

Systems of Renewal Equations on the Line

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Abstract. Systems of integral equations of renewal type on the whole line are considered. Using Banach-algebraic techniques, we study the asymptotic properties of the solutions and give rather general estimates of the rates of convergence.

1. Introduction

Consider the system of renewal equations

$$(1) \quad z_i(x) = g_i(x) + \sum_{j=1}^n \int_{\mathbb{R}} z_j(x-u) F_{ij}(du), \quad i = 1, \dots, n,$$

where $g_i \in L_1(\mathbb{R})$, $i = 1, \dots, n$, and $\mathbf{F} = (F_{ij})$ is a matrix of nonnegative finite measures on \mathbb{R} . Denote by $\mathbf{a} = (a_1, \dots, a_n)^T$ the column vector with coordinates a_1, \dots, a_n . In matrix notation, the system (1) takes the form

$$(2) \quad \mathbf{z}(x) = \mathbf{g}(x) + \mathbf{F} * \mathbf{z}(x),$$

where, by definition, $(\mathbf{F} * \mathbf{z}(x))_i$ is the sum on the right-hand side of (1).

The objectives of the present paper are (i) to obtain a Stone-type decomposition for the matrix renewal measure on the whole line (see Definition 2 in Section 3) and (ii) to apply the resulting decomposition to the study of the asymptotic properties of the solution to (2), deriving rather general submultiplicative rates of convergence for $\mathbf{z}(x)$ as $x \rightarrow \pm\infty$. The results obtained will generalize those of [29], where the one-dimensional ($n = 1$) case was considered. It should be noted that the proofs from the one-dimensional case by no means automatically apply to the case of systems

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of renewal equations, and, as pointed out in [5, Chapter 8]: “... new and powerful methods are required to handle the questions of asymptotic behavior of the solutions.”

Systems of renewal equations on the *whole* line do not seem to have been considered previously. As far as results on the half-axis are concerned, the reader is referred to [5, 8, 9, 3, 22, 23, 4, 10, 28].

2. Preliminaries

Let \mathbf{B} be a square matrix of order n and let μ_1, \dots, μ_n be its eigenvalues. The maximum of the $|\mu_j|$ is called the *spectral radius* of \mathbf{B} . By the Perron-Frobenius theorem [18, Theorem 9.2.1], every nonnegative irreducible matrix \mathbf{B} has a positive eigenvalue r of multiplicity 1, which is equal to its spectral radius; moreover, there exist positive right and left eigenvectors corresponding to r . If we drop the irreducibility condition, then the following assertion holds [18, Theorem 9.3.1]: Every nonnegative matrix \mathbf{B} has a real eigenvalue r , equal to its spectral radius, and there exists a normalized nonnegative right eigenvector corresponding to r .

We shall need some knowledge from the theory of Banach algebras of measures. In what follows, a key role will be played by Banach algebras of measures with submultiplicative weight functions.

DEFINITION 1. A function $\varphi(x)$, $x \in \mathbb{R}$, is called *submultiplicative* if $\varphi(x)$ is finite, positive, Borel-measurable and $\varphi(0) = 1$, $\varphi(x+y) \leq \varphi(x)\varphi(y)$ $\forall x, y \in \mathbb{R}$.

A few examples of such functions on $\mathbb{R}_+ := [0, \infty)$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\alpha)$ with $c > 0$ and $\alpha \in (0, 1)$; $\varphi(x) = \exp(rx)$ with $r \in \mathbb{R}$. Moreover, if $R(x)$, $x \in \mathbb{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent α (i.e., $R(tx)/R(x) \rightarrow t^\alpha$ for $t > 0$ as $x \rightarrow \infty$ [12, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0, \infty)$ such that $c_1 R(x) \leq \varphi(x) \leq c_2 R(x)$ for all $x \geq x_0$, where c_1 and c_2 are some positive constants [25, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

It is well known [15, Section 7.6] that

$$(3) \quad -\infty < r_1 := \lim_{x \rightarrow -\infty} \frac{\log \varphi(x)}{x} = \sup_{x < 0} \frac{\log \varphi(x)}{x} \\ \leq \inf_{x > 0} \frac{\log \varphi(x)}{x} = \lim_{x \rightarrow \infty} \frac{\log \varphi(x)}{x} =: r_2 < \infty$$

and $M(h) := \sup_{|x| \leq h} \varphi(x) < \infty \forall h > 0$.

Consider the collection $S(\varphi)$ of all complex-valued measures κ such that $\|\kappa\|_\varphi := \int_{\mathbb{R}} \varphi(x) |\kappa|(dx) < \infty$; here $|\kappa|$ stands for the total variation of κ . The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [15, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ , i.e., the measure of unit mass concentrated at the origin.

Define the Laplace transform of a measure κ as $\hat{\kappa}(s) := \int_{\mathbb{R}} \exp(sx) \kappa(dx)$. Relation (3) implies that the Laplace transform of any $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip

$$\Pi(r_1, r_2) := \{s \in \mathbb{C} : r_1 \leq \Re s \leq r_2\}.$$

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ -finite measure with the density $v(x; \nu) := \nu((x, \infty))$ for $x \geq 0$ and $v(x; \nu) := -\nu((-\infty, x])$ for $x < 0$. In case $\int_{\mathbb{R}} |x| |\nu|(dx) < \infty$, $T\nu$ is a finite measure whose Laplace transform is given by $(T\nu)^\wedge(s) = [\hat{\nu}(s) - \hat{\nu}(0)]/s$, $\Re s = 0$, the value $(T\nu)^\wedge(0)$ being defined by continuity as $\int_{\mathbb{R}} x \nu(dx) < \infty$.

The absolutely continuous part of an arbitrary distribution F will be denoted by F_c and its singular component, by F_s : $F_s = F - F_c$.

If \mathbf{F} is a matrix of nonnegative measures, then \mathbf{F}_s will denote the matrix whose entries are the singular components of the corresponding elements of \mathbf{F} . Similar conventions will be tacitly made for integrals, inequalities and the like, where matrices and vectors are involved.

3. A Stone-Type Decomposition

We first recall Stone's decomposition of the renewal measure in the one-dimensional case. Let F be a probability distribution on \mathbb{R} with $\mu =$

$\int_{\mathbb{R}} x F(dx) \in (0, +\infty]$, and let $H = \sum_{n=0}^{\infty} F^{n*}$ be the corresponding renewal measure; here $F^{1*} := F$, $F^{(n+1)*} := F * F^{n*}$, $n \geq 1$, and $F^{0*} := \delta$. Suppose F is *spread-out*, i.e., for some $m \geq 1$, F^{m*} has a nonzero absolutely continuous component. Stone [30] showed that then there exists a decomposition $H = H_1 + H_2$, where H_2 is a finite measure and H_1 is absolutely continuous with bounded continuous density $h(x)$ such that $\lim_{x \rightarrow +\infty} h(x) = \mu^{-1}$ and $\lim_{x \rightarrow -\infty} h(x) = 0$.

Consider the renewal equation

$$(4) \quad z(t) = g(t) + \int_{\mathbb{R}} z(t - y) F(dy) =: g(t) + z * F(t),$$

where $g \in L_1(\mathbb{R})$ and F is a spread-out probability distribution on \mathbb{R} with positive mean μ . The function $z(t) := g * H(t) + c$ is clearly a solution to (4); here c is any constant. So the asymptotic properties of the solutions to (4) are essentially those of the convolution $g * H(t)$. As pointed out in [2] and implemented in [1], Stone’s decomposition allows us to obtain an elegant proof of the following statement:

$$(5) \quad g * H(t) \rightarrow \begin{cases} \mu^{-1} \int_{\mathbb{R}} g(x) dx & \text{as } t \rightarrow +\infty, \\ 0 & \text{as } t \rightarrow -\infty, \end{cases}$$

provided that $g(x)$ is bounded and $\lim_{|x| \rightarrow \infty} g(x) = 0$.

Define the convolution of matrix measures as follows. If μ and ν are two arbitrary matrices of complex-valued measures, then $\mu * \nu := (\sum_{k=1}^n \mu_{ik} * \nu_{kj})$, provided the usual convolutions on the right-hand side make sense. Let \mathbf{I} be the unit matrix.

DEFINITION 2. A *matrix renewal measure* is a matrix of measures $\mathbf{H} := \sum_{k=0}^{\infty} \mathbf{F}^{k*}$; here $\mathbf{F}^{1*} := \mathbf{F}$, $\mathbf{F}^{(k+1)*} := \mathbf{F}^{k*} * \mathbf{F} = \mathbf{F} * \mathbf{F}^{k*}$, $\mathbf{F}^{0*} := \delta \mathbf{I}$.

In this section, we shall obtain an abstract Banach-algebraic version of Stone’s decomposition for the matrix renewal measure \mathbf{H} . Here we will assume that $\mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} \in (0, +\infty)$. By choosing specific Banach algebras of measures, we can apply the result to the study of the rates of convergence in a multidimensional analog of (5). The case $\mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} = +\infty$ does not seem to admit a Banach-algebraic treatment and will be dealt with by traditional methods in a forthcoming paper.

Let $S(r_1, r_2)$ be the Banach algebra $S(\varphi)$ with $\varphi(x) = \max(e^{r_1 x}, e^{r_2 x})$, where $r_1 \leq 0 \leq r_2$. We shall also consider Banach algebras \mathcal{A} of measures such that (i) $\mathcal{A} \subset S(r_1, r_2)$ and (ii) each homomorphism $\mathcal{A} \mapsto \mathbb{C}$ is the restriction to \mathcal{A} of some homomorphism $S(r_1, r_2) \mapsto \mathbb{C}$. Property (ii) can be restated as follows: Each maximal ideal M of \mathcal{A} is of the form $M_1 \cap \mathcal{A}$, where M_1 is a maximal ideal of $S(r_1, r_2)$. It follows from the general theory of Banach algebras that if $\nu \in \mathcal{A}$ is invertible in $S(r_1, r_2)$, then $\nu^{-1} \in \mathcal{A}$. There is already a sizeable pool of Banach algebras of measures satisfying the above properties (for an incomplete list, see, e.g., [7, 11, 20, 13, 24]). In some cases, it is the asymptotic behavior of $\nu((x, x + h])$ that characterizes the elements of \mathcal{A} ; otherwise, a distinguishing feature of the elements of \mathcal{A} is their specific density behavior (or tail behavior). Finally, one can take as \mathcal{A} the Banach algebra $S(\varphi)$ with a general submultiplicative $\varphi(x)$. Denote by L the restriction of Lebesgue measure to $[0, \infty)$.

THEOREM 1. *Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite non-negative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible with spectral radius $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue 1 of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} \in (0, +\infty)$, $\varrho[(\mathbf{F}^{m*})_s^\wedge(r_i)] < 1$, $i = 1, 2$, for some integer $m \geq 1$, and that in the strip $\Pi(r_1, r_2)$ there are no nonzero roots of the characteristic equation*

$$(6) \quad \det(\mathbf{I} - \hat{\mathbf{F}}(s)) = 0.$$

Let \mathcal{A} be a Banach algebra with properties (i) and (ii). Suppose, finally, that $\mathbf{F}, T\mathbf{F} \in \mathcal{A}$. Then the matrix renewal measure $\mathbf{H} = \sum_{n=0}^\infty \mathbf{F}^{n}$ admits a Stone-type decomposition $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, where $\mathbf{H}_2 \in \mathcal{A}$ and $\mathbf{H}_1 = \mathbf{r}\mathbf{l}L/\mu + rT\mathbf{H}_2$ for some $r > r_2$. If, in addition, $T^2\mathbf{F} \in \mathcal{A}$, then $\mathbf{H}_1 - \mathbf{r}\mathbf{l}L/\mu \in \mathcal{A}$.*

PROOF. We shall split the proof into several lemmas. Denote by $\hat{\mathbf{M}}(s)$ the adjugate matrix of $\mathbf{I} - \hat{\mathbf{F}}(s)$. We note that $\det(\mathbf{I} - \hat{\mathbf{F}}(s))$ is a linear combination of products of n factors which are elements of $\mathbf{I} - \hat{\mathbf{F}}(s)$. Hence $\det(\mathbf{I} - \hat{\mathbf{F}}(s))$ is the Laplace transform of some real-valued measure, say α . By the same reason, the entries of $\hat{\mathbf{M}}(s)$ are the Laplace transforms of real-valued measures belonging to \mathcal{A} , and we will denote by $\mathbf{M} = (M_{ij})$ the corresponding matrix of measures.

LEMMA 1. *Under the hypotheses of Theorem 1, $T\alpha \in \mathcal{A}$ and $TM \in \mathcal{A}$, i.e., $TM_{ij} \in \mathcal{A}$ for all i, j .*

PROOF OF LEMMA 1. It suffices to prove that if ν and κ are any two measures such that $\int_{\mathbb{R}} |x|e^{\gamma x} |\nu|(dx) < \infty$, $\int_{\mathbb{R}} |x|e^{\gamma x} |\kappa|(dx) < \infty$ and $\nu, \kappa, T\nu, T\kappa \in \mathcal{A}$, then $T\nu * \kappa \in \mathcal{A}$. But this follows immediately from the equality $T(\nu * \kappa) = (T\nu) * \kappa + \hat{\nu}(0)T\kappa$ implied by

$$\frac{\hat{\nu}(s)\hat{\kappa}(s) - \hat{\nu}(0)\hat{\kappa}(0)}{s} = \frac{[\hat{\nu}(s) - \hat{\nu}(0)]\hat{\kappa}(s)}{s} + \frac{\hat{\nu}(0)[\hat{\kappa}(s) - \hat{\kappa}(0)]}{s}, \quad \Re s = 0.$$

Lemma 1 is proved. \square

Let $r > r_2$ and $d(s) := (s - r) \det(\mathbf{I} - \hat{\mathbf{F}}(s))/s = (s - r)\hat{\alpha}(s)/s$, $s \in \Pi(r_1, r_2)$, where $d(0) := -r\hat{\alpha}'(0)$. By Lemma 1, $d(s)$ is the Laplace transform of the measure $D := \alpha - rT\alpha \in \mathcal{A}$ since

$$(7) \quad d(s) = \hat{\alpha}(s) - r\hat{\alpha}(s)/s = \hat{\alpha}(s) - r(T\alpha)^\wedge(s) = \hat{D}(s).$$

LEMMA 2. *Under the hypotheses of Theorem 1, D is invertible in \mathcal{A} .*

PROOF OF LEMMA 2. It suffices to prove invertibility in $S(r_1, r_2)$. Let \mathcal{M} be the space of maximal ideals of the Banach algebra $S(r_1, r_2)$. Each $M \in \mathcal{M}$ induces a homomorphism $h : S(r_1, r_2) \rightarrow \mathbb{C}$ and M is the kernel of h . Denote by $\nu(M)$ the value of h at $\nu \in S(r_1, r_2)$. An element $\nu \in S(r_1, r_2)$ has an inverse if and only if $\nu(M) \neq 0$ for all $M \in \mathcal{M}$.

The space \mathcal{M} is split into two sets: \mathcal{M}_1 is the set of those maximal ideals which do not contain the collection $L(r_1, r_2)$ of all absolutely continuous measures from $S(r_1, r_2)$, and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism induced by M is of the form $h(\nu) = \hat{\nu}(s_0)$, where $r_1 \leq \Re s_0 \leq r_2$. In this case, $M = \{\nu \in S(r_1, r_2) : \hat{\nu}(s_0) = 0\}$ [15, Chapter IV, Section 4]. If $M \in \mathcal{M}_2$, then $\nu(M) = 0 \forall \nu \in L(r_1, r_2)$.

We now show that $D(M) \neq 0$ for each $M \in \mathcal{M}$, thus establishing the existence of $D^{-1} \in S(r_1, r_2)$. Actually, if $M \in \mathcal{M}_1$, then, for some $s_0 \in \{r_1 \leq \Re s \leq r_2\}$, we have $D(M) = \hat{D}(s_0) \neq 0$. Now let $M \in \mathcal{M}_2$. By Theorem 1 of [20],

$$(8) \quad h(\nu) = \int_{\mathbb{R}} \chi(x, \nu) \exp(\beta x) \nu(dx), \quad \nu \in S(r_1, r_2),$$

where β is a real number such that $r_1 \leq \beta \leq r_2$, and the function $\chi(x, \nu)$ of two variables is a generalized character; here we mention only one property of a generalized character to be used later: $|\nu| - \text{ess sup}_{x \in \mathbb{R}} |\chi(x, \nu)| \leq 1$. By the multiplicative property of the functional $\nu \rightarrow \nu(M)$, $\nu \in S(r_1, r_2)$, we have $\mathbf{F}(M)^m = \mathbf{F}^{m*}(M) = (\mathbf{F}^{m*})_s(M)$. Denote $\Theta = (\Theta_{ij}) := (\mathbf{F}^{m*})_s$. By (8), we have, for some $\beta \in [r_1, r_2]$,

$$(9) \quad |\Theta_{ij}(M)| = \left| \int_{\mathbb{R}} \chi(x, \Theta_{ij}) \exp(\beta x) \Theta_{ij}(dx) \right| \leq \hat{\Theta}_{ij}(\beta).$$

Choose $\lambda \in [0, 1]$ such that $\beta = \lambda r_1 + (1 - \lambda)r_2$. Since $\hat{\Theta}_{ij}(\xi)$ is a convex function of ξ , we have $\hat{\Theta}_{ij}(\beta) \leq \lambda \hat{\Theta}_{ij}(r_1) + (1 - \lambda)\hat{\Theta}_{ij}(r_2)$. If \mathbf{A} and \mathbf{B} are matrices such that $|\mathbf{A}| \leq \mathbf{B}$, then $\varrho(\mathbf{A}) \leq \varrho(\mathbf{B})$ [16, Theorem 8.1.18]. Also, it is easily verified that if $\mathbf{B} = \lambda \mathbf{B}_1 + (1 - \lambda)\mathbf{B}_2$, $\lambda \in [0, 1]$, then $\varrho(\mathbf{B}) \leq \lambda \varrho(\mathbf{B}_1) + (1 - \lambda)\varrho(\mathbf{B}_2)$. It follows from (9) that

$$\varrho[\Theta(M)] \leq \varrho[\hat{\Theta}(\beta)] \leq \lambda \varrho[\hat{\Theta}(r_1)] + (1 - \lambda)\varrho[\hat{\Theta}(r_2)] < 1.$$

Consequently, $\varrho[\mathbf{F}(M)^m] < 1$. It follows that $\varrho[\mathbf{F}(M)]$, being equal to the m -th root of $\varrho[\mathbf{F}(M)^m]$, is also less than 1. Since $T\alpha \in L(r_1, r_2)$, (7) implies $D(M) = \alpha(M) = \det(\mathbf{I} - \mathbf{F}(M)) \neq 0$. So $D(M) \neq 0$ for all $M \in \mathcal{M}$. This means that $\exists D^{-1} \in S(r_1, r_2)$. Lemma 2 is proved. \square

Consider the auxiliary matrix

$$(10) \quad \mathbf{q}(s) := \frac{s}{s - r} [\mathbf{I} - \hat{\mathbf{F}}(s)]^{-1}, \quad s \in \Pi(r_1, r_2),$$

where $\mathbf{q}(0) := -\hat{\mathbf{M}}(0)/[r\hat{\alpha}'(0)]$.

LEMMA 3. Under the hypotheses of Theorem 1, $\mathbf{q}(s)$ is the Laplace transform matrix of some $\mathbf{Q} \in \mathcal{A}$.

PROOF OF LEMMA 3. It follows from (10) that $\mathbf{q}(s) = [1/d(s)]\hat{\mathbf{M}}(s)$. Consequently, by Lemma 2, $\mathbf{q}(s) = \hat{\mathbf{Q}}(s)$, where $\mathbf{Q} = D^{-1} * \mathbf{M} \in \mathcal{A}$, the entries of \mathbf{Q} being the convolutions of D^{-1} with the corresponding elements of \mathbf{M} . Lemma 3 is proved. \square

For $s \in \Pi(r_1, r_2) \setminus \{0\}$, we have

$$(11) \quad [\mathbf{I} - \hat{\mathbf{F}}(s)]^{-1} = \frac{s - r}{s} \hat{\mathbf{Q}}(s) = -\frac{r\hat{\mathbf{Q}}(0)}{s} - \frac{r[\hat{\mathbf{Q}}(s) - \hat{\mathbf{Q}}(0)]}{s} + \hat{\mathbf{Q}}(s).$$

We will show that (11) implies the following representation for \mathbf{H} :

$$(12) \quad \mathbf{H} = r\hat{\mathbf{Q}}(0)L - rT\mathbf{Q} + \mathbf{Q} = -\frac{\hat{\mathbf{M}}(0)}{\hat{\alpha}'(0)}L - rT\mathbf{Q} + \mathbf{Q}.$$

To this end, we shall need some knowledge from the theory of tempered distributions and more lemmas. Denote by \mathcal{S}_1 the space of rapidly decreasing functions in \mathbb{R} and by \mathcal{S}'_1 the dual space (the space of tempered distributions) [21, Chapter 7]. It is well known that if β is a σ -finite nonnegative measure such that

$$(13) \quad \int_{\mathbb{R}} \frac{\beta(dx)}{(1 + x^2)^k} < \infty$$

for some integer $k > 0$, then $\beta \in \mathcal{S}'_1$; here the tempered distribution β is defined by

$$\beta(\phi) := \int_{\mathbb{R}} \phi(x) \beta(dx), \quad \phi \in \mathcal{S}_1.$$

LEMMA 4. *Let β be a σ -finite nonnegative measure such that $\beta([x, x + 1]) \leq c < \infty \forall x \in \mathbb{R}$. Then (13) holds with $k = 1$.*

PROOF OF LEMMA 4. We have

$$\int_{\mathbb{R}} \frac{\beta(dx)}{1 + x^2} \leq \sum_{k=-\infty}^{-1} \frac{\beta([k, k + 1])}{1 + (k + 1)^2} + \sum_{k=0}^{\infty} \frac{\beta([k, k + 1])}{1 + k^2} \leq 2c \sum_{k=0}^{\infty} \frac{1}{1 + k^2} < \infty. \quad \square$$

LEMMA 5. *Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite nonnegative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue 1 of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} \in (0, +\infty)$. Then all the entries H_{ij} of the matrix renewal measure \mathbf{H} satisfy condition (13) for $k = 1$, so that the H_{ij} are tempered distributions.*

PROOF OF LEMMA 5. Although Lemma 5 is intuitive, especially in the light of a corresponding one-dimensional result [12, Chapter XI, Section 9, Theorem 1], its rigorous proof is rather involved. If \mathbf{F} is concentrated on $[0, \infty)$, then the asymptotic behavior of $\mathbf{H}((x, x + h])$ as $x \rightarrow +\infty$ is described in [9] for *nonlattice* \mathbf{F} . A distinctive feature of *lattice* matrices \mathbf{F} is the fact that all \mathbf{F}^{k*} are concentrated on sets of Lebesgue measure zero. In the case under consideration, the above property of lattice matrices is violated since $\varrho[(\mathbf{F}^{m*})_s(\mathbb{R})] < 1$ for some $m \geq 1$. Hence \mathbf{F} is nonlattice. Suppose \mathbf{F} is concentrated on $(0, \infty)$. Then, by Theorem 2.1 of [9], there exists $\lim_{x \rightarrow +\infty} \mathbf{H}((x, x + h]) := \mathbf{A}(h) < \infty$ for all $h > 0$. In particular, this means that

$$(14) \quad \left(\sup_{x \in \mathbb{R}} H_{ij}((x, x + h]) \right) := \mathbf{K}(h) < \infty.$$

We will show that this inequality is also valid when \mathbf{F} is concentrated on \mathbb{R} . Suppose, for the time being, that $\mathbf{F}(\mathbb{R})$ is a primitive [16, Definition 8.5.0] stochastic matrix, i.e., $\sum_{j=1}^n F_{ij}(\mathbb{R}) = 1 \ \forall i$. We shall use a basic factorization identity [19, (2.9)] without describing the probabilistic meaning of its components, since it is of no importance for our purposes. We have

$$\mathbf{I} - z\hat{\mathbf{F}}(s) = [\mathbf{I} - \hat{\mathbf{G}}_-(z, s)][\mathbf{I} - \hat{\mathbf{G}}_+(z, s)], \quad z \in (0, 1], \quad \Re s = 0,$$

where $\hat{\mathbf{G}}_{\pm}(z, s)$, as functions of s , are matrices of Laplace transforms of non-negative finite measures. The measures involved are concentrated on $(0, \infty)$ and $(-\infty, 0]$, respectively, and they continuously depend on z . Moreover, $\varrho[\hat{\mathbf{G}}_{\pm}(z, 0)] \leq z$. Hence, for $z \in (0, 1)$, we have

$$(15) \quad \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(s)^k = \sum_{k=0}^{\infty} z^k \hat{\mathbf{G}}_+(z, s)^k [\mathbf{I} - \hat{\mathbf{G}}_-(z, s)]^{-1}.$$

The matrix $\hat{\mathbf{F}}_-(s) := \mathbf{I} - \hat{\mathbf{G}}_-(1, s)$ is invertible in the sense that $[\mathbf{I} - \hat{\mathbf{G}}_-(1, s)]^{-1}$ is the matrix of Laplace transforms of finite (and even non-negative) measures. Let \mathbf{F}_-^{-1} be the corresponding matrix of measures, i.e., $\hat{\mathbf{F}}_-^{-1}(s) = [\mathbf{I} - \hat{\mathbf{G}}_-(1, s)]^{-1}$. Passing in (15) from Laplace transforms to measures and letting $z \uparrow 1$, we obtain

$$(16) \quad \mathbf{H}(A) = \mathbf{H}_1 * \mathbf{F}_-^{-1}(A), \quad A \in \mathcal{B},$$

where \mathcal{B} is the σ -algebra of all Borel subsets of \mathbb{R} and \mathbf{H}_1 is the matrix renewal measure induced by $\mathbf{F}_+(A) := \mathbf{G}_+(1, A)$, $A \in \mathcal{B}$. Notice that \mathbf{H}_1 is concentrated on $[0, \infty)$. In order to apply Theorem 2.1 of [9] to describe the asymptotic behavior of $\mathbf{H}_1((x, x + h])$, we have to show that \mathbf{F}_+ satisfies all the necessary requirements, namely, (i) $\varrho[\mathbf{F}_+(\mathbb{R})] = 1$, (ii) $\mathbf{F}_+(\mathbb{R})$ is irreducible, and (iii) \mathbf{F}_+ is nonlattice. In the notation of [6], $\mathbf{F}_+(\mathbb{R}) = M_+(0)$. Hence, by Lemma 2 (a) of [6], $\mathbf{F}_+(\mathbb{R})$ is a stochastic matrix, so that $\varrho[\mathbf{F}_+(\mathbb{R})] = 1$. Next, as pointed out in [6, the proof of Theorem 2], a Markov chain with transition matrix $Q = \mathbf{F}_+(\mathbb{R})$ can have only one class of essential states, which in the language of matrix theory means that $\mathbf{F}_+(\mathbb{R})$ is irreducible. Finally, the nonlattice property of \mathbf{F}_+ follows from the fact that $(\mathbf{F}_+^{k*})_s(\mathbb{R}) \neq \mathbf{F}_+^{k*}(\mathbb{R})$ for some $k \geq 1$. To prove this, we argue by contradiction. Suppose $(\mathbf{F}_+^{k*})_s(\mathbb{R}) = \mathbf{F}_+^{k*}(\mathbb{R}) \forall k$, and let $A \in \mathcal{B}$ be a set of Lebesgue measure zero such that all $(\mathbf{F}_+^{k*})_s$, $k = 1, 2, \dots$ are concentrated on A . Since $\mathbf{r} := (1, \dots, 1)^T$ is a right invariant vector of $\mathbf{F}_+(\mathbb{R})$, we have

$$(17) \quad \mathbf{H}_1(A)\mathbf{r} = \sum_{k=0}^{\infty} \mathbf{F}_+^{k*}(A)\mathbf{r} = \sum_{k=0}^{\infty} (\mathbf{F}_+^{k*})_s(\mathbb{R})\mathbf{r} = (\infty, \dots, \infty)^T.$$

Denote by \mathbf{F}_- a matrix of finite measures such that $\hat{\mathbf{F}}_-(s) = \mathbf{I} - \hat{\mathbf{G}}_-(1, s)$. In view of (16), we have

$$(18) \quad \begin{aligned} \mathbf{H}_1(A) &= \mathbf{H} * \mathbf{F}_-(A) = \left(\sum_{k=0}^{\infty} \mathbf{F}^{mk*} * \sum_{k=0}^{m-1} \mathbf{F}^{k*} * \mathbf{F}_- \right) (A) \\ &\leq \sum_{k=0}^{\infty} \left(\mathbf{F}^{mk*} \right)_s(\mathbb{R}) \sum_{k=0}^{m-1} \mathbf{F}^{k*}(\mathbb{R}) |\mathbf{F}_-(\mathbb{R})| \\ &\leq \sum_{k=0}^{\infty} [(\mathbf{F}^{m*})_s(\mathbb{R})]^k \sum_{k=0}^{m-1} \mathbf{F}^{k*}(\mathbb{R}) |\mathbf{F}_-(\mathbb{R})| < \infty. \end{aligned}$$

The last inequality follows from $\varrho[(\mathbf{F}_+^{m*})_s(\mathbb{R})] < 1$. Actually,

$$[\mathbf{I} - (\mathbf{F}^{m*})_s(\mathbb{R})]^{-1} = \sum_{k=0}^{\infty} [(\mathbf{F}^{m*})_s(\mathbb{R})]^k < \infty.$$

The contradiction between (17) and (18) proves that \mathbf{F}_+ is nonlattice. Applying [9, Theorem 2.1] to \mathbf{F}_+ , we see that (14) is valid for \mathbf{H}_1 with some constant matrix $\mathbf{K}_1(h)$. It follows from (16) that

$$(19) \quad \sup_{x \in \mathbb{R}} H_{ij}((x, x + h]) \leq \sum_{k=1}^n [\mathbf{K}_1(h)]_{ik} [\mathbf{F}^{-1}(\mathbb{R})]_{kj},$$

i.e., \mathbf{H} satisfies (14). Let now $\mathbf{F}(\mathbb{R})$ be a primitive but not necessarily stochastic matrix. We set $\mathbf{J} := \text{diag}(r_1, \dots, r_n)$, $\mathbf{G} := \mathbf{J}^{-1}\mathbf{F}\mathbf{J}$, and $\mathbf{H}_G := \sum_{k=0}^{\infty} \mathbf{G}^{k*}$. Since $\mathbf{G}(\mathbb{R}) = \mathbf{J}^{-1}\mathbf{F}(\mathbb{R})\mathbf{J}$ is a primitive stochastic matrix, \mathbf{H}_G satisfies (19). But then

$$(20) \quad \mathbf{H}((x, x + h]) = \mathbf{J}^{-1}\mathbf{H}_G((x, x + h])\mathbf{J} \leq \mathbf{J}^{-1}\mathbf{K}_G(h)\mathbf{J} < \infty.$$

Finally, let $\mathbf{F}(\mathbb{R})$ be an irreducible matrix with imprimitivity index $d > 1$ [14]. Rearranging, if necessary, the same rows and columns of $\mathbf{F}(\mathbb{R})^d$, we can decompose it into d primitive block matrices of spectral radius 1. Without loss of generality, we may assume that such a decomposition has already been achieved:

$$\mathbf{F}(\mathbb{R})^d = \begin{pmatrix} \mathbf{F}^{(1)}(\mathbb{R}) & 0 & \dots & 0 \\ 0 & \mathbf{F}^{(2)}(\mathbb{R}) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{F}^{(d)}(\mathbb{R}) \end{pmatrix},$$

where $\mathbf{F}^{(i)}(\mathbb{R})$ are primitive matrices. It follows that all the block matrices $\mathbf{F}^{(i)}$ are of the same type as \mathbf{F} itself; namely, there exists an integer $q \geq 1$ such that $\varrho[(\mathbf{F}^{(i)q*})_s(\mathbb{R})] < 1 \forall i$ [28, the proof of Lemma 2]. Put $\mathbf{H}^{(0)} := \sum_{k=0}^{\infty} \mathbf{F}^{kd*}$ and $\mathbf{H}^{(j)} := \mathbf{F}^{j*} * \mathbf{H}^{(0)}$, $j = 1, \dots, d - 1$. It is clear that $\mathbf{H} = \sum_{j=0}^{d-1} \mathbf{H}^{(j)}$ and

$$\mathbf{H}^{(0)} = \begin{pmatrix} \mathbf{H}_1^{(0)} & 0 & \dots & 0 \\ 0 & \mathbf{H}_2^{(0)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{H}_d^{(0)} \end{pmatrix},$$

where $\mathbf{H}_i^{(0)}$ are the matrix renewal measures induced by $\mathbf{F}^{(i)}$, $i = 1, \dots, d$. Since each of the $\mathbf{H}_i^{(0)}$ satisfies a relation similar to (14), the same is true

for $\mathbf{H}^{(0)}$. Next, relations of type (19) with \mathbf{F}_-^{-1} replaced by \mathbf{F}^{j*} hold for the $\mathbf{H}^{(j)}$. Finally, “summing up” the corresponding inequalities for $\mathbf{H}^{(j)}$ over j from 0 to $d - 1$, we obtain (14) in the general case. To complete the proof of Lemma 5, it now remains to apply Lemma 4. \square

Denote by $\mathcal{F}(u)$ the Fourier transform of $u \in \mathcal{S}'_1$: $\mathcal{F}(u)(\phi) := u(\mathcal{F}(\phi))$, $\phi \in \mathcal{S}_1$, where

$$\mathcal{F}(\phi)(t) := (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(x) \exp(-itx) dx, \quad t \in \mathbb{R}.$$

Let ν be a σ -finite measure defining an element in \mathcal{S}'_1 . For arbitrary $a \in \mathbb{R}$, we set $\nu_a(A) := \nu(A - a)$, $A \in \mathcal{B}$. Define the element $\Delta_a \nu \in \mathcal{S}'_1$ by $\Delta_a \nu := \nu - \nu_a$. Then $\mathcal{F}(\Delta_a \nu) = [1 - \exp(-iat)]\mathcal{F}(\nu)$. If ν and κ are any two measures which define tempered distributions and for which the convolution $\nu * \kappa$ makes sense, then obviously $\Delta_a(\nu * \kappa) = \nu * (\Delta_a \kappa) = (\Delta_a \nu) * \kappa$.

LEMMA 6 [26, Lemma 3]. *Let ν be a finite measure. Then the tempered distribution $\mathcal{F}(\Delta_a T\nu)$ may be identified with the function*

$$(2\pi)^{-1/2}(1 - e^{-iax})[\hat{\nu}(-ix) - \hat{\nu}(0)]/(-ix), \quad x \in \mathbb{R}.$$

It is also clear that $\Delta_a L$ is Lebesgue measure on the interval $[0, a]$ and, therefore, the tempered distribution $\mathcal{F}(\Delta_a L)$ can be identified with the function $[1 - \exp(-iat)]/[it(2\pi)^{1/2}]$, $t \in \mathbb{R}$.

We now turn to (11). Put $s = -ix$, $x \in \mathbb{R}$, multiply both sides by $(2\pi)^{-1/2}(1 - e^{-iax})\phi(x)$, $\phi \in \mathcal{S}_1$, and then integrate the resulting equality over the whole line \mathbb{R} . This yields

$$\begin{aligned} (21) \quad & (2\pi)^{-1/2} \int_{\mathbb{R}} (1 - e^{-iax})[\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \phi(x) dx \\ & = (2\pi)^{-1/2} \int_{\mathbb{R}} r\hat{\mathbf{Q}}(0) \frac{1 - e^{-iax}}{ix} \phi(x) dx \\ & - (2\pi)^{-1/2} \int_{\mathbb{R}} (1 - e^{-iax}) \frac{r[\hat{\mathbf{Q}}(-ix) - \hat{\mathbf{Q}}(0)]}{-ix} \phi(x) dx \\ & + (2\pi)^{-1/2} \int_{\mathbb{R}} (1 - e^{-iax}) \hat{\mathbf{Q}}(-ix) \phi(x) dx. \end{aligned}$$

LEMMA 7. Under the hypotheses of Theorem 1, the left-hand side of (21) is equal to $\mathcal{F}(\Delta_a \mathbf{H})(\phi)$.

PROOF OF LEMMA 7. Let $z \in (0, 1)$. Define the matrix measure \mathbf{H}_z by

$$\mathbf{H}_z(A) := \sum_{k=0}^{\infty} z^k \mathbf{F}^{k*}(A), \quad A \in \mathcal{B}.$$

Since $\varrho[z\mathbf{F}(\mathbb{R})] < 1$, the measure \mathbf{H}_z is finite and $\hat{\mathbf{H}}_z(s) = [\mathbf{I} - z\hat{\mathbf{F}}(s)]^{-1}$, $\Re s = 0$. It follows that the tempered distribution $\mathcal{F}(\mathbf{H}_z)$ may be identified with the function $(2\pi)^{-1/2}[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1}$, $x \in \mathbb{R}$. Clearly, $\mathbf{H}_z \rightarrow \mathbf{H}$ in \mathcal{S}'_1 as $z \rightarrow 1-$, and hence $\Delta_a \mathbf{H}_z \rightarrow \Delta_a \mathbf{H}$ as $z \rightarrow 1-$. Next, $\mathcal{F}(\Delta_a \mathbf{H}_z) \rightarrow \mathcal{F}(\Delta_a \mathbf{H})$ as $z \rightarrow 1-$. Now to complete the proof of Lemma 7, it suffices to justify the passage to the limit as $z \rightarrow 1-$ in

$$(22) \quad \mathcal{F}(\Delta_a \mathbf{H}_z)(\phi) = (2\pi)^{-1/2} \int_{\mathbb{R}} (1 - e^{-iax}) [\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} \phi(x) dx.$$

Subtracting the left-hand side of (21) from the right-hand side of (22) and multiplying both sides of the resulting equation by $(2\pi)^{1/2}$, we have

$$(23) \quad \begin{aligned} \mathbf{A}(z) &:= \int_{\mathbb{R}} (1 - e^{-iax}) \{ [\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} - [\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \} \phi(x) dx \\ &= \int_{\mathbb{R}} (1 - e^{-iax}) (z - 1) \hat{\mathbf{F}}(-ix) [\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} \\ &\quad \times [\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \phi(x) dx. \end{aligned}$$

As $z \rightarrow 1-$, the integrand converges to zero for all $x \neq 0$. Now, in order to prove $\lim_{z \rightarrow 1-} \mathbf{A}(z) = \mathbf{0}$, it remains to find an integrable majorant. By Theorem 8.6.1 of [16], $\sum_{j=0}^k \mathbf{F}(\mathbb{R})^j / k \rightarrow \mathbf{rl}$ as $k \rightarrow \infty$. Applying the Tauberian theorem for power series [12, Chapter XIII, Section 5, Theorem 5], we have, as $z \rightarrow 1-$,

$$(24) \quad |(z - 1)\hat{\mathbf{F}}(-ix)[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1}| \leq \mathbf{F}(\mathbb{R})(1 - z) \sum_{j=0}^{\infty} z^j \mathbf{F}(\mathbb{R})^j \rightarrow \mathbf{rl}.$$

It follows that the left-hand side of (24) is bounded for all $x \in \mathbb{R}$ and for all $z < 1$ sufficiently close to 1. Let us bound the expression

$$\mathbf{B}(x) := (1 - e^{-iax})[\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} = \frac{(1 - e^{-iax})\hat{\mathbf{M}}(-ix)}{\hat{\alpha}(-ix)}.$$

In a sufficiently small neighborhood of zero, we have

$$(25) \quad |\mathbf{B}(x)| \leq \frac{|ax||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)x + o(x)|} \leq \frac{|ax||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)||x| - |o(x)|} \leq \frac{2|a||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)|}.$$

It remains to bound $\mathbf{B}(x)$ outside a neighborhood of zero. We have

$$(26) \quad |\mathbf{B}(x)| \leq 2|\mathbf{M}|(\mathbb{R}) / \inf_{|x| \geq \varepsilon} |\hat{\alpha}(-ix)|.$$

The function $\hat{\alpha}(-ix)$, $x \in \mathbb{R}$, is continuous. By assumption, $|\hat{\alpha}(-ix)| \neq 0 \forall x \neq 0$. We will show that $\liminf_{|x| \rightarrow \infty} |\hat{\alpha}(-ix)| > 0$, thus establishing

$$(27) \quad \inf_{|x| \geq \varepsilon} |\hat{\alpha}(-ix)| > 0.$$

We argue by contradiction. Assume that

$$(28) \quad \liminf_{|x| \rightarrow \infty} |\hat{\alpha}(-ix)| = 0.$$

Then there exists a sequence $\{x_k\}$ such that $|x_k| \rightarrow \infty$, $\hat{\alpha}(-ix_k) \rightarrow 0$ and $\hat{\mathbf{F}}(-ix_k) \rightarrow \mathbf{G}$ as $k \rightarrow \infty$, where \mathbf{G} is a numerical matrix with $\varrho(\mathbf{G}) \geq 1$ since $\det(\mathbf{I} - \mathbf{G}) = \lim_{k \rightarrow \infty} \hat{\alpha}(-ix_k) = 0$. Let m be the number appearing in the statement of Theorem 1. Then $\hat{\mathbf{F}}(-ix_k)^m \rightarrow \mathbf{G}^m$ as $k \rightarrow \infty$. By the Riemann-Lebesgue lemma, $\lim_{k \rightarrow \infty} \left\{ \hat{\mathbf{F}}(-ix_k)^m - [(\mathbf{F}^{m*})_s]^\wedge(-ix_k) \right\} = \mathbf{0}$, and hence $|\mathbf{G}| \leq (\mathbf{F}^{m*})_s(\mathbb{R})$, whence $\varrho(\mathbf{G}^m) \leq \varrho[(\mathbf{F}^{m*})_s(\mathbb{R})] < 1$. On the other hand, $\varrho(\mathbf{G}^m) = [\varrho(\mathbf{G})]^m \geq 1$. The contradiction shows that the assumption (28) is false. It follows from (24)–(27) that the integrand in (23) is bounded entrywise by a matrix of functions $\phi(x)\mathbf{K}$, where $\mathbf{K} > \mathbf{0}$ is a constant matrix. Lemma 7 is proved. \square

LEMMA 8. *Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite non-negative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and*

$\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue 1 of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} \in (0, +\infty)$. Then, for any fixed $h > 0$,

$$\mathbf{H}((x, x + h]) \rightarrow \mathbf{0} \quad \text{as } x \rightarrow -\infty.$$

PROOF OF LEMMA 8. This is done by retracing the proof of Lemma 5 and applying, wherever necessary, the dominated convergence theorem. First, suppose $\mathbf{F}(\mathbb{R})$ is a primitive stochastic matrix. We have $\lim_{x \rightarrow -\infty} \mathbf{H}_1((x - y, x - y + h]) = \mathbf{0} \ \forall y \in \mathbb{R}$ since \mathbf{H}_1 is concentrated on $(0, \infty)$. By dominated convergence, the assertion of the lemma now follows from (16) and from (14) with \mathbf{H} replaced by \mathbf{H}_1 . If $\mathbf{F}(\mathbb{R})$ is primitive but not necessarily stochastic, then the assertion of the lemma is implied by the equality in (20). Consider now the general case. By the above, $\lim_{x \rightarrow -\infty} \mathbf{H}^{(0)}((x, x + h]) = \mathbf{0}$. Applying again the dominated convergence theorem, we have $\lim_{x \rightarrow -\infty} \mathbf{H}^{(j)}((x, x + h]) = \mathbf{0}, j = 1, \dots, d - 1$. and hence $\mathbf{H}((x, x + h]) \rightarrow \mathbf{0}$ as $x \rightarrow -\infty$. Lemma 8 is proved. \square

We return to the proof of Theorem 1. It follows from (11) and Lemmas 3–7 that $\mathcal{F}(\Delta_a \mathbf{H}) = r \hat{\mathbf{Q}}(0) \mathcal{F}(\Delta_a L) - \mathcal{F}(r \Delta_a T \mathbf{Q}) + \mathcal{F}(\Delta_a \mathbf{Q})$. Passing over from Fourier transforms to their inverse images, we obtain

$$(29) \quad \Delta_a \mathbf{H} = r \hat{\mathbf{Q}}(0) \Delta_a L - \Delta_a r T \mathbf{Q} + \Delta_a \mathbf{Q}.$$

Let $\mathcal{D}(\mathbb{R})$ be the space of all infinitely differentiable functions with compact supports. Any tempered distribution is completely determined by its values at functions $\phi \in \mathcal{D}(\mathbb{R})$ since $\mathcal{D}(\mathbb{R})$ is dense in \mathcal{S}_1 [21, Theorem 7.10]. Let ϕ be an arbitrary element of $\mathcal{D}(\mathbb{R})$ whose support is contained in a finite interval $[c, d]$. Apply both sides of (29) to ϕ . The left-hand side becomes $\int_{\mathbb{R}} \phi(x) \mathbf{H}(dx) - \int_{\mathbb{R}} \phi(x+a) \mathbf{H}(dx)$. The latter integral is bounded in absolute value by the matrix

$$\mathbf{H}([c - a, d - a]) \max\{|\phi(x)| : x \in [c, d]\},$$

which, by Lemma 8, tends to the null matrix $\mathbf{0}$ as $a \rightarrow +\infty$. Next,

$$\int_{\mathbb{R}} r \hat{\mathbf{Q}}(0) \phi(x) \Delta_a L(dx) = r \hat{\mathbf{Q}}(0) \int_0^a \phi(x) dx \rightarrow r \hat{\mathbf{Q}}(0) \int_0^\infty \phi(x) dx$$

as $a \rightarrow +\infty$ and

$$\Delta_a rT\mathbf{Q}(\phi) = \int_{\mathbb{R}} \phi(x) rT\mathbf{Q}(dx) - \int_{\mathbb{R}} \phi(x+a) rT\mathbf{Q}(dx);$$

the latter integral is bounded in absolute value by

$$rT\mathbf{Q}([c-a, d-a]) \max\{|\phi(x)| : x \in [c, d]\} \rightarrow \mathbf{0} \quad \text{as } a \rightarrow +\infty$$

since \mathbf{Q} is a matrix of finite measures and

$$|T\mathbf{Q}([c-a, d-a])| = \left| \int_{c-a}^{d-a} \mathbf{Q}((-\infty, x]) dx \right| \leq (d-c) |\mathbf{Q}|((-\infty, d-a]).$$

Finally,

$$\Delta_a \mathbf{Q}(\phi) = \int_{\mathbb{R}} \phi(x) \mathbf{Q}(dx) - \int_{\mathbb{R}} \phi(x+a) \mathbf{Q}(dx).$$

The latter integral is bounded in absolute value by

$$\mathbf{Q}([c-a, d-a]) \max\{|\phi(x)| : x \in [c, d]\} \rightarrow \mathbf{0} \quad \text{as } a \rightarrow +\infty$$

since \mathbf{Q} is a matrix of finite measures. Summing up, we arrive at the equality $\mathbf{H}(\phi) = r\hat{\mathbf{Q}}(0)L(\phi) - rT\mathbf{Q}(\phi) + \mathbf{Q}(\phi) \forall \phi \in \mathcal{D}(\mathbb{R})$, whence (12) follows.

LEMMA 9. *Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite non-negative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Assume that $\hat{\mathbf{F}}'(0) := (\hat{F}'_{ij}(0))$ is a matrix with finite entries. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue 1 of $\hat{\mathbf{F}}'(0)$ in such a way that $\mathbf{l}\mathbf{r} = 1$. Then*

$$(30) \quad -\frac{\hat{\mathbf{M}}(0)}{\hat{\alpha}'(0)} = \frac{\mathbf{r}\mathbf{l}}{\mathbf{l}\hat{\mathbf{F}}'(0)\mathbf{r}}.$$

PROOF OF LEMMA 9. We shall suitably modify the arguments employed in the proofs of Lemmas 8 and 6 in [17]. Unlike [17], here we shall use the fact that, because of the irreducibility of $\hat{\mathbf{F}}(0)$, $\sum_{j=1}^k \hat{\mathbf{F}}(q)^j/k \rightarrow \mathbf{r}\mathbf{l}$

as $k \rightarrow \infty$ [16, Theorem 8.6.1], and we will also replace the column vector \mathbf{e} with unit coordinates by the right eigenvector \mathbf{r} . We have $\hat{\mathbf{M}}(0) = c\mathbf{r}\mathbf{l}$, where

$$c = \lim_{z \rightarrow 1^-} \det(\mathbf{I} - z\hat{\mathbf{F}}(0))/(1 - z) > 0.$$

Indeed, for $0 < z < 1$, the adjugate matrix $\hat{\mathbf{M}}_z(0)$ of $\mathbf{I} - z\hat{\mathbf{F}}(0)$ is equal to

$$(31) \quad \det(\mathbf{I} - z\hat{\mathbf{F}}(0)) \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k = \frac{\det(\mathbf{I} - z\hat{\mathbf{F}}(0))}{1 - z} (1 - z) \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k.$$

The characteristic polynomial $f(\lambda)$ of $\hat{\mathbf{F}}(0)$ is equal to $\det(\lambda\mathbf{I} - \hat{\mathbf{F}}(0)) = \prod_{j=1}^k (\lambda - \lambda_j)^{m_j}$, where λ_j are the eigenvalues of $\hat{\mathbf{F}}(0)$ and m_j are their multiplicities, $j = 1, \dots, k$, $\sum_{j=1}^k m_j = n$, the eigenvalue $\lambda_1 = 1$ having multiplicity 1. Hence

$$\det(\mathbf{I} - z\hat{\mathbf{F}}(0)) = z^n f(1/z) = z^n \prod_{j=1}^k (1/z - \lambda_j)^{m_j} = (1 - z) \prod_{j=2}^k (1 - z\lambda_j)^{m_j}.$$

Thus, $c := \prod_{j=2}^k (1 - \lambda_j)^{m_j} > 0$ since $\lambda_j \neq 1 \ \forall j > 1$ and if λ_j is a complex eigenvalue, then $\exists \lambda_i = \overline{\lambda_j}$ with $m_i = m_j$, whence $(1 - \lambda_j)^{m_j} (1 - \lambda_i)^{m_i} = |1 - \lambda_j|^{2m_j}$. By the Tauberian theorem for power series, $(1 - z) \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k \rightarrow \mathbf{r}\mathbf{l}$ as $z \rightarrow 1^-$ [12, Chapter XIII, Section 5, Theorem 5]. Therefore, it follows from (31) that

$$(32) \quad \hat{\mathbf{M}}(0) = \lim_{z \rightarrow 1^-} \hat{\mathbf{M}}_z(0) = c\mathbf{r}\mathbf{l}.$$

Differentiating $[\mathbf{I} - \hat{\mathbf{F}}(s)]\hat{\mathbf{M}}(s) = \hat{\alpha}(s)\mathbf{I}$ at $s = 0$, we obtain $-\hat{\mathbf{F}}'(0)\hat{\mathbf{M}}(0) + [\mathbf{I} - \hat{\mathbf{F}}(0)]\hat{\mathbf{M}}'(0) = \hat{\alpha}'(0)\mathbf{I}$. Multiply both sides of the last equality first by \mathbf{l} from the left and then by \mathbf{r} from the right. Taking into account (32) and $\mathbf{l}\mathbf{r} = 1$, we arrive at

$$(33) \quad -\mathbf{l}\hat{\mathbf{F}}'(0)\hat{\mathbf{M}}(0)\mathbf{r} = -\mathbf{l}\hat{\mathbf{F}}'(0)c\mathbf{r} = \hat{\alpha}'(0).$$

Now (30) follows from (32) and (33). Lemma 9 is proved. \square

By Lemma 9, $r\hat{\mathbf{Q}}(0) = -\hat{\mathbf{M}}(0)/\hat{\alpha}'(0) = \mathbf{rl}/\mu$. In order to complete the proof of Theorem 1, it now remains to put $\mathbf{H}_1 := (\mathbf{rl}/\mu)L - rT\mathbf{Q}$ and $\mathbf{H}_2 := \mathbf{Q}$ (see (12)).

REMARK 1. In the case $r_1 = r_2 = 0$, the requirement that in $\Pi(r_1, r_2)$ there be no nonzero roots of (6) is superfluous as is shown by the following lemma.

LEMMA 10. *Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite non-negative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Assume that $\varrho[(\mathbf{F}^{m*})_s(\mathbb{R})] < 1$ for some integer $m \geq 1$. Then $\det(\mathbf{I} - \hat{\mathbf{F}}(s)) \neq 0$ for all $s \neq 0$ such that $\Re s = 0$.*

PROOF OF LEMMA 10. Assume the contrary, i.e., $\det(\mathbf{I} - \hat{\mathbf{F}}(s_0)) = 0$ for some $s_0 \neq 0$ with $\Re s_0 = 0$. This means that 1 is an eigenvalue of $\hat{\mathbf{F}}(s_0)$. By Theorem 8.4.5 of [16], there exist numbers $\theta_1, \dots, \theta_n \in \mathbb{R}$ such that $\hat{\mathbf{F}}(s_0) = \mathbf{D}\mathbf{F}(\mathbb{R})\mathbf{D}^{-1}$, where $\mathbf{D} = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$, and hence $\hat{\mathbf{F}}(s_0)^m = \mathbf{D}\mathbf{F}(\mathbb{R})^m\mathbf{D}^{-1}$. By assumption, $\mathbf{F}^{m*} = \mathbf{F}^{(1)} + \mathbf{F}^{(2)}$, where $\mathbf{F}^{(1)}$ is a nonzero matrix of absolutely continuous nonnegative measures, so that

$$\mathbf{F}(\mathbb{R})^m = \mathbf{D}^{-1}\hat{\mathbf{F}}^{(1)}(s_0)\mathbf{D} + \mathbf{D}^{-1}\hat{\mathbf{F}}^{(2)}(s_0)\mathbf{D},$$

which is impossible by the following reason. Suppose $F_{jk}^{(1)}(\mathbb{R}) \neq 0$. Then

$$\left| \hat{F}_{jk}^{(1)}(s_0) \right| = \left| \int_{\mathbb{R}} e^{s_0 x} F_{jk}^{(1)}(dx) \right| < F_{jk}^{(1)}(\mathbb{R})$$

and hence

$$\begin{aligned} [\mathbf{D}^{-1}\hat{\mathbf{F}}(s_0)^m\mathbf{D}]_{jk} &= \left| e^{-i\theta_j} \hat{F}_{jk}^{(1)}(s_0)e^{i\theta_k} + e^{-i\theta_j} \hat{F}_{jk}^{(2)}(s_0)e^{i\theta_k} \right| \\ &\leq \left| \hat{F}_{jk}^{(1)}(s_0) \right| + \left| \hat{F}_{jk}^{(2)}(s_0) \right| < F_{jk}^{(1)}(\mathbb{R}) + F_{jk}^{(2)}(\mathbb{R}) = (\mathbf{F}^{m*})_{jk}(\mathbb{R}). \end{aligned}$$

This contradiction proves Lemma 10. \square

4. Solutions and Rates of Convergence

Consider first the simplest case $\mathcal{A} = S(0, 0)$, i.e., \mathcal{A} is the algebra of finite measures. Suppose $\mathbf{g} \in L_1(\mathbb{R})$. It is clear that under the hypotheses of Theorem 1 with $T^2\mathbf{F} \in \mathcal{A}$, $\mathbf{z}(x) := \mathbf{H} * \mathbf{g}(x)$ is a solution to (2); moreover, if \mathbf{c} is an arbitrary right eigenvector of $\mathbf{F}(\mathbb{R})$ corresponding to the eigenvalue 1, then $\mathbf{z}(x) + \mathbf{c}$ is also a solution. So the asymptotic properties of solutions to (2) are those of the convolution $\mathbf{H} * \mathbf{g}(x)$. The decomposition of \mathbf{H} provided by Theorem 1 allows us to write $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_i := \mathbf{H}_i * \mathbf{g}$, $i = 1, 2$. It follows that $\mathbf{z}_2 \in L_1(\mathbb{R})$ and \mathbf{z}_1 is a continuous function such that $\mathbf{z}_1(x) \rightarrow \mathbf{r}\mathbf{l} \int_{\mathbb{R}} \mathbf{g}(y) dy / \mu$ as $x \rightarrow +\infty$ and $\mathbf{z}_1(x) \rightarrow (0, \dots, 0)^T$ as $x \rightarrow -\infty$. If, in addition, $\mathbf{g}(x)$ is bounded and $\lim_{|x| \rightarrow \infty} \mathbf{g}(x) = (0, \dots, 0)^T$, then $\lim_{|x| \rightarrow \infty} \mathbf{z}_2(x) = (0, \dots, 0)^T$, and the solution \mathbf{z} has the following property:

$$(34) \quad \mathbf{z}(x) \rightarrow \begin{cases} \mathbf{r}\mathbf{l} \int_{\mathbb{R}} \mathbf{g}(y) dy / \mu & \text{as } x \rightarrow +\infty, \\ (0, \dots, 0)^T & \text{as } x \rightarrow -\infty. \end{cases}$$

In this section, we will limit ourselves to obtaining submultiplicative rates of convergence in (34) by means of the Stone-type decomposition of Theorem 1 with $\mathcal{A} := S(\varphi)$. Let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function such that $r_1 \leq 0 \leq r_2$. By Theorem 1 of [20], $S(\varphi)$ satisfies properties (i) and (ii) of the preceding section, and hence Theorem 1 applies. We note some nuances. Let F be a nonnegative finite measure. Relation $TF \in S(\varphi)$ implies $F \in S(\varphi)$. Actually,

$$\begin{aligned} \int_0^\infty \varphi(x) F((x, \infty)) dx &\geq \sum_{k=0}^\infty \inf_{x \in [k, k+1)} \varphi(x) F((k+1, k+2]) \\ &\geq \frac{1}{M(1)} \sum_{k=0}^\infty \int_{k+1}^{k+2} \varphi(x) F(dx) = \frac{1}{M(1)} \int_1^\infty \varphi(x) F(dx). \end{aligned}$$

Since, obviously, $\int_0^1 \varphi(x) F(dx) < \infty$, we have $\int_0^\infty \varphi(x) F(dx) < \infty$. Similarly, $\int_{-\infty}^0 \varphi(x) F(dx) < \infty$. Therefore, instead of the hypotheses \mathbf{F} , $T\mathbf{F} \in S(\varphi)$ in Theorem 1, we may assume only $T\mathbf{F} \in S(\varphi)$. Similarly, the set of conditions \mathbf{F} , $T\mathbf{F}$, $T^2\mathbf{F} \in S(\varphi)$ may be replaced by $T^2\mathbf{F} \in S(\varphi)$. Suppose now that $\varphi(x) / \exp(r_1x)$ is nonincreasing on $(-\infty, 0)$ and $\varphi(x) / \exp(r_2x)$ is nondecreasing on $[0, \infty)$. Theorem 3 of [27] implies that

if $r_1 = 0 = r_2$ and $\int_{\mathbb{R}}(1 + |x|)^k \varphi(x) F(dx) < \infty$ for some integer $k \geq 1$, or if $r_1 < 0 = r_2$ and $\int_0^\infty (1 + x)^k \varphi(x) F(dx) < \infty$, or if $r_1 = 0 < r_2$ and $\int_{-\infty}^0 (1 + |x|)^k \varphi(x) F(dx) < \infty$, then $T^k F \in S(\varphi)$. If $r_1 < 0 < r_2$, then $F \in S(\varphi) \Rightarrow T^k F \in S(\varphi) \forall k \geq 1$ [27, Theorem 2]. Suppose now that $r_1 = 0 = r_2$. Then, instead of the hypotheses $\mathbf{F}, T\mathbf{F} \in S(\varphi)$ in Theorem 1, we may assume only $\mathbf{F} \in S(\varphi_1)$, where $\varphi_1(x) := (1 + |x|)\varphi(x)$. Similarly, the set of conditions $\mathbf{F}, T\mathbf{F}, T^2\mathbf{F} \in S(\varphi)$ may be replaced by $\mathbf{F} \in S(\varphi_2)$, where $\varphi_2(x) := (1 + |x|)^2\varphi(x)$. In the latter case, H_2 will be in $S(\varphi_1)$. Suppose $r_1 < 0 < r_2$. Then the set of conditions $\mathbf{F}, T\mathbf{F}, T^2\mathbf{F} \in S(\varphi)$ may be replaced by $\mathbf{F} \in S(\varphi)$. The intermediary cases $r_1 < 0 = r_2$ and $r_1 = 0 < r_2$ are dealt with in a similar way.

THEOREM 2. *Let $\varphi(x)$ be a submultiplicative function such that $r_1 \leq 0 \leq r_2$, and let $\mathbf{g}(x), x \in \mathbb{R}$, be a Borel-measurable vector function such that (a) $\mathbf{g} \in L_1(\mathbb{R})$, (b) $\mathbf{g} \cdot \varphi \in L_\infty(\mathbb{R})$, (c) $\mathbf{g}(x)\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ outside a set of Lebesgue measure zero, and (d) $\varphi(t) \int_t^\infty |\mathbf{g}(x)| dx \rightarrow 0$ as $t \rightarrow +\infty$ and $\varphi(t) \int_{-\infty}^t |\mathbf{g}(x)| dx \rightarrow 0$ as $t \rightarrow -\infty$. Let \mathbf{F} be an $n \times n$ -matrix whose elements are finite nonnegative measures on \mathbb{R} . Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue 1 of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l} \int_{\mathbb{R}} x \mathbf{F}(dx) \mathbf{r} \in (0, +\infty)$, $\varrho[(\mathbf{F}^{m*})_s^\wedge(r_i)] < 1$, $i = 1, 2$, for some integer $m \geq 1$, and that in the strip $\Pi(r_1, r_2)$ there are no nonzero roots of the characteristic equation (6) distinct from zero. Suppose $T^2\mathbf{F} \in S(\varphi)$. Then, as t approaches $\pm\infty$ outside a set of Lebesgue measure zero,*

$$\sup_{\alpha: |\alpha| \leq |\mathbf{g}|} \left| \mathbf{H} * \alpha(t) - \frac{\mathbf{r}\mathbf{l}}{\mu} \int_{\mathbb{R}} \alpha(x) dx \right| = o\left(\frac{1}{\varphi(t)}\right)$$

and $\sup_{\alpha: |\alpha| \leq |\mathbf{g}|} |\mathbf{H} * \alpha(t)| = o(1/\varphi(t))$, respectively, the $\alpha(x)$ being Borel-measurable vector functions on \mathbb{R} .

PROOF. By Theorem 1 with $\mathcal{A} = S(\varphi)$, both $\mathbf{H}_1 - \mathbf{r}\mathbf{l}L/\mu$ and \mathbf{H}_2 are elements of $S(\varphi)$. Choose $\tilde{\mathbf{g}} \in L_1(\mathbb{R})$ such that $\tilde{\mathbf{g}} = \mathbf{g}$ a.e., $\sup_{x \in \mathbb{R}} |\tilde{\mathbf{g}}(x)|\varphi(x) < \infty$, and $\tilde{\mathbf{g}}(x)\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ in the usual sense. It suffices to put $\tilde{\mathbf{g}}(x) = 0$ on $\{x \in \mathbb{R} : |\mathbf{g}(x)|\varphi(x) > \|\mathbf{g} \cdot \varphi\|_\infty\}$ and on a set, say B , of Lebesgue measure zero such that $\lim_{x \notin B, |x| \rightarrow \infty} \mathbf{g}(x)\varphi(x) = 0$;

otherwise, $\tilde{\mathbf{g}}(x) := \mathbf{g}(x)$. By Fubini's theorem, the sets

$$A_1 := \left\{ x : \left| \mathbf{H}_1 - \frac{\mathbf{r}L}{\mu} \right| * |\tilde{\mathbf{g}}|(x) \neq \left| \mathbf{H}_1 - \frac{\mathbf{r}L}{\mu} \right| * |\mathbf{g}|(x) \right\}$$

and $A_2 := \{x : |\mathbf{H}_2| * |\tilde{\mathbf{g}}|(x) \neq |\mathbf{H}_2| * |\mathbf{g}|(x)\}$ are both of Lebesgue measure zero. Set $A := A_1 \cup A_2$. We have

$$\varphi(t)|\mathbf{H}_2| * |\tilde{\mathbf{g}}|(t) \leq \int_{\mathbb{R}} |\mathbf{H}_2|(dx)\varphi(t-x)\varphi(x)|\tilde{\mathbf{g}}(t-x)|.$$

By dominated convergence, the right-hand side tends to zero as $|t| \rightarrow \infty$, and so does the left-hand side. Similarly, $\lim_{|t| \rightarrow \infty} \varphi(t)|\mathbf{H}_1 - \mathbf{r}L/\mu| * |\tilde{\mathbf{g}}|(t) = 0$. Hence both $\varphi(t)|\mathbf{H}_2| * |\mathbf{g}|(t)$ and $\varphi(t)|\mathbf{H}_1 - \mathbf{r}L/\mu| * |\mathbf{g}|(t)$ tend to zero as $|t| \rightarrow \infty$, remaining outside the set A of Lebesgue measure zero. The first assertion of the theorem now follows from the obvious inequality

$$\begin{aligned} \left| \mathbf{H} * \alpha(t) - \frac{\mathbf{r}l}{\mu} \int_{\mathbb{R}} \alpha(x) dx \right| \\ \leq \left| \mathbf{H}_1 - \frac{\mathbf{r}L}{\mu} \right| * |\mathbf{g}|(t) + |\mathbf{H}_2| * |\mathbf{g}|(t) + \frac{\mathbf{r}l}{\mu} \int_t^\infty |\mathbf{g}(x)| dx \end{aligned}$$

and condition (d). The case $t \rightarrow -\infty$ is dealt with in a similar way. \square

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