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Large Deviation Principles for a Type of Diffusion Processes on Euclidean Space

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Abstract. We consider a class of diffusion processes on Euclidean spaces, with the drift terms not weaker than linear order. We show large deviation principles for empirical distributions of positions for pinned processes under some explicit conditions in terms of the coefficients of the generator. Such a problem was discussed by Donsker-Varadhan [2] under some implicit conditions.

1. Introduction

Consider the stochastic differential equation (SDE) on Euclidean space $\mathbf{R}^d$ given by

$$dX_t^i = \sum_{j=1}^d \sigma_{ij}(X_t)dB_t^j + b_i(X_t)dt, \qquad i = 1, \cdots, d, \tag{1.1}$$

where $(B_t^1, \cdots, B_t^d)$ is a d-dimensional Brownian motion. We assume that the coefficients $\sigma = (\sigma_{ij})_{i,j=1}^d$ and $b = (b_1, \cdots, b_d)$ satisfy the following:

(A1) $\sigma \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^{d^2})$ and $a = \sigma^t\sigma$ is uniformly elliptic, *i.e.*, there exist $c_1, c_2 > 0$ such that

$$c_1 \sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le c_2 \sum_{i=1}^d \xi_i^2, \qquad \text{for all } x, \xi \in \mathbf{R}^d.$$

(A2) $b \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$ and there exists a $c_3 > 0$ such that

$$\sum_{i,j=1}^d \xi_i\xi_j \nabla_i b_j(x) \le c_3 \sum_{i=1}^d \xi_i^2, \qquad \text{for all } x, \xi \in \mathbf{R}^d.$$

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(A3) There exist $c_4, c_5 > 0$ such that $x \cdot b(x) \le c_4 - c_5|x|^2$ for any $x \in \mathbf{R}^d$.

Let P_x denote the distribution of the solution of (1.1) with $X_0 = x$. Then (P_x, X_t) is the diffusion with generator $L_0 = \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b \cdot \nabla$. We will prove that (P_x, X_t) has a unique invariant probability μ and $P_t(x, dy) := P_x(X_t \in dy)$ has a positive smooth density with respect to μ. (See Corollary 2.2 and Lemma 2.3). So we can define the pinned measure $P_x(\,\cdot\,\Big| X_t = y)$ for all $t > 0$ and $x, y \in \mathbf{R}^d$.

For $t > 0$, let $L_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$, where $\delta_\cdot$ denotes the delta measure. Let $\mu_t^{x,y}$ be the probability given by

$$\mu_t^{x,y}(A) = P_x(L_t \in A \Big| X_t = y).$$

Since the diffusion (P_x, X_t) has a unique invariant probability μ, we see by the ergodic theorem that $\mu_t^{x,y}$ converges to δ_μ as $t \to \infty$ weakly in $\wp(\mathbf{R}^d)$, where $\wp(\mathbf{R}^d)$ is the metric space consisting of all probabilities on $\mathbf{R}^d$ endowed with the Prohorov metric.

We say that $\mu_t^{x,y}$ satisfies the large deviation principle (LDP) with rate function I if I is lower semi-continuous and

$$-\inf_{A^0} I \le \liminf_{t\to\infty}\frac{1}{t}\log\mu_t^{x,y}(A) \le \limsup_{t\to\infty}\frac{1}{t}\log\mu_t^{x,y}(A) \le -\inf_{\overline{A}} I$$

for any $A \in \mathcal{B}(\wp(\mathbf{R}^d))$. Here $\mathcal{B}(\cdot)$ is the Borel σ-field of $\cdot$, and A^0 and $\overline{A}$ means the interior and the closure of A, respectively. I is said be to a good rate function if, in addition, $\{\nu \in \wp(\mathbf{R}^d); I(\nu) \le \ell\}$ is compact in $\wp(\mathbf{R}^d)$ for all $\ell \ge 0$.

THEOREM 1.1. *$\mu_t^{x,y}$ satisfies the LDP with rate function I given by*

$$\begin{aligned} I(\nu) &= \sup\Big\{-\int_{\mathbf{R}^d}\frac{L_0 u}{u}d\nu; u \in C^\infty(\mathbf{R}^d), u > 0, \frac{L_0 u}{u} \text{ is bounded}\Big\}, \\ &\quad \nu \in \wp(\mathbf{R}^d), \end{aligned}$$

for any $x, y \in \mathbf{R}^d$, and I is a good rate function.

The condition (A3) implies that the drift has an enough effect to attract the process toward the origin, which guarantees the existence of the invariant probability. We do not know if (A3) is optimal or not. The following example shows that the existence of an invariant probability is not sufficient for the LDP with good rate function: let $d = 1$, $a = 1$ and $b(x) = -(1 + |x|^2)^{-1/2}x$. Then the only candidate for the rate function is not a good rate function. Actually, the invariant probability is $\pi(dx) = \text{const} \times e^{-(1+|x|^2)^{-1/2}}dx$, and the rate function, if exists, is the entropy function I given by $I(f^2 d\pi) = \int |\nabla f|^2 d\pi$ for $f^2 d\pi \in \wp(\mathbf{R})$, which does not have compact level sets. (See, for example, Deuschel-Stroock [3]).

The LDP problem for compact state spaces is initiated by the celebrated work Donsker-Varadhan [1]. As for non-compact state spaces, Donsker-Varadhan [2] considered the same problem for Markov processes on complete separable metric spaces. They showed the LDP under some technical assumptions. Especially, to get the upper bound, they need the following condition:

(D-V) There exist a function $V(x)$ and a sequence $\{u_n; n \in \mathbf{N}\} \subset C^2(\mathbf{R}^d)$ such that

(1) $\{x \in \mathbf{R}^d : V(x) \geq \ell\}$ is a compact set for any $\ell \in \mathbf{R} \cup \{+\infty\}$,

(2) $u_n(x) \geq 1$ for all $n \in \mathbf{N}$ and all $x \in \mathbf{R}^d$,

(3) $\sup_{x \in W} \sup_{n \in \mathbf{N}} u_n(x) < \infty$ for each compact set $W \subset \mathbf{R}^d$,

(4) $\lim_{n \to \infty} \left(\frac{L_0 u_n}{u_n}\right)(x) = V(x)$ for each $x \in \mathbf{R}^d$,

(5) $\sup_{n \in \mathbf{N}, x \in \mathbf{R}^d} \left(\frac{L_0 u_n}{u_n}\right)(x) < \infty$.

We remark that, although Donsker-Varadhan [2] checked that the Ornstein-Uhlenbeck process satisfies (D-V), it is not easy to prove (D-V) for diffusions given by SDEs in general. Our purpose is to give an explicit condition for the LDP in terms of the coefficients of SDEs.

We remark that (D-V) follows from the following (see Appendix):

(A3') There exist constants $c_4, c_5, c_6 > 0$ and $\gamma_2 \geq \gamma_1 > 1$ satisfying $\gamma_2 < 2\gamma_1 - 1$ such that $x \cdot b(x) \leq c_4 - c_5|x|^{\gamma_1+1}$ and $|b(x)| + |\nabla b(x)| \leq c_6(1+|x|^{\gamma_2})$ for any $x \in \mathbf{R}^d$, where $\nabla b(x)$ is the matrix $(\frac{\partial}{\partial x_i} b_j(x))_{i,j=1}^d$.

The condition (A3') is more rigid than (A3) and excludes the Ornstein-Uhlenbeck process. It seems hard to deduce (D-V) from (A3). So we develop a new technique to prove the LDP under (A3).

It is standard to use spectral decomposition and measure change to prove the LDP. Unfortunately, one can not carry out these on $C_b(\mathbf{R}^d)$ (see Remark 2). To overcome this difficulty, we introduce new spaces

$$B_\alpha^0 := \{f \in C(\mathbf{R}^d); \|f\|_{B_\alpha^0} := \sup_{x\in\mathbf{R}^d} (1+|x|^2)^{-\frac{\alpha}{2}}|f(x)| < \infty\}, \qquad \alpha \geq 0.$$

Considering $P_t^c f(x) = E^{P_x}[e^{\int_0^t c(X_s)ds} f(X_t)]$ on B_α^0 in stead of $C_b(\mathbf{R}^d)$ enables us to use the spectral decomposition and the measure change argument (see Section 2 and Section 3 for the details).

The organization of the paper is as follows: In Section 2 we will show the boundedness of $\{P_t\}_{t\geq 0}$ on B_α^0 and give some basic facts. In Section 3 we will discuss measure changes. In Section 4 we will proof the exponential tightness of $\{\mu_t^{x,y}\}_{t\geq 0}$. The proof of Theorem 1.1 will be given in Sections 5 and 6.

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2. Boundedness of P_t

Define $\psi(x) = (1+|x|^2)^{\frac{1}{2}}$, $x \in \mathbf{R}^d$. For any $\alpha \in \mathbf{R}$, let $K_\alpha^1 = \frac{2c_4+c_2d+(\alpha-2)^+c_2}{2c_5}$ and $K_\alpha^2 = 2^{\alpha/2}\big(1+(K_\alpha^1)^{\alpha/2}\big)$. Our main result of this section is the following:

Lemma 2.1. *For any $\alpha \geq 2$, $x \in \mathbf{R}^d$ and $t > 0$,*

$$E^{P_x}\left[|X_t|^\alpha\right]^{\frac{2}{\alpha}} \leq \max\left\{e^{-2c_5t}|x|^2 + K_\alpha^1(1-e^{-2c_5t}), K_\alpha^1\right\}. \tag{2.1}$$

$$\psi^{-\alpha}P_t\psi^\alpha \leq K_\alpha^2. \tag{2.2}$$

PROOF. By Ito's formula and (1.1), we have for any $\alpha \geq 2$

$$
\begin{aligned}
(2.3) \qquad d|X_t|^\alpha &= \alpha|X_t|^{\alpha-2}\sum_{i,j=1}^d X_t^i\sigma_{ij}(X_t)dB_t^j \\
&\quad +\Big(\alpha|X_t|^{\alpha-2}X_t\cdot b(X_t)+\frac{\alpha}{2}|X_t|^{\alpha-2}\mathrm{tr}a(X_t) \\
&\qquad +\frac{1}{2}\alpha(\alpha-2)|X_t|^{\alpha-4}\sum_{i,j=1}^d X_t^iX_t^ja_{ij}(X_t)\Big)dt.
\end{aligned}
$$

Let $v_{\alpha,x}(t) = E^{P_x}\left[|X_t|^\alpha\right]$. Then (2.3) together with (A1) and (A3) implies

$$
\frac{d}{dt}\left(v_{\alpha,x}(t)^{\frac{2}{\alpha}}\right) \leq (2c_4+c_2d+(\alpha-2)c_2)-2c_5v_{\alpha,x}(t)^{\frac{2}{\alpha}} = 2c_5\left(K_\alpha^1 - v_{\alpha,x}(t)^{\frac{2}{\alpha}}\right).
$$

Solving this and noting that $v_{\alpha,x}(0) = |x|^\alpha$, we obtain (2.1). (2.2) is immediate from (2.1). □

Lemma 2.1 gives us the tightness of $\{P_t(y,\cdot)\}_{t\geq 0}$, which combined with the positivity of the transition probability yields the following:

COROLLARY 2.2. *The diffusion process* (P_x, X_t) *has a unique invariant probability* μ*, which has all moments finite.*

By Kusuoka-Liang [5], we deduce from (A1) and (A2) the following

LEMMA 2.3 $P_t(x,dy) = P_x(X_t \in dy)$ *has a smooth positive density* $p_t(x,y)$ *with respect to* μ*, and*

$$
\sup_{x\in\mathbf{R}^d,|y|\leq r} p_t(x,y) < \infty, \qquad \textit{for any } r,t>0.
$$

3. Measure Changes

For $\varphi \in C_b(\mathbf{R}^d)$ define the transition kernel $\{P_t^\varphi\}_{t\geq 0}$ by

$$
P_t^\varphi(x,A) = E^{P_x}\left[\exp(\int_0^t \varphi(X_u)du)1_A(X_t)\right], \qquad x\in\mathbf{R}^d, A\in\mathcal{E}(\mathbf{R}^d), t\geq 0.
$$

We use the same symbol $\{P_t^\varphi\}_{t\ge 0}$ to denote the corresponding semi-group. When $\varphi = 0$, we write P_t^φ as P_t for the sake of simplicity.

We have by (2.2) that, if $\alpha \ge 2$, then $\{\psi^{-\alpha} P_t^\varphi \psi^\alpha\}_{t\ge 0}$ is a semi-group on $C_b(\mathbf{R}^d)$, whose infinitesimal generator is $L_0 + \alpha A \cdot \nabla + B_\alpha + \varphi$, where $A = \left(\psi^{-1} \sum_{i=1}^d a_{ij} \nabla_i \psi\right)_{i,j=1}^d$ and $B_\alpha(x) = \psi(x)^{-\alpha} L_0 \psi^\alpha(x)$. By assumption, $\limsup_{|x|\to\infty} B_\alpha(x) \le -c_5 \alpha$. So there exists an $\alpha = \alpha(\varphi) \ge 2$ such that $\limsup_{|x|\to\infty} B_\alpha(x) + \varphi(x) + ||\varphi||_\infty \le -c_5$.

We have by (2.2) that $\psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}$ maps $C_b(\mathbf{R}^d)$ to $C_b(\mathbf{R}^d)$ for any $t > 0$ and

$$(3.1) \qquad ||\psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}||_{op} \le e^{t||\varphi||_\infty} C_{\alpha(\varphi)},$$

where $\|\cdot\|_{op}$ means $\|\cdot\|_{C_b(\mathbf{R}^d)\to C_b(\mathbf{R}^d)}$. Also,

$$||\psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}||_{op} \ge \psi^{-\alpha(\varphi)}(0) P_t^\varphi \psi^{\alpha(\varphi)} 1(0) \ge e^{-t||\varphi||_\infty}.$$

So its logarithmic spectral radius $\Lambda^\varphi = \lim_{t\to\infty} \frac{1}{t} \log ||\psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}||_{op}$ on $C_b(\mathbf{R}^d)$ is well-defined and satisfies $|\Lambda^\varphi| \le ||\varphi||_\infty$. Let

$$(3.2) \qquad \overline{Q_t^\varphi} = e^{-\Lambda^\varphi t} \psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}.$$

Then by (3.1) and $|\Lambda^\varphi| \le ||\varphi||_\infty$ we have

$$(3.3) \qquad \sup_{0\le s\le t} \|\overline{Q_s^\varphi}\|_{op} < \infty, \qquad \text{for all } t \in [0,\infty).$$

We remark that the infinitesimal generator of $\overline{Q_t^\varphi}$ is $L_0 + \alpha(\varphi) A \cdot \nabla + B_{\alpha(\varphi)} + \varphi - \Lambda^\varphi$. $\limsup_{|x|\to\infty} B_{\alpha(\varphi)}(x) + \varphi(x) - \Lambda^\varphi \le -c_5$ by the definition of $\alpha(\varphi)$ and Λ^φ. So there exist two functions $B_1^\varphi, B_2^\varphi \in C(\mathbf{R}^d)$ such that $B_{\alpha(\varphi)} + \varphi - \Lambda^\varphi = B_1^\varphi + B_2^\varphi$, $B_1^\varphi(x) \to 0$ as $|x| \to \infty$, and $B_2^\varphi(x) \le -c_5$ for any $x \in \mathbf{R}^d$. We have

$$(3.4) \qquad \overline{Q_t^\varphi} = Q_t^{\varphi,1} + \int_0^t \overline{Q_{t-s}^\varphi} B_1^\varphi Q_s^{\varphi,1} ds, \qquad \text{for any } t > 0,$$

where $Q_t^{\varphi,1}$ is the semi-group with generator $L_0 + \alpha(\varphi) A \cdot \nabla + B_2^\varphi$.

Lemma 3.1. (1) $\|Q_t^{\varphi,1}\|_{op} \le e^{-c_5 t}$ *for any* $t > 0$,
(2) $Q_t^{\varphi,1}$ *has a continuous density* $q_t^{\varphi,1}(x,y)$ *with respect to* μ *for any* $t > 0$,

(3) *for any* $\phi \in C_b(\mathbf{R}^d)$ *satisfying* $\phi(x) \to 0$ *as* $|x| \to \infty$, $\phi Q_s^{\varphi,1}$ *is a compact operator for any* $s > 0$.

PROOF. Let $\{\widetilde{Q_t^{\varphi,1}}\}_{t\geq 0}$ be the semi-group corresponding to $L_0 + \alpha(\varphi) A \cdot \nabla$. Then $||\widetilde{Q_t^{\varphi,1}}||_{op} \leq 1$ and $Q_t^{\varphi,1} \leq e^{-c_5 t}\widetilde{Q_t^{\varphi,1}}$, which yields assertion (1). (2) and (3) follow from a routing argument. □

Notice that the set of compact operators is closed with respect to Riemann integral. Therefore, by $B_1^\varphi(x) \to 0$(as $|x| \to \infty$) and (3.3), Lemma 3.1 implies the following:

LEMMA 3.2. $\int_0^t \overline{Q_{t-s}^\varphi} B_1^\varphi Q_s^{\varphi,1} ds$ *is a compact operator for any* $t > 0$.

Lemma 3.1 and (3.4) give us the following.

LEMMA 3.3. *For any* $\varphi \in C_b(\mathbf{R}^d)$, *there exist* $N \in \mathbf{N}$, $\lambda_1, \cdots, \lambda_N \in \mathbf{C}$ *with* $\Re\lambda_i = 0$, $i = 1, \cdots, N$, *finite dimensional spectral projections* $E_1, \cdots, E_N$ *and a semi-group* $\{\widetilde{Q_t^\varphi}\}_{t\geq 0}$ *on* $C_b(\mathbf{R}^d)$ *such that* $||\widetilde{Q_t^\varphi}||_{op} \to 0$ *exponentially fast as* $t \to \infty$, $E_1, \cdots, E_N$ *and* $\widetilde{Q_t^\varphi}$ *are orthogonal to each other and*

$$\overline{Q_t^\varphi} = \sum_{i=1}^N e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} & \frac{t^{n_i-2}}{(n_i-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} E_i + \widetilde{Q_t^\varphi} \tag{3.5}$$

as operators on $C_b(\mathbf{R}^d)$, *where* n_i *is the dimension of* E_i, $i = 1, \cdots, N$.

We refer to [4] for the definitions of the spectral projections and orthogonal.

PROOF. By using the same method as in the proof of [5, Proposition 5.4] and the paragraph following it, there exist $N \in \mathbf{N}$, $\lambda_1, \cdots, \lambda_N \in \mathbf{C}$ and corresponding finite dimensional spectral projections $E_1, \cdots, E_N$ such that

1. $\overline{Q_t^\varphi} E_i = E_i \overline{Q_t^\varphi}$ for all $t > 0$ and $i = 1, \cdots, N$, and
2. $||\overline{Q_t^\varphi} - \sum_{i=1}^N \overline{Q_t^\varphi} E_i||_{op} \to 0$ exponentially as $t \to \infty$.

As a semi-group on $E_i(C_b(\mathbf{R}^d))$, $\overline{Q_t^{\varphi}}\Big|_{E_i(C_b(\mathbf{R}^d))}$ is uniformly continuous. Therefore, by [4, Theorem VIII.1.2], there exists a bounded operator A_i such that

$$\overline{Q_t^{\varphi}}\Big|_{E_i(C_b(\mathbf{R}^d))} = e^{tA_i}.$$

The spectrum of A_i is λ_i. So by taking an orthonormal base of $E_i(C_b(\mathbf{R}^d))$ suitably, and by dividing $E_i(C_b(\mathbf{R}^d))$ into perpendicular parts if necessary, we may and do assume that

$$A_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}.$$

Hence

$$\overline{Q_t^{\varphi}}\Big|_{E_i(C_b(\mathbf{R}^d))} = e^{\lambda_i t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_i-3}}{(n_i-3)!} & \frac{t^{n_i-2}}{(n_i-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \square$$

Let $C_\infty(\mathbf{R}^d)$ denote the set of the continuous functions that converge to 0 at ∞.

LEMMA 3.4. (1) *$\overline{Q_t^{\varphi}}$ maps $C_\infty(\mathbf{R}^d)$ into $C_\infty(\mathbf{R}^d)$,*
(2) $E_1, \cdots, E_N$ map $C_\infty(\mathbf{R}^d)$ into $C_\infty(\mathbf{R}^d)$.

PROOF. Choose any $f \in C_\infty(\mathbf{R}^d)$ and fix it. For any $\varepsilon > 0$, there exists a $R > 0$ such that $|f(y)| \leq \varepsilon$ for $|y| > R$. So by (3.2),

$$\begin{aligned} \overline{Q_t^{\varphi}} f(x) &\leq e^{2t||\varphi||_\infty} \Big(\psi^{-\alpha(\varphi)}(x) E^{P_x} \left[\psi^{\alpha(\varphi)}(X_t); |X_t| \leq R \right] ||f||_\infty \\ &\qquad + \psi^{-\alpha(\varphi)}(x) E^{P_x} \left[\psi^{\alpha(\varphi)}(X_t) f(X_t); |X_t| > R \right] \Big) \\ &\leq e^{2t||\varphi||_\infty} \Big(||f||_\infty \psi^{\alpha(\varphi)}(R) \psi^{-\alpha(\varphi)}(x) \\ &\qquad + \varepsilon \psi^{-\alpha(\varphi)}(x) E^{P_x} \left[\psi^{\alpha(\varphi)}(X_t) \right] \Big). \end{aligned}$$

By Lemma 2.1, $\psi^{-\alpha(\varphi)}(x)E^{P_x}\left[\psi^{\alpha(\varphi)}(X_t)\right], x \in \mathbf{R}^d$, is bounded, which completes the proof of assertion (1). Assertion (2) follows from (1) and the definition of $E_i, i = 1, \cdots, N$. □

LEMMA 3.5. *Let* $n_1, \cdots, n_N$ *be the ones defined in Lemma* 3.3. *Then* $n_i = 1$ *for any* $i \in \{1, \cdots, N\}$ *with* $E_i(C_\infty(\mathbf{R}^d)) \neq \{0\}$. *Therefore, there exist bounded linear functionals* $a_1, \cdots, a_N$ *and* $f_1, \cdots, f_N \in C_\infty(\mathbf{R}^d)$ *such that* $a_i(f_j) = \delta_{ij}$, $\widetilde{Q_t^\varphi} f_i = 0$ *for* $i, j = 1, \cdots, N$ *and*

$$\overline{Q_t^\varphi} f = \sum_{i=1}^{N} e^{\lambda_i t} a_i(f) f_i + \widetilde{Q_t^\varphi} f, \qquad \text{for any } f \in C_\infty(\mathbf{R}^d).$$

PROOF. Let $n = \max\{n_1, \cdots, n_N\}$. Without loss of generality, we may and do assume that $n = n_1$. Let $a = \frac{2\pi}{-i\lambda_1}$.

We first show that there exists a $\widetilde{g} \in C_\infty(\mathbf{R}^d, \mathbf{R}^+)$ such that $\overline{Q_a^\varphi}\widetilde{g} = \widetilde{g}$. Since $\frac{1}{m}\sum_{k=1}^m e^{ibk}$ converges as $m \to \infty$ for any $b \in \mathbf{R}$, we have by Lemma 3.4 that for any $f \in C_\infty(\mathbf{R}^d)$, $\frac{1}{m}\sum_{k=1}^m (ak)^{-n}\overline{Q_{ak}^\varphi} f$ converges in $C_\infty(\mathbf{R}^d)$ as $n \to \infty$. Define the operator $A : C_\infty(\mathbf{R}^d) \to C_\infty(\mathbf{R}^d)$ by

$$Af = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} (ak)^{-n} \overline{Q_{ak}^\varphi} f, \qquad f \in C_\infty(\mathbf{R}^d).$$

$Af \geq 0$ for any $f \geq 0$, and there exists a $g \in C_\infty(\mathbf{R}^d)$ such that $Ag \neq 0$. So $A|g| \geq |Ag| \geq 0$ and $A|g| \not\equiv 0$. By definition $\overline{Q_a^\varphi}A = A\overline{Q_a^\varphi} = A$, so $\overline{Q_a^\varphi}A|g| = A|g|$, which implies that $A|g|(x) > 0$ for any $x \in \mathbf{R}^d$. Therefore, we have found a $\widetilde{g} = A|g| \in C_\infty(\mathbf{R}^d, \mathbf{R}^+)$ such that $\overline{Q_a^\varphi}\widetilde{g} = \widetilde{g}$.

Let $C_0(\mathbf{R}^d)$ denote the set of continuous functions with compact support. For any $f \in C_0(\mathbf{R}^d)$, since $\widetilde{g}(x) > 0$ for any $x \in \mathbf{R}^d$, there exists a constant $c_f > 0$ such that $|f| \leq c_f \widetilde{g}$. Notice that by Lemma 3.4, $c_7 := \sup_{t \in (0,a]} ||Q_t \widetilde{g}||_\infty < \infty$. For any $t > 0$, let $\widetilde{t} = \inf\{s > 0; a|(t-s)\} \in (0, a]$. Since the operator $\overline{Q_t^\varphi}$ is monotone nondecreasing, we have that

$$\overline{Q_t^\varphi}|f| \leq c_f \overline{Q_t^\varphi}\widetilde{g} = c_f \overline{Q_{\widetilde{t}}^\varphi}\widetilde{g} \leq c_f c_7.$$

i.e., $\overline{Q_t^\varphi}|f|, t > 0$, is bounded. This is true for any $f \in C_0(\mathbf{R}^d)$. Therefore, n_i must be 1 for any $i = 1, \cdots, N$ with $E_i^\varphi C_0(\mathbf{R}^d) \neq \{0\}$. In other words, for any $i = 1, \cdots N$ with $n_i \neq 1$, we must have $E_i^{\varphi} C_0(\mathbf{R}^d) = \{0\}$. Since the operator E_i^{φ} is bounded and $C_0(\mathbf{R}^d)$ is dense in $C_\infty(\mathbf{R}^d)$, this implies that $E_i^{\varphi} C_\infty(\mathbf{R}^d) = \{0\}$. This gives us our first assertion.

The others are now easy. □

LEMMA 3.6. *There exist a* $h^{\varphi} \in C_\infty(\mathbf{R}^d)$ *and a probability* ν^{φ} *such that* $supp(\nu^{\varphi}) = \mathbf{R}^d$, $\{\overline{Q_t^{\varphi}}\}_{t>0}$ *is invariant under* ν^{φ} *and*

$$\overline{Q_t^{\varphi}} f = h^{\varphi} \int_{\mathbf{R}^d} f d\nu^{\varphi} + \widetilde{Q_t^{\varphi}} f, \qquad \text{for any } f \in C_b(\mathbf{R}^d),$$

where $\widetilde{Q_t^{\varphi}}$ *is the same as in Lemma 3.3.* $\widetilde{Q_t^{\varphi}}$ *is orthogonal to* $h^{\varphi}\langle \cdot, d\nu^{\varphi}\rangle$, *hence* h^{φ} *is the only positive eigenfunction of* $\overline{Q_t^{\varphi}}$.

PROOF. For $i = 1, \cdots, N$, let $f_i \in C_\infty(\mathbf{R}^d)$ be the eigenfunction corresponding to λ_i, as given in Lemma 3.5. Then $\overline{Q_t^{\varphi}} |f_i| \geq |f_i|$. So by Lemma 3.5, $\frac{1}{T}\int_0^T \overline{Q_t^{\varphi}} |f_i| dt$ converges in $C_\infty(\mathbf{R}^d)$ to a non-trivial limit as $T \to \infty$. Write the limit as $\widetilde{f_i}$. So $\overline{Q_t^{\varphi}} \widetilde{f_i} = \widetilde{f_i}$. Let $h = \sum_{i=1}^N \widetilde{f_i}$. Then $\overline{Q_t^{\varphi}} h = h$ for any $t > 0$. So $h \geq 0$ and $h \not\equiv 0$ imply that $h(x) > 0$ for any $x \in \mathbf{R}^d$. Choose a $x \in \mathbf{R}^d$ and fix it. We have that $\frac{1}{T}\int_0^T \overline{Q_t^{\varphi}} h(x) dt = h(x) > 0$. Let $B_r = \{y; |y| \leq r\}$, $r > 0$. Then

$$\text{(3.6)} \qquad \frac{1}{T}\int_0^T \overline{Q_t^{\varphi}}(x, B_r) \leq \frac{h(x)}{\inf_{B_r} h} < \infty, \qquad \text{for any } r > 0.$$

On the other hand, $h1_{B_r^c} \to 0$ in $C_\infty(\mathbf{R}^d)$ as $r \to \infty$, hence $\overline{Q_t^{\varphi}}(h1_{B_r^c}) = \sum_{i=1}^N e^{\lambda_i t} a_i(h1_{B_r^c}) + \widetilde{Q_t^{\varphi}}(h1_{B_r^c}) \to 0$ uniformly in $t > 0$ as $r \to \infty$. Therefore, there exists a $r_0 > 0$ such that for any $r > r_0$, $\sup_{t>0} \overline{Q_t^{\varphi}}(h1_{B_r^c})(x) < \frac{h(x)}{2}$, hence $\frac{1}{T}\int_0^T \overline{Q_t^{\varphi}}(h1_{B_r})(x) dt = \frac{1}{T}\int_0^T \overline{Q_t^{\varphi}} h(x) dt - \frac{1}{T}\int_0^T \overline{Q_t^{\varphi}}(h1_{B_r^c})(x) dt > \frac{h(x)}{2} > 0$. So

$$\text{(3.7)} \qquad \frac{1}{T}\int_0^T \overline{Q_t^{\varphi}}(x, B_r) > \frac{h(x)}{2\sup_{B_r} h} > 0, \qquad \text{for any } r > r_0.$$

(3.6) and (3.7) give us that $\frac{1}{T}\int_0^T \overline{Q_t^{\varphi}}(x, dy) dt$ converges to a non-trivial measure. Write the limit as ν^{φ}. It is obvious that $\{\overline{Q_t^{\varphi}}\}$ is ν^{φ}-invariant. So $supp\nu^{\varphi} = \mathbf{R}^d$.

Notice that $f_1, \cdots, f_N$ are all integrable with respect to ν^φ. Also, for any $f \in C_\infty(\mathbf{R}^d)$, we have by the definition of $\widetilde{Q_t^\varphi}$ that $\frac{1}{T}\int_0^T \overline{Q_s^\varphi}\widetilde{Q_t^\varphi} f(x)ds = \frac{1}{T}\int_0^T \widetilde{Q_{t+s}^\varphi} f(x)ds \to 0$ as $T \to \infty$. Therefore, $\widetilde{Q_t^\varphi} f$ is integrable with respect to ν^φ and $\int_{\mathbf{R}^d} \widetilde{Q_t^\varphi} f d\nu^\varphi = 0$ for any $f \in C_\infty(\mathbf{R}^d)$ and $t > 0$.

For $i = 1, \cdots, N$, we have $\overline{Q_t^\varphi} f_i = e^{\lambda_i t} f_i$. If f_i could not be written as a complex number times a positive valued function, then $\overline{Q_t^\varphi}|f_i| \geq |f_i|$ and the equality does not always holds. Since $supp\nu^\varphi = \mathbf{R}^d$, this implies that $\int_{\mathbf{R}^d} \overline{Q_t^\varphi}|f_i|d\nu^\varphi > \int_{\mathbf{R}^d} |f_i|d\nu^\varphi$, which contradicts with the fact that $\overline{Q_t^\varphi}$ is ν^φ-invariant. Therefore, by ignoring the constant times, any eigenfunction must be in $C_\infty(\mathbf{R}^d; \mathbf{R}^+)$. Hence $\lambda_i = 0$ for $i = 1, \cdots, N$. If $N \geq 2$, then there would exist $f_1, f_2 \in C_\infty(\mathbf{R}^d, \mathbf{R}^+)$ such that $f_1 \neq f_2$ and $\overline{Q_t^\varphi} f_i = f_i$, $i = 1, 2$. So there exists a constant $a \in \mathbf{R}$ such that $f_1 - af_2$ is neither always positive nor always negative, and $\overline{Q_t^\varphi}(f_1 - af_2) = (f_1 - af_2)$. This contradicts with the fact that any eigenfunction must be in $C_\infty(\mathbf{R}^d, \mathbf{R}^+)$, as we just proved. Therefore, $N = 1$ and $\lambda_1 = 0$.

Write f_1 as h^φ. For any $f \in C_\infty(\mathbf{R}^d)$, we have that $\overline{Q_t^\varphi} f = a_1(f)h^\varphi + \widetilde{Q_t^\varphi}$. Both sides above are integrable with respect to ν^φ, $\{\overline{Q_t^\varphi}\}_{t\geq 0}$ is ν^φ-invariant, and $\widetilde{Q_t^\varphi}$ is orthogonal to $a_1(\cdot)h^\varphi$. Therefore, $\int_{\mathbf{R}^d} f d\nu^\varphi = \int_{\mathbf{R}^d} \overline{Q_t^\varphi} f d\nu^\varphi = a_1(f)\int_{\mathbf{R}^d} h^\varphi d\nu^\varphi$. So by re-normalizing h^φ if necessary, we have that $a_1(f) = \int_{\mathbf{R}^d} f d\nu^\varphi$, $f \in C_\infty(\mathbf{R}^d)$.

We show that ν^φ is actually a finite measure, so by re-normalizing, we may assume that it is a probability. There exist bounded functions $f_n \in C_\infty(\mathbf{R}^d; \mathbf{R}^+)$, $n \in \mathbf{N}$, such that $||f_n||_\infty \leq 1$ and $f_n(x) \uparrow 1$ as $n \to \infty$ for any $x \in \mathbf{R}^d$. Choose any $x \in \mathbf{R}^d$ and fix it. By Lemma 2.1,

$$\begin{aligned}\int_{\mathbf{R}^d} f_n d\nu^\varphi &= h^\varphi(x)^{-1}\left(\overline{Q_1^\varphi} f_n(x) - \widetilde{Q_1^\varphi} f_n(x)\right)\\ &\leq h^\varphi(x)^{-1}\left(e^{2||\varphi||_\infty}\psi^{-\alpha}(x)E^{P_x}[\psi^\alpha(X_1)] + \|\widetilde{Q_1^\varphi}\|_{op}\right) < \infty.\end{aligned}$$

Therefore, 1 is also integrable with respect to ν^φ, *i.e.*, ν^φ is a finite measure.

The only thing left is to extend the results above to the whole $C_b(\mathbf{R}^d)$. For any $f \in C_b(\mathbf{R}^d)$, there exist functions $f_n \in C_\infty(\mathbf{R}^d)$ such that $||f_n||_\infty \leq ||f||_\infty$ for any $n \in \mathbf{N}$ and $f_n(x) \to f(x)$ as $n \to \infty$ for any $x \in \mathbf{R}^d$. $\left|\overline{Q_t^\varphi} f_n - (\int_{\mathbf{R}^d} f_n d\nu^\varphi)h^\varphi\right| = |\widetilde{Q_t^\varphi} f_n| \leq \|\widetilde{Q_1^\varphi}\|_{op}||f||_\infty$ for any $t > 0$ and $n \in \mathbf{N}$. Notice that $\overline{Q_t^\varphi} f_n(x) \to \overline{Q_t^\varphi} f(x)$ for any $x \in \mathbf{R}^d$ and $\int_{\mathbf{R}^d} f_n d\nu^\varphi \to \int_{\mathbf{R}^d} f d\nu^\varphi$

as $n \to \infty$. Take $n \to \infty$ in the inequality above, and we get our assertion. □

COROLLARY 3.7. *Use the same notations as in Lemma* 3.6. *Then*

$$P_t^\varphi f = e^{\Lambda^\varphi t}\psi^{\alpha(\varphi)}h^\varphi \int_{\mathbf{R}^d} f\psi^{-\alpha(\varphi)}d\nu^\varphi + e^{\Lambda^\varphi t}\psi^{\alpha(\varphi)}\widetilde{Q_t^\varphi}(\psi^{-\alpha(\varphi)}f)$$

for any $f \in C_b(\mathbf{R}^d)$ *and* $t > 0$.

We also have the following:

COROLLARY 3.8. *If there exist a* $\lambda \in \mathbf{R}$ *and a* $f \in C(\mathbf{R}^d, \mathbf{R}^+)$ *such that* $P_t^\varphi f = e^{\lambda t} f$ *for any* $t > 0$, *then* $\lambda \geq \Lambda^\varphi$.

PROOF. Since $P_t^\varphi f = e^{\lambda t} f$ for any $t > 0$, we get from the definition of $\overline{Q_t^\varphi}$ that

$$\overline{Q_t^\varphi}(\psi^{-\alpha(\varphi)}f) = e^{\lambda t}e^{-\Lambda^\varphi t}f, \qquad \text{for any } t > 0.$$

On the other hand, choose any $\widetilde{f} \in C_b(\mathbf{R}^d, \mathbf{R}^+)$ such that $\widetilde{f} \leq \psi^{-\alpha(\varphi)}f$, and choose any $x \in \mathbf{R}^d$. We have by Lemma 3.6 that

$$\begin{aligned}\overline{Q_t^\varphi}(\psi^{-\alpha(\varphi)}f)(x) \geq \overline{Q_t^\varphi}\widetilde{f}(x) &= \Big(\int_{\mathbf{R}^d}\widetilde{f}d\nu^\varphi\Big)h^\varphi(x) + \widetilde{Q_t^\varphi}\widetilde{f}(x) \\ &\to \Big(\int_{\mathbf{R}^d}\widetilde{f}d\nu^\varphi\Big)h^\varphi(x) > 0\end{aligned}$$

as $t \to \infty$. These give us that $\lambda \geq \Lambda^\varphi$. □

REMARK 1. When $\varphi = 0$, we can take $\alpha(0) = 2$. Since $P_t 1 = 1$, it is easy that $\Lambda^0 = 0$, $h^0 = \psi^{-2}$ and $\nu^0 = \psi^2\mu$.

REMARK 2. For general $\varphi \in C_b(\mathbf{R}^d)$, we may not expect the (unique) positive eigenfunction of P_t^φ to be bounded. For example, let $b(x) = -x$, $\alpha \in \mathbf{R}$, and $\varphi_\alpha(x) = -\psi^{-\alpha}(x)\left(\frac{1}{2}\Delta\psi^\alpha(x) - x\cdot\nabla\psi^\alpha(x)\right)$. Then φ_α is in $C_b(\mathbf{R}^d)$, and $(\frac{1}{2}\Delta + b\cdot\nabla + \varphi_\alpha)\psi^\alpha = 0$. So ψ^α is a positive eigenfunction of P_t^φ, which is certainly not bounded if $\alpha > 0$.

Now, we can define a set of probabilities $\{Q_x^\varphi\}_{x\in\mathbf{R}^d}$ on $(\Omega, \mathcal{F})$ such that

$$Q_x^\varphi(A) = \frac{e^{-\Lambda^\varphi t}}{h^\varphi(x)}\psi^{-\alpha(\varphi)}(x)E^{P_x}\left[1_A(X_t)\exp(\int_0^t \varphi(X_u)du)\psi^{\alpha(\varphi)}(X_t)h^\varphi(X_t)\right]$$

for any $x \in \mathbf{R}^d$, $t \geq 0$ and $A \in \mathcal{F}_t$. Let $\{Q_t^{\varphi}\}$ be the corresponding semigroup of bounded linear operators on $C_b(\mathbf{R}^d)$.

Notice that by Lemma 3.6,

$$\begin{aligned} Q_t^{\varphi} f(x) &= \frac{1}{h^{\varphi}(x)} \overline{Q_t^{\varphi}}(f h^{\varphi})(x) \\ &= \int_{\mathbf{R}^d} f h^{\varphi} d\nu^{\varphi} + \frac{1}{h^{\varphi}(x)} \widetilde{Q_t^{\varphi}}(f h^{\varphi})(x), \qquad f \in C_b(\mathbf{R}^d). \end{aligned}$$

The (unique) invariant probability of $\{Q_t^{\varphi}\}$ is $\pi^{\varphi} := h^{\varphi}\nu^{\varphi}$.

4. Exponential Tightness

A family of probabilities $\{\nu_t\}_{t\geq 0}$ is said to be exponentially tight if for any $L \geq 0$, there exists a compact C_L such that $\lim_{t\to\infty} \frac{1}{t} \log \nu_t(C_L^c) \leq -L$.

Our main result of this section is the following:

Proposition 4.1. *$\{\mu_t^{x,y}\}_{t\geq 0}$ is exponentially tight for any $x, y \in \mathbf{R}^d$.*

We first prepare several lemmas. Notice that by Stirling's formula

$$K_{\alpha}^3 := \sum_{n=0}^{\infty} \frac{1}{n!} \left(2K_{\alpha n}^1\right)^{\frac{\alpha n}{2}} < \infty, \qquad \text{for any } \alpha \in (0,2),$$

where $K_{\alpha n}^1$ is as defined in Section 2. So we have the following by Lemma 2.1:

Lemma 4.2. *For any $\alpha \in (0,2)$, there exist constants $C_\alpha > 0$ and $T_\alpha \in \mathbf{N}$ such that*

$$E^{P_x}\left[e^{|X_t|^{\alpha}}\right] \leq C_\alpha e^{\frac{1}{2}|x|^{\alpha}}, \qquad \textit{for any } t \geq T_\alpha - 1 \textit{ and } x \in \mathbf{R}^d.$$

Proof. For any $m \in \mathbf{N}$, there exists a constant $C(m) < 1$ such that $\sum_{n=0}^{m} \frac{1}{n!} z^n < C(m) e^z$ for any $z \geq 0$. Hence

$$e^z \leq (1 - C(m))^{-1}\Big(1 + \sum_{n=m+1}^{\infty} \frac{1}{n!} z^n\Big), \qquad \text{for any } z \geq 0.$$

For $\alpha \in (0,2)$, let $m = [\frac{2}{\alpha}]$. Then by Lemma 2.1,

$$E^{P_x}\left[|X_t|^{\alpha n}\right] \le \left(e^{-2c_5 t}|x|^2 + K^1_{\alpha n}\right)^{\frac{\alpha n}{2}} \le 2^{\frac{\alpha n}{2}} e^{-\alpha n c_5 t}|x|^{\alpha n} + (K^1_{\alpha n})^{\frac{\alpha n}{2}}$$

for any $n \ge m+1$, $x \in \mathbf{R}^d$ and $t > 0$. Therefore,

$$\begin{aligned} E^{P_x}\left[e^{|X_t|^\alpha}\right] &\le (1 - C(m))^{-1}\Big(1 + \sum_{n=m+1}^{\infty} \frac{1}{n!} E^{P_x}\left[|X_t|^{\alpha n}\right]\Big) \\ &\le (1 - C(m))^{-1}(K^3_\alpha + 2)\exp(e^{-\alpha c_2 t} 2^{\alpha/2}|x|^\alpha). \end{aligned}$$

Choosing $T_\alpha \in \mathbf{N}$ so that $e^{-\alpha c_2 t}2^{\alpha/2} \le \frac{1}{2}$ for any $t \ge T_\alpha - 1$ completes the proof. □

LEMMA 4.3. *For any $C \in (0, \frac{c_5}{c_2})$, there exists a constant $\xi_C > 0$ such that*

$$E^{P_x}\left[e^{C|X_t|^2}\right] \le e^{C|x|^2} e^{\xi_C t}, \qquad \textit{for any } x \in \mathbf{R}^d \textit{ and } t > 0.$$

PROOF. For any $C \in (0, \frac{c_5}{c_2})$, by (A3), there exists a $\xi_C > 0$ such that $2Cx \cdot b(x) + Cc_2 d + 2C^2 c_2|x|^2 \le \xi_C$ for any $x \in \mathbf{R}^d$. Therefore, we have by Ito's formula that $\frac{d}{dt}E^{P_x}\left[e^{C|X_t|^2}\right] \le \xi_C E^{P_x}\left[e^{C|X_t|^2}\right]$. This and $E^{P_x}\left[e^{C|X_0|^2}\right] = e^{C|x|^2}$ give us our assertion. □

LEMMA 4.4. *For any $\alpha \in (0,2)$, there exists a constant $p_\alpha > 0$ such that*

$$\sup_{t>0, x\in W} \left(E^{P_x}\left[e^{p_\alpha \int_0^t |X_s|^\alpha ds}\right]\right)^{1/t} < \infty, \qquad \textit{for any } W \subset\subset \mathbf{R}^d.$$

Here $\subset\subset$ means compact subset.

PROOF. For any $t > 0$, let $n = [t] \in \mathbf{N} \cup \{0\}$. Choose and fix any $\alpha \in (0,2)$, and let $C_\alpha > 0$ and $T_\alpha \in \mathbf{N}$ be as in Lemma 4.2. Also, fix any $C < \frac{c_5}{c_2} \wedge \frac{1}{2}$. Then

$$\begin{aligned} E^{P_x}\left[e^{\frac{C}{2T_\alpha}\int_0^t |X_s|^\alpha ds}\right] &\le E^{P_x}\left[e^{\frac{C}{2T_\alpha}\int_0^{(n+1)T_\alpha} |X_s|^\alpha ds}\right] \\ &\le E^{P_x}\left[e^{\frac{C}{T_\alpha}\int_0^{T_\alpha} |X_s|^\alpha ds}\right]^{\frac{1}{2}} \cdot E^{P_x}\left[e^{\frac{1}{2T_\alpha}\int_{T_\alpha}^{(n+1)T_\alpha} |X_s|^\alpha ds}\right]^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

We estimate the first term on the right hand side of (4.1). Since $z^\alpha \le 1+z^2$ for any $z \ge 0$, we have by Jensen's inequality and Lemma 4.3

$$(4.2)\quad E^{P_x}\left[e^{\frac{C}{T_\alpha}\int_0^{T_\alpha}|X_s|^\alpha ds}\right] \le e^C \frac{1}{T_\alpha}\int_0^{T_\alpha} E^{P_x}\left[e^{C|X_s|^2}\right] ds \le e^C e^{C|x|^2} e^{\xi_C T_\alpha}.$$

As for the second term on the right hand side of (4.1), we see by Hölder's inequality

$$(4.3)\quad E^{P_x}\left[e^{\frac{1}{2T_\alpha}\int_{T_\alpha}^{(n+1)T_\alpha}|X_s|^\alpha ds}\right] \le \prod_{r=0}^{T_\alpha-1} E^{P_x}\left[e^{\frac{1}{2}\sum_{k=1}^n \int_{kT_\alpha+r}^{kT_\alpha+r+1}|X_s|^\alpha ds}\right]^{1/T_\alpha}.$$

On the other hand, by Schwartz's inequality and Jensen's inequality,

$$\begin{aligned} & E^{P_x}\left[e^{\frac{1}{2}\int_{T_\alpha+r-1}^{T_\alpha+r}|X_s|^\alpha ds} e^{\frac{1}{2}|X_{T_\alpha+r}|^\alpha}\right] \\ \le\ & \left(\int_{T_\alpha+r-1}^{T_\alpha+r} E^{P_x}\left[e^{|X_s|^\alpha}\right] ds \cdot E^{P_x}\left[e^{|X_{T_\alpha+r}|^\alpha}\right]\right)^{\frac{1}{2}} \\ \le\ & C_\alpha e^{\frac{1}{2}|x|^\alpha}. \end{aligned}$$

Here we used Lemma 4.2 for the second line. This combined with the Markovian property implies by induction that

$$E^{P_x}\left[e^{\frac{1}{2}\sum_{k=1}^n \int_{kT_\alpha+r}^{kT_\alpha+r+1}|X_s|^\alpha ds} \cdot e^{\frac{1}{2}|X_{nT_\alpha+r+1}|^\alpha}\right] \le C_\alpha^n e^{\frac{1}{2}|x|^\alpha}, \quad n \in \mathbf{N}.$$

In particular,

$$(4.4)\qquad E^{P_x}\left[e^{\frac{1}{2}\sum_{k=1}^n \int_{kT_\alpha+r}^{kT_\alpha+r+1}|X_s|^\alpha ds}\right] \le C_\alpha^n e^{\frac{1}{2}|x|^\alpha}.$$

By (4.3) and (4.4),

$$(4.5)\qquad E^{P_x}\left[e^{\frac{1}{2T_\alpha}\int_{T_\alpha}^{(n+1)T_\alpha}|X_s|^\alpha ds}\right] \le C_\alpha^n e^{\frac{1}{2}|x|^\alpha}.$$

(4.1), (4.2) and (4.5) complete the proof. □

Lemma 4.5. *For any* $\alpha \in (0,2)$, *there exists a constant* $\widetilde{p_\alpha} > 0$ *such that*

$$\sup_{x,y\in W}\sup_{t>0}\left(E^{P_x}\left[e^{\widetilde{p_\alpha}\int_0^t |X_s|^\alpha ds}\middle| X_t = y\right]\right)^{1/t} < \infty, \qquad \textit{for any } W \subset\subset \mathbf{R}^d.$$

Proof. Choose any $C \in (0, p_\alpha \wedge \frac{c_5}{2c_2})$ and fix it, where p_α is as in Lemma 4.4. We have by Lemma 4.3 that

$$\begin{aligned} E^{P_z}\left[e^{C\int_0^1 |X_s|^\alpha ds}\right] &\le e^C \int_0^1 E^{P_z}\left[e^{C|X_s|^2}\right] ds \\ &\le e^{C+\xi_C} e^{C|z|^2}, \qquad \text{for any } z \in \mathbf{R}^d. \end{aligned}$$

By Lemma 2.3, we have that $c_8 := \sup_{x\in\mathbf{R}^d, y\in W} p_1(x,y) < \infty$. Therefore,

$$\begin{aligned} & E^{P_x}\left[e^{\frac{C}{2}\int_0^t |X_s|^\alpha ds}\middle| X_t = y\right] \\ \le\ & c_8 E^{P_x}\left[e^{C\int_0^{t-1} |X_s|^\alpha ds}\right]^{1/2} \cdot E^{P_x}\left[(e^{C+\xi_C} e^{C|X_{t-1}|^2})^2\right]^{1/2} \\ \le\ & c_8 E^{P_x}\left[e^{C\int_0^{t-1} |X_s|^\alpha ds}\right]^{1/2} \cdot e^{C+\xi_C} e^{C|x|^2} e^{\xi_{2C} t/2}, \qquad \text{for any } y \in \mathbf{R}^d. \end{aligned}$$

This and Lemma 4.4 give us our assertion. □

Proof of Proposition 4.1. Choose an $\alpha \in (0,2)$ and fix it. Let $\widetilde{p_\alpha} > 0$ be as in Lemma 4.5. Define $V : \mathbf{R}^d \to \mathbf{R}, x \mapsto \widetilde{p_\alpha}|x|^\alpha$. By Lemma 4.5, there exists a constant $c_9 > 0$ such that $E^{P_x}\left[\exp\left(\int_0^t V(X_s)ds\right)\middle| X_t = y\right] \le e^{c_9 t}$ for any $t > 0$. For any $l > 0$, there exists a $k_l \in \mathbf{N}$ such that $\inf_{B_{k_l}^c} V \ge l^2$. Therefore,

$$\begin{aligned} P_x\left(L_t(B_{k_l}^c) > \frac{1}{l}\middle| X_t = y\right) &\le e^{-\frac{t}{l}\left(\inf_{B_{k_l}^c} V\right)} E^{P_x}\left[e^{\int_0^t V(X_s)ds}\middle| X_t = y\right] \\ &\le e^{-(l-c_9)t}. \end{aligned}$$

For any $L \in \mathbf{N}$, let $C_L = \cap_{l\ge L}\left\{\nu \in \wp(\mathbf{R}^d); \nu(B_{k_l}) \ge 1 - \frac{1}{l}\right\}$. Then C_L is compact and

$$P_x\left(L_t \in C_L^c \middle| X_t = y\right) \le \sum_{l=L}^\infty e^{-(l-c)t} \le \frac{e^{-(L-c_9)t}}{1-e^{-1}}, \qquad \text{for any } t \ge 1.$$

This gives us our assertion. □

5. Proof of Upper Bound

We show the upper bound in this section (See Lemma 5.3 below).

First, we have the following. As the proof is easy, we omit it here.

LEMMA 5.1. *For any* $\varphi \in C_b(\mathbf{R}^d)$, P_t^φ *has a strictly positive continuous density* $p_t^\varphi(x,y)$ *with respect to* μ, *and* $|\log p_t^\varphi(x,y)| \leq ||\varphi||_\infty t + |\log p_t(x,y)|$. *In particular,* $p_1^\varphi(\cdot,y)$ *is bounded for any* $y \in \mathbf{R}^d$.

Let $\Lambda^\varphi = \lim_{t\to\infty} \frac{1}{t} \log ||\psi^{-\alpha(\varphi)} P_t^\varphi \psi^{\alpha(\varphi)}||_{op}$ as before. Notice that $p_t^\varphi(x,y) = P_{t-1}^\varphi(p_1^\varphi(\cdot,y))(x)$, and $p_1^\varphi(\cdot,y)$ is bounded. So we have the following by Corollary 3.7:

LEMMA 5.2. *For any* $x, y \in \mathbf{R}^d$, $\frac{1}{t}\log p_t^\varphi(x,y) \to \Lambda^\varphi$ *as* $t \to \infty$. *In particular,* $\frac{1}{t}\log p_t(x,y) \to 0$ *as* $t \to \infty$.

We have by Lemma 5.2

$$\begin{aligned} \frac{1}{t}\log \int_{\wp(\mathbf{R}^d)} e^{t\langle\varphi,\nu\rangle} \mu_t^{x,y}(d\nu) &= \frac{1}{t}\log E^{P_x}\left[e^{\int_0^t \varphi(X_s)ds} \middle| X_t = y\right] \\ &= \frac{1}{t}\log \frac{p_t^\varphi(x,y)}{p_t(x,y)} \to \Lambda^\varphi \end{aligned}$$

as $t \to \infty$. This is true for any $\varphi \in C_b(\mathbf{R}^d)$. Let

$$\Lambda^*(\nu) = \sup\{\int_{\mathbf{R}^d} \phi d\nu - \Lambda^\phi; \phi \in C_b(\mathbf{R}^d)\}, \qquad \nu \in \wp(\mathbf{R}^d).$$

Then we get the following by Proposition 4.1 and Deuschel-Stroock [3]:

LEMMA 5.3. Λ^* *is a non-negative, lower semi-continuous, convex function, and*

$$\limsup_{t\to\infty} \frac{1}{t}\log \mu_t^{x,y}(C) \leq -\inf_{\nu\in C} \Lambda^*(\nu), \qquad \textit{for any closed sets } C \subset \wp(\mathbf{R}^d).$$

6. Proof of Lower Bound

We prove the lower bound in this section. First, we have the following law of large numbers.

REMARK 3. For any $\varphi \in C_b(\mathbf{R}^d)$, $W \subset\subset \mathbf{R}^d$ and $\varepsilon > 0$,

$$Q_x^\varphi(L_t \in B(\pi^\varphi, \varepsilon) \Big| X_t = y) \to 1$$

as $t \to \infty$ uniformly in $x, y \in W$, where π^φ is the probability defined at the end of Section 3, and $B(\nu, \varepsilon)$ means the set $\{\eta \in \wp(\mathbf{R}^d) \Big| dist(\nu, \eta) < \varepsilon\}$ for any $\nu \in \wp(\mathbf{R}^d)$ and $\varepsilon > 0$.

LEMMA 6.1. *For any* $\varphi \in C_b(\mathbf{R}^d)$, $W \subset\subset \mathbf{R}^d$ *and* $\varepsilon > 0$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x,y\in W} P_x(L_t \in B(\mu^\varphi, \varepsilon) \Big| X_t = y) \geq -\Lambda^*(\mu^\varphi).$$

PROOF. Choose $\varphi \in C_b(\mathbf{R}^d)$ and fix it. For any $\delta > 0$, there exists an $\varepsilon_1 \in (0, \varepsilon)$ such that $|\int_{\mathbf{R}^d} \varphi d\nu - \int_{\mathbf{R}^d} \varphi d\pi^\varphi| \leq \delta$ for any $\nu \in \wp(\mathbf{R}^d)$ with $dist(\nu, \pi^\varphi) \leq \varepsilon_1$. Therefore, by Remark 3,

$$\begin{aligned}
& \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x,y\in W} P_x(L_t \in B(\pi^\varphi, \varepsilon) \Big| X_t = y) \\
\geq\ & \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x,y\in W} \left(e^{\Lambda^\varphi t} E^{Q_x^\varphi} \left[e^{-t\int_{\mathbf{R}^d} \varphi dL_t} 1_{\{L_t \in B(\pi^\varphi, \varepsilon_1)\}} \Big| X_t = y \right] \right) \\
\geq\ & \Lambda^\varphi - \int_{\mathbf{R}^d} \varphi d\pi^\varphi - \delta \\
\geq\ & -\Lambda^*(\pi^\varphi) - \delta.
\end{aligned}$$

Let $\delta \to 0$, and we get our assertion. □

REMARK 4. From the proof of Lemma 6.1, we see that $\Lambda^\phi - \int_{\mathbf{R}^d} \phi d\pi^\varphi \geq \Lambda^\varphi - \int_{\mathbf{R}^d} \varphi d\pi^\varphi$ for any $\phi \in C_b(\mathbf{R}^d)$. Therefore, $\Lambda^*(\pi^\varphi) = \int_{\mathbf{R}^d} \varphi d\pi^\varphi - \Lambda^\varphi$.

Let us define J in the following way. For any $\nu \in \wp(\mathbf{R}^d)$, let

$$\begin{aligned}
J_\varepsilon(\nu) \ = \ & \inf \Big\{ \sum_{i=1}^n \eta_i \Lambda^*(\pi^{\phi_i}); n \in \mathbf{N}, \eta_i \geq 0, \sum_{i=1}^n \eta_i = 1, \\
& \qquad \phi_i \in C_b(\mathbf{R}^d), dist(\nu, \sum_{i=1}^n \eta_i \pi^{\phi_i}) \leq \varepsilon \Big\}, \qquad \varepsilon > 0,
\end{aligned}$$

and let

$$J(\nu) = \lim_{\varepsilon \to 0} J_\varepsilon(\nu) = \sup_{\varepsilon > 0} J_\varepsilon(\nu).$$

Also, let $J(\nu) = +\infty$ for any $\nu \in \mathcal{M}(\mathbf{R}^d) \setminus \wp(\mathbf{R}^d)$, where $\mathcal{M}(\mathbf{R}^d)$ denotes the set of all signed measures on $\mathbf{R}^d$ with finite total variations.

Notice that Λ^* is always non-negative, and $\wp(\mathbf{R}^d)$ is complete with respect to the topology given by $\nu_n \to \nu \Leftrightarrow \int_{\mathbf{R}^d} f(d\nu_n - d\nu) \to 0$ for any $f \in C_b(\mathbf{R}^d)$, $\nu_n, \nu \in \mathcal{M}(\mathbf{R}^d)$. So we have the following result. The proof is easy and will be omitted.

Lemma 6.2. *$J : \mathcal{M}(\mathbf{R}^d) \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semi-continuous.*

Let J^* be the Legendre transfer of J given by $J^*(\phi) = \sup\{\int_{\mathbf{R}^d} \phi d\nu - J(\nu); \nu \in \mathcal{M}(\mathbf{R}^d)\}$, $\phi \in C_b(\mathbf{R}^d)$. Then we have the following.

Lemma 6.3. *$\Lambda^\phi = J^*(\phi)$ for any $\phi \in C_b(\mathbf{R}^d)$, and $J(\nu) = \Lambda^*(\nu)$ for any $\nu \in \wp(\mathbf{R}^d)$.*

Proof. Λ^* is convex and lower semi-continuous. So it is easy by the definition of J that $J(\nu) \geq \Lambda^*(\nu)$ for any $\nu \in \wp(\mathbf{R}^d)$. Therefore, for any $\nu \in \wp(\mathbf{R}^d)$ and $\phi \in C_b(\mathbf{R}^d)$, $J(\nu) \geq \Lambda^*(\nu) \geq \int_{\mathbf{R}^d} \phi d\nu - \Lambda^\phi$, hence $\Lambda^\phi \geq \int_{\mathbf{R}^d} \phi d\nu - J(\nu)$. On the other hand, it is easy from the definition of J that $J(\pi^\phi) \leq \Lambda^*(\pi^\phi)$, which is equal to $\int_{\mathbf{R}^d} \phi d\pi^\phi - \Lambda^\phi$ as mentioned in Remark 4. So $\Lambda^\phi \leq \int \phi d\pi^\phi - J(\pi^\phi)$. Therefore, $\Lambda^\phi = \sup\{\int_{\mathbf{R}^d} \phi d\nu - J(\nu); \nu \in \wp(\mathbf{R}^d)\} = \sup\{\int_{\mathbf{R}^d} \phi d\nu - J(\nu); \nu \in \mathcal{M}(\mathbf{R}^d)\} = J^*(\phi)$ for any $\phi \in C_b(\mathbf{R}^d)$. This gives us our first assertion.

The second assertion is now easy by Lemma 6.2 and [3, Theorem 2.2.15]. □

Lemma 6.4.

$$\inf_{t > T_0, x, y \in W} p_t(x, y) > 0$$

for any $W \subset\subset \mathbf{R}^d$ and $T_0 > 0$.

Proof. We have by Lemma 2.1 (with $\alpha = 2$) that

$$\sup_{s > 0, x \in W} P_x(|X_s| > r) \leq \sup_{s > 0, x \in W} \frac{1}{r^2} E^{P_x}[|X_s|^2] \leq \frac{1}{r^2} \max \left\{ \sup_{x \in W} |x|^2, K_2^1 \right\}.$$

Therefore, there exists a constant $r > 0$ such that $\sup_{s>0,x\in W} P_x(|X_s| > r) < \frac{1}{2}$, hence $\inf_{s>0,x\in W} P_x(|X_s| \le r) > \frac{1}{2}$. Therefore,

$$\begin{aligned} p_t(x,y) &= \int_{\mathbf{R}^d} p_{t-T_0}(x,z)p_{T_0}(z,y)\mu(dz) \\ &\ge \inf_{|z|\le r, y\in W} p_{T_0}(z,y) \times P_x(|X_{t-T_0}| \le r) \\ &\ge \frac{1}{2} \inf_{|z|\le r, y\in W} p_{T_0}(z,y) > 0, \qquad \text{for any } t > T_0, x, y \in W. \ \Box \end{aligned}$$

Lemma 6.5.

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x(L_t \in B(\nu,\varepsilon) \Big| X_t = y) \ge -\Lambda^*(\nu)$$

for any $\nu \in \wp(\mathbf{R}^d)$, $\varepsilon > 0$ *and* $x, y \in \mathbf{R}^d$.

Proof. Let $L_{t_1,t_2} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \delta_{X_s} ds$, $0 \le t_1 < t_2 < \infty$. For any $n \in \mathbf{N}$, $\eta_i \in [0,1], \phi_i \in C_b(\mathbf{R}^d), i = 1, \cdots, n$, satisfying $\sum_{i=1}^n \eta_i = 1$ and $dist(\nu, \sum_{i=1}^n \eta_i \pi^{\phi_i}) < \frac{\varepsilon}{2}$, we have that

$$\begin{aligned} \bigcap_{i=1}^n \left\{ L_{\sum_{j=0}^{i-1}\eta_j t, \sum_{j=0}^{i}\eta_j t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} &\subset \left\{ L_t \in B(\sum_{i=1}^n \eta_i \pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} \\ &\subset \left\{ L_t \in B(\nu,\varepsilon) \right\}. \end{aligned}$$

Choose and fix any $W \subset\subset \mathbf{R}^d$ with $\pi(W) > 0$ and $x, y \in W$. Write $x_0 = x$ and $x_n = y$. We have from Markov property that

$$\begin{aligned} & P_x(L_t \in B(\nu,\varepsilon) \Big| X_t = y) \\ \ge\ & P_x\left(\bigcap_{i=1}^n \left\{ L_{\sum_{j=0}^{i-1}\eta_j t, \sum_{j=0}^{i}\eta_j t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) \right\} \Big| X_t = y \right) \\ \ge\ & \int_{x_1,\cdots,x_{n-1}\in W} \prod_{i=1}^n P_{x_{i-1}}\Big(L_{\eta_i t} \in B(\pi^{\phi_i}, \frac{\varepsilon}{2}) \Big| X_{\eta_i t} = x_i \Big) \\ & \qquad \times \prod_{i=1}^n p_{\eta_i t}(x_{i-1}, x_i)\pi(dx_1)\cdots\pi(dx_{n-1}) \end{aligned}$$

$$\geq \prod_{i=1}^{n}\Big(\inf_{x_{i-1},x_i\in W} P_{x_{i-1}}\left(L_{\eta_i t}\in B(\pi^{\phi_i},\frac{\varepsilon}{2})\Big|X_{\eta_i t}=x_i\right)\Big) \\ \times \prod_{i=1}^{n}\left(\inf_{x_{i-1},x_i\in W} p_{\eta_i t}(x_{i-1},x_i)\right)\pi(W)^{n-1}.$$

Notice that $\inf_{t>1}\prod_{i=1}^{n}\left(\inf_{x_{i-1},x_i\in W} p_{\eta_i t}(x_{i-1},x_i)\right) > 0$ by Lemma 6.4. Therefore, by Lemma 6.1,

$$\begin{aligned} &\liminf_{t\to\infty}\frac{1}{t}\log P_x(L_t\in B(\nu,\varepsilon)\Big|X_t=y) \\ \geq\ &\liminf_{t\to\infty}\sum_{i=1}^{n}\frac{1}{t}\log\left(\inf_{x_{i-1},x_i\in C} P_{x_{i-1}}\left(L_{\eta_i t}\in B(\pi^{\phi_i},\frac{\varepsilon}{2})\Big|X_{\eta_i t}=x_i\right)\right) \\ \geq\ &-\sum_{i=1}^{n}\eta_i\Lambda^*(\pi^{\phi_i}). \end{aligned}$$

Take infimum with respect to $n\in\mathbf{N}$ and $\eta_i,\phi,i=1,\cdots,n$, and we get from the definitions of $J_{\varepsilon/2}$ and J that

$$\liminf_{t\to\infty}\frac{1}{t}\log P_x(L_t\in B(\nu,\varepsilon)\Big|X_t=y)\geq -J_{\varepsilon/2}(\nu)\geq -J(\nu).$$

This combined with Lemma 6.3 gives us our assertion. □

Therefore, we get the following lower bound.

Lemma 6.6.

$$\liminf_{t\to\infty}\frac{1}{t}\log P_x(L_t\in G\Big|X_t=y)\geq -\inf_{\nu\in G}\Lambda^*(\nu)$$

for any open subset $G\subset\wp(\mathbf{R}^d)$.

Finally, we prove the following identification of the rate function.

Lemma 6.7. $I=\Lambda^*=J$.

Proof. The second equality is done in Lemma 6.3. We show the first one.

First, for any $\varphi \in C^\infty(\mathbf{R}^d) \cap C_b(\mathbf{R}^d)$, by Corollary 3.7, there exists a unique positive $h^\varphi \in C(\mathbf{R}^d)$ such that $P_t^\varphi h^\varphi = e^{\Lambda^\varphi t} h^\varphi$ for any $t > 0$. So $h^\varphi \in C^\infty(\mathbf{R}^d)$ and $L_0 h^\varphi + \varphi h^\varphi = \Lambda^\varphi h^\varphi$. Define L^φ by $L^\varphi f = (h^\varphi)^{-1}(L_0 + \varphi - \Lambda^\varphi)(h^\varphi f) = (h^\varphi)^{-1} L_0(h^\varphi f) + (\varphi - \Lambda^\varphi) f$. L^φ is nothing but the infinitesimal generator of $\{Q_x^\varphi\}$.

$$
\begin{aligned}
I(\nu) &= \sup\{-\int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu; u \in C^\infty(\mathbf{R}^d), u > 0, \frac{L_0 u}{u} \text{ is bounded}\} \\
&= \int_{\mathbf{R}^d} \varphi d\nu - \Lambda^\varphi \\
&\quad + \sup\{-\int_{\mathbf{R}^d} \frac{L^\varphi((h^\varphi)^{-1}u)}{(h^\varphi)^{-1}u} d\nu; u \in C^\infty(\mathbf{R}^d), u > 0, \\
&\qquad\qquad \frac{L^\varphi((h^\varphi)^{-1}u)}{(h^\varphi)^{-1}u} \text{ is bounded}\} \\
&\geq \int_{\mathbf{R}^d} \varphi d\nu - \Lambda^\varphi.
\end{aligned}
$$

This holds for any $\varphi \in C^\infty(\mathbf{R}^d) \cap C_b(\mathbf{R}^d)$, hence for any $\varphi \in C_b(\mathbf{R}^d)$. Therefore, $I(\nu) \geq \Lambda^*(\nu)$.

We next show the opposite inequality. For any $u \in C^\infty(\mathbf{R}^d, \mathbf{R}^+)$ such that $\frac{L_0 u}{u}$ is bounded, let $\varphi_u = -\frac{L_0 u}{u}$. Then since $(L_0 - \varphi_u)u = 0$, we have $P_t^{\varphi_u} u = u$ for any $t > 0$. hence by Corollary 3.8, $\Lambda^{\varphi_u} \leq 0$. Therefore, $-\int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu \leq -\int_{\mathbf{R}^d} \frac{L_0 u}{u} d\nu - \Lambda^{-\frac{L_0 u}{u}} \leq \Lambda^*(\nu)$. Take supreme with respect to u, and we get $I(\nu) \leq \Lambda^*(\nu)$. This completes the proof of our assertion. □

These finish the proof of Theorem 1.1.

Appendix: Deduction of (D-V) from (A3')

In this Appendix, we sketch the proof of the fact that the condition (D-V) is satisfied under the assumption (A3').

We assume (A3') throughout this appendix.

By (A3'), we have that $2\gamma_1 - \gamma_2 - 1 > 0$. Therefore, there exists an $a_0 \in (0, 2\gamma_1 - \gamma_2 - 1)$. Let $\alpha > 0$ be any constant, and let $\varphi(x) = |x|^{a_0}$, $x \in \mathbf{R}^d$. Then we have by Liang [7] the following: (1) $\Lambda^{P,\varphi}$ is well-defined and finite, (2) there exists a unique (up to constant times) $h^\varphi \in B_\alpha^0$ such that $h^\varphi = e^{-\Lambda^\varphi t} P_t^\varphi h^\varphi$ for any $t > 0$, and h^φ does not depend on $\alpha > 0$, (3)

$(h^\varphi)^{-1} \in B^0_\alpha$ and $\nabla h^\varphi \in B^0_{\gamma_2-\gamma_1+1+\beta+\alpha+a_0}$ for any $\beta > 0$. Let $u_n = \psi^\alpha h^\varphi$, $n \in \mathbf{N}$, and let $V = \Lambda^\varphi - \varphi + \frac{L_0\psi^\alpha}{\psi^\alpha} + \frac{\sum_{i,j=1}^d a_{ij}\nabla_i\psi^\alpha\nabla_j h^\varphi}{\psi^\alpha h^\varphi}$. By taking constant times if necessary, we may assume that the second condition of (D-V) is satisfied. The third and the forth conditions are trivial. Notice that by (A3'), there exist constants $A_1, A_2 > 0$ such that $\psi^{-\alpha}(x)L_0\psi^\alpha(x) \le A_1 - A_2\alpha|x|^{\gamma_1-1}$ for any $x \in \mathbf{R}^d$. Also, since $a_{ij}, i, j = 1, \cdots, d$, are bounded, there exists a constant $A_3 > 0$ such that

$$\left|\frac{\sum_{i,j=1}^d a_{ij}\nabla_i\psi^\alpha\nabla_j h^\varphi}{\psi^\alpha h^\varphi}\right| \le \alpha A_3 \frac{|x||\nabla h^\varphi|}{\psi^2 h^\varphi} \in B^0_{\gamma_2-\gamma_1+\beta+2\alpha+a_0}.$$

Since $a_0 < \gamma_1 - 1 - (\gamma_2 - \gamma_1)$ by our assumption, there exist positive constants $\alpha, \beta > 0$ such that $\gamma_2 - \gamma_1 + \beta + 2\alpha + a_0 < \gamma_1 - 1$. Therefore, $V(x) \searrow -\infty$ as $|x| \nearrow +\infty$, hence V satisfies the first condition of (D-V). The fifth one is now also easy.

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