dfrac{1}{r+\gamma}$ *and* $\dfrac{1}{r} = 1 - \dfrac{1}{p} - \dfrac{1}{q}$.

Proof. By Lemma 2, we get

$$\begin{aligned} \big|\, \mathrm{E}[U(X)V] \big| \;\le\; & 8(C_0+1)\big(\, \mathrm{E}[\sup_{x\in S} |U(x)|^p]^{1/p} + 1\big) \\ & \times \mathrm{E}[|V|^q]^{1/q} \big\{\varepsilon + \varepsilon^{1-r-\gamma} \alpha(\mathcal{A}, \mathcal{B})\big\}. \end{aligned}$$

The assertion now follows by taking $\varepsilon = \alpha(\mathcal{A}, \mathcal{B})^{\varrho}$. $\square$

We denote by $\mathcal{A} \vee \mathcal{B}$ the smallest σ-algebra which includes both $\mathcal{A}$ and $\mathcal{B}$. The following lemma is obtained by Lemma 3 and the arguments in the proof of Lemma 2 in [5].

Lemma 4. *Let* $1 < p, q, r < \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} < 1$. *Let* $U, V : S \times \Omega \longrightarrow \mathbb{R}$ *be continuous random functions such that* $U(x)$ *and* $V(x)$ *are* $\mathcal{A}$ *and* $\mathcal{B}$*-measurable respectively and* $\mathrm{E}[U(x)] = 0$ *for each* $x \in S$*, and* $X : \Omega \longrightarrow S$, $Z : \Omega \longrightarrow \mathbb{R}$ *be* $\mathcal{C}$*-measurable random variables. Suppose that there exist positive constants* C_0, u^*, v^* *and* γ *such that*

$$\sup_{\varepsilon>0} \varepsilon^{\gamma} \big\{N(\varepsilon, p; U) + N(\varepsilon, q; V)\big\} \le C_0, \tag{2.13}$$

$$\mathrm{E}[\sup_{x\in S} |U(x)|^p]^{1/p} \le u^* \tag{2.14}$$

and

$$\mathrm{E}[\sup_{x\in S}|V(x)|^q]^{1/q} \le v^*. \tag{2.15}$$

Then there exists a constant $C>0$ depending only on C_0, u^*, v^* and γ such that

$$\big|\,\mathrm{E}[\Xi(X)Z]\big| \le C\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C})^{\varrho'}\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{\varrho'}, \tag{2.16}$$

where $\Xi(x) = U(x)V(x) - E[U(x)V(x)]$, $\varrho' = \dfrac{1}{2s+4\gamma}$ *and* $\dfrac{1}{s} = 1-\dfrac{1}{p}-\dfrac{1}{q}-\dfrac{1}{r}$.

Proof. Set $\tilde\varepsilon = \dfrac{\varepsilon}{2(u^*+v^*)}$. Let $t\ge 1$ be such that $\dfrac{1}{t} = \dfrac{1}{p}+\dfrac{1}{q}$. Then we have

$$N(\varepsilon,t;\Xi) \le N(\tilde\varepsilon,p;U)N(\tilde\varepsilon,q;V). \tag{2.17}$$

Indeed, if we let $(S_i)_{i=1}^{N(\tilde\varepsilon,p,U)}$ and $(\tilde S_j)_{j=1}^{N(\tilde\varepsilon,q,V)}$ be $(\tilde\varepsilon,p,U)$-net and $(\tilde\varepsilon,p,U)$-net respectively, then the Hölder inequality implies

$$\begin{aligned}
&\mathrm{E}\Big[\max_{i,j}\sup_{x,y\in S_i\cap\tilde S_j}|\Xi(x)-\Xi(y)|^t\Big]^{1/t}\\
\le\;& 2\Big\{\,\mathrm{E}\Big[\sup_{x\in S}|U(x)|^t\max_j\sup_{x,y\in\tilde S_j}|V(x)-V(y)|^t\Big]^{1/t}\\
&\quad+\mathrm{E}\Big[\max_i\sup_{x,y\in S_i}|U(x)-U(y)|^t\sup_{x\in S}|V(x)|^t\Big]^{1/t}\Big\}\\
\le\;& 2\Big\{u^*\,\mathrm{E}\Big[\max_j\sup_{x,y\in\tilde S_j}|V(x)-V(y)|^q\Big]^{1/q}\\
&\quad+\mathrm{E}\Big[\max_i\sup_{x,y\in S_i}|U(x)-U(y)|^p\Big]^{1/p}v^*\Big\}\\
\le\;& 2(u^*+v^*)\tilde\varepsilon = \varepsilon.
\end{aligned}$$

Thus $(S_i\cap\tilde S_j)_{i=1,\dots,N(\tilde\varepsilon,p;U),j=1,\dots,N(\tilde\varepsilon,q;V)}$ is an (ε,t,Ξ)-net. This implies (2.17).

So we get

$$N(\varepsilon,t;\Xi) \le 2^{2\gamma}(u^*+v^*)^{2\gamma}C_0^2\varepsilon^{-2\gamma}. \tag{2.18}$$

Then, using Lemma 3 with Ξ substituted for U, we have

$$(2.19)\quad \begin{aligned}\big|\,\mathrm{E}[\Xi(X)Z]\big| &\le C_1\big(\,\mathrm{E}[\sup_{x\in S}|\Xi(x)|^t]^{1/t}+1\big)\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C})^{\varrho''}\\ &\le 2C_1(u^*v^*+1)\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C})^{2\varrho'}\end{aligned}$$

for some $C_1>0$ depending only on C_0, u^*, v^* and $\gamma>0$.

On the other hand, using Lemma 3 with $V(X)Z$ substituted for V, we have

$$(2.20)\quad \begin{aligned}\big|\,\mathrm{E}[U(X)V(X)Z]\big| &\le C_2(u^*+1)\,\mathrm{E}[|V(X)Z|^{t'}]^{1/t'}\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{\varrho''}\\ &\le C_2(u^*+1)v^*\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{2\varrho'}.\end{aligned}$$

for some $C_2>0$ depending only on C_0 and $\gamma>0$, where $\dfrac{1}{t'}=\dfrac{1}{q}+\dfrac{1}{r}$ and $\varrho''=\dfrac{1}{s+\gamma}$.

Set $W(x)=\mathrm{E}[U(x)V(x)]$. By Lemma 1, we see

$$|W(x)|\le 8u^*v^*\alpha(\mathcal{A},\mathcal{B})^{1-1/t}\le 8u^*v^*\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{2\varrho'}$$

for each $x\in S$. Thus

$$(2.21)\qquad \big|\,\mathrm{E}[W(X)Z]\big|\le 8u^*v^*\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{2\varrho'}.$$

By (2.19), (2.20) and (2.21), it follows that

$$\begin{aligned}\big|\,\mathrm{E}[\Xi(X)Z]\big| &= \big|\,\mathrm{E}[\Xi(X)Z]\big|^{1/2}\big|\,\mathrm{E}[\Xi(X)Z]\big|^{1/2}\\ &\le C_3\,\mathrm{E}[|Z|^r]^{1/r}\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C})^{\varrho'}\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{\varrho'}\end{aligned}$$

for some $C_3>0$ depending only on C_0, u^*, v^* and $\gamma>0$. This implies the assertion. □

3. Proof of Theorem 1

Let $\varphi_M\in C^\infty(\mathbb{R}^d;\mathbb{R})$ be such that $0\le\varphi_M\le 1$,

$$\varphi_M(x)=\begin{cases}1 & \text{if } |x|\le M/2\\ 0 & \text{if } |x|\ge M,\end{cases}$$

and the gradient of $\varphi_M(x)$ is bounded uniformly in $x \in \mathbb{R}^d$ and $M \geq 1$. Define the truncated functions $F_k^{n,M}(w) = (F_k^{n,M,i}(w))_{i=1}^d$ and $G_k^{n,M}(w) = (G_k^{n,M,i}(w))_{i=1}^d$ by

$$F_k^{n,M}(w) = \varphi_M(w(k/n))F_k^n(w), \quad G_k^{n,M}(w) = \varphi_M(w(k/n))G_k^n(w).$$

We also define the stochastic process $X_t^{n,M} = (X_t^{n,M,i})_{i=1}^d$ by (1.1) and (1.2) for which F_k^n and G_k^n are replaced by $F_k^{n,M}$ and $G_k^{n,M}$.

To make notations simple, we set $H_k^{n,M,i}(w) = F_k^{n,M,i}(w) + \frac{1}{\sqrt{n}}G_k^{n,M,i}(w)$. Then $X_t^{n,M,i}$ satisfies the following equation

$$X_{(k+1)/n}^{n,M,i} - X_{k/n}^{n,M,i} = \frac{1}{\sqrt{n}}H_k^{n,M}(X^{n,M}). \tag{3.1}$$

PROPOSITION 1. *For each $\omega \in \Omega^n$, if $|X_t^{n,M}(\omega)| \leq M$, then $|X_s^{n,M}(\omega)| \leq M$ for any $s \in [0,t]$.*

PROOF. We prove the contraposition of the assertion. Suppose that $|X_s^{n,M}| > M$ holds for some $s \in [0,t]$. Let $k = [ns]$. If $|X_{k/n}^{n,M}| > M$, we have $|X_t^{n,M}| = |X_s^{n,M}| > M$ obviously. So we may suppose $|X_{k/n}^{n,M}| \leq M$.

Then we see $|X_{(k+1)/n}^{n,M}| > M$. Indeed, if $|X_{(k+1)/n}^{n,M}| \leq M$, then $|X_s^{n,M}| \leq M$ holds by the convexity of the set $\{x \in \mathbb{R}^d; |x| \leq M\}$, and this contradicts the supposition. So $X_t^{n,M}$ is in $\{uX_s^{n,M} + (1-u)X_{(k+1)/n}^{n,M}; 0 \leq u \leq 1\} \subset \{uX_s^{n,M} + (1-u)X_{k/n}^{n,M}; u \geq 1\}$. Since $|X_{k/n}^{n,M}| \leq M$ and $|X_s^{n,M}| > M$ hold, we have $|uX_s^{n,M} + (1-u)X_{k/n}^{n,M}| > M$ for each $u \geq 1$. Thus $|X_t^{n,M}| > M$ holds and we obtain the assertion. □

By Proposition 1, the assumption [A3] and the definition of $X_t^{n,M}$, we see that $X_t^{n,M}$ is $\mathcal{F}_{0,[nt]}^n$-measurable and that there exists a constant $C(M) > 0$ such that

$$\sum_{m=0}^{2} \mathrm{E}^n\Big[\big|\nabla^m F_k^{n,M,i}(X^{n,M})\big|_{L_{k/n}^m}^{p_0}\Big] \leq C(M) \tag{3.2}$$

and

$$\sum_{m=0}^{1} \mathrm{E}^{n}\Big[\big|\nabla^m G_k^{n,M,i}(X^{n,M})\big|_{L^m_{k/n}}^{p_0}\Big] \le C(M) \tag{3.3}$$

for $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$.

Let

$$Y_k^{n,M}(u,t) = X^{n,M}_{t\wedge(k/n)} + u\big(X^{n,M}_{t\wedge((k+1)/n)} - X^{n,M}_{t\wedge(k/n)}\big), \quad u \in [0,1].$$

Easily we have

$$Y_k^{n,M}(u,t) = \begin{cases} X_t^{n,M} & \text{if } t \le \dfrac{k}{n} \\ X^{n,M}_{k/n+u(t-k/n)} & \text{if } \dfrac{k}{n} < t \le \dfrac{k+1}{n} \\ X^{n,M}_{(k+u)/n} & \text{if } \dfrac{k+1}{n} < t. \end{cases} \tag{3.4}$$

By Lemma 3 and Lemma 4, we obtain the following two propositions.

PROPOSITION 2. *Let $1 < q < \infty$ be such that $\dfrac{1}{q} \le \dfrac{1}{2}\Big(1 + \dfrac{1}{p_0}\Big)$, and let $U : C([0,\infty);\mathbb{R}^d) \times \Omega^n \longrightarrow \mathbb{R}$ be such that $U(w)$ is $\mathcal{F}^n_{k,\infty}$-measurable and $\mathrm{E}^n[U(w)] = 0$ for each $w \in \mathcal{C}^d_M$, and $V : \Omega^n \longrightarrow \mathbb{R}$ be an $\mathcal{F}^n_{0,l}$-measurable random variable. Suppose that there exists a constant $C_0 = C_0(M) > 0$ such that*

$$\sup_{\varepsilon>0} \varepsilon^{\gamma} N_n(\varepsilon, M; U) \le C_0. \tag{3.5}$$

Then there exists a constant $C > 0$ depending only on M and C_0 such that for all $l \le k$, $u \in [0,1]$ and $\beta = (\beta^1, \dots, \beta^d) \in \mathbb{Z}^d_+$ with $|\beta| = \beta^1 + \cdots + \beta^d \le 2$

$$\begin{aligned} &\big|\,\mathrm{E}^n[U^M_\beta(Y^{n,M}_l(u,\cdot))V]\big| \\ \le\ & C(\mathrm{E}^n[\sup_{|w|_\infty \le M} |U(w)|^{p_0}]^{1/p_0} + 1)\,\mathrm{E}^n[|V|^q]^{1/q} \alpha^{\varrho_0}_{k-l}, \end{aligned} \tag{3.6}$$

where $U^M_\beta(w) = D^\beta \varphi_M(w(k/n))U(w)$ and $D^\beta = \dfrac{\partial^{|\beta|}}{\partial x^{\beta^1} \cdots \partial x^{\beta^d}}$.

PROOF. Define $\hat{Y}_l^{n,M}(u,t)$ and $\hat{V}$ by

$$\hat{Y}_l^{n,M}(u,t) = \begin{cases} Y_l^{n,M}(u,t) & \text{if } \left|X_{(l+u)/n}^{n,M}\right| \le M \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

and

$$\hat{V} = \begin{cases} V & \text{if } \left|X_{(l+u)/n}^{n,M}\right| \le M \\ 0 & \text{otherwise}. \end{cases}$$

By (3.4) and Proposition 1, we see that $|\hat{Y}_l^{n,M}(u,t)| \le M$ for all $t \ge 0$ almost surely and

$$\mathrm{E}^n[U_\beta^M(Y_l^{n,M}(u,\cdot))V] \;=\; \mathrm{E}^n[U(\hat{Y}_l^{n,M}(u,\cdot))D^\beta\varphi_M(X_{(l+u)/n}^{n,M})\hat{V}]. \tag{3.8}$$

Using Lemma 3, we see that

$$\begin{aligned} &\left|\mathrm{E}^n[U(\hat{Y}_l^{n,M}(u,\cdot))D^\beta\varphi_M(X_{(l+u)/n}^{n,M})\hat{V}]\right| \\ \le\;& 16(C_0+1)\big(\mathrm{E}^n\big[\sup_{|w|_\infty\le M}|U(w)|^{p_0}\big]^{1/p_0}+1\big) \\ &\times \mathrm{E}^n[|D^\beta\varphi_M(X_{(l+u)/n}^{n,M})\hat{V}|^q]^{1/q}\alpha_{k-l}^{\varrho_0'}, \end{aligned}$$

where $\varrho_0' = \dfrac{1}{s_0'+\gamma}$ and $\dfrac{1}{s_0'} = 1-\dfrac{1}{p_0}-\dfrac{1}{q}$. Since $s_0' \le 2s_0$ holds, which implies $\varrho_0' \ge 2\varrho_0$, and $D^\beta\varphi_M$ is bounded uniformly in x, we have our assertion. □

PROPOSITION 3. *Let $U,V : C([0,\infty);\mathbb{R}^d)\times\Omega^n \longrightarrow \mathbb{R}$ be such that $U(w)$ and $V(w)$ are $\mathcal{F}_{k,k}^n$ and $\mathcal{F}_{l,l}^n$-measurable respectively and $\mathrm{E}^n[U(w)]=0$ for each $w\in\mathcal{C}_M^d$, and $Z:\Omega^n\longrightarrow\mathbb{R}$ be an $\mathcal{F}_{0,m}^n$-measurable random variable. Suppose that there exists $C_0=C_0(M)>0$ such that*

$$\sup_{\varepsilon>0}\varepsilon^\gamma\big\{N_n(\varepsilon,M;U)+\varepsilon^\gamma N_n(\varepsilon,M;V)\big\}\le C_0, \tag{3.9}$$

$$\mathrm{E}^n\Big[\sup_{|w|_\infty\le M}|U(w)|^{p_0}\Big]^{1/p_0}\le C_0 \tag{3.10}$$

and

$$\mathrm{E}^n\Big[\sup_{|w|_\infty\le M}|V(w)|^{p_0}\Big]^{1/p_0}\le C_0. \tag{3.11}$$

Then there exists a constant $C > 0$ depending only on M and C_0 such that for all $m \le l \le k$, $u \in [0,1]$ and $\beta, \beta' \in \mathbb{Z}_+^d$ with $|\beta| + |\beta'| \le 2$

$$\left| \mathrm{E}^n[\Xi^M_{\beta,\beta'}(Y^{n,M}_m(u,\cdot))Z] \right| \le C\,\mathrm{E}^n[|Z|^{p_0}]^{1/p_0} \alpha^{\varrho_0}_{k-l} \alpha^{\varrho_0}_{l-m},$$

where $\Xi^M_{\beta,\beta'}(w) = D^\beta \varphi_M(w(k/n)) D^{\beta'} \varphi_M(w(l/n)) \Xi(w)$, $\Xi(w) = U(w)V(w) - \mathrm{E}^n[U(w)V(w)]$.

PROOF. Define $\hat{Z}$ by

$$\hat{Z} = \begin{cases} Z & \text{if } \left|X^{n,M}_{(m+u)/n}\right| \le M \\ 0 & \text{otherwise}. \end{cases}$$

Then we have

$$\begin{aligned} &\mathrm{E}^n[\Xi^M(Y^{n,M}_m(u,\cdot))Z] \\ = \ &\mathrm{E}^n[\Xi(\hat{Y}^{n,M}_m(u,\cdot)) D^\beta \varphi_M(X^{n,M}_{(m+u)/n}) D^{\beta'} \varphi_M(X^{n,M}_{(m+u)/n}) \hat{Z}], \end{aligned} \tag{3.12}$$

where $\hat{Y}^{n,M}_m(u,t)$ is given by (3.7). Using Lemma 4, we see that there exists $C_1 > 0$ depending only on M and C_0 such that

$$\begin{aligned} &\left| \mathrm{E}^n[\Xi(\hat{Y}^{n,M}_m(u,\cdot)) \varphi_M(X^{n,M}_{(m+u)/n})^2 \hat{Z}] \right| \\ \le \ & C_1\,\mathrm{E}^n[|D^\beta \varphi_M(X^{n,M}_{(m+u)/n}) D^{\beta'} \varphi_M(X^{n,M}_{(m+u)/n}) \hat{Z}|^{p_0}]^{1/p_0} \alpha^{\varrho_0}_{k-l} \alpha^{\varrho_0}_{l-m}. \end{aligned}$$

Then we have our assertion. □

Let $Q^{n,M}$ be the probability measure induced by $X^{n,M}$ on $C([0,\infty);\mathbb{R}^d)$.

PROPOSITION 4. *The family of measures $(Q^{n,M})_n$ is tight for each fixed $M > |x_0|$.*

PROOF. Take any $T > 0$. Let $0 \le s < t < u \le T$, $0 < \delta_0 < \dfrac{p_0 - 3}{2} \wedge 1$ and set

$$J^n_0 = \mathrm{E}^n[|X^{n,M,i}_u - X^{n,M,i}_t|^2 |X^{n,M,i}_t - X^{n,M,i}_s|^{1+\delta_0}].$$

By the argument in [1], [5] and [16], it suffices to show that there exists a constant $C_0 = C_0(M,T) > 0$ which is independent of s, t, u and n such that

$$J^n_0 \le C_0 |u - s|^{1+1/q_0}, \tag{3.13}$$

where $q_0 = \dfrac{p_0}{1+\delta_0}$.

First we consider the case of $u - s < 1/n$. In this case, it follows that $[ns]+1 = [nt] = [nu]$ or $[ns] = [nt] = [nu]-1$.

If $[ns]+1 = [nt] = [nu]$, by assumption [A3] and Proposition 1, we have

$$
\begin{aligned}
(3.14)\quad J_0^n &= \mathrm{E}^n\Big[\big|\sqrt{n}(u-t)H_{[nt]}^{n,M}(X^{n,M})\big|^2 \\
&\qquad \times\Big|\frac{1}{\sqrt{n}}(nt-[nt])H_{[nt]}^{n,M}(X^{n,M}) \\
&\qquad +\frac{1}{\sqrt{n}}(1-ns+[ns])H_{[ns]}^{n,M}(X^{n,M})\Big|^{1+\delta_0}\Big] \\
&= (\sqrt{n})^{1-\delta_0}|u-s|^2\,\mathrm{E}^n\big[\big|H_{[nt]}^{n,M,i}(X^{n,M})\big|^2 \\
&\qquad \times\big\{(nt-[nt])H_{[nt]}^{n,M,i}(X^{n,M}) \\
&\qquad +(1-ns+[ns])H_{[ns]}^{n,M,i}(X^{n,M})\big\}^2\big] \\
&\le (\sqrt{n})^{1-\delta_0}|u-s|^2\big\{E^n[|H_{[nt]}^{n,M,i}(X^{n,M})|^{p_0}]^{(3+\delta_0)/p_0} \\
&\qquad +E^n[|H_{[nt]}^{n,M,i}(X^{n,M})|^{p_0}]^{2/p_0} \\
&\qquad \times E^n[|H_{[ns]}^{n,M,i}(X^{n,M})|^{p_0}]^{(1+\delta_0)/p_0}\big\} \\
&\le C_1(\sqrt{n})^{1-\delta_0}|u-s|^2 \le C_1|u-s|^{(3+\delta_0)/2} \le C_2|u-s|^{1+1/q_0}
\end{aligned}
$$

for some $C_1 = C_1(M) > 0$ and $C_2 = C_2(M,T) > 0$.

If $[ns] = [nt] = [nu]-1$, the similar calculation gives us the following estimation

$$J_0^n \le C_3|u-s|^{1+1/q_0}$$

for some $C_3 = C_3(M,T) > 0$. So the inequality (3.13) holds when $u - s < 1/n$.

Next we consider the case of $u-s \ge 1/n$. We will show that there exists a constant $C_4 = C_4(M,T) > 0$ such that

$$(3.15)\qquad \mathrm{E}^n[|X_v^{n,M,i} - X_r^{n,M,i}|^2\Phi] \le C_4|u-s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0}$$

for each $r, v \in [s,u]$ with $r \le v$ and each $\mathcal{F}^n_{0,([nr]-1)\vee 0}$-measurable non-negative random variable Φ.

Since we have

$$\begin{aligned} & |X_v^{n,M,i} - X_r^{n,M,i}|^2 \\ \le\ & 3\Big\{|X_{([nv]+1)/n}^{n,M,i} - X_v^{n,M,i}|^2 + |X_r^{n,M,i} - X_{[nr]/n}^{n,M,i}|^2 \\ & \qquad + \Big|\sum_{k=[nr]}^{[nv]}\big(X_{(k+1)/n}^{n,M,i} - X_{k/n}^{n,M,i}\big)\Big|^2\Big\} \end{aligned}$$

and the following equality

$$\Big(\sum_{l=1}^k x_l\Big)^2 = \sum_{l=1}^k x_l^2 + 2\sum_{l=1}^k x_l(x_1 + \cdots + x_l), \quad x_1, \ldots, x_k \in \mathbb{R}, \tag{3.16}$$

it follows that

$$\mathrm{E}^n[|X_v^{n,M,i} - X_r^{n,M,i}|^2\Phi] \le 6(J_1^n + J_2^n + J_3^n + J_4^n + J_5^n),$$

where

$$\begin{aligned} J_1^n &= \mathrm{E}^n[|X_{([nv]+1)/n}^{n,M,i} - X_v^{n,M,i}|^2\Phi], \\ J_2^n &= \mathrm{E}^n[|X_r^{n,M,i} - X_{[nr]/n}^{n,M,i}|^2\Phi], \\ J_3^n &= \frac{1}{n}\sum_{k=[nr]}^{[nv]} \mathrm{E}^n\big[\big|H_k^{n,M,i}(X^{n,M})\big|^2\Phi\big], \\ J_4^n &= \frac{1}{\sqrt{n}}\sum_{k=[nr]}^{[nv]} \big|\mathrm{E}^n[F_k^{n,M,i}(X^{n,M})(X_{k/n}^{n,M,i} - X_{[nr]/n}^{n,M,i})\Phi]\big|, \\ J_5^n &= \frac{1}{n}\sum_{k=[nr]}^{[nv]} \big|\mathrm{E}^n[G_k^{n,M,i}(X^{n,M})(X_{k/n}^{n,M,i} - X_{[nr]/n}^{n,M,i})\Phi]\big|. \end{aligned}$$

Since $\dfrac{2}{p_0} + \dfrac{1}{q_0} < 1$, we have

$$J_1^n \le \frac{1}{n}([nv] + 1 - v)^2\,\mathrm{E}^n[|H_{[nv]}^{n,M,i}(X^{n,M})|^{p_0}]^{2/p_0}\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \tag{3.17}$$

$$\le C_5 \times \frac{1}{n}\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \le C_5|u - s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \tag{3.18}$$

for some $C_5 = C_5(M) > 0$. Similarly we have

$$J_2^n \le C_6|u-s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \tag{3.19}$$

for some $C_6 = C_6(M) > 0$. We also have

$$\begin{aligned} J_3^n &\le C_7 \cdot \frac{[nv]-[nr]+1}{n}\,\mathrm{E}^n[\Phi^{p_0}]^{1/p_0} \\ &\le C_7\Big(|v-r| + \frac{2}{n}\Big)\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \le 3C_7|u-s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \end{aligned} \tag{3.20}$$

for some $C_7 = C_7(M) > 0$.

To estimate J_4^n, using Taylor's theorem (Theorem 1.43 in [12]), we have

$$\begin{aligned} &\mathrm{E}^n[F_k^{n,M,i}(X^{n,M})(X_{k/n}^{n,M,i} - X_{[nr]/n}^{n,M,i})\Phi] \\ =& \sum_{l=[nr]}^{k-1} \Big\{ \mathrm{E}^n\big[F_k^{n,M,i}(X_{\cdot\wedge((l+1)/n)}^{n,M})\big(X_{(l+1)/n}^{n,M,i} - X_{l/n}^{n,M,i}\big)\Phi\big] \\ &\qquad + \mathrm{E}^n\big[\big(F_k^{n,M,i}(X_{\cdot\wedge((l+1)/n)}^{n,M}) \\ &\qquad - F_k^{n,M,i}(X_{\cdot\wedge(l/n)}^{n,M})\big)\big(X_{l/n}^{n,M,i} - X_{[nr]/n}^{n,M,i}\big)\Phi\big]\Big\} \\ =& \frac{1}{\sqrt{n}} \sum_{l=[nr]}^{k-1} \{\Lambda_{k,l}^{n,(1)} + \Lambda_{k,l}^{n,(2)} + \Lambda_{k,l}^{n,(3)}\}, \end{aligned}$$

where

$$\Lambda_{k,l}^{n,(1)} = \mathrm{E}^n\big[\varphi_M(X_{(l+1)/n}^{n,M})F_k^{n,i}(X_{\cdot\wedge((l+1)/n)}^{n,M})H_l^{n,M,i}(X^{n,M})\Phi\big],$$

$$\begin{aligned} \Lambda_{k,l}^{n,(2)} &= \sum_{j=1}^{d}\int_0^1 \mathrm{E}^n\big[\frac{\partial}{\partial x^j}\varphi_M(Y_l^{n,M}(u,k/n))F_k^{n,i}(Y_l^{n,M}(u,\cdot)) \\ &\qquad \times H_l^{n,M,j}(X^{n,M})\big(X_{l/n}^{n,M,i} - X_{[nr]/n}^{n,M,i}\big)\Phi\big]\,du, \\ \Lambda_{k,l}^{n,(3)} &= \sum_{j=1}^{d}\int_0^1 \mathrm{E}^n\big[\varphi_M(Y_l^{n,M}(u,k/n))\nabla F_k^{n,i}(Y_l^{n,M}(u,\cdot);I_l^n e_j) \\ &\qquad \times H_l^{n,M,j}(X^{n,M})\big(X_{l/n}^{n,M,i} - X_{(nr]/n}^{n,M,i}\big)\Phi\big]\,du. \end{aligned}$$

Let r_0 be such that $\dfrac{1}{r_0} = \dfrac{1}{p_0} + \dfrac{1}{q_0}$. Since

$$\frac{1}{2}\Big(1 + \frac{1}{p_0}\Big) - \frac{1}{r_0} = \frac{p_0 - 3 - 2\delta_0}{2p_0} > 0, \tag{3.21}$$

using Proposition 2 with $U = F^{n,i}_k$, $V = H^{n,M,i}_l(X^{n,M})$ and $u = 1$, we have

$$\begin{aligned} |\Lambda^{n,(1)}_{k,l}| &\leq C_8\big(\mathrm{E}^n[\sup_{|w|_\infty \leq M} |F^{n,i}_k(w)|^{p_0}]^{1/p_0} + 1\big) \\ &\quad \times \mathrm{E}^n[|H^{n,M,i}_l(X^{n,M})\Phi|^{r_0}]^{1/r_0}\alpha^{\varrho_0}_{k-l} \\ &\leq C_9\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0}\alpha^{\varrho_0}_{k-l}. \end{aligned} \tag{3.22}$$

for some $C_8, C_9 > 0$ depending only on M.

Also we see

$$\begin{aligned} &\mathrm{E}^n[|H^{n,M,j}_l(X^{n,M})(X^{n,M,i}_{l/n} - X^{n,M,i}_{[nr]/n})\Phi|^{r_0}]^{1/r_0} \\ =\ &\mathrm{E}^n[|\varphi_M(X^{n,M}_{l/n})H^{n,j}_l(X^{n,M})(X^{n,M,i}_{l/n} - X^{n,M,i}_{[nr]/n})\Phi|^{r_0}]^{1/r_0} \\ \leq\ &M\,\mathrm{E}^n[|\varphi_M(X^{n,M}_{l/n})H^{n,j}_l(X^{n,M})\Phi|^{r_0}]^{1/r_0} \\ \leq\ &M\,\mathrm{E}^n[|H^{n,M,j}_l(X^{n,M})|^{p_0}]^{1/p_0}\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0}. \end{aligned} \tag{3.23}$$

Then, using Proposition 2 again, we have

$$|\Lambda^{n,(2)}_{k,l}|,\ |\Lambda^{n,(3)}_{k,l}| \leq C_{10}\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0}\alpha^{\varrho_0}_{k-l} \tag{3.24}$$

for some $C_{10} = C_{10}(M) > 0$. Thus

$$\begin{aligned} J^n_4 &\leq C_{11} \times \frac{1}{n}\sum_{k=[nr]}^{[nv]}\sum_{l=[nr]}^{k-1}\mathrm{E}^n[\Phi^{q_0}]^{1/q_0}\alpha^{\varrho_0}_{k-l} \\ &\leq 3C_{11}\Big(\sum_{k=1}^{\infty}\alpha^{\varrho_0}_k\Big)|u-s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \end{aligned} \tag{3.25}$$

for some $C_{11} = C_{11}(M) > 0$.

By the similar calculation of (3.23), we have

$$J^n_5 \leq C_{12}|u-s|\,\mathrm{E}^n[\Phi^{q_0}]^{1/q_0} \tag{3.26}$$

for some $C_{12} = C_{12}(M) > 0$. Then the inequality (3.15) holds.

Using (3.15) with $v = u, r = t$ and $\Phi = |X_t^{n,M,i} - X_s^{n,M,i}|^{1+\delta_0} 1_{\{|X^{n,M}|_{[nt]/n}| \leq M\}}$, we get

$$J_0^n \leq C_4 |u-s| \,\mathrm{E}^n[|X_t^{n,M,i} - X_s^{n,M,i}|^{p_0} 1_{\{|X_{[nt]/n}^{n,M}| \leq M\}}]^{1/q_0}. \tag{3.27}$$

Using (3.15) again with $v = [nt]/n, r = s$ and $\Phi = 1$, we get

$$\mathrm{E}^n[|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^2] \leq C_4 |u-s|. \tag{3.28}$$

Thus

$$\begin{aligned}
& \mathrm{E}^n[|X_{t/n}^{n,M,i} - X_s^{n,M,i}|^{p_0} 1_{\{|X_{[nt]/n}^{n,M}| \leq M\}}] \\
\leq\ & C_{13} \Big\{ \mathrm{E}^n[|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^{p_0} 1_{\{|X_{[nt]/n}^{n,M}| \leq M\}}] \\
& \qquad + \mathrm{E}^n[|X_t^{n,M,i} - X_{[nt]/n}^{n,M,i}|^{p_0} 1_{\{|X_{[nt]/n}^{n,M}| \leq M\}}] \Big\} \\
\leq\ & C_{14} \Big\{ M^{p_0-2} \,\mathrm{E}^n[|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^2] \\
& \qquad + \frac{1}{(\sqrt{n})^{p_0}} (nt - [nt]) \,\mathrm{E}^n[|H_{[nt]}^{n,M,i}(X^{n,M})|^{p_0}] \Big\} \\
\leq\ & C_{15} \Big(|u-s| + \frac{1}{(\sqrt{n})^{p_0}} \Big) \leq 2C_{15} |u-s|
\end{aligned}$$

for some $C_{13}, C_{14}, C_{15} > 0$ depending only on M. Thus the inequality (3.13) holds also when $u - s \geq 1/n$. This completes the proof of Proposition 4. $\square$

By Proposition 4, for any subsequence $(n_k)_k$, there is a further subsequence $(n_{k_l})_l$ such that $Q^{n_{k_l},M}$ converges weakly to some probability measure Q^M on $C([0,\infty);\mathbb{R}^d)$ as $l \to \infty$ for each fixed $M > 1 + |x_0|$.

Proposition 5. $Q^M(\mathcal{C}_M^d) = 1$.

Proof. For each $T > 0$, it follows that

$$\begin{aligned}
& Q^M(\sup_{0 \leq t \leq T} |w(t)| > M) \\
=\ & \lim_{\varepsilon \searrow 0} Q^M(\sup_{0 \leq t \leq T} |w(t)| > M + \varepsilon) \\
\leq\ & \lim_{\varepsilon \searrow 0} \liminf_{n \to \infty} P^{n_{k_j}}(\sup_{0 \leq t \leq T} |X_t^{n,M}| > M + \varepsilon).
\end{aligned} \tag{3.29}$$

Here we see

$$
\begin{aligned}
& P^n(\sup_{0\le t\le T}|X^{n,M}_t| > M+\varepsilon) \\
\le\ & P^n(|X^{n,M}_{k/n}| \le M,\ |X^n_{k/n}| + \frac{1}{\sqrt{n}}|H^{n,M}_k(X^{n,M})| > M+\varepsilon \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{for some } k = 0,\dots,[nT]) \\
\le\ & \sum_{k=0}^{[nT]} P^n(|H^{n,M}_k(X^{n,M})| \ge \varepsilon\sqrt{n}) \le C_0 \times \frac{1}{\varepsilon^3\sqrt{n}}
\end{aligned}
$$

for some $C_0 = C_0(M,T) > 0$. Thus

$$
Q^M(\sup_{0\le t\le T}|w(t)| > M) = 0, \quad T > 0. \tag{3.30}
$$

This implies the assertion. □

Next we define functions $a^{M,ij}(t,w)$ and $b^{M,i}(t,w)$ by

$$
\begin{aligned}
a^{M,ij}(t,w) &= \varphi_M(w(t))^2 a^{ij}(t,w) \\
b^{M,i}(t,w) &= \varphi_M(w(t))b^i_0(t,w) + \sum_{j=1}^d \Big\{\varphi_M(w(t))^2 B^{ij}(t,w) \\
&\qquad + \varphi_M(w(t))\frac{\partial}{\partial x^j}\varphi_M(w(t))A^{ij}(t,w)\Big\}
\end{aligned}
$$

and let

$$
\mathscr{L}^M f(t,w) = \frac{1}{2}\sum_{i,j=1}^d a^{M,ij}(t,w)\frac{\partial^2}{\partial x^i\partial x^j}f(w(t)) + \sum_{i=1}^d b^{M,i}(t,w)\frac{\partial}{\partial x^i}f(w(t))
$$

for $f \in C^2(\mathbb{R}^d)$.

PROPOSITION 6. *Q^M is a solution of the martingale problem associated with the generator $\mathscr{L}^M$ and starting at x_0.*

By Proposition 5, in order to prove Proposition 6, it suffices to show that

$$
\begin{aligned}
& \mathrm{E}^{Q^M}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\dots,w(s_N))] \\
=\ & \mathrm{E}^{Q^M}[\int_s^t \mathscr{L}^M f(u,w)du\Phi(w(s_1),\dots,w(s_N))]
\end{aligned} \tag{3.31}
$$

for any C^∞ function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ with compact support, integer N, real numbers $0 \le s_1 < \dots < s_N \le s < t$ and bounded continuous function $\Phi : (\mathbb{R}^N)^m \longrightarrow \mathbb{R}$. Until Proposition 14, we omit the M in $X_t^{n,M}$ and $Y_k^{n,M}(u,t)$ as long as there is no misunderstanding, and simply denote (n_{k_l}) by (n).

Since f and Φ are bounded, it follows that

$$\begin{aligned}(3.32)\qquad & \mathrm{E}^{Q^{n,M}}[(f(w(t)) - f(w(s)))\Phi(w(s_1), \dots, w(s_N))] \\ & \longrightarrow \ \mathrm{E}^{Q^M}[(f(w(t)) - f(w(s)))\Phi(w(s_1), \dots, w(s_N))].\end{aligned}$$

On the other hand, Taylor's theorem implies

$$\begin{aligned}(3.33)\qquad & \mathrm{E}^{Q^{n,M}}[(f(w(t)) - f(w(s)))\Phi(w(s_1), \dots, w(s_N))] \\ = \ & K_1^n + K_2^n + K_3^n + K_4^n + \frac{1}{2}K_5^n + K_6^n + \frac{1}{2}K_7^n + \frac{1}{2}K_8^n,\end{aligned}$$

where

$$\begin{aligned}
K_1^n &= \mathrm{E}^n[(f(X_t^n) - f(X_{[nt]/n}^n))\Phi(X_{s_1}^n, \dots, X_{s_N}^n)], \\
K_2^n &= \mathrm{E}^n[(f(X_{[ns]/n}^n) - f(X_s^n))\Phi(X_{s_1}^n, \dots, X_{s_N}^n)], \\
K_3^n &= \frac{1}{\sqrt{n}} \sum_{i=1}^{d} \sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{k/n}^n) F_k^{n,M,i}(X^n)\Phi(X_{s_1}^n, \dots, X_{s_N}^n)], \\
K_4^n &= \frac{1}{n} \sum_{i=1}^{d} \sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{k/n}^n) G_k^{n,M,i}(X^n)\Phi(X_{s_1}^n, \dots, X_{s_N}^n)], \\
K_5^n &= \frac{1}{n} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X_{k/n}^n) \\
&\qquad\qquad\qquad \times F_k^{n,M,i}(X^n) F_k^{n,M,j}(X^n)\Phi(X_{s_1}^n, \dots, X_{s_N}^n)], \\
K_6^n &= \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X_{k/n}^n) \\
&\qquad\qquad\qquad \times F_k^{n,M,i}(X^n) G_k^{n,M,j}(X^n)\Phi(X_{s_1}^n, \dots, X_{s_N}^n)],
\end{aligned}$$

$$K_7^n = \frac{1}{n^2}\sum_{i,j=1}^{d}\sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X_{k/n}^n) \\ \times G_k^{n,M,i}(X^n) G_k^{n,M,j}(X^n)\Phi(X_{s_1}^n,\dots,X_{s_N}^n)],$$

$$K_8^n = \frac{1}{n\sqrt{n}}\sum_{i,j,\nu=1}^{d}\sum_{k=[ns]}^{[nt]-1}\int_0^1 (1-u)^2 \mathrm{E}^n[\frac{\partial^3}{\partial x^i \partial x^j \partial x^\nu} f(Y_k^n(u,k/n)) \\ \times H_k^{n,M,i}(X^n) H_k^{n,M,j}(X^n) H_k^{n,M,\nu}(X^n)\Phi(X_{s_1}^n,\dots,X_{s_N}^n)]du.$$

PROPOSITION 7. *$K_j^n \longrightarrow 0$ as $n \to \infty$, $j = 1,2,6,7,8$.*

PROOF. By (3.2) and (3.3), we have

$$|K_6^n| \le \frac{1}{n\sqrt{n}}\sum_{k=[ns]}^{[nt]-1} C(M,f,\Phi) \longrightarrow 0$$

for some constant $C(M,f,\Phi) > 0$. Similarly we get $K_7^n \longrightarrow 0$ and $K_8^n \longrightarrow 0$. Taylor's theorem implies

$$\begin{aligned} |K_1^n| &\le \frac{1}{\sqrt{n}}\sum_{i=1}^{d}\int_0^1 \mathrm{E}^n\Big[\Big|\frac{\partial}{\partial x^i} f\big(Y_{[nt]}^n(u,t)\big)(nt-[nt]) H_{[nt]}^{n,M,i}(X^n)\Phi\Big|\Big]du \\ &\le \mathrm{const.} \times \frac{1}{\sqrt{n}} \longrightarrow 0. \end{aligned}$$

Similar arguments give us $K_2^n \longrightarrow 0$. Then we obtain the assertion. □

To treat the convergent of K_3^n, K_4^n and K_5^n, we will show the following three propositions.

PROPOSITION 8. *Let $U_k^n : C([0,\infty);\mathbb{R}^d) \times \Omega^n \longrightarrow \mathbb{R}$ be a continuously Fréchet differentiable random function such that $U_k^n(w)$ is $\mathcal{F}_{k,\infty}^n$-measurable and $\mathrm{E}^n[U_k^n(w)] = 0$ for each $w \in \mathcal{C}_M^d$, and $V^n : \Omega^n \longrightarrow \mathbb{R}$ be an $\mathcal{F}_{0,[ns]}^n$-measurable random variable. Suppose that there exists a constant $C_0 =$*

$C_0(M) > 0$ *such that*

$$\sup_{\varepsilon>0} \varepsilon^\gamma N_n(\varepsilon, M; U_k^n) \le C_0, \tag{3.34}$$
$$\sup_{l\le k} \sup_{\varepsilon>0} \varepsilon^\gamma N_n(\varepsilon, M; \nabla U_k^n(\cdot; I_l^n e_j)) \le C_0,$$

$$\sum_{m=0}^{1} \mathrm{E}^n\Big[\sup_{|w|_\infty \le M} |\nabla^m U_k^n(w)|_{L^m_{k/n}}^{p_0}\Big] \le C_0 \tag{3.35}$$

and

$$\mathrm{E}^n[|V^n|^{p_0/2}] \le C_0 \tag{3.36}$$

for any $j = 1, \dots, d, n \in \mathbb{N}$ *and* $k \in \mathbb{Z}_+$. *Then it holds that*

$$\frac{1}{n}\sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[D^\beta \varphi_M(X^n_{k/n}) U_k^n(X^n) V^n] \longrightarrow 0, \quad n \to \infty \tag{3.37}$$

for $\beta \in \mathbb{Z}_+^d$ *with* $|\beta| \le 1$.

PROOF. By Taylor's theorem, we have

$$\begin{aligned}
&\mathrm{E}^n[D^\beta \varphi_M(X^n_{k/n}) U_k^n(X^n) V^n] \\
= &\sum_{l=[ns]}^{k-1} \mathrm{E}^n[\{D^\beta \varphi_M(X^n_{(l+1)/n}) U_k^n(X^n_{\cdot\wedge((l+1)/n)}) \\
&\qquad\qquad - D^\beta \varphi_M(X^n_{l/n}) U_k^n(X^n_{\cdot\wedge(l/n)})\} V^n] \\
&+ \mathrm{E}^n[D^\beta \varphi_M(X^n_{[ns]/n}) U_k^n(X^n_{\cdot\wedge([ns]/n)}) V^n] \\
= &\frac{1}{\sqrt{n}} \sum_{i=1}^{d} \sum_{l=[ns]}^{k-1} \int_0^1 \Big\{ \mathrm{E}^n[\frac{\partial}{\partial x^i} D^\beta \varphi_M(Y_l^{n,M}(u, k/n)) \\
&\qquad\qquad \times U_k^n(Y_l^{n,M}(u,\cdot)) H_l^{n,M,i}(X^n) V^n] \\
&\qquad + \mathrm{E}^n[D^\beta \varphi_M(Y_l^{n,M}(u, k/n)) \\
&\qquad\qquad \times \nabla U_k^n(Y_l^{n,M}(u,\cdot); I_l^n e_i) H_l^{n,M,i}(X^n) V^n]\Big\} du \\
&+ \mathrm{E}^n[D^\beta \varphi_M(X^n_{[ns]/n}) U_k^n(X^n_{\cdot\wedge([ns]/n)}) V^n].
\end{aligned}$$

By Proposition 2, we see that

$$
\begin{aligned}
(3.38)\quad & \big|\,\mathrm{E}^{n}[\frac{\partial}{\partial x^i} D^\beta \varphi_M(Y_l^{n,M}(u,k/n))U_k^n(Y_l^{n,M}(u,\cdot))H_l^{n,M,i}(X^n)V^n]\big| \\
\leq\ & C_1 \alpha_{k-l}^{\varrho_0}, \\
(3.39)\quad & \big|\,\mathrm{E}^{n}[D^\beta \varphi_M(Y_l^{n,M}(u,k/n))\nabla U_k^n(Y_l^{n,M}(u,\cdot);I_l^n e_i)H_l^{n,M,i}(X^n)V^n]\big| \\
\leq\ & C_1 \alpha_{k-l}^{\varrho_0}
\end{aligned}
$$

and

$$
(3.40)\qquad \big|\,\mathrm{E}^{n}[D^\beta \varphi_M(X_{[ns]/n}^n)U_k^n(X_{\cdot\wedge([ns])/n}^n)V^n]\big| \leq C_1 \alpha_{k-[ns]}^{\varrho_0}
$$

for some $C_1 > 0$ depending only on M and C_0. Thus

$$
\begin{aligned}
& \frac{1}{n}\sum_{k=[ns]}^{[nt]-1} \big|\,\mathrm{E}^{n}[D^\beta \varphi_M(X_{k/n}^n)U_k^n(X^n)V^n]\big| \\
\leq\ & 2C_1 d \times \frac{1}{n}\sum_{k=[ns]}^{[nt]-1}\Big\{\sum_{l=[ns]}^{k-1}\frac{1}{\sqrt{n}}\alpha_{k-l}^{\varrho_0} + \alpha_{k-[ns]}^{\varrho_0}\Big\} \\
\leq\ & 2C_1 d\Big(\sum_{k=1}^{\infty}\alpha_k^{\varrho_0}\Big)(t+1)\times\frac{1}{\sqrt{n}} \ \longrightarrow\ 0, \quad n\to\infty.
\end{aligned}
$$

Then we obtain the assertion. □

PROPOSITION 9. *Let $U_k^n, V_k^n : C([0,\infty);\mathbb{R}^d)\times\Omega^n \longrightarrow \mathbb{R}$ be such that $U_k^n(w)$ and $V_k^n(w)$ are $\mathcal{F}_{k,k}^n$-measurable and continuously Fréchet differentiable random functions such that $\mathrm{E}^n[U_k^n(w)] = 0$ for each $w \in \mathcal{C}_M^d$, and $Z^n : \Omega^n \longrightarrow \mathbb{R}$ be an $\mathcal{F}_{0,[ns]}^n$-measurable random variable. Suppose that there exists a constant $C_0 = C_0(M) > 0$ such that*

$$
(3.41)\qquad \sup_{\varepsilon>0}\varepsilon^\gamma\big\{N_n(\varepsilon,M;U_k^n)+N_n(\varepsilon,M;V_k^n)\big\}\leq C_0,
$$

$$
\begin{aligned}
(3.42)\qquad & \sup_{l\leq k}\sup_{\varepsilon>0}\varepsilon^\gamma\big\{N_n(\varepsilon,M;\nabla U_k^n(\cdot;I_l^n e_j)) \\
& \qquad +N_n(\varepsilon,M;\nabla V_k^n(\cdot;I_l^n e_j))\big\}\leq C_0,
\end{aligned}
$$

$$
\begin{aligned}
(3.43)\qquad & \sum_{m=0}^{1}\mathrm{E}^n\Big[\sup_{|w|_\infty\leq M}|\nabla^m U_k^n(w)|_{L_{k/n}^m}^{p_0}\Big] \\
\leq\ & C_0,\ \sum_{m=0}^{1}\mathrm{E}^n\Big[\sup_{|w|_\infty\leq M}|\nabla^m V_k^n(w)|_{L_{k/n}^m}^{p_0}\Big]\leq C_0
\end{aligned}
$$

and

$$\mathrm{E}^n[|Z^n|^{p_0}] \le C_0 \tag{3.44}$$

for any $j = 1, \dots, d, n \in \mathbb{N}$ *and* $k \in \mathbb{Z}_+$. *Then it holds that*

$$\text{(i)} \quad \frac{1}{n}\sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[D^\beta \varphi_M(X^n_{k/n}) D^{\beta'}\varphi_M(X^n_{k/n})\Xi^n_{kk}(X^n)Z^n] \longrightarrow 0, \tag{3.45}$$

$$\begin{aligned}\text{(ii)} \quad \frac{1}{n}\sum_{k=[ns]}^{[nt]-1}\sum_{l=[ns]}^{k-1} \mathrm{E}^n[D^\beta \varphi_M(X^n_{l/n}) \\ \times D^{\beta'}\varphi_M(X^n_{l/n})\Xi^n_{kl}(X^n_{\cdot\wedge(l/n)})Z^n] \longrightarrow 0\end{aligned} \tag{3.46}$$

as $n \to \infty$ *for* $\beta, \beta' \in \mathbb{Z}^d_+$ *with* $|\beta| + |\beta'| \le 1$, *where* $\Xi^n_{kl}(w) = U^n_k(w)V^n_l(w) - \mathrm{E}^n[U^n_k(w)V^n_l(w)]$.

Proof. By Taylor's theorem, we have

$$\begin{aligned}
&\mathrm{E}^n[D^\beta\varphi_M(X^n_{l/n})D^{\beta'}\varphi_M(X^n_{l/n})\Xi^n_{kl}(X^n_{\cdot\wedge(l/n)})Z^n] \\
= &\sum_{m=[ns]}^{l-1} \mathrm{E}^n[\{D^\beta\varphi_M(X^n_{(m+1)/n})D^{\beta'}\varphi_M(X^n_{(m+1)/n})\Xi^n_{kl}(X^n_{\cdot\wedge((m+1)/n)}) \\
&\qquad -D^\beta\varphi_M(X^n_{m/n})D^{\beta'}\varphi_M(X^n_{m/n})\Xi^n_{kl}(X^n_{\cdot\wedge(m/n)})\}Z^n] \\
&+\mathrm{E}^n[D^\beta\varphi_M(X^n_{[ns]/n})D^{\beta'}\varphi_M(X^n_{[ns]/n})\Xi^n_{kl}(X^n_{\cdot\wedge([ns])/n})Z^n] \\
= &\frac{1}{\sqrt{n}}\sum_{i=1}^{d}\sum_{m=[ns]}^{l-1}\int_0^1 \Big\{ \mathrm{E}^n\Big[\Big\{\frac{\partial}{\partial x^i}D^\beta\varphi_M D^{\beta'}\varphi_M \\
&\qquad +D^\beta\varphi_M\frac{\partial}{\partial x^i}D^{\beta'}\varphi_M\Big\}(Y^{n,M}_m(u,l/n)) \\
&\qquad \times\Xi^n_{kl}(Y^{n,M}_m(u,\cdot))H^{n,M,i}_m(X^n)Z^n\Big] \\
&+\mathrm{E}^n\big[D^\beta\varphi_M(Y^{n,M}_m(u,l/n))D^{\beta'}\varphi_M(Y^{n,M}_m(u,l/n)) \\
&\qquad \times\nabla\Xi^n_{kl}(Y^{n,M}_m(u,\cdot);I^n_m e_i)H^{n,M,i}_m(X^n)Z^n\big]\Big\}du \\
&+\mathrm{E}^n[D^\beta\varphi_M(X^n_{[ns]/n})D^{\beta'}\varphi_M(X^n_{[ns]/n})\Xi^n_{kl}(X^n_{\cdot\wedge([ns])/n})Z^n].
\end{aligned}$$

Since

$$\begin{aligned} (3.47) \qquad & \nabla \Xi^n_{kl}(w; I^n_m e_i) \\ = \ & \nabla U^n_k(w; I^n_m e_i) V^n_l(w) - \mathrm{E}^n[\nabla U^n_k(w; I^n_m e_i) V^n_l(w)] \\ & + U^n_k(w) \nabla V^n_l(w; I^n_m e_i) - \mathrm{E}^n[U^n_k(w) \nabla V^n_l(w; I^n_m e_i)] \end{aligned}$$

holds, using Proposition 3, we get

$$\begin{aligned} (3.48) \qquad & \left| \mathrm{E}^n[D^\beta \varphi_M(X^n_{l/n}) D^{\beta'} \varphi_M(X^n_{l/n}) \Xi^n_{kl}(X^n_{\cdot\wedge(l/n)}) Z^n] \right| \\ \leq \ & C_1 \Big\{ \frac{1}{\sqrt{n}} \sum_{m=[ns]}^{l-1} \alpha^{\varrho_0}_{k-l} \alpha^{\varrho_0}_{l-m} + \alpha^{\varrho_0}_{k-l} \alpha^{\varrho_0}_{l-[ns]} \Big\} \end{aligned}$$

for some $C_1 > 0$ depending only on M and C_0. In particular it follows that

$$\begin{aligned} (3.49) \qquad & \left| \mathrm{E}^n[D^\beta \varphi_M(X^n_{k/n}) D^{\beta'} \varphi_M(X^n_{k/n}) \Xi^n_{kk}(X^n) Z^n] \right| \\ \leq \ & C_1 \Big\{ \frac{1}{\sqrt{n}} \sum_{m=[ns]}^{k-1} \alpha^{\varrho_0}_{k-m} + \alpha^{\varrho_0}_{k-[ns]} \Big\}. \end{aligned}$$

Thus we have

$$\begin{aligned} & \frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \left| \mathrm{E}^n[D^\beta \varphi_M(X^n_{k/n}) D^{\beta'} \varphi_M(X^n_{k/n}) \Xi^n_{kk}(X^n) Z^n] \right| \\ \leq \ & 2C_1 \Big(\sum_{k=1}^{\infty} \alpha^{\varrho_0}_k \Big) (t+1) \times \frac{1}{\sqrt{n}} \ \longrightarrow \ 0, \quad n \to \infty \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \left| \mathrm{E}^n[D^\beta \varphi_M(X^n_{l/n}) D^{\beta'} \varphi_M(X^n_{l/n}) \Xi^n_{kl}(X^n_{\cdot\wedge(l/n)}) Z^n] \right| \\ \leq \ & 2C_1 \Big(\sum_{k=1}^{\infty} \alpha^{\varrho_0}_k \Big)^2 (t+1) \times \frac{1}{\sqrt{n}} \ \longrightarrow \ 0, \quad n \to \infty. \end{aligned}$$

Then we obtain the assertion. □

PROPOSITION 10. *Let* $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be a continuously differentiable function such that* $\psi(x) = 0$ *for any* $x \in \mathbb{R}^d$ *with* $|x| > M$ *and* g^n :

$\mathbb{Z}_+ \times C([0,\infty);\mathbb{R}^d) \longrightarrow \mathbb{R}$, $g : [0,\infty) \times C([0,\infty);\mathbb{R}^d) \longrightarrow \mathbb{R}$ *be functionals. Suppose that* $g^n(k,\cdot)$ *is* $\mathcal{B}_{k/n}$*-measurable and continuous, and that there exists a constant* $C_0 = C_0(M) > 0$ *such that*

$$\sup_{|w|_\infty \le M} |g^n(k,w)| \le C_0 \tag{3.50}$$

for each $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}_+$. *Moreover suppose*

$$\sup_{w\in K} |g^n([nt],w) - g(t,w)| \longrightarrow 0, \quad n \to \infty \tag{3.51}$$

for each $K \in \mathcal{K}^d$ *and* $t \ge 0$. *Then it holds that*

$$\begin{aligned} &\frac{1}{n}\sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\psi(X^n_{k/n})g^n(k,X^n)\Phi(X^n_{s_1},\dots,X^n_{s_N})] \\ &\longrightarrow \int_s^t \mathrm{E}^{Q^M}[\psi(w(u))g(u,w)\Phi(w(s_1),\dots,w(s_N))]du, \quad n \to \infty \end{aligned} \tag{3.52}$$

PROOF. Denote the left-hand side of (3.52) by K^n. Define L^n and S^n by

$$L^n = \int_s^t \mathrm{E}^n[\psi(X^n_{k/n})g^n([nu],X^n)\Phi(X^n_{s_1},\dots,X^n_{s_N})]du$$

and

$$S^n = \int_s^t \mathrm{E}^n[\psi(X^n_u)g(u,X^n)\Phi(X^n_{s_1},\dots,X^n_{s_N})]du.$$

Then we have

$$\begin{aligned} |K^n - L^n| &\le C_0\int_s^t \mathrm{E}^n[|\psi(X^n_u) - \psi(X^n_{[nu]/n})|\cdot|\Phi|]du \\ &\le \mathrm{const.} \times \frac{1}{\sqrt{n}}\sum_{i=1}^d \int_s^t\int_0^1 \mathrm{E}^n\Big[\Big|\frac{\partial}{\partial x^i}\psi(Y^n_{[nu]}(v,u)) \\ &\qquad\qquad \times(nu-[nu])H^{n,M,j}_{[nu]}(X^n)\Big|\Big]dvdu \\ &\le \mathrm{const.} \times \frac{1}{\sqrt{n}} \longrightarrow 0. \end{aligned}$$

Next we will show

$$L^n - S^n \longrightarrow 0. \tag{3.53}$$

Take any $\varepsilon > 0$. Then, by Proposition 4, there exists a compact set $K \subset C([0,\infty);\mathbb{R}^d)$ such that

$$\inf_n Q^{n,M}(K) > 1 - \varepsilon. \tag{3.54}$$

Set $K_M = K \cap \mathcal{C}^d_M$. Then, by Proposition 1, we have

$$\begin{aligned}
&\big| \mathrm{E}^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\big| \\
\leq\ & \mathrm{const.} \times \Big\{ \sup_{w \in K_M} |g^n([nu], w) - g(u, w)| \\
&\qquad + \big| \mathrm{E}^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n)); X^n \notin K]\big|\Big\} \\
\leq\ & \mathrm{const.} \times \Big\{ \sup_{w \in K_M} |g^n([nu], w) - g(u, w)| \\
&\qquad + \sup_{|w|_\infty \leq M} \big\{|g^n([nu], w)| + |g(u, w)|\big\}\varepsilon\Big\}.
\end{aligned}$$

for each $u \in [s,t]$. Since $K_M \in \mathcal{K}^d$ holds, by (3.50), we have

$$\limsup_{n\to\infty} \big| \mathrm{E}^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\big| \leq \mathrm{const.} \times \varepsilon. \tag{3.55}$$

Thus

$$\lim_{n\to\infty} \big| \mathrm{E}^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\big| = 0 \tag{3.56}$$

for each $u \in [s,t]$. By (3.50) again and the bounded convergence theorem, we get

$$\begin{aligned}
&|L^n - S^n| \\
\leq\ & \int_s^t \big| \mathrm{E}^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\big| du \ \longrightarrow\ 0.
\end{aligned} \tag{3.57}$$

Since

$$F(w) = \int_s^t \psi(w(u)) g(u, w) \Phi(w(s_1), \dots, w(s_N)) du$$

is continuous and Proposition 1 implies

$$Q^{n,M}(|F(w)| \le C_1) = 1 \tag{3.58}$$

for each $n \in \mathbb{N}$, where

$$C_1 = C_0|t-s| \sup_{|x|\le M} |\psi(x)| \sup_{y_1,\dots,y_N\in\mathbb{R}^d} |\Phi(y_1,\dots,y_N)|,$$

using the continuous mapping theorem, we get

$$S^n \longrightarrow \int_s^t \mathrm{E}^{Q^M}[\psi(w(u))g(u,w)\Phi(w(s_1),\dots,w(s_N))]du.$$

This completes the proof of Proposition 10. □

By Proposition 8, 9(i) and 10, we have the following.

Proposition 11.

(i) $$K_4^n \longrightarrow \sum_{i=1}^{d}\int_s^t \mathrm{E}^{Q^M}\Big[\frac{\partial}{\partial x^i} f(w(u))\varphi_M(w(u)) \times b_0^i(u,w)\Phi(w(s_1),\dots,w(s_N))\Big]du,$$

(ii) $$K_5^n \longrightarrow \sum_{i,j=1}^{d}\int_s^t \mathrm{E}^{Q^M}\Big[\frac{\partial^2}{\partial x^i\partial x^j} f(w(u))\varphi_M(w(u))^2 \times a_0^{ij}(u,w)\Phi(w(s_1),\dots,w(s_N))\Big]du$$

as $n \to \infty$.

Next we calculate the limit of K_3^n. Using Taylor's theorem, we have

$$K_3^n = K_{3,1}^n + K_{3,2}^n + K_{3,3}^n + K_{3,4}^n + K_{3,5}^n + K_{3,6}^n + K_{3,7}^n + K_{3,8}^n,$$

where

$$K_{3,1}^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{d}\sum_{k=[ns]}^{[nt]-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{[ns]/n}^n)\varphi_M(X_{[ns]/n}^n)F_k^{n,i}(X_{\cdot\wedge([ns]/n)}^n)\Phi],$$

$$K_{3,2}^n = \frac{1}{n}\sum_{i,j=1}^{d}\sum_{k=[ns]}^{[nt]-1}\sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial^2}{\partial x^i\partial x^j} f(X_{l/n}^n)\varphi_M(X_{l/n}^n)^2 \times F_k^{n,i}(X_{\cdot\wedge(l/n)}^n)F_l^{n,j}(X^n)\Phi],$$

$$K_{3,3}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)^2$$
$$\times F_k^{n,i}(X_{\cdot\wedge(l/n)}^n) G_l^{n,j}(X^n)\Phi],$$

$$K_{3,4}^n = \frac{1}{n} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)$$
$$\times \frac{\partial}{\partial x^j} \varphi_M(X_{l/n}^n) F_k^{n,i}(X_{\cdot\wedge(l/n)}^n) F_l^{n,j}(X^n)\Phi],$$

$$K_{3,5}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)$$
$$\times \frac{\partial}{\partial x^j} \varphi_M(X_{l/n}^n) F_k^{n,i}(X_{\cdot\wedge(l/n)}^n) G_l^{n,j}(X^n)\Phi],$$

$$K_{3,6}^n = \frac{1}{n} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)^2$$
$$\times \nabla F_k^{n,i}(X_{\cdot\wedge(l/n)}^n; I_l^n e_j) F_l^{n,j}(X^n)\Phi],$$

$$K_{3,7}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^n[\frac{\partial}{\partial x^i} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)^2$$
$$\times \nabla F_k^{n,i}(X_{\cdot\wedge(l/n)}^n; I_l^n e_j) G_l^{n,j}(X^n)\Phi],$$

$$K_{3,8}^n = \frac{1}{n\sqrt{n}} \sum_{i,j,\nu=1}^{d} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \int_0^1 (1-u)\,\mathrm{E}^n[\eta_{kl}^{n,M,ij\nu}(Y_l^n(u,\cdot))$$
$$\times H_l^{n,M,j}(X^n) H_l^{n,M,\nu}(X^n)\Phi]du$$

and

$$\begin{aligned}
\eta_{kl}^{n,M,ij\nu}(w) \quad = \quad & \frac{\partial^3}{\partial x^i \partial x^j \partial x^\nu} f(w(l/n)) F_k^{n,M,i}(w) \\
& + \frac{\partial^2}{\partial x^i \partial x^j} f(w(l/n)) \nabla F_k^{n,M,i}(w; I_l^n e_\nu) \\
& + \frac{\partial^2}{\partial x^i \partial x^\nu} f(w(l/n)) \nabla F_k^{n,M,i}(w; I_l^n e_j) \\
& + \frac{\partial}{\partial x^i} f(w(l/n)) \nabla^2 F_k^{n,M,i}(w; I_l^n e_j, I_l^n e_\nu).
\end{aligned}$$

PROPOSITION 12. *$K^n_{3,j} \longrightarrow 0$ as $n \to \infty$, $j = 1, 3, 5, 7, 8$.*

PROOF. Applying Proposition 2 with $U = F^{n,i}_k$ and $V = \dfrac{\partial}{\partial x^i} f(X^n_{[ns]/n})\Phi$, we have

$$|K^n_{3,1}| \le \text{const}\,. \times \frac{1}{\sqrt{n}} \sum_{k=[ns]}^{[nt]-1} \alpha^{\varrho_0}_{k-[ns]} \le \text{const}\,. \times \Big(\sum_{k=0}^{\infty} \alpha^{\varrho_0}_k\Big) \frac{1}{\sqrt{n}} \longrightarrow 0.$$

Applying Proposition 2 again with $U = F^{n,i}_k$ and $V = \dfrac{\partial^2}{\partial x^i \partial x^j} f(X^n_{l/n}) \varphi_M(X^n_{l/n}) G^{n,j}_l(X^n)\Phi$, we have

$$|K^n_{3,3}| \le \text{const}\,. \times \frac{1}{n\sqrt{n}} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \alpha^{\varrho_0}_{k-l} \le \text{const}\,. \times \Big(\sum_{k=0}^{\infty} \alpha^{\varrho_0}_k\Big) \frac{1}{\sqrt{n}} \longrightarrow 0.$$

Similarly we have $K^n_{3,5} \longrightarrow 0$ and $K^n_{3,7} \longrightarrow 0$. Since $\eta^{n,M,ij\nu}_{kl}(w)$ is the finite sum of the following terms

$$D^{\beta} f(w(l/n)) D^{\beta'} \varphi_M(w(k/n)) U(w)$$

with $\beta, \beta' \in \mathbb{Z}^d_+$ and $U(w) = F^{n,i}_k(w), \nabla F^{n,i}_k(w; I^n_l e_j)$ or $\nabla^2 F^{n,i}_k(w; I^n_l e_j, I^n_l e_\nu)$, by Proposition 2, it follows that $K^n_{3,8} \longrightarrow 0$. Then we obtain the assertion. □

For $K^n_{3,2}, K^n_{3,4}$ and $K^n_{3,6}$, we will show the following proposition.

PROPOSITION 13. *Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $\psi(x) = 0$ for any $x \in \mathbb{R}^d$ with $|x| > M$, and $\xi^n_{k,l} : C([0,\infty); \mathbb{R}^d) \longrightarrow \mathbb{R}$, $k, l \in \mathbb{Z}_+$, $\Xi : [0,\infty) \times C([0,\infty); \mathbb{R}^d) \longrightarrow \mathbb{R}$ be functionals. Suppose that $\xi^n_{k,l}$ is $\mathcal{B}_{l/n}$-measurable and continuous, and that there exists a constant $C_0 = C_0(M) > 0$ such that*

$$\sum_{k=1}^{\infty} \sup_{l \in \mathbb{Z}_+} \sup_{|w|_\infty \le M} |\xi^n_{k,l}(w)| \le C_0 \tag{3.59}$$

for each $n \in \mathbb{N}$. Moreover suppose

$$\sup_{w \in K} \Big| \sum_{k=1}^{\infty} \xi^n_{k,[nt]}(w) - \Xi(t,w) \Big| \longrightarrow 0, \quad n \to \infty \tag{3.60}$$

for each $K \in \mathcal{K}^d$ and $t \geq 0$. Then it holds that

$$\begin{aligned} &\frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{k-1} \mathrm{E}^{\,n}[\psi(X^n_{l/n}) \xi^n_{k-l,l}(X^n) \Phi(X^n_{s_1}, \dots, X^n_{s_N})] \\ &\longrightarrow \int_s^t \mathrm{E}^{\,Q^M}[\psi(w(u)) \Xi(u,w) \Phi(w(s_1), \dots, w(s_N))] du, \quad n \to \infty. \end{aligned} \tag{3.61}$$

PROOF. Denote the left-hand side of (3.61) by U^n and set

$$V^n = \frac{1}{n} \sum_{l=[ns]}^{[nt]-1} \sum_{k=1}^{\infty} \mathrm{E}^{\,n}[\psi(X^n_{l/n}) \xi^n_{k,l}(X^n) \Phi(X^n_{s_1}, \dots, X^n_{s_N})].$$

Since Fubini's theorem implies

$$U^n = \frac{1}{n} \sum_{l=[ns]}^{[nt]-2} \sum_{k=1}^{[nt]-l-1} \mathrm{E}^{\,n}[\psi(X^n_{l/n}) \xi^n_{k,l}(X^n) \Phi(X^n_{s_1}, \dots, X^n_{s_N})], \tag{3.62}$$

we have

$$\begin{aligned} &|U^n - V^n| \\ \leq\ & C_1(M, \psi, \Phi) \Big\{ \frac{1}{n} + \int_s^t \sum_{k=[nt]-[nu]}^{\infty} \sup_{l \in \mathbb{Z}_+} \sup_{|w|_\infty \leq M} |\xi^n_{k,l}(w)| du \Big\} \end{aligned} \tag{3.63}$$

for some $C_1(M, \psi, \Phi) > 0$. By (3.59), the integrand in the right-hand side of (3.63) is bounded and converges to zero as $n \to \infty$ for $u \in [s,t)$. Thus, using the bounded convergence theorem, we have

$$U^n - V^n \longrightarrow 0. \tag{3.64}$$

Since Proposition 10 implies

$$V^n \longrightarrow \int_s^t \mathrm{E}^{\,Q^M}[\psi(w(u)) \Xi(u,w) \Phi(w(s_1), \dots, w(s_N))] du, \tag{3.65}$$

we have our assertion. □

PROPOSITION 14.

(i) $$K^n_{3,2} \longrightarrow \sum_{i,j=1}^{d} \int_s^t \mathrm{E}^{Q^M}[\frac{\partial^2}{\partial x^i x^j} f(w(u))\varphi_M(w(u))^2 \times A^{ij}(u,w)\Phi(w(s_1),\dots,w(s_N))]du,$$

(ii) $$K^n_{3,4} \longrightarrow \sum_{i,j=1}^{d} \int_s^t \mathrm{E}^{Q^M}[\frac{\partial}{\partial x^i} f(w(u))\varphi_M(w(u))\frac{\partial}{\partial x^j}\varphi_M(w(u)) \times A^{ij}(u,w)\Phi(w(s_1),\dots,w(s_N))]du,$$

(iii) $$K^n_{3,6} \longrightarrow \sum_{i,j=1}^{d} \int_s^t \mathrm{E}^{Q^M}[\frac{\partial}{\partial x^i} f(w(u))\varphi_M(w(u))^2\varphi_M(w(u)) \times B^{ij}(u,w)\Phi(w(s_1),\dots,w(s_N))]du$$

as $n \to \infty$.

PROOF. Define $\xi^{n,ij}_{k,l}$ by

$$\xi^{n,ij}_{k,l} = \mathrm{E}^n\Big[F^{n,i}_{k+l}\Big(w\Big(\cdot \wedge \frac{l}{n}\Big)\Big)F^{n,j}_l(w)\Big].$$

By assumption [A7], we have

$$\sup_{w\in K}\Big|\sum_{k=1}^{\infty}\xi^{n,ij}_{k,[nt]}(w) - A^{n,ij}(t,w)\Big| \longrightarrow 0, \quad n\to\infty \tag{3.66}$$

for any $K \in \mathcal{K}^d$ and $t \geq 0$.

By Proposition 9, it follows that

$$K^n_{3,2} - K^n_{3,2,1} \longrightarrow 0, \quad n\to\infty \tag{3.67}$$

where

$$K^n_{3,2,1} = \frac{1}{n}\sum_{i,j=1}^{d}\sum_{k=[ns]}^{[nt]-1}\sum_{l=[ns]}^{k-1}\mathrm{E}^n[\frac{\partial^2}{\partial x^i x^j} f(X^n_{l/n})\varphi_M(X^n_{l/n})^2 \times \xi^{n,ij}_{k-l,l}(X^n)\Phi(X^n_{s_1},\dots,X^n_{s_N})].$$

Since Lemma 1 implies

$$|\xi^{n,ij}_{k,l}(w)| \leq 8\,\mathrm{E}^n[|F^{n,i}_{k+l}(w)|^3]^{1/3}\,\mathrm{E}^n[|F^{n,j}_l(w)|^3]^{1/3}\alpha_k^{1/3},$$

we have

$$\sum_{k=1}^{\infty} \sup_{l\in\mathbb{Z}_+} \sup_{|w|_\infty\le M} |\xi_{k,l}^{n,ij}(w)| \le C_0 \sum_{k=1}^{\infty} \alpha_k^{1/3} \tag{3.68}$$

for some $C_0 = C_0(M) > 0$. Then, applying Proposition 13, we get

$$\begin{aligned} K_{3,2,1}^n \quad \longrightarrow \quad \sum_{i,j=1}^{d} \int_s^t \mathrm{E}^{Q^M}[&\frac{\partial^2}{\partial x^i x^j} f(w(u))\varphi_M(w(u))^2 \\ &\times A^{ij}(u,w)\Phi(w(s_1),\dots,w(s_N))]du. \end{aligned} \tag{3.69}$$

Then we obtain the assertion (i).

The assertions (ii) and (iii) follow by the same way. □

By Proposition 7, 11, 12 and 14, it follows that

$$\begin{aligned} &\mathrm{E}^{Q^{n,M}}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\dots,w(s_N))] \\ &\longrightarrow \mathrm{E}^{Q^M}[\int_s^t \mathscr{L}^M f(u,w)du\Phi(w(s_1),\dots,w(s_N))]. \end{aligned} \tag{3.70}$$

The equality (3.31) now follows by (3.32) and (3.70). This completes the proof of Proposition 6.

PROPOSITION 15. *The family of measures $(Q^M)_{M>1+|x_0|}$ is tight on $C([0,\infty);\mathbb{R}^d)$.*

PROOF. We define the matrix $\sigma^M(t,w) = (\sigma^{M,ij}(t,w))_{i,j=1}^d$ by $\sigma^M(t,w) = \varphi_M(w(t))a^{1/2}(t,w)$, where $a^{1/2}(t,w)$ is the square root matrix of $a(t,w)$. By Proposition 6, there exists the weak solution $(\Omega^M, \mathcal{F}^M, (\mathcal{F}_t^M)_t, P^M, (B_t^M)_t, (X_t^M)_t)$ of the following stochastic differential equation

$$\left\{ \begin{array}{l} dX_t^M = \sigma^M(t,X^M)dB_t^M + b^M(t,X^M)dt \\ X_0^M = x_0 \end{array} \right. \tag{3.71}$$

such that the distribution of X^M under P^M is equal to Q^M.

Let $T > 0$. We will show that there exists a constant $C_0(T) > 0$ such that

$$\mathrm{E}^M[\sup_{0\le t\le T} |X_t^M|^4] \le C_0(T) \tag{3.72}$$

Fix any $R > 0$ and define the stopping time τ_R and the function $m_R(t)$ by

$$\tau_R = \inf\{t \in \mathbb{R}_+ \ ; \ |X_t^M| \geq R\}.$$

and

$$m_R(t) = \mathrm{E}^M[\sup_{0\leq s\leq t} |X_{s\wedge\tau_R}^M|^4],$$

where E^M denotes the expectation under P^M.

By the continuity of X^M, we see that $\tau_R \longrightarrow \infty$ as $R \to \infty$ almost surely under P^M. By the assumption [A8], the Hölder inequality and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
m_R(t) &\leq C_1\Big\{ \mathrm{E}^M\Big[\sup_{0\leq s\leq t}\Big|\int_0^{s\wedge\tau_R} \sigma^M(u, X^M) dB_u^M\Big|^4\Big] \\
&\quad + \mathrm{E}^M\Big[\sup_{0\leq s\leq t}\Big|\int_0^{s\wedge\tau_R} b^M(u, X^M) du\Big|^4\Big]\Big\} \\
&\leq C_1\Big\{ t\,\mathrm{E}^M\Big[\int_0^t 1_{\{s\leq\tau_R\}}|\sigma^M(s, X^M)|^4 ds\Big] \\
&\quad + t^3\,\mathrm{E}^M\Big[\int_0^t 1_{\{s\leq\tau_R\}}|b^M(s, X^M)|^4 ds\Big]\Big\} \\
&\leq C_2(T)\,\mathrm{E}^M\Big[\int_0^t 1_{\{s\leq\tau_R\}}(1 + \sup_{0\leq u\leq s}|X_u^M|)^4 ds\Big] \\
&\leq C_3(T)\Big\{1 + \int_0^t m_R(s) ds\Big\}
\end{aligned}$$

for each $t \leq T$ and for some constants $C_1, C_2(T), C_3(T) > 0$. Applying the Gronwall inequality, we see

$$\sup_{0\leq t\leq T} m_R(t) \leq C_4(T) \tag{3.73}$$

for some $C_4(T) > 0$. Letting $R \to \infty$, we get (3.72) by Fatou's lemma.

Then, using the Hölder inequality and the Burkholder-Davis-Gundy inequality again, we have

$$\begin{aligned}
&\mathrm{E}^{P^M}[|X_t^M - X_s^M|^4] \\
&\leq C_1\Big\{ \mathrm{E}^M\Big[\Big|\int_0^t 1_{\{u\geq s\}}\sigma^M(u, X^M) dB_u^M\Big|^4\Big] \\
&\qquad + \mathrm{E}^M\Big[\Big|\int_s^t b^M(u, X^M) du\Big|^4\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq C_1\Big\{|t-s|\,\mathrm{E}^{M}[\int_s^t |\sigma^M(u,X^M)|^4 du] \\
&\qquad\qquad + |t-s|^3\,\mathrm{E}^{M}[\int_s^t |b^M(u,X^M)|^4 du]\Big\} \\
&\leq C_5(T)|t-s|\int_s^t \big(1+\mathrm{E}^{M}[\sup_{0\leq v\leq u}|X^M_v|^4]\big)du \leq C_0(T)C_5(T)|t-s|^2
\end{aligned}
$$

for some $C_5(T) > 0$. Obviously $Q^M(w \in C([0,\infty);\mathbb{R}^d); w(0) = x_0) = 1$ holds for all M. Then, using theorem 2.3 in [13], we obtain the tightness of $(Q^M)_{M>1+|x_0|}$. $\square$

PROOF OF THEOREM 1. Proposition 15 implies that for any subsequence $(M_k)_k$, there exists a further subsequence $(M_{k_l})_l$ such that $Q^{M_{k_l}}$ converges to some probability measure Q^* on $C([0,\infty);\mathbb{R}^d)$.

Take M_0 large enough so that the support of f is contained in $\{x \in \mathbb{R}^d; |x| \leq M_0/2\}$. Since $\mathscr{L}^M f = \mathscr{L} f$ holds for $M > M_0$, by (3.31), it follows that

$$
\begin{aligned}
&\mathrm{E}^{Q^{M_{k_l}}}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\dots,w(s_N))] \\
= \;&\mathrm{E}^{Q^{M_{k_l}}}[\int_s^t \mathscr{L} f(u,w)du\Phi(w(s_1),\dots,w(s_N))]
\end{aligned}
\tag{3.74}
$$

for $M_{k_l} > M_0$. Letting $l \to \infty$, we see that Q^* is a solution of the martingale problem associated with the generator $\mathscr{L}$. Moreover, by the assumption [A10], Q^* equals to Q and is independent of a subsequence $(M_{k_l})_l$. Then it follows that Q^M converges weakly to Q on $C([0,\infty);\mathbb{R}^d)$ as $M \to \infty$.

Finally, repeating the arguments in [5] p.119-120, we show that Q^n converges weakly to Q on $C([0,\infty);\mathbb{R}^d)$. This completes the proof of Theorem 1. $\square$

4. Proof of Theorem 2

To prove Theorem 2, we will show two lemmas below. Let $(\Omega,\mathcal{F},P)$ be a probability space and (S,d) be a metric space.

LEMMA 5. *Let* $1 < p, q < \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} < 1$ *and* $U : S \times \Omega \longrightarrow \mathbb{R}$ *be a continuous random function such that* $U(x)$ *is* $\mathcal{A}$*-measurable and*

$\mathrm{E}[U(x)] = 0$ *for each* $x \in S$*, and* $X : \Omega \longrightarrow S$*,* $V : \Omega \longrightarrow \mathbb{R}$ *be* $\mathcal{B}$*-measurable random variables. Suppose that there exist positive constants* C_0 *and* γ *such that*

$$\sup_{\varepsilon>0} \varepsilon^\gamma \log N(\varepsilon, p; U) \le C_0. \tag{4.1}$$

Then for each $\varrho \in (0, 1/\gamma)$ *there exists a constant* $C > 0$ *depending only on* p, q, γ, ϱ *and* C_0 *such that*

$$\begin{aligned} &\big|\,\mathrm{E}[U(X)V]\big| \\ &\le\ C\big(\,\mathrm{E}[\sup_{x\in S}|U(x)|^p]^{1/p} + 1\big)\,\mathrm{E}[|V|^q]^{1/q}\Big(\frac{1}{\log(1/\alpha(\mathcal{A},\mathcal{B}))}\Big)^\varrho. \end{aligned}\tag{4.2}$$

Proof. We may assume that the right-hand side of (4.2) is finite. Set $\xi = \dfrac{1}{\log(1/\alpha(\mathcal{A},\mathcal{B}))}$. Using Lemma 2 with $\varepsilon = \xi^\varrho$, we have

$$\begin{aligned} \big|\,\mathrm{E}[U(X)V]\big| \ &\le\ 8\big(\,\mathrm{E}[\sup_{x\in S}|U(x)|^p]^{1/p} + 1\big) \\ &\qquad \times \mathrm{E}[|V|^q]^{1/q}\big(\xi^\varrho + \xi^{(1-r)\varrho}\exp(C_0\xi^{-\varrho\gamma} - \xi^{-1})\big), \end{aligned}\tag{4.3}$$

where $\dfrac{1}{r} = 1 - \dfrac{1}{p} - \dfrac{1}{q}$. Since $\varrho\gamma \in (0,1)$ and $\xi \in (0,1)$, there is a constant $C_1 > 0$ which depends only on p, q, γ, ϱ and C_0 such that

$$\xi^{(1-r)\varrho}\exp(C_0\xi^{-\varrho\gamma} - \xi^{-1}) \le C_1\xi^\varrho. \tag{4.4}$$

By (4.3) and (4.4), we obtain our assertion. □

Lemma 6. *Let* $1 < p, q, r < \infty$ *be such that* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} < 1$*. Let* $U, V : S \times \Omega \longrightarrow \mathbb{R}$ *be continuous random functions such that* $U(x)$ *and* $V(x)$ *are* $\mathcal{A}$ *and* $\mathcal{B}$*-measurable respectively and* $\mathrm{E}[U(x)] = 0$ *for each* $x \in S$*, and* $X : \Omega \longrightarrow S$*,* $Z : \Omega \longrightarrow \mathbb{R}$ *be* $\mathcal{C}$*-measurable random variables. Suppose that there exist positive constants* $C_0, u^*, v^* > 0$ *and* γ *such that*

$$\sup_{\varepsilon>0} \varepsilon^\gamma\big\{\log N(\varepsilon, p; U) + \log N(\varepsilon, q; V)\big\} \le C_0, \tag{4.5}$$

$$\mathrm{E}[\sup_{x\in S}|U(x)|^p]^{1/p} \le u^* \tag{4.6}$$

and

$$\mathrm{E}[\sup_{x\in S}|V(x)|^q]^{1/q} \le v^*. \tag{4.7}$$

Then for each $\varrho' \in \left(0, \frac{1}{2\gamma}\right)$ *there exists a constant* $C > 0$ *depending only on* $p, q, r, \gamma, \varrho', u^*, v^*$ *and* C_0 *such that*

$$\begin{aligned} &\bigl|\,\mathrm{E}[\Xi(X)Z]\bigr| \\ \le\ & C\,\mathrm{E}[|Z|^r]^{1/r}\Bigl(\frac{1}{\log(1/\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C}))}\Bigr)^{\varrho'}\Bigl(\frac{1}{\log(1/\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C}))}\Bigr)^{\varrho'}, \end{aligned} \tag{4.8}$$

where $\Xi(x) = U(x)V(x) - E[U(x)V(x)]$.

PROOF. By (2.17), we have

$$\sup_{\varepsilon>0}\varepsilon^\gamma \log N(\varepsilon, p; \Xi) \le 2^{\gamma+1}C_0(u^*+v^*)^\gamma. \tag{4.9}$$

Then, by Lemma 5, we see that

$$\bigl|\,\mathrm{E}[\Xi(X)Z]\bigr| \le C_1\,\mathrm{E}[|Z|^r]^{1/r}\Bigl(\frac{1}{\log(1/\alpha(\mathcal{A}\vee\mathcal{B},\mathcal{C}))}\Bigr)^{2\varrho'} \tag{4.10}$$

for some $C_1 = C_1(p, q, r, \gamma, \varrho', u^*, v^*, C_0) > 0$. By Lemma 1 and Lemma 5, we have

$$\begin{aligned} &\bigl|\,\mathrm{E}[\Xi(X)Z]\bigr| \\ \le\ & C_2\,\mathrm{E}[|Z|^r]^{1/r}\Bigl\{\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C})^{1-1/p-1/q} + \Bigl(\frac{1}{\log(1/\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C}))}\Bigr)^{2\varrho'}\Bigr\} \end{aligned} \tag{4.11}$$

for some $C_2 = C_2(p, q, r, \gamma, \varrho', u^*, v^*, C_0) > 0$. Since there is $C_3 = C_3(p, q, \varrho') > 0$ such that

$$t^{1-1/p-1/q} \le C_3\Bigl(\frac{1}{\log(1/t)}\Bigr)^{2\varrho'} \tag{4.12}$$

for all $t \in (0, 1/4]$, we get

$$\bigl|\,\mathrm{E}[\Xi(X)Z]\bigr| \le C_2(C_3+1)\,\mathrm{E}[|Z|^r]^{1/r}\Bigl(\frac{1}{\log(1/\alpha(\mathcal{A},\mathcal{B}\vee\mathcal{C}))}\Bigr)^{2\varrho'}. \tag{4.13}$$

By (4.10) and (4.13), we obtain the assertion. □

By Lemma 5, Lemma 6 and the same arguments in the proof of Theorem 1, we obtain Theorem 2.

5. Appendix

5.1. Sufficient conditions for $[A9]$

Let $a(t,w) = (a^{ij}(t,w))_{ij=1}^{d}$ and $b(t,w) = (b^i(t,w))_{i=1}^{d}$ be as in $[A8]$, and let $\sigma(t,w) = (\sigma^{ij}(t,w))_{i,j=1}^{d} = a^{1/2}(t,w)$. It is well-known that if we assume the Lipschitz condition of $\sigma^{ij}(t,w)$ and $b^i(t,w)$, then the condition $[A9]$ holds. In fact, the local Lipschitz continuity of $b^i(t,w)$ is obtained by $[A3]$ and $[A5]$. In this section we introduce the sufficient condition under which $\sigma^{ij}(t,w)$ is Lipschitz continuous.

$[A10]$ $a^{ij}(t,w)$ is twice continuously Fréchet differentiable in w for each $t \geq 0$, and for each $T > 0$ there exists a positive constant $C(T) > 0$ such that

$$|\nabla_w^2 a^{ij}(t,w)|_{L_t^2} \leq C(T) \tag{5.1}$$

for each $t \in [0,T]$ and $w \in C([0,\infty);\mathbb{R}^d)$, where $\nabla_w^2 a^{ij}(t,w)$ denotes the second Fréchet derivative of $a^{ij}(t,w)$ with respect to w.

Here we remark that since $a^{ij}(t,\cdot)$ is measurable with respect to $\mathcal{B}_t$, we can regard $\nabla_w^2 a^{ij}(t,w)$ as the element of L_t^2 for each fixed $t \geq 0$.

Theorem 3. *Assume $[A1]-[A8]$ and $[A10]$. Then the conclusion of Theorem 1 holds.*

Proof. Let $\sigma(t,w) = a^{1/2}(t,w)$. To check the condition $[A9]$, it suffices to show that for each $M > 0$ and $T > 0$ there exists a constant $C_0 = C_0(M,T) > 0$ such that

$$|\sigma^{ij}(t,w) - \sigma^{ij}(t,w')| \leq C_0 \sup_{0\leq s\leq t} |w(s) - w'(s)|, \tag{5.2}$$

$$|b^i(t,w) - b^i(t,w')| \leq C_0 \sup_{0\leq s\leq t} |w(s) - w'(s)| \tag{5.3}$$

for any $t \in [0,T]$ and $w, w' \in \mathcal{C}_M^d$.

By $[A3]$, we have

$$|\nabla_w b_0^{n,i}(k,w)|_{L^1_{k/n}} \le \mathrm{E}^n[|\nabla G_k^{n,i}(w)|_{L^1_{k/n}}] \le C_1, \quad k \in \mathbb{Z}_+, \ w \in \mathcal{C}_M^d \tag{5.4}$$

for some $C_1 = C_1(M) > 0$. Moreover, by $[A3]$, $[A5]$ and Lemma 1, we have

$$\begin{aligned}
&|\nabla_w B^{n,ij}(k,w)|_{L^1_{k/n}} \\
\le\ & \sum_{l=1}^{\infty} \Big\{ \mathrm{E}^n\Big[\Big|\nabla^2 F_{k+l}^{n,i}\Big(w\Big(\cdot \wedge \frac{k}{n}\Big)\Big)\Big|^3_{L^2_{k/n}}\Big]^{1/3} \mathrm{E}^n[|F_k^{n,j}(w)|^3]^{1/3} \\
&\qquad + \mathrm{E}^n\Big[\Big|\nabla F_{k+l}^{n,i}\Big(w\Big(\cdot \wedge \frac{k}{n}\Big)\Big)\Big|^3_{L^1_{k/n}}\Big]^{1/3} \mathrm{E}^n[|\nabla F_k^{n,j}(w)|^3_{L^1_{k/n}}]^{1/3}\Big\} \alpha_l^{1/3} \\
\le\ & C_2 \sum_{l=1}^{\infty} \alpha_l^{1/3}, \quad k \in \mathbb{Z}_+, \ w \in \mathcal{C}_M^d
\end{aligned} \tag{5.5}$$

for some $C_2 = C_2(M) > 0$. By (5.4) and (5.5), we get (5.3).

To see (5.2), we introduce the following theorem (Theorem 5.2.3 in [14]).

THEOREM 4. *Let $f(t,x) = (f^{ij}(t,x))_{i,j=1}^d : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ be a symmetric non-negative definite matrix-valued function. Suppose that $f^{ij}(t,x)$ is twice continuously differentiable in x for each $t \ge 0$ and that there is a positive constant $C(T)$ such that*

$$\Big|\frac{\partial^2}{\partial x^2} f^{ij}(t,x)\Big| \le C(T) \tag{5.6}$$

for each $t \in [0,T]$, $x \in \mathbb{R}$ and $i,j = 1,\dots,d$. Then it holds that

$$|g^{ij}(t,x) - g^{ij}(t,y)| \le d\sqrt{2C(T)}|x-y| \tag{5.7}$$

for each $t \in [0,T]$ and $x, y \in \mathbb{R}$, where $g(t,x) = f^{1/2}(t,x)$.

For each fixed $T > 0$ and $w, w' \in C([0,\infty);\mathbb{R}^d)$, define the functions $f, g : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ by $f(t,x) = a(t, w' + x(w - w'))$ and $g(t,x) = f^{1/2}(t,x)$. By $[A10]$, $f(t,x)$ is twice continuously differentiable in x for each t and

$$\begin{aligned}
\Big|\frac{d^2}{dx^2} f^{ij}(t,x)\Big| &= |\nabla_w^2 a^{ij}(t, w' + x(w-w'); w-w', w-w')| \\
&\le C_4 \sup_{0\le s\le t} |w(s) - w'(s)|^2, \quad t \in [0,T], \ x \in \mathbb{R}
\end{aligned} \tag{5.8}$$

for some $C_4(T) > 0$. Then Theorem 4 implies

$$|\sigma^{ij}(t,w) - \sigma^{ij}(t,w')| = |g^{ij}(t,1) - g^{ij}(t,0)| \le d\sqrt{2C_4} \sup_{0\le s\le t} |w(s) - w'(s)|.$$

This implies (5.2). Then the condition $[A9]$ holds and we obtain the conclusion. □

5.2. Sufficient conditions for [A4] and [B4]

In this section we provide sufficient conditions under which $[A4]$ and $[B4]$ are filled.

Let $\varepsilon > 0$, (S,d) be a metric space and A be a totally bounded subset of S. We say that a family of sets $(A_i)_{i=1}^m$ is an ε-net of A if $A \subset \bigcup_{i=1}^m A_i$ and $\sup_{x,y\in A_i} d(x,y) < \varepsilon$ for each $i = 1,\dots,m$. We denote by $\hat{N}(\varepsilon; A, d)$ the minimum of cardinals of ε-nets of A in the metric d.

THEOREM 5. *Let $(\Omega,\mathcal{F},P)$ be a probability space, $p \ge 1$, (S,d) be a metric space, $(B,||\cdot||_B)$ be a Banach space and A be a totally bounded subset of B. Let $f : B\times\Omega \longrightarrow \mathbb{R}$ be a continuously Fréchet differentiable random function and $u : S \longrightarrow B$ be a continuous function such that $u(x) \in A$ for any $x \in S$. Suppose that there exists a positive constant C_0 such that*

$$\mathrm{E}[\sup_{y\in\tilde{A}} ||\nabla f(y)||_{B^*}^p]^{1/p} \le C_0, \tag{5.9}$$

where $\tilde{A}$ is a convex hull of A and

$$||\nabla f(y)||_{B^*} = \sup_{z\in B, z\ne 0} \frac{|\nabla f(y;z)|}{||z||_B}, \quad y \in B.$$

Then for any $\varepsilon > 0$

$$N(\varepsilon,p;U) \le \hat{N}(\varepsilon/C_0; A, d_B), \tag{5.10}$$

where $U(x,\omega) = f(u(x),\omega)$ and $d_B(y,y') = ||y-y'||_B$, $y,y' \in B$.

PROOF. Let $(A_i)_{i=1}^m$ be an ε-net of A. We define $S_i \subset S$ by

$$S_i = \{x \in S\ ;\ u(x) \in A_i\}.$$

Then we have

$$(5.11) \qquad S = \bigcup_{i=1}^{m} S_i$$

and for each $x, x' \in S_i$

$$\begin{aligned} |U(x) - U(x')| &\leq \int_0^1 ||\nabla f(tu(x) + (1-t)u(x'))||_{B^*} dt ||u(x) - u(x')||_B \\ &\leq \sup_{y \in \tilde{A}} ||\nabla f(y)||_{B^*} \times \varepsilon. \end{aligned}$$

Then we have

$$(5.12) \qquad \mathrm{E}[\max_{i=1,\dots,m} \sup_{x,x' \in S_i} |U(x) - U(x')|^p]^{1/p} \leq C_0 \varepsilon.$$

By (5.11) and (5.12), we see that $(S_i)_{i=1}^m$ is an $(C_0\varepsilon, p, U)$-net of S. Then we obtain the assertion. □

Let B be a Banach space and $\mathcal{B}(B)$ be a Borel field of B. By Theorem 5, under suitable conditions, we can check conditions $[A4]$ and $[B4]$ when $F_k^{n,i}$ and $G_k^{n,i}$ are represented in the following form

$$(5.13) \quad F_k^{n,i}(w, \omega) = f_k^{n,i}(u(k/n, w), \omega), \quad G_k^{n,i}(w, \omega) = g_k^{n,i}(v(k/n, w), \omega),$$

where $f_k^{n,i}(x, \omega), g_k^{n,i}(x, \omega) : B \times \Omega^n \longrightarrow \mathbb{R}$ be $\mathcal{B}(B) \otimes \mathcal{F}^n$-measurable random functions and $u(t, w), v(t, w) : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \longrightarrow B$ be $(\mathcal{B}_t)_t$-adapted (i.e. $u(t, \cdot)$ and $v(t, \cdot)$ are $\mathcal{B}_t$-measurable for each $t \geq 0$) deterministic functions.

We also have the condition $[A4]$ when the image spaces of $F_k^{n,i}$ and $G_k^{n,i}$ are finite dimensional in $L^{p_0}(\Omega^n)$. Let $p \geq 1$, $(\Omega, \mathcal{F}, P)$ be a probability space, (S, d) be a metric space and $U : S \times \Omega \longrightarrow \mathbb{R}$ be a continuous random function which satisfies $\mathrm{E}[|U(x)|^p] < \infty$ for any $x \in S$. We define the metric space $(\mathcal{S}_p(U), d_p)$ by

$$\mathcal{S}_p(U) = \{U(x) \in L^p(\Omega) \; ; \; x \in S\}$$

and $d_p(X, Y) = \mathrm{E}[|X - Y|^p]^{1/p}$.

THEOREM 6. *Suppose that there are constants $\gamma \in (0, p/2)$, $C_0 > 0$ and $C_1 > 0$ such that*

$$\sup_{\varepsilon>0} \varepsilon^{\gamma} \hat{N}(\varepsilon; \mathcal{S}_p(U), d_p) \le C_0 \tag{5.14}$$

and

$$\mathrm{E}[\sup_{x\in S} |U(x)|^p] \le C_1. \tag{5.15}$$

Then for each $\lambda \in \Big(0, \dfrac{p-2\gamma}{p}\Big)$ there exists a constant $C > 0$ which depends only on p, γ, λ, C_0 and C_1 such that

$$\sup_{\varepsilon>0} \varepsilon^{\gamma/\lambda} N(\varepsilon, p; U) \le C. \tag{5.16}$$

PROOF. Define $F : \mathcal{S}_p(U) \times \Omega \longrightarrow \mathbb{R}$ by $F(X, \omega) = X(\omega)$. Then we have

$$\mathrm{E}[|F(X) - F(Y)|^p] = \mathrm{E}[|X - Y|^p] = d_p(X, Y)^p \tag{5.17}$$

for any $X, Y \in \mathcal{S}_p(U)$. By (5.14), (5.17) and the similar arguments in the proof of Theorem 1.4.1 in [7], we see that there exist a continuous modification $\tilde{F}$ of F and a constant $C_2 > 0$ depending only on p, γ, λ and C_0 such that

$$\mathrm{E}\Big[\sup_{X,Y\in\mathcal{S}_p(U), 0<d_p(X,Y)<1} \Big|\frac{\tilde{F}(X) - \tilde{F}(Y)}{d_p(X, Y)^{\lambda}}\Big|^p\Big] \le C_2. \tag{5.18}$$

Define the random variable K by

$$K = \sup_{X,Y\in\mathcal{S}_p(U), X\neq Y} \frac{|\tilde{F}(X) - \tilde{F}(Y)|}{d_p(X, Y)^{\lambda}}.$$

Then it holds that

$$\mathrm{E}[|K|^p] \le 2^{p-1} C_1 + C_2. \tag{5.19}$$

Thus, for each subsets $S_1, \dots, S_m \subset \mathcal{S}_p(U)$, we have

$$\begin{aligned}
& \mathrm{E}[\max_{i=1,\dots,m} \sup_{x,y \in S_i} |U(x) - U(y)|^p]^{1/p} \\
= \;& \mathrm{E}[\max_{i=1,\dots,m} \sup_{x,y \in S_i} |\tilde{F}(U(x)) - \tilde{F}(U(y))|^p]^{1/p} \\
\leq \;& \mathrm{E}[|K|^p]^{1/p} \max_{i=1,\dots,m} \sup_{x,y \in S_i} d_p(U(x), U(y))^\lambda \\
\leq \;& C_3 \max_{i=1,\dots,m} \sup_{x,y \in S_i} \mathrm{E}[|U(x) - U(y)|^p]^{\lambda/p},
\end{aligned}$$

where $C_3 = (2^{p-1}C_1 + C_2)^{1/p}$. So we get

$$N(\varepsilon, p; U) \leq \hat{N}(\varepsilon^{1/\lambda}/C_3; \mathcal{S}_p(U), d_p) \tag{5.20}$$

for any $\varepsilon > 0$. Then we have

$$\sup_{\varepsilon > 0} \varepsilon^{\gamma/\lambda} N(\varepsilon, p; U) \leq C_3^\gamma \sup_{\varepsilon > 0} \varepsilon^\gamma \hat{N}(\varepsilon; \mathcal{S}_p(U), d_p) \leq C_3^\gamma C_0. \tag{5.21}$$

This implies our assertion. □

By Theorem 6, we can check $[A4]$ under the following condition $[A4']$.

$[A4']$ For some $\gamma_2 \in (0, p_0/2)$, (1.6)−(1.10) hold with γ_2 and $\tilde{N}_n(\varepsilon, M; U)$ instead of γ_0 and $N_n(\varepsilon, M; U)$, where $\tilde{N}_n(\varepsilon, M; U)$ is the smallest integer m such that there exist sets $S_1, \dots, S_m$ which satisfy $\mathcal{C}_M^d = \bigcup_{i=1}^m S_i$ and

$$\sup_{x,y \in S_i} \mathrm{E}^n[|U(x) - U(y)|^{p_0}]^{1/p_0} < \varepsilon$$

for each $i = 1, \dots, m$.

5.3. Examples

In this section, we give two examples of Theorem 2. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\xi_k = (\xi_k^i)_{i=1}^{m_1}$, $k \in \mathbb{Z}_+$, be an m_1-dimensional stationary Gaussian process.

(a.) Let $f(x) = (f^i(x))_{i=1}^d : \mathbb{R}^{m_2} \longrightarrow \mathbb{R}^d$, $u(t, x, y) = (u^i(t, x, y))_{i=1}^{m_2}$:

$[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{m_3} \longrightarrow \mathbb{R}^{m_2}$ and $\psi(x) = (\psi^i(x))_{i=1}^{m_3} : \mathbb{R}^{m_1} \longrightarrow \mathbb{R}^{m_3}$ be Borel measurable functions. Let $\Psi(t,w,y) = (\Psi^i(t,w,y))_{i=1}^{m_2}$ and $h(t,w,y) = (h^i(t,w,y))_{i=1}^{d}$ be such that

$$\Psi^i(t,w,y) = \int_0^t u^i(s, w(t-s), \psi(y))ds$$

and

$$h^i(t,w,y) = f^i(\Psi(t,w,y)).$$

We define $F_k^{n,i}(w)$ and $G_k^{n,i}(w)$ by

$$G_k^{n,i}(w) = \mathrm{E}[h^i(k/n, w, \xi_k)] \tag{5.22}$$

and

$$F_k^{n,i}(w) = h^i(k/n, w, \xi_k) - G_k^{n,i}(w). \tag{5.23}$$

We introduce the following conditions.

$[C1]$ $f^i(x)$ is three times continuously differentiable in x. Moreover $u(t,x,y)$ is three times continuously differentiable in x and y, and all derivatives are continuous in t.

$[C2]$ It holds that

$$\sum_{|\beta|\le 3} \sup_{x\in\mathbb{R}^{m_2}} |D^\beta f^i(x)| < \infty, \tag{5.24}$$

$$\sum_{|\beta|+|\beta'|\le 2} \int_0^\infty \sup_{x\in\mathbb{R}^d, y\in\mathbb{R}^{m_3}} |D_x^\beta D_y^{\beta'} u^j(t,x,y)|dt < \infty \tag{5.25}$$

and

$$\sup_{x\in\mathbb{R}^{m_1}} |\psi^\nu(x)| < \infty \tag{5.26}$$

for each $i = 1,\dots,d$, $j = 1,\dots,m_2$ and $\nu = 1,\dots,m_3$.

$[C3]$ Let $\mathcal{G}_{k,l} = \sigma(\xi_\nu^i\ ;\ i = 1,\dots,d,\ k \le \nu \le l)$ and

$$\beta_k = \sup_l \sup\{|P(A\cap B) - P(A)P(B)|\ ;\ A \in \mathcal{G}_{0,l},\ B \in \mathcal{G}_{k+l,\infty}\}.$$

Then for some $\varrho_4 \in (0, 1/2)$

$$\sum_{k=1}^{\infty} \Big(\frac{1}{\log(1/\beta_k)}\Big)^{\varrho_4} < \infty. \tag{5.27}$$

Define $\hat{b}^i(t, w)$ and $\eta_k^{ij}(t, w)$ by

$$\hat{b}^i(t, w) = \mathrm{E}[h^i(t, w, \xi_0)] \tag{5.28}$$

and

$$\eta_k^{ij}(t, w) = \mathrm{E}[h^i(t, w, \xi_k)h^j(t, w, \xi_0)] - \hat{b}^i(t, w)\hat{b}^j(t, w), \tag{5.29}$$

and $\hat{a}^{ij}(t, w)$ by

$$\hat{a}^{ij}(t, w) = \eta_0^{ij}(t, w) + \sum_{k=1}^{\infty} \big\{\eta_k^{ij}(t, w) + \eta_k^{ji}(t, w)\big\}. \tag{5.30}$$

Let

$$\hat{\mathscr{L}} f(t, w) = \frac{1}{2}\sum_{i,j=1}^{d} \hat{a}^{ij}(t, w)\frac{\partial^2}{\partial x^i \partial x^j} f(w(t)) + \sum_{i=1}^{d} \hat{b}^i(t, w)\frac{\partial}{\partial x^i} f(w(t)) \tag{5.31}$$

for $f \in C^2(\mathbb{R}^d)$.

THEOREM 7. *Assume* $[C1] - [C3]$. *Then the conclusion of Theorem 1 holds replacing* $\mathscr{L}$ *with* $\hat{\mathscr{L}}$.

PROOF. We will check that $F_k^{n,i}$ and $G_k^{n,i}$ satisfy the assumptions of Theorem 2. $[A1] - [A3]$, $[B5]$ and $[A6]$ are obvious.

PROPOSITION 16. *The condition* $[B4]$ *holds with* $\gamma_1 = 1$.

PROOF. Let $U(w, \omega) = h^i(t, w, \xi_k(\omega))$. We define $g(v, \omega) : \hat{\mathcal{C}}_R \times \Omega \longrightarrow \mathbb{R}$ by $g(v, \omega) = f^i(v(\psi(\xi_k(\omega))))$, where $\hat{\mathcal{C}}_R = C(K_R; \mathbb{R}^{m_1})$, $K_R = \{x \in \mathbb{R}^{m_3}\ ;\ |x| \le R\}$ and $R = \sum_{i=1}^{m_3} \sup_{x \in \mathbb{R}^{m_1}} |\psi^i(x)|$. We also define $\tilde{\Psi}(t, w, y) = (\tilde{\Psi}^j(t, w, y))_{j=1}^{m_2} : [0, \infty) \times \mathcal{C}_M^d \times K_R \longrightarrow \mathbb{R}^{m_2}$ by

$$\tilde{\Psi}^j(t, w, y) = \int_0^t u^j(s, w(t - s), y)ds.$$

Then it follows that

$$U(w,\omega) = g(\tilde{\Psi}(t,w,\cdot),\omega). \tag{5.32}$$

By $[C2]$, we see that there is a constant $C_0 > 0$ such that

$$\sum_{j=1}^{m_2} \sum_{|\beta|\le 1} |D_y^{\beta} \tilde{\Psi}^j(t,w,y)| \le C_0, \quad w \in \mathcal{C}_M^d,\ y \in K_R. \tag{5.33}$$

Then we have

$$\tilde{\Psi}(t,w,\cdot) \in A_R, \quad w \in \mathcal{C}_M^d, \tag{5.34}$$

where

$$\begin{aligned} A_R \ = \ \Big\{ & v \in \hat{\mathcal{C}}_R\ ;\ v \text{ is continuously differentiable and} \\ & \qquad \sum_{j=1}^{m_2} \sum_{|\beta|\le 1} \sup_{|y|\le R} |D^{\beta} v^j(y)| \le C_0 \Big\}. \end{aligned}$$

$[C2]$ also implies

$$|\nabla g(v,\omega;\tilde{v})| \le C_1 \sum_{j=1}^{m_2} \sup_{|y|\le R} |\tilde{v}^j(y)|, \quad v,\tilde{v} \in A_R,\ \omega \in \Omega \tag{5.35}$$

for some $C_1 > 0$. Then, by Theorem 5, we get

$$N(\varepsilon,p,M;U) \le \hat{N}(\varepsilon/C_1; A_R, d_\infty) \tag{5.36}$$

for each $M > 0$ and $p \ge 1$, where $d_\infty(v,v') = \sup_{y \in K_R} |v(y) - v'(y)|$ and $N(\varepsilon,p,M;U)$ is the minimum of cardinals of (ε,p,U)-nets of $\mathcal{C}_M^d$.

Moreover, by Theorem XIII in [8], we have

$$\log \hat{N}(\varepsilon/C_1; A_R, d_\infty) \le C_1 C_2 \varepsilon^{-1} \tag{5.37}$$

for some $C_2 > 0$ depending only on R and C_0. Then we get

$$\log N(\varepsilon,p,M;U) \le C_3 \varepsilon^{-1} \tag{5.38}$$

for some $C_3 > 0$ with $U(w,\omega) = h^i(t,w,\xi_k(\omega))$.

Similarly we see that (5.38) holds with $U(w,\omega)=\nabla_w h^i(t,w,\xi_k(\omega);I_l^n e_j)$ and $U(w,\omega)=\nabla_w^2 h^i(t,w,\xi_k(\omega);I_l^n e_j,I_l^n e_\nu)$. Then we obtain the assertion. $\square$

To check the condition $[A7]$, we will show the following proposition.

PROPOSITION 17. *For each $K\in\mathcal{K}^d$, $t\geq 0$ and $k\in\mathbb{Z}_+$, it holds that*

$$\sup_{w\in K,y\in\mathbb{R}^{m_1}}\Big|\Psi^i\Big(\frac{[nt]+k}{n},w\Big(\cdot\wedge\frac{[nt]}{n}\Big),y\Big) -\Psi^i(t,w,y)\Big|\longrightarrow 0,\quad n\to\infty. \tag{5.39}$$

PROOF. Let

$$\delta_T(s;w) = \sup\{|w(r)-w(r')|\ ;\ 0\leq r,r'\leq T,\ |r-r'|\leq s\},\qquad s,T>0,\ w\in C([0,\infty);\mathbb{R}).$$

Then we have

$$\begin{aligned}
&\sup_{w\in K,y\in\mathbb{R}^{m_1}}\Big|\Psi^i\Big(\frac{[nt]+k}{n},w\Big(\cdot\wedge\frac{[nt]}{n}\Big),y\Big)-\Psi^i(t,w,y)\Big|\\
\leq\ &\int_t^{([nt]+k)/n}\sup_{x,y}|u^i(s,x,y)|ds\\
&+\sum_{j=1}^d\int_0^t\sup_{x,y}\Big|\frac{\partial}{\partial x^j}u^i(s,x,y)\Big|\\
&\times\sup_{w\in K}\Big|w^j\Big(\Big(\frac{[nt]+k}{n}-s\Big)\wedge\frac{[nt]}{n}\Big)-w^j(t-s)\Big|ds\\
\leq\ &\int_t^{([nt]+k)/n}\sup_{x,y}|u^i(s,x,y)|ds\\
&+\sum_{j=1}^d\int_0^t\sup_{x,y}\Big|\frac{\partial}{\partial x^j}u^i(s,x,y)\Big|ds\sup_{w\in K}\delta_t\Big(\frac{k+1}{n};w^j\Big).
\end{aligned}$$

Since K is compact, we see that

$$\sup_{w\in K}\delta_t\Big(\frac{k+1}{n};w^j\Big)\longrightarrow 0,\quad n\to\infty,\ k\in\mathbb{Z}_+. \tag{5.40}$$

Then we have the assertion. $\square$

Define $a_0^{n,ij}(k,w), b_0^{n,i}(k,w), A^{n,ij}(k,w)$ and $B^{n,ij}(k,w)$ as in $[A7]$.

PROPOSITION 18. *It holds that*

(i) $\sup_{w\in K} |a_0^{n,ij}([nt],w) - \eta_0^{ij}(t,w)| \longrightarrow 0,$

(ii) $\sup_{w\in K} |b_0^{n,i}([nt],w) - \hat{b}^i(t,w)| \longrightarrow 0,$

(iii) $\sup_{w\in K} |A^{n,ij}([nt],w) - \hat{A}^{ij}(t,w)| \longrightarrow 0,$

(iv) $\sup_{w\in K} |B^{n,ij}([nt],w)| \longrightarrow 0$

for each $t \geq 0$ and $K \in \mathcal{K}^d$, where $\hat{A}^{ij}(t,w) = \sum_{k=1}^{\infty} \eta_k^{ij}(t,w)$.

PROOF. By Proposition 17, we get

$$\begin{aligned}
&\mathrm{E}[\sup_{w\in K} |h^i([nt]/n, w, \xi_k) - h^i(t,w,\xi_k)|] \\
\leq\ & \sum_{j=1}^{m_2} \sup_x \Big|\frac{\partial}{\partial x^j} f^i(x)\Big| \\
&\times \mathrm{E}\Big[\Big|\sup_{w\in K, y\in\mathbb{R}^{m_1}} \Big|\Psi^j\Big(\frac{[nt]}{n}, w\Big(\cdot \wedge \frac{[nt]}{n}\Big), y\Big) - \Psi^j(t,w,y)\Big|\Big] \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. Then we have the assertion (ii). Moreover this implies

$$\begin{aligned}
&\sup_{w\in K} |a_0^{n,ij}([nt],w) - \eta_0^{ij}(t,w)| \\
\leq\ & 2\big\{\sup_x |f^i(x)|\, \mathrm{E}[\sup_{w\in K} |h^j([nt]/n, w, \xi_k) - h^j(t,w,\xi_k)|] \\
&\quad + \sup_x |f^j(x)|\, \mathrm{E}[\sup_{w\in K} |h^i([nt]/n, w, \xi_k) - h^i(t,w,\xi_k)|]\big\} \longrightarrow 0, \quad n \to \infty.
\end{aligned}$$

Then the assertion (i) holds.

Since ξ_k is stationary, we have

$$A^{n,ij}([nt],w) = \sum_{l=1}^{\infty} \hat{\eta}_l^{n,ij}([nt],w), \tag{5.41}$$

where

$$\begin{aligned}\hat{\eta}^{n,ij}_l(k,w) &= \mathrm{E}\Big[h^i\Big(\frac{k+l}{n}, w\Big(\,\cdot\wedge\frac{k}{n}\Big),\xi_l\Big)h^j\Big(\frac{k}{n},w,\xi_0\Big)\Big] \\ &\quad -\mathrm{E}\Big[h^i\Big(\frac{k+l}{n}, w\Big(\,\cdot\wedge\frac{k}{n}\Big),\xi_l\Big)\Big]\,\mathrm{E}\Big[h^j\Big(\frac{k}{n},w,\xi_0\Big)\Big].\end{aligned}$$

By Proposition 17, we have

$$\begin{aligned}&\sup_{w\in K}|\hat{\eta}^{n,ij}_k([nt],w)-\eta^{ij}_k(t,w)| \\ &\le 2\Big\{\sum_{\nu=1}^{m_2}\sup_x\Big|\frac{\partial}{\partial x^\nu}f^i(x)\Big|\sup_x|f^j(x)| \\ &\quad\times\sup_{w\in K,y\in\mathbb{R}^{m_2}}\Big|\Psi^\nu\Big(\frac{[nt]+k}{n},w\Big(\,\cdot\wedge\frac{[nt]}{n}\Big),y\Big)-\Psi^\nu(t,w,y)\Big| \\ &\quad+\sup_x|f^i(x)|\,\mathrm{E}[\sup_{w\in K}|h^j([nt]/n,w,\xi_0)-h^j(t,w,\xi_0)|]\Big\} \\ &\longrightarrow 0,\quad n\to\infty\end{aligned}$$

for each $k\in\mathbb{Z}_+$ and $t\ge 0$. Moreover, using Lemma 1, we have

$$\sup_{w\in K}|\hat{\eta}^{n,ij}_k([nt],w)-\eta^{ij}_k(t,w)|\le 16\sup_x|f^i(x)|\sup_x|f^j(x)|\beta_k, \tag{5.42}$$

and $[C3]$ implies

$$\sum_{k=1}^\infty \beta_k<\infty. \tag{5.43}$$

Thus the dominated convergence theorem implies

$$\begin{aligned}&\sup_{w\in K}|A^{n,ij}([nt],w)-\hat{A}^{ij}(t,w)| \\ &\le \sum_{k=1}^\infty\sup_{w\in K}|\hat{\eta}^{n,ij}_k([nt],w)-\eta^{ij}_k(t,w)|\longrightarrow 0,\quad n\to\infty.\end{aligned} \tag{5.44}$$

This implies the assertion (iii).

Since

$$\begin{aligned}&\nabla_w h^i\Big(\frac{[nt]+k}{n},w\Big(\,\cdot\wedge\frac{[nt]}{n}\Big),y;I^n_{[nt]}e_j\Big) \\ &=\sum_{\nu=1}^{m_2}\frac{\partial}{\partial x^\nu}f^i\Big(\Psi\Big(\frac{[nt]+k}{n},w\Big(\,\cdot\wedge\frac{[nt]}{n}\Big),y\Big)\Big)\end{aligned}$$

$$\times \int_0^{k/n} \frac{\partial}{\partial x^j} u^\nu\Big(\frac{[nt]+k}{n}, w\Big(\Big(\frac{[nt]+k}{n} - s\Big) \wedge \frac{[nt]}{n}\Big), y\Big) I^n_{[nt]}$$
$$\times \Big(\frac{[nt]+k}{n} - s\Big) ds,$$

we have

$$\text{(5.45)} \qquad \sup_{w \in K} |B^{n,ij}([nt], w)| \leq 8 \sum_{\nu=1}^{m_2} \sup_x \Big| \frac{\partial}{\partial x^\nu} f^i(x) \Big| \sup_x |f^j(x)| \times \sum_{k=1}^{\infty} \int_0^{k/n} \sup_{x,y} \Big| \frac{\partial}{\partial x^j} u^\nu(s,x,y) \Big| ds \beta_k.$$

Then, [C2], (5.43) and the dominated convergence theorem imply the assertion (iv). □

By Proposition 18, we see that $[A7]$ holds. Obviously $\hat{a}^{ij}$ and $\hat{b}^i$ satisfies the condition $[A8]$ and $[A10]$. Then, using Theorem 3, we obtain Theorem 7. □

(b.) Let $f(x) = (f^i(x))_{i=1}^d : \mathbb{R}^{m_2} \longrightarrow \mathbb{R}^d$, $u(t,x,y) = (u^i(t,x,y))_{i=1}^{m_2} : [0,\infty) \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_1} \longrightarrow \mathbb{R}^{m_2}$, and $\psi(t,x) = (\psi^i(t,x))_{i=1}^{m_3} : [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^{m_3}$ be Borel measurable functions. Let $\Psi(t,w,y) = (\Psi^i(t,w,y))_{i=1}^{m_2}$ and $h(t,w,y) = (h^i(t,w,y))_{i=1}^d$ be such that

$$\Psi^i(t,w,y) = \int_0^t u^i\Big(s, \int_s^t \psi(r, w(r)) dr, y\Big) ds$$

and

$$h^i(t,w,y) = f^i(\Psi(t,w,y)).$$

We define $F_k^{n,i}(w)$ and $G_k^{n,i}(w)$ by (5.22) and (5.23). We introduce the following conditions.

$[D1]$ $f^i(x)$ is three times continuously differentiable in x. Moreover $u(t,x,y)$ (respectively, $\psi^i(t,x)$) is three times (respectively, twice) continuously differentiable in x, and all derivatives are continuous in t.

$[D2]$ It holds that

$$\sum_{|\beta|\leq 3} \sup_{x\in\mathbb{R}^{m_2}} |D^\beta f^i(x)| < \infty, \tag{5.46}$$

$$\sum_{|\beta|\leq 2} \int_0^\infty \sup_{x\in\mathbb{R}^{m_3}, y\in\mathbb{R}^{m_1}} |D_x^\beta u^j(t,x,y)|dt < \infty \tag{5.47}$$

and

$$\sum_{|\beta|\leq 2} \int_0^\infty \sup_{x\in\mathbb{R}^d} |D_x^\beta \psi^\nu(t,x)|dt < \infty \tag{5.48}$$

for each $i = 1,\dots,d$, $j = 1,\dots,m_2$ and $\nu = 1,\dots,m_3$.

THEOREM 8. *Assume* $[D1], [D2]$ *and* $[C3]$. *Then the conclusion of Theorem 1 holds replacing* $\mathscr{L}$ *with* $\hat{\mathscr{L}}$ *which is defined by* (5.28)-(5.31).

Theorem 8 is obtained by the similar arguments in the proof of Theorem 7. So we will check only the condition $[B4]$.

PROPOSITION 19. *The condition* $[B4]$ *holds with* $\gamma_1 = 1$.

PROOF. Let $U(w,\omega) = h^i(t,w,\xi_k(\omega))$ and $\tilde{\mathcal{C}}_t = C([0,t];\mathbb{R}^{m_3})$. We define $\varphi(w) = (\varphi^j(w))_{j=1}^{m_3} : C([0,\infty);\mathbb{R}^d) \longrightarrow \tilde{\mathcal{C}}_t$ and $g(v,\omega) : \tilde{\mathcal{C}}_t \times \Omega \longrightarrow \mathbb{R}$ by

$$\big(\varphi^j(w)\big)(s) = \int_s^t \psi^j(r,w(r))dr$$

and

$$g(v,\omega) = f^i\Big(\int_0^t u\Big(s,v(s),\xi_k(\omega)\Big)ds\Big).$$

Then it follows that

$$U(w,\omega) = g(\varphi(w),\omega). \tag{5.49}$$

Set

$$C_0 = \sum_{j=1}^{m_3} \sum_{|\beta|\leq 1} \int_0^\infty \sup_{x\in\mathbb{R}^d} |D_x^\beta \psi^j(s,x)|ds.$$

By $[D2]$, we see that C_0 is finite and

$$\varphi(w) \in \tilde{A}_t, \quad w \in C([0,\infty);\mathbb{R}^d), \tag{5.50}$$

where

$$\tilde{A}_t = \Big\{ v \in \tilde{\mathcal{C}}_t \ ; \ v \text{ is continuously differentiable and } \sum_{j=1}^{m_3} \Big(\sup_{0\le s\le t} |v^j(s)| + \sup_{0\le s\le t} \Big| \frac{d}{ds} v^j(s) \Big| \Big) \le C_0 \Big\}.$$

Moreover we have

$$|\nabla g(v,\omega;\tilde{v})| \le C_1 \sum_{j=1}^{m_3} \sup_{0\le s\le t} |\tilde{v}^j(s)|, \quad v,\tilde{v} \in \tilde{\mathcal{C}}_t, \ \omega \in \Omega \tag{5.51}$$

for some $C_1 > 0$. Then we have the assertion by the same arguments in the proof of Proposition 16. □

References

[1] Billingsley, P., Convergence of probability measures 2nd. edition, Wiley-Interscience (1999).

[2] Borodin, A. N., A limit theorem for solutions of differential equations with random right-hand side, Theor. Prob. Appl. **22** (1977), 482–497.

[3] Fujiwara, T., Limit theorems for random difference equations driven by mixing processes, J. Math. Kyoto Univ. **32** (1992), 763–795.

[4] Ibragimov, I. A. and Yu. V. Linnik, Independent and stationary sequences of random variables, Groningen : Wolters-Noordhoff (1971).

[5] Kesten, H. and G. C. Papanicolaou, A limit theorem for turbulent diffusion, Comm. Math. Phys. **65** (1979), 97–128.

[6] Khasminskii, R. Z., A limit theorem for solutions of differential equations with a random right-hand sides, Theor. Prob. Appl. **11** (1966), 390–406.

[7] Kunita, H., Lecture on stochastic flows and applications, Tata Institute of Fundamental Research (1986).

[8] Kolmogorov, A. N. and V. M. Tihomirov, ε-entropy and ε-capacity of sets in functional spaces, Uspehi Mat. Nauk **14** (1959), 3–86, English transl., Amer. Math. Soc. Transl. (2) **17** (1961), 277–364.

[9] Kushner, H. J. and Hai-Huang, On the weak convergence of a sequence of general stochastic difference equations to a diffusion, SIAM J. Appl. Math. **40** (1981), 528–541.

[10] Papanicolaou, G. C. and W. Kohler, Asymptotic theory of mixing stochastic ordinary differential equations, Comm. Pure Appl. Math. **27** (1974), 641–668.

[11] Papanicolaou, G. C., Stroock, D. W. and S. R. S. Varadhan, Martingale approach to some limit theorems, in Statistical Mechanics and Dynamical Systems, Duke Univ. Conf. Turbulence, Duke Univ. Math. Ser. 3, Durham, N.C. (1977).

[12] Schwartz, J. T., Nonlinear functional analysis, Gordon and Breach Science Publishers (1969).

[13] Stroock, D. W. and S. R. S. Varadhan, Diffusion processes with continuous coefficients I, Comm. Pure Appl. Math. **22** (1969), 345–400.

[14] Stroock, D. W. and S. R. S. Varadhan, Multidimensional diffusion processes, Springer-Verlag, Berlin (1979).

[15] Watanabe, H., Diffusion approximations of some stochastic difference equations, Pinsky, ed., Stochastic Analysis and Application, Marcel Dekker, New York (1984).

[16] Watanabe, H., Diffusion approximations of some stochastic difference equations II, Hiroshima Math. J. **14** (1984), 15–34.

[17] Watanabe, H., Diffusion approximations of some stochastic difference equations revisited, Stoch. Proc. Appl. **29** (1988), 147–154.

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Graduate School of Mathematical Sciences
The University of Tokyo
Meguro-ku, Komaba 3-8-1
Tokyo 153-8914, Japan