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A Remark on Whittaker Functions on $Sp(2, \mathbb{R})$

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Abstract. Let π be a principal series representation of $G = Sp(2, \mathbb{R})$ induced from the maximal parabolic subgroup of G with non-abelian unipotent radical. We show that Whittaker functions on G corresponding to a certain K-finite vector of π have simple integral expressions of Mellin-Barnes type. As an application, we compute the real component of Novodvorsky's zeta integral for $GSp(2) \times GL(2)$ in a special case.

§0. Introduction

Whittaker functions on real reductive groups play fundamental roles in the (archimedean local) theory of automorphic forms. Hence various aspects of them are studied by many authors. Among them, Bump [B, Ch.II] presents an explicit formula of the Whittaker function on $GL(3, \mathbb{R})$ arising from the spherical principal series representation. His formula is given by a certain integral of Mellin-Barnes type.

Our aim in this paper is to remark that Whittaker functions on the symplectic group $G = Sp(2, \mathbb{R})$ of rank two have the same kind of integral expression if they arise from the P_1-principal series representation of G (actually we shall discuss a special case, because the other cases are essentially the same). Here the P_1-principal series representation is the principal series representation of G induced from the maximal parabolic subgroup P_1 of G with non-abelian unipotent radical. Note that Miyazaki and Oda ([M-O1], [M-O2]) have already obtained an integral expression of Eulerian type for the Whittaker function on G associated with the P_1-principal series representation. They deduced their formula by constructing a system of differential equations satisfied by the Whittaker function. Our explicit formula (Theorem 3) is obtained from the same system of differential equations, too.

The explicit formula that we present here seems natural from the viewpoint of automorphic L-functions. To illustrate the utility of our formula, we

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compute the local component of Novodvorsky's zeta integral for $GSp(2) \times GL(2)$ ([No1],[No2, §3]) at the real place in §3 (We add §3 in the process of revising this paper).

§1. Preliminaries

(1.1) The symplectic group. Let G be the real symplectic group $Sp(2,\mathbb{R})$ of rank two, which is defined by

$$Sp(2,\mathbb{R}) := \left\{ g \in M(4,\mathbb{R}) \,\middle|\, {}^t g J_4 g = J_4 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}.$$

Here, I_2 is the identity matrix of degree 2. We denote the Lie algebra of G by $\mathfrak{g}$. We fix a maximal compact subgroup K of G as follows:

$$K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(2,\mathbb{R}) \,\middle|\, A, B \in M(2,\mathbb{R}) \right\}.$$

It is isomorphic to the unitary group $U(2) := \{g \in GL(2,\mathbb{C}) \mid {}^t\overline{g}g = I_2\}$. Take a maximal split torus A of G as $A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) | a_i > 0 \ (i = 1, 2)\}$. We also fix a maximal unipotent subgroup N of G as follows:

$$N := \left\{ n(n_0,n_1,n_2,n_3) = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \,\middle|\, n_i \in \mathbb{R} \right\}.$$

Then we have an Iwasawa decomposition $G = NAK$.

(1.2) Whittaker functions. Let η be a character of N. It is written as

$$\eta : N \ni n(n_0,n_1,n_2,n_3) \mapsto \exp(2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3)) \in \mathbb{C}^{(1)}$$

with some $c_0, c_3 \in \mathbb{R}$. Throughout this paper, we assume that η is non-degenerate, that is, $c_0 c_3 \neq 0$. We denote by $C_\eta^\infty(N\backslash G)$ the space of C^∞-functions $W : G \to \mathbb{C}$ satisfying

$$W(ng) = \eta(n)W(g), \quad \text{for all } (n,g) \in N \times G.$$

By the right regular action of G, $C^\infty_\eta(N\backslash G)$ is a smooth G-module. Let (π, H_π) be an (irreducible) admissible representation of G. We denote its underlining $(\mathfrak{g}, K)$-module by the same symbol. By *an algebraic Whittaker functional*, we understand a homomorphism ψ of $(\mathfrak{g}, K)$-modules

$$\psi : H_\pi \to C^\infty_\eta(N\backslash G).$$

DEFINITION 1. Let $\psi : H_\pi \hookrightarrow C^\infty_\eta(N\backslash G)$ be an algebraic Whittaker functional for π. For a K-finite vector $v \in H_\pi$, we call its image $W_v := \psi(v) \in C^\infty_\eta(N\backslash G)$ by ψ *a Whittaker function corresponding to* $v \in H_\pi$.

(1.3) P_1-principal series representations of G. The simple Lie group $G = Sp(2,\mathbb{R})$ has, up to conjugation, two maximal parabolic subgroups: one with abelian unipotent radical and the other with non-abelian unipotent radical. The latter is called the *Jacobi parabolic subgroup* of G and denoted by P_1. A Langlands decomposition $P_1 = M_1A_1N_1$ of P_1 is given by

$$M_1 := \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array} \right) \middle| \epsilon \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\};$$

$$A_1 := \{\mathrm{diag}(t, 1, t^{-1}, 1) | t > 0\}; \quad N_1 := \{n(n_0, n_1, n_2, 0) \in N | n_i \in \mathbb{R}\}.$$

A discrete series representation (σ, V_σ) of the semisimple part M_1 of P_1 is of the form $\sigma = \epsilon \boxtimes D_k (|k| \geqslant 2)$, where $\epsilon : \{\pm 1\} \to \mathbb{C}^*$ is a character, D_k is the discrete series representation of $SL(2,\mathbb{R})$ with Blattner parameter k, that is, the extremal weight vector v of D_k satisfies

$$D_k(\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix})v = e^{\sqrt{-1}kx}v, \qquad x \in \mathbb{R}.$$

For each $\nu_1 \in \mathbb{C}$, we define a quasi-character $\exp(\nu_1)$ of A_1 by

$$\exp(\nu_1)(a_1) = t^{\nu_1}, \quad \text{for} \quad a_1 = \mathrm{diag}(t, 1, t^{-1}, 1) \in A_1.$$

We call an induced representation

$$I(P_1; \sigma, \nu_1) := C^\infty\text{-}\mathrm{Ind}^G_{P_1}(\sigma \otimes \exp(\nu_1 + 2) \otimes 1_{N_1})$$

the P_1-*principal series representation* of G. The representation space of $I(P_1;\sigma,\nu_1)$ is given by

$$\big\{F : G \longrightarrow V_\sigma |\ C^\infty\text{-class}, F(m_1a_1n_1g) = \sigma(m_1)\exp(\nu_1+2)(a_1)F(g),\\ \forall(m_1,a_1,n_1,g) \in M_1 \times A_1 \times N_1 \times G\big\},$$

on which G acts by right translation. Set $\gamma_1 := \mathrm{diag}(-1,1,-1,1) \in G$. We say the P_1-principal series representation $I(P_1;\epsilon\otimes D_{-k},\nu_1)$ is *even* (resp. *odd*) if $\epsilon(\gamma_1) = (-1)^k$ (resp. $\epsilon(\gamma_1) = (-1)^{k+1}$).

§2. Explicit Formulae of Whittaker Functions

Throughout this section, suppose that $\pi = I(P_1,\epsilon\otimes D_{-k},\nu_1)$ $(k \geqslant 2)$ is an even P_1-principal series representation of G. Then there exists a unique, up to constant multiple, non-zero element v_0 of H_π satisfying

$$\pi(\begin{pmatrix} A & B \\ -B & A\end{pmatrix})v_0 = \det(A+\sqrt{-1}B)^{-k}v_0, \qquad \forall\begin{pmatrix} A & B \\ -B & A\end{pmatrix} \in K.$$

We call $v_0 \in H_\pi$ *the corner vector of* π. Let W_{v_0} be a Whittaker function on G corresponding to $v_0 \in H_\pi$. By the Iwasawa decomposition $G = NAK$, the function W_{v_0} is uniquely determined by the restriction $W_{v_0}|_A$ to A. Miyazaki and Oda construct a system of partial differential equations satisfied by $W_{v_0}|_A$. In order to write the system, we introduce a new coordinate $x = (x_1,x_2)$ on $A = \{\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1}) | a_i > 0\ (i = 1,2)\}$ by

$$x_1 := \sqrt{4\pi^3c_0^2|c_3|}\,a_1, \quad x_2 := \sqrt{4\pi|c_3|}\,a_2.$$

Let $\partial_i := x_i\frac{\partial}{\partial x_i}$ $(i = 1,2)$ be the Euler operators for this coordinate.

PROPOSITION 2 ([M-O1, Proposition 7.1, Theorem 8.1]). (i) *Let $W_{v_0} \in C^\infty_\eta(N\backslash G)$ be a Whittaker function corresponding to the corner vector $v_0 \in H_\pi$. Define $h(x) \in C^\infty(A)$ by*

$$W_{v_0}\big(\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})\big) = x_1^{k+1}x_2^k\exp(\mathrm{sgn}(c_3)x_2^2/2)h(x_1,x_2).$$

Then $h(x)$ satisfies

$$\left[\partial_1\partial_2 + 4(x_1/x_2)^2\right]h(x) = 0; \tag{1}$$

$$[(\partial_1+\partial_2+k-1+\nu_1)(\partial_1+\partial_2+k-1-\nu_1) \\ +2\,\mathrm{sgn}(c_3)x_2^2\partial_2\big]\,h(x) = 0. \tag{2}$$

(ii) *If* $c_0 \neq 0, c_3 < 0$, *then the system of differential equations* (1) *and* (2) *has a unique, up to constant multiple, solution* $h(x) \in C^\infty(A)$ *such that the corresponding* $W_{v_0}(g)$ *is of moderate growth. The solution is given by*

$$h(x) = \int_0^\infty t^{-k+1/2} W_{0,\nu_1}(t) \exp\Big(-\frac{t^2}{16x_2^2} - \frac{16x_1^2}{t^2}\Big)\frac{dt}{t}. \tag{3}$$

Here $W_{0,\nu_1}(t)$ *is the usual Whittaker function* ([W-W, Ch.16]).
(iii) *If* $c_0 \neq 0, c_3 > 0$, *then there are no solutions for the system of differential equations* (1) *and* (2) *such that the corresponding* $W_{v_0}(g)$ *is of moderate growth.*

We shall supply another integral expression of (3), which is the main result of this paper:

Theorem 3. *Assume* $c_0 \neq 0, c_3 < 0$ *and let* $h(x)$ *be the solution of the system of differential equations* (1) *and* (2) *with the property in Proposition* 2 (ii). *Take a pair* (σ_1, σ_2) *of two real numbers satisfying*

$$\mathrm{Re}(\sigma_1 + \sigma_2 \pm \nu_1 - k + 1) > 0, \quad \textit{and} \quad \sigma_1 > 0 > \sigma_2. \tag{4}$$

Then we have

$$h(x) = C \times \frac{1}{(2\pi\sqrt{-1})^2} \int_{L(\sigma_1)} \int_{L(\sigma_2)} M(h; s_1, s_2) x_1^{-s_1} x_2^{-s_2}\, ds_1\, ds_2 \tag{5}$$

with some constant $C \in \mathbb{C}^\times$ *and*

$$\begin{aligned} M(h; s_1, s_2) := {}& \Gamma\Big(\frac{s_1 + s_2 + \nu_1 - k + 1}{2}\Big)\Gamma\Big(\frac{s_1 + s_2 - \nu_1 - k + 1}{2}\Big) \\ & \times \Gamma\Big(\frac{s_1}{2}\Big)\Gamma\Big(\frac{-s_2}{2}\Big). \end{aligned} \tag{6}$$

Here, the path of integration $L(\sigma_j)$ $(j = 1, 2)$ *is the vertical line from* $\sigma_j - \sqrt{-1}\infty$ *to* $\sigma_j + \sqrt{-1}\infty$.

Proof. We begin with recalling that for any real numbers α, β satisfying $0 < \alpha < \beta$ there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\begin{aligned} |\Gamma(\sigma + \sqrt{-1}\tau)| \leqslant C_{\alpha,\beta} \exp\big(-(\pi - \epsilon)|\tau|/2\big), \\ \forall(\sigma, \tau) \in [\alpha, \beta] \times \mathbb{R}, \quad \forall \epsilon > 0. \end{aligned} \tag{7}$$

This is an easy consequence of the Stirling formula (see [A, Ch.5, §2.5]). From this, we know that the integral (5) converges absolutely and does not depend on (σ_1, σ_2) satisfying (4). Further, by virtue of the formula (7), we can interchange differentiation and integration legitimately. Hence, the differential equations (1) and (2) are transformed into

$$s_1 s_2 M(h; s_1, s_2) + 4M(h; s_1 + 2, s_2 - 2) = 0,$$

and

$$\begin{aligned}&(s_1 + s_2 - k + 1 - \nu_1)(s_1 + s_2 - k + 1 + \nu_1)M(h; s_1, s_2)\\ &+ 2(s_2 + 2)M(h; s_1, s_2 + 2) = 0,\end{aligned}$$

respectively. It is easy to see that $M(h; s_1, s_2)$ in the Theorem is a solution of these difference equations. By using (7) again, we know that for each (σ_1, σ_2) satisfying (4) there exists a constant $C_{\sigma_1,\sigma_2} > 0$ such that

$$|h(x_1, x_2)| \leqslant C_{\sigma_1,\sigma_2} x_1^{-\sigma_1} x_2^{-\sigma_2}. \tag{8}$$

Hence the function $W_{v_0}(g)$ corresponding to the solution (5) is of moderate growth. □

REMARK. (i) We can prove this theorem from (3) by computing

$$\int_0^\infty \int_0^\infty \Big\{ \int_0^\infty t^{-k+1/2} W_{0,\nu_1}(t) \exp\Big(-\frac{t^2}{16x_2^2} - \frac{16x_1^2}{t^2}\Big)\frac{dt}{t}\Big\} x_1^{s_1} x_2^{s_2} \frac{dx_1}{x_1}\frac{dx_2}{x_2}.$$

(ii) Suppose that (π, H_π) is an odd P_1-principal series representation of G or a large discrete series representation of G. Then the Whittaker function corresponding to a certain K-finite vector in H_π satisfies essentially the same system of differential equations as that in Proposition 2 (see [M-O2],[O, Lemma (8.1)]).

§3. An Application to Novodvorsky's Zeta Integral for $GSp(2) \times GL(2)$

We shall recall the definition of the real component $Z^{(\infty)}(s)$ of Novodvorsky's zeta integral for $GSp(2) \times GL(2)$ in a rather special setting. For more details, see [No1],[No2], and [S].

Let $(\pi, H_\pi) = I(P_1, \epsilon \otimes D_{-k}, \nu_1)$ $(k \geqslant 2)$ be an even P_1-principal series representation of G and $v_0 \in H_\pi$ the corner vector of π. We take an algebraic Whittaker functional $\psi : H_\pi \to C^\infty_\eta(N\backslash G)$ such that $\psi(v_0)(g)$ is of moderate growth, which is unique up to constant multiple. Without any loss of generality we may assume that $(c_0, c_3) = (-1, -1)$. Put $\mathcal{W}_G(g) := \psi(v_0)(g)$. This is nothing but the Whittaker function considered in the previous section. We introduce the group $G_1 = GSp^+(2, \mathbb{R})$ by

$$G_1 = GSp^+(2, \mathbb{R}) := \{g \in GL(4, \mathbb{R}) |\, {}^t g J_4 g = \nu(g) J_4 \text{ for some } \nu(g) > 0\}.$$

We extend the function $\mathcal{W}_G$ to a function $\mathcal{W}_{G_1}$ on G_1 by setting $\mathcal{W}_{G_1}(tg) := \mathcal{W}_G(g)$ for $g \in G$ and $t > 0$. We also need a Whittaker function on $G' = GL^+(2, \mathbb{R}) := \{g' \in GL(2, \mathbb{R}) |\det g' > 0\}$. We extend the discrete series representation D_k of $SL(2, \mathbb{R})$ to a representation of G' by letting $\begin{pmatrix} t & \\ & t \end{pmatrix}$ $(t > 0)$ act on the representation space of D_k trivially. This extended representation is denoted by $\tilde{D}_k$. Define a function $\mathcal{W}_{G'}$ on G' by

$$\mathcal{W}_{G'}(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) := e^{2\pi\sqrt{-1}x} y_1^{k/2} e^{-2\pi y_1} e^{\sqrt{-1}k\theta},$$

for $x \in \mathbb{R}$, $y_1, y_2 > 0$, and $\theta \in \mathbb{R}$. We find that $\mathcal{W}_{G'}$ is a Whittaker function on G' corresponding to the extremal weight vector in $\tilde{D}_k$. Then the real component $Z^{(\infty)}(s)$ $(s \in \mathbb{C})$ of Novodvorsky's zeta integral for $GSp(2) \times GL(2)$ is defined by

$$\begin{aligned} Z^{(\infty)}(s) :=& \pi^{-s-k/2} \Gamma(s + \frac{k}{2}) \int_0^\infty y_1^{s-2} \frac{dy_1}{y_1} \int_0^\infty y_2^{2s-2} \frac{dy_2}{y_2} \\ & \times \mathcal{W}_{G_1}\Big(\begin{pmatrix} y_1 y_2^2 & & & \\ & y_1 y_2 & & \\ & & 1 & \\ & & & y_2 \end{pmatrix}\Big) \mathcal{W}_{G'}(\begin{pmatrix} y_1 y_2 & \\ & y_2 \end{pmatrix}). \end{aligned}$$

By using Theorem 3, we obtain the following

Proposition 4. *The integral $Z^{(\infty)}(s)$ converges absolutely for $\mathrm{Re}(s) > |\mathrm{Re}(\nu_1)|/2$ and is, up to constant multiple, equal to*

$$\Gamma_{\mathbb{C}}(s + \frac{\nu_1}{2}) \Gamma_{\mathbb{C}}(s - \frac{\nu_1}{2}) \Gamma_{\mathbb{C}}(s + \frac{\nu_1}{2} + k - 1) \Gamma_{\mathbb{C}}(s - \frac{\nu_1}{2} + k - 1).$$

Here we set $\Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$.

PROOF. It is readily seen that

$$\begin{aligned} Z^{(\infty)}(s) =& C \times \pi^{-s-k/2}\Gamma(s+\frac{k}{2}) \int_0^\infty y_1^{s+3(k-1)/2}\frac{dy_1}{y_1}\int_0^\infty y_2^{2s+k-1}\frac{dy_2}{y_2} \\ & \times \exp(-4\pi y_1) h(\sqrt{4\pi^3}y_1^{1/2}y_2, \sqrt{4\pi}y_1^{1/2}), \end{aligned}$$

with some constant $C \in \mathbf{C}^\times$. By the estimate (8), we know that $Z^{(\infty)}(s)$ converges absolutely for $\mathrm{Re}(s) > |\,\mathrm{Re}(\nu_1)|/2$. To prove the second assertion, we may assume that $\mathrm{Re}(s)$ is sufficiently large. Then we have, up to constant multiple,

$$\begin{aligned} Z^{(\infty)}(s) =& \frac{\pi^{-s-k/2}\Gamma(s+\frac{k}{2})}{(2\pi\sqrt{-1})^2}(4\pi)^{-s-3(k-1)/2} \\ & \times \int_0^\infty y_2^{2s+k-1}\frac{dy_2}{y_2}\int_{L(\sigma_1)} \pi^{-s_1}y_2^{-s_1}ds_1 \\ & \times \int_{L(\sigma_2)} M(h; s_1, s_2)\Gamma(s+\frac{3(k-1)-s_1-s_2}{2})ds_2. \end{aligned}$$

We can evaluate the integral with respect to ds_2 by using Barnes' first lemma ([W-W, p.289]). The rest of computation is easy. □

REMARK. The analogous computation for the spherical principal series representations of $GSp(2)$ and $GL(2)$ is carried out by Niwa ([Nw, Theorem 3]).

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