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Uniqueness of Crapper's Pure Capillary Waves of Permanent Shape

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Abstract. Two-dimensional water-waves of permanent shape with constant propagation speed are considered under the assumption that the gravity is neglected and only the surface tension is taken into account. Crapper's solutions, which are exact solutions of the governing equations, are proved to be unique among those which satisfy a certain positivity property.

1. Introduction

We consider two-dimensional water-waves on the surface of irrotational flow of incompressible inviscid fluid. Only waves of permanent shape with constant propagation speed are considered. In the present paper, the surface tension is the only force acting on the fluid, hence in particular, the gravity is neglected.

The shape of a water-wave is determined by solving a free boundary problem for the Laplace equation with a nonlinear boundary condition which is derived from Bernoulli's theorem. The problem is then transformed, by a certain change of variables, to a nonlinear boundary value problem for an analytic function defined in the unit disk in the complex plane (see, for instance, [3] or [6]). It is well-known that Crapper [4] found a family of exact solutions represented in terms of elementary functions. The purpose of the present paper is to prove, under a certain positivity condition, that there does not exist a solution other than Crapper's solutions.

The nonlinear boundary value problem for analytic functions can further be re-written (see [6]) as a problem to look for a 2π-periodic function θ

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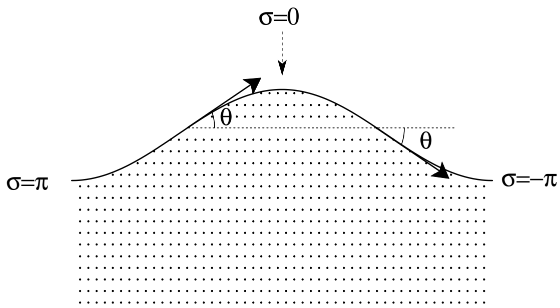

Fig. 1. θ represents the angle between the tangent and the horizontal line. σ is a Lagrange variable along the wave profile.

satisfying $\int_{-\pi}^{+\pi} \theta(\sigma)\mathrm{d}\sigma = 0$ and

$$q\frac{\mathrm{d}\theta}{\mathrm{d}\sigma} = -\sinh(H\theta) \qquad (-\pi \le \sigma \le \pi). \tag{1}$$

Here H is the Hilbert transform (or can be called the conjugate operator; its concrete form will be given in the next section) and q is a nondimensionalized surface tension coefficient. See [6] for more detail. The unknown θ represents the angle between the tangent at the free boundary and the horizontal line. See Fig. 1. σ is a Lagrange variable along the free boundary.

Once $\theta(\sigma)$ is known by (1), we can determine the wave profile $(x(\sigma), y(\sigma))$ $(-\pi \le \sigma \le \pi)$. In fact it is known ([6]) that

$$\frac{\mathrm{d}x}{\mathrm{d}\sigma} = -\frac{L}{2\pi}e^{-\tau(\sigma)}\cos\theta(\sigma), \qquad \frac{\mathrm{d}y}{\mathrm{d}\sigma} = -\frac{L}{2\pi}e^{-\tau(\sigma)}\sin\theta(\sigma), \tag{2}$$

where L denotes the wave length and $\tau = H\theta$. The right hand sides are known once θ is known. Thus, after integrating in σ, we have a parametric representation of the free boundary $\{ (x(\sigma), y(\sigma))\ ;\ -\pi \le \sigma \le \pi \}$. Consequently what remains to be done is to solve the equation (1). Wave profiles of Crapper's waves can be found in [3, 4, 6].

Crapper [4] found a family of exact solutions, which in our notation are written as follows (see [6]): the solutions are parameterized by $A \in (-1, 1)$, and (q, θ) is represented as

$$q = \frac{1 + A^2}{1 - A^2}, \tag{3}$$

$$(4) \qquad \theta(\sigma) = -2\arctan\left(\frac{2A\sin\sigma}{1-A^2}\right) = -4\left(A\sin\sigma + \frac{A^3}{3}\sin 3\sigma + \frac{A^5}{5}\sin 5\sigma + \cdots\right).$$

The Hilbert transform of this θ is given by

$$(5) \qquad H\theta(\sigma) = \log\frac{1+A^2+2A\cos\sigma}{1+A^2-2A\cos\sigma} = 4\left(A\cos\sigma + \frac{A^3}{3}\cos 3\sigma + \frac{A^5}{5}\cos 5\sigma + \cdots\right).$$

Other solutions are obtained by replacing q and $\theta(\sigma)$ by q/n and $\theta(n\sigma)$, respectively, where n is a positive integer. These are solutions to (1), as is verified in an elementary way ([6]). A solution given by (3) and (4) was called in [6] Crapper's solution of mode one. A solution of the form $(q/n, \theta(n\sigma))$ was called Crapper's solution of mode n. These solutions are plotted in Fig. 2. As the graph shows, Crapper's waves bifurcate from the trivial solution $\theta \equiv 0$.

Henceforth $H\theta$ is denoted by τ:

$$\tau = H\theta.$$

It was conjectured in [6] that these were the only possible solutions of (1) and no other solution of (1) would exist. Numerical computation in

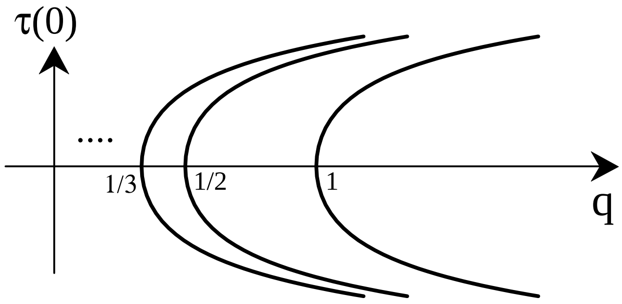

Fig. 2. The bifurcation diagram of Crapper's waves. Solutions of mode n bifurcate at $q = 1/n$.

[6] seems to support this conjecture but, as far as the author knows, no mathematical proof seems to have been published up until now. Our aim in the present paper is to prove the uniqueness under a certain positivity assumption.

The contents of the present paper is as follows. In section 2, we recall some facts about Crapper's solutions. The main theorem is stated and proved in section 3. Some comments on the general case are given in section 4.

2. Preliminary Results

We recall some mathematical facts, all of which can be found with proof in [6].

R1 H is explicitly written as

$$Hf(\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\left(\frac{\sigma - s}{2}\right) f(s)\,\mathrm{d}s,$$

where Cauchy's principal value is taken. This transform is also characterized as

$$H\left(\sum_{n=1}^{\infty}\left(a_n \sin n\sigma + b_n \cos n\sigma\right)\right) = \sum_{n=1}^{\infty}\left(- a_n \cos n\sigma + b_n \sin n\sigma\right),$$

where a_n's and b_n's are real constants. Note finally that $H^2 = -I$, where I is an identity operator.

R2 Any solution $\theta(\sigma)$ of (1), hence $\tau(\sigma)$, too, are infinitely many times differentiable in σ.

R3 $(H\frac{\mathrm{d}}{\mathrm{d}\sigma})^{-1}$ is an integral operator. More specifically, if $H\frac{\mathrm{d}}{\mathrm{d}\sigma}f = g$ for odd functions f and g, then we have

$$f(\sigma) = \int_0^{\pi} G(\sigma, s)g(s)\mathrm{d}s \qquad (0 \le \sigma \le \pi) \tag{6}$$

where

$$\begin{aligned} G(\sigma, s) &= \frac{1}{\pi}\log\left|\frac{\sin\frac{\sigma+s}{2}}{\sin\frac{\sigma-s}{2}}\right| \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\sigma)\sin(ns)}{n} \qquad (0 \le \sigma, s \le \pi). \end{aligned} \tag{7}$$

It can be proved that G is positive for all $0 < \sigma, s < \pi$.

In what follows, we assume that θ *is odd in* σ. This assumption implies that *the wave profile is symmetric with respect to the crest*, see [6].

We now prove that $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ and $\sin\theta$ are eigenfunctions of a certain eigenvalue problem. We differentiate (1) to obtain

$$q\frac{\mathrm{d}^2\theta}{\mathrm{d}\sigma^2} = -\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\cosh\tau.$$

This, together with $H^2 = -I$, implies that $f = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ is a solution to the following equation:

$$H\frac{\mathrm{d}}{\mathrm{d}\sigma}f = \frac{1}{q}\left(\cosh\tau\right)f. \tag{8}$$

We next show that $\sin\theta$, too, satisfies the same equation. To this end, we note that

$$H\frac{\mathrm{d}}{\mathrm{d}\sigma}\sin\theta = H\left(\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right) = -\frac{1}{q}H\left(\cos\theta\sinh\tau\right). \tag{9}$$

Note that H sends the real part of an analytic function to the imaginary part. Then, since $\sin(\theta+\mathrm{i}\tau) = \sin\theta\cosh\tau + \mathrm{i}\cos\theta\sinh\tau$ is analytic, we have

$$\cos\theta\sinh\tau = H\left(\sin\theta\cosh\tau\right).$$

It therefore follows from (9) that

$$H\frac{\mathrm{d}}{\mathrm{d}\sigma}\sin\theta = \frac{1}{q}\cosh\tau\sin\theta. \tag{10}$$

Since both $\sin\theta$ and $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ are odd functions of σ, (8) and (10) imply that they are solutions of

$$qf(\sigma) = \int_0^\pi G(\sigma,s)\cosh(\tau(s))f(s)\mathrm{d}s. \tag{11}$$

Let us define an operator L by

$$Lf(\sigma) = \int_0^\pi G(\sigma,s)\cosh(\tau(s))f(s)\mathrm{d}s. \tag{12}$$

(Here the function τ is fixed.) Then both $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}$ and $\sin\theta$ are eigenfunctions of L with q the eigenvalue.

3. Proof of Uniqueness

We now recall an important tool in functional analysis. Let E be the Banach space

$$E = \{ f \in C[0, \pi] \ ; \ f(0) = f(\pi) = 0 \ \}$$

with the usual maximum norm $\|f\| = \max_{0 \le \sigma \le \pi} |f(\sigma)|$. Further, let K be defined by

$$K = \{ f \in E \ ; \ f(\sigma) \ge 0 \ \ (0 \le \sigma \le \pi \ ) \ \}.$$

K is called the positive cone. With this setting, we make the following definition:

DEFINITION 1. Let w_0 be an element of $K \setminus \{0\}$. A bounded linear operator $L : E \to E$ is called w_0-positive if for every $u \in K \setminus \{0\}$ a positive integer n and positive numbers α, β can be chosen in such a way that $\alpha w_0 \le L^n u \le \beta w_0$ everywhere in $[0, \pi]$.

THEOREM 1. *Let $L : E \to E$ be a compact linear operator. Suppose that $Lf \in K$ for all $f \in K$. Suppose also that there exists $w_0 \in K \setminus \{0\}$ such that L is w_0-positive. We finally assume that L has a positive eigenvalue λ_0 with an eigenvector $f \in K \setminus \{0\}$. Then the eigenvalue λ_0 is simple.*

This theorem is a special case of Theorem 2.10 of [5, page 76].

We now use **Theorem 1** to prove the uniqueness of Crapper's waves under one of the following assumptions:

A1 $0 \le \theta(\sigma) \le \pi$ everywhere in $0 \le \sigma \le \pi$;

A2 $\frac{d\tau}{d\sigma}(\sigma) \ge 0$ everywhere in $0 \le \sigma \le \pi$.

Of course, **A1** implies that $\sin\theta(\sigma) \ge 0$ everywhere in $0 \le \sigma \le \pi$. Note that Crapper's solutions of mode one with $A < 0$ satisfy both **A1** and **A2**.

THEOREM 2. *Suppose that a solution θ of* (1) *satisfies either* **A1** *or* **A2**. *Then it is given by* (3) *and* (4) *with an appropriate $A \in (-1, 0]$.*

PROOF. L in (12) satisfies the assumptions in **Theorem 1**, verification of which is easy except for the w_0-positivity. To prove w_0-positivity, we set $w_0(\sigma) = \sin\sigma$. We note that $Lu(\sigma) > 0$ for all $\sigma \in (0, \pi)$ and $Lu(0) =$

$Lu(\pi) = 0$ if $u \in K \setminus \{0\}$. If Lu is of C^1-class and $\frac{\mathrm{d}Lu}{\mathrm{d}\sigma}(0) > 0$, $\frac{\mathrm{d}Lu}{\mathrm{d}\sigma}(\pi) < 0$ are satisfied, then it is easy to prove that $Lu \geq \alpha \sin\sigma$ for some $\alpha > 0$. However, Lu is not necessarily of C^1-class if u is merely continuous. We can nevertheless prove the $\sin\sigma$-positivity by considering L^2u. We note that $Lu = \left(H\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^{-1}(\cosh(\tau)u)$ is Hölder continuous with any exponent < 1. This implies, in particular, that $L^2u \in C^1[0,\pi]$. Note also that, if v is Hölder continuous, $v(0) = 0$, and $v \in K \setminus \{0\}$, then we have by (7)

$$\left.\frac{\mathrm{d}Lv}{\mathrm{d}\sigma}\right|_{\sigma=0} = \frac{1}{\pi}\int_0^\pi \cot\left(\frac{s}{2}\right)\cosh(\tau(s))\, v(s)\mathrm{d}s > 0.$$

Similarly we have $\left.\frac{\mathrm{d}Lv}{\mathrm{d}\sigma}\right|_{\sigma=\pi} < 0$. With these in mind, it is not difficult to verify that, for all $u \in K \setminus \{0\}$, $L^2u \geq \alpha\sin\sigma$ with some $\alpha > 0$. The proof of $L^2u \leq \beta\sin\sigma$ is elementary.

Since, by the assumption, $\sin\theta$ or $\mathrm{d}\tau/\mathrm{d}\sigma$ is nonnegative everywhere, q is a simple eigenvalue by **Theorem 1**. Consequently there exists a constant k such that

$$\frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = k\sin\theta, \tag{13}$$

which is nothing but the bifurcation problem considered in section 3 of Toland [7]. In the paper he considered the nonlinear equation $H\frac{\mathrm{d}f}{\mathrm{d}\sigma} = \sin f$ and showed that all the solutions were concretely written. Specifically, any solution f of $H\frac{\mathrm{d}f}{\mathrm{d}\sigma} = \sin f$ was shown to be either the following f_1 or f_2:

$$f_1(\sigma) = \pm 2\tan^{-1}(\sigma + a) + 2\pi n, \tag{14}$$

where a is a real constant and n is an integer;

$$f_2(\sigma) = 2\tan^{-1}\left(\gamma^{-1}\tan\delta\sigma\right) - 2\tan^{-1}\left(\gamma\tan\delta\sigma\right), \tag{15}$$

where γ and δ are real constants.

Note that f_2 can be rewritten as

$$f_2(\sigma) = 2\tan^{-1}\left(\frac{1}{2}(\gamma^{-1} - \gamma)\sin 2\delta\sigma\right).$$

f_1 is not periodic and does not suit our solutions; we have accordingly

$$\theta(\sigma) = 2\tan^{-1}\left(\frac{1}{2}(\gamma^{-1} - \gamma)\sin 2\delta k\sigma\right).$$

Since θ is 2π-periodic, $2\delta k$ must be an integer. Let it be denoted by n. We then obtain

$$\theta(\sigma) = -2\tan^{-1}\left(\frac{1}{2}(\gamma - \gamma^{-1})\sin n\sigma\right).$$

Note, on the other hand, that $\sin\theta \geq 0$ for all $\sigma \in [0, \pi]$. (This follows from the assumptions **A1, A2** and (13).) We must therefore have $n = \pm 1$. We consider the case of $n = 1$, since the other case is proved in the same way. Note that $2A/(1 - A^2)$ runs monotonically from $-\infty$ to ∞ as A runs from -1 to 1. Accordingly there is exactly one $A \in (-1, 1)$ such that

$$\frac{1}{2}\left(\gamma - \frac{1}{\gamma}\right) = \frac{2A}{1 - A^2}.$$

Since we have assumed that $\theta \geq 0$, it holds that $\gamma - \gamma^{-1} < 0$. Consequently, we have $A = (\gamma - 1)/(\gamma + 1)$ if $0 < \gamma < 1$ and $A = (1 + \gamma)/(1 - \gamma)$ if $\gamma < -1$. We have thus get to Crapper's solution. This ends the proof. □

4. Comments on the General Case

If all the eigenvalues of L are simple, we can prove the uniqueness without assuming that $\sin\theta \geq 0$. However, the general theory does not seem to guarantee the simpleness of the eigenvalues other than the one corresponding to a positive eigenfunction.

Toland [7] made the following interesting discovery. Consider

$$H\frac{\mathrm{d}f}{\mathrm{d}\sigma} = \sin f, \tag{16}$$

which was called in [7] the Peierls-Nabarro equation. Then he showed that there are two solutions, say g_1 and g_2, of

$$H\frac{\mathrm{d}g}{\mathrm{d}\sigma} = -g + g^2 \tag{17}$$

such that $\frac{\mathrm{d}f}{\mathrm{d}\sigma} = g_1 - g_2$. On the other hand, the complete set of solutions of (17) are known in [1, 2]. This leads him to have the formula (14) and (15). Since the difference between (16) and (1) is just a difference of sin and sinh, it it natural to imagine that Toland's method with a possible modification enables us to write any solution of (1) by those of (17). We tried this but could not derive a necessary formula. This is why we employed a somewhat clumsy approach in the previous sections. Thus it was necessary for us to assume **A1** or **A2**. Although we could not prove, we believe that the uniqueness holds true without assuming such a positivity assumption.

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