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A Note on Class One Whittaker Functions on $SO_o(2,q)$

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Abstract. We give explicit formulas for class one Whittaker functions on $SO_o(2,q)$ by examing the system of partial differential equations for their radial parts.

§0. Introduction

As is well known, Whittaker functions on real reductive Lie groups are closely related to some fundamental aspects in the theory of automorphic forms; Fourier expansions and construction of automorphic L-functions are typical examples. From representation theoretical points of view, the theory of Whittaker functions has been studied by many authors. In particular, for split groups, Jacquet ([4]) introduced integral expressions for Whittaker functions for principal series representation and which was later generalized to non-split groups. However the original form of Jacquet's integral is sometimes not so convenient for applications to number theory.

When we consider the class one Whittaker functions, that is, Whittaker functions for class one principal series representations, Jacquet's integral can be easily evaluated by using modified K-Bessel functions for real rank one Lie groups. For higher rank case, the first result was on $GL(3,\mathbf{R})$ due to Bump ([1]) and Vinogradov-Tahtajan ([17]). They found a integral expression involving products of K-Bessel functions. Further, Stade ([15]) extended their results and found a remarkable integral expression for the class one Whittaker function on $GL(n,\mathbf{R})$ and applied it for the computations of the gamma factors of certain automorphic L-functions. On the other hand Niwa ([7]) and Proskurin ([11]) obtained integral expressions in case of $Sp(2,\mathbf{R})$ and $Sp(2,\mathbf{C})$, respectively.

In this paper we study class one Whittaker functions on $SO_o(2,q)$ ($q \geq 3$), which are related to wave forms on Hermitian symmetric spaces of type IV. Note that $\mathfrak{so}(2,3) \cong \mathfrak{sp}(2,\mathbf{R})$, $\mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$. In addition to the

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direct manipulation of Jacquet's integral, we study a system of partial differential equations for the radial parts of class one Whittaker functions and give a fundamental system of solutions in terms of the generalized hypergeometric series ${}_3F_2(1)$ (Theorem 4.1). We believe that such explicit formulas of fundamental solutions will be useful for the construction of Poincaré series (cf. [14]).

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§1. Definition of Whittaker Functions

Let G be a real connected semisimple Lie group with the Lie algebra $\mathfrak{g}$. Fix a maximal compact subgroup K of G and put $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. We fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. For nonzero $\alpha \in \mathfrak{a}^*$, put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X, \ \forall H \in \mathfrak{a}\}$. We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the restricted root system and fix a positive system Δ^+ in Δ. If we put $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, then we have the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Put $N = \exp(\mathfrak{n})$ and $A = \exp(\mathfrak{a})$ and hence $G = NAK$, the Iwasawa decomposition of G. We denote by W the Weyl group of the root system Δ. We also put $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} (\dim \mathfrak{g}_\alpha)\alpha$.

Now we recall the definition of class one principal series representations of G. Denote by $\mathfrak{a}_{\mathbf{C}}$ the complexifications of $\mathfrak{a}$ and take $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. Let H_{π_ν} be the space of smooth functions ϕ on G satisfying

$$\phi(mang) = a^{\nu+\rho}\phi(g),$$

for $m \in M$, $a \in A$, $n \in N$ and $g \in G$. The group G acts on H_{π_ν} by right translation and define the representation of G on H_{π_ν}. We call this induced representation $\pi_\nu = \mathrm{Ind}_{MAN}^G(1_M \otimes a^{\nu+\rho} \otimes 1_N)$ the *class one principal series representation of* G. Define an element 1_ν in H_{π_ν} by $1_\nu(g) = a(g)^{\nu+\rho}$ with $g = n(g)a(g)k(g)$ $(n(g) \in N, a(g) \in A, k(g) \in K)$. Then $1_\nu(namgk) = a^{\nu+\rho}1_\nu(g)$ for $n \in N, a \in A, m \in M, g \in G$ and $k \in K$, that is, 1_ν is a K-fixed vector in H_{π_ν}.

Let $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{a}_{\mathbf{C}})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{a}_{\mathbf{C}}$, respectively. Set $U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \mathrm{Ad}(k)X = X, \ \ \forall k \in K\}$.

Let p be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ along the decomposition

$$U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{a}_{\mathbf{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k}).$$

Define the automorphism γ of $U(\mathfrak{a}_{\mathbf{C}})$ by $\gamma(H) = H + \rho(H)$ for $H \in \mathfrak{a}_{\mathbf{C}}$. For $\nu \in \mathfrak{a}_{\mathbf{C}}^*$, define an algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$ by $\chi_\nu(z) = \nu(\gamma \circ p(z))$ for $z \in U(\mathfrak{g}_{\mathbf{C}})^K$. Note that χ_ν is trivial on $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ and the restriction of χ_ν to the center $Z(\mathfrak{g}_{\mathbf{C}})$ of $U(\mathfrak{g}_{\mathbf{C}})$ coincides with the infinitesimal character of the class one principal series representation π_ν.

Let η be a unitary character of N and $C_\eta^\infty(N\backslash G/K)$ be the space of C^∞-functions on G satisfying $f(ngk) = \eta(n)f(g)$ for $n \in N$, $g \in G$ and $k \in K$.

Definition 1.1. For a unitary character η of N and an algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$, we call

$$\mathrm{Wh}(\nu,\eta) = \{f \in C_\eta^\infty(N\backslash G/K) \mid zf = \chi_\nu(z)f, \quad \forall z \in U(\mathfrak{g}_{\mathbf{C}})^K\}.$$

the space of *class one Whittaker functions* on G. We also denote by $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ the subspace of moderate growth functions in $\mathrm{Wh}(\nu,\eta)$ ([18]).

Remark. In case of $G = SO_o(2,q)$ $(q \geq 3)$,

$$\mathrm{Wh}(\nu,\eta) = \{f \in C_\eta^\infty(N\backslash G/K) \mid zf = \chi_\nu(z)f, \quad \forall z \in \mathbf{C}[C_2, C_4]\},$$

where C_2 and C_4 are generators of $Z(\mathfrak{g}_{\mathbf{C}})$ with degree 2 and 4 respectively ([6, §4]).

Theorem 1.2 (Hashizume [3]). *Under the above notation*

$$\dim_{\mathbf{C}} \mathrm{Wh}(\nu,\eta) = |W|.$$

Moreover,

$$\dim_{\mathbf{C}} \mathrm{Wh}(\nu,\eta)^{\mathrm{mod}} \leq 1$$

and the unique element in $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ *is given by Jacquet's integral*:

$$J_\nu^\eta(g) = \int_N 1_\nu(s_0^{-1}ng)\eta^{-1}(n)dn.$$

Here s_0 *is the longest element in* W *and* dn *is the normalized Haar measure on* N *as in* [3, §1].

§2. Structure Theory for $SO_o(2,q)$ $(q \geq 3)$

We give explicit descriptions of the symbols introduced in §1 in our case $G = SO_o(2,q)$, the identity component of $SO(2,q)$.

$$SO(2,q) = \left\{ g \in SL(2+q,\mathbf{R}) \,\middle|\, {}^t g I_{2,q} g = I_{2,q} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_q \end{pmatrix} \right\},$$

$$\mathfrak{g} = \mathfrak{so}(2,q) = \{ X \in M_{2+q}(\mathbf{R}) \mid {}^t X I_{2,q} + I_{2,q} X = 0 \},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \,\middle|\, X_1 = -{}^t X_1 \in M_2(\mathbf{R}),\ X_2 = -{}^t X_2 \in M_q(\mathbf{R}) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ {}^t X & 0 \end{pmatrix} \,\middle|\, X \in M_{2,q}(\mathbf{R}) \right\},$$

$$\mathfrak{a} = \mathbf{R}A_1 \oplus \mathbf{R}A_2 \quad \text{with} \quad A_1 = E_{1,q+2} + E_{q+2,1},\ A_2 = E_{2,q+1} + E_{q+1,2},$$

$$K = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \,\middle|\, k_1 \in SO(2),\ k_2 \in SO(q) \right\},$$

$$A = \{ \exp(\log(a_1)A_1 + \log(a_2)A_2) \mid a_1, a_2 > 0 \},$$

$$\Delta = \Delta(\mathfrak{g},\mathfrak{a}) = \{ \pm e_1, \pm e_2, \pm e_1 \pm e_2 \} \quad \text{with}$$

$$e_i(a_1 A_1 + a_2 A_2) = a_i \ (i = 1,2),$$

$$\Delta^+ = \{ e_1, e_2, e_1 \pm e_2 \},$$

$$\mathfrak{g}_{e_1} = \oplus_{i=1}^{q-2} \mathbf{R} X_i, \quad \mathfrak{g}_{e_2} = \oplus_{i=1}^{q-2} \mathbf{R} Y_i, \quad \mathfrak{g}_{e_1 - e_2} = \mathbf{R} Z_1,$$

$$\mathfrak{g}_{e_1+e_2} = \mathbf{R} Z_2, \quad \text{with}$$

$$X_i = E_{1,i+2} + E_{i+2,1} - E_{i+2,q+2} + E_{q+2,i+2} \quad (1 \leq i \leq q-2),$$

$$Y_i = E_{2,i+2} + E_{i+2,2} - E_{i+2,q+1} + E_{q+1,i+2} \quad (1 \leq i \leq q-2),$$

$$Z_1 = (-E_{1,2} - E_{1,q+1} + E_{2,1} - E_{2,q+2} - E_{q+1,1} + E_{q+1,q+2} - E_{q+2,2} - E_{q+2,q+1})/2,$$

$$Z_2 = (-E_{1,2} + E_{1,q+1} + E_{2,1} - E_{2,q+2} + E_{q+1,1} - E_{q+1,q+2} - E_{q+2,2} + E_{q+2,q+1})/2,$$

($E_{i,j}$ is the matrix with 1 at (i,j) entry and 0 elsewhere),

$$W = \mathfrak{S}_2 \ltimes (\mathbf{Z}/2\mathbf{Z})^2, \quad s_0 = I_{2,q},$$

$$\eta(Z_1) = 2\sqrt{-1}\eta_1, \quad \eta(Y_i) = 2\sqrt{-1}\xi_i, \quad ([\mathfrak{n},\mathfrak{n}] = \mathfrak{g}_{e_1} \oplus \mathfrak{g}_{e_1+e_2}),$$

put $\eta_2 := \sqrt{\sum_{i=1}^{q-2} \xi_i^2}$,

$\rho = (q/2, q/2-1)$,
$\nu = (\nu_1, \nu_2)$,
$a^{\nu+\rho} = a_1^{\nu_1+q/2} a_2^{\nu_2+q/2-1} \quad (a = \exp(\log(a_1)A_1 + \log(a_2)A_2))$.

§3. System of Differential Equations

In this section we write down a system of partial differential equations satisfied by class one Whittaker functions. Because of the Iwasawa decomposition $G = NAK$, the value of $f \in C_\eta^\infty(N\backslash G/K)$ is determined by its restriction $\mathrm{res}|_A(f)$ to A. We call $f|_A := \mathrm{res}|_A(f)$ the *radial part* of f. For a linear operator $D : C_\eta^\infty(N\backslash G/K) \to C_\eta^\infty(N\backslash G/K)$, there exists a linear operator $R(D) : C^\infty(A) \to C^\infty(A)$ satisfying $R(D) \circ \mathrm{res}|_A = \mathrm{res}|_A \circ R(D)$. We also call $R(D)$ the *radial part* of D. Any $X \in \mathfrak{g}$ can be regarded as a differential operator on $C^\infty(G)$ by $(Xf)(g) := \frac{d}{dt}(f(g \exp tX))\big|_{t=0}$ for $f \in C^\infty(G)$, and we extend it to the action of $U(\mathfrak{g}_{\mathbf{C}})$ as usual manner. Then the following can be easily shown.

Lemma 3.1. *Let $\phi = f|_A$ be the radial part of $f \in C_\eta^\infty(N\backslash G/K)$. Then*

$$(R(A_i)\phi)(a) = a_i \frac{\partial}{\partial a_i}\phi(a) \quad (i = 1, 2), \tag{1}$$

$$(R(X_i)\phi)(a) = 0, \tag{2}$$

$$(R(Y_i)\phi)(a) = 2\sqrt{-1}\,\xi_i a_2\, \phi(a), \tag{3}$$

$$(R(Z_1)\phi)(a) = 2\sqrt{-1}\,\eta_1 a_1 a_2^{-1}\, \phi(a), \tag{4}$$

$$(R(Z_2)\phi)(a) = 0. \tag{5}$$

In the same way as in [6], we can deduce the following:

Theorem 3.2. *Let $f \in \mathrm{Wh}(\nu, \eta)$ be a class one Whittaker function on $SO_o(2, q)$. We introduce new variables given by $y = (y_1, y_2) = (a_1/a_2, a_2)$ and put $f|_A(y) = y_1^{q/2} y_2^{q-1}\phi(y)$. If we denote $\partial_i = y_i \frac{\partial}{\partial y_i}$ then $\phi(y)$ satisfies*

$$[2\partial_1^2 + \partial_2^2 - 2\partial_1\partial_2 - 8\eta_1^2 y_1^2 - 4\eta_2^2 y_2^2 - (\nu_1^2 + \nu_2^2)]\phi(y) = 0, \tag{3.1}$$

$$\begin{aligned}[(\partial_2^2 - 2\partial_1\partial_2 - \nu_1^2 + \nu_2^2)(\partial_2^2 - 2\partial_1\partial_2 + \nu_1^2 - \nu_2^2) - 16\eta_1^2 y_1^2 \partial_2^2 \\ - 8\eta_2^2 y_2^2(\partial_2^2 - 2\partial_1\partial_2 - 2\partial_1 + 2\partial_2 + 2) + 16\eta_2^4 y_2^4]\phi(y) = 0.\end{aligned} \tag{3.2}$$

Proof. By using Lemma 3.1, we can compute the radial parts $R(C_2)$ and $R(C_4)$. The explicit formulas of C_2 and C_4 are as follows ([6, Proposition 4.1]):

$$\begin{aligned} C_2 = A_1^2 + A_2^2 - qA_1 - (q-2)A_2 + 2(Z_1Z_{-1} + Z_2Z_{-2}) \\ + \sum_i (X_iX_{-i} + Y_iY_{-i}) - \sum_{i<j} K_{i,j}^2 \end{aligned}$$

and

$$\begin{aligned} C_4 = {} & 4A_1^2A_2^2 - 8A_1A_2(Z_1Z_{-1} - Z_2Z_{-2}) \\ & + 4(Z_1^2Z_{-1}^2 + Z_2^2Z_{-2}^2) - 8Z_1Z_2Z_{-1}Z_{-2} \\ & - \sum_{i\neq j}(X_{-i}^2X_j^2 + Y_{-i}^2Y_j^2) + 2\sum_{i<j}(X_iX_jX_{-i}X_{-j} + Y_iY_jY_{-i}Y_{-j}) \\ & - 2\sum_{i\neq j}(Y_iY_{-j}X_iX_{-j} + Y_iY_{-j}X_jX_{-i}) + 4\sum_{i\neq j} Y_jY_{-j}X_iX_{-i} \\ & + 4\sum_i (A_1^2Y_iY_{-i} + A_2^2X_iX_{-i}) \\ & + 4\sum_i (Z_1Z_{-1} + Z_2Z_{-2})(X_iX_{-i} + Y_iY_{-i}) \\ & + 4\sum_i (Z_{-1}Z_{-2}X_i^2 + Z_1Z_2X_{-i}^2 - Z_1Z_{-2}Y_i^2 - Z_{-1}Z_2Y_{-i}^2) \\ & + 4\sum_i (A_1 - A_2)(Z_{-2}Y_iX_i + Z_2Y_{-i}X_{-i}) \\ & + 4\sum_i (A_1 + A_2)(Z_{-1}Y_{-i}X_i + Z_1Y_iX_{-i}) \\ & - 4\sum_{i<j}(A_1^2 + A_2^2)K_{i,j}^2 - 8\sum_{i<j}(Z_1Z_{-1} + Z_2Z_{-2})K_{i,j}^2 \\ & - 4\sum_{i\neq j}(A_1X_iX_{-j} + A_2Y_iY_{-j})K_{i,j} - 4\sum_{i<j,\, l\neq i,j}(X_lX_{-l} + Y_lY_{-l})K_{i,j}^2 \\ & + 4\sum_{i<j,\, l\neq i,j}(X_iX_{-j} + X_jX_{-i} + Y_iY_{-j} + Y_jY_{-i})K_{i,l}K_{j,l} \\ & - 4\sum_{i\neq j}(Z_1Y_jX_{-i} + Z_2X_{-i}Y_{-j} + Z_{-1}X_jY_{-i} + Z_{-2}X_jY_i)K_{i,j} \end{aligned}$$

$$
\begin{aligned}
&+4\sum_{i<j<l<m}(K_{i,j}^2K_{l,m}^2+K_{i,l}^2K_{j,m}^2+K_{i,m}^2K_{j,l}^2)\\
&-8\sum_{i<j<l<m}(K_{i,l}K_{j,l}K_{i,m}K_{j,m}\\
&\qquad\qquad+K_{i,j}K_{l,j}K_{i,m}K_{l,m}+K_{i,j}K_{j,m}K_{i,l}K_{m,l})\\
&-4(q-2)A_1^2A_2-4qA_1A_2^2\\
&-4(q-1)A_1Z_1Z_{-1}-4(q-1)A_1Z_2Z_{-2}\\
&-4(q-5)A_2Z_1Z_{-1}-4(q+1)A_2Z_2Z_{-2}\\
&-2(q-1)\sum_i A_1X_iX_{-i}-4(q-2)\sum_i A_1Y_iY_{-i}\\
&-4(q-3)\sum_i A_2X_iX_{-i}-2(q-3)\sum_i A_2Y_iY_{-i}\\
&+4q\sum_{i<j}A_1K_{i,j}^2+4(q-2)\sum_{i<j}A_2K_{i,j}^2\\
&+\sum_{i<j}\{2(q-5)(X_iX_{-j}+Y_iY_{-j})+2(q-3)(X_jX_{-i}+Y_jY_{-i})\}K_{i,j}\\
&+2(q-3)\sum_i(Z_1Y_iX_{-i}+Z_2Y_{-i}X_{-i}-Z_{-1}Y_{-i}X_i-Z_{-2}Y_iX_i)\\
&+4\sum_{i<j<l<m}(K_{i,j}K_{i,l}K_{j,l}+K_{i,j}K_{m,i}K_{j,m}\\
&\qquad\qquad+K_{i,l}K_{i,m}K_{l,m}+K_{j,l}K_{m,j}K_{m,l})\\
&-\frac{1}{3}(q^2-49q+96)A_1^2-\frac{1}{3}(q^2+11q-36)A_2^2+4(q-2)^2A_1A_2\\
&-\frac{2}{3}(q-1)(q-12)(Z_1Z_{-1}+Z_2Z_{-2})+\frac{1}{3}(q^2-25q+192)\sum_{i<j}K_{i,j}^2\\
&-\frac{1}{3}(q^2-37q+108)\sum_i X_iX_{-i}-\frac{1}{3}(q-4)(q-9)\sum_i Y_iY_{-i}\\
&+\frac{1}{3}(q^3-37q^2+156q-168)A_1+\frac{1}{3}(q-2)(q-4)(q-9)A_2.
\end{aligned}
$$

Here $K_{i,j}=E_{i+2,j+2}-E_{j+2,i+2}\in\mathfrak{k}$ $(1\le i<j\le q-2)$. Since f is annihilated by $U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k}$, $\mathfrak{g}_{e_1}U(\mathfrak{g}_{\mathbf{C}})$ and $\mathfrak{g}_{e_1+e_2}U(\mathfrak{g}_{\mathbf{C}})$, we have

$$(R(C_2)f)(a)=\big(R(A_1^2+A_2^2-qA_1-(q-2)A_2+Y+2Z_1^2)f\big)(a)$$

and

$$\begin{aligned}(R(C_4)f)(a) = \big(&R(4A_1^2A_2^2 - 8Z_1^2A_1A_2 + 4Z_1^4 + 4YA_1^2 + 4Z_1^2Y \\ &+ 4qZ_1^2A_1 + 4(q-2)Z_1^2A_2 - 4(q-2)A_1^2A_2 - 4qA_1A_2^2 - 4qYA_1 \\ &- \tfrac{1}{3}q(q-1)A_1^2 - \tfrac{1}{3}(q-1)(q-12)A_2^2 + 4q(q-2)A_1A_2 \\ &- \tfrac{2}{3}(7q-19)qZ_1^2 - \tfrac{1}{3}(q-1)(q-12)Y + \tfrac{1}{3}q^2(q-1)A_1 \\ &+ \tfrac{1}{3}(q-1)(q-2)(q-12)A_2)f\big)(a).\end{aligned}$$

Here $Y = \sum_{i=1}^{q-2} Y_i^2$. Combined with Lemma 3.1 and

$$\begin{aligned}\chi_\nu(C_2) &= \nu_1^2 + \nu_2^2 - \frac{q^2}{2} + q - 1, \\ \chi_\nu(C_4) &= 4\nu_1^2\nu_2^2 - \frac{1}{3}(4q^2 - 13q + 12)(\nu_1^2 + \nu_2^2) \\ &\quad + \frac{1}{12}(5q^4 - 30q^3 + 80q^2 - 100q + 48),\end{aligned}$$

([6, Lemma 6.5]) we get a system of differential equations. Note that the equation (3.2) can be deduced from $R(C_4')f = \chi_\nu(C_4')f$ with $C_4' = C_4 - C_2^2 + \frac{5}{3}(q-1)(2q-3)C_2$. $\square$

Remark. (1) By taking notice that q does not appear in (3.1) and (3.2), we can see that class one Whittaker function on $SO_o(2,q)$ differs from on $SO_o(2,3)$ only by a simple factor $(a_1a_2)^{(q-3)/2}$.
(2) If we denote by $P_1\phi(y) = 0$ and $P_2\phi(y) = 0$ the equations (3.1) and (3.2) respectively, we can check that P_1 and P_2 are commutative. Then our calculation is correct from the result of Oshima and Sekiguchi ([10]) which implies that P_2 can be uniquely (modulo P_1, P_1^2) determined from the commuting relation $[P_1, P_2] = 0$ (and the invariace under the action of the Weyl group W). See also the result of Ochiai ([9]).
(3) If we substitute $q = 3, 4$ in the above system, it agrees with the system of differential equations of principal series Whittaker functions on $Sp(2,\mathbf{R})$ and $SU(2,2)$ obtained by Miyazaki, Oda ([8]) and Hayata ([5]) respectively. But they studied more general Whittaker functions not only for the class one principal series.

§4. Explicit Formulas of Whittaker Functions

We first consider formal power series solutions around $(y_1, y_2) = (0, 0)$ and construct the space of solutions. Our formulas include (terminating) generalized hypergeometric series ${}_3F_2(1)$ (cf. [16]). We put

$$\phi(y) = \sum_{m,n\geq 0} c_{m,n}(|\eta_1|y_1)^{2m+\tau_1}(\eta_2 y_2)^{2n+\tau_2} \quad (c_{0,0} \neq 0).$$

Since the characteristic indices satisfy

$$\begin{aligned} &2\tau_1^2 + \tau_2^2 - 2\tau_1\tau_2 - (\nu_1^2 + \nu_2^2) = 0, \\ &(\tau_2^2 - 2\tau_1\tau_2 - \nu_1^2 + \nu_2^2)(\tau_2^2 - 2\tau_1\tau_2 + \nu_1^2 - \nu_2^2) = 0 \end{aligned}$$

from Theorem 3.2, we have

$$\begin{aligned} (\tau_1, \tau_2) &= \{w(\nu_1, \nu_1 + \nu_2) \mid w \in W\} \\ &= \{(\nu_1, \nu_1 \pm \nu_2), (-\nu_1, -\nu_1 \pm \nu_2), \\ &\qquad (\nu_2, \pm\nu_1 + \nu_2), (-\nu_2, \pm\nu_1 - \nu_2)\}. \end{aligned}$$

In case of $(\tau_1, \tau_2) = (\nu_1, \nu_1 + \nu_2)$ we have the recurrence relations

$$\{2m^2 + n^2 - 2mn + (\nu_1 - \nu_2)m + \nu_2 n\}c_{m,n} - c_{m,n-1} - 2c_{m-1,n} = 0 \tag{4.1}$$

and

$$\begin{aligned} &\{2n^2 - 4mn - 2(\nu_1 + \nu_2)m + 2\nu_2 n - \nu_1^2 + \nu_2^2\} \\ &\quad \cdot \{n^2 - 2mn - (\nu_1 + \nu_2)m + \nu_2 n\}c_{m,n} \\ &\quad - \{4n^2 - 8mn - 4(\nu_1 + \nu_2 - 1)m + 4(\nu_2 - 1)n \\ &\qquad - \nu_1^2 + \nu_2^2 - 2\nu_2 + 2\}c_{m,n-1} \\ &\quad + 2c_{m,n-2} - 2(2n + \nu_1 + \nu_2)^2 c_{m-1,n} = 0. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we get

THEOREM 4.1. *We assume that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers. Let*

$$\phi_{(\nu_1,\nu_2)}(y) = \sum_{m,n\geq 0} {}_3F_2\left(\begin{matrix} -m, \ -n - \frac{\nu_1+\nu_2}{2}, \ n + \frac{\nu_1+\nu_2}{2} + 1 \\ \frac{\nu_1-\nu_2}{2} + 1, \ \frac{\nu_1+\nu_2}{2} + 1 \end{matrix} \middle| \ 1 \right) \cdot \frac{(|\eta_1|y_1)^{2m+\nu_1}(\eta_2 y_2)^{2n+\nu_1+\nu_2}}{m!\, n!\, (\nu_1 + 1)_m (\nu_2 + 1)_n}.$$

Then the set $\{\phi_{w(\nu_1,\nu_2)}(y) \mid w \in W\}$ forms a basis of the space of solutions of the system in Theorem 3.2. Here $(a)_k = \Gamma(a+k)/\Gamma(a)$ and

$$
{}_pF_q\left(\begin{matrix} a_1, \ \ldots, \ a_p \\ b_1, \ \ldots, \ b_q \end{matrix} \,\middle|\, z \right) = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},
$$

the generalized hypergeometric function.

Proof (cf. [16]). Since the relation (4.1) determines $c_{m,n}$ uniquely up to constant $c_{0,0}$, we have only to prove that the above $\phi_{(\nu_1,\nu_2)}(y)$ satisfies the recurrence relations (4.1) and (4.2). We first put

$$
\begin{aligned}
d_{m,n} :=& \frac{(\frac{\nu_1-\nu_2}{2}+1)_m(\frac{\nu_1+\nu_2}{2}+1)_m}{m!\,(\nu_1+1)_m} \\
&\cdot {}_3F_2\left(\begin{matrix} -m, \ -n-\frac{\nu_1+\nu_2}{2}, \ n+\frac{\nu_1+\nu_2}{2}+1 \\ \frac{\nu_1-\nu_2}{2}+1, \ \frac{\nu_1+\nu_2}{2}+1 \end{matrix} \,\middle|\, 1 \right) \\
=& \frac{(n+\nu_1+1)_m(n+\nu_1+\nu_2+1)_m}{m!\,(\nu_1+1)_m} \\
&\cdot {}_3F_2\left(\begin{matrix} -m, \ -n-\frac{\nu_1+\nu_2}{2}, \ -m-\nu_1 \\ -m-n-\nu_1, \ -m-n-\nu_1-\nu_2 \end{matrix} \,\middle|\, 1 \right).
\end{aligned}
$$

Here we used the formula

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} -m, \ a, \ b \\ c, \ d \end{matrix} \,\middle|\, 1 \right) =& \frac{(c-a)_m(d-a)_m}{(c)_m(d)_m} \\
&\cdot {}_3F_2\left(\begin{matrix} -m, \ a, \ a+b-c-d-m+1 \\ a-c-m+1, \ a-d-m+1 \end{matrix} \,\middle|\, 1 \right)
\end{aligned}
$$

([12, 7.4.4.81]). Then our task is to show

$$
\begin{aligned}
&\{2m^2+n^2-2mn+(\nu_1-\nu_2)m+\nu_2 n\}d_{m,n} \\
&\quad -2(m+\tfrac{\nu_1-\nu_2}{2})(m+\tfrac{\nu_1+\nu_2}{2})d_{m-1,n}-n(n+\nu_2)d_{m,n-1}=0.
\end{aligned}
\tag{4.3}
$$

If we use the relation

$$
\begin{aligned}
(1-z)^{-a'}{}_2F_1\left(\begin{matrix} a, \ b \\ c \end{matrix} \,\middle|\, 1 \right) =& \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{z^m}{m!} \\
&\cdot {}_3F_2\left(\begin{matrix} -m, \ 1-c-m, \ a' \\ 1-a-m, \ 1-b-m \end{matrix} \,\middle|\, 1 \right)
\end{aligned}
$$

for $|z| < 1$ ([2, p.187]), then we have

$$\begin{aligned} F_n(z) &:= \sum_{m=0}^{\infty} d_{m,n} z^m \\ &= (1-z)^{n+(\nu_1+\nu_2)/2} {}_2F_1\left(\begin{matrix} n+\nu_1+1,\ n+\nu_1+\nu_2+1 \\ \nu_1+1 \end{matrix} \middle| \, z \right). \end{aligned}$$

Therefore to show (4.3), it is sufficient to verify that

$$\begin{aligned} &\Big[2(1-z)z^2\frac{d^2}{dz^2} + \{-2(\nu_1+3)z + (\nu_1-\nu_2-2n+2)\}z\frac{d}{dz} \\ &\quad - 2\big(\frac{\nu_1+\nu_2}{2}+1\big)\big(\frac{\nu_1-\nu_2}{2}+1\big) + n(n+\nu_2)\Big]F_n(z) \\ &\quad - n(n+\nu_2)F_{n-1}(z) = 0. \end{aligned} \tag{4.4}$$

If we write $\delta_n(z) = (1-z)^{n+(\nu_1+\nu_2)/2}$ and $F_n(z) = \delta_n(z)p_n(z)$, then

$$\delta_n(z)^{-1}\frac{dF_n}{dz} = \Big[\frac{d}{dz} - \big(n + \frac{\nu_1+\nu_2}{2}\big)\frac{1}{1-z}\Big]p_n(z),$$

$$\begin{aligned} &\delta_n(z)^{-1}z(1-z)\frac{d^2F_n}{dz^2} \\ &= \Big[z(1-z)\frac{d^2}{dz^2} - 2\big(n+\frac{\nu_1+\nu_2}{2}\big)z\frac{d}{dz} \\ &\qquad + \big(n+\frac{\nu_1+\nu_2}{2}\big)\big(n+\frac{\nu_1+\nu_2}{2}-1\big)\frac{z}{1-z}\Big]p_n(z) \\ &= \Big[\{-(\nu_1+1) + (\nu_1+3)z\}\frac{d}{dz} \\ &\qquad + \{(n+\nu_1+1)(n+\nu_1+\nu_2+1) \\ &\qquad + \big(n+\frac{\nu_1+\nu_2}{2}\big)\big(n+\frac{\nu_1+\nu_2}{2}-1\big)\frac{z}{1-z}\}\Big]p_n(z) \end{aligned}$$

by the hypergeometric equation. Then we can see that (4.4) is equivalent to

$$\begin{aligned} &\big[(2n+\nu_1+\nu_2)(1-z)z\frac{d}{dz} + \{-n(n+\nu_2)(1-z) \\ &\qquad - (2n+\nu_1+\nu_2)(2n+\nu_1+\nu_2+1)z\}\big]p_n(z) + n(n+\nu_2)p_{n-1}(z) = 0, \end{aligned}$$

and which is easily shown by the power series expansion of $p_n(z) = {}_2F_1(z)$. The recurrence (4.2) can be checked in the same way. □

Now we treat the Jacquet's integral to find a unique element in $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ (Theorem 1.2), and which is suitable linear combinations of $a_1^{q/2}a_2^{q/2-1}\phi_{w(\nu_1,\nu_2)}(a)$ $(w\in W)$. Though, in view of remark after Theorem 3.2, we have only to consider the Jacquet's integral for $SO_o(2,3)$. In case of $q=3$, we can write an element of N as

$$\begin{aligned}
&n(n_0,n_1,n_2,n_3)\\
&=\exp(n_0Y_1+(n_2+n_0n_3/2)X_1+n_3Z_1+(n_1+n_0n_2+n_0^2n_3/6)Z_2)\\
&=\begin{pmatrix} 1 & & & & \\ & 1+\frac{n_0^2}{2} & n_0 & -\frac{n_0^2}{2} & \\ & n_0 & 1 & -n_0 & \\ & \frac{n_0^2}{2} & n_0 & 1-\frac{n_0^2}{2} & \\ & & & & 1 \end{pmatrix}\\
&\qquad\cdot\begin{pmatrix} 1+\frac{n_2^2-n_1n_3}{2} & -\frac{n_1+n_3}{2} & n_2 & \frac{n_1-n_3}{2} & \frac{-n_2^2+n_1n_3}{2} \\ \frac{n_1+n_3}{2} & 1 & 0 & 0 & -\frac{n_1+n_3}{2} \\ n_2 & 0 & 1 & 0 & -n_2 \\ \frac{n_1-n_3}{2} & 0 & 0 & 1 & \frac{-n_1+n_3}{2} \\ \frac{n_2^2-n_1n_3}{2} & -\frac{n_1+n_3}{2} & n_2 & \frac{n_1-n_3}{2} & 1-\frac{n_2^2-n_1n_3}{2} \end{pmatrix}.
\end{aligned}$$

Then $\eta(n(n_0,n_1,n_2,n_3))=\exp(2\sqrt{-1}(\eta_1n_3+\eta_2n_0))$. We denote

$$a=a(a_1,a_2)=\exp(\log(a_1)A_1+\log(a_2)A_2)\in A.$$

To write down the Jacquet's integral J_ν^η for $SO_o(2,3)$, we consider the Iwasawa decomposition $n'\cdot a(a_1',a_2')\cdot k$ $(n\in N,k\in K)$ of $s_0^{-1}\cdot n(n_0,n_1,n_2,n_3)\cdot a(a_1,a_2)$. Since $(s_0n'a')^{-1}\cdot n\cdot a\in K$, we get

$$a_1'=a_1a_2/\Delta_1,\ a_2'=\Delta_1/\Delta_2,$$

with

$$\begin{aligned}
\Delta_1&=\{a_1^4a_2^2+n_3^2a_1^2a_2^4+2n_2^2a_1^2a_2^2+n_1^2a_1^2+(n_1n_3-n_2^2)^2a_2^2\}^{1/2},\\
\Delta_2&=a_1^2a_2^2+n_0^2a_1^2+(n_0n_3+n_2)^2a_2^2+(n_0n_2+n_1)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
J_\nu^\eta(a)&=(a_1a_2)^{\nu_1+3/2}\\
&\quad\cdot\int_{\mathbf{R}^4}\Delta_1^{-\nu_1+\nu_2-1}\Delta_2^{-\nu_2-1/2}\exp\bigl(-2\sqrt{-1}(\eta_1n_3+\eta_2n_0)\bigr)dn_0dn_1dn_2dn_3.
\end{aligned}$$

If we change the variables $(t_1, t_2) = (\sqrt{a_1 a_2}, \sqrt{a_1 a_2^{-1}})$, we can see that it becomes the Jacquet's integral for $Sp(2, \mathbf{R})$ Whittaker function (The maximal split torus of $Sp(2, \mathbf{R})$ is $\{\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1}) \mid t_1, t_2 > 0\}$). If we first integrate with respect to n_0, then

$$
\begin{aligned}
J_\nu^\eta(a) &= 2^{\nu_2+1}\pi^{1/2}\eta_2^{-\nu_2-1}(\Gamma(\nu_2+1/2))^{-1}a_1^{-2\nu_2} \\
&\quad\cdot \int_{\mathbf{R}^3} \Delta_3^{-1/2}\Delta_2^{-\nu_1-\nu_2-1}K_{-\nu_2}\Big(2\eta_2 a_1 \frac{\Delta_1}{\Delta_3}\Big) \\
&\qquad\cdot \exp\Big(2\eta_2\sqrt{-1}\,\frac{n_2(n_3 a_1 a_2 + n_1 a_1 a_2^{-1})}{\Delta_3} - 2\sqrt{-1}\eta_1 n_3\Big) dn_1 dn_2 dn_3.
\end{aligned}
$$

Here $\Delta_3 = a_1^3 a_2^{-1} + n_3^2 a_1 a_2 + n_2^2 a_1 a_2^{-1}$ and $K_\nu(z)$ is the modified Bessel function. This expression is similar to the formula of Proskurin ([11, p.161(2.4.40)]) for $Sp(2, \mathbf{C})$-Whittaker function. By following the same manipulation as in his one ([11, pp.162-166]), we finally reach the following integral expression.

THEOREM 4.2. *For* $\mathrm{Re}(s_1) > |\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_2)|$ *and* $\mathrm{Re}(s_2) > |\mathrm{Re}(\nu_1 \pm \nu_2)/2|$, *the radial part of class one Whittaker function on* $SO_o(2, q)$ *is*

$$
\begin{aligned}
W_{(\nu_1,\nu_2)}^\eta(a) &= \Big(|\eta_1|\frac{a_1}{a_2}\Big)^{(-\nu_1-\nu_2+q)/2}(\eta_2 a_2)^{(\nu_1+\nu_2)/2+q-1} \\
&\quad\cdot \int_0^\infty\int_0^\infty K_{(\nu_1-\nu_2)/2}\big(2|\eta_1|\frac{a_1}{a_2}\sqrt{(1+1/x)(1+1/y)}\big) \\
&\qquad\qquad\cdot K_{(\nu_1+\nu_2)/2}\big(2\eta_2 a_2\sqrt{1+x+y}\big) \\
&\qquad\qquad\cdot \Big(\frac{x^2y^2}{1+x+y}\Big)^{(\nu_1+\nu_2)/4}\Big(\frac{x(1+x)}{y(1+y)}\Big)^{(\nu_1-\nu_2)/4}\frac{dx}{x}\frac{dy}{y}
\end{aligned}
$$

up to constant. Further

$$
\begin{aligned}
&W_{(\nu_1,\nu_2)}^\eta(a) \\
&\quad= \sum_{w\in W} w\Big(\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\big(-\frac{\nu_1+\nu_2}{2}\big)\Gamma\big(-\frac{\nu_1-\nu_2}{2}\big)\Big)M_{w(\nu_1,\nu_2)}^\eta(a).
\end{aligned}
$$

Here $M_{w(\nu_1,\nu_2)}^\eta(a) = \frac{1}{4}(|\eta_1|a_1/a_2)^{q/2}(\eta_2 a_2)^{q-1}\phi_{w(\nu_1,\nu_2)}(a)$.

PROOF. The latter identity was shown by Hashizume ([3, Theorem 7.8]) in more general setting, though we can prove it directly by considering a Mellin-Barnes integral expression for $W^{\eta}_{(\nu_1,\nu_2)}(a)$ and residue calculus (cf. [14]). Let

$$\begin{aligned} W^{\eta}_{(\nu_1,\nu_2)}(a) &= (|\eta_1|a_1/a_2)^{q/2}(\eta_2 a_2)^{q-1}\tilde{W}^{\eta}_{(\nu_1,\nu_2)}(a) \\ &= (|\eta_1|y_1)^{q/2}(\eta_2 y_2)^{q-1}\tilde{W}^{\eta}_{(\nu_1,\nu_2)}(y), \end{aligned}$$

and

$$V_{(\nu_1,\nu_2)}(s_1,s_2) = \int_0^\infty\int_0^\infty \tilde{W}^{\eta}_{(\nu_1,\nu_2)}(y)(|\eta_1|y_1)^{s_1}(\eta_2 y_2)^{s_2}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

be the double Mellin transform of $\tilde{W}^{\eta}_{(\nu_1,\nu_2)}$. If we change the order of integration and use the formulas

$$\int_0^\infty K_\nu(ax)x^{s-1}dx = 2^{s-2}a^{-s}\Gamma\big(\frac{s-\nu}{2}\big)\Gamma\big(\frac{s+\nu}{2}\big)$$

for $\mathrm{Re}(s) > |\mathrm{Re}(\nu)|$, $a > 0$ and

$$\begin{aligned} &\int_0^\infty\int_0^\infty x^{a-1}y^{b-1}(1+x)^{-c}(1+x)^{-d}(1+x+y)^{-e}dxdy \\ &\quad = \frac{\Gamma(a)\Gamma(b)\Gamma(-a+c+e)\Gamma(-b+d+e)}{\Gamma(c+e)\Gamma(d+e)}\,{}_3F_2\left(\begin{matrix} a,\ b,\ e \\ c+e,\ d+e\end{matrix}\ \middle|\ 1\right) \end{aligned}$$

for $\mathrm{Re}(c+e) > \mathrm{Re}(a) > 0$, $\mathrm{Re}(d+e) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c+d+e-a-b) > 0$, then $V_{(\nu_1,\nu_2)}(s_1,s_2)$ becomes

$$\begin{aligned} &2^{-2}\,\Gamma\big(\frac{s_1+\nu_1}{2}\big)\Gamma\big(\frac{s_1-\nu_1}{2}\big)\Gamma\big(\frac{s_1+\nu_2}{2}\big)\Gamma\big(\frac{s_1-\nu_2}{2}\big) \\ &\cdot\Gamma\big(\frac{s_2}{2}\big)\Gamma\big(\frac{s_2+\nu_1+\nu_2}{2}\big)\Gamma\big(\frac{s_2+\nu_1-\nu_2}{2}\big)\Gamma\big(\frac{s_2-\nu_1+\nu_2}{2}\big) \\ &\cdot\Big(\Gamma\big(\frac{s_1+s_2+\nu_1}{2}\big)\Gamma\big(\frac{s_1+s_2+\nu_2}{2}\big)\Big)^{-1} \\ &\cdot{}_3F_2\left(\begin{matrix} \frac{s_1+\nu_1}{2},\ \frac{s_1+\nu_2}{2}\ \frac{s_2+\nu_1+\nu_2}{2} \\ \frac{s_1+s_2+\nu_1}{2},\ \frac{s_1+s_2+\nu_2}{2}\end{matrix}\ \middle|\ 1\right) \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&= 2^{-2}\,\Gamma\big(\frac{s_1+\nu_1}{2}\big)\Gamma\big(\frac{s_1-\nu_1}{2}\big)\Gamma\big(\frac{s_1+\nu_2}{2}\big)\Gamma\big(\frac{s_1-\nu_2}{2}\big)\\
&\quad\cdot\Gamma\big(\frac{s_2+\nu_1+\nu_2}{2}\big)\Gamma\big(\frac{s_2-\nu_1-\nu_2}{2}\big)\Gamma\big(\frac{s_2+\nu_1-\nu_2}{2}\big)\Gamma\big(\frac{s_2-\nu_1+\nu_2}{2}\big)\\
&\quad\cdot\Big(\Gamma\big(\frac{s_1+s_2+\nu_1}{2}\big)\Gamma\big(\frac{s_1+s_2-\nu_1}{2}\big)\Big)^{-1}\\
&\quad\cdot {}_3F_2\left(\begin{matrix}\frac{s_2}{2},\ \frac{s_1+\nu_2}{2},\ \frac{s_1-\nu_2}{2}\\ \frac{s_1+s_2+\nu_1}{2},\ \frac{s_1+s_2-\nu_1}{2}\end{matrix}\ \middle|\ 1\right).
\end{aligned}
\tag{4.6}
$$

Here we used the formula

$$
{}_3F_2\left(\begin{matrix}a,\ b,\ c\\ d,\ e\end{matrix}\ \middle|\ 1\right) = \frac{\Gamma(d)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(d-c)}\,{}_3F_2\left(\begin{matrix}e-a,\ e-b,\ c\\ d+e-a-b,\ e\end{matrix}\ \middle|\ 1\right)
$$

for $\mathrm{Re}(d+e-a-b-c), \mathrm{Re}(d-c) > 0$ ([12, 7.4.4.1]) with $a = (s_2+\nu_1+\nu_2)/2$, $b = (s_1+\nu_1)/2$, $c = (s_1+\nu_2)/2$, $d = (s_1+s_2+\nu_2)/2$, $e = (s_1+s_2+\nu_1)/2$.

We notice that $V_{(\nu_1,\nu_2)}(s_1,s_2)$ is invariant under the change $(\nu_1,\nu_2) \to (\nu_2,\nu_1)$ by (4.5) and $(\nu_1,\nu_2) \to (-\nu_1,\nu_2)$ by (4.6), thus, under the action of the Weyl group W to the parameter (ν_1,ν_2) of the principal series.

Mellin inversion formula implies

$$
\begin{aligned}
\tilde{W}^{\eta}_{\nu}(y) &= \frac{1}{(2\pi\sqrt{-1})^2}\\
&\quad\cdot\int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} V_{(\nu_1,\nu_2)}(s_1,s_2)(|\eta_1|y_1)^{-s_1}(\eta_2y_2)^{-s_2}ds_1ds_2.
\end{aligned}
$$

Here the paths of integration are taken as to the right of all poles of $M(s_1,s_2)$. Now we move the paths to the left and evaluate the residue at the poles

$$
\{(s_1,s_2) = w(-2m-\nu_1, -2n-(\nu_1+\nu_2)) \mid w \in W,\ m,n \in \mathbf{N}\}
$$

of $M(s_1,s_2)$ then we get the assertion. For example,

$$
\begin{aligned}
&\mathrm{Res}_{(s_1,s_2)=(-2m-\nu_1,-2n-(\nu_1+\nu_2))}(V_{(\nu_1,\nu_2)}(s_1,s_2)(|\eta_1|y_1)^{-s_1}(\eta_2y_2)^{-s_2})\\
&= \frac{\Gamma(-m-\nu_1)\Gamma(-m-\frac{\nu_1+\nu_2}{2})\Gamma(-m-\frac{\nu_1-\nu_2}{2})}{4\Gamma(-m-n-\nu_1)\Gamma(-m-n-\frac{\nu_1+\nu_2}{2})}\\
&\quad\cdot\Gamma(-n-\nu_1)\Gamma(-n-\nu_2)\Gamma\big(-n-\frac{\nu_1+\nu_2}{2}\big)
\end{aligned}
$$

$$
\cdot \frac{(-1)^{m+n}}{m!\,n!}\, {}_3F_2\left(\begin{matrix} -m,\ -n\ \ -m-\frac{\nu_1-\nu_2}{2} \\ -m-n-\nu_1,\ -m-n-\frac{\nu_1+\nu_2}{2} \end{matrix} \,\middle|\, 1 \right)
\cdot (|\eta_1|y_1)^{2m+\nu_1}(\eta_2 y_2)^{2n+\nu_1+\nu_2}
$$

$$
= 2^{-2}\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma\Big(\frac{-\nu_1+\nu_2}{2}\Big)\Gamma\Big(\frac{-\nu_1-\nu_2}{2}\Big)
\cdot {}_3F_2\left(\begin{matrix} -m,\ -n-\frac{\nu_1+\nu_2}{2},\ n+\frac{\nu_1+\nu_2}{2}+1 \\ \frac{\nu_1-\nu_2}{2}+1,\ \frac{\nu_1+\nu_2}{2}+1 \end{matrix} \,\middle|\, 1 \right)
\cdot \frac{(|\eta_1|y_1)^{2m+\nu_1}(\eta_2 y_2)^{2n+\nu_1+\nu_2}}{m!\,n!\,(\nu_1+1)_m(\nu_2+1)_n}.
$$

Here we used

$$
{}_3F_2\left(\begin{matrix} -m,\ a,\ b \\ c,\ d \end{matrix} \,\middle|\, 1 \right) = \frac{(c-a)_m(b)_m}{(c)_m(d)_m}\, {}_3F_2\left(\begin{matrix} -m,\ d-b,\ 1-c-m \\ 1-b-m,\ a-c-m+1 \end{matrix} \,\middle|\, 1 \right)
$$

([12, 7.4.4.87]) with $a=-n-\frac{\nu_1+\nu_2}{2}$, $b=n+\frac{\nu_1+\nu_2}{2}+1$, $c=\frac{\nu_1-\nu_2}{2}+1$, $d=\frac{\nu_1+\nu_2}{2}+1$ and $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$. □

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