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On the Stability of Homogeneous Vector Bundles

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Abstract. Let G be a connected semisimple linear algebraic group over an algebraically closed field k and $P \subset G$ a parabolic subgroup without any simple factor. Let V be an irreducible P–module and $E_P(V) = (G \times V)/P$ the associated vector bundle over G/P. We prove that $E_P(V)$ is stable with respect to any polarization on G/P. In [Um] this was proved under the assumption that the characteristic of k is zero and the question was asked whether it remains valid when the characteristic is positive.

1. Introduction

We begin by recalling a result due to Hiroshi Umemura.

Let k be an algebraically closed field. Let G be a connected semisimple linear algebraic group over k and $P \subset G$ a reduced proper parabolic subgroup without any simple factor. The projection $G \longrightarrow G/P$ defines a principal P–bundle over G/P which will be denoted by E_P. Let $\rho : P \longrightarrow \mathrm{GL}(V)$ be an irreducible P–module and

$$\mathbb{V} := E_P(V) = \frac{G \times V}{P}$$

the vector bundle associated to the P–bundle E_P for the P–module V.

The main theorem of [Um] says that the vector bundle $\mathbb{V}$ is stable with respect to any polarization on G/P provided the characteristic of k is zero [Um, page 136, Theorem 2.4].

Umemura asks the question whether the theorem remains valid if the characteristic of k is positive (see the end of the introduction in [Um, page 131]).

The aim here is to show that the above question has an affirmative answer. We prove that the vector bundle $\mathbb{V}$ is stable for any algebraically closed field k (Theorem 2.1).

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2. Homogeneous Bundles

Let k an algebraically closed field and G a connected semisimple linear algebraic group over k. Let

$$P \subsetneq G$$

be a (reduced) proper parabolic subgroup. We will further assume that P does not contain any simple factor. This means that the intersection of $P/Z(G)$ with any simple factor $G' \subset G/Z(G)$, where $Z(G)$ is the center of G, is a proper parabolic subgroup of G'.

Let

$$R_u(P) \subset P$$

be the *unipotent radical.* So $R_u(P)$ is a connected normal unipotent subgroup and the quotient $P/R_u(P)$, which is called the *Levi quotient*, is reductive (see [Hu, page 125]).

Let

$$\rho : P \longrightarrow \mathrm{GL}(V)$$

be a finite dimensional irreducible left P–module. Therefore,

$$\rho(R_u(P)) = e\,. \tag{1}$$

Indeed, if $\rho(R_u(P))$ is a nontrivial unipotent subgroup, then $\rho(R_u(P))$ preserves a nontrivial filtration of V. Since $R_u(P)$ is a normal subgroup of P, this filtration is also preserved by $\rho(P)$. But this contradicts the assumption that ρ is irreducible. Therefore, (1) is valid.

As before, let

$$E_P(V) = \frac{G \times V}{P} \tag{2}$$

be the vector bundle over G/P associated to the principal P–bundle E_P, defined by the projection $G \longrightarrow G/P$, for the P–module V. We recall that the action of any $h \in P$ on $G \times V$ sends any $(g\,, v) \in G \times V$ to $(gh\,, \rho(h^{-1})v)$.

For notational convenience the vector bundle $E_P(V)$ defined in (2) will also be denoted by $\mathbb{V}$.

Fix an ample line bundle ξ on G/P. Note that the left–translation action of G on G/P induces the trivial action of G on $\mathrm{Pic}(G/P)$. The *degree* of a coherent sheaf F on G/P is defined to be the cycle class

$$\mathrm{degree}(F) := [c_1(F)c_1(\xi)^{\dim G/P - 1}] \in \mathbb{Z}\,.$$

We recall that a torsionfree coherent sheaf E over G/P is called *stable* (respectively, *semistable*) if for any nonzero coherent proper subsheaf $F \subsetneq E$, with E/F torsionfree, the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)}$$

(respectively, $\text{degree}(F)/\text{rank}(F) \leq \text{degree}(E)/\text{rank}(E)$) is valid. Also, E is called *polystable* if it is a direct sum of stable sheaves of same degree/rank quotient (see [Ko, Ch. V, §7]).

THEOREM 2.1. *The vector bundle* $\mathbb{V} := E_P(V)$ *(defined in* (2)*) is stable with respect to any polarization on* G/P*.*

PROOF. We will first prove that $\mathbb{V} := E_P(V)$ is semistable.

Let $F \subset \mathbb{V}$ be the maximal semistable subsheaf. So F is the first nonzero term in the Harder–Narasimhan filtration of $\mathbb{V}$ (see [Ko, page 174, Theorem 7.15]).

The left–translation action of G on G/P lifts to an action of G on $\mathbb{V}$. More precisely, construct the diagonal action of G on $G \times V$ using the left–translation action of G on itself and the trivial action of G on V. This action of G on $G \times V$ clearly commutes with the action of P on $G \times V$ in (2). Therefore, we get an action of G on the quotient $\mathbb{V} := E_P(V) = (G \times V)/P$.

Since the Picard group $\text{Pic}(G/P)$ is discrete and G is connected, the induced action of G on $\text{Pic}(G/P)$ is the trivial action. Since the polarization ξ is left invariant by the action of G on G/P, the uniqueness of the Harder–Narasimhan filtration implies that the subsheaf F is left invariant by the action of G on $\mathbb{V}$. As the action of G on G/P is transitive, this immediately implies that F is a subbundle of the vector bundle $\mathbb{V}$ and the fiber

$$F_{eP} \subset \mathbb{V}_{eP} = E_P(V)_{eP}$$

over the point $eP \in G/P$ is left invariant by the action of the isotropy subgroup $P \subset G$ for eP; here $e \in G$ is the identity element.

Note that the fiber $\mathbb{V}_{eP}$ considered above is canonically identified with V by sending any $v \in V$ to the equivalence class of $(e, v) \in G \times V$ (the equivalence relation is defined by the action of P). Furthermore, this identification $\mathbb{V}_{eP} := E_P(V)_{eP} = V$ takes the action of P on the P–module V to the action of the isotropy subgroup P (of eP) on the fiber $\mathbb{V}_{eP}$.

Since V is an irreducible P–module, we conclude that

$$F_{eP} = \mathbb{V}_{eP} := E_P(V)_{eP}\,.$$

Therefore, $F = \mathbb{V}_{eP}$, and hence the vector bundle $\mathbb{V}$ is semistable.

We will next show that the vector bundle $\mathbb{V}$ is polystable.

Let

$$S \subset \mathbb{V}$$

be the *socle* of the vector bundle $\mathbb{V}$. So S is generated by all polystable subsheaves W of $\mathbb{V}$ with

$$\frac{\text{degree}(W)}{\text{rank}(W)} = \frac{\text{degree}(\mathbb{V})}{\text{rank}(\mathbb{V})}$$

(see [MR, Section 2], [AB, Section 2]). We recall that S is the unique maximal polystable subsheaf of the semistable vector bundle $\mathbb{V}$ with same degree/rank quotient as that of $\mathbb{V}$, and furthermore, S is left invariant by any automorphism of the vector bundle $\mathbb{V}$ [MR, page 164, Lemma 2.2].

Therefore, repeating the above argument for the semistability of $\mathbb{V}$ we conclude that $S = \mathbb{V}$. Hence the vector bundle $\mathbb{V}$ is polystable. So $\mathbb{V}$ is a direct sum of stable vector bundles.

Therefore, to complete the proof of the theorem it suffices to show that

$$\dim H^0(G/P,\, \text{End}(\mathbb{V})) = 1\,. \tag{3}$$

Let

$$W \subseteq \text{End}(\mathbb{V}) \tag{4}$$

be the coherent subsheaf generated by the global sections of $\text{End}(\mathbb{V})$. So W is globally generated.

The action of G on $\text{End}(\mathbb{V})$ induced by the action of G on $\mathbb{V}$ evidently leaves invariant the subsheaf W in (4).

So W is associated to the P–bundle E_P for the left P–module

$$W_{eP} \subset \text{End}(\mathbb{V})_{eP} = \text{End}(V)\,.$$

We will now need a couple of results.

Lemma 2.2. *Let V' be a subquotient of the left P–module* End(V). *Then*

$$c_1(E_P(V')) \,=\, 0\,,$$

where $E_P(V') \,=\, (G \times V')/P$ is the vector bundle over G/P associated to the principal P–bundle E_P for the P–module V'.

Proof. To prove the lemma, let $Z \,\subset\, L(P)$ be the reduced center. Since V is an irreducible $L(P)$–module (recall that the unipotent radical $R_u(P)$ acts trivially on an irreducible P–module), the action of Z on V is as scalar multiplications (Schur's lemma). Hence Z acts trivially on End(V). Consequently, the action of Z on V' is trivial.

Note that the quotient $L(P)/Z$ is semisimple; in particular, $L(P)/Z$ does not admit any nontrivial character. Therefore, the line bundle $\bigwedge^{\rm top} E_P(V')$, which is associated to the P–bundle E_P for the left P–module $\bigwedge^{\rm top} V'$, is trivial. Indeed, since Z acts trivially on $\bigwedge^{\rm top} V'$, the P–module $\bigwedge^{\rm top} V'$ is defined by a character of $L(P)/Z$; hence $\bigwedge^{\rm top} V'$ is a trivial P–module. Therefore, we have $c_1(E_P(V')) \,=\, 0$. This completes the proof of the lemma. $\square$

Proposition 2.3. *Let E be a globally generated vector bundle of rank n over G/P with $c_1(E) \,=\, 0$. Then E is isomorphic to the trivial vector bundle of rank n.*

Proof. To prove the proposition, fix a closed point $x \,\in\, G/P$. Fix sections

$$s_i \,\in\, H^0(G/P,\, E)\,,$$

$1 \leq i \leq n$, such that $\{s_i(x)\}_{i=1}^n$ is a basis of the fiber E_x.

Let $\phi_i \,:\, \mathcal{O}_{G/P} \,\longrightarrow\, E$ be the homomorphism defined by the section s_i using the natural identification $\mathrm{Hom}_{\mathcal{O}_{G/P}}(\mathcal{O}_{G/P}\,, E) \,=\, H^0(G/P,\, E)$ Now consider the homomorphism of vector bundles

$$\phi \,:=\, \bigoplus_{i=1}^n \phi_i \,:\, \mathcal{O}_{G/P}^{\oplus n} \,\longrightarrow\, E\,.$$

Let

$$\wedge^n\phi \,\in\, H^0(G/P,\, \wedge^n E) \tag{5}$$

be the section defined by ϕ using the fact that $\bigwedge^n \mathcal{O}_{G/P}^{\oplus n} = \mathcal{O}_{G/P}$.

The homomorphism ϕ is an isomorphism over a Zariski open dense subset of G/P containing x. Therefore, $\bigwedge^n \phi$ in (5) is a nonzero section.

Since $c_1(\bigwedge^n E) = c_1(E) = 0$, the nonzero section $\bigwedge^n \phi$ must be nowhere zero. Indeed, the effective divisor over which $\bigwedge^n \phi$ vanishes defines $c_1(\bigwedge^n E)$.

As the section $\bigwedge^n \phi$ vanishes nowhere we conclude that the homomorphism ϕ is an isomorphism over G/P. This completes the proof of the proposition. □

Continuing with the proof of the theorem, Lemma 2.2 and Proposition 2.3 together imply that the vector bundle W in (4) is isomorphic to a trivial vector bundle.

Let

$$0 = U_0 \subset U_1 \subset \cdots \subset U_k := W_{eP} \tag{6}$$

be a filtration of left P–modules such that each subsequent quotient U_i/U_{i-1}, $i \in [1\,,k]$, is an irreducible P–module. So the filtration (6) defines a filtration of subbundles

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k := W\,, \tag{7}$$

where F_i is the vector bundle over G/P associated to the principal P–bundle E_P for the P–module U_i.

Note that, as before, $U_i = (F_i)_{eP}$ by sending any $u \in U_i$ to the equivalence class of $(e\,,u) \in G \times U_i$. Our aim is to show that each U_i is a trivial P–module.

Lemma 2.4. *For each $i \in [1\,,k]$, the quotient F_i/F_{i-1} in (7) is a trivial vector bundle.*

Proof. To prove the lemma first consider the vector bundle W/F_i. This vector bundle is associated to the P–bundle E_P for the P–module U_k/U_i (see (6)). Also, note that W/F_i is globally generated since W is so. Now using Lemma 2.2 and Proposition 2.3 we conclude that W/F_i is a trivial vector bundle.

Since W and W/F_i are both trivial, it follows immediately that the vector bundle F_i is also trivial. Since F_i and F_{i-1} are both trivial, the quotient bundle F_i/F_{i-1} is also trivial. This completes the proof of the lemma. □

The assumption that the parabolic subgroup P does not contain any simple factor is used in the proof of the following lemma.

LEMMA 2.5. *For each $i \in [1\,,k]$, the quotient U_i/U_{i-1} in (6) is a trivial P–module.*

PROOF. To prove the lemma, first consider the vector bundle F_i/F_{i-1} in (7) which is associated to the principal P–bundle E_P for the P–module U_i/U_{i-1} in (6). Lemma 2.4 says that F_i/F_{i-1} is a trivial vector bundle. From this it follows that the action of P on U_i/U_{i-1} extends to an action of G on U_i/U_{i-1}. To prove this, consider the action of G on $H^0(G/P,\, F_i/F_{i-1})$ constructed using the action of G on F_i/F_{i-1}. Since the evaluation homomorphism

$$H^0(G/P,\, F_i/F_{i-1}) \longrightarrow (F_i/F_{i-1})_{eP} = U_i/U_{i-1}$$

is an isomorphism (as F_i/F_{i-1} is trivial), this gives an action of G on U_i/U_{i-1}. The restriction of this action to $P \subset G$ clearly coincides with the one defined by the P–module structure of U_i/U_{i-1}.

If $Q \subset G'$ is a proper parabolic subgroup of a simple group G' and V' a nontrivial G'–module, then the vector space V' has a nontrivial filtration preserved by Q. Since $P \subset G$ is a parabolic subgroup without any simple factor, any nontrivial left G–module V_1 has the property that there is a nontrivial filtration of the vector space V_1 which is preserved by P. Therefore, the given condition that U_i/U_{i-1} is an irreducible P–module implies that the above constructed action of G on U_i/U_{i-1} is the trivial action. Consequently, U_i/U_{i-1} is a trivial P–module. This completes the proof of the lemma. □

Continuing with the proof of the theorem, since $L(P)$ is reductive and each quotient U_i/U_{i-1} in (6) is a trivial $L(P)$–module (Lemma 2.5), it follows immediately that $U_k = W_{eP}$ itself is a trivial $L(P)$–module. Indeed, there is no nontrivial homomorphism from a reductive group to a unipotent group.

Since V is an irreducible $L(P)$–module, the Schur's lemma says that the space of $L(P)$–invariants in $\mathrm{End}(V)$ is one–dimensional (generated by scalar multiplications). We have proved that $U_k = W_{eP}$ in (6) is a trivial submodule of the $L(P)$–module $\mathrm{End}(V)$. Consequently, we have $\dim W_{eP} \leq 1$. This immediately implies that (3) is valid. This completes the proof of the theorem. $\square$

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