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*A New Proof of a Theorem of J. M. Montesinos**

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Abstract. If M denotes a closed orientable 3-manifold, a Dehn sphere Σ [Pa] in M will be a 2-sphere immersed in M with only double curve and triple point singularities. The sphere Σ fills M [M2] if it defines a cell decomposition of M. In [M2] it is proved that every closed orientable 3-manifold has a nulhomotopic filling Dehn sphere, and Johansson diagrams [J] of Dehn spheres are proposed as a new method for representing all closed, orientable 3-manifolds. In the present paper another proof of this theorem is given and an algorithm for obtaining Johansson diagrams of closed orientable 3-manifolds from their Heegaard diagrams is developed in detail. Some examples are given.

1. Introduction

Through this paper we will work in the differentiable category.

Let A and B be two sets. For a map $f : A \to B$ the *singular values* or *singularities* of f are the points $y \in B$ with $\#\left\{f^{-1}(y)\right\} > 1$, and the *singular points* of f are the inverse images by f of the singularities. The *singular set* $S(f)$ of f is the set of singular points of f in A, and the *singularity set* $\bar{S}(f)$ is the set of singularities of f in B. Of course, $f(S(f)) = \bar{S}(f)$.

Let M be a closed, orientable 3-manifold.

A *surface* is a compact orientable 2-manifold which might have non-empty boundary and more than one connected component.

A subset $\Sigma \subset M$ is a *Dehn surface* in M [Pa] if there exists a compact surface S and a transverse immersion $f : S \to M$ such that $\Sigma = f(S)$ and the singular set of f does not contain points of the boundary ∂S of S. In this situation we say that f *parametrizes* Σ. If S is a 2-sphere then Σ is a *Dehn sphere*. Moreover, if S if a connected surface of genus $g \geq 0$ without boundary, then Σ is a *genus g Dehn surface*.

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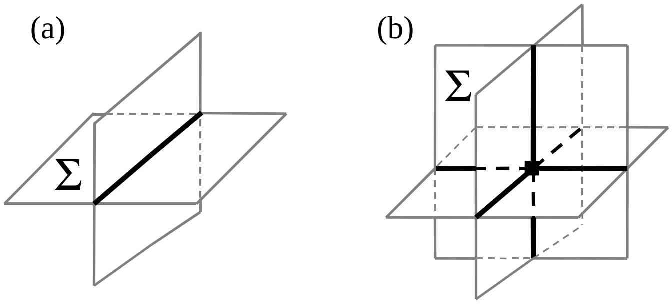

Fig. 1. Double and triple points.

Let Σ be a Dehn surface in M, and let S be a compact surface and $f : S \to M$ a transverse immersion parametrizing Σ. Then, for any $y \in M$ is $\#\left\{f^{-1}(y)\right\} \leq 3$ [He]. The singularities are divided into *double points*, with $\#\left\{f^{-1}(y)\right\} = 2$, and *triple points* with $\#\left\{f^{-1}(y)\right\} = 3$. A small neighbourhood of a double or a triple point looks like in Figures 1(a) and 1(b) respectively. We say that two different points of S are *related* by f if their respective images by f coincide. Then, each point of S is related with 0 points of S if it is a non-singular point of f; it is related with 1 point of S if its image by f is a double point of Σ; and it is related with 2 points of S if its image by f is a triple point of Σ. If $f(x)$ is a triple point of Σ, then two branches of $S(f)$ cross at $x \in S$ and we say that x is a *double point of the singular set* of f. The inverse set of a triple point of Σ under the map f consists of three such double points of $S(f)$. Because $\bar{S}(f)$ only depends on Σ, it will be denoted sometimes as the *singularity set of* Σ. A *double curve* of Σ is the image of an immersion $\gamma : S^1 \to M$ contained in the singularity set of f [S1], [S2]. Such a γ is not unique, and it is called a *parametrization* of the double curve $\gamma\left(S^1\right)$. The singularity set of f is the union of the double curves of Σ. Because S is compact Σ has a finite number of double curves, and since M is orientable, each double curve has exactly two preimage curves in S (see [J], part I, p. 319, 2 Satz). Therefore, if we take a parametrization $\bar{\gamma} : S^1 \to M$ of a double curve of Σ, there are exactly two different immersions $\gamma, \gamma' : S^1 \to S$ such that $f \circ \gamma = f \circ \gamma' = \bar{\gamma}$. We say that the parametrized curves γ and γ' are *lifted curves* of $\bar{\gamma}$ under f and also that they are *sister curves* under f. The singular set of f together with the information about how its points are related by f is the *Johansson diagram* of Σ [M2].

A *complete parametrization* of the singularity set $\bar{S}(f)$ of f is a set $\bar{\mathfrak{D}} = \{\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_m\}$ of immersions from S^1 into M such that: (i) each $\bar{\alpha}_i$ parametrizes a double curve of Σ; (ii) $\bar{\alpha}_i(S^1) \neq \bar{\alpha}_j(S^1)$ if $i \neq j$; and (iii) $\bar{S}(f) = \bigcup_{i=1}^{m} \bar{\alpha}_i(S^1)$. If $\bar{\mathfrak{D}}$ is a complete parametrization of $\bar{S}(f)$ and we denote by $\mathfrak{D}$ the set of all lifted curves of the curves of $\bar{\mathfrak{D}}$, the map $\tau : \mathfrak{D} \to \mathfrak{D}$ that assigns to each curve of $\mathfrak{D}$ its sister curve under f defines a free involution of $\mathfrak{D}$. The Johansson diagram of f can be defined as the pair $(\mathfrak{D}, \tau)$, because it contains all the information about the singular set $S(f)$ and about how the points of $S(f)$ are identified by the map f: two different points $A, B \in S$ verify $f(A) = f(B)$ if and only if there is a parametrized curve $\alpha \in \mathfrak{D}$ and a $z \in S^1$ with $A = \alpha(z)$ and $B = \tau\alpha(z)$ (compare [Pa], p. 6). A different notation is used in [C].

We will generalize to general surfaces a definition given in [M2] for Dehn spheres:

Definition 1. Let Σ be a Dehn surface in M. Then Σ fills M if it defines a cell decomposition of M in which the 0-skeleton is the set of triple points of Σ; the 1-skeleton is the set of double curves of Σ; and the 2-skeleton is Σ itself.

In other words, Σ fills M if and only if

1. $M - \Sigma$ is a disjoint union of open 3-balls,
2. $\Sigma - \{\text{double curves of } \Sigma\}$ is a disjoint union of open 2-dimensional disks, and
3. $\{\text{double curves of } \Sigma\} - \{\text{triple points of } \Sigma\}$ is a disjoint union of open intervals.

In [Ha] it is stated that every homotopy sphere has a filling Dehn sphere. In [F-R] it is proved that every closed orientable 3-manifold has a Dehn sphere Σ such that $M - \Sigma$ is a disjoint union of open 3-balls. In [M2] is proved:

Theorem 2 ([M2]). *Every closed orientable 3-manifold has a nulhomotopic filling Dehn sphere.*

The proof in [M2] of this theorem is constructive and gives an algorithm for obtaining the Johansson diagram of a filling Dehn sphere of any 3-manifold M from a Heegaard diagram of M. A Dehn sphere is *nulhomotopic* if one (and hence any) of its parametrizations is nulhomotopic. In [M2] no mention is made of the fact that the filling Dehn sphere there constructed is nulhomotopic, but we remark that this is the case. This is an important qualification in view of [V1], [V2], in which a complete set of moves relating nulhomotopic filling Dehn spheres for closed orientable 3-manifolds is given.

As it is pointed out in [M2] (see also [V1]), for a given diagram in S^2 it is possible to know if it is the Johansson diagram for a filling Dehn sphere Σ in some 3-manifold M. If this is the case, it is possible also to reconstruct such M from the given diagram. So, Johansson diagrams are a suitable way for representing all closed, orientable 3-manifolds and it is interesting to further study them. A *Johansson representation* of M [M2] is the Johansson diagram of a filling Dehn sphere of M.

In this paper we give another constructive proof of Theorem 2. From the same starting point of [M2], we construct here a simpler filling Dehn sphere (with less triple points). This construction also gives an algorithm for obtaining a Johansson representation out from a Heegaard diagram. In Section 2 we introduce some generic modifications of Dehn surfaces that we will use later. These modifications are a simple generalization of one special case of the *surgeries* of immersed surfaces introduced in [B]. In Section 3 we give the proof of Theorem 2 focusing only in the existence part. The constructive part of the same theorem is studied in detail in Sections 4 and 5. Specifically we construct a filling Dehn sphere Σ of M, obtaining also its Johansson diagram. These results are resumed in Section 6 in an algorithm that provides automatically a Johansson representation of M from any Heegaard diagram of M. This algorithm is applied in Section 7 to some examples.

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2. Piping Dehn Surfaces

An essential tool in our construction will be the modifications for Dehn surfaces by surgery explained in detail in [B], specially the modifications by surgery of type 1 that we will call simply *pipings* (see [R-S], p. 67). For more details, see [B].

An *arc* in a 2- or 3-manifold N is the image of an embedding from the unit interval $[0,1]$ into N.

Let Σ be a Dehn surface in M. Let S be a compact surface and $f : S \to M$ a transverse immersion parametrizing Σ.

DEFINITION 3. An arc $\bar{\delta} \subset \Sigma$ is a type 1 arc in Σ if:

(i) $\bar{\delta}$ has no tangent intersections with the double curves of Σ;

(ii) the endpoints $\bar{X}$ and $\bar{Y}$ of $\bar{\delta}$ are double points of Σ; and

(iii) $\bar{\delta}$ contains no triple point of Σ.

Let $\bar{\delta}$ be a type 1 arc in Σ. Because of the property (i) of the previous definition, $\bar{\delta}$ contains a finite number of double points of Σ. It will be *simple* if the unique double points that it contains are its endpoints and *non-simple* otherwise. Because $\bar{\delta}$ is an embedded interval in Σ, if it contains k double points of Σ, $k \in \mathbb{N}$, then the inverse image by f of $\bar{\delta}$ is composed by an arc δ

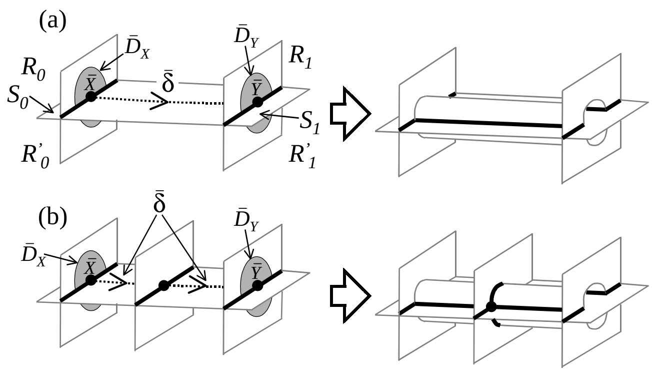

Fig. 2. Simple and non-simple pipings along $\bar{\delta}$.

in S such that $f(\delta) = \bar{\delta}$ and k isolated points of the singular set of f. These k isolated points are related (in the precise sense defined before) by f with the intersecting points of δ with $S(f)$. Let $X, Y \in S$ be the endpoints of δ with $f(X) = \bar{X}$ and $f(Y) = \bar{Y}$, and $X', Y' \in S$ their related (isolated) points by f respectively. Consider two small closed disks $D_X, D_Y \subset S$ with X', Y' in their respective interiors and their images $\bar{D}_X := f(D_X), \bar{D}_Y := f(D_Y)$ by f.

We can obtain a new Dehn surface Σ' taking the image by f of $S - (D_X \cup D_Y)$ (which is not the same thing as $\Sigma - (\bar{D}_X \cup \bar{D}_Y)$) and attaching to it a small "tube" parallel to δ along the two circles $\partial\bar{D}_X$ and $\partial\bar{D}_Y$ as in Figure 2. It is clear that it can be done in such a way that the resulting surface is still smoothly (and transversely) immersed in M.

Definition 4. In this situation, we say that Σ' is obtained by piping Σ along $\bar{\delta}$.

A piping in a Dehn surface is *simple* or *non-simple* if the corresponding type 1 arc $\bar{\delta}$ is simple or non-simple respectively. After a non-simple piping, a new double curve and two new triple points appear close to each inner double point of $\bar{\delta}$.

Note that the Dehn surface obtained after a piping could be non-orientable. This cannot happen if we connect two different orientable components of Σ, and this will be always the case in this paper.

If Σ is a filling Dehn surface of M and the surface Σ' obtained by piping Σ along $\bar{\delta}$ is still a filling Dehn surface of M, we will say that $\bar{\delta}$ is *filling-preserving.*

After performing a simple piping, two pairs of 3-dimensional regions (denoted R_0, R_1, and R'_0, R'_1 in Figure 2(a)) of $M - \Sigma$; a pair of 2-dimensional regions (denoted S_0, S_1 in Figure 2(a)) of $\Sigma - \{\text{double curves of } \Sigma\}$; and two additional pairs, become connected. This is not the case for non-simple pipings, where the intermediate sheets of Σ hinder these new connections. In particular we have the following trivial result.

Proposition 5. *Let Σ be a filling Dehn surface of M, and $\bar{\delta}$ a type 1 arc in Σ.*

Then, if $\bar{\delta}$ is non-simple, then it is filling-preserving.

A particular case in which we can assure that simple pipings are filling-preserving is the following.

PROPOSITION 6. *Let $\Sigma, \Sigma_1, ..., \Sigma_m$ be a collection of Dehn surfaces in M with $\Sigma = \Sigma_1 \cup ... \cup \Sigma_m$ and such that Σ fills M. Assume that for some particular $j \in \{1, ..., m\}$ the Dehn surface Σ_j is the image of an embedding $f_j : S_j \to M$ from a 2-sphere S_j into M and Σ_j bounds a ball B_j. If $\bar{\delta}$ is a simple type 1 arc in Σ such that $\bar{\delta} \cap B_j$ is precisely one endpoint of $\bar{\delta}$, then $\bar{\delta}$ is filling-preserving.*

PROOF. Assume, as we can because $\#\left\{\bar{\delta} \cap \Sigma_j\right\} = 1$ (endpoint of $\bar{\delta}$), that $\bar{\delta}$ is the arc of Figure 2(a) and in this figure the vertical sheet of the immersed surface Σ containing $\bar{D}_Y$ belongs to Σ_j and the other two sheets (the vertical one containing $\bar{D}_X$ and the horizontal one) belong to the rest of Σ. The arc $\bar{\delta}$ must be contained in the closure of one of the two components M_1 and M_2 of $M - \Sigma_j$. If, for example, $\bar{\delta}$ is contained in the closure of M_1, then in Figure 2(a) the point $\bar{X}$ is an interior point of M_1, and so it is $R_0, R'_0, S_0 \subset M_1$. But in this case we have that $R_1, R'_1, S_1 \subset M_2 = B_j$ and so $\bar{\delta}$ is filling-preserving. $\square$

We end this section by making some remarks.

For filling Dehn surfaces non-simple pipings always preserve fillingness, but they *increase the complexity* of the filling Dehn surface because the filling Dehn surface so obtained have more triple points than the original one. On the other hand, simple pipings do not always preserve the filling property, but they always preserve the number of triple points. Another kind of surgery (*type 2 surgery*) introduced in [B] that *reduces* the number of triple points is used in [M2].

We observe, in passing, that the minimal number of triple points of filling Dehn surfaces satisfying some particular property can be in some cases a topological invariant of the manifold. We can define, for example the *triple point number* of a closed orientable 3-manifold as the minimal number of triple points of all its filling Dehn surfaces, the *genus g triple point number* as the minimal number of triple points of all its genus g filling Dehn surfaces, or the *nulhomotopic genus g triple point number* as the minimal number of triple points of all its nulhomotopic genus g filling Dehn surfaces. All of them are topological invariants of the 3-manifold and give a kind of measure of the *complexity* of the manifold in the same way as the Heegaard genus, for example. If we have a filling Dehn surface Σ in a 3-manifold, using the type 2 surgery mentioned in the previous paragraph perhaps we can reduce the number of triple points of Σ, but increasing the genus of the filling Dehn surface. So there is some relation between the different genus g triple point numbers that would be interesting to clarify.

We can define that a genus g filling Dehn surface Σ of a 3-manifold M is *minimal* if there is no other genus g filling Dehn surface of M with less triple points than Σ. Minimal Dehn surfaces, in particular minimal filling Dehn spheres, should have interesting properties, and their classification is another interesting problem. The classification of minimal Dehn spheres has been solved for S^3 in [S2]. In that work, A. Shima gives in a different context six examples of Dehn spheres in S^3 with only 2 triple points, and three of these six examples fill S^3. These three filling examples are minimal according to the definition given here because, as it is pointed out in [Ha], p. 105, any Dehn sphere in a closed orientable 3-manifold has an even number of triple points. It can be deduced by the main theorem of [S2] that these three filling examples given there are the unique possible minimal filling Dehn spheres in S^3.

3. Proof of Theorem 2

DEFINITION 7. A Dehn surface $\Sigma \subset M$ that fills M is called a *filling collection of spheres* in M if there exists a surface S which is a disjoint union of a finite number of 2-spheres and a transverse immersion $f : S \to M$ parametrizing Σ.

THEOREM 8. *If M has a filling collection of spheres then M has a filling Dehn sphere.*

PROOF. Let Σ be a filling collection of spheres in M. Then there exists an $m \in \mathbb{N}$, $m > 0$ and a disjoint union of m 2-spheres $S = \coprod_{i=1}^{m} S_i^2$, such that $\Sigma = f(S)$ for a transverse immersion $f : S \to M$.

If $m = 1$ there is nothing to do, so assume that $m > 1$.

The filling Dehn surface Σ is the union of the m Dehn spheres $\Sigma_1, ..., \Sigma_m$, where $\Sigma_i := f\left(S_i^2\right)$ for $i = 1, ..., m$.

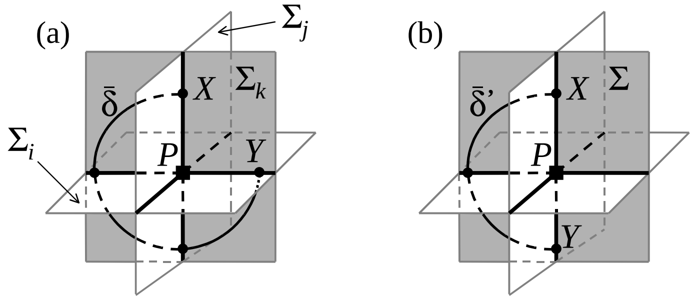

Fig. 3.

Because M is connected, the 2-skeleton of any cell decomposition of M is connected, and then Σ is connected. This implies that for a given $i \in \{1, ..., m\}$ there is at least one $j \in \{1, ..., m\}$ different of i such that $\Sigma_i \cap \Sigma_j \neq \emptyset$. Because Σ is a Dehn surface, Σ_i and Σ_j share at least one double curve α of Σ, and since Σ fills M, α must contain at least one triple point P of Σ.

At the triple point P the two Dehn spheres Σ_i, Σ_j intersect with other Dehn sphere Σ_k (which could coincide with Σ_i or Σ_j) of the collection Σ as in Figure 3(a). The arc $\bar{\delta}$ of Figure 3(a) is a non-simple type 1 arc of Σ and so it is filling-preserving by Proposition 5. Then, by piping Σ along $\bar{\delta}$, we obtain again a filling Dehn surface Σ' which is still a filling collection of spheres. The filling Dehn surface Σ' is the union of $m - 1$ Dehn spheres because after the piping along $\bar{\delta}$ the two Dehn spheres Σ_i and Σ_j become a unique Dehn sphere.

We have shown that if M have a filling collection of m spheres, with $m > 1$, then it has a filling collection of $m - 1$ spheres. Applying the same argument inductively we conclude at last that M has a filling Dehn sphere. □

A construction similar to that of the previous proof gives the following theorem:

Theorem 9. *If M has a filling Dehn sphere, then M has a genus g filling Dehn surface for all $g \geq 1$.*

Proof. Let Σ be a filling Dehn sphere of M. As we have indicated in the last paragraph of Section 2, the filling Dehn sphere Σ has at least 2 triple points. Let P be a triple point of Σ.

A small neighbourhood of P looks like Figure 3(b). The arc $\bar{\delta}'$ of Figure 3(b) is a non-simple type 1 arc in Σ, and so it is filling-preserving by Proposition 5. Because of the choice of the arc $\bar{\delta}'$ the Dehn surface Σ' obtained by piping Σ along $\bar{\delta}'$ is orientable. Then, Σ' is a genus 1 filling Dehn surface of M. Repeating this operation we can construct filling Dehn surfaces of M of arbitrary genus. □

Note that, in the previous proof, the arc $\bar{\delta}'$ of Figure 3(b) can be as small as we want. This implies that if the filling Dehn sphere Σ of M is nulhomotopic, then the filling Dehn surface Σ' obtained by piping Σ along $\bar{\delta}'$ is also nulhomotopic. Furthermore, the filling Dehn surfaces constructed repeating this operation will be again nulhomotopic, so we have:

COROLLARY 10. *If M has a nulhomotopic filling Dehn sphere, then M has a nulhomotopic genus g filling Dehn surface for all $g \geq 1$.*

In [M2] a filling pair of Dehn spheres is constructed for every manifold using a Heegaard splitting of the 3-manifold, and then a unique filling Dehn sphere is obtained by piping those two Dehn spheres. We will give here a different construction.

PROOF OF THEOREM 2.

Let M be a closed orientable 3-manifold.

As in [F-R] and [M2], we start with a Heegaard diagram of M. We use the Waldhausen notation [W] (see also [M2]).

Let (M, S) be a Heegaard splitting of M of genus greater than 0. Thus S is a closed orientable surface embedded in M of genus $g > 0$ that separates M out into two handlebodies V and W of genus g with $V \cap W = \partial V = \partial W = S$. Take $v = \{v_1, ..., v_g\}$ and $w = \{w_1, ..., w_g\}$ complete systems of meridian disks for V and W respectively. Thus, $(S, \partial v, \partial w)$ is a Heegaard diagram of M, where $\partial v = \{\partial v_1, ..., \partial v_g\}$ and $\partial w = \{\partial w_1, ..., \partial w_g\}$. As in [M2], we can assume that the systems of simple closed curves ∂v and ∂w cut transversely in S and that $\partial v \cup \partial w$ *fills* S, that is, $\partial v \cup \partial w$ defines a cell decomposition of S (see [M1], p. 118). If ∂v and ∂w cut transversely in S, then $\partial v \cup \partial w$ fills S if and only if $S - \partial v \cup \partial w$ is a disjoint union of open disks.

Step 1. *Constructing a Dehn sphere* Σ_V *from the surface* S *and the system of disks* v.

Let $V_1, ..., V_g$ be a set of closed 3-balls embedded in V, and let denote $\sigma_i := \partial V_i$ for $i = 1, ..., g$. Assume that: (i) the members of $V_1, ..., V_g$ are pairwise disjoint; (ii) $(v_i, \partial v_i) \subset (V_i, \sigma_i)$; and (iii) $V_i \cap S = \sigma_i \cap S = \partial v_i$ for $i = 1, ..., g$ (Figure 4(b)). The union $S \cup \sigma_1 \cup ... \cup \sigma_g$ is a non-transversely immersed 2-sphere in M. After a slight modification of $S \cup \sigma_1 \cup ... \cup \sigma_g$ near the ∂v_i's we obtain a transversely immersed 2-sphere Σ_V in M whose singularity set is precisely ∂v (Figure 4(c)).

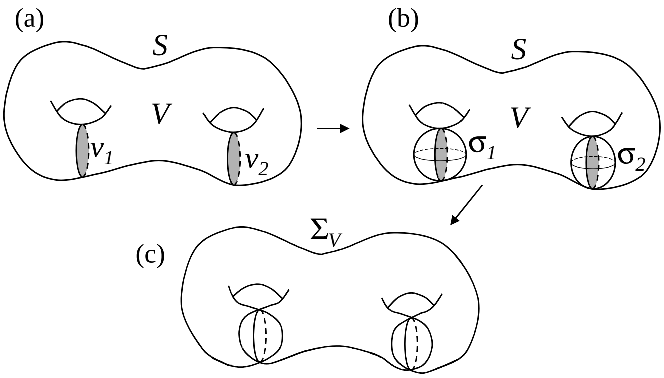

Fig. 4. Step 1.

Step 2. *Inflating the system of disks* w.

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ be the closed disk with center at the origin and radius 1, and $\mathbb{S}^1 = \partial\mathbb{D}^2$. For a small $\varepsilon > 0$ fixed, consider the set

$$\mathbb{B}^\varepsilon = \left\{\mathbf{x} \in \mathbb{R}^3 : d\left(\mathbf{x}, \mathbb{D}^2 \times \{0\}\right) \leq \varepsilon\right\},$$
$$\mathbb{S}^\varepsilon = \left\{\mathbf{x} \in \mathbb{R}^3 : d\left(\mathbf{x}, \mathbb{D}^2 \times \{0\}\right) = \varepsilon\right\},$$

where d denotes the euclidean distance in $\mathbb{R}^3$. Let $\mathbb{A}^\varepsilon$ be the closure of $\mathbb{B}^\varepsilon - \left(\mathbb{D}^2 \times [-\varepsilon, \varepsilon]\right)$. Then $\mathbb{B}^\varepsilon$ is the union of $\mathbb{A}^\varepsilon$ and $\mathbb{D}^2 \times [-\varepsilon, \varepsilon]$ and $\mathbb{A}^\varepsilon \cap \left(\mathbb{D}^2 \times [-\varepsilon, \varepsilon]\right) = \mathbb{S}^1 \times [-\varepsilon, \varepsilon]$ (see Figure 5). Now, we take a collection

$$b_1, ..., b_g : \mathbb{B}^\varepsilon \to M$$

of embeddings from $\mathbb{B}^\varepsilon$ into M *thickening* w. That is:

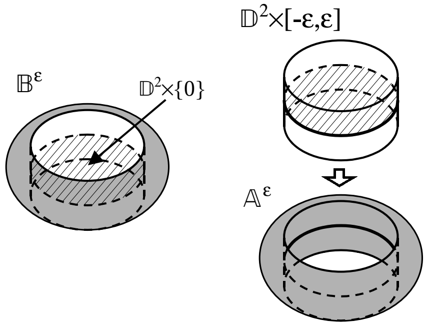

Fig. 5. The flattened 3-ball $\mathbb{B}^{\varepsilon}$.

(i) the images $B_1, ..., B_g$ of the maps $b_1, ..., b_g$ are pairwise disjoint;

(ii) $b_j\left(\mathbb{D}^2 \times \{0\}\right) = w_j$, $b_j\left(\mathbb{D}^2 \times [-\varepsilon, \varepsilon]\right) = B_j \cap W$ and $b_j\left(\mathbb{A}^{\varepsilon}\right) = B_j \cap V$, for all $j \in \{1, ..., g\}$

We can imagine each B_j as a car wheel with the *rim* $(b_j\left(\mathbb{D}^2 \times [-\varepsilon, \varepsilon]\right))$ inside W and the *tyre* $(b_j\left(\mathbb{A}^{\varepsilon}\right))$ inside V (Figure 6). Put $\Sigma_j := b_j\left(\mathbb{S}^{\varepsilon}\right) = \partial B_j$ for $j = 1, ..., g$. For each $j = 1, ..., g$, we choose b_j in such a way that Σ_V and Σ_j intersect transversely and Σ_j is "as close to w_j as necessary" to avoid extra intersections between B_j and Σ_V.

In this way, if ∂w_j has n intersection points with the system of curves ∂v, B_j is divided by Σ_V into $2n+1$ different 3-balls (Figure 7(c)): one for $B_j \cap W$, one for each intersection point of ∂w_j with ∂v, corresponding to the intersection of B_j with the V_i's, and another one for each segment into which ∂w_j is divided by ∂v, corresponding to the components of $(B_j \cap V) - (V_1 \cup ... \cup V_g)$.

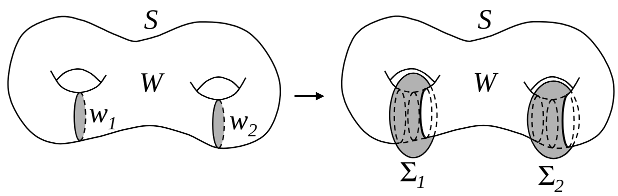

Fig. 6. Inflating w.

Figure 7 depicts what happens near one of the disks of w during Steps 1 and 2.

Now, $\Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g$ is a filling collection of spheres in M because, by construction, the complementary set of $\Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g$ in M is a union of disjoint open 3-balls; the set

$$(\Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g) - \{\text{double curves of } \Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g\}$$

is a union of disjoint open disks because $\partial v \cup \partial w$ fills S; and the set

$$\{\text{double curves of } \Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g\} \\ - \{\text{triple points of } \Sigma_V \cup \Sigma_1 \cup ... \cup \Sigma_g\}$$

is a union of disjoint open intervals.

Then, in view of Theorem 8 we can construct a filling Dehn sphere Σ of M piping Σ_V with $\Sigma_1, ..., \Sigma_g$.

For all $j \in \{1, ..., g\}$ we can shrink the 2-sphere Σ_j to a point inside the 3-ball B_j. Using these deformations of $\Sigma_1, ..., \Sigma_g$ the filling Dehn sphere Σ constructed can be continuously deformed into Σ_V. Finally, Σ_V itself can be (continuously) deformed inside V into a 2-sphere Σ_0 standardly embedded in V (Σ_0 bounds a 3-ball in V). Therefore, Σ is nulhomotopic.

This ends the proof of Theorem 2. □

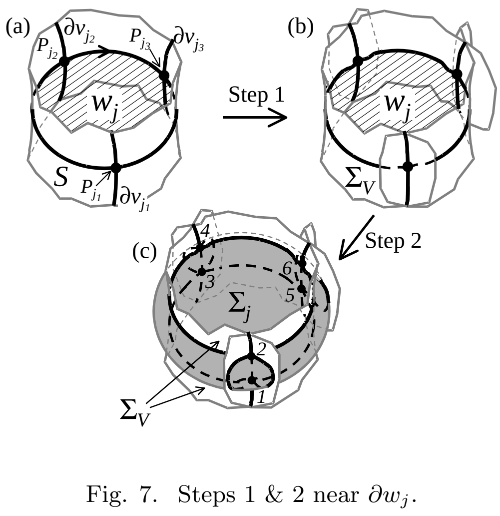

Fig. 7. Steps 1 & 2 near ∂w_j.

In fact, the 2-spheres $\Sigma_1, ..., \Sigma_g$ of the previous proof satisfy the hypothesis of Proposition 6. Then we can construct Σ using only simple pipings, as will be done in Section 5. In Section 5 a concrete detailed procedure for constructing Σ will provide us with an algorithm to obtaining the Johansson diagram of Σ from the Heegaard diagram $(S, \partial v, \partial w)$.

4. Singular and Singularity Sets

To create the algorithm referred to above (giving Σ and a Johansson diagram from a Heegaard diagram of M in an automatic way) we will make a more detailed study of the construction made in the Proof of Theorem 2. We keep notation.

Let S^2_{2g} be a compact $2g$-holed 2-sphere, and let $\pi : S^2_{2g} \to S$ be a surjective immersion such that (i) the image by π of the boundary ∂S^2_{2g} of S^2_{2g} is the system of curves ∂v and (ii) the restriction of π to the interior of S^2_{2g} is an homeomorphism onto $S - \partial v$. We can always obtain such π. With these assumptions, the restriction of π to ∂S^2_{2g} must be a non-connected 2:1 covering onto ∂v, and we say that S^2_{2g} is obtained *cutting S along ∂v*. Two points or two connected components of ∂S^2_{2g} are *related* if they have the same image by π. One standard method of representing the Heegaard diagram $(S, \partial v, \partial w)$ is taking S^2_{2g} with information about how its boundary points are related, and drawing over it $\pi^{-1}(\partial w)$. We will always represent Heegaard diagrams in this way in the examples of Section 7. Because $(S, \partial v, \partial w)$ is a filling Heegaard diagram, each curve of ∂w intersects a curve of ∂v, and so ∂v divides ∂w into a collection of arcs with endpoints in ∂v. Two such arcs are *consecutive* if they share an endpoint. The inverse image by π of ∂w is then a collection of disjoint arcs with their endpoints in ∂S^2_{2g} (see Figures 14(a), 16(a), and 18(a)). We also say that two arcs of $\pi^{-1}(\partial w)$ are *consecutive* if their images by π are consecutive. For two consecutive arcs λ, λ' of $\pi^{-1}(\partial w)$ there is an endpoint of λ which is related with an endpoint of λ'. Therefore, the set of endpoints of $\pi^{-1}(\partial w)$ is composed by a union of pairs of related points of ∂S^2_{2g}. If $\#\{\partial w_j \cap \partial v\} = 1$ then $\pi^{-1}(\partial w_j)$ is one single arc which is consecutive to itself (see Figure 14(a)).

Attaching a closed disk to each boundary component of S^2_{2g}, we obtain a 2-sphere S^2. In this way, the $2g$-holed 2-sphere S^2_{2g} is now a subset of the 2-sphere S^2, and the closure of $S^2 - S^2_{2g}$ is a disjoint union of $2g$ closed disks.

The $2g$ simple closed curves which compose ∂S^2_{2g} in S^2 (with the information about how their points are related by π) can be interpreted as the Johansson diagram of a transverse immersion $f_V : S^2 \to M$ such that $f_V\left(S^2\right) = \Sigma_V$ and that coincides with π when restricted to ∂S^2_{2g}. The collection of arcs in S^2_{2g} which corresponds to $\pi^{-1}(\partial w)$ can be seen now as $f_V^{-1}(\partial w)$.

Take $j \in \{1, ..., g\}$ fixed.

Assume that $\#\{\partial w_j \cap \partial v\} = n$, with $n \geq 1$. Then, after Steps 1 and 2 of the Proof of Theorem 2, $\Sigma_V \cap \Sigma_j$ is composed by n double curves and $2n$ triple points (see Figure 7). Take the embedding $b_j : \mathbb{B}^\varepsilon \to M$ that inflates w_j, and the image set B_j of b_j. Let f_j be the restriction of b_j to $\mathbb{S}^\varepsilon$, and let $f_V + f_j : S^2 \coprod \mathbb{S}^\varepsilon \to M$ be the immersion from the disjoint union of S^2 and $\mathbb{S}^\varepsilon$ that coincides with f_V and f_j when restricted to S^2 and $\mathbb{S}^\varepsilon$ respectively. It is clear that $f_V + f_j$ parametrizes the Dehn surface $\Sigma_V \cup \Sigma_j$. The singularity set of $\Sigma_V \cup \Sigma_j$ is

$$\partial v \cup (\Sigma_V \cap \Sigma_j)\,,$$

The inverse image by f_V of B_j is a collection of n disjoint closed disks, each one with an arc of $f_V^{-1}(\partial w_j)$ in its interior (see Figure 8). The boundary of the union of these closed disks is the inverse image by f_V of the double curves of $\Sigma_V \cap \Sigma_j$, it coincides with $f_V^{-1}(\Sigma_j)$, and it is a collection of n disjoint simple closed curves surrounding the arcs of $f_V^{-1}(\partial w_j)$. We will call *surrounding curves* to the curves of $f_V^{-1}(\Sigma_j)$, and we define that surrounding curves that surround consecutive arcs of $f_V^{-1}(\partial w_j)$ are also *consecutive*.

The inverse image by f_j of the double curves of $\Sigma_V \cap \Sigma_j$ (coincides with $f_j^{-1}(\Sigma_V)$ and) is a collection of n closed curves (which are simple if $n > 1$) arranged in a *necklace* in $\mathbb{S}^\varepsilon$ as in Figure 8 (see also Figure 7). Each of these curves will be called a *necklace curve* and it is the sister curve under $f_V + f_j$ of one of the surrounding curves of $f_V^{-1}(\partial w_j)$ in S^2. Two necklace curves in $\mathbb{S}^\varepsilon$ are *consecutive* if their sister curves in S^2 are consecutive surrounding curves, and this occurs if and only if they intersect transversely in two (or four if $n = 2$) points (Figure 8).

The singular set of $f_V + f_j$ is the union of the inverse images by f_V and

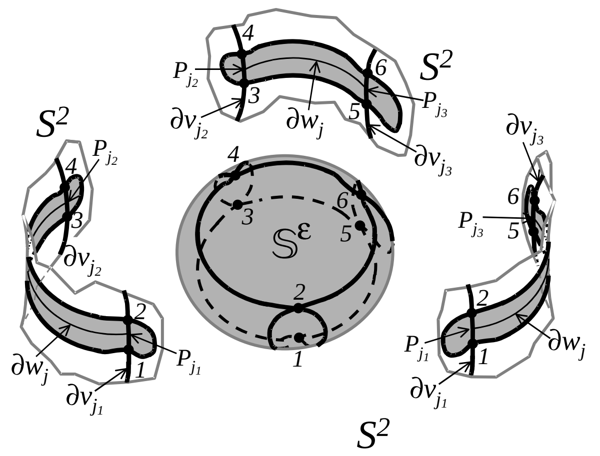

Fig. 8. Inverse image of Figure 7(c).

f_j of the singularity set of $\Sigma_V \cup \Sigma_j$, that is

$$\partial S^2_{2g} \cup \{\text{surrounding curves of } \partial w_j\} \subset S^2 \text{ and}$$
$$\{\text{necklace curves}\} \subset \mathbb{S}^\varepsilon.$$

Each triple point of $\Sigma_V \cup \Sigma_j$ has one preimage point in $\mathbb{S}^\varepsilon$ by f_j which is an intersecting point of two necklace curves, and two preimage points in S^2 by f_V lying in consecutive surrounding curves (Figures 7(c) and 8). So, each double point of the singular set of $f_V + f_j$ that belongs to a surrounding curve β has a related double point in a surrounding curve β' consecutive with β and another related double point in $\mathbb{S}^\varepsilon$ which is an intersection point of the necklace curves $\tau\beta, \tau\beta'$ sisters of β, β' under $f_V + f_j$ respectively (Figure 8).

5. Piping

Finally, we select an arc along which we perform the piping between Σ_V and Σ_j. We refer to Figures 9 and 10.

Choose in S^2 as *piping point* an intersecting point K of ∂S^2_{2g} with one of the surrounding curves α (Figure 9), and take $\bar{K} := f_V(K)$ (Figure 10(a)).

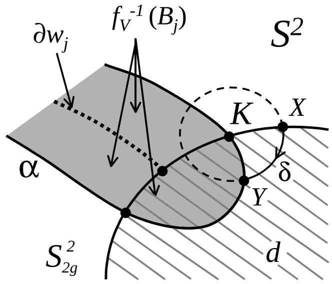

Fig. 9. The piping point.

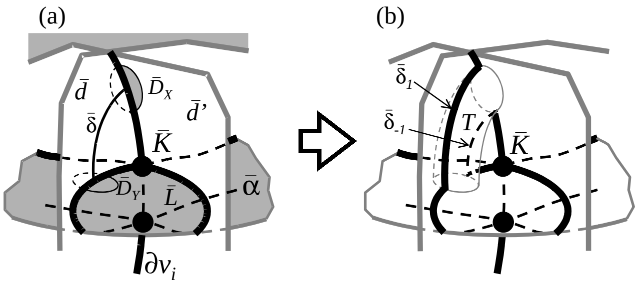

Fig. 10. Piping Σ_V with Σ_j along $\bar{\delta}$.

A small circle around K is divided by ∂S^2_{2g} and α into four arcs, and exactly one of them, say δ, is not contained in $S^2_{2g} \cup f_V^{-1}(B_j)$ (Figure 9).

Let $\Sigma_V \# \Sigma_j$ be the Dehn sphere obtained after piping $\Sigma_V \cup \Sigma_j$ along the type 1 arc $\bar{\delta} := f(\delta)$ (Figure 10).

The singularity set of $\Sigma_V \# \Sigma_j$ will coincide with the singularity set of $\Sigma_V \cup \Sigma_j$ except in a small neighbourhood of the arc $\bar{\delta}$. We will study now what happens in this small neighbourhood of $\bar{\delta}$. First, we need to give names to some of the elements involved in our present situation.

Notation 11.

- $\tau\alpha \subset \mathbb{S}^\varepsilon$ of Figure 11(e) is the necklace curve which is sister under $f_V + f_j$ of the surrounding curve α of Figure 9, and $\bar{\alpha} := f_V(\alpha) = f_j(\tau\alpha)$ of Figure 10(a) is their common image curve in M.

- $\bar{K}$ belongs to ∂v_i (Figure 10(a)) for some $i \in \{1, ..., g\}$. Let d, d' be the two closed disks attached to S^2_{2g} along $\pi^{-1}(\partial v_i)$ and assume that K belongs to ∂d (Figures 9, 11(b) and 11(c)). Let $\bar{d}, \bar{d}' \subset \Sigma_V$ be the images by f_V of d, d' respectively (Figure 10(a)).

- $X \in \partial d, Y \in \alpha$ are the endpoints of δ (Figure 9 and 11(b)), and $X' \in \partial d'$ (Figure 11(c)) and $Y' \in \tau\alpha$ (Figure 11(e)) are the two related points by $f_V + f_j$ of X, Y respectively.

- The necklace curves divide the 2-sphere $\mathbb{S}^\varepsilon$ into two n-gons, n 4-gons and n 2-gons (see Figure 8). Let L be the 2-gon having K and Y' on its boundary (Figure 11(e)) and let $\bar{L} = f_j(L)$ (Figure 10(a)).

To construct $\Sigma_V \# \Sigma_j$ we take in $S^2 \coprod \mathbb{S}^\varepsilon$ two small closed disks D_X, D_Y with X' and Y' in their respective interiors and attach to $f_V(S^2 - int(D_X)) \cup f_j(\mathbb{S}^\varepsilon - int(D_Y))$ a "tube" T parallel to $\bar{\delta}$ along $\partial\bar{D}_X$ and $\partial\bar{D}_Y$, where $\bar{D}_X = f_V(D_X), \bar{D}_Y = f_j(D_Y)$, as in Figure 10. The intersection of T with $\Sigma_V \cup \Sigma_j$ is the union of $\partial\bar{D}_X, \partial\bar{D}_Y$ and two arcs $\bar{\delta}_{-1}, \bar{\delta}_1$ parallel to $\bar{\delta}$ (Figure 10(b)). The disks $\bar{D}_X$ and $\bar{D}_Y$ intersect the singularity set of $\Sigma_V \cup \Sigma_j$ in two small arcs $\bar{a}_X \subset \partial v$ and $\bar{a}_Y \subset \bar{\alpha}$ respectively. The singular-

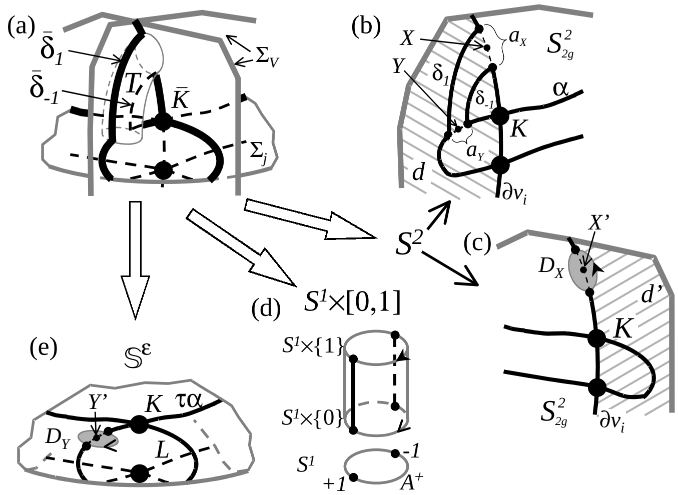

Fig. 11. Inverse image for Figure 10(b).

ity set of $\Sigma_V \# \Sigma_j$ is the result of substituting $\bar{\delta}_{-1} \cup \bar{\delta}_1$ for $\bar{a}_X \cup \bar{a}_Y$ in the singularity set of $\Sigma_V \cup \Sigma_j$.

Let $S^2 \# \mathbb{S}^\varepsilon$ be the abstract 2-sphere obtained from the disjoint union of $S^2 - int\,(D_X)$ and the *necklace disk* $\mathbb{S}^\varepsilon - int\,(D_Y)$ with an annulus $S^1 \times I$, where $I = [0,1]$, identifying the "top edge" $S^1 \times \{1\}$ with ∂D_X and the "bottom edge" $S^1 \times \{0\}$ with ∂D_Y (Figure 11). If the identifications are coherent with the piping, we can see $\Sigma_V \# \Sigma_j$ as the image of an immersion $F : S^2 \# \mathbb{S}^\varepsilon \to M$ which coincides with $f_V + f_j$ when restricted to $\left(S^2 - int\,(D_X)\right) \coprod \left(\mathbb{S}^\varepsilon - int\,(D_Y)\right)$ respectively, and such that $T = F\left(S^1 \times I\right)$. Taking S^1 as the complex numbers of norm equal to 1, we can assume that $F\left(\{-1\} \times I\right) = \bar{\delta}_{-1}$ and $F\left(\{1\} \times I\right) = \bar{\delta}_1$. If $A^+ \subset S^1$ is the half-circle with $\operatorname{Re} z \geq 0$ we can see in Figure 11 that if, for example, $A^+ \times \{0\}$ is identified with $\partial D_Y \cap L$, then $A^+ \times \{1\}$ must be identified with $\partial D_X \cap d'$. So assume that this is the case.

If $\delta_{-1}, \delta_1 \subset d, a_X \subset \partial d, a_Y \subset \alpha$ are the arcs such that

$$f_V\left(\delta_{-1}\right) = \bar{\delta}_{-1},\ f_V\left(\delta_1\right) = \bar{\delta}_1,\ f_V\left(a_X\right) = \bar{a}_X,\ f_V\left(a_Y\right) = \bar{a}_Y$$

(Figure 11(b)), then the singular set $S(F)$ is composed by (i) the part of the singular set of $f_V + f_j$ lying in $\left(S^2 - int\,(D_X)\right) \coprod \left(\mathbb{S}^\varepsilon - int\,(D_Y)\right)$, removing $a_X \cup a_Y$ and attaching $\delta_{-1} \cup \delta_1$; and (ii) $\{-1, 1\} \times I \subset S^1 \times I$ (see Figure 11).

In general, for an arc δ connecting two curves α, β on a surface as in Figure 12(a), to *pipe α and β along δ* is to replace two small arcs of α and β respectively containing the endpoints of δ with two arcs parallel to δ (Figure 12(b)).

Remark 12.

1. The configuration of the necklace curves in the necklace disk *only depends on the number $n = \#\{\partial w_j \cap \partial v\}$*. If the arc δ is chosen as

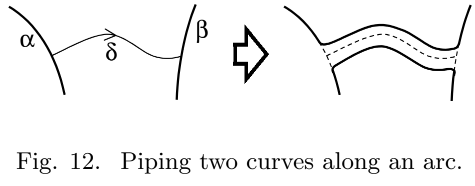

Fig. 12. Piping two curves along an arc.

it has been indicated, then the necklace disk $\mathbb{S}^\varepsilon - int\,(D_Y)$ does not depend on the piping point neither on other elements of the Heegaard diagram. (In Figure 13 we can see how looks like the *necklace disk* $\mathbb{S}^\varepsilon - int\,(D_Y)$ for different values of n.)

2. We obtain an equivalent singular set for F by shrinking $S^1 \times I$ to a circle projecting onto the first factor.

Then, the singular set of F can be obtained identifying in the union of $S^2 - int\,(D_X)$ and $\mathbb{S}^\varepsilon - int\,(D_Y)$ the boundaries ∂D_X and ∂D_Y in such a way that $\partial D_X \cap d'$ is identified with $\partial D_Y \cap L$. Due to the symmetry of the necklace disks (Figure 13), this condition is sufficient for doing the identification without any ambiguity. We will denote by C_n the necklace disk for each $n \in \mathbb{N}$ (see Figure 13).

As we have defined in Section 1, the Johansson diagram of $\Sigma_V \# \Sigma$ is given by $S(F)$ plus the information about how their points are related by F. The points of ∂S^2_{2g} which have not been affected by the piping (that is, outside D_X and a_X) are related by F in the same way as they were related by f_V, so we only need to know how to get the identification information for the surrounding curves and the curves of the necklace disk. This remaining

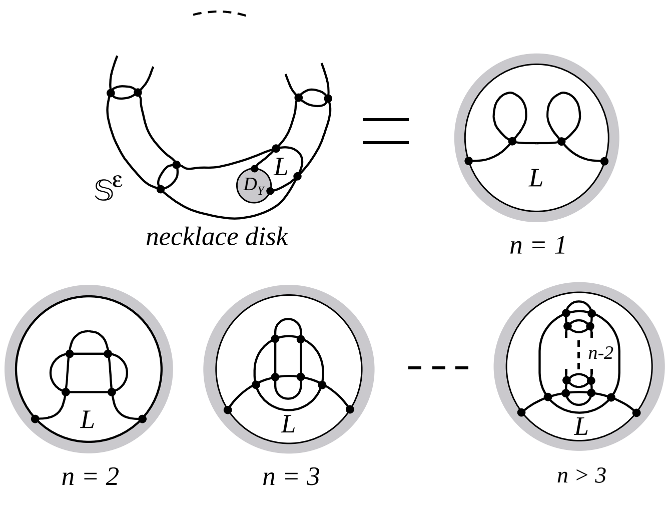

Fig. 13. The necklace disk C_n for different values of n.

identification information can be deduced inductively following the obvious rule that sister curves of a Johansson diagram must pass through related double points of the diagram. The necklace disk C_n contains an arc (*necklace arc*, which corresponds to $\tau\alpha - D_Y$ and has self-intersections if $n = 1$), and $n - 1$ necklace curves (Figure 13). The curve γ of the diagram obtained by piping ∂d and α along δ must be sister under F of the curve composed by the union of $\partial d' - D_X$ and the necklace arc. This allows us to determine the related points by F of the double points of the necklace arc. Remember that intersecting necklace curves are sisters of consecutive surrounding curves, so looking for the consecutive surrounding curves of α, we can determine the related points by F of the double points of the necklace curves that intersect with the necklace arc. Continuing in this way we can obtain all the identification information for the Johansson diagram of F. This will be clarified in the examples at the end of the paper.

Piping in the same way Σ_V with Σ_j for all $j \in \{1, ..., g\}$, a filling Dehn sphere Σ of M is constructed. The modifications made in the Heegaard diagram $(S, \partial v, \partial w)$ by which we obtained the Johansson diagram of $\Sigma_V \# \Sigma_j$ took place in a neighbourhood of ∂w_j which can be as small as we want. So we can perform the same modifications for all $j \in \{1, ..., g\}$ independently. The diagram so obtained is the Johansson diagram of the filling Dehn sphere Σ.

6. The Algorithm

Essentially, the modifications that we make in Section 5 in the Heegaard diagram $(S, \partial v, \partial w)$ of M to obtain a Johansson representation of M can be resumed in three steps. Remember that we assume that the Heegaard diagram $(S, \partial v, \partial w)$ fills S as it has been defined in Section 3. If we are given a non-filling Heegaard diagram, first of all we must modify it as in steps 3 and 4 of [M1], p. 119, to obtain a filling one.

F1. *Pass from S to S^2*: cut S along ∂v to obtain S^2_{2g} and *fill the holes* (attach disks to ∂S^2_{2g}) to obtain S^2.

We keep the *identification information* of ∂S^2_{2g}, that is, the information about how the curves of ∂S^2_{2g} are identified in pairs for reconstructing S (how their points are *related*). The system (of g simple closed curves in S) ∂w becomes a system of disjoint arcs in $S^2_{2g} \subset S^2$

whose set of boundary points is composed by pairs of related points of ∂S^2_{2g}.

F2. *Surround the arcs of ∂w in S^2.* Take a collection of disjoint simple closed curves in S^2 satisfying: (i) each one bounds a disk with exactly one arc of ∂w in its interior; (ii) each one has 4 intersecting points with ∂S^2_{2g}; and (iii) the set of intersecting points of the surrounding curves with ∂S^2_{2g} must be also composed by pairs of related points of the Heegaard diagram.

F3. *Piping.* For each $\partial w_j \in \partial w$, choose as *piping point* a point K of intersection of one of the surrounding curves α of ∂w_j with ∂S^2_{2g}. Take a *piping arc* δ near K as indicated in Figure 9. If X is the endpoint of δ which lies in ∂S^2_{2g} and $X' \in \partial S^2_{2g}$ is the related point of X in the Heegaard diagram $(S, \partial v, \partial w)$, take a small *piping disk* $D_X \subset S^2$ with X' in its interior as in Figure 11(c). Let d' be the "attached disk" to S^2_{2g} of step F1 such that $X' \in \partial d'$. Pipe ∂S^2_{2g} with α along δ (Figure 12) and replace the piping disk with the necklace disk C_{n_j} of Figure 13 for $n_j = \# \{\partial w_j \cap \partial v\}$ in such a way that $\partial C_{n_j} \cap L$ coincides with $\partial D_X \cap d'$.

The diagram in S^2 so constructed is the singular set of an immersion $f : S^2 \to M$ whose image is a filling Dehn sphere of M. The related points of the Heegaard diagram which have not been modified in step F3 are also related by f. From this information we can deduce how are related by f the rest of the points of the diagram and so we obtain the Johansson diagram of a filling Dehn sphere of M.

7. Examples

We will apply the algorithm to some examples.

7.1. $M = S^3$

Let start with the standard genus 1 Heegaard diagram $(S, \partial v, \partial w)$ of the 3-sphere S^3. This diagram is a filling Heegaard diagram, so we can apply the algorithm to it.

Steps F1&F2. *From S to S^2. Surround ∂w.*

Cutting S along ∂v the 2-holed sphere of Figure 14(a) is obtained. For reconstructing S, the two connected components of ∂S^2_2 must be identified in such a way that the points labeled with the same name coincide and the orientations of the two components agree. After filling the two holes of S^2_2 with the disks d, d' and surrounding the arc corresponding to ∂w_1 by the curve α, we obtain the picture of Figure 14(b).

Step F3. *Piping:* we choose the piping point as indicated in Figure 14(b). The corresponding piping arc δ, the endpoints X, Y of δ, the related point X' of X and the piping disk D_X around X' appear also in Figure 14(b). Now, we must pipe ∂S^2_2 with α along δ and replace the piping disk with the necklace disk C_n of Figure 13 for $n = \#\{\partial w_1 \cap \partial v\} = 1$ in such a way that $\partial C_1 \cap L$ coincides with $\partial D_X \cap d'$.

We obtain the diagram of Figure 15(a), where there are only two curves $\gamma, \tau\gamma$ which must be sisters. Following the rule that sister curves must pass through related double points of the diagram, it is easily checked that the unique coherent way of labeling the double points inside the necklace disk is as it is indicated on this figure.

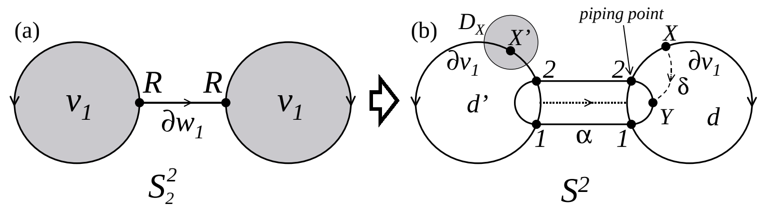

Fig. 14.

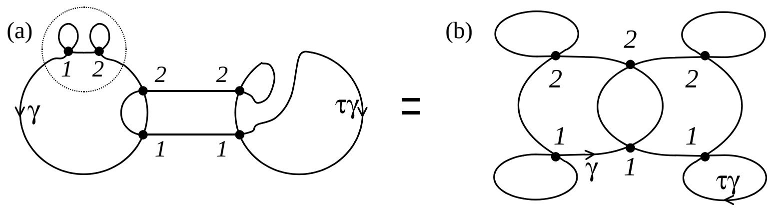

Fig. 15. Johansson representation of S^3.

The Johansson representation of S^3 we have found (Figure 15(a)) has only two triple points. It coincides with A. Shima's Example 1.5 of [S2] and turns out to be equivalent to a diagram (Figure 15(b)) given as an example in [J]. (So, this historically important Johansson diagram is one of the simplest!)

7.2. $M = \mathbb{R}P^3$

The genus one Heegaard diagram $(S, \partial v, \partial w)$ of the real projective space $\mathbb{R}P^3$ depicted in Figure 16(a) is also a filling Heegaard diagram, so we can apply our construction to it.

Steps F1&F2. *From S to S^2. Surround ∂w.*

Cutting S along ∂v the 2-holed sphere of Figure 16(a) is obtained. After filling the two holes of S^2_2 with the disks d, d' and surrounding the two arcs of ∂w_1 by the curves α_1, α_2, we obtain the picture of Figure 16(b), where related points of the surrounding curves are equally labeled.

Step F3. *Piping:* we choose the piping point as indicated in Figure 16(b). The corresponding piping arc δ, the endpoints X, Y of δ, the related point X' of X and the piping disk D_X around X' appear also in Figure 16(b). Now, we pipe ∂S^2_2 with α_2 along δ and we replace the piping disk with the necklace disk C_n of Figure 13 for $n = \#\{\partial w_1 \cap \partial v\} = 2$ in such a way that $\partial C_2 \cap L$ coincides with $\partial D_X \cap d'$.

We obtain the diagram of Figure 17. Following the rule that sister curves must pass thorough related double points of the diagram, it is easily checked that the unique coherent way of labeling the double points inside the necklace disk is as it is indicated on the figure.

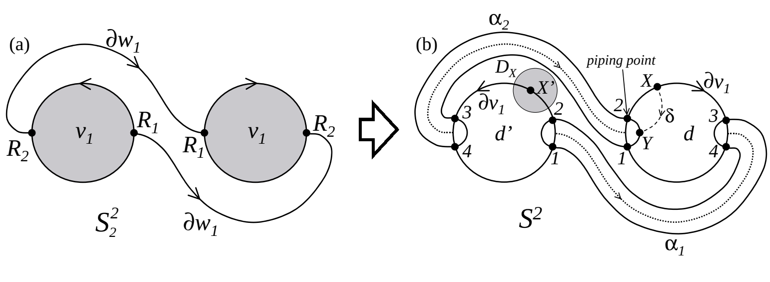

Fig. 16.

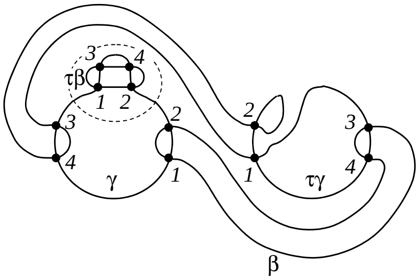

Fig. 17. Johansson representation of $\mathbb{R}P^3$.

7.3. Poincaré Homology 3-sphere

The classical genus two Heegaard diagram $(S, \partial v, \partial w)$ of the Poincaré homology sphere appeared in [Po] is depicted in Figure 18(a). This is again a filling Heegaard diagram, so we can apply our algorithm to it.

Steps F1&F2. *From* S *to* S^2. *Surround* ∂w.

Cutting S along ∂v the 4-holed sphere of Figure 18(a) is obtained. Here, for reconstructing S, the connected components $+A$ and $-A$ of ∂S_4^2 must be identified in such a way that points equally labeled coincide and the same must be done with $+B$ and $-B$. The dashed and non-dashed arcs represents the two different curves of ∂w respectively. Assume that the non-dashed arcs correspond to ∂w_1 and the dashed arcs correspond to ∂w_2. After filling the four holes of S_4^2 with the disks d_A, d'_A, d_B, d'_B and surrounding the arcs of ∂w, we obtain the picture of Figure 18(b), where we have labeled the points of the surrounding curves in such a way that related points are equally labeled.

Step F3. *Piping:* we choose the two piping points for ∂w_1 and ∂w_2 as indicated in Figure 18(b). The respective piping arcs δ_1 and δ_2, and piping disks D_1 and D_2 appear in the same figure. Now, we have to pipe the corresponding curves of the diagram along δ_1 and δ_2, and replace the piping disks with their corresponding necklace disks as indicated in F3 of Section 6.

Using that $n_1 = \#\{\partial w_1 \cap \partial v\} = 7$ and $n_2 = \#\{\partial w_1 \cap \partial v\} = 5$, we obtain the Johansson diagram of Figure 19, where the unique coherent way of labeling the double points inside the necklace disks is as it has been

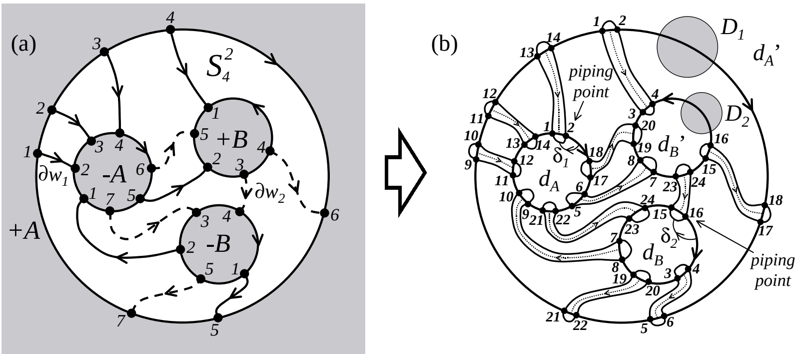

Fig. 18.

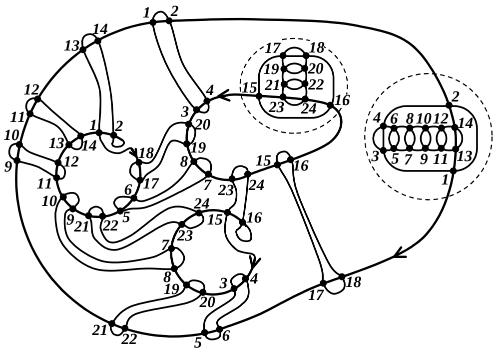

Fig. 19. Poincaré Homology 3-sphere.

indicated on the figure. The filling Dehn sphere that it represents has 24 triple points and 12 double curves.

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