# Elementary Moduli Space of Triangles and Iterative Processes

By H. Nakamura and K. Oguiso

**Abstract.** We regard the unit disk of the complex plane as the moduli space of the similarity classes of triangles. A certain plane geometric operation on triangles may then be interpreted as a rotation of the moduli disk. We examine conditions under which the operation has a finite period, and discuss examples and related topics.

## §1. Introduction

Iterative processes on triangles are studied in several contemporary articles (e.g. Shapiro [1], Chang-Davis [2], Davis [3,4]), where standard approach seems to have been making use of circulant matrices.

In this article, we propose another approach to such sorts of problems — the *moduli approach.* In fact, the similarity classes of triangles are naturally parametrized by the points of the unit disk $\mathcal{D} = \{w \in \mathbb{C}; |w| < 1\}$ (§2) and a certain type of plane geometric operator acts on $\mathcal{D}$ as its anticlockwise rotation around the origin through a certain angle (§3). We will investigate the conditions under which our operator has a finite period (§3, §4, §5). Finally we consider cases where a "reversely similar" triangle appears after iterative actions of this operator (§6).

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## §2. Moduli Space $\mathcal{D}$

A triangle is a subset $\Delta = \{a, b, c\}$ of $\mathbb{C}$ with its cardinality 3 which satisfies the condition : $(a - b)/(c - b) \notin \mathbb{R}$. The transformation group $G = \{f : \mathbb{C} \to \mathbb{C} \mid f(z) = \alpha z + \beta(\alpha, \beta \in \mathbb{C}, \alpha \neq 0)\}$ acts naturally on the set $T$ of all triangles. We call each orbit ($\in S := T/G$) a(n ordinary) similarity class of triangles. The similarity class to which a triangle $\Delta$ belongs will be denoted by $[\Delta]$.

*Notation* 1.

$$\begin{aligned} \rho \quad &:= \quad \exp(2\pi i/6), \ \omega := \exp(2\pi i/3), \\ \mathcal{H} \quad &:= \quad \{z \in \mathbb{C}; \operatorname{Im} z > 0\} : \text{the upper half plane in variable } z. \\ \mathcal{B} \quad &:= \quad \{Z \in \mathbb{C}; |Z| < 1\} : \text{the unit disk in variable } Z. \\ \mathcal{D} \quad &:= \quad \{w \in \mathbb{C}; |w| < 1\} : \text{the unit disk in variable } w. \end{aligned}$$

DEFINITION 2. We define the maps $\lambda : \mathcal{H} \to \mathcal{H}$, $\mu : \mathcal{B} \to \mathcal{B}$, $f : \mathcal{H} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{D}$ by

$$\lambda(z) = \frac{1}{1 - z} \ , \ \mu(Z) = \omega Z \ , \ f(z) = \frac{\omega - \rho z}{z + \omega} \ , \ g(Z) = Z^3.$$

The bijectivity of $f$, the surjectivity of $g$ and the commutativity of Diagram 1 below follow from simple calculations.

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\ f\ } & \mathcal{B} & \xrightarrow{\ g\ } & \mathcal{D} \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle\mu}\downarrow & & \| \\ \mathcal{H} & \xrightarrow{\ f\ } & \mathcal{B} & \xrightarrow{\ g\ } & \mathcal{D} \end{array} \qquad \text{Diagram 1}$$

The operator $\lambda$ (resp. $\mu$) generates an automorphism group of $\mathcal{H}$ (resp. $\mathcal{B}$) of order 3, whose orbits will be called $\lambda$-orbits (resp. $\mu$-orbits).

Now we define the surjection $\phi : \mathcal{H} \to S$ by

$$\phi(z) = [\{0, 1, z\}].$$

It is easy to see that for any triangle $\Delta \in T$ the fiber $\phi^{-1}([\Delta])$ coincides with exactly one $\lambda$-orbit. It is also clear that for any $w \in \mathcal{D}$ the fiber $g^{-1}(w)$ forms a single $\mu$-orbit; hence there arises a bijection $\psi : \mathcal{D} \to S$ which makes Diagram 2 commute.

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow[\sim]{f} & \mathcal{B} \\
\phi \downarrow & & g \downarrow \\
S & \xleftarrow[\psi]{} & \mathcal{D}
\end{array}
\qquad \text{Diagram 2}
$$

DEFINITION 3. Through the above bijection $\psi$, we call $\mathcal{D}$ the moduli space of the similarity classes of triangles.

REMARK 4. For any real number $x \in \mathcal{D}$, $\psi(x)$ is a similarity class of isosceles triangles. In particular, $\psi(0)$ is the similarity class of regular triangles. The similarity classes of rectangular triangles are parameterized by

$$
\left\{ 2\cos\left(\frac{\theta - \pi}{3}\right) - \sqrt{4\cos^2\left(\frac{\theta - \pi}{3}\right) - 1} \right\}^3 e^{i\theta} \quad (0 \le \theta < 2\pi).
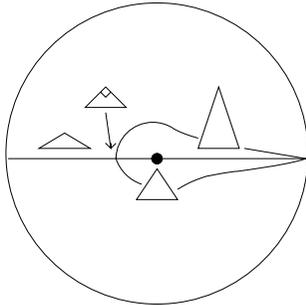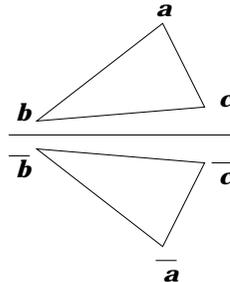$$



Figure 1a



Figure 1b

Two "complex conjugate" triangles $\{a, b, c\}$ and $\{\bar{a}, \bar{b}, \bar{c}\}$ are generally not similar due to our definition of (ordinary) similarity class of triangles. For each similarity class $[\Delta] = [\{a, b, c\}]$, the conjugate similarity class $\overline{[\Delta]} := [\{\bar{a}, \bar{b}, \bar{c}\}]$ is clearly well defined. If $\Delta_1 \in [\Delta]$ and $\Delta_2 \in \overline{[\Delta]}$ for some $\Delta$, then $\Delta_1$ and $\Delta_2$ are called reversely similar (Fig.1b). The reverse similarity will be studied later in §6.

## §3.    Operator $T_{p,q}$

Let $\Delta = \{a, b, c\}$ be a triangle and let $p, q$ be real numbers which satisfy $(p, q) \neq (1/2, 1/2)$ and $pq \neq 1$. We put

$$u_1 := pc + (1-p)b, \ u_2 := pa + (1-p)c, \ u_3 := pb + (1-p)a,$$
$$v_1 := qb + (1-q)c, \ v_2 := qc + (1-q)a, \ v_3 := qa + (1-q)b,$$

and

$$a' := \text{ the intersection point of the lines } \overline{bu_2} \text{ and } \overline{av_1},$$
$$b' := \text{ the intersection point of the lines } \overline{cu_3} \text{ and } \overline{bv_2},$$
$$c' := \text{ the intersection point of the lines } \overline{au_1} \text{ and } \overline{cv_3}.$$
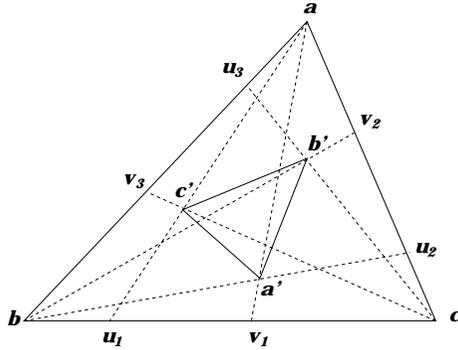


Figure 2

Let us show that $\{a', b', c'\}$ becomes a triangle. Since the process described above belongs to the similarity geometry, it suffices to consider the case where $a = z$, $b = 0$, $c = 1$ $(z \in \mathcal{H})$. By using the well known theorem of Menelaos, we get the equations

$$(\#): \begin{cases} a' := (1-q)(pz + (1-p))/(1-pq), \\ b' := (1-p)((1-q)z + q)/(1-pq), \\ c' := \{(1-p)qz + (1-q)p\}/(1-pq). \end{cases}$$

Our assumption on the pair $(p, q)$ insures that the cardinality of $\{a', b', c'\}$ is equal to 3. In fact, $(\#)$ implies

$$a' - b' = \{(1-q)(2p-1)z + (1-p)(1-2q)\}/(1-pq),$$
$$c' - b' = \{(1-p)(2q-1)z + (p-q)\}/(1-pq).$$

Therefore, to show the non-degeneracy of $\{a', b', c'\}$, it suffices to see the non-vanishing of

$$
\begin{aligned}
D: \;&=\; \{(1-q)(2p-1)\}\{(p-q)\} - \{(1-p)(1-2q)\}\{(1-p)(2q-1)\} \\
&=\; 4p^2q^2 - 6p^2q - 6pq^2 + 3p^2 + 3q^2 + 7pq - 3p - 3q + 1.
\end{aligned}
$$

If we introduce $u := p + q$, $v := pq$, then we see $D = 3u^2 + 4v^2 - 6uv + v - 3u + 1$. This quadratic form of elliptic type can easily be estimated under the condition (*) : $u^2 - 4v \geq 0$. In fact, the solution of the equations $\frac{\partial D}{\partial u} = \frac{\partial D}{\partial v} = 0$ is $(u, v) = (3/2, 1)$, which does not satisfy (*). Hence, $D$ takes the minimal value on the boundary set $\{(u, v); u^2 - 4v = 0\}$. Put $v = u^2/4$, then $D = (u - 1)^2(u - 2)^2/4$. From this, it follows that $D$ is always positive, because the cases of $p = q = 1/2$ and of $pq = 1$ are excluded by our assumption on $(p, q)$. Thus, $(a' - b')/(c' - b')$ belongs to $\mathcal{H}$, and especially $\Delta' := \{a', b', c'\}$ becomes triangle.

So we shall define the map $S_{p,q} : S \to S$ and $T_{p,q} : \mathcal{D} \to \mathcal{D}$ by

$$
S_{p,q}([\Delta]) := [\Delta'], \; T_{p,q} := \psi^{-1} \circ S_{p,q} \circ \psi,
$$

where "$\circ$" denotes the composition of maps.

$$
\begin{array}{ccc}
S & \xrightarrow{\;S_{p,q}\;} & S \\
\psi \uparrow & & \uparrow \psi \\
\mathcal{D} & \xrightarrow{\;T_{p,q}\;} & \mathcal{D}
\end{array}
\qquad \text{Diagram 3}
$$

THEOREM 1.   *For any $w \in \mathcal{D}$,*

$$
T_{p,q}(w) = \frac{t(p,q)}{|t(p,q)|} w,
$$

*where $t(p, q) = \{(p - 1)(2q - 1)\rho - (q - 1)(2p - 1)\}^6$.*

PROOF.   Let $a, b, c$ be $z, 0, 1$ respectively ($z \in \mathcal{H}$) and $\Delta$ be the triangle $\{a, b, c\}$. Suppose that $\Delta' = \{a', b', c'\}$ is given by (#). It is clear that $[\Delta']$ is equal to $[\{0, 1, (a' - b')/(c' - b')\}]$, where

$$
\frac{a' - b'}{c' - b'} = \frac{(1-q)(2p-1)z + (1-p)(1-2q)}{(1-p)(2q-1)z + (p-q)}.
$$

This is already known to belong to $\mathcal{H}$ . So we define $k(z)$ to be the right hand side of this equation. Then,

$$f \circ k \circ f^{-1}(Z) = \frac{(-4pq + 3p + 3q - 2)\rho + 2pq - 3p + 1}{(2pq - 3q + 1)\rho + 2pq - 3p + 1} Z.$$

Multiplying $-(1 + \rho)/3$ to both the numerator and the denominator above, we may also write

$$f \circ k \circ f^{-1}(Z) = \frac{(p - 1)(2q - 1)\rho - (q - 1)(2p - 1)}{(p - 1)(2q - 1)\rho^{-1} - (q - 1)(2p - 1)} Z.$$

Since $\rho^{-1}$ and $\rho$ are complex conjugate to each other, the coefficient $A_{p,q}$ of $Z$ in $f \circ k \circ f^{-1}(Z)$ has absolute value 1. This together with the right commutativity of Diagram 4 concludes the statement of the theorem. $\square$

$$
\begin{array}{ccccc}
\mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D} \\
{\scriptstyle k}\downarrow & {\scriptstyle \cdot A_{p,q}}\downarrow & & {\scriptstyle \cdot A_{p,q}^3}\downarrow & \\
\mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D}
\end{array}
\qquad \text{Diagram 4}
$$

By Theorem 1, the operator $T_{p,q}$ acts on $\mathcal{D}$ as the anticlockwise rotation around the origin through the angle $\arg(t(p, q))$.

COROLLARY 1.　$T_{p,q}$ and $T_{r,s}$ are commutative :

$$T_{p,q} \circ T_{r,s} = T_{r,s} \circ T_{p,q}.$$

PROOF.　This is clear from the above description. $\square$

COROLLARY 2.

(1) $T_{q,p} = T_{p,q}{}^{-1}$;

(2) $T_{\frac{3}{2}-p, \frac{3}{2}-q} = T_{p,q}{}^{-1}$.

PROOF. Put $s(p,q) = (p-1)(2q-1)\rho - (q-1)(2p-1)$ so that $s(p,q)^6 = t(p,q)$. Then, it is easy to see that

$$s(p,q) = s((3/2) - q, (3/2) - p) = -\rho \cdot \overline{s(q,p)}.$$

The corollary follows immediately from this. □

In our formulation of the iterative process $S_{p,q}$ on triangles described above, we are mainly interested in the conditions for $(p,q)$ under which the cyclic group $\{T_{p,q}{}^n | n \in \mathbb{Z}\}$ acting on $\mathcal{D}$ becomes finite.

DEFINITION 5. We define the period of $T_{p,q}$ to be the order of the cyclic group $\{T_{p,q}{}^n | n \in \mathbb{Z}\} \subset Aut(\mathcal{D})$. If it is not finite, the period is set to be $\infty$.

LEMMA 6. *Let $K(\subset \mathbb{C})$ be an algebraic number field which is stable under the complex conjugation. For $x \in K$, the following two conditions are equivalent.*

(a) *There exists an integer $n$ such that $(x/|x|)^n = 1$.*

(b) *$(x/|x|)^2$ is a root of unity in $K$ .*

PROOF. Clear from the fact that $|x|^2 = x\bar{x} \in K$. □

THEOREM 2. *Let $(p,q)$ be any pair of real numbers satisfying $(p,q) \neq (1/2, 1/2)$ and $pq \neq 1$. Then,*

(1) *The period of $T_{p,q}$ is equal to 1 if and only if at least one of the following equations holds: $2p = 1, 2q = 1, p = 1, q = 1, p = q$.*

(2) *The period of $T_{p,q}$ is equal to 2 if and only if at least one of the following equations holds: $2pq - 3q + 1 = 0, 2pq - 3p + 1 = 0, 4pq - 3p - 3q + 2 = 0$.*

(3) *If $p, q$ are rational numbers satisfying neither (1) nor (2), then the period of $T_{p,q}$ is $\infty$ .*

PROOF. Put $A := (p-1)(2q-1), B := -(q-1)(2p-1)$. Then $T_{p,q} : \mathcal{D} \to \mathcal{D}$ is the rotation around 0 through the angle $6 \arg(A\rho + B)$.

(1) The period of $T_{p,q}$ is 1 if and only if $\arg(A\rho + B) = \pi n/3$ ($n = 0, 1, 2, \cdots$). This is equivalent to say that $A : B = 1 : 0$ or $0 : 1$ or $1 : (-1)$. (See Figure 3 below).

(2) The period of $T_{p,q}$ is 2 if and only if $\arg(A\rho + B) = \pi(2n + 1)/6$ ($n = 0, 1, 2, \cdots$). This is equivalent to say $A : B = 1 : 1$, $2 : (-1)$ or $1 : (-2)$.

(3) For rational numbers $p, q$, the number $A\rho + B$ belongs to the quadratic field $\mathbb{Q}(\rho)$. Since the roots of unity in $\mathbb{Q}(\rho)$ are $\rho^m$ ($m = 0, 1, 2, \cdots, 5$),
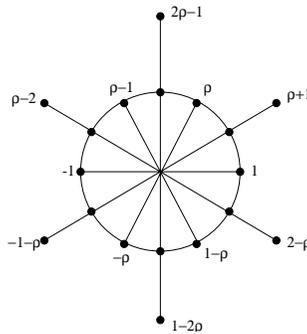


Figure 3
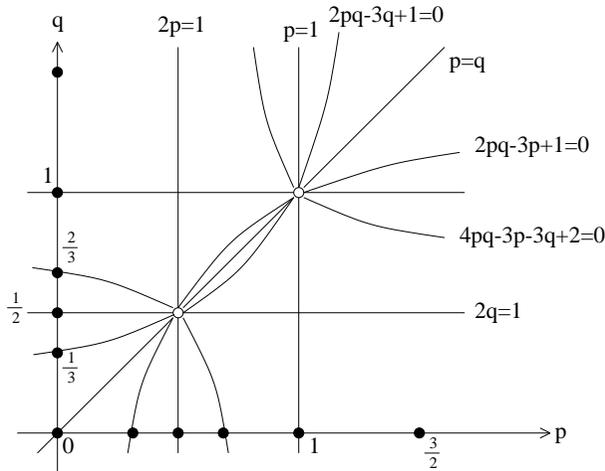


Figure 4

the above Lemma implies that, in order for the period of $T_{p,q}$ to be finite, $\arg(A\rho + B)$ should be $\pi n/6$ $(n = 0, 1, 2, \cdots)$ i.e. only in the cases of (1),(2). This verifies the statement. $\square$

## §4. Special Case $(p = 0)$

Let $\Delta = \{a, b, c\}$ be a triangle. If we set $p = 0$ at the beginning stage of §3, then $a', b', c'$ are the division points of the sides $\overline{bc}, \overline{ca}, \overline{ab}$ respectively with the ratio $(1 - q) : q$. This case has been studied by many authors and some results are generalized for "$p$-affine $n$-gons" in Shapiro[1]. In this section, applying the result of §3, we give a necessary and sufficient condition for $T_{0,q}$ to have a finite period.
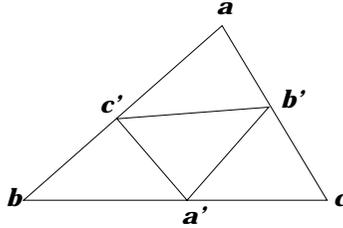


Figure 5

Note that we already know by Theorem 2 that a rational number $q$ for which $T_{0,q}$ has a finite period is one of $0, 1/3, 1/2, 2/3, 1$.

THEOREM 3. *For any given real number $q$, let $\theta \in (-1, 1)$ be the unique number such that $q = \frac{1}{2}(1 + \frac{\sqrt{3}}{3}\tan\frac{\pi\theta}{2})$. Then $T_{0,q}$ is the anticlockwise rotation of $\mathcal{D}$ around the origin through the angle $3\pi\theta$. Hence $T_{0,q}$ has a finite period if and only if $\theta \in \mathbb{Q}$. Furthermore, when $\theta$ is $n/m$ $(m \neq 0, (m,n) = 1)$ the period of $T_{0,q}$ equals $|\frac{2m}{(3,m)(2,n)}|$, where $(\,,\,)$ denotes the greatest common divisor.*

PROOF. Put $p = 0$ in Theorem 1, then $T_{0,q}$ becomes the rotation of $\mathcal{D}$ through an angle of $6\arg(\omega - q\sqrt{3}i)$. Let $\theta$ satisfy:

$$-1 < \theta < 1 \text{ and } \arg(\omega - q\sqrt{3}i) = \pi + \frac{\pi\theta}{2}.$$

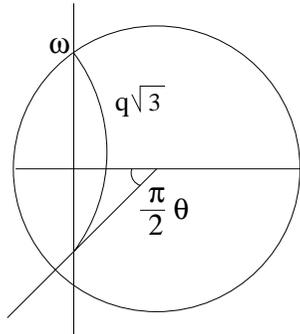Then $q = \frac{1}{2}(1 + \frac{\sqrt{3}}{3}\tan\frac{\pi\theta}{2})$.

Figure 6

Obviously, there exists a natural number $k$ with $k(\pi + \pi\theta/2) \equiv 0 \pmod{2\pi}$ iff $\theta$ belongs to $\mathbb{Q}$. If $\theta = n/m$ (irreducible fraction), then

$$6\arg(\omega - q\sqrt{3}i) \equiv 3\pi n/m \pmod{2\pi}.$$

Hence the last statement also follows. □

COROLLARY 3. *For any natural number $N$ there exists a real number $q$ such that the period of $T_{0,q}$ exactly equals $N$.*

PROOF. Using the notations of Theorem 3, we argue case by case as follows.

(a) When $3 \nmid N, 2 \nmid N$, we may put $m = N, n = 2$ and $\theta = 2/N$.

(b) When $3|N, 2 \nmid N$, we may put $m = 3N, n = 2$ and $\theta = 2/3N$.

(c) When $3 \nmid N, 2|N$, we may put $m = N/2, n = 1$ and $\theta = 2/N$.

(d) When $3|N, 2|N$, we may put $m = 3N/2, n = 1$ and $\theta = 2/3N$. □

The following proposition answers to an exercise raised in the article by Shapiro[1] (16.(2)).

PROPOSITION 7. *For $T_{0,r} \circ T_{0,s} = id$, it is necessary and sufficient that the pair $(r, s)$ satisfies at least one of the following equations : $r + s - 1 = 0, 3rs - r - s = 0, 3rs - 2r - 2s + 1 = 0$.*

PROOF. By Corollary 2 we may consider the case where $T_{0,r} = T_{s,0}$. Since we have : $t(0,r) = \{-(2r-1)\rho + (r-1)\}^6$ and $t(s,0) = \{-(s-1)\rho + (2s-1)\}^6$, it is necessary and sufficient for $T_{0,r} = T_{s,0}$ that

$$\arg\{(2r-1)\rho + (1-r)\} \equiv \arg\{(s-1)\rho - (2s-1)\} \quad (\text{mod } \pi/3).$$

The proposition follows from the following simple fact: For any real numbers $A, B, C, D$ with $(A^2 + B^2 \neq 0, C^2 + D^2 \neq 0)$, $\arg(A\rho + B) \equiv \arg(C\rho + D)$ (mod $\pi/3$) if and only if $AD - BC = 0$ or $AD + BD + AC = 0$ or $BD + AC + BC = 0$. (See Figure 3). □

## §5.  Special Case $(p + q = 1)$

If $p + q = 1$ at the beginning stage of §3, then $u_1, u_2, u_3$ coincide with $v_1, v_2, v_3$ respectively. So the triangle $\{a', b', c'\}$ is given as in Figure 7.
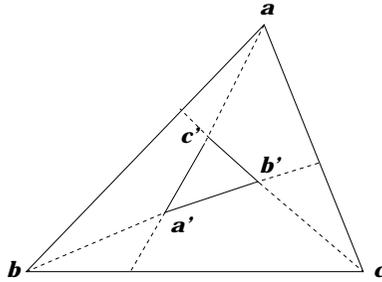


Figure 7

Apply Theorem 1 to this case, then, for any $w \in \mathcal{D}$,

$$T_{1-q,q}(w) = \frac{t(1-q,q)}{|t(1-q,q)|} w,$$

where $t(1-q,q)$ is equal to $\{(1-2q)(q\rho + (1-q))\}^6$. In this case we have

$$\lim_{q \to \frac{1}{2}} \frac{\{q\rho + (1-q)\}^6}{|q\rho + (1-q)|^6} = -1.$$

So we may define the operator $T_{\frac{1}{2},\frac{1}{2}}$ by

$$T_{\frac{1}{2},\frac{1}{2}}(w) = -w \quad (w \in \mathcal{D}).$$

Theorem 4.    *For any given real number $q$, take the unique $\theta$ such that $-1 < \theta < 1$ and $2q = (1 + \sqrt{3}\tan\frac{\pi\theta}{2})$. Then $T_{1-q,q}$ is the anticlockwise rotation of $\mathcal{D}$ around the origin through $\pi(1+3\theta)$. Hence it is necessary and sufficient for $T_{1-q,q}$ to have a finite period that $\theta$ belongs to $\mathbb{Q}$. Furthermore if $\theta = n/m$ ($m \neq 0, (m,n) = 1$), the period of $T_{1-q,q}$ equals $|\frac{2m}{(3n+m,2)(m,3)}|$, where ( , ) is the greatest common divisor.*

Proof.    Put $\arg(q\rho + 1 - q) = \frac{\pi}{6} + \frac{\pi}{2}\theta$ ($-1 < \theta < 1$). Then, it follows that

$$q = \frac{1}{2}(1 + \sqrt{3}\tan\frac{\pi\theta}{2}).$$

(Cf. Figure 8). We see that $6\arg(q\rho + 1 - q)$ equals $6(\frac{\pi}{6} + \frac{\pi}{2}\theta) = \pi(1 + 3\theta)$ modulo $2\pi$. If $\theta = n/m$ (irreducible fraction), then $\pi(1 + 3\theta)$ equals $(3n + m)\pi/m$, which verifies the last statement. $\square$
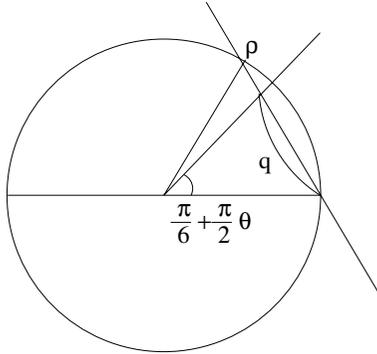


Figure 8

## §6.    Reverse Similarity

Let $\Delta = \Delta_0$ be a triangle and choose triangles $\Delta_k \in S_{p,q}{}^k([\Delta])$ for $k = 1, 2, 3, \cdots$. In this section we give some conditions under which there appears a triangle $\Delta_n$ ($n > 0$) in the above sequence $\{\Delta_k; 0 < k < \infty\}$ such that $\Delta_n$ is reversely similar to $\Delta_0$.

THEOREM 5. *Let $p, q$ be rational numbers which satisfy the assumption of §3. Assume that there exists $n(> 0)$ such that $\Delta_n$ and $\Delta_0$ are reversely similar. Then one of the followings holds.*

(1) *$\Delta_{2i}(i = 0, 1, 2, \cdots)$ are (ordinarily) similar, $\Delta_{2i+1}(i = 0, 1, 2, \cdots)$ are (ordinarily) similar and $\Delta_0$ and $\Delta_1$ are reversely similar.*

(2) *For any $m(\neq n, 0)$, $\Delta_m$ is neither ordinarily nor reversely similar to $\Delta_0$.*

PROOF. If (2) does not hold, then there exists $m$ ($m \neq n, 0 < m < \infty$) such that $\Delta_m$ is ordinarily similar to $\Delta_0$ or $\Delta_n$. Note that we may assume that $\Delta$ is not a regular triangle. Since $p$ and $q$ are rational, the period of $T_{p,q}$ is 1 or 2 by Theorem 2 (3). Hence (1) holds. $\square$

THEOREM 6. *Let $p, q$ be real numbers which satisfy the assumption of §3 and $n$ be any given natural number. Then there are infinitely many similarity classes $[\Delta] \in S$ such that $\Delta$ and $\Delta_n$ are reversely similar.*

PROOF. By assumption, we can take an angle $\theta$ such that

$$n \arg(t(p, q)) + \theta \equiv -\theta \pmod{2\pi}.$$

Since there are infinitely many $w \in \mathcal{D}$ such that $\arg(w) = \theta$, the statement follows. $\square$

## §7. Examples

In this section we describe a few concrete examples. For simplicity we write $T_{p,q}(\Delta) = \Delta'$ when $S_{p,q}([\Delta]) = [\Delta']$.

*Example* 1. The most typical examples are the cases where $(p, q) = (0, 1/2)$ and $(0, 1/3)$. Let $\Delta$ be a triangle $[A, B, C]$. The operator $T_{0,\frac{1}{2}}(\Delta)$ is the midpoint triangle of $\Delta$ which is clearly similar to $\Delta$ (Figure 9a). The operator $T_{0,\frac{1}{3}}$ is described as in Fig.9b (See Shapiro[1]). The period of $T_{0,\frac{1}{2}}$ equals 1 and that of $T_{0,\frac{1}{3}}$ is equal to 2 (cf. Theorem 2).
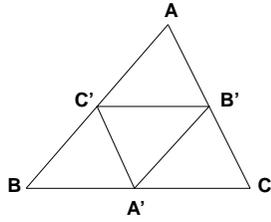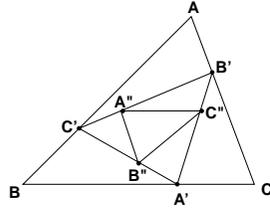
Figure 9a



Figure 9b

*Example* 2.   In the notation of Theorem 3, we put $\theta = -2/5$. Then $p = 0, q \doteqdot 0.2903$. Put $\Delta = \Delta_0 = \{0, 1, i\}$ and $\Delta_k = T_{p,q}{}^k(\Delta_0)$ $(k = 1, 2, \cdots)$. By observing the behavior of $T_{p,q}$ in our $\mathcal{D}$, we easily see the following facts (1) and (2). (Note that $\Delta$ is now isosceles):

(1)  $\Delta_i$ and $\Delta_j$ are ordinarily similar iff $i \equiv j \pmod 5$.

(2)  $\Delta_i$ and $\Delta_j$ are reversely similar iff $i + j \equiv 0 \pmod 5$.
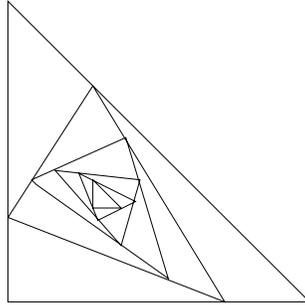


Figure 10

*Example* 3.   The three cases of Proposition 7 in §4.

(1)  $r + s = 1$ $(r = 3/7, s = 4/7)$ : Figure 11a.

(2)  $3rs - r - s = 0$ $(r = 3/7, s = 3/2)$ : Figure 11b.

(3)  $3rs - 2r - 2s + 1 = 0$ $(r = 3/7, s = 1/5)$ : Figure 11c.

In these cases it is easily proven by Euclidean geometry that the corresponding sides are parallel.
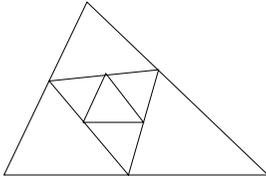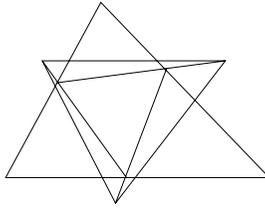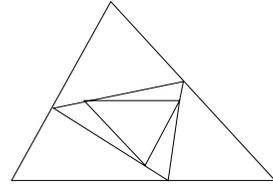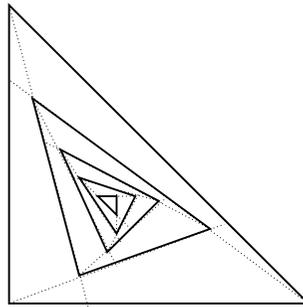
Figure 11a          Figure 11b          Figure 11c



Figure 12

*Example* 4. In the notation of Theorem 4, we put $\theta = 1/6$. Then $q \doteqdot 0.7321$. The period of $T_{q,1-q}$ equals 4.

*Example* 5. We describe examples of (1) and (2) in Theorem 5.

(1) $\Delta_0 = \{0, 1, \sqrt{2}i\}$, $p = 0$, $q = 2/3$: $\psi^{-1}([\Delta_0])$ is on the imaginary axis in $\mathcal{D}$. Hence $\Delta_n$ $(n \geq 0)$ are divided into two similarity classes : $\{\Delta_0, \Delta_2, \cdots\}$ and $\{\Delta_1, \Delta_3, \cdots\}$. $\Delta_0$ is reversely similar to $\Delta_1$ (Figure 13a).

(2) $\Delta_0 = \{A, B, C\} = \{0, 1, \sqrt{3}i\}$, $p = 0$, $q = 3/4$: In this case $\Delta_0$ and $\Delta_1$ are reversely similar and $\Delta_m$ $(m > 1)$ is similar to neither $\Delta_0$ nor $\Delta_1$ (Figure 13b).
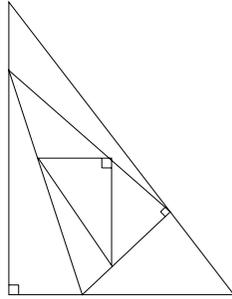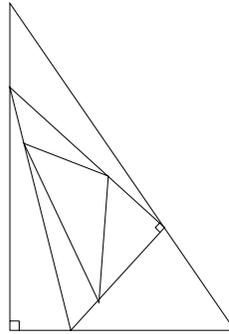
Figure 13a



Figure 13b

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