THE $(\mathfrak{g}, K)$-MODULE STRUCTURE OF PRINCIPAL SERIES AND RELATED WHITTAKER FUNCTIONS OF $SU(2,2)$

(和訳・$SU(2,2)$ の主系列の $(\mathfrak{g}, K)$-加群構造と関連する WHITTAKER 関数)

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Introduction .

This is an overview of my thesis which consists of two papers.

Our principal research interest lies in the theory of automorphic forms of many variables, especially in the spherical functions over real semisimple Lie groups, and their applications to the theory of automorphic forms. Though this is a natural and fundamental problem, precise studies for the case of higher-rank Lie groups are very recent.

Over the recent decades representation theory gained a central role in modern mathematics, linking such areas as number theory, differential equations, algebraic and arithmetic geometry and theory of automorphic forms. In particular, the Whittaker models are one of the main ingredients of the theory in Fourier expansions of automorphic form at cusps. In this sense, explicit knowledge of Whittaker functions is very important for deeper studies of automorphic forms.

My research in doctoral course focuses on various Whittaker models of the principal series representations of $SU(2,2)$ obtained by parabolic induction. The main object of this paper is the space of algebraic Whittaker vectors attached to the principal series representations of $SU(2,2)$, parabolically induced with respect to the minimal parabolic subgroup P_{min}. In this setting, firstly, we describe completely the whole structure of the $(\mathfrak{g}, K)$-modules of these representations in the first part of thepapers, entitled "The $(\mathfrak{g}, K)$-module structures of principal series of $SU(2,2)$".

Secondly, we obtain various integral expressions of some smooth Whittaker functions with certain K-types to provide an explicit form of generators of the space of algebraic Whittaker vectors locally in the paper entitled "Explicit evaluation of certain Jacquet integrals on $SU(2,2)$".

In more detail:

The Purpose. Let (π, H_π) be an irreducible principal series representations of G, parabolically induced with respect to the minimal parabolic subgroup P_{min} with Langlands decomposition $P_{min} = MAN$. For a continuous unitary character $\eta : N \to U(1)$ of N, let $C^\infty_\eta(N \setminus G)$ be the subspace of $C^\infty(G)$ consisting of functions f such that

$$f(ng) = \eta(n) f(g) \qquad \text{for any } n \in N, g \in G.$$

Regarding $C^\infty_\eta(N \setminus G)$ as a $(\mathfrak{g}, K)$-module via right regular action of G, one has a natural map

$$Hom_G(H^\infty_\pi, C^\infty_\eta(N \setminus G)) \to Hom_{(\mathfrak{g},K)}(H_\pi|_K, C^\infty_\eta(N \setminus G)) \tag{1}$$

Our main object is the right hand side of (1), *i.e.*, the space of algebraic Whittaker vectors of π introduced by Kostant. We are interested to study the both spaces explicitely.

For the left had side of (1), H. Jacquet introduced a functional on H^∞_π which defines an intertwiner from π to $C^\infty_\eta(N \setminus G)$. The image of this intertwiner is a Whittaker model of π. The local multiplicity one theorem of Shalika at the archimedean place implies the uniqueness of such kind of functionals. Note also that Wallach reformulated this result in a slightly different but useful manner, i.e., in terms of "moderate growth condition".

Part 1. The first part of my thesis gives the whole structure of the $(\mathfrak{g}, K)$-modules of the principal series representations of G, where K is the standard fixed maximal compact subgroup of G and $\mathfrak{g}$ is the Lie algebra of G. To describe the $\mathfrak{g}_{\mathbb{C}}$-action on the subspace of K-finite vectors in π explicitly, the notion of "marked basis" introduced in is crucial and let us specify it for the closed subspace $L^2_{(M,\sigma)}(K)$ of $L^2(K)$ which is a realization of the representation space H_π, where

$$L^2_{(M,\sigma)}(K) = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k) \text{ for } m \in M, k \in K(a.e.)\}.$$

Lemma 1. *Denote by $H_\pi(\tau)$ the τ-isotypic component realized in $L^2(K)$ for each simple K-module τ occurs in π. Then there is a finite set of pairs (W_α, B_α), where $W_\alpha \cong \tau$ as K-modules and B_α is a finite set consisting of some functions $f_{\alpha 1}, ..., f_{\alpha n}$ on K in $H_\pi(\tau)$ which forms a standard basis for W_α, such that*

$$H_\pi(\tau) = \sum_\alpha W_\alpha \text{ and } f_{\alpha j}(1) = \delta_{\alpha j}.$$

Here $n = \dim(\tau)$, $\alpha = 1, 2, ..., [\pi|_K : \tau]$ and 1 is the unit in K.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. The adjoint representation of K on the complexification $\mathfrak{p}_\mathbb{C}$ of $\mathfrak{p}$ splits into two irreducible components, namely, the holomorphic part $\mathfrak{p}_+$ and the antiholomorphic part $\mathfrak{p}_-$.

Then the template of our main result is

$$\mathcal{C}_{[\pm,\pm;\pm]}\mathbf{S}^{(m)} = \mathbf{S}^{(m')}\Gamma_{[\pm,\pm;\pm]}.$$

Here $\mathbf{S}^{(m)}$ is the matrix consisting of functions $f_{\alpha j}$ in the lemma above, $\mathcal{C}_{[\pm,\pm;\pm]}$ is a matrix with entries either in $\mathfrak{p}^+$ or in $\mathfrak{p}^-$, and $\Gamma_{[\pm,\pm;\pm]}$ is a constant matrix whose entries consists of linear forms in the parameters of the representation.

Let us recall the Casimir equation for the Casimir operator $\mathcal{C}$:

$$\mathcal{C}v = \gamma(\mathcal{C})v,$$

where γ is the infinitesimal character and v is a differential vector. Our formula is a "covariant" analogue of this.

Part 2. The second part focuses on the explicit integral expressions of some Whittaker functions of G. In a number of applications (say, to have explicit Γ-factors of automorphic L-functions), it is very important to have explicit integral expressions of Whittaker functions for deeper studies. In particular, our formulas, arising some standard principal series representations of G, provide an explicit form of generators of the space of algebraic Whittaker vectors locally.

In Jacquet defined the continuous functional $J_{\sigma,\nu}$ on the space of differentiable functions of H_π realized in $L^2(K)$ satisfying $J_{\sigma,\nu}(\pi(n)f) = \eta(n)J_{\sigma,\nu}(f)$, that is

$$J_{\sigma,\nu}(f) = \int_N \eta(n)^{-1}a(s^*n)^{\nu+\rho}f(k(s^*n))dn$$

for a differentiable function f in $L^2_\sigma(K)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_K(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K, by M and s^* is an element of M^* mapping to the longest element $s \in W(A)$.

Multiplicity one theorem tells that there is at most one intertwiner (up to constant) from the space of K-finite vectors of π into the subspace $A_\eta(N\backslash G)$ of moderate growth functions in $C^\infty_\eta(N \setminus G)$. If it exists, then the construction is as follows: for each differentiable $f \in L^2_\sigma(K)$ it associates a function $J_f(g)$ in $C^\infty_\eta(N \setminus G)$ defined by

$$J_f(g) = J_{\sigma,\nu}(\pi(g)f), \ (g \in G).$$

These functions $J_f(g)$ are of moderate growth on G, and in particular so on the subgroup A. We want to have an explicit formula for the A-radial part of $J_f(g)$ with f belonging to a special K-type τ in π. The main body of our formula is

$$\frac{1}{\Gamma(\frac{\nu_1+1}{2})\Gamma(\frac{\nu_2+1}{2})\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)}\times$$

$$\left(\frac{y_1}{y_2}\right)^{\frac{\nu_2}{2}}\int_0^\infty\int_0^\infty K_{\frac{\nu_1}{2}}\left(2y_2\sqrt{\frac{(1+x)(1+y)}{xy}}\right)K_{\frac{\nu_2}{2}}\left(2y_1\sqrt{1+x+y}\right)$$

$$\left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}}\left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2}{4}}\frac{dx}{x}\frac{dy}{y}$$

which is rapidly decreasing at infinity for each variable of $(y_1, y_2) \in A$. More specifically, our formulas are expressed in terms of the modified Bessel functions, to obtain their Mellin-Barnes integral representations. As a consequences, we obtain explicitly the system of generators of the space of algebraic Whittaker vectors, around zero.

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PART 1

THE $(\mathfrak{g}, K)$-MODULE STRUCTURES OF PRINCIPAL SERIES OF $SU(2,2)$

THE $(\mathfrak{g}, K)$-MODULE STRUCTURE OF PRINCIPAL SERIES OF $SU(2,2)$

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ABSTRACT. We explicitly describe the $(\mathfrak{g}_{\mathbb{C}}, K)$-module structures of the principal series representations of $SU(2,2)$ associated with a maximal parabolic subgroup.

Introduction

The purpose of this paper is to describe explicitly the $(\mathfrak{g}, K)$-module structure of the principal series representations of $SU(2,2)$, parabolically induced with respect to the minimal parabolic subgroup P_{min}.

This is motivated by the problem of the determination of the precise formulas for various spherical models of the standard representations. Among others we are interested in the Whittaker models (Bayarmagnai [1], Hayata [3], Ishii [5], Miyazaki-Oda [10]). Our basic concern is in arithmetic of automorphic forms. However, in our case, we should also recall that the group $SU(2,2)$, which is locally isomorphic to the conformal group $SO(4,2)$, plays a very important role in physics. Our method of proof is similar to that of a recent paper of Oda [12], which describes the $(\mathfrak{g}, K)$-module structure of standard representations of $Sp(2,\mathbb{R})$. Namely we utilize the concept of simple K-modules with marking, to overcome the problem of multiplicities in K-types.

Our main results are Theorem 3.5 and Theorem 3.6 which are shortly explained below. The template of the formulas is the following:

$$\mathcal{C}_{[\pm,\pm;\pm]}\mathbf{S}^{(m)} = \mathbf{S}^{(m')}\Gamma_{[\pm,\pm;\pm]}.$$

Here $\mathbf{S}^{(m)}$ is the matrix consisting of elementary functions in the representation identified with a closed subspace of $L^2(K)$, $\mathcal{C}_{[\pm,\pm;\pm]}$ is a matrix with entries either in $\mathfrak{p}^+$ or in $\mathfrak{p}^-$, and $\Gamma_{[\pm,\pm;\pm]}$ is a constant matrix whose entries consists of linear forms in the parameters of the representation. The last is called a matrix of intertwining constants.

Let us recall the Casimir equation for the Casimir operator $\mathcal{C}$:

$$\mathcal{C}v = \gamma(\mathcal{C})v,$$

where γ is the infinitesimal character and v is a differential vector. Our formula is a "covariant" analogue of this. The details of each symbol is explained in the text.

This paper is arranged as follows. In the section 1, we establish our notation and define the class of the principal series representations of $SU(2,2)$ corresponding to the minimal parabolic subgroup P_{min}. The marked basis for each K-isotypic component in the principal series representation is introduced in terms of the elementary functions in the section 2. We begin section

3 by computing the Clebsch-Gordan coefficients of finite dimensional representations of K (Propositions 3.1 and 3.2). Then we shall determine our main result concerning the $\mathfrak{g}_{\mathbb{C}}$-module (Theorems 3.5 and 3.6), and finally give some examples.

We want to refer to the former results on $(\mathfrak{g}, K)$-module structures: Klimyk-Gruber [6], [7], Molchanov [11], Thieleker [13], Howe [4], and Lee-Loke [8]. Their interests are mainly to study the composition series of $(\mathfrak{g}, K)$-modules for degenerate principal series representations which are K-multiplicity free, except for [4] and [8].

The method of [4] for $GL(3, \mathbb{R})$ is to find nice elements in the enveloping algebra $U(\mathfrak{g})$ to generate the K-types in a principal series representation, hence it is different from our results. The paper [8] is most similar to ours, but this also considers the composition series of degenerate principal series.

The result of the papers of Yamashita [15],[16] also gives some structure of the composition series of the principal series representations of $SU(2,2)$ by direct determination of intertwining operators.

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1. Preliminaries

1.1 The group $SU(2,2)$. In this paper, the group $SU(2,2)$ is the special unitary group of signature $(+2,-2)$ associated to the Hermitian form $\langle\ ,\ \rangle$ defined on $\mathbb{C}^4$ by

$$\langle z, w\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 - \bar{z}_3 w_3 - \bar{z}_4 w_4$$

for $z = (z_1, z_2, z_3, z_4)$ and $w = (w_1, w_2, w_3, w_4)$. In terms of matrices, the group consists of all matrices $g \in SL_4(\mathbb{C})$ that satisfy the following identity:

$$^t\bar{g} I_{2,2} g = I_{2,2},$$

where $I_{2,2} = \mathrm{diag}(1,1,-1,-1)$. It is a standard fact that $G = SU(2,2)$ is a quasi-split real semisimple group of real rank two.

Let θ be a Cartan involution given by

$$\theta(g) =^t \bar{g}^{-1},\ g \in G.$$

Then the fixed point set $K = G^\theta$ is the standard maximal compact subgroup $S(U(2) \times U(2))$ of G. The group $K = S(U(2) \times U(2))$ can be represented by matrices

$$\begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \in G,$$

where $k_1, k_2 \in U(2)$ and $\det(k_1 k_2) = 1$.

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2,2)$ of G is the set of matrices $X \in M_4(\mathbb{C})$ such that $^t\bar{X} I_{2,2} + I_{2,2} X = 0$ and $tr(X) = 0$. We let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigen-spaces of the differential of θ, respectively. Then we have

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & \\ & X_3 \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) : X_1, X_3 \in \mathfrak{u}(2) \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} & X_2 \\ {}^t\bar{X}_2 & \end{pmatrix} \in M_4(\mathbb{C}) : X_2 \in M_2(\mathbb{C}) \right\}.$$

For $x \in M_2(\mathbb{C})$ we set

$$p_+(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ and } p_-(x) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

Let $H_i = p_+(e_{ii}) + p_-(e_{ii})$ $(i = 1, 2)$, where e_{ij} the matrix unit of $M_2(\mathbb{R})$ with 1 in the (i, j)-entry and zero elsewhere. Then the space $\mathfrak{a}$ spanned by H_1, H_2 over $\mathbb{R}$ is a maximally abelian subalgebra of $\mathfrak{p}$. Let $\{\lambda_1, \lambda_2\}$ be a basis of the dual space $\mathfrak{a}^*$ such that $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system for $\Phi(\mathfrak{g}, \mathfrak{a})$ is of type C_2, namely

$$\Phi(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

Choose $\lambda_1 - \lambda_2$ and $2\lambda_2$ as simple roots of $\Phi(\mathfrak{g}, \mathfrak{a})$. Denote by E_{ij} the matrix units in $M_4(\mathbb{C})$ for $0 \le i, j \le 4$. Then the corresponding root spaces of dimension two and one are given by

$$\mathfrak{g}_{\lambda_1 - \lambda_2} = \mathbb{R} \cdot E_1 \oplus \mathbb{R} \cdot E_2 \text{ and } \mathfrak{g}_{2\lambda_2} = \mathbb{R} \cdot E_0,$$

where $E_0 = \kappa^{-1} E_{24} \kappa, E_1 = \kappa^{-1}(E_{12} - E_{43})\kappa$ and $E_2 = \kappa^{-1}(iE_{12} + iE_{43})\kappa$. Here

$$\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}$$

with $i = \sqrt{-1}$.

We put $A = \exp(\mathfrak{a})$, $M = Z_A(K)$, and choose a minimal parabolic subgroup P_{min} with Langlands decomposition $P_{min} = MAN$.

Here N is the maximal unipotent subgroup of G and an element $n = n(n_0, n_1, n_2, n_3) \in N$ takes the form:

$$\kappa^{-1} \begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{n}_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \kappa$$

for $n_1, n_3 \in \mathbb{R}$, $n_0, n_2 \in \mathbb{C}$.

1.2 The K-modules. Let (τ, V_τ) be an irreducible representation of K. The fact is that the dimension of V_τ is finite and τ is unitary. To clarify K-action on V_τ it is enough to consider the $\mathfrak{k}_{\mathbb{C}}$-action on that vector space.

Note that the group $\tilde{K} = SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ is a twofold covering of K with a projection given by

$$pr(g_1, g_2; u) = \operatorname{diag}(ug_1, u^{-1}g_2),$$

where $g_1, g_2 \in SU(2)$ and $u \in \mathbb{C}^{(1)}$. The kernel of this homomorphism is

$$\operatorname{Ker}(pr) = \{\pm(1_2, 1_2; 1)\}.$$

Let $(\operatorname{sym}^m, V_m)$ be the m-th symmetric tensor representation of the group $SU(2)$. Then the unitary dual of K can be parameterized by the set

$$\hat{K} = \{(\tau_{[m_1, m_2;\ l]}, V_{m_1 m_2}) \mid m_1, m_2 \in \mathbb{N} \cup 0,\ l \in \mathbb{Z},\ m_1 + m_2 + l \in 2\mathbb{Z}\}.$$

Here $V_{m_1m_2}$ is the outer tensor product of the spaces V_{m_1} and V_{m_2}, and if $g_1, g_2 \in SU(2)$ and $u \in \mathbb{C}^{(1)}$, then the action is

$$\tau_{[m_1,m_2;\,l]}(g_1, g_2; u) = \mathrm{sym}^{m_1}(g_1) \otimes \mathrm{sym}^{m_2}(g_2) \otimes u^l.$$

We fix now a basis for $\mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}(K)_{\mathbb{C}}$:

$$h^1 = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, \qquad h^2 = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \qquad I_{2,2} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix},$$
$$e^1_{\pm} = \begin{pmatrix} e_{\pm} & 0 \\ 0 & 0 \end{pmatrix}, \qquad e^2_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & e_{\pm} \end{pmatrix},$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we need the following well known fact for the determination of the K-action on the principal series representations which will be defined in the subsection 1.5.

Lemma 1.1. *Let $(\tau_{[m_1,m_2;l]}, V_{m_1m_2}) \in \hat{K}$. Then there is a basis*

$$\{f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$$

for $V_{m_1m_2}$ such that $I_{2,2}f_{pq} = lf_{pq}$ and

$$\begin{aligned} &h^1(f_{pq}) = (2p - m_1)f_{pq}, && h^2(f_{pq}) = (2q - m_2)f_{pq}, \\ &e^1_+(f_{pq}) = (m_1 - p)f_{p+1,q}, && e^2_+(f_{pq}) = (m_2 - q)f_{p,q+1}, \\ &e^1_-(f_{pq}) = pf_{p-1,q}, && e^2_-(f_{pq}) = qf_{p,q-1}, \\ &I_{2,2}f_{pq} = lf_{pq}. \end{aligned}$$

Throughout this paper by basis for $\tau_{[m_1,m_2,l]}$ we mean a basis that satisfying the above lemma. For a simple K-module τ, we can normalize the one dimensional space of K-homomorphisms of τ into itself by the following definition.

Definition 1.1. *A simple K-module τ equipped with a canonical basis is called a* marked simple K-module *or a* simple K-module with marking.

1.3 Iwasawa decomposition. The set $\{E_{i,j+2}, E_{i+2,j} \mid i,j = 1,2\}$ forms a basis of the 8-dimensional vector space $\mathfrak{p}_{\mathbb{C}}$ and one has

$$E_{i,j+2} = p_+(e_{ij}) \qquad \text{and} \qquad E_{i+2,j} = p_-(e_{ij}),$$

where $i, j = 1, 2$.

Lemma 1.2. *Put*

$$\begin{aligned} E_{2\lambda_1} &= \kappa^{-1}E_{13}\kappa, \; E^1_{\lambda_1+\lambda_2} = \kappa^{-1}E_{14}\kappa, \; E^1_{\lambda_1-\lambda_2} = \kappa^{-1}E_{43}\kappa, \\ E_{2\lambda_2} &= \kappa^{-1}E_{24}\kappa, \; E^2_{\lambda_1+\lambda_2} = \kappa^{-1}E_{23}\kappa, \; E^2_{\lambda_1-\lambda_2} = \kappa^{-1}E_{12}\kappa. \end{aligned}$$

Then we have

$$p_{\pm}(e_{ii}) = \tfrac{1}{2}(\mp 2\sqrt{-1}E_{2\lambda_i} + H_i \pm \tfrac{1}{2}(I_{2,2} - \epsilon(i)(h^1 - h^2))),$$
$$p_{\pm}(e_{ij}) = \tfrac{1}{2}(-\epsilon(i)E^j_{\lambda_1-\lambda_2} \mp \sqrt{-1}E^i_{\lambda_1+\lambda_2}) - \epsilon(j)\begin{cases} e^j_{\epsilon(j)}, & \text{if } (+) \\ e^i_{-\epsilon(i)}, & \text{if } (-) \end{cases}$$

where $\epsilon(i) := \mathrm{sign}(-1)^i \quad (i \neq j,\ i,j \in \{1,2\})$.

Proof. We can show this by direct computation. □

1.4 The adjoint representation. Now we consider the adjoint representation Ad of K on the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$. It splits into two K-irreducible components, namely, the holomorphic part $\mathfrak{p}_+$ generated by the set of matrix units $\{E_{ij} \mid i = 1,2, j = 3,4\}$ and the antiholomorphic part $\mathfrak{p}_-$ generated by the set $\{E_{ij} \mid i = 3,4, j = 1,2\}$ over $\mathbb{C}$. Moreover, we have:

Lemma 1.3. (cf. [3,3.10]) *The linear maps from $\mathfrak{p}_+$ and $\mathfrak{p}_-$ to V_{11} given by*

$$(E_{23}, E_{13}, E_{24}, E_{14}) \to (f_{00}, f_{10}, -f_{01}, -f_{11})$$

and

$$(E_{41}, E_{31}, E_{42}, E_{32}) \to (f_{00}, f_{01}, -f_{10}, -f_{11}),$$

respectively, induce the K-isomorphisms

$$(Ad, \mathfrak{p}_+) \cong (\tau_{[1,1;2]}, V_{11}) \text{ and } (Ad, \mathfrak{p}_-) \cong (\tau_{[1,1;-2]}, V_{11}).$$

1.5 Principal series representations. In this paper we will be dealing with principal series representations which are parabolically induced with respect to the minimal parabolic subgroup. We take a moment here to review basic definition to that of $SU(2,2)$.

Let P_{min} be a minimal parabolic subgroup of G with Langlands decomposition $P_{min} = MAN$ with $M = Z_K(A)$. In particular, by setting $i = \sqrt{-1}$, each element of M can be represented by a matrix

$$[\exp(i\theta)]\gamma^j \text{ for some } \theta \in \mathbb{R} \text{ and } j = 0, 1,$$

where γ is the diagonal matrix $\operatorname{diag}(1, -1, 1, -1)$ and $[\exp(i\theta)]$ stands for

$$\operatorname{diag}(\exp(i\theta), \exp(-i\theta), \exp(i\theta), \exp(-i\theta)).$$

Let χ be a character of the multiplicative group $\{\pm 1\}$. For an integer s, we define a unitary character of M by

$$\sigma_{\chi,s}([\exp(i\theta)]\gamma^j) = \chi(-1)^j \exp(is\theta).$$

Given a complex valued real linear form $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$ on $\mathfrak{a}$, define a character e^μ of A by

$$e^\mu(a) = \exp(\mu_1 a_1 + \mu_2 a_2),$$

for $a = \exp(a_1 H_1 + a_2 H_2) \in A$. We extend it to a character of AN so that the restriction to N is trivial. Define an admissible character of P_{min} by tensoring these characters. Then one has the induced representation

$$\pi = \operatorname{Ind}_{P_{min}}^G (\sigma_{\chi,s} \otimes e^{\mu+\rho} \otimes 1_N)$$

and call it *the principal series representation* of G. Here ρ denotes the half sum of the positive roots of $\Phi(\mathfrak{g}, \mathfrak{a})$. Now look at the compact realization of π. Then representation space H_π of π can be realized on the Hilbert space

$$L^2_{\sigma_{\chi,s}}(K) = \{f \in L^2(K) \mid f(mk) = \sigma_{\chi,s}(m) f(k) \text{ for } m \in M, k \in K, \ a.e.\}$$

with G-action defined by

$$(\pi(g)f)(x) = e^{\mu+\rho}(a(xg)) f(k(xg)), \ x \in K, g \in G,$$

where $xg = n(xg)a(xg)m(xg)k(xg)$ is the Iwasawa decomposition of the element xg.

2. The structure of K-types of the principal series representation

In this section we express the K-isotypic components of H_π in terms of the elementary functions obtained from the tautological representation of $SU(2)$. Combining it with Lemma 1.1, the K-module structure on H_π^K is described explicitly.

2.1 Elementary functions in $L^2(K)$. We begin this subsection by reviewing the parametrization of the unitary dual of $SU(2)$. Let $S(x)$ ($x \in SU(2)$) be a square matrix function associated to $SU(2)$ given by

$$S(x) = \begin{pmatrix} s_1(x) & s_2(x) \\ -\bar{s}_2(x) & \bar{s}_1(x) \end{pmatrix}, \quad \text{with } \det(S(x)) = 1.$$

Then we have $S(xy) = S(x)S(y)$ and $s_i(-x) = -s_i(x)$ for $i = 1, 2$. For the independent parameters X and Y, we put

$$X' = Xs_1(x) - Y\bar{s}_2(x) \text{ and } Y' = Xs_2(x) + Y\bar{s}_1(x).$$

Then for each positive integer $n \geq 1$, there is a linear transformation

$$\mathrm{Sym}^{(n)}(S(x)) = \begin{bmatrix} s_{nn}^{(n)}(x) & \cdots & s_{n0}^{(n)}(x) \\ \vdots & \ddots & \vdots \\ s_{0n}^{(n)}(x) & \cdots & s_{00}^{(n)}(x) \end{bmatrix} = \{s_{ij}^{(n)}(x)\}_{0\leq i,j\leq n}$$

between the homogeneous forms of (X, Y) and (X', Y') of degree n via

$$((X')^n, (X')^{n-1}Y', ..., (Y')^n) = (X^n, X^{n-1}Y, ..., Y^n) \cdot \mathrm{Sym}^{(n)}(S(x)).$$

We recall the following well-known observation without proof.

Lemma 2.1. *The $n+1$ entries of each i-th row vector of $\mathrm{Sym}^{(n)}(S(x))$ make a canonical basis of the irreducible right $SU(2)$-representation of dimension $n+1$ in $L^2(SU(2))$. In particular, we have*

1. $\mathrm{Sym}^{(n)}(S(xy)) = \mathrm{Sym}^{(n)}(S(x))\mathrm{Sym}^{(n)}(S(y)), \quad x, y \in SU(2),$
2. $\mathrm{Sym}^{(n)}(S(x)) = \mathrm{diag}_{0\leq i\leq n}(e^{\sqrt{-1}t(n-2i)})$ *if* $x = \mathrm{diag}(e^{\sqrt{-1}t}, e^{-\sqrt{-1}t})$

with $t \in \mathbb{R}$.

2.2 Elementary functions in $L^2(\tilde{K})$. Fix positive integers m_1, m_2 and an integer l. For each quadruple $(i, j, p, q) \in \mathbb{Z}_+^4$ such that $i, p \leq m_1$ and $j, q \leq m_2$, we associate a $\mathbb{C}$-valued function on $\tilde{K}$ by

$$S_{ij,pq}(g_1, g_2, u) = s_{ip}^{(m_1)}(g_1)s_{jq}^{(m_2)}(g_2)u^l,$$

where $g_1, g_2 \in SU(2)$ and $u \in \mathbb{C}^{(1)}$. For a fixed pair (i, j), a space $W_{ij}^{(m)}$ generated by the set of functions $\{S_{ij,pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$ is a $\tilde{K}$-module if we define an action τ_m, ($m = [m_1, m_2; l]$), of $\tilde{K}$ by

$$\tau_m(g_1, g_2; u)S_{ij,pq}(x, y; v) = S_{ij,pq}(xg_1, yg_2; vu)$$

for $g_1, g_2, x, y \in SU(2)$ and $u, v \in \mathbb{C}^{(1)}$. Note that, by Lemma 2.1, we have that $(\tau_m, W_{00}^{(m)}) \cong (\tau_m, W_{ij}^{(m)})$ for each pair (i, j). Furthermore, when we regard $L^2(\tilde{K})$ as a right $\tilde{K}$-module, its the τ_m-isotypic component is just the direct sum of all these spaces $W_{ij}^{(m)}$.

2.3 K-isotypic components of the principal series representations. For $x \in SU(2)$, Lemma 2.1 implies that

$$\mathrm{Sym}^{(n)}(S(-x)) = (-1)^n \mathrm{Sym}^{(n)}(S(x)),$$

and so $S_{ij,pq}(k) = S_{ij,pq}(-(1_2, 1_2; 1)k)$ for $k \in \tilde{K}$ when $m_1 + m_2 + l \in 2\mathbb{Z}$. Therefore, in this case, the functions $S_{ij,pq}(k)$ are well defined on K. Thus, we can derive that the unitary dual $\hat{K}$ is parameterized by the set

$$\{(\tau_m, W_{00}^{(m)}) \mid m = [m_1, m_2; l], m_1 + m_2 + l \in 2\mathbb{Z}\}.$$

Note also that Lemma 2.1 shows $S_{ij,pq}(k) = \delta_{ij,pq}$ at the point $k = 1_4$. This property will be used several times later. In fact, we have constructed a unique basis for each K-isotypic component of $L^2(K)$. We now summarize the main properties of this basis in the following proposition.

Proposition 2.2. *Let $H(\tau)$ be the τ-isotypic component of $L^2(K)$ corresponding to $\tau \in \hat{K}$ of dimension n. Then there exists a unique square matrix function $\mathbf{S}^{(\tau)}(k)$ on K of size n with entries in $H(\tau)$,*

$$\mathbf{S}^{(\tau)}(k) = \begin{bmatrix} f_{11}(k) & \cdots & f_{n1}(k) \\ \vdots & \ddots & \vdots \\ f_{1n}(k) & \cdots & f_{nn}(k) \end{bmatrix} = \{f_{ij}(k)\}_{1 \le i,j \le n},$$

satisfying the following two conditions:

1. $\mathbf{S}^{(\tau)}(1_K) = \mathrm{diag}(1, ..., 1) \in M_n(\mathbb{C})$,

2. For each α, the column vector $\{f_{\alpha 1}(k), ..., f_{\alpha n}(k)\}$ induces a canonical basis for τ. Moreover, we have

$$H(\tau) = \bigoplus_{\alpha} W_\alpha,$$

where W_α denotes the space spanned by the functions $f_{\alpha 1}(k), ..., f_{\alpha n}(k)$.

Proof. It is enough to show the uniqueness. Assume that there exist two matrices $\mathbf{F}^{(\tau)}(k) = \{f_{ij}(k)\}$ and $\mathbf{G}^{(\tau)}(k) = \{g_{ij}(k)\}$ satisfying the required conditions. Denote by F_α the K-isomorphism between τ and the space spanned by $\{f_{\alpha j}(k), ..., f_{\alpha n}(k)\}$. Similarly, we define G_α for the α-th column of $\mathbf{G}^{(\tau)}(k)$. As a result, we obtain two ordered bases $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ for the n-dimensional vector space $\mathrm{Hom}_K(\tau, H(\tau))$. Then we have the n by n matrix $A = \{a_{\alpha\beta}\}$, the change of coordinate matrix, such that

$$F_\alpha = \sum_\beta a_{\alpha\beta} G_\beta.$$

For a canonical basis $\{f_\gamma\}$ of τ, one obtains

$$f_{\alpha\gamma}(k) = F_\alpha(f_\gamma) = \sum_\beta a_{\alpha\beta} G_\beta(f_\gamma) = \sum_\beta a_{\alpha\beta} f_{\beta\gamma}(k).$$

Evaluation at the point 1_K shows that $a_{\alpha\gamma} = \delta_{\alpha\gamma}$, and hence $F_\alpha = G_\alpha$ for each α.

If $v \neq 0 \in W_\alpha \cap W_\beta$, then $Kv = W_\alpha = W_\beta$. Schur's lemma and the second condition imply that $\alpha = \beta$. Hence we have the direct sum decomposition of $H(\tau)$. □

Set $\sigma = \sigma_{\chi,s}$. Since $L^2_\sigma(K) \subset L^2(K)$, as a right unitary representation of K, it has an irreducible decomposition of $K \times K$-bimodules

$$L^2_\sigma(K) \cong \hat{\oplus}_{\tau \in \hat{K}} \{(\tau^* \mid_M)[\sigma^{-1}] \otimes \tau\}$$

by the Peter-Weyl theorem. Here $(\tau^* \mid M)[\sigma^{-1}]$ is the σ^{-1}-isotypic component in $\tau^* \mid_M$. Hence one can explicitly describe the K-isotypic components of the principal series representation π.

Lemma 2.3. (cf. [3,3.6]) *Assume* $m_1+m_2 \geq |s|$ *and* $l \equiv 2m_2+s+1-\chi(-1)$ (mod 4). *Then the* τ_m*-isotypic component* $H_\pi(\tau_m)$ *in the principal series representation* π *is isomorphic to*

$$\oplus_\gamma W^{(m)}_\gamma \text{ with } \gamma = (t, (m_1+m_2+s)/2 - t)),$$

where t *runs over integers satisfying,*

$$\begin{cases} 0 \leq t \leq (m_1+m_2+s)/2, & \text{if } s < \min(m_1-m_2, m_2-m_1) \\ (m_1-m_2+s)/2 \leq t \leq m_1, & \text{if } s \geq \max(m_2-m_1, m_1-m_2) \end{cases}$$

and when $\min(m_1-m_2, m_2-m_1) \leq s < \max(m_1-m_2, m_2-m_1)$

$$\begin{cases} 0 \leq t \leq m_1, & \text{if } m_1 < m_2 \\ (m_1-m_2+s)/2 \leq t \leq (m_1+m_2+s)/2, & \text{if } m_1 > m_2. \end{cases}$$

Extending the notion given in Definition 1.1 slightly, we can define a set of markings for each isotypic companent of $L^2(K)$.

Definition 2.1. *Let* (τ_m, V_m) *be an irreducible representation of* K *with parametrization* $m = [m_1, m_2; l]$. *For each possible pair* (i,j), *the marking on the simple* K*-module* $(\tau_m, W^{(m)}_{ij})$ *specified by the basis*

$$\{S_{ij,pq}(k) \in L^2_\sigma(K) \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$$

is called the marking by elementary functions.

Conventions. Fix π and a marked simple K-module τ_m in $\pi \mid_K$ with parametrization $m = [m_1, m_2; l]$. Denote by $I(\pi, \tau_m)$ the set of all γ such that $\gamma = (t, (m_1+m_2+s)/2 - t))$ as in Lemma 2.3 and $W^{(m)}_\gamma$ occurs in $\pi \mid_K$. Then the multiplicity $\mathrm{m}(\pi, \tau_m)$ of τ_m in $\pi \mid_K$ is the cardinality of the finite set $I(\pi, \tau_m)$.

When $\gamma \in I(\pi, \tau_m)$, there is a K-isomorphism from V_m onto $W^{(m)}_\gamma$ by sending the set of marked basis onto the set of marked elementary functions and hence denote this K-isomorphism by $[\gamma]$.

3. The $(\mathfrak{g}_\mathbb{C}, K)$-module structure

In this section we investigate the action $\mathfrak{g} = \mathrm{Lie}(G)$ (or $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$) on the subspace $H_{\pi,K}$ of the K-finite vectors in the representation space H_π. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_\mathbb{C}$.

3.1 Clebsch-Gordan coefficients. We recall that the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$ splits into two irreducible components, namely the holomorphic part $\mathfrak{p}_+$ and the antiholomorphic part $\mathfrak{p}_-$. Let (τ_m, V_m) be an irreducible representation of K with parametrization $(m = [m_1, m_2; l])$.

$\mathfrak{p}_+$-side. By the well known Clebsch-Gordan theorem and Lemma 1.3, the irreducible components in the K-module $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ are precisely the K-representations

$$\{\ \tau_{[m_1+e_1, m_2+e_2;\ l+2]}\ \mid e_1, e_2 \in \{\pm 1\}\},$$

and we will denote these by $\tau_{[\pm,\pm;+]}$ or $\tau_{[e_1,e_2;+]}$ respectively.

For a fixed pair (e_1, e_2), $e_j \in \{\pm 1\}$ with $j = 1, 2$, we define $\mathbf{c}_t^j$ by

$$\mathbf{c}_t^j = \frac{t}{m_j + 1}\ (0 \leq t \leq m_j + e_j).$$

When $\tau_{[e_1,e_2;+]}$ is non zero, we now express the canonical basis vectors of $\tau_{[e_1,e_2;+]}$ in terms of the basis vectors of $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ induced from those of $\mathfrak{p}_+$ and τ. In this case, denote by $I_{[\pm,\pm;+]}$ a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m)$, which is unique up to constant multiple. More precisely, we have

Proposition 3.1. *The image of the (p,q)-th canonical basis vector f'_{pq} of $\tau_{[e_1,e_2;+]}$ under the K-homomorphism $I_{[e_1,e_2;+]}$ is given by*

i. If $(e_1, e_2) = (-1, -1)$ then

$$E_{23} \otimes f_{p+1q+1} - E_{13} \otimes f_{pq+1} + E_{24} \otimes f_{p+1q} - E_{14} \otimes f_{pq}\ ,$$

ii. If $(e_1, e_2) = (+1, -1)$ then

$$(1 - \mathbf{c}_p^1)(E_{23} \otimes f_{pq+1} + E_{24} \otimes f_{pq}) + \mathbf{c}_p^1(E_{13} \otimes f_{p-1q+1} + E_{14} \otimes f_{p-1q}),$$

iii. If $(e_1, e_2) = (-1, +1)$ then

$$(1 - \mathbf{c}_q^2)(E_{13} \otimes f_{pq} - E_{23} \otimes f_{p+1q}) + \mathbf{c}_q^2(E_{24} \otimes f_{p+1q-1} - E_{14} \otimes f_{pq-1}),$$

iv. If $(e_1, e_2) = (+1, +1)$ then

$$\begin{aligned} &-(1 - \mathbf{c}_q^2)((1 - \mathbf{c}_p^1)E_{23} \otimes f_{pq} + \mathbf{c}_p^1 E_{13} \otimes f_{p-1q}) \\ &+\mathbf{c}_q^2((1 - \mathbf{c}_p^1)E_{24} \otimes f_{pq-1} + \mathbf{c}_p^1 E_{14} \otimes f_{p-1q-1}) \end{aligned}$$

where $0 \leq p \leq m_1 + e_1$ and $0 \leq q \leq m_2 + e_2$, respectively.

Proof. Denote by u_{pq} the element in $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ defined in our Proposition. To prove $I_{[e_1,e_2;+]}(f'_{pq}) = u_{pq}$, it is enough to show that the correspondence $f_{pq} \to u_{pq}$ is a K-module homomorphism by utilizing the infinitesimal representation of K. Hence we only consider the first case as an example. We now claim that the weight of the vector $u_{m_1-1m_2-1}$ given by

$$E_{23} \otimes f_{m_1m_2} - E_{13} \otimes f_{m_1-1m_2} + E_{24} \otimes f_{m_1m_2-1} - E_{14} \otimes f_{m_1-1m_2-1}$$

is the same as that of $f_{m_1-1m_2-1}$ in $\tau_{[-,-;+]}$. Note that the algebra generated by h^1, h^2 and $I_{2,2}$ form a Cartan subalgebra. Moreover, it is clear that

$I_{2,2} \cdot u_{m_1-1m_2-1} = (l+2)u_{m_1-1m_2-1}$. By Lemma ?? and Lemma 1.3, it follows that

$$\begin{aligned} h^1 \cdot E_{14} \otimes f_{m_1-1m_2-1} &= (1+2(m_1-1)-m_1)E_{14} \otimes f_{m_1-1m_2-1}, \\ h^1 \cdot E_{13} \otimes f_{m_1-1m_2} &= (1+2(m_1-1)-m_1)E_{13} \otimes f_{m_1-1m_2}, \\ h^1 \cdot E_{24} \otimes f_{m_1m_2-1} &= (-1+2m_1-m_1)E_{24} \otimes f_{m_1m_2-1}, \\ h^1 \cdot E_{23} \otimes f_{m_1m_2} &= (m_1+1-2)E_{23} \otimes f_{m_1m_2}. \end{aligned}$$

Hence the eigenvalue of u_{m_1-1,m_2-1} under h^1 is just m_1-1. Similarly, one can check that the eigenvalue via h^2 is equal to m_2-1. The next claim is

$$u_{p-1,q} = \frac{e_-^1 \cdot u_{p,q}}{p}$$

for all possible values of (p,q). By using Lemma ?? and Lemma 1.3 again, we obtain that

$$\begin{aligned} e_-^1 \cdot E_{23} \otimes f_{p+1q+1} &= (p+1)E_{23} \otimes f_{pq+1}, \\ e_-^1 \cdot E_{13} \otimes f_{pq+1} &= E_{23} \otimes f_{pq+1} + pE_{13} \otimes f_{p-1q+1}, \\ e_-^1 \cdot E_{24} \otimes f_{p+1q} &= (p+1)E_{24} \otimes f_{pq}, \\ e_-^1 \cdot E_{14} \otimes f_{pq} &= E_{24} \otimes f_{pq} + p \cdot E_{14} \otimes f_{p-1q}. \end{aligned}$$

Hence the claim follows from the above. Similarly, for all possible indices (p,q), we can show that $u_{pq-1} = e_-^2 \cdot u_{pq}/q$. Therefore the natural correspondence $f_{pq} \to u_{pq}$ gives a non zero K-isomorphism. □

$\mathfrak{p}_-$-side. Since $(Ad, \mathfrak{p}_-) \cong \tau_{[1,1;-2]}$, the tensor product $\mathfrak{p}_- \otimes_{\mathbb{C}} \tau_m$ has four irreducible K-components:

$$\{\ \tau_{[m_1+e_1,m_2+e_2;l-2]} \mid e_1, e_2 \in \{\pm 1\}\}$$

and we will denote these by $\tau_{[e_1,e_2;-]}$ respectively. Let $I_{[e_1,e_2;-]}$ be a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, \mathfrak{p}_- \otimes_{\mathbb{C}} \tau_m)$ when $\tau_{[e_1,e_2;-]}$ is non zero. Similar to the previous Proposition, we have the following:

Proposition 3.2. *The image of the (p,q)-th canonical basis vector f'_{pq} of $\tau_{[e_1,e_2;-]}$ under the K-homomorphism $I_{[e_1,e_2;-]}$ is given by*

i. If $(e_1,e_2) = (-1,-1)$ then

$$E_{41} \otimes f_{p+1q+1} + E_{42} \otimes f_{pq+1} - E_{31} \otimes f_{p+1q} - E_{32} \otimes f_{pq}\ ,$$

ii. If $(e_1,e_2) = (+1,-1)$ then

$$(1-\mathbf{c}_p^1)(E_{31} \otimes f_{pq} - E_{41} \otimes f_{pq+1}) + \mathbf{c}_p^1(E_{42} \otimes f_{p-1q+1} - E_{32} \otimes f_{p-1q}),$$

iii. If $(e_1,e_2) = (-1,+1)$ then

$$(1-\mathbf{c}_q^2)(E_{42} \otimes f_{pq} + E_{41} \otimes f_{p+1q}) + \mathbf{c}_q^2(E_{31} \otimes f_{p+1q-1} + E_{32} \otimes f_{pq-1}),$$

iv. If $(e_1,e_2) = (+1,+1)$ then

$$\begin{aligned} &-(1-\mathbf{c}_q^2)((1-\mathbf{c}_p^1)E_{41} \otimes f_{pq} - \mathbf{c}_p^1 E_{42} \otimes f_{p-1q}) \\ &-\mathbf{c}_q^2((1-\mathbf{c}_p^1)E_{31} \otimes f_{pq-1} - \mathbf{c}_p^1 E_{32} \otimes f_{p-1q-1}), \end{aligned}$$

where $0 \le p \le m_1 + e_1$ and $0 \le q \le m_2 + e_2$, respectively.

Proof. The proof is quite similar to that of Proposition 3.1. □

3.2 Matrix form of the Clebsch-Gordan decompositions. For the further convenience, it is useful to describe the K-isomorphisms $I_{[e_1,e_2;\pm]}$ described in Propositions 3.1 and 3.2 in terms of the canonical basis of V_m.

To the set of all canonical basis $\{f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$ of the simple K-module V_m, we associate a row vector of size $(m_1+1)(m_2+1)$ with entries f_{pq} given by

$$\mathbf{F}_\tau = (f_{00}, f_{01} \ldots, f_{0m_2}, f_{10}, f_{11}, \ldots, f_{m_1,m_2-1}, f_{m_1 m_2}).$$

$\mathfrak{p}_+$-side. Define a matrix $\mathcal{C}_{[-,-;+]} = \{C_{ij}\}$ of size $(m_1 m_2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{14}, \\
C_{m_2p+q+1,(m_2+1)p+q+2} &= -E_{13}, \\
C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= E_{24}, \\
C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{23},
\end{aligned}$$

for each $0 \leq p \leq m_1 - 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,-;+]} = \{C_{ij}\}$ of size $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_p^1)E_{24}, \\
C_{m_2p+q+1,(m_2+1)p+q+2} &= (1-\mathbf{c}_p^1)E_{23}, \\
C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= \mathbf{c}_p^1 E_{14}, \\
C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= \mathbf{c}_p^1 E_{13},
\end{aligned}$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[-,+;+]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_q^2)E_{13}, \\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -\mathbf{c}_q^2 E_{14}, \\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= -(1-\mathbf{c}_q^2)E_{23}, \\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= \mathbf{c}_q^2 E_{24},
\end{aligned}$$

for $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,+;+]} = \{C_{ij}\}$ of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-\mathbf{c}_p^1)(1-\mathbf{c}_q^2)E_{23}, \\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= (1-\mathbf{c}_p^1)\mathbf{c}_q^2 E_{24}, \\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= -\mathbf{c}_p^1(1-\mathbf{c}_q^2)E_{13}, \\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= \mathbf{c}_p^1 \mathbf{c}_q^2 E_{14},
\end{aligned}$$

for each $0 \leq p \leq m_1 + 1$ and $0 \leq q \leq m_2 + 1$, but all other entries are 0. Then Proposition 3.1 reads as the following proposition .

Proposition 3.3. *Let $\mathcal{C}_{[e_1,e_2;+]}$, $\mathbf{F}_\tau$ be as above. Then for each pair e_1, e_2 the simple K-module $V_{[e_1,e_2;+]}$ is generated by the entries of the matrix $\mathcal{C}_{[e_1,e_2;+]}{}^{\mathbf{t}}\mathbf{F}_\tau$. Moreover, these entries make a set of canonical basis.*

Proof. Note that for the (i,j)-th entry of $\mathcal{C}_{[e_1,e_2;+]}$, the index i indicates the i-th coordinate in $\mathbf{F}_{[\mathbf{e_1},\mathbf{e_2};+]}$ and the index j indicates the j-th coordinate in $\mathbf{F}_\tau$. The i-th coordinate in $\mathbf{F}_{[\mathbf{e_1},\mathbf{e_2};+]}$ is uniquely expressed as

$$i = (m_2 + 1 + e_2)p + q + 1$$

for some pair (p,q) so that $0 \le p \le m_1+e_1$ and $0 \le q \le m_2+e_2$. Hence it is just the (p,q)-th canonical basis vector in $\tau_{[e_1,e_2;+]}$ by definition of $\mathcal{C}_{[e_1,e_2;+]}$. Similarly, the j-th coordinate in $\mathbf{F}_\tau$ corresponds to the (p,q)-th basis vector in τ. Thus the proposition follows from Proposition 3.1. □

$\mathfrak{p}_-$**-side.** Define a matrix $\mathcal{C}_{[-,-;-]} = \{C_{ij}\}$ of size $m_1 m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{32}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= E_{42}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= -E_{31}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{41}, \end{aligned}$$

for $0 \le i \le m_1-1$ and $0 \le q \le m_2-1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,-;-]} = \{C_{ij}\}$ of size $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_p^1)E_{31}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= -(1-\mathbf{c}_p^1)E_{41}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= -\mathbf{c}_p^1 E_{32}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= \mathbf{c}_p^1 E_{42}, \end{aligned}$$

for $0 \le p \le m_1+1$ and $0 \le q \le m_2-1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[-,+;-]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_q^2)E_{42}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= \mathbf{c}_q^2 E_{32}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= (1-\mathbf{c}_q^2)E_{41}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= \mathbf{c}_q^2 E_{31}, \end{aligned}$$

for $0 \le p \le m_1-1$ and $0 \le q \le m_2+1$, but all other entries are 0.

Define a matrix $\mathcal{C}_{[+,+;-]} = \{C_{ij}\}$ of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-\mathbf{c}_p^1)(1-\mathbf{c}_q^2)E_{41}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -(1-\mathbf{c}_p^1)\mathbf{c}_q^2 E_{31}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= \mathbf{c}_p^1(1-\mathbf{c}_q^2)E_{42}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= \mathbf{c}_p^1\mathbf{c}_q^2 E_{32}, \end{aligned}$$

for each $0 \le p \le m_1+1$ and $0 \le q \le m_2+1$, but all other entries are 0. Then Proposition 3.2 reads as the following proposition.

Proposition 3.4. *Let $\mathcal{C}_{[e_1,e_2;-]}, \mathbf{F}_\tau$ be as above. Then for each pair e_1, e_2 the simple K-module $V_{[e_1,e_2;-]}$ is generated by the entries of the matrix $\mathcal{C}_{[e_1,e_2;-]}{}^{\mathbf{t}}\mathbf{F}_\tau$. Moreover, these entries make a set of canonical basis.*

Proof. The proof is similar to that of Proposition 3.3. □

3.3 The Dirac-Schmid operators. In this subsection we discuss the main result of this paper, that is, to compute the matrix forms of intertwining constants explicitly.

$\mathfrak{p}_+$**-side.** Note that the homomorphisms $[\gamma]$ with $\gamma \in I(\pi, \tau_m)$ defined in the section 2 form a basis of the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and hence we fix this basis for each τ_m in π. Take an element $i \in \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$, then the $(\mathfrak{g}, K)$-module property of H_π^K gives us the canonical surjective K-homomorphism

$$\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \to \mathfrak{p}_+ \mathrm{Im}(\tau_m).$$

For the K-module $\tau_{[e_1,e_2;+]}$, by composing this K-homomorphism with the injection $\tau_{[e_1,e_2;+]} \subset \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$, we obtain a $\mathbb{C}$-linear map ϕ

$$\phi : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \to \mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]})),$$

which is determining the action of $\mathfrak{p}_+$ on H_π^K.

Our goal is to determine the matrix representation $\Gamma_{[e_1,e_2;+]}$ of ϕ *i.e.*, to find a matrix $\Gamma_{[e_1,e_2;+]}$ such that

$$\phi\Big(\sum_{\gamma \in I(\pi,\tau_m)} [\gamma] \Big) = \Big(\sum_{\gamma' \in I(\pi,\tau_{m'})} [\gamma'] \Big) \times \Gamma_{[e_1,e_2;+]},$$

where $m' = [e_1, e_2; +]$. Therefore we have to compute the image (under ϕ) of the K-isomorphism $[\gamma] : \tau_m \to W_\gamma^{(m)}$ for each $\gamma \in I(\pi, \tau_m)$, that is, to express the K-homomorphism ϕ_γ in the commutative diagram

$$\begin{array}{ccc} \tau_{[e_1,e_2;+]} & \longrightarrow & \mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \\ & \searrow^{\phi_\gamma} & \downarrow [\gamma] \\ & & \mathfrak{p}_+ W_\gamma^{(m)} \longrightarrow H_\pi(\tau_{[e_1,e_2;+]}) \end{array}$$

Diagram 1.

in terms of the fixed basis $[\gamma']$ with $\gamma' \in I(\pi, \tau_{[e_1,e_2;+]})$.

Set $\nu = (m_1 + m_2 + s)/2$. For each τ_m, we regard the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ as a subspace of the $\nu + 1$-dimensional vector space $\mathrm{Hom}_K(\tau_m, \oplus_\gamma W_\gamma^{(m)})$ with γ running over all nonnegative positive integer pairs (t_1, t_2) such that $t_1 + t_2 = \nu$ and hence define $\Gamma_{[e_1,e_2;+]}$ as a matrix of size $(\nu + 1 + (e_1 + e_2)/2) \times (\nu + 1)$.

Remark 3.1. *For fixed* e_1, e_2, *we remark that* $\Gamma_{[e_1,e_2;\pm]}$ *is a matrix of size* $I(\pi, \tau_{[e_1,e_2;\pm]}) \times I(\pi, \tau)$ *but is represented here as an embedded one inside of a matrix of size* $(\nu+1+(e_1+e_2)/2) \times (\nu+1)$. *Note that the explicit formula of* $m(\pi, \tau_{[e_1,e_2;\pm]})$ *seems to be involved.*

Fix a K-module τ_m with $m = [m_1, m_2; l]$. Set $r = (s + l)/2$ and $m' = [m_1 + e_1, m_2 + e_2; l + 2]$. In the following list, we use the coefficients $\mathbf{c}_p^1$ and $\mathbf{c}_q^2$ defined in subsection 3.1.

1. Define a matrix $\Gamma_{[-,-;+]} = \{a_{ij}\}_{0 \le i \le \nu-1, 0 \le j \le \nu}$ of size $\nu \times (\nu + 1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t-1,t} &= a_t && \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t - 1, \nu - t) \in I(\pi, \tau_{m'}), \\ a_{t,t} &= b_t && \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t - 1) \in I(\pi, \tau_{m'}), \end{aligned}$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t),$$
$$b_t = -\frac{1}{2}(\mu_1 - 1 - m_2 + r - 2t)$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

2. Define a matrix $\Gamma_{[+,+;+]} = \{a_{ij}\}_{0 \le i \le \nu+1, 0 \le j \le \nu}$ of size $(\nu + 2) \times (\nu + 1)$ so that its all non zero entries are given by

$$a_{t,t} = a_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t + 1) \in I(\pi, \tau_{m'}),$$
$$a_{t+1,t} = b_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t + 1, \nu - t) \in I(\pi, \tau_{m'}),$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}_t^1)\mathbf{c}_{\nu-t+1}^2,$$
$$b_t = -\frac{1}{2}(\mu_1 + 3 + 2m_1 + m_2 + r - 2t)\mathbf{c}_{t+1}^1(1 - \mathbf{c}_{\nu-t}^2)$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

3. Define a square matrix $\Gamma_{[-,+;+]} = \{a_{ij}\}_{0 \le i \le \nu, 0 \le j \le \nu}$ of size $(\nu+1) \times (\nu+1)$ so that its all non zero entries are given by

$$a_{t-1,t} = a_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t - 1, \nu - t + 1) \in I(\pi, \tau_{m'})$$
$$a_{t,t} = b_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}),$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)\mathbf{c}_{\nu-t+1}^2,$$
$$b_t = \frac{1}{2}(\mu_1 + 1 + m_2 + r - 2t)(1 - \mathbf{c}_{\nu-t}^2)$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

4. Define a square matrix $\Gamma_{[+,-;+]} = \{a_{ij}\}_{0 \le i \le \nu, 0 \le j \le \nu}$ of size $(\nu+1) \times (\nu+1)$ so that its all non zero entries are given by

$$a_{t,t} = a_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}),$$
$$a_{t+1,t} = b_t \qquad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t + 1, \nu - t - 1) \in I(\pi, \tau_{m'}),$$

where

$$a_t = \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}_t^1),$$
$$b_t = \frac{1}{2}(\mu_1 + 1 + 2m_1 - m_2 + r - 2t)\mathbf{c}_{t+1}^1$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

Our main result is these constructions of $\Gamma_{[e_1,e_2;+]}$. In the following, we show that these matrices are the desired ones.

Theorem 3.5. *Let (e_1, e_2) be a pair so that $e_1, e_2 \in \{\pm 1\}$. Then the matrix $\Gamma_{[e_1,e_2;+]}$ defined above is the $\mathbb{C}$-linear homomorphism between the vector spaces $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]}))$.*

Proof. We only consider the case $(e_1, e_2) = (-1, -1)$, because the remaining cases are proved similarly. Set $m' = [m_1 - 1, m_2 - 1; l + 2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1, the image of a fixed basis element $f_{pq}^{(m')}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f_{pq}^{(m')}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S_{\gamma',pq}^{(m')}(x).$$

Note that we omit the index (m) of basis vectors for only τ_m *i.e.*, write f_{pq} instead of $f_{pq}^{(m)}$. Consider the above expression at $x = 1_4$, by using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$, we then get

$$\phi_\gamma(f_{pq}^{(m')})(1_4) = c_{\gamma'}, \text{ if } \gamma' = (p,q).$$

On the other hand, the commutativity of the Diagram 1 and Proposition 3.1 imply that $\phi_\gamma(f_{pq}^{(m')})$ is equal to

$$E_{23}S_{\gamma,p+1q+1}(k) - E_{13}S_{\gamma,pq+1}(k) + E_{24}S_{\gamma,p+1q}(k) - E_{14}S_{\gamma,pq}(k).$$

Note that $XS_{\gamma,pq}(k)\mid_{k=1_4} = 0$ for any $X \in \mathfrak{n}$. By considering the Iwasawa decomposition of E_{ij} $(i = 1,2, j = 3,4)$ given in Lemma 1.2, one can calculate that

$$\begin{aligned}(E_{13}S_{\gamma,pq})(1_4) &= \frac{1}{2}\Big(H_1 + \frac{1}{2}(I_{2,2} + h^1 - h^2)\Big)S_{\gamma,pq}(k)\mid_{k=1_4} \\ &= \frac{1}{4}(2\mu_1 + 6 + l + (2p - m_1) - (2q - m_2))S_{\gamma,pq}(1_4), \\ (E_{24}S_{\gamma,pq})(1_4) &= \frac{1}{2}\Big((H_2 + \frac{1}{2}(I_{2,2} - h^1 + h^2)\Big)S_{\gamma,pq}(k)\mid_{k=1_4} \\ &= \frac{1}{4}\Big(2\mu_2 + 2 + l - (2p - m_1) + (2q - m_2)\Big)S_{\gamma,pq}(1_4),\end{aligned}$$

$$\begin{aligned}(E_{14}S_{\gamma,(pq)})(1_4) &= -e_+^2 S_{\gamma,pq}(k)\mid_{k=1_4} = (q - m_2)S_{\gamma,pq+1}(1_4), \\ (E_{23}S_{\gamma,pq})(1_4) &= e_-^1 S_{\gamma,pq}(k)\mid_{k=1_4} = pS_{\gamma,p-1q}(1_4).\end{aligned}$$

Combining these observations, we obtain that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\begin{aligned}&\frac{1}{2}\Big(\mu_2 + q - p + \frac{m_1 - m_2 + l}{2}\Big)S_{\gamma,p+1q}(1_4) + S_{\gamma,pq+1}(1_4)\times \\ &\Big(-\frac{1}{2}\Big(\mu_1 + 2 + p - q + \frac{m_2 - m_1 + l}{2}\Big) + p + 1 - (q - m_2)\Big)\end{aligned}$$

Using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$ again, one has

$$\gamma' \text{ is equal to } \gamma - (1,0) \text{ or } \gamma - (0,1)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\Big[\mu_2 + 1 + m_1 + \frac{s + l}{2} - 2t\Big]$$

or

$$c_{\gamma'} = -\frac{1}{2}\Big[\mu_1 - 1 - m_2 + \frac{l + s}{2} - 2t\Big],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[-,-;+]}$ with ϕ. □

$\mathfrak{p}_-$-side. By the same computation as the case $\mathfrak{p}_+$-side we obtain similar results for the matrix form of the $\mathbb{C}$-linear map

$$\Gamma_{[e_1,e_2;-]} : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \to \mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]})).$$

1. Define a matrix $\Gamma_{[-,-;-]} = \{a_{ij}\}_{0\le i\le \nu-1, 0\le j\le \nu}$ of size $\nu \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t,\nu-t-1) \in I(\pi,\tau_{m'}),\\ a_{t-1,t} &= b_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t-1,\nu-t) \in I(\pi,\tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2+1-m_1-r+2t),\\ b_t &= -\frac{1}{2}(\mu_1-1-2m_1-m_2-r+2t) \end{aligned}$$

for $\gamma = (t,\nu-t) \in I(\pi,\tau)$.

2. Define a matrix $\Gamma_{[+,+;-]} = \{a_{ij}\}_{0\le i\le \nu+1, 0\le j\le \nu}$ of size $(\nu+2) \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t+1,t} &= a_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t+1,\nu-t) \in I(\pi,\tau_{m'}),\\ a_{t,t} &= b_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t,\nu-t+1) \in I(\pi,\tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2+1-m_1-r+2t)\mathbf{c}^1_{t+1}(1-\mathbf{c}^2_{\nu-t}),\\ b_t &= -\frac{1}{2}(\mu_1+3+m_2-r+2t)(1-\mathbf{c}^1_t)\mathbf{c}^2_{\nu-t+1} \end{aligned}$$

for $\gamma = (t,\nu-t) \in I(\pi,\tau)$.

3. Define a square matrix $\Gamma_{[-,+;-]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1) \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t,\nu-t) \in I(\pi,\tau_{m'}),\\ a_{t-1,t} &= b_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t-1,\nu-t+1) \in I(\pi,\tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2+1-m_1-r+2t)(1-\mathbf{c}^2_{\nu-t}),\\ b_t &= \frac{1}{2}(\mu_1+1-2m_1+m_2-r+2t)\mathbf{c}^2_{\nu-t+1} \end{aligned}$$

for $\gamma = (t,\nu-t) \in I(\pi,\tau)$.

4. Define a square matrix $\Gamma_{[+,-;-]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1) \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t+1,t} &= a_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t+1,\nu-t-1) \in I(\pi,\tau_{m'}),\\ a_{t,t} &= b_t && \text{if } (t,\nu-t) \in I(\pi,\tau_m),\ (t,\nu-t) \in I(\pi,\tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2+1-m_1-r+2t)\mathbf{c}^1_{t+1},\\ b_t &= \frac{1}{2}(\mu_1+1-m_2-r+2t)(1-\mathbf{c}^1_t) \end{aligned}$$

for $\gamma = (t,\nu-t) \in I(\pi,\tau)$.

Thus we have the following results similar to that of $\mathfrak{p}_+$-side.

Theorem 3.6. *Let (e_1, e_2) be a pair so that $e_1, e_2 \in \{\pm 1\}$. Then the matrix $\Gamma_{[e_1,e_2;-]}$ defined above is the $\mathbb{C}$-linear homomorphism between the vector spaces $Hom_K(\tau_m, H_\pi(\tau_m))$ and $Hom_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]}))$.*

Proof. Set $m' = [m_1 + e_1, m_2 + e_2; l - 2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1 , the image of a fixed basis element $f_{pq}^{(m')}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f_{pq}^{(m')}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S_{\gamma',pq}^{(m')}(x).$$

On the other hand, the commutativity of the Diagram 1 and Proposition 3.2 imply that $\phi_\gamma(f_{pq}^{(m')})$ is equal to

$$E_{41} S_{\gamma,p+1q+1}(k) + E_{42} S_{\gamma,pq+1}(k) - E_{31} S_{\gamma,p+1q}(k) - E_{32} S_{\gamma,pq}(k).$$

Combining the fact $X S_{\gamma,pq}(k) \mid_{k=1_4} = 0$ for any $X \in \mathfrak{n}$ and the Iwasawa decomposition of E_{ji} $(i = 1, 2, j = 3, 4)$ given in Lemma 1.2, one can also calculate that

$$\begin{aligned}
(E_{31} S_{\gamma,pq})(1_4) &= \frac{1}{2}\Big(H_1 - \frac{1}{2}(I_{2,2} + h^1 - h^2)\Big) S_{\gamma,pq}(k) \mid_{k=1_4} \\
&= \frac{1}{4}(2\mu_1 + 6 - l - (2p - m_1) + (2q - m_2)) S_{\gamma,pq}(1_4), \\
(E_{42} S_{\gamma,pq})(1_4) &= \frac{1}{2}\Big(H_2 - \frac{1}{2}(I_{2,2} - h^1 + h^2)\Big) S_{\gamma,pq}(k) \mid_{k=1_4} \\
&= \frac{1}{4}\Big(2\mu_2 + 2 - l + (2p - m_1) - (2q - m_2)\Big) S_{\gamma,pq}(1_4),
\end{aligned}$$

$$\begin{aligned}
(E_{32} S_{\gamma,pq})(1_4) &= -e_+^1 S_{\gamma,pq}(k) \mid_{k=1_4} = (p - m_1) S_{\gamma,p+1q}(1_4), \\
(E_{41} S_{\gamma,pq})(1_4) &= e_-^2 S_{\gamma,pq}(k) \mid_{k=1_4} = (q + a_2) S_{\gamma,pq-1}(1_4).
\end{aligned}$$

It follows that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\begin{aligned}
\Big(-\frac{1}{2}\Big(\mu_1 + q - p - \frac{m_2 - m_1 + l}{2}\Big) + q + m_1 - p\Big) S_{\gamma,p+1q}(1_4) \\
- \frac{1}{2}\Big(\mu_2 + p - q - \frac{m_1 - m_2 + l}{2}\Big) S_{\gamma,pq+1}(1_4).
\end{aligned}$$

As seen in the previous theorem

$$\gamma' \text{ is equal to } \gamma - (0,1) \text{ or } \gamma - (1,0)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\Big[\mu_2 + 1 - m_1 - r + 2t\Big]$$

or

$$c_{\gamma'} = -\frac{1}{2}\Big[\mu_1 - 1 - 2m_1 - m_2 - r + 2t\Big],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[e_1,e_2;-]}$ with ϕ. □

3.4. **Matrix representations.** We now describe the relations between the matrices $\mathcal{C}_{[e_1,e_2;\pm]}$ and $\Gamma_{[e_1,e_2;\pm]}$ in terms of the marked elementary basis functions in the K-isotypic component of π. Fix τ_m with $m = [m_1, m_2; l]$. For a pair (i, j) such that $i + j = \nu$ and $i, j \in \mathbb{Z}_+$, we define a row matrix $\mathbf{F}^{(m)}_{(i,j)}$ of size $1 \times (m_1 + 1)(m_2 + 1)$ with entries in the set of all marked elementary functions of $W^{(m)}_{ij}$ introduced in Definition 2.1 as follows

$$\mathbf{F}^{(m)}_{\gamma} = (S_{\gamma,00}, S_{\gamma,01}..., S_{\gamma,0m_2}, S_{\gamma,10}, S_{\gamma,11}, ..., S_{\gamma,m_1(m_2-1)}, S_{\gamma,m_1m_2})$$

with $\gamma = (i, j)$. To the K-isotypic component of τ_m in π we associate a matrix $\mathbf{S}^{(m)}$ of size $(m_1 + 1)(m_2 + 1) \times (\nu + 1)$ such that the non zero columns are those ${}^{\mathbf{t}}\mathbf{F}^{(m)}_{\gamma}$ with entries in the K-isotypic component $H_\pi(\tau_m)$, that is,

$$\mathbf{S}^{(m)} = [{}^{\mathbf{t}}\mathbf{F}^{(m)}_{(0,\nu)}, ..., {}^{\mathbf{t}}\mathbf{F}^{(m)}_{(\nu,0)}],$$

where the symbol ^{t} is the transpose and $\mathbf{F}^{(m)}_{\gamma} = \mathbf{0}$ when $\gamma \notin I(\pi, \tau_m)$.

Now we are in a position to state the main result which includes all results in this paper.

Theorem 3.8. *Let $\tau_{[e_1,e_2;\pm]}$ be a simple K-submodule of the K-module $\mathfrak{p}_\pm \otimes_{\mathbb{C}} \tau_m$ for a given simple K-module τ_m and the K-module* $(\mathrm{Ad}, \mathfrak{p}_\pm)$. *Then we have that*

$$\mathcal{C}_{[e_1,e_2;\pm]}\mathbf{S}^{(m)} = \mathbf{S}^{([e_1,e_2;\pm])}\Gamma_{[e_1,e_2;\pm]},$$

where the product of the entries of matrices of the left hand side is the differential operation.

3.5. **Examples of contiguous relations and their composites.** Here are some examples of contiguous relations along the multiplicity one K-types in a given principal series representation π. We refer the reader to [5] for further reference and contiguous relations.

Let $\tau = \tau_{[m_1,m_2;l]}$ be a K-submodule of $\pi = \mathrm{Ind}^G_P(\sigma_{s,e} \otimes e^{\mu+\rho} \otimes 1_N)$. Then Lemma 2.2 implies that $[\pi\mid_K: \tau] = 1$ if and only if

$$\mid s \mid = m_1 + m_2 \text{ and } l = 2m_2 + s + 1 - e(-1)\ (mod\ 4).$$

Hence, in this case, we may assume that the size of the matrices $\Gamma_{[+,-;\pm]}, \Gamma_{[+,-;\pm]}$ are just 1×1 *i, e.*, they are constants and $\Gamma_{[+,+;\pm]}$ is of size 2×1, because the other entries are zero. Although there is no $\Gamma_{[-,-;\pm]}$, since $\tau_{[-,-;\pm]}$ does not occur in π.

Note that $H_\pi(\tau) \cong W^{(m)}_{(m_1,m_2)}$ if $s \geq 0$ and $H_\pi(\tau) \cong W^{(m)}_{(0,0)}$ if $s \leq 0$. Put

$$\nu_1 = \frac{l + m_1 - m_2}{2} \quad \text{and} \quad \nu_2 = \frac{l + m_2 - m_1}{2}.$$

Formula 3.9. *Assume $s \geq 0$. Then we have*

$$\mathcal{C}_{[+,-;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} = \frac{1}{2}(\mu_1 + 1 + \nu_1){}^{\mathbf{t}}\mathbf{F}^{\tau[+,-;+]}_{(+,-)},$$
$$\mathcal{C}_{[-,+;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} = \frac{1}{2}(\mu_2 + 1 + \nu_2){}^{\mathbf{t}}\mathbf{F}^{\tau[-,+;+]}_{(-,+)},$$
$$\mathcal{C}_{[+,-;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} = \frac{1}{2}(\mu_2 + 1 - \nu_2){}^{\mathbf{t}}\mathbf{F}^{\tau[+,-;-]}_{(+,-)},$$
$$\mathcal{C}_{[-,+;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{(m_1,m_2)} = \frac{1}{2}(\mu_1 + 1 - \nu_1){}^{\mathbf{t}}\mathbf{F}^{\tau[-,+;-]}_{(-,+)}.$$

Here the symbol $(\pm, \pm)$ means $(m_1 \pm 1, m_2 \pm 1)$, respectively.

Formula 3.10. *Assume $s \leq 0$ and set $n = (0,0)$. Then we have*

$$\mathcal{C}_{[+,-;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} = \frac{1}{2}(\mu_2 + 1 + \nu_1){}^{\mathbf{t}}\mathbf{F}^{\tau[+,-;+]}_{n},$$
$$\mathcal{C}_{[-,+;+]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} = \frac{1}{2}(\mu_1 + 1 + \nu_2){}^{\mathbf{t}}\mathbf{F}^{\tau[-,+;+]}_{n},$$
$$\mathcal{C}_{[+,-;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} = \frac{1}{2}(\mu_1 + 1 - \nu_2){}^{\mathbf{t}}\mathbf{F}^{\tau[+,-;-]}_{n},$$
$$\mathcal{C}_{[-,+;-]}{}^{\mathbf{t}}\mathbf{F}^{\tau}_{n} = \frac{1}{2}(\mu_2 + 1 - \nu_1){}^{\mathbf{t}}\mathbf{F}^{\tau[-,+;-]}_{n}.$$

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PART 2

EXPLICIT EVALUATION OF CERTAIN JACQUET INTEGRALS ON $SU(2,2)$

EXPLICIT EVALUATION OF CERTAIN JACQUET INTEGRALS ON $SU(2,2)$

G. BAYARMAGNAI

ABSTRACT. We give explicit formulas for certain Jacquet integrals on some standard principal series representations of the group $SU(2,2)$.

Introduction. The main object of this paper is to obtain explicit integral expressions of some Whittaker functions on $G = SU(2,2)$. More specifically we evaluate the Jacquet integrals with certain K-types belonging to a principal series representation, parabolically induced by the minimal parabolic subgroup of G.

The Whittaker models are one of the main ingredients in the theory of Fourier expansions of automorphic forms at some cusps. In this sense, explicit knowledge of Whittaker functions is very important for deeper studies of automorphic forms.

H. Jacquet [6] introduced a functional on the space of differentiable vectors in a given representation π of G which defines an intertwiner from its representation space to the space of smooth functions f on G satisfying $f(ng) = \eta(n)f(g)$ for all $(n,g) \in N \times G$, where η is a unitary character of the standard maximal unipotent subgroup N of G. The image of this intertwiner is a Whittaker model of π. The local multiplicity one theorem of Shalika [12] at the archimedean place implies the uniqueness of such kind of functionals when the representation π is irreducible admissible. Note also that Wallach [15, §8] reformulated this result in a slightly different but useful manner, i.e., in terms of "moderate growth condition". When π is given by a standard model on $L^2(K)$, the unique functional is realized by the Jacquet integral. We want to compute for special vectors in $L^2(K)$.

Our method of evaluation of Jacquet integral is based on that of Proskurin [11], similarly as in Ishii [5]. Main results of the paper, described in Theorems 3.2, 3.3 and 4.2, show that the Whittaker function corresponding to certain K-type of π is expressed in terms of the modified Bessel function and hence we obtain its Mellin-Barnes integral representation. Since the restricted root system of $SU(2,2)$ is the same type as that of $Sp(2,\mathbb{R})$ except for

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multiplicities, our results resemble to those of [5]. But because our group is non-split, it is much more involved from technical viewpoints.

In this paper we discuss only "very small" K-types in some standard principal series representations of G. But combined with results of the other paper [2], we can expect to handle other K-types in the same representation.

We want to refer to the meaning in physics of the group $SU(2,2)$ which is locally isomorphic to the conformal group $SO(4,2)$: this group was the group of symmetry of massless free particles [16]; also the Lie algebra $\mathfrak{su}(2,2)$ was the spectrum generating algebra of the hydrogen atom. Related to these topics, there is a very general result on the minimal representation of $O(p,q)$ by Kobayashi-Ørsted [7].

However the group $SO(4,2)$ now becomes fundamental in the conjecture of AdS/CFT correspondence [1]. Though the situation is not clear, our result is very rare results on special functions in *"two variables"* related to spherical functions on $SO(4,2)$ in the literature. So this might bring some new aspects which were not found in the case of the minimal representations.

For other Lie groups, there are related works by Bump [3] on $GL(3)$, Stade [13] on $GL(n)$ and Vinogradov and L. Tahtajan [14] on $SL(3)$.

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1. Basic notions

1.1. **The group $SU(2,2)$.** Let G denote the special unitary group of signature (2,2) and K be a maximal compact subgroup of G given by the fixed part $K = G^\theta$ of the Cartan involution $\theta(g) =^t \bar{g}^{-1}, g \in G$:

$$K = S(U(2)\times U(2)) = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : \ k_1, k_2 \in U(2), \det(k_1k_2) = 1 \right\}.$$

The associated Lie algebras are

$$\mathfrak{g} = \mathfrak{su}(2,2) = \{X \in M_4(\mathbb{C}) \mid I_{2,2}X +^t \bar{X} I_{2,2} = 0,\ Tr(X) = 0\},$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} : -^t\bar{X}_i = X_i \in M_2(\mathbb{C}), i = 1,2 \right\}.$$

Denoting by $\mathfrak{p}$ the (-1)-eigenspace of the differential of θ, we have a Cartan (symmetric) decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Let $H_i = E_{i,2+i} + E_{2+i,i} (i = 1, 2)$, where E_{ij} is the matrix unit with 1 in the (i, j)-entry and zero elsewhere. A subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ spanned by H_1, H_2 over $\mathbb{R}$ is maximally abelian and any element $a = a(t_1, t_2)$ of its Lie group $A = \exp(\mathfrak{a})$ can be expressed as

$$\exp(t_1 H_1 + t_2 H_2) = \sum_{i=1}^{2} \Big\{ \cosh(t_i)(E_{ii} + E_{i+2i+2}) + \sinh(t_i)(E_{ii+2} + E_{i+2i}) \Big\}$$

with $t_1, t_2 \in \mathbb{R}$.

Let $\{\lambda_1, \lambda_2\}$ be a basis of the dual space $\mathfrak{a}^*$ such that $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system $\Phi(\mathfrak{g}, \mathfrak{a})$ is of type C_2, namely

$$\Phi = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

Choose $\lambda_1 - \lambda_2$ and $2\lambda_2$ as simple roots of $\Phi(\mathfrak{g}, \mathfrak{a})$. Put

$$E_0 = \kappa^{-1}(E_{12} - E_{43})\kappa, \quad E_1 = i\kappa^{-1}(E_{12} + E_{43})\kappa, \quad E_2 = \kappa^{-1}E_{24}\kappa,$$
$$F_0 = \kappa^{-1}(E_{14} + E_{23})\kappa, \quad F_1 = i\kappa^{-1}(E_{14} - E_{23})\kappa, \quad F_2 = \kappa^{-1}E_{13}\kappa,$$

with $i = \sqrt{-1}$ and $\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_2 & 1_2 \\ -i1_2 & i1_2 \end{pmatrix}$.

Then the corresponding root spaces of positive roots in $\Phi(\mathfrak{g}, \mathfrak{a})$ are given by

$$\mathfrak{g}_{\lambda_1-\lambda_2} = E_0 \cdot \mathbb{R} \oplus E_1 \cdot \mathbb{R}, \qquad \mathfrak{g}_{2\lambda_2} = E_2 \cdot \mathbb{R},$$
$$\mathfrak{g}_{\lambda_1+\lambda_2} = F_0 \cdot \mathbb{R} \oplus F_1 \cdot \mathbb{R}, \qquad \mathfrak{g}_{2\lambda_1} = F_2 \cdot \mathbb{R}.$$

Let $\mathfrak{n}$ be a subalgebra defined by $\mathfrak{n} = \sum_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$. We now describe elements of a maximal unipotent subgroup N of G given by $N = \exp(\mathfrak{n})$.

Lemma 1.1. *Let E_i, F_i be as above and set $X = x_0 E_0 + y_0 E_1$ and $Y = x_2 F_0 + y_2 F_1 + x_1 F_2 + x_3 E_2$ for $x_i, y_j \in \mathbb{R}$ $(i = 0, 1, 2, 3,\ j = 0, 2)$. Then*

$$\exp(X + Y) = \exp(X) \exp(Y - \frac{1}{2}[X, Y] - \frac{1}{3} XYX).$$

Proof. To see this, it suffices to verify relations $X^2 = Y^2 = YXY = 0$. □

The Killing form $B(X, Y) = tr(\mathrm{ad}X \cdot \mathrm{ad}Y)$, $(X, Y \in \mathfrak{g})$ and Cartan involution θ of $\mathfrak{g}$ induce an inner product $\langle , \rangle$ of $\mathfrak{g}$ via

$$\langle X, Y \rangle = -B(X, Y^\theta), \ (X, Y \in \mathfrak{g}).$$

Then one has that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ if $\alpha \neq \beta$, because of the involution θ.

Lemma 1.2. *The vectors E_i, F_i $(i = 0, 1, 2)$ of the subspace $\mathfrak{n}$ of $\mathfrak{g}$ defined above are an orthogonal basis of $\mathfrak{n}$ with respect to the inner product $\langle , \rangle$.*

Proof. For the orthogonality of the basis of $\mathfrak{n}$, it suffices to show that

$$\langle E_0, E_1\rangle = \langle F_0, F_1\rangle = 0,$$

Recall that $\mathrm{ad}E_0 \cdot \mathrm{ad}E_1^\theta$ sends the subspace $\mathfrak{g}_\lambda$ ($\lambda \in \Phi(\mathfrak{g}, \mathfrak{a})$) into itself. By setting $A = -\mathrm{ad}E_0 \cdot \mathrm{ad}E_1^\theta$, we give the list of all non zero restrictions of A to the subspaces $\mathfrak{g}_\lambda$ of $\mathfrak{g}$:

$$A|_{\mathfrak{g}_{\lambda_1+\lambda_2}} = A|_{\mathfrak{g}_{-\lambda_1-\lambda_2}} = \frac{1}{2^3}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \; A|_{\mathfrak{a}+\mathfrak{m}} = \frac{1}{2^4}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}.$$

Hence $tr(\mathrm{ad}E_0 \cdot \mathrm{ad}E_1^\theta) = 0$ which follows that E_0 and E_1 are orthogonal. Similarly F_0 is orthogonal to F_1. □

We may regard $\mathfrak{n}$ as the vector space $\mathbb{R}^6$. Define a map $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$\phi(x) = (x_1, x_2, x_3 - \frac{x_1x_4 + x_2x_5}{2} + \frac{(x_1^2 + x_2^2)x_6}{3}, x_4 - x_1x_6, x_5 - x_2x_6, x_6)$$

for $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$.

Then ϕ is a diffeomorphism and its Jacobian determinant is 1. We now denote i-th coordinate function of ϕ by ϕ_i for $1 \le i \le 6$ and put

$$n_0 = \phi_1(x) + \sqrt{-1}\phi_2(x), \qquad n_1 = \phi_3(x),$$
$$n_2 = \phi_4(x) + \sqrt{-1}\phi_5(x), \qquad n_3 = \phi_6(x).$$

Then the maximal unipotent group N of G can be written:

$$N = \left\{ \kappa^{-1}\left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\bar{n}_0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right)\kappa \;\middle|\; \begin{array}{c} n_1, n_3 \in \mathbb{R}, \\ n_0, n_2 \in \mathbb{C} \end{array} \right\}.$$

Since

$$\mathfrak{g}_{\lambda_1+\lambda_2} = [\mathfrak{g}_{\lambda_1-\lambda_2}, \mathfrak{g}_{2\lambda_2}] \text{ and } \mathfrak{g}_{2\lambda_1} = [\mathfrak{g}_{\lambda_1-\lambda_2}, \mathfrak{g}_{\lambda_1+\lambda_2}],$$

any character η of N is uniquely determined by the values of $E_i (i = 0, 1, 2)$. Put

$$c_0 = \sqrt{-1}\eta(E_0), \; c_1 = \sqrt{-1}\eta(E_1) \text{ and } c_2 = \sqrt{-1}\eta(\mathrm{E}_2)$$

with $c_0, c_1, c_2 \in \mathbb{C}$. Then these numbers are real when η is unitary and therefore such η is given by

$$\eta(n) = \exp(2\sqrt{-1}(\mathrm{Re}(\bar{c}n_0) + c_2n_3)), \quad n = n(n_0, n_1, n_2, n_3) \in N$$

for a real number c_2 and $c = c_0 + \sqrt{-1}c_1 \in \mathbb{C}$.

Conventions. We say that the character η of N is nondegenerate if both $c_0^2 + c_1^2$ and c_2 are non-zero. Throughout this paper, we shall fix a nondegenerate character η of N.

1.2. **Principal series representations.** Let P be a minimal parabolic subgroup of G with Langlands decomposition $P = MAN$ with $M = Z_A(K)$. In particularly, the subgroup M of P is given by

$$M = \{[e^{\sqrt{-1}\theta}]\gamma^j \mid \theta \in \mathbb{R}, j \in \{0,1\}\}$$

where $\gamma = \mathrm{diag}(1,-1,1,-1) \in G$ and

$$[e^{\sqrt{-1}\theta}] = \mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}).$$

For a pair $n \in \mathbb{Z}$ and a character ε of the group $\mu_2 = \{\pm 1\}$, we define a unitary character of M as

$$\sigma_{n,\varepsilon}([e^{\sqrt{-1}\theta}]\gamma^j) = \varepsilon(-1)^j e^{\sqrt{-1}n\theta}.$$

Denote by ρ the half sum of the positive restricted roots, i.e., $\rho = 3\lambda_1 + \lambda_2$, and define a quasi-character $e^{\nu+\rho}$ of A:

$$e^{\nu+\rho}(a) = e^{(\nu+\rho)\log(a)} \qquad (\nu = (\nu_1,\nu_2) \in (\mathfrak{a}_{\mathbb{C}})^*).$$

We extend it to a character of AN so that the restriction to N is trivial. Define an admissible character of P by tensoring these characters of M and AN. Then we get the induced representation (π, H_π) usually denoted by $\pi = \mathrm{ind}_P^G(\sigma_{n,\varepsilon} \otimes e^{\nu+\rho} \otimes 1_N)$ and called the *principal series representation* of G. Set $e = (1-\varepsilon(-1))/2$.

Definition 1.3. *Let u be integer. Then we define a K-module structure τ_u on $V_u \cong \mathbb{C}$ by the action*

$$\tau_u(k)v = det(k_2)^u v, \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K, v \in \mathbb{C}$$

and denote by f_u the canonical generator of V_u.

Lemma 1.4. *Let $\pi = \mathrm{ind}_P^G(\sigma_{0,\varepsilon} \otimes e^{\nu+\rho} \otimes 1_N)$ and τ_u be as above. Then τ_u is a K-submodule of $\pi\mid_K$ if and only if $u \equiv e \pmod 2$. In this case τ_u occurs exactly once.*

Proof. By Frobenius reciprocity we have that $[\pi\mid_K : \tau_u] = [\tau_u \mid_M : \sigma_{0,\varepsilon}]$. Hence the multiplicity is at most one. By considering the action of M on V_u we get the assumption on u as required. □

Assumption. When we consider the principal series representation $\pi = \mathrm{ind}_P^G(\sigma_{0,\varepsilon} \otimes e^{\mu+\rho} \otimes 1_N)$, throughout this paper, we assume that

$$\nu_1 + 1 + e, \nu_2 + 1 + e \text{ and } \nu_1 \pm \nu_2 \text{ are not integers.}$$

1.3. **The Jacquet integral.** Let $\sigma = \sigma_{n,e}$. By definition the principal series representation π of G can be realized on the Hilbert space

$$L^2_\sigma(K) = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k), m \in M, k \in K \ (a.e)\}$$

with G-action defined by

$$(\pi(g)f)(k) = a(kg)^{\nu+\rho} f(k(kg)), \ k \in K, g \in G,$$

where $kg = n(kg)a(kg)k(kg)$ is the Iwasawa decomposition of the element kg.

In [6] Jacquet defined the continuous functional $J_{\sigma,\nu}$ on the space of differentiable functions of $L^2_\sigma(K)$ satisfying $J_{\sigma,\nu}(\pi(n)f) = \eta(n)J_{\sigma,\nu}(f)$, that is

$$J_{\sigma,\nu}(f) = \int_N \eta(n)^{-1} a(s^*n)^{\nu+\rho} f(k(s^*n))dn$$

for a differentiable function f in $L^2_\sigma(K)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_K(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K, by M and s^* is an element of M^* mapping to the longest element $s \in W(A)$.

Multiplicity one theorem tells that there is at most one intertwiner (up to constant) from the space of K-finite vectors of π into the subspace $A_\eta(N\backslash G)$ of moderate growth functions [15, 8.1] in $C^\infty_\eta(N \setminus G)$. If exist, then the construction is as follows: for each differentiable $f \in L^2_\sigma(K)$ it associates a function $J_f(g)$ in $C^\infty_\eta(N \setminus G)$ defined by

$$J_f(g) = J_{\sigma,\nu}(\pi(g)f), \ (g \in G).$$

These $J_f(g)$ functions are of moderate growth on G, and in particular so on the subgroup A. We want to have an explicit formula for the A-radial part of $J_f(g)$ with f belongs to a special K-type τ in π.

2. Preliminaries

2.1. **Classical formulas.** In this section we collect some classical formulas and their combinations which is used in our evaluation. Let $K_\nu(z)$ be the Bessel function defined for $\nu, z \in \mathbb{C}$, by the integral

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \exp\Big(-(t+t^{-1})\frac{z}{2}\Big)t^\nu \frac{dt}{t}. \tag{1}$$

Our object is to evaluate the integral $J_{f_u}(g)$, further denote it by $J_u(g)$, in terms of the modified Bessel functions of the second order $K_\nu(z)$ when $u = 0, \pm1, \pm2$.

We recall the Euler integral of the second kind in the form

$$\Gamma(\nu) = c^\nu \int_0^\infty \exp(-ct)t^\nu \frac{dt}{t} \tag{2}$$

for $c \in \mathbb{R}_{>0}$ and $\mathrm{Re}(\nu) > 0$.

For $a, b, c, \in \mathbb{R}^*$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and $n \in \mathbb{N}$, we set

$$F^{(n)}_{(a,b)} = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{a}\right) \int_{\mathbb{R}} x^n \exp(-ax^2 + 2\sqrt{-1}bx)dx,$$

and

$$G^{(n)}_{(a,b,c)} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\exp(-c(x^2+y^2) - a(\alpha x + \beta y)^2 + 2\sqrt{-1}b(\alpha x + \beta y))}{(\alpha x + \beta y)^{-n}} dxdy.$$

We need the following formulas.

Proposition 2.1. *Let $a, c \in \mathbb{R}^*_+$ and $b \in \mathbb{R}$. Then*

$$(3) \quad F^{(0)}_{(a,b)} = 1, \qquad F^{(1)}_{(a,b)} = \frac{b}{a}\sqrt{-1}, \qquad F^{(2)}_{(a,b)} = \frac{a - 2b^2}{2a^2},$$

$$(4) \quad G^{(0)}_{(a,b,c)} = \frac{\pi \exp(\frac{-b^2}{a+c})}{(c^2+ac)^{\frac{1}{2}}}, \quad \frac{G^{(1)}_{(a,b,c)}}{G^{(0)}_{(a,b,c)}} = \frac{b\sqrt{-1}}{a+c}, \quad \frac{G^{(2)}_{(a,b,c)}}{G^{(0)}_{(a,b,c)}} = \frac{a+c-2b^2}{2(a+c)^2}.$$

Proof. By [3.4.11], we have that

$$\int_{\mathbb{R}} \exp(-ax^2 + 2\sqrt{-1}bx)dx = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(-\frac{b^2}{a}\right).$$

Then (3) can be verified by applying the operators $\partial/\partial a$ and $\partial/\partial b$ to both sides of the above formula. The first formula in (4) follows from the first one in (3) and using a similar argument as above, we can derive other formulas. □

2.2. **The first modification of the radial part of Jacquet integrals.** For our purposes, it will be enough to consider the A-radial part of the Jacquet integral because of the Iwasawa decomposition.

We put $a_i = \exp(t_i)$ for the element $a = a(t_1, t_2)$ of the $\mathbb{R}$-split torus A. For a fixed pair $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, by definition of the character $e^{\nu+\rho}$, one has

$$\begin{aligned} e^{\nu+\rho}(a) =& (\cosh(t_1) + \sinh(t_1))^{\nu_1+3}(\cosh(t_2) + \sinh(t_2))^{\nu_2+1} \\ =& a_1^{\nu_1+3} a_2^{\nu_2+1}. \end{aligned}$$

In our case $s^* = I_{2,2}$ and hence by setting $a(s^{-1}n) = a(t_1', t_2')$, one can see that

$$a_1' = 1/\sqrt{\Delta_1} \text{ and } a_2' = \sqrt{\Delta_1/\Delta_2},$$

where $a_i' = \exp(t_i')$, $(i = 1, 2)$. Here the Δ_1, Δ_2 are as follows:

$$\Delta_1 = 1 + n_1^2 + \bar{n}_2 n_2 + (\bar{n}_0 n_2 + n_0 \bar{n}_2)(n_1 + n_3) + \bar{n}_0 n_0 (1 + \bar{n}_2 n_2 + n_3^2),$$

$$\Delta_2 = 1 + n_1^2 + 2n_2\bar{n}_2 + n_3^2 + (n_1 n_3 - n_2 \bar{n}_2)^2$$

for $n = n(n_0, n_1, n_2, n_3) \in N$.

For convenience we shall rewrite Δ_1 in terms of Δ_2 and Δ_3, where Δ_3 denotes the sum $1 + n_2\bar{n}_2 + n_3^2$.

Lemma 2.2. *Put $n_i = x_i + \sqrt{-1}y_i$ with $x_i, y_i \in \mathbb{R}$ $(i = 0, 2)$. Then we have the following identities for Δ_1 and Δ_2:*

$$\Delta_1\Delta_3 = (X_0^2 + Y_0^2)\Delta_3^2 + \Delta_2$$

with $(X_0, Y_0) = \left(x_0 + \dfrac{n_1 + n_3}{\Delta_3}x_2, y_0 + \dfrac{n_1 + n_3}{\Delta_3}y_2\right)$.

$$(1 + n_3^2)\Delta_2 = (1 + N_1^2)\Delta_3^2, \text{ with } N_1 = \frac{(1 + n_3^2)n_1 - n_2\bar{n}_2n_3}{\Delta_3}.$$

Proof. (a). To prove this part, by direct computation, one can see that

$$\Delta_2 = (1 + n_1^2 + n_2\bar{n}_2)\Delta_3 - (n_1 + n_3)^2 n_2\bar{n}_2,$$

and hence (a) is immediate.

(b). It is straightforward to check that $\sqrt{\Delta_2}$ is the complex norm of

$$(1 - n_1n_3 + n_2\bar{n}_2) + \sqrt{-1}(n_1 + n_3).$$

The lemma follows. □

For an integer u, define a function $f_u(k)$ on K by

$$f_u(k) := \det(k_2), \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K.$$

Lemma 2.3. *The function $f_u(k)$ belongs to $L^2_{\sigma(0,e)}(K)$. In particular, we have*

$$f_u(k(I_{2,2}n)) = \left(\frac{1 - n_1n_3 + n_2\bar{n}_2 + \sqrt{-1}(n_1 + n_3)}{1 - n_1n_3 + n_2\bar{n}_2 - \sqrt{-1}(n_1 + n_3)}\right)^{\frac{u}{2}}$$

for $n = (n_0, n_1, n_2, n_3) \in N$.

Proof. For the factor $k(I_{2,2}n)$ of the Iwasawa decomposition of $I_{2,2}n$ with $n \in N$, there are $k_1, k_2 \in U(2)$ such that $k(I_{2,2}n) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K$. Put $N_1 = \begin{pmatrix} n_1 & n_2 \\ \bar{n}_2 & n_3 \end{pmatrix}$ for $n = n(n_0, n_1, n_2, n_3) \in N$. One can see that

$$\frac{\det(k_1)}{\det(k_2)} = \frac{\det(1 - \sqrt{-1}N_1)}{\det(1 + \sqrt{-1}\mathrm{N}_1)}.$$

Since $\det(k_1)\det(k_2) = 1$, the function f_u has the required expression. □

Note that the K-submodule in $L^2_{\sigma_{(0,e)}}(K)$ generated by $f_u(k)$ is isomorphic to V_u when u satisfying the condition in Lemma 1.4. By setting $J_u := J_{f_u}$ for Jacquet function J_{f_u}, the function $J_u(a)$ on A is given by the integral expression

$$a^{\rho-\nu}\int_N a(I_{2,2}n)^{\nu+\rho}\exp(-2\sqrt{-1}\Big(\frac{a_1}{a_2}\mathrm{Re}(\bar{c}n_0)+c_2a_2^2n_3\Big))f_u(k(I_{2,2}n))dn$$

for a character η depending on $c\in\mathbb{C}$ and $c_2\in\mathbb{R}$. For future convenience, we choose a new coordinate

$$y=(y_1,y_2)=\Big(\frac{a_1}{a_2},a_2^2\Big).$$

Thus, we can summarize the following lemma.

Lemma 2.4. *The radial part of the moderate growth Whittaker function $W_{(\nu_1,\nu_2)}(y;u)=y_1^3y_2^2\tilde{W}_{(\nu_1,\nu_2)}(y;u)$ (up to constant) associated with the K-type τ_u can be written in the form*

$$\begin{aligned}\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;u)=&y_1^{-\nu_1}y_2^{-\frac{\nu_1+\nu_2}{2}}\int_N\Delta_1^{-\frac{\nu_1-\nu_2}{2}-1}\Delta_2^{-\frac{\nu_2+1}{2}}\\&\times\exp(-2\sqrt{-1}\Big(y_1\mathrm{Re}(\bar{c}n_0)+c_2y_2n_3\Big)f_u(k(I_{2,2}n))dn,\end{aligned}$$

where dn is a multiplicative Haar measure on N.

3. Explicit formulas.

In this section we consider the integral J_u when $u=0,\pm1,\pm2$. Actually the results corresponding to $u=0,\pm1$ are quite similar to that integrals on $Sp(2,\mathbb{R})$ in [5] which could be explained by the coincidence of the restricted root system of type C_2. Throughout this paper we denote by I the interval $[0,\infty)$.

Now we shall give a normalization of Haar measure of N. In the section 1, the subalgebra $\mathfrak{n}$ is regarded as $\mathbb{R}^6$ with coordinates $(\phi_i)_{1\le i\le 6}$. Let $d\phi$ be the corresponding Lebesgue measure on $\mathfrak{n}$. Since the exponential map of $\mathfrak{n}$ onto N is an analytic isomorphism, there exists a unique Haar measure dn on N that corresponds to $d\phi$.

Set $n_i=x_i+\sqrt{-1}y_i$ $(i=0,2)$. For $\mu_1,\mu_2\in\mathbb{C}$ and nondegenerated unitary character η such that $c_0^2+c_1^2=1$ and $c_2=\pm1$, let us evaluate

$$J=\int_{\mathbb{R}^6}\Delta_1^{\mu_1}\Delta_2^{\mu_2}\exp(-2\sqrt{-1}(c_0x_0A_1+c_1y_0A_1-n_3A_2))dn$$

where $dn=dx_0dy_0dn_1dx_2dy_2dn_3$ and A_1,A_2 are positive real parameters.

Lemma 3.1. *We have that the integral J defined above is equal to*

$$\frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)}\int_{\mathbb{R}^4}\int_0^\infty\int_0^\infty \frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{1+n_3^2}\exp\left(-\frac{A_1^2}{t_1}-\frac{\Delta_2}{\Delta_3^2}(t_1+t_2)\right)$$
$$\exp\left(2\sqrt{-1}\{-n_3A_2+\frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1\}\right)\Delta_3^{\mu_1+2\mu_2+1}dn$$

with $dn = dN_1dx_2dy_2dn_3\frac{dt_1}{t_1}\frac{dt_2}{t_2}$

Proof. Firstly we change the system variable from

$$(x_0,y_0,x_2,y_2,n_1,n_3) \text{ to } (X_0,Y_0,x_2,y_2,N_1,n_3).$$

Here X_0, Y_0 and N_1 are defined in Lemma 2.2. Then

$$dx_0dy_0dx_2dy_2dn_1dn_3 = \frac{\Delta_3}{1+n_3^2}dX_0dY_0dx_2dy_2dN_1dn_3.$$

Moreover

$$(c_0x_0+c_1y_0)A_1 = (c_0X_0+c_1Y_0)A_1 - \frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1.$$

We apply all these replacement for the integration of J together with the insertion of

$$(\Delta_1/\Delta_3)^{\mu_1} = \frac{1}{\Gamma(-\mu_1)}\int_0^\infty \exp(-\Delta_1t_1/\Delta_3)t_1^{-\mu_1}\frac{dt_1}{t_1}$$

which is the Euler integral of the second kind (2). Then J is equal to

$$\frac{1}{\Gamma(-\mu_1)}\int_I\int_{\mathbb{R}^6}\exp\left(-(X_0^2+Y_0^2)t_1-\frac{\Delta_2}{\Delta_3^2}t_1\right)\exp(-2\sqrt{-1}(c_0X_0+c_1Y_0)A_1)$$
$$\exp\left(2\sqrt{-1}(\pm n_3A_2+\frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1)\right)\frac{\Delta_2^{\mu_2}\Delta_3^{\mu_1+1}}{t_1^{\mu_1}(1+n_3^2)}dn$$

with $dX_0dY_0dx_2dy_2dN_1dn_3\frac{dt_1}{t_1}$.

Note here that we use the equation

$$\Delta_1/\Delta_3 = X_0^2+Y_0^2+\Delta_2/\Delta_3^2$$

in Lemma 2.2. Now we can execute the integrations with respect to the variables X_0, Y_0 applying the formula (3) with $n=0$ to get

$$J = \frac{\pi}{\Gamma(-\mu_1)}\int_{\mathbb{R}^4}\int_I \exp\left(2\sqrt{-1}\{\pm n_3A_2+\frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1\}\right)$$
$$\exp\left(-\frac{A_1^2}{t_1}-\frac{\Delta_2}{\Delta_3^2}t_1\right)\Delta_2^{\mu_2}\Delta_3^{\mu_1+1}\frac{t_1^{-\mu_1-1}}{1+n_3^2}\frac{dt_1}{t_1}dN_1dx_2dy_2dn_3\frac{dt_1}{t_1}.$$

To finish the proof we remove the factor $\Delta_2^{\mu_2}$ by applying the formula (2) again

$$\Delta_2^{\mu_2} = \frac{\Delta_3^{2\mu_2}}{\Gamma(-\mu_2)} \int_0^\infty \exp(-\Delta_2 t_2/\Delta_3^2) t_2^{-\mu_2} \frac{dt_2}{t_2}.$$

This completes the proof of our Lemma. □

3.1. The standard cases $|u| \leq 1$. In this subsection we discuss the main results of this paper. These standard cases seem to be very useful for the the Jacquet vectors corresponding to the minimal K-types of other principal series representations. Let

$$\Gamma(s_1, s_2) = \frac{\Gamma_\pm(s_1,\nu_1)\Gamma_\pm(s_1,\nu_2)\Gamma_\pm(s_2,(\nu_1+\nu_2)/2)\Gamma_\pm(s_2,(\nu_1-\nu_2)/2)}{\Gamma_\pm(s_1+s_2,\nu_1+\nu_2)\Gamma_\pm(s_1+s_2,\nu_1-\nu_2)}$$

with

$$\Gamma_\pm(s,t) := \Gamma\left(\frac{s+t}{2}\right)\Gamma\left(\frac{s-t}{2}\right)$$

for suitable $s_i, \nu_i \in \mathbb{C}, (i = 1, 2)$.

Set $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Let us begin with the case $u = 0$, i.e.,the class one case.

Theorem 3.2. *Let $\pi = \mathrm{Ind}_P^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$ be an irreducible representation. For a nondegenerated unitary character η of N we have the following assertions on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; 0) = y_1^3 y_2^2 \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$. The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ has the following integral expressions:*

1. *We have*

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0) = \int_0^\infty \int_0^\infty K_{\frac{\nu_1+\nu_2}{2}}(2\sqrt{t_2/t_1}) K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1 t_2})$$

$$\exp\left(-|c_2|y_2 t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2 + c_1^2)|c_2|\frac{y_1^2 y_2}{t_2}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

2. *The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ is identified with*

$$\left(\frac{y_1}{y_2}\right)^{\frac{\nu_2}{2}} \int_0^\infty int_0^\infty K_{\frac{\nu_1}{2}}(X) K_{\frac{\nu_2}{2}}(Y) \left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}} \left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2}{4}} \frac{dx}{x}\frac{dy}{y}$$

with $X = 2|c_2|y_2\left(\frac{(1+x)(1+y)}{xy}\right)^{\frac{1}{2}}$ and $Y = 2(c_0^2+c_1^2)^{1/2} y_1 (1+x+y)^{\frac{1}{2}}$,

3. *The Mellin-Barnes's integral expression of $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ is*

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0) = \frac{1}{(2\sqrt{-1})^2} \int_{s_1}\int_{s_2} V_{(\nu_1,\nu_2)}(s_1, s_2) y_1^{-s_1} y_2^{-s_2} ds_1 ds_2.$$

Here the paths of integrations are the vertical lines from $\alpha_i - \sqrt{-1}\infty$ to $\alpha_i + \sqrt{-1}\infty$ with real number α_i such that

$$\alpha_1 > |\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_1)|, \ \alpha_2 > |\mathrm{Re}(\nu_1+\nu_2)|/2, |\mathrm{Re}(\nu_1-\nu_2)|/2$$

and the integrand $V_{(\nu_1,\nu_2)}(s_1,s_2)$ is equal to

$$\Gamma(s_1,s_2) \times {}_3F_2\left(\begin{matrix} \frac{s_1}{2}, \frac{s_2}{2} + \frac{\nu_2-\nu_1}{4}, \frac{s_2}{2} - \frac{\nu_2-\nu_2}{4} \\ \frac{s_2+s_1}{2} + \frac{\nu_2+\nu_1}{4}, \frac{s_2+s_1}{2} - \frac{\nu_1+\nu_2}{4} \end{matrix} \middle| 1 \right).$$

Proof. In order to get the desired result we shall evaluate the integration J in Lemma 3.1 with the assumption for η, because of $f_0 = 1$ and Lemma 2.4.

Step 1. Integration for N_1.

To integrate J in the statement of Lemma 3.1 with respect to N_1, we use the expression of Δ_2 in Lemma 2.2 and apply (3) with

$$(n,a,b) = \left(0, \frac{t_1+t_2}{1+n_3^2}, \frac{c_0x_2+c_1y_2}{1+n_3^2}A_1\right).$$

Then we find that

$$J = \frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4}\int_{I^2} \exp\left(2\sqrt{-1}\{-n_3A_2 + \frac{n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1\}\right)$$
$$\exp\left(-\frac{A_1^2}{t_1} - \frac{P}{1+n_3^2} - \frac{(c_0x_0+c_1x_2)^2}{P(1+n_3^2)}A_1^2\right)\left(\frac{\pi}{P(1+n_3^2)}\right)^{\frac{1}{2}} \frac{\Delta_3^{\mu_1+2\mu_2+1}}{t_1^{\mu_1+1}t_2^{\mu_2}}dn$$

with $P = t_1 + t_2$.

Step 2. Integration for n_2.

We apply (2) with $(c,\nu) = \left(\frac{A_1^2\Delta_3}{1+n_3^2}, -\mu_1 - 2\mu_2 - 1\right)$ to rewrite $\Delta_3^{\mu_1+2\mu_2+1}$ as

$$\frac{(A_1^{-2}(1+n_3^2))^{\mu_1+2\mu_2+1}}{\Gamma(-\mu_1-2\mu_2-1)} \int_0^\infty \exp\left(-A_1^2t_3 - \frac{A_1^2t_3(x_2^2+y_2^2)}{1+n_3^2}\right)t_3^{-\mu_1-2\mu_2-1}\frac{dt_3}{t_3}.$$

Substitute this into the last expression of J and using (4) for the variables x_2 and y_2 by choosing

$$(c,a,b) = \left(\frac{A_1^2t_3}{1+n_3^2}, \frac{A_1^2}{P(1+n_3^2)}, \frac{n_3A_1}{1+n_3^2}\right).$$

Thus we can rewrite J as

$$\frac{\pi^{\frac{5}{2}}A_1^{-2\mu_1-4\mu_2-4}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1-2\mu_2-1)} \int_{\mathbb{R}}\int_{I^3} \exp(-2\sqrt{-1}n_3A_2)\frac{(1+n_3^2)^{\mu_1+2\mu_2+\frac{3}{2}}}{(Pt_3+1)^{\frac{1}{2}}}$$
$$\exp\left(-\frac{A_1^2}{t_1} - \frac{P}{1+n_3^2} - A_1^2t_3 - \frac{n_3^2P}{(1+n_3^2)(Pt_3+1)}\right)\frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{t_3^{\mu_1+2\ mu_2+\frac{3}{2}}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}dn_3.$$

Changing to the variables (u_1, u_2, u_3) from (t_1, t_2, t_3) defined through

$$u_1 = t_3 + \frac{1}{P}, u_2 = \frac{t_3 P}{(1+n_3^2)}, u_3 = \frac{t_2}{t_1},$$

the integration J has the following expression

$$J = \frac{\pi^{\frac{3}{2}} A_1^{-2\mu_1 - 4\mu_2 - 4}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1 - 2\mu_2 - 1)} \int_{\mathbb{R}} \int_{I^3} \frac{(1+u_3)^{\mu_1+\mu_2+1} Q^{\mu_2}}{u_1^{\mu_2+\frac{1}{2}} u_2^{\mu_1+2\mu_2+\frac{3}{2}} u_3^{\mu_2}}$$
$$\exp(-2\sqrt{-1} n_3 A_2) \exp\Big(-u_1 A_1^2 \Big(1 + \frac{u_3}{Q}\Big) - \frac{1+u_2}{u_1}\Big) dn_3 \frac{du_1}{u_1}\frac{du_2}{u_2}\frac{du_3}{u_3}$$

with $Q = 1 + (1+n_3^2)u_2$.

Step 3. Integration for n_3.

Before performing this step, we again change the variables by the rule

$$u_1 = \frac{1+x}{A_1^2 A_2} t_2, \ u_2 = x, \ u_3 = y + \frac{xy}{1+x} n_3^2$$

to get

$$J = \frac{\pi^{\frac{3}{2}} A_1^{-2\mu_1 - 2\mu_2 - 3} A_2^{\mu_2 + \frac{1}{2}}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1 - 2\mu_2 - 1)} \int_{\mathbb{R}} \int_{I^3} \exp\Big(-\frac{1+x+y}{A_2} t_2 - \frac{A_1^2 A_2}{t_2}\Big)$$
$$\exp(-2\sqrt{-1} n_3 A_2) \frac{((1+x)(1+y) + xyn_3^2)^{\mu_1+\mu_2+1}}{(x(1+x))^{\mu_1+\mu_2+\frac{3}{2}} (xy)^{\mu_2} t_2^{\mu_2+\frac{1}{2}}} dn_3 \frac{dx}{x}\frac{dy}{y}\frac{dt_2}{t_2}.$$

By suitable substitution, from (3) one can derive that

(5)

$$\int_{\mathbb{R}} (ax^2 + b)^{\nu} \exp(2\sqrt{-1} cx) dx = \operatorname{sgn}(c) \frac{\sqrt{\pi/a}}{\Gamma(-\nu)} \int_0^{\infty} \exp\Big(-bt - \frac{c^2}{at}\Big) t^{-\nu-\frac{1}{2}} \frac{dt}{t}.$$

for $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$. Apply this formula to the above expression of J by choosing

$$(x, a, b, c, t) = \Big(n_3, \frac{x}{1+x}, \frac{1+y}{y}, A_2, t_1 A_2\Big)$$

and put

$$(\mu_1, \mu_2, A_1, A_2) = \Big(\frac{-\nu_1 + \nu_2 - 2}{2}, \frac{-\nu_2 - 1}{2}, y_1\sqrt{c_0^2 + c_1^2}, y_2|c_2|\Big).$$

We then arrive at an evaluation of $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$, that is:

$$\frac{\pi^3 sgn(c_2)(c_0^2 + c_1^2)^{\frac{\nu_1}{2}} |c_2|^{\frac{\nu_1 - \nu_2}{2}} y_2^{-\nu_2}}{\Gamma(\frac{\nu_1+1}{2})\Gamma(\frac{\nu_2+1}{2})\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)} \int_{I^4} x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}$$
$$\exp\Big(-|c_2| y_2 \Big(\frac{1+x+y}{c_2^2 y_2^2} t_2 + (c_0^2 + c_1^2)\ fracy_1^2 t_2 + \frac{1+y}{y} t_1 + \frac{1+x}{x t_1}\Big)\Big) dn.$$

with $dn = \dfrac{dx}{x}\dfrac{dy}{y}\dfrac{dt_1}{t_1}\dfrac{dt_2}{t_2}$.

The integrand in the above integral expression of $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$ is rapidly decreasing at both zero and infinity for each variable of x, y, t_1 and t_2. Hence the integral converges and $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$ is well defined to the whole plane $\mathbb{C}^2$ if $\frac{\nu_1+1}{2}, \frac{\nu_2+1}{2}, \frac{\nu_1-\nu_2+2}{2}, \frac{\nu_1+\nu_2+2}{2}$ are not negative integers simultaneously.

By a simple substitution, (1) can be transformed in the form

$$(6) \qquad K_\nu(2\sqrt{ab}) = \frac{1}{2}(a/b)^{\frac{\nu}{2}} \int_0^\infty \exp(-ax - b/x)x^\nu \frac{dx}{x}, \qquad a, b \in \mathbb{R}_{>0}$$

To get the first expression in our theorem we apply (6), for the variables x and y, with

$$(a,b,\nu) = \Big(\frac{t_2}{|c_2|y_2}, \frac{|c_2|y_2}{t_1}, \frac{\nu_1+\nu_2}{2}\Big) \text{ and } \Big(\frac{t_2}{|c_2|y_2}, |c_2|t_1y_2, \frac{\nu_2-\nu_1}{2}\Big).$$

2. In the above expression of $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$ we again utilize (6) for the variables t_1 and t_2 by choosing (a,b,ν) as

$$\left(|c_2|y_2\frac{1+y}{y}, |c_2|y_2\frac{1+x}{x}, \frac{\nu_1}{2}\right) \text{ and } \left(\frac{1+x+y}{|c_2|y_2}, (c_0^2+c_1^2)|c_2|y_1^2y_2, \frac{\nu_2}{2}\right),$$

respectively. Then we obtain the second expression.
3. For this one, the method of proof is similar to that of [5]. $\square$

We now turn to the discussion of non class one case i.e., $u = \pm 1$.

Theorem 3.3. *Let $\pi = \mathrm{Ind}_P^G(\sigma_{(0,-1)} \otimes e^{\nu+\rho} \otimes 1_N)$ be an irreducible representation with $\nu = (\nu_1,\nu_2) \in \mathbb{C}^2$. For a normalized character η of N we have the following assertions on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1,y_2;\lambda) = y_1^3y_2^2\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\lambda)$. The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\lambda)$ has the following integral expressions:*

1. *For $u = \pm 1$, we have that $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\lambda)$ is equal to*

$$\frac{y_1y_2}{2^2}\int_0^\infty\int_0^\infty K_{\frac{\nu_1+\nu_2}{2}}\left(2\sqrt{\frac{t_2}{t_1}}\right) K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1t_2}) \left(\sqrt{\frac{t_1}{t_2}} - \frac{u}{\sqrt{t_1t_2}}\right)$$
$$\times \exp\left(-|c_2|y_2t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2+c_1^2)|c_2|\frac{y_1^2y_2}{t_2}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

2. *The function* $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\lambda)$ *is identified with*

$$y_1^{\frac{\nu_2+1}{2}} y_2^{-\frac{\nu_2-1}{2}} \int_0^\infty \int_0^\infty K_{\frac{\nu_2-1}{2}}(Y) \left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}} \left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2-1}{4}}$$
$$\times \left(y\Big(\frac{x(1+x)}{y(1+y)}\Big)^{1/4} K_{\frac{\nu_1+1}{2}}(X) + ux\Big(\frac{y(1+y)}{x(1+x)}\Big)^{1/4} K_{\frac{\nu_1-1}{2}}(X) \right) \frac{dx}{x}\frac{dy}{y}$$

with $X = 2|c_2|y_2 \left(\frac{(1+x)(1+y)}{xy}\right)^{\frac{1}{2}}$ *and* $Y = 2(c_0^2+c_1^2)^{\frac{1}{2}} y_1(1+x+y)^{\frac{1}{2}}$,

3. *The Mellin-Barnes's integral expression of* $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\lambda)$ *is*

$$\frac{1}{(2\sqrt{-1})^2} \int_{s_1}\int_{s_2} (V^1_{(\nu_1,\nu_2)}(s_1,s_2) - \lambda V^2_{(\nu_1,\nu_2)}(s_1,s_2)) y_1^{-s_1} y_2^{-s_2} ds_1 ds_2$$

Here the paths of integrations are the vertical lines from $\alpha_i - \sqrt{-1}\infty$ *to* $\alpha_i + \sqrt{-1}\infty$ *with real number* α_i *such that*

$$\alpha_1 > |\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_1)|, \ \alpha_2 > |\mathrm{Re}(\nu_1+\nu_2)|/2, |\mathrm{Re}(\nu_1-\nu_2)|/2$$

and the integrand $V^1_{(\nu_1,\nu_2)}(s_1,s_2)$ *is equal to* $\dfrac{(\frac{s_1-1}{2})\Gamma_\pm(s_2+1,\frac{\nu_1-\nu_2}{2})}{\Gamma_\pm(s_2,\frac{\nu_1-\nu_2}{2})}$ *times*

$$\Gamma(s_1,s_2) \times {}_3F_2 \left(\begin{matrix} \frac{s_1}{2}, \frac{s_2+1}{2} + \frac{\nu_2-\nu_1}{4}, \frac{s_2+1}{2} - \frac{\nu_2-\nu_2}{4} \\ \frac{s_1+s_2+1}{2} + \frac{\nu_2+\nu_1}{4}, \frac{s_1+s_2+1}{2} - \frac{\nu_1+\nu_2}{4} \end{matrix} \middle| 1 \right)$$

and $V^2_{(\nu_1,\nu_2)}(s_1,s_2)$ *is equal to* $\Gamma_\pm(s_2+1,\frac{\nu_1+\nu_2}{2})/\Gamma_\pm(s_2,\frac{\nu_1+\nu_2}{2})$ *times*

$$\Gamma(s_1,s_2) \times {}_3F_2 \left(\begin{matrix} \frac{s_1-1}{2}, \frac{s_2}{2} + \frac{\nu_2-\nu_1}{4}, \frac{s_2}{2} - \frac{\nu_2-\nu_2}{4} \\ \frac{s_2+s_1}{2} + \frac{\nu_2+\nu_1}{4}, \frac{s_2+s_1}{2} - \frac{\nu_1+\nu_2}{4} \end{matrix} \middle| 1 \right).$$

Proof. 1. In this case, the integrand f_u in Lemma 2.4 is

$$\begin{aligned} f_u(n) &= (1 + n_2\bar{n}_2 - n_1n_3 + u\sqrt{-1}(n_1+n_3))/\Delta_2^{\frac{1}{2}} \\ &= (1 - N_1n_3 + u\sqrt{-1}(N_1+n_3))\Delta_3/((1+n_3^2)\Delta_2^{\frac{1}{2}}) \end{aligned}$$

with $u = \pm 1$, and it does not depend X_0 and Y_0. By Lemma 3.1, we evaluate J_u to get the first part of this theorem. Here

$$\begin{aligned} J_u =& \frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4}\int_{I^2} \frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{1+n_3^2} \exp\Big(-\frac{A_1^2}{t_1} - \frac{\Delta_2}{\Delta_3^2}(t_1+t_2)\Big) \\ & \exp\Big(2\sqrt{-1}(-n_3A_2 + \frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1)\Big)\Delta_3^{\mu_1+2\mu_2+2} f_u dn \end{aligned}$$

As we have seen in the previous theorem, the integrations for N_1 and n_2 can be done as well by applying the formulas (3) and (4). For the integration for n_3, we use

$$\int_{\mathbb{R}} x(ax^2+b)^{\nu}\exp(2\sqrt{-1}cx)dx = \frac{\sqrt{-1}c(\pi/a)^{\frac{1}{2}}}{a\Gamma(-\nu)}\int_I \exp\Big(-bt-\frac{c^2}{at}\Big)t^{-\nu-\frac{3}{2}}\frac{dt}{t}.$$

for $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$. Then we get

$$\begin{aligned}\tilde{W}_{\nu}(y;u) =& \frac{2^{-4}(c_0^2+c_1^2)^{\nu_1}|c_2|^{\nu_2}y_1^4y_2^{-\nu_2+3}}{\Gamma(\frac{\nu_1}{2}+1)\Gamma(\frac{\nu_2}{2}+1)\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)}\int_{I^4} x^{\frac{\nu_1+\nu_2}{2}}y^{\frac{\nu_2-\nu_1}{2}}\\ &\exp\Big(-|c_2|y_2\Big(\frac{1+y}{y}t_1+\frac{1+x}{xt_1}+\frac{1+x+y}{c_2^2y_2^2}t_2+(c_0^2+c_1^2)\frac{y_1^2}{t_2}\Big)\Big)\\ &t_1^{\frac{\nu_1}{2}}t_2^{\frac{\nu_2}{2}}\left(\sqrt{\frac{t_1}{t_2}}+\frac{u}{\sqrt{t_1t_2}}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dx}{x}\frac{dy}{y}.\end{aligned}$$

Note here that the above integral converges and therefore the function $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;u)$ is well defined, because of the assumption for the pair (ν_1,ν_2). Hence our theorem follows. □

Remark 3.4. *In Theorem 3.3 above, we write the constants $c_i, (i = 0,1,2)$. It looks like superfluous, because replacing $|c_2|y_2$ by y_2 we can erase this constant. However if one try to discuss other K-types which are not handled in this paper, sometime the derivatives with respect to these parameters are crucial.*

4. Explicit formula, the case $u = \pm 2$.

The feature of the case $u = \pm 2$ is that the K-types corresponding to $u = \pm 2$ and $u = 0$ belongs to the same principal series representation $\pi = \mathrm{Ind}_P^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$. This cases do not seem to appear in the literature. In this subsection we handle this case. Note that one can do this by using $(\mathfrak{g}, K)$-module structure of the principal series representation of $SU(2,2)$ computed in [2]. We may normalize the nondegenerated unitary character η of N so that $c_0^2+c_1^2 = c_2 = 1$ without loss of generality and call it *normalized* character.

4.1. Evaluation of Jacquet integrals.

First of all we consider an evaluation of integrals with a certain integrand that is closely related to the case $u = \pm 2$.

For $\nu_1, \nu_2 \in \mathbb{C}$ and normalized character η of N let us evaluate

$$J_\lambda = A_1^{-\nu_1} A_2^{-\frac{\nu_1+\nu_2}{2}} \int_{\mathbb{R}^6} \exp\Big(-2\sqrt{-1}(c_0x_0A_1 + c_1y_0A_1 \pm n_3A_2)\Big) \\ \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1} \Delta_2^{\frac{\nu_2+1}{2}} F_\lambda(n) dx_0dy_0dn_1dx_2dy_2dn_3,$$

where A_1, A_2 are positive real parameters and the integrand $F_\lambda(n)$ is

$$F_\lambda(n) = \frac{1}{\Delta_2}\Big((1 + n_2\bar{n_2} - n_1n_3)^2 + \lambda\sqrt{-1}(n_1 + n_3)(1 + n_2\bar{n_2} - n_1n_3)\Big),$$

where $n_2 = x_2 + \sqrt{-1}y_2$ and $\lambda = \pm 1$.

Lemma 4.1. *Let J_λ be as above. Then the function $\tilde{J}_\lambda = A_1^{-3}A_2^{-2}J_\lambda$ is proportional to*

$$A_2^{-\nu_2} \int_{I^4} \exp\left(-\frac{1+x+y}{A_2}t_2 - \frac{1+y}{y}A_2t_1 - A_2\frac{1+x}{xt_1} - \frac{A_1^2A_2}{t_2}\right) \\ \frac{x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}}}{t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}} \left(A_1^2A_2^2\frac{t_1}{t_2} + \frac{\nu_1+1}{4} - \frac{1+y}{2y}t_1A_2 - \lambda\Big(\frac{A_1^2A_2^2}{t_2} - \frac{A_2}{2}\Big)\right) dX$$

where $dX = dx/x \cdot dy/y \cdot dt_1/t_1 \cdot dt_2/t_2$.

Proof. Recalling Lemma 3.1, we have

$$J_\lambda = \frac{\pi A_1^{-\nu_1} A_2^{-\frac{\nu_1+\nu_2}{2}}}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4} \int_0^\infty \int_0^\infty \frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{1+n_3^2} \exp\left(-\frac{A_1^2}{t_1} - \frac{\Delta_2}{\Delta_3^2}(t_1+t_2)\right) \\ \exp\Big(2\sqrt{-1}(n_3A_2 + \frac{N_1+n_3}{1+n_3^2}(c_0x_2 + c_1y_2)A_1)\Big)\Delta_3^{\mu_1+2\mu_2+1}F_\lambda(n)dn,$$

because the function $F_\lambda(n)$ does not depend on the variable n_0.

In terms of variables N_1, n_3 and Δ_3, the function $F_\lambda(n)$ is expressed by

$$F_\lambda(n) = \frac{\Delta_3^2}{(1+n_3^2)^2}(1 - n_3N_1)(1 - n_3N_1 + \sqrt{-1}\lambda(N_1 + n_3)).$$

Thus we are now in a position to perform the transformations with respect to the variables N_1, n_2 and n_3 as we have seen in the previous cases. In this manner, the integral J_λ can be identified with

$$\frac{\pi^3 A_2^{-\nu_2}}{\Gamma(\frac{\nu_1+1}{2}+1)\Gamma(\frac{\nu_2+1}{2}+1)\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)} \int_I\int_I\int_I\int_I t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}} \\ \exp\left(-\frac{1+x+y}{A_2}t_2 - \frac{1+y}{y}A_2t_1 - A_2\frac{1+x}{xt_1} - \frac{A_1^2A_2}{t_2}\right) x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} \\ \left(\frac{t_1}{t_2}A_1^2A_2^2 + \frac{\nu_1+1}{4} - \frac{1+y}{2y}t_1A_2 - \lambda\left(\frac{A_1^2A_2^2}{t_2} - \frac{A_2}{2}\right)\right) \frac{dx}{x}\frac{dy}{y}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

□

Let us consider the following theorem to get the Mellin-Barnes integral expression for the case $u = \pm 2$.

Theorem 4.2. *For a normalized character η of the unipotent group N, $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $u = \pm 2$, on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; u) = y_1^3 y_2^2 \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$*

(1). We have

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u) = y_1^{\frac{\nu_2}{2}} y_2^{-\frac{\nu_2}{2}} \int_0^\infty \int_0^\infty \left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}} \left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2}{4}}$$
$$\times \Big\{ K_{\frac{\nu_1}{2}}(X)\Big(\{2uy_2 - (\nu_1+1)(\nu_2-1)\}K_{\frac{\nu_2}{2}}(Y) - 2uy_2 Y K_{\frac{\nu_2}{2}-1}(Y)\Big)$$
$$+ \frac{2y}{1+y} XYK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}-1}(Y) - 2XK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}}(Y)\Big\}\frac{dx}{x}\frac{dy}{y},$$

with $X = 2y_2((1+1/x)(1+1/y))^{\frac{1}{2}}$ and and $Y = 2y_1(1+x+y)^{\frac{1}{2}}$.

(2). the function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ is equal to

$$\int_{I^2} \exp\Big(-y_2 t_1 - \frac{y_2}{t_1} - \frac{t_2}{y_2} - \frac{y_1^2 y_2}{t_2}\Big) K_{\frac{\nu_1+\nu_2}{2}}\Big(2\sqrt{\frac{t_2}{t_1}}\Big)\Big\{-4t_1 y_2 K_{\frac{\nu_2-\nu_1-2}{2}}(2\sqrt{t_1 t_2})$$
$$+ K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1 t_2})\Big((\nu_1+1)(1-\nu_2) + \Big(\frac{2y_1^2 y_2^2}{t_2} - y_2\Big)(4t_1 - u)\Big)\Big\}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Proof. By putting $u = 2\lambda$, one has that

$$f_u(n) = 2F_\lambda(n) - 1.$$

Change A_i by y_i in the expression of J_λ for $i = 1, 2$, then we obtain

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u) = 2\tilde{J}_\lambda(y_1, y_2) - \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0).$$

By making a similar computation as in 3.1, we also obtain that the corresponding integral with respect to Δ_2 in the above expression is equal to $(\nu_1+1)(\nu_2+1)$ times

$$y_1^{\frac{\nu_2}{2}+3} y_2^{-\frac{\nu_2}{2}+2} \int_{I^2} \Big(\frac{x(1+x)}{y(1+y)}\Big)^{\frac{\nu_1}{4}} \Big(\frac{x^2y^2}{1+x+y}\Big)^{\frac{\nu_2}{4}} K_{\frac{\nu_1}{2}}(X)K_{\frac{\nu_2}{2}}(Y)\frac{dx}{x}\frac{dy}{y}.$$

Hence the desired result follows from (*) in the evaluation 3.1 and above integral representation.

(b).

$$(a, b, \nu) = \Big(\frac{1+y}{y}y_2, \frac{1+x}{x}y_2, \frac{\nu_1}{2}\Big) \text{ and } (a, b, \nu) = \Big(\frac{1+x+y}{y_2}, y_1^2 y_2, \frac{\nu_2}{2}\Big),$$

we obtain that $\tilde{W}_{(\nu_1,\nu_2)}(y)y_2^{\nu_2}$ is equal to

$$\frac{1}{4}\int_{I^4} x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} \exp\Big(-y_2(-\frac{1+y}{y}t_1 - \frac{1+x}{xt_1} - (1+x+y)\frac{t_2}{y_2^2} - \frac{y_1^2}{t_2})\Big)$$
$$\Big(-(\nu_1+1)(\nu_2-1) + 8\frac{y_1^2y_2^2t_1}{t_2} - 4t_1y_2\frac{1+y}{y} + uy_2 - \frac{2uy_1^2y_2^2}{t_2}\Big)t_1^{\frac{\nu_1}{2}}t_2^{\frac{\nu_2}{2}}\,dz$$

with $dz = dt_1/t_1dt_2/t_2dx/xdy/y$. To finish the proof we again apply (6) for the variables x and y. Then we get the desired result. □

4.2. **Mellin-Barnes integral representation.** Let us compute the double Mellin transformation

$$V(s_1,s_2) = \int_0^\infty\int_0^\infty \tilde{W}_{(\nu_1,\nu_2)}(y;\pm 2)y_1^{s_1}y_2^{s_2}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

of $\tilde{W}_{(\nu_1,\nu_2)}(y;\pm 2)$ from the previous theorem. Then, by applying Mellin inversion to this, we get the desired result as in the following theorem. We use the following notations:

$$b = \frac{s_2}{2} + \frac{\nu_1-\nu_2}{4}, c = \frac{s_2}{2} + \frac{\nu_2-\nu_1}{4}, d = \frac{s_2}{2} + \frac{\nu_1+\nu_2}{4}, e = \frac{s_2}{2} - \frac{\nu_1+\nu_2}{4}.$$

and $a = s_1/2$.

Lemma 4.3. *We have*
1.

$$\frac{y}{1+y}XYK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}-1}(Y) = \frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2} V^1_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2$$

where

$$V^1_{(\nu_1,\nu_2)}(s_1,s_2) = \frac{4abc\Gamma(s_1,s_2)}{(a+d)(a+e)}\cdot {}_3F_2\left(\begin{array}{c} a+1,b+1,c+1 \\ d+1,e+1\end{array}\middle| 1\right),$$

2.

$$-(\nu_1+1)(\nu_2-1)K_{\frac{\nu_1}{2}}(X)K_{\frac{\nu_2}{2}}(Y) - 2XK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}}(Y) =$$
$$\frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2} V^2_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2$$

where

$$V^2_{(\nu_1,\nu_2)}(s_1,s_2) = \Gamma(s_1,s_2)(-(\nu_1+1)(\nu_2-1)-2e)\cdot {}_3F_2\left(\begin{array}{c} a,b,c \\ d,e\end{array}\middle| 1\right),$$

3.

$$K_{\frac{\nu_1}{2}}(X)\Big(K_{\frac{\nu_2}{2}}(Y) - YK_{\frac{\nu_2}{2}-1}(Y)\Big)\frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2} V^3_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2$$

where

$$V^3_{(\nu_1,\nu_2)}(s_1,s_2) = \Gamma(s_1, s_2+1)\cdot(1-a)\cdot {}_3F_2\left(\begin{array}{c} a+\frac{1}{2}, b, c+\frac{1}{2} \\ d, e+\frac{1}{2}\end{array}\middle| 1\right).$$

Here the pats s_1, s_2 are the same with those defined in Theorem 3.1.

Proof. 1. Utilizing the formulas

$$\int_I K_\nu(ax)x^s \frac{dx}{x} = 2^{s-2}a^{-s}\Gamma_\pm(s,\nu), \text{ for } a>0, \mathrm{Re}(s) > \mid \mathrm{Re}(\nu) \mid$$

and

$$\int_{I^2} \frac{x^a y^b (1+x+y)^{-e}}{(1+x)^c(1+y)^d}\frac{dx}{x}\frac{dy}{y} = \frac{\Gamma(a)\Gamma(b)\Gamma(c+e-a)\Gamma(d+e-b)}{\Gamma(c+e)\Gamma(d+e)} \times {}_3F_2\left(\begin{array}{c} a, b, e \\ c+e, d+e\end{array}\middle| 1\right)$$

for $\mathrm{Re}(c+e) > \mathrm{Re}(a) > 0, \mathrm{Re}(d+e) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c+d+e-a-b) > 0$, we get

$$V(s_1,s_2) = 2^{-4}\Gamma(d)\Gamma(c+1)\Gamma(e)\Gamma(b+1)\Gamma(a+\frac{\nu_2}{2})\Gamma(a+1)\Gamma(a-\frac{\nu_1}{2})$$
$$\Gamma(a+\frac{\nu_1}{2})[\Gamma(a+c)\Gamma(a+d+1)]^{-1}{}_3F_2\left(\begin{array}{c} d, c+1, a+\frac{\nu_2}{2} \\ a+c, a+d+1\end{array}\middle| 1\right).$$

Apply it with in this form to the Thomae's transformation for the hypergeometric series ${}_3F_2$, then we get 1. Similarly we obtain the other cases. □

By collecting the partial representations of $\tilde{W}_{(\nu_1,\nu_2)}(y;\pm 2)$ in the above lemma, we have

Theorem 4.4. *Let $V^i_{(\nu_1,\nu_2)}(s_1,s_2)$ be the function defined above for each $i \in \{1,2,3\}$. Then we have that $\tilde{W}_{(\nu_1,\nu_2)}(y;\pm 2)$ is equal to 2^{-4} times*

$$\int_{s_1}\int_{s_2}\Big(V^1_{(\nu_1,\nu_2)}(s_1,s_2) - V^2_{(\nu_1,\nu_2)}(s_1,s_2) \pm 2y_2V^3_{(\nu_1,\nu_2)}(s_1,s_2)\Big)y_1^{-s_1}y_2^{-s_2}ds_1ds_2.$$

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