On the existence of homomorphisms between principal series of complex semisimple Lie groups

（複素半単純リー群の主系列表現の間の準同型の存在について）

阿部 紀行

On the existence of homomorphisms between principal series of complex semisimple Lie groups

Noriyuki Abe

ABSTRACT. We determine when there exists a nonzero homomorphism between principal series representations of a complex semisimple Lie group. We also determine the condition for the existence of nonzero homomorphisms between twisted Verma modules.

§1. Introduction

Let G be a complex semisimple Lie group. Then the principal series representations of G are defined and play an important role in the representation theory of G. One of a fundamental problem about principal series is a description of the space of homomorphisms between such representations (cf. [Žel75, p. 720, II]). In this paper, we determine when there exists a nonzero homomorphism between principal series representations of a complex semisimple Lie group. We also determines the existence of homomorphisms between twisted Verma modules. This gives a generalization of results of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71].

We state our main results. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ its Cartan subalgebra, Δ the root system for $(\mathfrak{g}, \mathfrak{h})$ and W the Weyl group of Δ. By the Killing form we identify $\mathfrak{g}$ with $\mathfrak{g}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$. Then the Killing form also defines a non-degenerate bilinear form on $\mathfrak{g}^*$. We denote this form by $\langle \cdot, \cdot \rangle$. For $\alpha \in \mathfrak{h}^*$, put $\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$ and $s_\alpha(\lambda) = \lambda - \langle \check{\alpha}, \lambda \rangle \alpha$. Take a positive system $\Delta^+ \subset \Delta$. Then Δ^+ determines a Borel subalgebra $\mathfrak{b}$. Put $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Let $\mathcal{O}$ be the Bernstein-Gelfand-Gelfand category [BGG76, Definition 1] for $(\mathfrak{g}, \mathfrak{b})$ and $M(\lambda)$ the Verma module with highest weight $\lambda - \rho$ for $\lambda \in \mathfrak{h}^*$ where ρ is the half sum of positive roots. Fix an involution σ of $\mathfrak{g}$ such that $\sigma|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. The category $\mathcal{O}$ has a dualizing functor δ defined by $\delta M = \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C})_{\mathfrak{h}\text{-finite}}$ where the action is given by $(Xf)(m) = f(-\sigma(X)m)$. Put $\mathfrak{k} = \{(X, \sigma(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}$. For $M, N \in \mathcal{O}$, we define the $\mathfrak{g} \oplus \mathfrak{g}$-module $L(M, N) = \operatorname{Hom}_{\mathbb{C}}(M, N)_{\mathfrak{k}\text{-finite}}$ where the action is given by $((X, Y)f)(m) = \sigma(X)f(-Ym)$. Then under some identification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g} \oplus \mathfrak{g}$, the principal representations of G are $L(\lambda, \mu) = L(M(-\mu), \delta M(-\lambda))$. This is an object of $\mathcal{H}$ where $\mathcal{H}$ is a category of Harish-Chandra modules.

For $\lambda \in \mathfrak{h}^*$, let Δ_λ be the integral root system of λ, W_λ the Weyl group of Δ_λ. Let $\mathcal{P}$ be the integral weight lattice of Δ. Then it is well-known that $W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathcal{P}\}$. Let w_λ be the longest element of W_λ. Put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. Then Δ_λ^+ determines the set of simple roots Π_λ. Put $S_\lambda = \{s_\alpha \mid \alpha \in \Pi_\lambda\}$ and $W_\lambda^0 = \{w \in W_\lambda \mid w\lambda = \lambda\}$. For $w \in W_\lambda$, let $\ell_\lambda(w)$ be a length of w as an element of W_λ.

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3–8–1 KOMABA, MEGURO-KU, TOKYO 153–8914, JAPAN.

E-mail address: abenori@ms.u-tokyo.ac.jp

2000 *Mathematics Subject Classification*. Primary 22E47, Secondary 17B10.

For a sequence of simple roots $\alpha_1, \dots, \alpha_l \in \Pi_\lambda$ and $\mu \in \mathfrak{h}^*$, we define a subset $A_{(s_{\alpha_1},\dots,s_{\alpha_l})}(\mu)$ of $\mathfrak{h}^*$ as follows. Put $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ for $i = 1, \dots, l$. For $\mu \in \mathfrak{h}^*$, put

$$A_{(s_{\alpha_1},\dots,s_{\alpha_l})}(\mu) = \left\{ \mu' \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} \text{for some } 1 \le i_1 < \cdots < i_r \le l,\ \mu' = s_{\beta_{i_r}} \cdots s_{\beta_{i_1}} \mu \text{ and} \\ \langle \check{\beta}_{i_k}, s_{\beta_{i_{k-1}}} \cdots s_{\beta_{i_1}} \mu \rangle \in \mathbb{Z}_{<0} \text{ for all } k = 1, \dots, r \end{array} \right\}$$

For a reduced expression $w = s_1 \cdots s_l \in W$, it will be proved that the set $A_{(s_1,\dots,s_l)}(\mu)$ is independent of the choice of a reduced expression (Lemma 2.3). We write $A_w(\mu)$ instead of $A_{(s_1,\dots,s_l)}(\mu)$.

Now we state the main theorems of this paper.

Theorem 1.1. *Let $\lambda \in \mathfrak{h}^*$, $\mu_1, \mu_2 \in \lambda + \mathcal{P}$ and $w, w' \in W_\lambda$. Assume that λ is dominant, i.e., $\langle \check{\alpha}, \lambda \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Delta^+$. Then $\operatorname{Hom}_{\mathcal{H}}(L(M(w_1\lambda), \delta M(\mu_1)), L(M(w_2\lambda), \delta M(\mu_2))) \neq 0$ if and only if $w_1^{-1} w_\lambda A_{w_\lambda w_1}(w_\lambda \mu_1) \cap W_\lambda^0 w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset$.*

Moreover, if $\operatorname{Hom}_{\mathcal{H}}(L(M(w_1\lambda), \delta M(\mu_1)), L(M(w_2\lambda), \delta M(\mu_2))) = 0$, then for all $k \in \mathbb{Z}_{\ge 0}$, we have $\operatorname{Ext}^k_{\mathcal{H}}(L(M(w_1\lambda), \delta M(\mu_1)), L(M(w_2\lambda), \delta M(\mu_2))) = 0$.

We can determine when there exists a nonzero homomorphisms between principal series representations of G from Theorem 1.1 (see Lemma 3.2).

Let T_w be the twisting functor for $w \in W$ [AL03, 6.2] and w_0 the longest element of W (see also Arkhipov [Ark04, Definition 2.3.4]).

Theorem 1.2. *We have $\operatorname{Hom}_{\mathcal{O}}(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) \neq 0$ if and only if $w_1 A_{w_1^{-1}}(\mu_1) \cap w_2 w_0 A_{w_0 w_2^{-1}}(w_0 \mu_2) \neq \emptyset$.*

Moreover, if $\operatorname{Hom}_{\mathcal{O}}(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) = 0$, then $\operatorname{Ext}^k_{\mathcal{O}}(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) = 0$ for all $k \in \mathbb{Z}_{\ge 0}$.

The proof of this theorem gives a new proof of the famous result of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71] about homomorphisms between Verma modules.

Acknowledgments

The author would like to thank his advisor Hisayosi Matumoto for reading a manuscript and giving comments. He is supported by the Japan Society for the Promotion of Science Research Fellowships for Young Scientists.

§2. General theory

We use the notation in Section 1. It is easy to prove the following lemma. We omit the proof.

Lemma 2.1. *Let $s_1, \dots, s_l, s'_1, \dots, s'_{l'} \in S_\lambda$ be simple reflections. Put $w = s_1 \cdots s_l$. Then we have $A_{(s_1,\dots,s_l,s'_1,\dots,s'_{l'})}(\mu) = \bigcup_{\mu' \in A_{(s_1,\dots,s_l)}(\mu)} w A_{(s'_1,\dots,s'_{l'})}(w^{-1}\mu')$.*

Fix a dominant $\lambda \in \mathfrak{h}^*$. Let $\mathcal{C}$ be an abelian category with enough injective objects, $\mathcal{D} \subset \mathfrak{h}^*$ a W_λ-stable subset. Let $\{M_\lambda(w, \mu) \mid w \in W, \mu \in \mathcal{D}\}$ be objects of $\mathcal{C}$ such that the following conditions are satisfied:

(A1) For $w \in W_\lambda$ and $w' \in W_\lambda^0$, $M_\lambda(ww', \mu) \simeq M_\lambda(w, \mu)$.

(A2) For $\alpha \in \Pi_\lambda$ such that $s_\alpha w > w$, if $\langle \check{\alpha}, \mu \rangle \notin \mathbb{Z}_{<0}$ then we have $M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, s_\alpha \mu)$.

(A3) For $\alpha \in \Pi_\lambda$ such that $s_\alpha w > w$, $\langle \alpha, w\lambda \rangle \neq 0$ and $\langle \check{\alpha}, \mu \rangle \in \mathbb{Z}_{<0}$ there exists an exact sequence $0 \to M_\lambda(w, \mu) \to M_\lambda(s_\alpha w, \mu) \to M_\lambda(w, s_\alpha \mu) \to M_\lambda(w, \mu) \to 0$.

(A4) We have $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu'), M_\lambda(e, \mu)) \neq 0$ if and only if $\mu \in W_\lambda^0 w_\lambda \mu'$.

(A5) We have $\mathrm{Ext}_{\mathcal{C}}^k(M_\lambda(w_\lambda, \mu'), M_\lambda(e, \mu)) = 0$ for $k > 0$.

Lemma 2.2. *Let $\alpha \in \Pi_\lambda$, $w \in W_\lambda$, $\mu \in \mathcal{D}$. Assume that $\langle \alpha, w\lambda \rangle = 0$. Then we have $M_\lambda(w, \mu) \simeq M_\lambda(w, s_\alpha \mu) \simeq M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(s_\alpha w, s_\alpha \mu)$.*

Proof. If necessary, replacing w by $s_\alpha w$, we may assume that $s_\alpha w < w$. By applying the condition (A1) as $w' = s_{w^{-1}\alpha}$, we get $M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, \mu)$ and $M_\lambda(s_\alpha w, s_\alpha \mu) \simeq M_\lambda(w, s_\alpha \mu)$. If $\langle \alpha, \mu \rangle \geq 0$, then $M_\lambda(w, \mu) \simeq M_\lambda(s_\alpha w, s_\alpha \mu)$ by the condition (A2). If $\langle \alpha, \mu \rangle \leq 0$, then $M_\lambda(w, s_\alpha \mu) \simeq M_\lambda(s_\alpha w, \mu)$ by the condition (A2). Hence we have $M_\lambda(w, \mu) \simeq M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, s_\alpha \mu) \simeq M_\lambda(s_\alpha w, s_\alpha \mu)$ for all μ. □

Lemma 2.3. *Let $w_2 \in W$, $w_2 = s_1 \cdots s_l$ be a reduced expression and $\mu_1, \mu_2 \in \mathcal{D}$. Then the following conditions are equivalent.*

(1) $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0$.

(2) There exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Ext}_{\mathcal{C}}^k(M_\lambda(w_\lambda, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0$.

(3) $\mu_1 \in w_\lambda W_\lambda^0 w_2^{-1} A_{(s_1, \ldots, s_l)}(\mu_2)$.

Proof. Obviously, (1) implies (2). We prove the lemma by induction on $\ell_\lambda(w_2)$. If $w_2 = e$, then the lemma follows from the conditions (A4) and (A5).

Assume that $\ell_\lambda(w_2) > 0$. Take $\alpha \in \Pi_\lambda$ such that $s_1 = s_\alpha$. First assume that $\langle \alpha, w_2\lambda \rangle = 0$. Then we have $W_\lambda^0 (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(\mu_2) = W_\lambda^0 (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(s_\alpha \mu_2)$ by Lemma 2.2 and induction hypothesis. By the definition, we have $A_{(s_\alpha)}(\mu_2) = \{\mu_2\}$ or $A_{(s_\alpha)}(\mu_2) = \{\mu_2, s_\alpha \mu_2\}$. Therefore $W_\lambda^0 w_2^{-1} A_{(s_1, \ldots, s_l)}(\mu_2) = W_\lambda^0 (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(\mu_2)$ by Lemma 2.1. This implies the lemma in the case of $\langle \alpha, w_2\lambda \rangle = 0$.

In the rest of this proof, we assume that $\langle \alpha, w_2\lambda \rangle \neq 0$. Assume that $\langle \check{\alpha}, \mu_2 \rangle \notin \mathbb{Z}_{<0}$, then, by the condition (A2), $M_\lambda(w_2, \mu_2) \simeq M_\lambda(s_\alpha w_2, s_\alpha \mu_2)$. Since $A_{(s_\alpha)}(\mu_2) = \{\mu_2\}$, we have $w_2^{-1} A_{(s_1, \ldots, s_l)}(\mu_2) = (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(s_\alpha \mu_2)$ by Lemma 2.1. Hence (1)–(3) are equivalent in this case.

Finally assume that $\langle \check{\alpha}, \mu_2 \rangle \in \mathbb{Z}_{<0}$. Then we have $A_{(s_\alpha)}(\mu_2) = \{\mu_2, s_\alpha \mu_2\}$. This implies that $w_2^{-1} A_{(s_1, \ldots, s_l)}(\mu_2) = (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(\mu_2) \cup (s_\alpha w_2)^{-1} A_{(s_2, \ldots, s_l)}(s_\alpha \mu_2)$ by Lemma 2.1. By the induction hypothesis, $\mu_1 \notin w_\lambda W_\lambda^0 w_2^{-1} A_{(s_1, \ldots, s_l)}(\mu_2)$ if and only if

$$\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = \mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)) = 0.$$

From the condition (A3), we have an exact sequence

$$0 \to M_\lambda(s_\alpha w_2, \mu_2) \to M_\lambda(w_2, \mu_2) \to M_\lambda(s_\alpha w_2, s_\alpha \mu_2) \to M_\lambda(s_\alpha w_2, \mu_2) \to 0. \tag{2.1}$$

Since the functor $\mathrm{Hom}_{\mathcal{C}}$ is left-exact, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}&(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)). \end{aligned}$$

If (3) does not hold, $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))=\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,s_\alpha\mu_2))=0$. Hence we have $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))=0$, i.e., (1) does not hold. Therefore, (1) implies (3).

Now assume that (1) does not hold, i.e., $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))=0$. We prove $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))=\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,s_\alpha\mu_2))=0$ and, for all $k\in\mathbb{Z}_{>0}$, $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))=0$. These imply the lemma.

By the exact sequence

$$0\to\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))\to\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2)),$$

we have $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))=0$. Hence we have $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))=0$ for all $k\in\mathbb{Z}_{\geq 0}$ by induction hypothesis. Put $L=\mathrm{Ker}(M_\lambda(s_\alpha w_2,s_\alpha\mu_2)\to M_\lambda(s_\alpha w_2,\mu_2))$. From an exact sequence (2.1), we have exact sequences

$$0\to M_\lambda(s_\alpha w,\mu_2)\to M_\lambda(w,\mu_2)\to L\to 0$$

and

$$0\to L\to M_\lambda(s_\alpha w_2,s_\alpha\mu_2)\to M_\lambda(s_\alpha w_2,\mu_2)\to 0.$$

Using $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,\mu_2))=0$ and the long exact sequences induced from there sequences, we have

$$\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))\simeq\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),L)\simeq\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,s_\alpha\mu_2)).$$

In particular, $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,s_\alpha\mu_2))\simeq\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))=0$. By induction hypothesis, we have $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(s_\alpha w_2,s_\alpha\mu_2))=0$ for all $k\in\mathbb{Z}_{>0}$. Hence we have $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_\lambda,\mu),M_\lambda(w_2,\mu_2))=0$ for all $k\in\mathbb{Z}_{>0}$. □

If for some abelian category $\mathcal{C}$ and some regular λ there exist objects which satisfy the conditions (A1–5), then the set $A_{(s_1,\dots,s_l)}(\mu)$ is independent of the choice of a reduced expression by Lemma 2.3. In the rest of this section, we assume it (It will be proved in Section 3). Put $A_{w_2}(\mu)=A_{(s_1,\dots,s_l)}(\mu)$.

Theorem 2.4. *Let $w_1,w_2\in W_\lambda$ and $\mu_1,\mu_2\in\mathcal{D}$. The following conditions are equivalent.*

(1) $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))\neq 0$.

(2) There exists $k\in\mathbb{Z}_{\geq 0}$ such that $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))\neq 0$.

(3) $w_1^{-1}w_\lambda A_{w_\lambda w_1}(w_\lambda\mu_1)\cap W^0_\lambda w_2^{-1}A_{w_2}(\mu_2)\neq\emptyset$.

Proof. We prove by backward induction on $\ell_\lambda(w_1)$. If $w_1=w_\lambda$, then from Lemma 2.3, (1)–(3) are equivalent. We use the similar argument in the proof of Lemma 2.3.

Take $\alpha\in\Pi_\lambda$ such that $s_\alpha w_1>w_1$. Put $\beta=-w_\lambda(\alpha)\in\Pi_\lambda$. We have $A_{w_\lambda w_1}(w_\lambda\mu_1)=\bigcup_{\mu_0\in A_{s_\beta}(w_\lambda\mu_1)}s_\beta A_{w_\lambda s_\alpha w_1}(s_\beta\mu_0)$ by Lemma 2.1. First assume that $\langle\alpha,w_1\lambda\rangle=0$. Then by Lemma 2.2, we have $M_\lambda(w_1,\mu_1)\simeq M_\lambda(s_\alpha w_1,\mu_1)\simeq M_\lambda(s_\alpha w_1,s_\alpha\mu_1)$. This implies the lemma.

In the rest of this proof, we assume that $\langle\alpha,w_1\lambda\rangle\neq 0$. First assume that $\langle\check{\alpha},\mu_1\rangle\notin\mathbb{Z}_{>0}$, then by the condition (A2), $M_\lambda(w_1,\mu_1)\simeq M_\lambda(s_\alpha w_1,s_\alpha\mu_1)$. Since $A_{s_\beta}(w_\lambda\mu_1)=\{w_\lambda\mu_1\}$, $A_{w_\lambda w_1}(w_\lambda\mu_1)=A_{w_\lambda s_\alpha w_1}(w_\lambda s_\alpha\mu_0)$. Hence we have the lemma.

Finally, we assume that $\langle\check{\alpha},\mu_1\rangle\in\mathbb{Z}_{>0}$. We have $w_1^{-1}w_\lambda A_{w_\lambda w_1}(w_\lambda\mu_1)\cap W_\lambda^0 w_2^{-1}A_{w_2}(\mu_2)\neq\emptyset$ if and only if $(s_\alpha w_1)^{-1}w_\lambda A_{w_\lambda s_\alpha w_1}(w_\lambda\mu_1)\cap W_\lambda^0 w_2^{-1}A_{w_2}(\mu_2)\neq\emptyset$ or $(s_\alpha w_1)^{-1}w_\lambda A_{w_\lambda s_\alpha w_1}(w_\lambda s_\alpha\mu_1)\cap W_\lambda^0 w_2^{-1}A_{w_2}(\mu_2)\neq\emptyset$ since $A_{s_\beta}(w_\lambda\mu_1)=\{w_\lambda\mu_1,w_\lambda s_\alpha\mu_1\}$. By the condition (A2), we have $M_\lambda(w_1,s_\alpha\mu_1)\simeq M_\lambda(s_\alpha w_1,\mu_1)$. Hence, there exists an exact sequence $0\to M_\lambda(s_\alpha w_1,\mu_1)\to M_\lambda(s_\alpha w_1,s_\alpha\mu_1)\to M_\lambda(w_1,\mu_1)\to M_\lambda(s_\alpha w_1,\mu_1)\to 0$ by the condition (A3). Therefore (1) implies (3).

Now assume (1) dose not hold, i.e., $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))=0$. We prove that $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(s_\alpha w_1,\mu_1),M_\lambda(w_2,\mu_2))=\mathrm{Hom}_{\mathcal{C}}(M_\lambda(s_\alpha w_1,s_\alpha\mu_1),M_\lambda(w_2,\mu_2))=0$ and, for all $k\in\mathbb{Z}_{>0}$, $\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))=0$. Since we have an exact sequence

$$0\longrightarrow\mathrm{Hom}_{\mathcal{C}}(M_\lambda(s_\alpha w_1,\mu_1),M_\lambda(w_2,\mu_2))\to\mathrm{Hom}_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2)),$$

we have $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(s_\alpha w_1,\mu_1),M_\lambda(w_2,\mu_2))=0$. Hence, by induction hypothesis, we have that $\mathrm{Ext}^k_{\mathcal{C}}((M_\lambda(s_\alpha w_1,\mu_1),M_\lambda(w_2,\mu_2))=0$. Therefore we have

$$\mathrm{Ext}^k_{\mathcal{C}}((M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))\simeq\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(s_\alpha w_1,s_\alpha\mu_1),M_\lambda(w_2,\mu_2)).$$

In particular, $\mathrm{Hom}_{\mathcal{C}}(M_\lambda(s_\alpha w_1,s_\alpha\mu_1),M_\lambda(w_2,\mu_2))=0$. Hence we have

$$\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(w_1,\mu_1),M_\lambda(w_2,\mu_2))\simeq\mathrm{Ext}^k_{\mathcal{C}}(M_\lambda(s_\alpha w_1,s_\alpha\mu_1),M_\lambda(w_2,\mu_2))=0.$$

□

§3. Proof of the main theorems

In this section, we prove Theorem 1.1 and Theorem 1.2 using the result of Section 2. First we consider the twisted Verma modules. Fix a regular dominant integral element λ. Put $\mathcal{C}=\mathcal{O}$, $\mathcal{D}=\mathfrak{h}^*$. Set $M_\lambda(w,\mu)=T_{w^{-1}w_0}M(w_0\mu)$.

Lemma 3.1. *The modules $\{M_\lambda(w,\mu)\}$ satisfy the conditions (A1–5).*

Proof. The condition (A1) is obvious since λ is regular. The conditions (A2) and (A3) are [AL03, Proposition 6.3]. Since $T_{w_0}M(w_0\mu)\simeq\delta M(\mu)$ [AL03, Corollary 5.1], we have $\mathrm{Hom}_{\mathcal{O}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu))=\mathrm{Hom}_{\mathcal{O}}(M(w_0\mu'),\delta M(\mu))$. Since $\mathrm{Hom}_{\mathcal{O}}(M(w_0\mu'),\delta M(\mu))\neq 0$ if and only if $w_0\mu'=\mu$, we have (A4). Moreover, we have $\bigoplus_{\mu'\in\mathfrak{h}^*}\mathrm{Ext}^k_{\mathcal{O}}(M(w_0\mu'),\delta M(\mu))\simeq H^k(\mathfrak{n},\delta M(\mu))\simeq H_k(\bar{\mathfrak{n}},M(\mu))=0$ where $\bar{\mathfrak{n}}$ is the nilradical of the opposite Borel subalgebra of $\mathfrak{b}$. Hence we have (A5). □

From Lemma 3.1 and Theorem 2.4, we have Theorem 1.2.

Next, we consider the principal series representations of G. This is a full-subcategory of $\mathfrak{g}\oplus\mathfrak{g}$-modules. We also regard $\mathcal{H}$ as a full-subcategory of $\mathfrak{g}$-bimodules.

Lemma 3.2. *Let $\lambda,\mu\in\mathfrak{h}^*$ such that $\lambda-\mu\in\mathcal{P}$, $w\in W$. Put $\Delta^-=-\Delta^+$ and $\Delta^-_\lambda=-\Delta^+_\lambda$.*

(1) There exists $w'\in W_\lambda$ such that $\Delta^+\cap(w'w^{-1})^{-1}\Delta^-\cap w\Delta_\lambda=\emptyset$.

(2) Take w' as in (1). Then we have $L(M(w\lambda),\delta M(w\mu))\simeq L(M(w'\lambda),\delta M(w'\mu))$.

PROOF. (1) Since $w^{-1}\Delta^+ \cap \Delta_\lambda$ is a positive system of Δ_λ, there exists $w' \in W_\lambda$ such that $w^{-1}\Delta^+ \cap \Delta_\lambda = (w')^{-1}\Delta_\lambda^+$. Since $(w')^{-1}\Delta_\lambda^- = (w')^{-1}(\Delta^- \cap \Delta_\lambda) = (w')^{-1}\Delta^- \cap \Delta_\lambda$, we have $\Delta^+ \cap (w'w^{-1})^{-1}\Delta^- \cap w\Delta_\lambda = w(w^{-1}\Delta^+ \cap (w')^{-1}\Delta^- \cap \Delta_\lambda) = w(w^{-1}\Delta^+ \cap (w')^{-1}\Delta_\lambda^- \cap \Delta_\lambda) = \emptyset$.

(2) By the condition of w', for all $\alpha \in \Delta^+ \cap (w'w^{-1})^{-1}\Delta^-$ we have $\langle \check{\alpha}, -w\lambda\rangle \notin \mathbb{Z}$. Hence by [Duf77, 4.8. Proposition] we have $L(M(w\lambda), \delta M(w\mu)) \simeq L(M(w'\lambda), \delta M(w'\mu))$. □

By Lemma 3.2, it is sufficient to study $\mathrm{Hom}_{\mathcal{H}}(L(M(w'\lambda), \delta M(\mu')), L(M(w\lambda), \delta M(\mu)))$ for dominant λ and $w, w' \in W_\lambda$. Moreover, we may assume $\mu \in W_\lambda\mu'$ since $L(M(w\lambda), \delta M(\mu)) = 0$ unless $w\lambda - \mu \in \mathcal{P}$. Fix such a λ and put $M_\lambda(w,\mu) = L(M(w\lambda), \delta M(\mu))$ for $\mu \in \lambda + \mathcal{P}$ and $w \in W_\lambda$. Put $\mathcal{D} = \lambda + \mathcal{P}$. For $\alpha \in \Pi_\lambda$, let C_α be Joseph's Enright functor [Jos82]. Recall that $M \in \mathcal{O}$ is called α-free if the canonical map $M \to C_\alpha M$ is injective.

Lemma 3.3. *Let $\mu \in \lambda + \mathcal{P}$ and $\alpha \in \Pi_\lambda$.*

(1) If $N \in \mathcal{O}$ is α-free and $\langle \check{\alpha}, \mu\rangle \in \mathbb{Z}_{\le 0}$ then $L(M_\lambda(s_\alpha\mu), C_\alpha N) \simeq L(M_\lambda(\mu), N)$.

(2) Let $w \in W_\lambda$. If $\langle \check{\alpha}, w\lambda\rangle \in \mathbb{Z}_{\le 0}$ and $\langle \check{\alpha}, \mu\rangle \in \mathbb{Z}_{\le 0}$, then $L(M(s_\alpha w\lambda), M(s_\alpha\mu)) \simeq L(M(w\lambda), M(\mu))$.

(3) Let $w \in W_\lambda$. If $\langle \check{\alpha}, w\lambda\rangle \in \mathbb{Z}_{\le 0}$ and $\langle \check{\alpha}, \mu\rangle \in \mathbb{Z}_{\ge 0}$, then $L(M(s_\alpha w\lambda), \delta M(s_\alpha\mu)) \simeq L(M(w\lambda), \delta M(\mu))$.

(4) We have $L(M(w_\lambda\lambda), \delta M(\mu)) \simeq L(M(\lambda), M(w_\lambda\mu))$.

(5) The modules $\{M_\lambda(w,\mu)\}$ satisfy the conditions (A1–5).

PROOF. (1) Put $M = M(\mu)$ and $M' = M(s_\alpha\mu)$ in [Jos82, 3.8. Lemma]. Then we get (1).

(2) Take $N = M(\mu)$ in (1) and use [Jos82, 2.5. Lemma].

(3) Let $\widetilde{\lambda} \in \lambda + \mathcal{P}$ be a regular element such that $\widetilde{\lambda}$ is dominant. Then by [Jos83, 2.5. Lemma], we have $C_\alpha \delta M(\widetilde{\lambda}) \simeq \delta M(s_\alpha\widetilde{\lambda})$. For $\mathfrak{g} \oplus \mathfrak{g}$-module N, let N^η be a $\mathfrak{g} \oplus \mathfrak{g}$-module where the action is twisted by $(X,Y) \mapsto (Y,X)$. Using [Jos82, 2.8], we have $L(M(\widetilde{\lambda}), C_\alpha\delta M(\mu)) \simeq L(M(s_\alpha\widetilde{\lambda}), \delta M(\mu)) \simeq L(M(\mu), \delta M(s_\alpha\widetilde{\lambda}))^\eta \simeq L(M(\mu), C_\alpha\delta M(\widetilde{\lambda}))^\eta$. Notice that $\delta M(s_\alpha\widetilde{\lambda})$ is α-free. Hence we have $L(M(\mu), C_\alpha\delta M(\widetilde{\lambda}))^\eta \simeq L(M(s_\alpha\mu), \delta M(\widetilde{\lambda}))^\eta \simeq L(M(\widetilde{\lambda}), \delta M(s_\alpha\mu))$ by (1). Therefore we have $C_\alpha\delta M(\mu) \simeq \delta M(s_\alpha\mu)$. We get (3) by (1).

(4) Take $w \in W_\lambda$ such that $\langle \check{\beta}, w\mu\rangle \in \mathbb{Z}_{\le 0}$ for all $\beta \in \Delta_\lambda^+$. Put $\mu_0 = w\mu$. Let $w = s_{\alpha_l}\cdots s_{\alpha_1}$ be a reduced expression. Then we have $\langle \check{\alpha}_i, s_{\alpha_i}\dots s_{\alpha_l}\mu\rangle \in \mathbb{Z}_{\ge 0}$ and $\langle \check{\alpha}_i, s_{\alpha_{i-1}}\dots s_{\alpha_1}w_\lambda\lambda\rangle \in \mathbb{Z}_{\le 0}$. Hence by (3), we have $L(M(w_\lambda\lambda), \delta M(\mu)) \simeq L(M(ww_\lambda\lambda), \delta M(\mu_0))$. Take a reduced expression of ww_λ and use (2), then we have $L(M(ww_\lambda\lambda), M(\mu_0)) \simeq L(M(\lambda), M(w_\lambda\mu))$ by the same argument. Since $\delta M(\mu_0) \simeq M(\mu_0)$, we have (4).

(5) The condition (A1) is obvious. The condition (A2) follows from (2) and (A3) from [Jos82, 4.7. Corollary]. To prove (A4) and (A5), we may assume that $\mu' \in W_\lambda\mu$. Let $\mu_1 \in W_\lambda\mu$ such that $\langle \beta, \mu_1\rangle \ge 0$ for all $\beta \in \Delta_\lambda^+$. Take $w, w' \in W_\lambda$ such that $\mu' = w'\mu_1$ and $\mu = w\mu_1$. Then by the argument in (4), we have $L(M(w_\lambda\lambda), \delta M(\mu')) \simeq L(M(w_\lambda(w')^{-1}w_\lambda\lambda), \delta M(w_\lambda\mu_1))$ and $L(M(\lambda), \delta M(\mu)) \simeq L(M(w^{-1}\lambda), \delta M(\mu_1))$. We prove (A4) and (A5). First we assume that μ_1

is regular. Then by (4), we have

$$\begin{aligned}
&\operatorname{Ext}^k_{\mathcal{H}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda(w')^{-1}w_\lambda\lambda),\delta M(w_\lambda\mu_1)),L(M(w^{-1}\lambda),\delta M(\mu_1)))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda(w')^{-1}w_\lambda\lambda),\delta M(w_\lambda\mu_1))^\eta,L(M(w^{-1}\lambda),\delta M(\mu_1))^\eta)\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda\mu_1),\delta M(w_\lambda(w')^{-1}w_\lambda\lambda)),L(M(\mu_1),\delta M(w^{-1}\lambda)))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(\mu_1),M((w')^{-1}w_\lambda\lambda)),L(M(\mu_1),\delta M(w^{-1}\lambda)))
\end{aligned}$$

By the Bernstein-Gelfand-Joseph-Enright equivalence [BG80, 5.9. Theorem], this space is isomorphic to $\operatorname{Ext}^k(M((w')^{-1}w_\lambda\lambda),\delta M(w^{-1}\lambda))$. Hence the proof is done in this case (see the proof of Lemma 3.3).

We prove (A4) and (A5) for general μ_1. Take a regular element $\mu_2\in\mu_1+\mathcal{P}$ such that for all $\beta\in\Delta^+_\lambda$. Let $T^{\mu_2}_{\mu_1}$ be the translation functor of $\mathcal{O}$ and $L^{\mu_2}_{\mu_1}$ the translation functor of $\mathcal{H}$ with respect to the left $\mathfrak{g}$-action. Then we have $L^{\mu_2}_{\mu_1}L(M,N)=L(M,T^{\mu_2}_{\mu_1}N)$ for $M,N\in\mathcal{O}$. Since $T^{\mu_2}_{\mu_1}$ commutes with δ, we have

$$\begin{aligned}
\operatorname{Ext}^k_{\mathcal{H}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu)) &\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda\lambda),T^{\mu_2}_{\mu_1}M(w'\mu_2)),L(M(\lambda),T^{\mu_2}_{\mu_1}\delta M(w\mu_2)))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L^{\mu_2}_{\mu_1}L(M(w_\lambda\lambda),M(w'\mu_2)),L(M(\lambda),T^{\mu_2}_{\mu_1}\delta M(w\mu_2)))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda\lambda),M(w'\mu_2)),L^{\mu_1}_{\mu_2}L(M(\lambda),T^{\mu_2}_{\mu_1}\delta M(w\mu_2)))\\
&\simeq \operatorname{Ext}^k_{\mathcal{H}}(L(M(w_\lambda\lambda),M(w'\mu_2)),L(M(\lambda),\delta T^{\mu_1}_{\mu_2}T^{\mu_2}_{\mu_1}M(w\mu_2)))
\end{aligned}$$

The module $T^{\mu_1}_{\mu_2}T^{\mu_2}_{\mu_1}M(w\mu_2)$ has a filtration $0=M_0\subset M_1\subset\cdots\subset M_r=T^{\mu_1}_{\mu_2}T^{\mu_2}_{\mu_1}M(w\mu_2)$ such that $\{M_i/M_{i-1}\mid 1\le i\le r\}=\{M(wv\mu_2)\mid v\in W^0_{\mu_1}\}$ [Jan79, 2.3 Satz (b)]. Since λ is dominant, $L(M(\lambda),\cdot)$ is an exact functor. Hence we have an exact sequence $0\to L(M(\lambda),M_i)\to L(M(\lambda),M_{i-1})\to L(M(\lambda),M_i/M_{i-1})\to 0$. Using the long exact sequence and the result in regular case, we have $\operatorname{Ext}^k_{\mathcal{H}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu))=0$ for $k>0$. Moreover, by the vanishing of the Ext-groups,

$$\begin{aligned}
&\dim\operatorname{Hom}_{\mathcal{H}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu))\\
&\qquad=\sum_{v\in W^0_{\mu_1}}\dim\operatorname{Hom}^k_{\mathcal{H}}(L(M(w_\lambda\lambda),M(w'\mu_2)),L(M(\lambda),\delta M(wv\mu_2))).
\end{aligned}$$

From this formula, we have $\operatorname{Hom}_{\mathcal{H}}(M_\lambda(w_\lambda,\mu'),M_\lambda(e,\mu))\neq 0$ if and only if $w'\in W^0_\lambda w_\lambda wv$ for some $v\in W^0_{\mu_1}$. This condition is equivalent to $\mu'=w'\mu_1\in W^0_\lambda w_\lambda w\mu_1=W^0_\lambda w_\lambda\mu$. □

From Lemma 3.3 and Theorem 2.4, we have Theorem 1.1.

References

[AL03] H. H. Andersen and N. Lauritzen, *Twisted Verma modules*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 1–26.

[Ark04] Sergey Arkhipov, *Algebraic construction of contragradient quasi-Verma modules in positive characteristic*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 27–68.

[BG80] J. N. Bernstein and S. I. Gel′fand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), no. 2, 245–285.

[BGG71] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, *Structure of representations that are generated by vectors of higher weight*, Funkcional. Anal. i Priložen. **5** (1971), no. 1, 1–9.

[BGG76] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand, *A certain category of $\mathfrak{g}$-modules*, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 1–8.

[Duf77] Michel Duflo, *Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple*, Ann. of Math. (2) **105** (1977), no. 1, 107–120.

[Jan79] Jens Carsten Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, vol. 750, Springer, Berlin, 1979.

[Jos82] A. Joseph, *The Enright functor on the Bernstein-Gel′fand-Gel′fand category $\mathcal{O}$*, Invent. Math. **67** (1982), no. 3, 423–445.

[Jos83] A. Joseph, *Completion functors in the $\mathcal{O}$ category*, Noncommutative harmonic analysis and Lie groups (Marseille, 1982), Lecture Notes in Math., vol. 1020, Springer, Berlin, 1983, pp. 80–106.

[Ver68] Daya-Nand Verma, *Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc. **74** (1968), 160–166.

[Žel75] D. P. Želobenko, *Elementary representations of a semisimple complex Lie group*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 707–722.