Inverse Source Problems for Diffusion Equations and Fractional Diffusion Equations

（拡散方程式及び非整数階拡散方程式に対する
ソース項決定逆問題）

坂本 健一

Inverse Source Problems for Diffusion Equations and Fractional Diffusion Equations

Ken-ichi Sakamoto

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Part I

Inverse Source Problems for Diffusion Equations

Chapter 1

Inverse heat source problem from time distributing overdetermination

1.1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider an initial-boundary value problem for a parabolic equation:

$$u_t(x,t) = \frac{1}{r}\Delta u(x,t) + f(x)h(x,t), \qquad x \in \Omega,\ t \in (0,T), \tag{1.1}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T), \tag{1.3}$$

where $R > 0$ is fixed, $r \in (0,R)$ is a parameter, and h is a given function on $\overline{\Omega} \times [0,T]$. We note that $\frac{1}{r}$ is a diffusion coefficient.

We discuss the following inverse problem:

Inverse Problem. Let $r > 0$ be fixed and let $\rho(x,t)$ be given. Determine $u(x,t) = u(r,f)(x,t)$ and $f(x)$, $x \in \Omega$, $t \in (0,T)$ satisfying (1.1) - (1.3) and

$$\int_0^T \rho(x,t)u(x,t)dt = \varphi(x), \qquad x \in \overline{\Omega}. \tag{1.4}$$

Here f is a spatially varying function in the source term, while $\rho(x,t)$ is a density function in the observation. As ρ, for example, we can take

$$\rho(x,t) = \rho_0(t)g(x),$$

where $\rho_0(t)$ is the t-intensity of spatial observation density $g(x)$. Another example is $\rho(x,t) = \rho_1\left(\frac{1}{c}x - \nu t\right)$ where $\nu \in \mathbb{R}^d$ with $|\nu| = 1$, is a fixed vector, $c > 0$ is a constant and ρ_1 is a function in $\mathbb{R}^d$ supported in a compact set. This function describes a moving sensor with velocity $c\nu$.

Similar kinds of inverse problems are discussed in Prilepko, Orlovsky and Vasin [42], but the weight function ρ is mainly assumed to be independent of x and the well-posedness in the sense of Hadamard is proved under some smallness assumptions (e.g., the smallness of diameter of Ω). In this chapter, ρ may depend on x, which can describe a moving sensor from the physical point of view.

We prove that the inverse problem is well-posed in the sense of Hadamard except for a finite set of r. This idea is based on Choulli and Yamamoto [6], where in place of (1.4), the final overdetermining observation $u(\cdot, T)$ is considered. As for inverse problems with final overdetermining observations, see Choulli and Yamamoto [5], Hoffmann and Yamamoto [15], Isakov [17], [18], Prilepko, Kostin and Tikhonov [41], Prilepko, Orlovsky and Vasin [42] and the references therein.

Our main tools are the theory of analytic perturbation of linear operators and the uniqueness in the inverse problem (1.1) - (1.4) for small $r > 0$.

The remainder of this chapter is composed of three sections. In Section 2, we state our main result. In Section 3, we reduce the inverse problem to a Fredholm equation of second kind, and in Section 4, we complete the proof of the main result.

1.2 Main results

We denote the Sobolev spaces by $H^l(\Omega)$ with $l > 0$ (e.g., Adams [1]) and set $H^{2,1}(\Omega \times (0,T)) = L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$. Let us define an operator A in $L^2(\Omega)$ by

$$(-Av)(x) = \Delta v(x), \quad x \in \Omega, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then A^{-1} exists and $A^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega))$. Here and henceforth $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to a Banach space Y, and by $\|\cdot\|_{\mathcal{L}(X,Y)}$ we denote the operator norm if we should specify the spaces X and Y.

Throughout this chapter, we assume:

$$\begin{aligned} &\text{(i)} \quad h \in H^1(0,T;L^\infty(\Omega)), \quad \rho \in L^2(0,T;W^{2,\infty}(\Omega)), \\ &\text{(ii)} \quad M^{-1} \in \mathcal{L}(H^2(\Omega)\cap H_0^1(\Omega), L^2(\Omega)). \end{aligned} \tag{1.5}$$

Here we set

$$Mf = \int_0^T \rho(\cdot,t)A^{-1}(h(\cdot,t)f)dt.$$

We note that $M \in \mathcal{L}(L^2(\Omega), H^2(\Omega)\cap H_0^1(\Omega))$.

We arbitrarily choose $r_0 > 0$ such that $0 < r_0 < R$ and set $I = (r_0, R)$. Here and henceforth C_j denotes positive constants which are independent of choices of $r \in I$ and f in (1.1), but may depend on h and ρ.

Then for every arbitrary fixed $r \in I$ and $f \in L^2(\Omega)$, by means of $h \in H^1(0,T;L^\infty(\Omega))$, there exists a unique solution

$$u = u(r,f) \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$$

to (1.1) - (1.3) (e.g., Theorem 1.1 on p.5 in Lions and Magenes [26], Theorem 3.5 (ii) on p.114 in Pazy [38]). Moreover this solution satisfies

$$\|u(r,f)\|_{H^{2,1}(\Omega\times(0,T))} \le C_1\|fh\|_{L^2(\Omega\times(0,T))}. \tag{1.6}$$

Now we are ready to state our main theorem.

Theorem 1.2.1 *There exists a finite set $E = E(h,\rho,I) \subset I$ satisfying: For $r \in I \setminus E$ and $\varphi \in H^2(\Omega)\cap H_0^1(\Omega)$, there exists a unique solution $\{u(r,f), f\} \in \{C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))\} \times L^2(\Omega)$ to (1.1) - (1.4). Moreover there exists a constant $C_2 > 0$ satisfying*

$$\|f\|_{L^2(\Omega)} + \|u(r,f)\|_{C([0,T];H^2(\Omega))} + \|u(r,f)\|_{C^1([0,T];L^2(\Omega))} \le C_2\|\varphi\|_{H^2(\Omega)}. \tag{1.7}$$

Our main result asserts the generic well-posedness in the sense of Hadamard, that is, the well-posedness holds except a finite set of values $\frac{1}{r}$ of diffusion coefficients. In general, the exceptional set E is not empty, as the following example shows.

Example 1.2.2 *Let* $\Omega = (0,1)$*,* $f(x) = \sin \pi x$ *and* $h(x,t) = 1$*,* $x \in \Omega, t > 0$*. We consider*

$$\begin{cases} u_t(x,t) = \dfrac{1}{r} u_{xx}(x,t) + \sin \pi x, & x \in \Omega,\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & x \in \Omega. \end{cases}$$

For any fixed $r > 0$*, we choose* $\rho = \rho(t)$ *such that* $\rho \in L^2(0,T)$*,*

$$\int_0^T \rho(t)dt = \int_0^T e^{-\frac{\pi^2}{r}t}\rho(t)dt = 1.$$

Then

$$\left\| \int_0^T \rho(t) A^{-1} f dt \right\|_{H^2(\Omega)} = \|A^{-1}f\|_{H^2(\Omega)} \geq C_3 \|f\|_{L^2(\Omega)},$$

that is, assumption (1.5) *holds. However we have* $u(x,t) = \frac{r}{\pi^2}\left(1 - e^{-\frac{\pi^2}{r}t}\right)\sin \pi x$ *and* $\int_0^T \rho(t)u(x,t)dt = 0$ *for* $x \in \overline{\Omega}$*. This shows that in our inverse problem, the uniqueness breaks for some value* $r > 0$*.*

We state three corollaries where the technical assumption (1.5) is satisfied.

Corollary 1.2.3 *Let* $\rho(x,t) = \rho_0(t)g(x)$ *with* $\rho_0 \in L^2(0,T)$*,* $g \in W^{2,\infty}(\Omega)$ *and* $h \in H^1(0,T; L^\infty(\Omega))$*. If*

$$\left| \int_0^T \rho(x,t)h(x,t)dt \right| > 0, \quad x \in \overline{\Omega}, \tag{1.8}$$

then (1.5) *holds.*

Corollary 1.2.4 *Let* $h(x,t) = h(t)$ *be* x*-independent and* $h \in H^1(0,T)$*,* $\rho \in L^2(0,T; W^{2,\infty}(\Omega))$*. Then* (1.8) *implies* (1.5)*.*

Corollary 1.2.5 *(moving sensor with high speed). Let* $\rho_1 \in W^{2,\infty}(\mathbb{R}^n)$*,* $\nu \in S^{d-1} \equiv \{x \in \mathbb{R}^d;\ |x| = 1\}$*,* $c > 0$*, and let us set*

$$\rho_c(x,t) = \rho_1\left(\frac{1}{c}x - \nu t\right).$$

We assume

$$\left| \int_0^T \rho_1(-\nu t)h(x,t)dt \right| > 0, \quad x \in \overline{\Omega}.$$

Then there exists $c_0 > 0$ *such that* (1.5) *holds for any* $c > c_0$*.*

In ρ_c, we notice that $c\nu$ corresponds to the velocity of the moving sensor and c is the speed. We conclude this section with the proofs of the corollaries.

Proof: (Proof of Corollary 1.2.3) We have

$$\begin{aligned}\int_0^T \rho(\cdot,t)A^{-1}(h(\cdot,t)f)dt =& g\int_0^T \rho_0(t)A^{-1}(h(\cdot,t)f)dt\\ =& gA^{-1}\left(\left(\int_0^T \rho_0(t)h(\cdot,t)dt\right)f\right).\end{aligned}$$

Since $\int_0^T \rho(\cdot,t)A^{-1}(h(\cdot,t)f)dt \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\|\Delta\eta\|_{L^2(\Omega)} \le C_5\|\eta\|_{H^2(\Omega)}$, setting $y = A^{-1}\left(\left(\int_0^T \rho_0(t)h(\cdot,t)dt\right)f\right) \in H^2(\Omega)\cap H_0^1(\Omega)$, we have

$$\begin{aligned}\left\|\int_0^T \rho(\cdot,t)A^{-1}(h(\cdot,t)f)dt\right\|_{H^2(\Omega)} \ge& C_5^{-1}\left\|\Delta\left[gA^{-1}\left(\left(\int_0^T \rho_0(t)h(\cdot,t)dt\right)f\right)\right]\right\|_{L^2(\Omega)}\\ =& C_5^{-1}\|g\Delta y + 2\nabla g\cdot\nabla y + y\Delta g\|_{L^2(\Omega)}.\end{aligned}$$

By (1.8), we notice that $|g| > 0$ on $\overline{\Omega}$. We set $Ly = \Delta y + 2\frac{\nabla g}{g}\cdot\nabla y + \frac{\Delta g}{g}y$ where the coefficients are in $L^\infty(\Omega)$ by $g \in W^{2,\infty}(\Omega)$. If 0 is not an eigenvalue of L with $\mathcal{D}(L) = H^2(\Omega)\cap H_0^1(\Omega)$, then we see that

$$\|y\|_{H^2(\Omega)} \le C_6\|Ly\|_{L^2(\Omega)}$$

(e.g., Gilbarg and Trudinger [12], Kato [20]). We can prove that 0 is not an eigenvalue as follows. Let $Ly = 0$ in Ω. Then $0 = gLy = \Delta(gy)$ in Ω. By $gy \in H^2(\Omega)\cap H_0^1(\Omega)$, this means that $gy = 0$ in Ω. By (1.8) we see that $y = 0$ in Ω.

Therefore we have

$$\begin{aligned}\left\|\int_0^T \rho(\cdot,t)A^{-1}(h(\cdot,t)f)dt\right\|_{H^2(\Omega)} &\ge C_5^{-1}\|gLy\|_{L^2(\Omega)} \ge C_5'^{-1}\|Ly\|_{L^2(\Omega)}\\ \ge C_5'^{-1}C_6^{-1}\|\Delta y\|_{L^2(\Omega)} = C_5'^{-1}C_6^{-1}&\left\|\left(\int_0^T \rho_0(t)h(\cdot,t)dt\right)f\right\|_{L^2(\Omega)} \ge C_7\|f\|_{L^2(\Omega)}\end{aligned}$$

again by (1.8). On the other hand, since A^{-1} is a surjection from $L^2(\Omega)$ onto $H^2(\Omega)\cap H_0^1(\Omega)$, we have $\mathcal{R}(M) = H^2(\Omega)\cap H_0^1(\Omega)$. Thus (1.5) holds true.

Proof: (Proof of Corollary 1.2.4) We have

$$\int_0^T \rho(\cdot,t)A^{-1}(h(t)f)dt = \left(\int_0^T \rho(\cdot,t)h(t)dt\right)A^{-1}f,$$

that is,

$$f = A\left(\frac{\int_0^T \rho(\cdot,t)A^{-1}(h(t)f)dt}{\int_0^T \rho(\cdot,t)h(t)dt}\right)$$

by (1.8). Thanks to $\rho \in L^2(0,T;W^{2,\infty}(\Omega))$, we obtain (1.5).

Proof: (Proof of Corollary 1.2.5) We set

$$(M_c f)(x) = \int_0^T \rho_1\left(\frac{x}{c} - \nu t\right) A^{-1}(h(\cdot,t)f)dt$$

and

$$(M_\infty f)(x) = \int_0^T \rho_1(-\nu t)A^{-1}(h(\cdot,t)f)dt.$$

By Corollary 1.2.3, we see that

$$\|M_\infty^{-1}\|_{\mathcal{L}(H^2(\Omega)\cap H_0^1(\Omega),L^2(\Omega))} \le C_8.$$

On the other hand, for $c \ge 1$, we have

$$\begin{aligned}
\|(M_\infty - M_c)f\|_{H^2(\Omega)}^2 \le & C_9\left\|\Delta\left(\int_0^T \left(\rho_1(-\nu t) - \rho_1\left(\frac{x}{c} - \nu t\right)\right) A^{-1}(h(\cdot,t)f)dt\right)\right\|_{L^2(\Omega)}^2 \\
\le C_9 \int_0^T & \left\|(\Delta\rho_1)\left(\frac{x}{c} - \nu t\right)\frac{1}{c^2}A^{-1}(h(\cdot,t)f) + 2(\nabla\rho_1)\left(\frac{x}{c} - \nu t\right)\frac{1}{c}\cdot\nabla A^{-1}(h(\cdot,t)f)\right. \\
& \left. - \left(\rho_1(-\nu t) - \rho_1\left(\frac{x}{c} - \nu t\right)\right) h(\cdot,t)f\right\|_{L^2(\Omega)}^2 dt \\
\le C_{10}\|f\|_{L^2(\Omega)}^2 & \int_0^T \left(\left(\frac{1}{c^2} + \frac{1}{c}\right)\|\rho_1\|_{W^{2,\infty}(\mathbb{R}^n)}\|h(\cdot,t)\|_{L^\infty(\Omega)}\right. \\
& \left. + \frac{\sup_{x\in\Omega}|x|}{c}\|\rho_1\|_{W^{2,\infty}(\mathbb{R}^n)}\|h(\cdot,t)\|_{L^\infty(\Omega)}\right)^2 dt \\
\le \frac{C_{11}}{c^2}&\|f\|_{L^2(\Omega)}^2.
\end{aligned}$$

Hence

$$\|M_\infty - M_c\|_{\mathcal{L}(L^2(\Omega),H^2(\Omega)\cap H_0^1(\Omega))} \le \frac{\sqrt{C_{11}}}{c}.$$

Since

$$M_c = M_\infty + (M_c - M_\infty) = M_\infty(I + M_\infty^{-1}(M_c - M_\infty)),$$

for sufficiently large $c > 0$, the Neumann series (e.g., Kato [20]) implies that $(I + M_\infty^{-1}(M_c - M_\infty))^{-1} \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$, so that $M_c^{-1} \in \mathcal{L}(H^2(\Omega)\cap H_0^1(\Omega), L^2(\Omega))$. Thus the proof of Corollary 1.2.5 is completed.

1.3 Reduction of the inverse problem to a Fredholm equation of second kind

We reduce the inverse problem to a Fredholm equation of second kind.

We notice that $-\frac{1}{r}A$ generates an analytic semigroup in $L^2(\Omega)$ (e.g., Pazy [38]) and represent the solution u to (1.1) - (1.3) by

$$u(t) = \int_0^t e^{-(t-s)\frac{A}{r}} h(s) f ds, \quad t > 0. \tag{1.9}$$

Here we write $u(t) = u(\cdot, t)$, etc. In terms of (1.9) and intergration by parts, we rewrite (1.4) as

$$\varphi(\cdot) = \int_0^T \rho(\cdot, t) u(\cdot, t) dt = \int_0^T \rho(t) u(t) dt$$
$$= \int_0^T \rho(t) \left(\int_0^t e^{-(t-s)\frac{A}{r}} h(s) f ds \right) dt$$
$$= \int_0^T \rho(t) \left\{ \left[rA^{-1} e^{-(t-s)\frac{A}{r}} (h(s) f) \right]_{s=0}^{s=t} - \int_0^t rA^{-1} e^{-(t-s)\frac{A}{r}} (h'(s) f) ds \right\} dt$$
$$= \int_0^T \rho(t) rA^{-1} (h(t) f) dt - \int_0^T \rho(t) rA^{-1} \left(e^{-t\frac{A}{r}} (h(0) f) + \int_0^t e^{-(t-s)\frac{A}{r}} (h'(s) f) ds \right) dt.$$

Here and henceforth we write $h'(t) = h_t(\cdot, t)$.

Hence

$$\frac{M^{-1}\varphi}{r} = f - M^{-1} \int_0^T \rho(t) A^{-1} \left(e^{-t\frac{A}{r}} (h(0) f) + \int_0^t e^{-(t-s)\frac{A}{r}} (h'(s) f) ds \right) dt.$$

We set

$$K_r f = M^{-1} \int_0^T \rho(t) A^{-1} \left(e^{-t\frac{A}{r}} (h(0) f) + \int_0^t e^{-(t-s)\frac{A}{r}} (h'(s) f) ds \right) dt \tag{1.10}$$

and

$$\Phi_r = \frac{M^{-1}\varphi}{r}.$$

Then we obtain a Fredholm equation of second kind:

$$f = K_r f + \Phi_r. \tag{1.11}$$

Lemma 1.3.1 *Let $r \in I$ be arbitrarily fixed.*
(i) Equation (1.11) *possesses a solution $f \in L^2(\Omega)$ if and only if $\{u(r,f), f\} \in \{C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))\} \times L^2(\Omega)$ satisfies* (1.1) - (1.4).
(ii) Equation (1.11) *possesses a unique solution if and only if there exists a unique solution $\{u(r,f), f\}$ to* (1.1) - (1.4).

Proof: (i) First assume that (1.11) possesses a solution $f \in L^2(\Omega)$. By Theorem 3.5 (ii) on p.114 in [38], there exists a unique solution $u(r,f) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ to (1.1) - (1.3) with this f. We have to prove that the solution $u(r,f)$ satisfies (1.4). We set

$$\int_0^T \rho(t) u(r,f)(t) dt = \varphi_1.$$

By $u(r,f) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, we can directly see $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. Let us multiply the both sides of (1.1) by the operator $\rho(t) A^{-1}$ and integrate the equation with respect to t from 0 to T. Then we have

$$\int_0^T \rho(t) A^{-1} u_t(r,f)(t) dt = -\frac{\varphi_1}{r} + Mf.$$

Since $\rho(t) A^{-1} u_t(r,f)(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for almost all $t \in (0,T)$ and $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, we see that

$$f = M^{-1} \int_0^T \rho(t) A^{-1} u_t(r,f)(t) dt + \frac{M^{-1}\varphi_1}{r}. \tag{1.12}$$

On the other hand, the solution $u_t(r,f) \in C([0,T]; L^2(\Omega))$ is given by

$$u_t(r,f)(\cdot,t) = e^{-t\frac{A}{r}}(fh(0)) + \int_0^t e^{-(t-s)\frac{A}{r}}(fh'(s)) ds, \qquad t \in (0,T) \tag{1.13}$$

(e.g., Theorem 3.5 (p.114) in [38]). Therefore by (1.11) and (1.12), we see that

$$\frac{M^{-1}\varphi}{r} = \frac{M^{-1}\varphi_1}{r} \qquad \text{in } L^2(\Omega).$$

Hence $\varphi = \varphi_1$.

Next the converse assertion of (i) is already seen in deriving (1.11).

(ii) Next we prove the part (ii). Let us assume that (1.11) possesses a unique solution $f \in L^2(\Omega)$. Contrarily suppose that both $\{u(r,f_1), f_1\}$ and $\{u(r,f_2), f_2\}$

are distinct solutions to (1.1) - (1.4). Let $f_1 = f_2$. Then $u(r, f_1) = u(r, f_2)$ because of the unique solvability of the direct problem (1.1) - (1.3). If $f_1 \neq f_2$, then f_1 and f_2 are solutions to (1.11) from the part (i), which contradicts the unique solvability of (3.3). Conversely we assume that the solution to (1.1) - (1.4) exists uniquely, say $\{u(r, f), f\}$. If f_1 and f_2 are distinct solutions to (1.11), then from the part (i), there exist the solutions $\{u(r, f_1), f_1\}$ and $\{u(r, f_2), f_2\}$ to (1.1) - (1.4), which contradicts the unique solvability to (1.1) - (1.4). Thus the proof of Lemma 1.3.1 is completed.

1.4 Proof of Theorem 1.2.1

We divide the proof into four steps.

First Step We show

Lemma 1.4.1 *For an arbitrarily fixed $f \in L^2(\Omega)$, the mapping*

$$K_r f : I \longrightarrow L^2(\Omega)$$

is real analytic in $r \in I$.

In order to prove Lemma 1.4.1, we prove the following lemma.

Lemma 1.4.2 *For an arbitrarily fixed $f \in L^2(\Omega)$, let us define the mapping*

$$\mathfrak{F} : I \longrightarrow H^{2,1}(\Omega \times (0, T))$$

by $\mathfrak{F}(r)(x, t) = u(r, f)(x, t)$. Then $\mathfrak{F}$ is real analytic in $r \in I$.

Proof: (Proof of Lemma 1.4.2.) The same results are proved in Choulli and Yamamoto [5] as Proposition 5 and Choulli and Yamamoto [6] as Lemma 3. For completeness, we repeat the proof here. We set $\widetilde{I} = (R^{-1}, r_0^{-1})$. Let us consider the following equations:

$$v_t(x, t) = s\Delta v(x, t) + f(x)h(x, t), \qquad x \in \Omega,\ t \in (0, T), \tag{1.14}$$

$$v(x, 0) = 0, \qquad x \in \Omega, \tag{1.15}$$

$$v(x, t) = 0, \qquad x \in \partial\Omega,\ t \in (0, T). \tag{1.16}$$

For $f \in L^2(\Omega)$ and $s \in \widetilde{I}$, system (1.14) - (1.16) possesses a unique solution $v(s,f) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. Since f is fixed, we set $v(s,f) = v(s)$ for simplicity. It is sufficient to prove that $v(s) : \widetilde{I} \longrightarrow H^{2,1}(\Omega \times (0,T))$ is real analytic in $s \in \widetilde{I}$. We denote the n-th derivative in $s \in \widetilde{I}$ of the mapping $v(s) : \widetilde{I} \longrightarrow H^{2,1}(\Omega \times (0,T))$ by $v^{(n)}(s)$.

For the real analyticity in $s \in \widetilde{I}$, it is sufficient to prove the following two things (e.g., pp.65-66 in John [19]):

(i) $v \in C^\infty(\widetilde{I}; H^{2,1}(\Omega \times (0,T)))$.

(ii) For every closed interval $J \subset \widetilde{I}$, there exist positive constants $\tilde{M} = \tilde{M}(f,h,J)$ and $\eta = \eta(f,h,J)$ satisfying

$$\|v^{(n)}(s)\|_{H^{2,1}(\Omega\times(0,T))} \le \tilde{M}\eta^n n!, \quad n \in \mathbb{N},\ s \in J. \tag{1.17}$$

Indeed we will prove that we can choose $\tilde{M} = \tilde{M}(f,h,\widetilde{I})$ and $\eta = \eta(f,h,\widetilde{I})$ which are independent of the interval J.

By induction, we will show that $v(s)$ is n-times differentiable for all $n \in \mathbb{N}$ in $s \in \widetilde{I}$, and the function $v^n(s)$ solves the following initial-boundary value problem:

$$v_t^{(n)}(s)(x,t) = s\Delta v^{(n)}(s)(x,t) + n\Delta v^{(n-1)}(s)(x,t), \quad x \in \Omega,\ t \in (0,T), \tag{1.18}$$

$$v^{(n)}(s)(x,0) = 0, \qquad x \in \Omega, \tag{1.19}$$

$$v^{(n)}(s)(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T), \tag{1.20}$$

and that $v^{(n)}(s)$ satisfies (ii) with $\eta = C_1$ which is the constant in (1.6).

First of all, by (1.6), we see that for all $s \in \widetilde{I}$, the solution $v(s)$ satisfies

$$\|v(s)\|_{H^{2,1}(\Omega\times(0,T))} \le C_1\|fh\|_{L^2(\Omega\times(0,T))} \equiv \tilde{M}. \tag{1.21}$$

We take a sufficiently small δ such that $s+\delta, s-\delta \in \widetilde{I}$. Then $v(s+\delta)$ is the solution to the following equations:

$$v_t(s+\delta)(x,t) = (s+\delta)\Delta v(s+\delta)(x,t) + f(x)h(x,t), \quad x \in \Omega, t \in (0,T), \tag{1.22}$$

$$v(s+\delta)(x,0) = 0, \qquad x \in \Omega, \tag{1.23}$$

$$v(s+\delta)(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T). \tag{1.24}$$

We show that $v(s)$ is continuous with respect to $s \in \widetilde{I}$. By (1.14) - (1.16) and (1.22) - (1.24), we see that $v(s+\delta) - v(s)$ solves the following equations:

$$(v(s+\delta) - v(s))_t(x,t) = s\Delta(v(s+\delta) - v(s))(x,t) + \delta\Delta v(s+\delta), \quad x \in \Omega,\ t \in (0,T), \tag{1.25}$$

$$(v(s+\delta) - v(s))(x,0) = 0, \quad x \in \Omega, \tag{1.26}$$

$$(v(s+\delta) - v(s))(x,t) = 0, \quad x \in \partial\Omega,\ t \in (0,T). \tag{1.27}$$

Therefore by (1.21), we obtain

$$\begin{aligned}\|v(s+\delta) - v(s)\|_{H^{2,1}(\Omega\times(0,T))} &\le C_1\|\delta\Delta v(s+\delta)\|_{L^2(\Omega\times(0,T))}\\ &\le C_1\tilde{M}|\delta| \to 0 \quad \text{as } \delta \to 0,\end{aligned} \tag{1.28}$$

which implies that $v(s)$ is continuous in $s \in \widetilde{I}$ with the norm in $H^{2,1}(\Omega \times (0,T))$.

By $w_1(s) \in H^{2,1}(\Omega \times (0,T))$ we denote the solution to (1.18) - (1.20) with $n = 1$. Now we will show $v^{(1)}(s) = w_1(s)$ and continuity of $w_1(s)$ in $s \in \widetilde{I}$ with the norm in $H^{2,1}(\Omega \times (0,T))$.

We set $g(s+\delta) = v(s+\delta) - v(s) - \delta w_1(s)$. Then by (1.14) - (1.16), (1.18) - (1.20) with $n = 1$, and (1.22) - (1.24), we see that $g(s+\delta)$ solves the following equations:

$$g_t(s+\delta)(x,t) = s\Delta g(s+\delta)(x,t) + \delta\Delta(v(s+\delta) - v(s))(x,t),\ x \in \Omega,\ t \in (0,T), \tag{1.29}$$

$$g(s+\delta)(x,0) = 0, \quad x \in \Omega, \tag{1.30}$$

$$g(s+\delta)(x,t) = 0, \quad x \in \partial\Omega,\ t \in (0,T). \tag{1.31}$$

By (1.6) and (1.28), we obtain

$$\begin{aligned}\|g(s+\delta)\|_{H^{2,1}(\Omega\times(0,T))} &\le C_1\|\delta\Delta(v(s+\delta) - v(s))\|_{L^2(\Omega\times(0,T))}\\ &\le C_1|\delta|\|v(s+\delta) - v(s)\|_{H^{2,1}(\Omega\times(0,T))} \le C_1^2\tilde{M}|\delta|^2.\end{aligned}$$

Therefore we have $v^{(1)}(s) = w_1(s)$. Furthermore by (1.18) - (1.20) with $n = 1$, we obtain

$$\|v^{(1)}(s)\|_{H^{2,1}(\Omega\times(0,T))} \le C_1\|\Delta v(s)\|_{L^2(\Omega\times(0,T))} \le C_1\tilde{M}. \tag{1.32}$$

Next, we prove the continuity of $v^{(1)}(s)$ for $s \in \widetilde{I}$ with the norm in $H^{2,1}(\Omega \times (0,T))$. By (1.18) - (1.20) with $n=1$, the function $v^{(1)}(s+\delta)$ gives the solution to the following equations:

$$
\begin{aligned}
&v_t^{(1)}(s+\delta)(x,t) = (s+\delta)\Delta v^{(1)}(s+\delta)(x,t) + \Delta v(s+\delta)(x,t), \quad x \in \Omega,\ t \in (0,T),\\
&v^{(1)}(s+\delta)(x,0) = 0, \qquad x \in \Omega,\\
&v^{(1)}(s+\delta)(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T).
\end{aligned}
$$

Therefore $(v^{(1)}(s+\delta) - v^{(1)}(s))$ is the solution to the following equations:

$$
\begin{aligned}
&(v^{(1)}(s+\delta) - v^{(1)}(s))_t(x,t) = s\Delta(v^{(1)}(s+\delta) - v^{(1)}(s))(x,t)\\
&\qquad +\Delta(v(s+\delta) - v(s))(x,t) + \delta\Delta v^{(1)}(s+\delta)(x,t), \quad x \in \Omega,\ t \in (0,T),\\
&(v^{(1)}(r+\delta) - v^{(1)}(s))(x,0) = 0, \qquad x \in \Omega,\\
&(v^{(1)}(s+\delta) - v^{(1)}(s))(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T).
\end{aligned}
$$

By (1.6), (1.28) and (1.32), we obtain

$$
\begin{aligned}
&\|v^{(1)}(s+\delta) - v^{(1)}(s)\|_{H^{2,1}(\Omega\times(0,T))}\\
&\qquad \le C_1\|\Delta(v(s+\delta) - v(s)) + \delta\Delta v^{(1)}(s+\delta)\|_{L^2(\Omega\times(0,T))}\\
&\qquad \le C_1(\|v(s+\delta) - v(s)\|_{H^{2,1}(\Omega\times(0,T))} + |\delta|C_1\tilde{M}) \to 0 \qquad \text{as } \delta \to 0.
\end{aligned}
$$

Therefore $v^{(1)}(s)$ is continuous in $s \in \widetilde{I}$. Thus we have verified that $v^{(1)}(s)$ is given by (1.18) - (1.20).

Next, we assume that for $n = m$, the function $v^{(m)}(s)$ satisfy (1.18) - (1.20) and (1.17) with $\eta = C_1$. By $w_{m+1}(s)$ we denote the solution to (1.18) - (1.20) with $n = m+1$. For $s+\delta, s-\delta \in \widetilde{I}$, we set

$$
\widetilde{g}(s+\delta)(x,t) = v^{(m)}(s+\delta)(x,t) - v^{(m)}(s)(x,t) - \delta w_{m+1}(s)(x,t).
$$

Then $\widetilde{g}(s+\delta)$ solves the following equations:

$$
\begin{aligned}
&\widetilde{g}_t(s+\delta)(x,t) = s\Delta\widetilde{g}(r+\delta)(x,t) + m\Delta(v^{(m-1)}(s+\delta) - v^{(m-1)}(s) - \delta v^{(m)}(s))(x,t)\\
&\qquad +\delta\Delta(v^{(m)}(s+\delta) - v^{(m)}(s))(x,t), \quad x \in \Omega,\ t \in (0,T),\\
&\widetilde{g}(s+\delta)(x,0) = 0, \qquad x \in \Omega,\\
&\widetilde{g}(s+\delta)(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T).
\end{aligned}
$$

Therefore by (1.6) we obtain

$$\begin{aligned}
&\|\widetilde{g}(s+\delta)\|_{H^{2,1}(\Omega\times(0,T))}\\
&\le C_1\|m\Delta(v^{(m-1)}(s+\delta)-v^{(m-1)}(s)-\delta v^{(m)}(s))\\
&+\delta\Delta(v^{(m)}(s+\delta)-v^{(m)}(s))\|_{L^2(\Omega\times(0,T))}\\
&\le C_1\{m\|v^{(m-1)}(s+\delta)-v^{(m-1)}(s)-\delta v^{(m)}(s))\|_{H^{2,1}(\Omega\times(0,T))}\\
&+|\delta|\|v^{(m)}(s+\delta)-v^{(m)}(s)\|_{H^{2,1}(\Omega\times(0,T))}\}.
\end{aligned}$$

By the assumption in the induction, we have $\left\|\frac{\widetilde{g}(s+h)}{\delta}\right\|_{H^{2,1}(\Omega\times(0,T))}=o(1)$ as $\delta\to 0$. Therefore $v^{(m+1)}(s)=w_{m+1}(s)$.

Moreover by (1.6) and (1.17) with $n=m$, we can obtain

$$\begin{aligned}
\|v^{(m+1)}(s)\|_{H^{2,1}(\Omega\times(0,T))}&\le C_1\|(m+1)\Delta v^{(m)}(s)\|_{L^2(\Omega\times(0,T))}\\
&\le C_1(m+1)\|v^{(m)}(s)\|_{H^{2,1}(\Omega\times(0,T))}\\
&\le \tilde{M}C_1^{m+1}(m+1)!.
\end{aligned}$$

In the same way as (1.28), we obtain

$$\begin{aligned}
&\|v^{(m+1)}(s+\delta)-v^{(m+1)}(s)\|_{H^{2,1}(\Omega\times(0,T))}\\
&\le C_1\|\delta\Delta v^{(m+1)}(s+\delta)+(m+1)\Delta(v^{(m)}(s+\delta)-v^{(m)}(s))\|_{L^2(\Omega\times(0,T))}\\
&\le C_1(|\delta|\|v^{(m+1)}(s+\delta)\|_{H^{2,1}(\Omega\times(0,T))}+(m+1)\|v^{(m)}(s+\delta)-v^{(m)}(s)\|_{H^{2,1}(\Omega\times(0,T))})\\
&\le C_1(|\delta|\tilde{M}C_1^{m+1}(m+1)!+(m+1)\|v^{(m)}(s+\delta)-v^{(m)}(s)\|_{H^{2,1}(\Omega\times(0,T))})\to 0,\ \text{as}\ \ \delta\to 0,
\end{aligned}$$

which implies the continuity of $v^{(m+1)}(s)$ for $s\in\widetilde{I}$.

Thus the proof of Lemma 1.4.2 is completed.

Proof: (Proof of Lemma 1.4.1.) By (1.10) and (1.13), we note

$$K_rf=M^{-1}\int_0^T\rho(t)A^{-1}u_t(r,f)(t)dt.$$

Therefore, for every arbitrary bounded closed interval $J\subset I$, we have

$$\begin{aligned}
&\|(K_rf)^{(n)}\|_{L^2(\Omega)}\\
&\le C_{12}'\|M^{-1}\|\|A^{-1}\|_{\mathcal{L}(L^2(\Omega),H^2(\Omega))}\int_0^T\|\rho(t)\|_{W^{2,\infty}(\Omega)}\|u_t^{(n)}(r,f)(t)\|_{L^2(\Omega)}dt\\
&\le C_{12}\|M^{-1}\|\|\|A^{-1}\|_{\mathcal{L}(L^2(\Omega),H^2(\Omega))}\|\rho\|_{L^2(0,T;W^{2,\infty}(\Omega))}\tilde{M}C_1^n n!,\quad n\in\mathbb{N},\quad r\in J,
\end{aligned}$$

which implies that $K_r f$ is real analytic in $r \in I$.

Second Step

Lemma 1.4.3 *$K_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact for $r \in I$.*

Proof: Let $f_n \to f_0$ weakly in $L^2(\Omega)$. Since $\mathcal{R}(e^{-\frac{tA}{r}}) \subset \mathcal{D}(A)$ for $t > 0$ and the embedding $\mathcal{D}(A) \longrightarrow L^2(\Omega)$ is compact, we see that $e^{-\frac{tA}{r}} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact for $t > 0$. Therefore $e^{-\frac{tA}{r}}(h(0)f_n) \longrightarrow e^{-\frac{tA}{r}}(h(0)f_0)$ in $L^2(\Omega)$. Moreover $A^{-1} : L^2(\Omega) \longrightarrow H^2(\Omega)$ is bounded, so that for $t > 0$ we have

$$\rho(t)A^{-1}e^{-\frac{tA}{r}}(h(0)f_n) \longrightarrow \rho(t)A^{-1}e^{-\frac{tA}{r}}(h(0)f_0) \quad \text{in } H^2(\Omega).$$

On the other hand, by $\rho \in L^2(0,T;W^{2,\infty}(\Omega))$, we have

$$\begin{aligned} &\|\rho(t)A^{-1}e^{-\frac{tA}{r}}(h(0)f_n)\|_{H^2(\Omega)} \le C'_{13}\|\rho(\cdot,t)\|_{W^{2,\infty}(\Omega)} \sup_{0\le t\le T}\|e^{-\frac{tA}{r}}(h(0)f_n)\|_{L^2(\Omega)} \\ \le\ & C_{13}\|\rho(\cdot,t)\|_{W^{2,\infty}(\Omega)} \end{aligned}$$

by $\sup_{n\in\mathbb{N}}\|f_n\|_{L^2(\Omega)} < \infty$. The Lebesgue theorem yields that

$$M^{-1}\int_0^T \rho(t)A^{-1}e^{-\frac{tA}{r}}(h(0)f_n)dt \longrightarrow M^{-1}\int_0^T \rho(t)A^{-1}e^{-\frac{tA}{r}}(h(0)f_0)dt$$

in $L^2(\Omega)$. Similarly we can prove that

$$\begin{aligned} & M^{-1}\int_0^T \rho(t)\left(\int_0^t e^{-\frac{(t-s)A}{r}}A^{-1}(h'(s)f_n)ds\right)dt \\ \longrightarrow\ & M^{-1}\int_0^T \rho(t)\left(\int_0^t e^{-\frac{(t-s)A}{r}}A^{-1}(h'(s)f_0)ds\right)dt \quad \text{in } L^2(\Omega). \end{aligned}$$

Thus the proof of Lemma 1.4.3 is completed.

Third Step

Lemma 1.4.4 *For $f \in L^2(\Omega)$ and $r > 0$, we have*

$$\|u_t(r,f)(\cdot,t)\|_{L^2(\Omega)} \le C_{14}(e^{-\frac{\lambda}{r}t} + \sqrt{r})\|f\|_{L^2(\Omega)}, \qquad t \in (0,T). \tag{1.33}$$

Proof: For simplicity we set $u(r,f)(x,t) = u(t)$. We note that there exist constants $C_{15} > 0$ and $\lambda > 0$ such that

$$\|e^{-tA}a\|_{L^2(\Omega)} \le C_{15}e^{-\lambda t}\|a\|_{L^2(\Omega)}, \quad a \in L^2(\Omega)$$

(e.g., Pazy [38]). Since $u_t \in C([0,T]; L^2(\Omega))$ is given by (1.13), we have

$$\begin{aligned}
\|u_t(t)\|_{L^2(\Omega)} &\le \|e^{-t\frac{A}{r}}fh(0)\|_{L^2(\Omega)} + \int_0^t \|e^{-(t-s)\frac{A}{r}}fh'(s)\|_{L^2(\Omega)}ds \\
&\le C_{15}e^{-\frac{\lambda}{r}t}\|h(0)\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)} + C_{15}\int_0^t e^{-\frac{\lambda(t-s)}{r}}\|h'(s)\|_{L^\infty(\Omega)}ds\|f\|_{L^2(\Omega)} \\
&\le C_{15}e^{-\frac{\lambda}{r}t}\|h(0)\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)} \\
&+ C_{15}\left(\int_0^t e^{-\frac{2\lambda(t-s)}{r}}ds\right)^{\frac{1}{2}}\|h'\|_{L^2(0,T;L^\infty(\Omega))}\|f\|_{L^2(\Omega)} \\
&\le C_{15}\left(e^{-\frac{\lambda}{r}t}\|h(0)\|_{L^\infty(\Omega)} + \left(\frac{r}{2\lambda}\right)^{\frac{1}{2}}\|h'\|_{L^2(0,T;L^\infty(\Omega))}\right)\|f\|_{L^2(\Omega)}
\end{aligned}$$

for $t \in (0,T)$. Thus the proof of Lemma 1.4.4 is completed.

Lemma 1.4.5 *There exists a small $r^* > 0$ such that for $0 < r < r^*$, there exists a constant $0 < \theta(r) < 1$ satisfying*

$$\|K_r f\|_{L^2(\Omega)} \le \theta(r)\|f\|_{L^2(\Omega)}, \qquad f \in L^2(\Omega).$$

Proof: For $f \in L^2(\Omega)$, by (1.10), (1.13) and (1.33), we have

$$\begin{aligned}
\|K_r f\|_{L^2(\Omega)} &\le C'_{16}\|M^{-1}\|\|A^{-1}\|_{\mathcal{L}(L^2(\Omega),H^2(\Omega))}\int_0^T \|\rho(t)\|_{W^{2,\infty}(\Omega)}\|u_t(t)\|_{L^2(\Omega)}dt \\
&\le C_{16}\int_0^T \|\rho(t)\|_{W^{2,\infty}(\Omega)}(e^{-\frac{\lambda}{r}t} + \sqrt{r})dt\|f\|_{L^2(\Omega)}.
\end{aligned}$$

Since

$$\lim_{r\to 0}\|\rho(t)\|_{W^{2,\infty}(\Omega)}(e^{-\frac{\lambda}{r}t} + \sqrt{r}) = 0$$

for $t \ne 0$ and

$$\|\rho(t)\|_{W^{2,\infty}(\Omega)}(e^{-\frac{\lambda}{r}t} + \sqrt{r}) \le C_{17}\|\rho(t)\|_{W^{2,\infty}(\Omega)}, \quad 0 \le t \le T,$$

and $\|\rho(\cdot)\|_{W^{2,\infty}(\Omega)} \in L^1(0,T)$, we see by the Lebesgue theorem that

$$\int_0^T \|\rho(t)\|_{W^{2,\infty}(\Omega)}(e^{-\frac{\lambda}{r}t} + \sqrt{r})dt = o(1)$$

as $r \to 0$. Thus the proof of Lemma 1.4.5 is completed.

Fourth Step

Now we complete the proof of Theorem 1.2.1. By Lemma 1.3.1, it is sufficient to prove that (1.11) is uniquely solvable. By Lemmata 1.4.1 and 1.4.3, we can apply the result on analytic perturbation to the operator $K_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ (e.g., Theorem 1.9 on p.370 in Kato [20]).

Then the following alternative holds.

(i)There exists a finite set $E = E(h, \rho, I) \subset I$ such that $1 \notin \sigma(K_r)$ for all $r \in I \setminus E$.

or

(ii)$1 \in \sigma(K_r)$ for all $r \in I$.

Lemma 1.4.5 implies that 1 can not be an eigenvalue of K_r for small r. Consequently the second alternative (ii) can not occur. We see that E is the set described in Theorem 1.2.1.

Finally we prove (1.7). Let $r \in I \setminus E$. By Lemma 1.4.3, we can apply the Fredholm alternative in $L^2(\Omega)$, and obtain

$$\|f\|_{L^2(\Omega)} \le C'_{18} \left\| \frac{M^{-1}\varphi}{r} \right\|_{L^2(\Omega)} \le C_{18} \|\varphi\|_{H^2(\Omega)}.$$

We apply Theorem 3.5 (ii) (p.114) in [38], and obtain (1.7).

Thus the proof of Theorem 1.2.1 is completed.

Chapter 2

Inverse source problem from instantaneous time distributing overdetermination

2.1 Introduction

Let us consider an initial-boundary value problem for a parabolic equation:

$$u_t(x,t) = \Delta u(x,t) + f(x)h(x,t), \qquad x \in \Omega,\ t \in (0,T), \tag{2.1}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{2.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T), \tag{2.3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain and $\partial\Omega$ is its boundary. Here Δ is the Laplacian, $T > 0$ is a constant and h is a given function on $\overline{\Omega} \times [0,T]$.

We discuss the following inverse problem:

Inverse Problem. Let $\gamma(x)$ be given. Determine $u(x,t) = u(f)(x,t)$ and $f(x)$, $x \in \overline{\Omega}$, $t \in [0,T]$ satisfying (2.1) - (2.3) and

$$u(x,\gamma(x)) = \varphi(x), \qquad x \in \overline{\Omega}. \tag{2.4}$$

Here the data φ is derived from an instantaneous time distributing observation on $\overline{\Omega}$. this observation is compared to the observation in Chapter 1:

$$\int_0^T \rho(x,t)u(x,t)dt = \tilde{\varphi}(x), \qquad x \in \overline{\Omega},$$

where $\rho(x,t)$ is given function. The data $\tilde{\varphi}(x)$ is interpreted as the average data of the solution $u(x,t)$ on each time interval for each point $x \in \overline{\Omega}$, whereas the data $\varphi(x)$ means the instantaneous time data of the solution $u(x,t)$ for each point $x \in \overline{\Omega}$. In the case of $\gamma(x) \equiv T$, the inverse problem has been studied by different author as an inverse problem with a final overdetermination. Concerning these inverse problems, we can refer to Choulli and Yamamoto [5], [6], Hoffmann and Yamamoto [15], Isakov [17], [18], Prilepko, Orlovsky and Vasin [42] and the references therein. In Hömberg and Yamamoto [16], the inverse problem for a time dependent source term from the observation $u(\tilde{\gamma}(t),t)$ has been studied.

In this chapter, we prove that the inverse problem has the following alternative:

(i) For an arbitrary data $u(x,\gamma(x)) = \varphi(x)$, there exist a unique solution $\{u(f), f\}$ to (2.1) - (2.4).

(ii) For the data $u(x,\gamma(x)) \equiv 0$, there exist a nontrivial solution $\{u(f), f\}$ to (2.1) - (2.4).

Our main tools are the a priori Hölder estimates, a semigroup approach and the Fredholm alternative theorem.

The remainder of this chapter is composed of two sections. In Section 2, we state our main result. In Section 3, we complete the proof of the main result.

2.2 Main results

In what follows, $\lambda \in (0,1)$ and $\Omega \subset \mathbb{R}^d$ is assume to be $C^{2+\lambda}$ -smooth. We denote the Hölder space by $C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$ ($C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$, etc.) with $\lambda \in (0,1)$ (e.g., Adams [1], Ladyzenskaja, Solonnikov, Ural'ceva [25]).

Throughout this chapter, we assume that the given functions h and γ satisfy

$$h,\ h_t,\ h_{tt} \in C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]),$$

$$h(x,0) = h_t(x,0) = 0,\ x \in \overline{\Omega}, \quad h_{tt}(x,0) = 0,\ x \in \partial\Omega, \tag{2.5}$$

$$\gamma \in C^3(\overline{\Omega}), \qquad 0 < \gamma(x) \leq T,\ x \in \overline{\Omega}, \tag{2.6}$$

$$|h(x,\gamma(x))| > 0,\ x \in \overline{\Omega}. \tag{2.7}$$

Here and henceforth C_j denotes positive constants which are independent of f in (2.1), but may depend on h and γ.

For $f \in C^\lambda(\overline{\Omega})$, there exists a unique solution $u(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$ to (2.1) - (2.3) satisfying the inequality:

$$\|u(f)\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])} \le C_1 \|f\|_{C^\lambda(\overline{\Omega})} \tag{2.8}$$

(e.g., Ladyzenskaja, Solonnikov, Ural'ceva [25]).

Now we are ready to state our main theorem.

Theorem 2.2.1 *The following alternative holds:*

> *(i) For* $\varphi \in C^{2+\lambda}(\overline{\Omega})$ *with* $\varphi|_{\partial\Omega} = 0$, *there exists a unique solution* $\{u(f), f\} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]) \times C^\lambda(\overline{\Omega})$ *to* (2.1) - (2.4).
>
> *(ii) For* $\varphi \equiv 0$, *there exists a solution* $\{u(f), f\} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times [0,T]) \times C^\lambda(\overline{\Omega})$ *such that* $u \not\equiv 0$ *and* $f \not\equiv 0$ *to* $(2.1) - (2.4)$.

Remark 2.2.2 If the uniqueness holds for the inverse problem (2.1) - (2.4), we can see that (ii) does not hold. Then the above theorem assures us that (i) is true. That is, the uniqueness of the solution $\{u(f), f\} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]) \times C^\lambda(\overline{\Omega})$ implies the existence of the solution to (2.1) - (2.4).

Remark 2.2.3 If the domain Ω is sufficientlly small (e.g., the smallness of diameter of Ω), we can prove that (i) is true. However, we omit the details here.

Remark 2.2.4 By setting a diffusion parameter in the equation (2.1), we can prove the analogous theorem to that of Chapter 1.

2.3 Proof of Theorem 2.2.1

For some fixed $\lambda \in (0,1)$, we choose $p > 1$ such that

$$1 - \lambda > \frac{n}{p}. \tag{2.9}$$

We define an operator $-A$ in $L^p(\Omega)$ by

$$(-Au)(x) = \Delta u(x), \ x \in \Omega, \qquad \mathcal{D}(A) = \{u \in W^{2,p}(\Omega); u|_{\partial\Omega} = 0\},$$

where $W^{2,p}(\Omega)$ is the Sobolev space. Then $-A$ generate an analytic semigroup in $L^p(\Omega)$ and we have the semigroup e^{-tA}.

Lemma 2.3.1 *For $f \in C^{\lambda}(\overline{\Omega})$, the unique solution $u(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])$ to (2.1) - (2.3) has the properties $u_t(f), u_{tt}(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])$ with $u_t|_{\partial\Omega} = 0$ and $u_{tt}|_{\partial\Omega} = 0$ respectively. Moreover, the following inequality holds:*

$$\|u_t(f)\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])} + \|u_{tt}(f)\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])} \leq C_2\|f\|_{C^{\lambda}(\overline{\Omega})}. \tag{2.10}$$

Proof: For simplicity we denote $u(f)(\cdot,t)$ and $h(\cdot,t)$ by $u(t)$ and $h(t)$ respectively. For $f \in C^{\lambda}(\overline{\Omega})$, the unique solution u is represented in the sense of the evolution equations in $L^p(\Omega)$ (e.g., Pazy [38]):

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}(t) &= -Au(t) + fh(t), \qquad t \in (0,T), \\ u(0) &= 0, \end{cases} \tag{2.11}$$

and this solutin $u \in C([0,T]; L^p(\Omega)) \cap C^1((0,T]; L^p(\Omega))$ is given by

$$u(t) = \int_0^t e^{-A(t-\eta)} fh(\eta)\mathrm{d}\eta. \tag{2.12}$$

On the other hand, since $fh_t \in C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])$, there exists a unique solution $v \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])$ to the following equations:

$$v_t(x,t) = \Delta v(x,t) + f(x)h_t(x,t), \qquad x \in \Omega,\ t \in (0,T), \tag{2.13}$$

$$v(x,0) = 0, \qquad x \in \Omega, \tag{2.14}$$

$$v(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T). \tag{2.15}$$

Moreover, by $h_t \in C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])$, the solution v satisfies

$$\|v\|_{C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega}\times[0,T])} \leq C_3\|f\|_{C^{\lambda}(\overline{\Omega})}. \tag{2.16}$$

Since $h(0) = 0$, this solution $v \in C([0,T]; L^p(\Omega)) \cap C^1((0,T]; L^p(\Omega))$ is given by

$$v(t) = \int_0^t e^{-A(t-\eta)} fh_\eta(\eta)\mathrm{d}\eta, \tag{2.17}$$

in the sense of evolution equations:

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= -Av(t) + fh_t(t), \qquad t \in (0,T), \\ v(0) &= 0. \end{cases} \tag{2.18}$$

We will show that $u_t(t) = v(t), t \in (0,T)$. Let us set

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = \tilde{v}(t). \tag{2.19}$$

Then (2.11), (2.12) and (2.19) imply

$$\tilde{v}(t) = -\int_0^t Ae^{-A(t-\eta)} fh(\eta)\mathrm{d}\eta + fh(t). \tag{2.20}$$

For sufficiently small $\varepsilon > 0$, we obtain

$$\begin{aligned} -\int_0^{t-\varepsilon} Ae^{-A(t-\eta)} fh(\eta)\mathrm{d}\eta &= -\int_0^{t-\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\eta}(e^{-A(t-\eta)}) fh(\eta)\mathrm{d}\eta \\ &= -e^{-A\varepsilon} fh(t-\varepsilon) + e^{-At} fh(0) + \int_0^{t-\varepsilon} e^{-A(t-\eta)} fh_\eta(\eta)\mathrm{d}\eta, \end{aligned} \tag{2.21}$$

by integration by parts since $fh, fh_t \in C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$. Since $h(0) = 0$ and $h \in C^{\lambda,\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$, we can make ε tend to 0 and obtain

$$-\int_0^t Ae^{-A(t-\eta)} fh(\eta)\mathrm{d}\eta = -fh(t) + \int_0^t e^{-A(t-\eta)} fh_\eta(\eta)\mathrm{d}\eta. \tag{2.22}$$

Consequently we see that $(u_t(t) =)\tilde{v}(t) = v(t), t \in (0,T)$ by the uniqueness of the solution. Similarly, we can prove that this property is true for the function u_{tt}. Thus the proof of Lemma 2.3.1 is completed.

Lemma 2.3.2

(i) For $\varphi \in C^{2+\lambda}(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$, $\{u(f), f\} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]) \times C^{\lambda}(\overline{\Omega})$ satisfies (2.1) - (2.4) *if and only if there exists a solution $f \in C^{\lambda}(\overline{\Omega})$ to the linear equation:*

$$f = Kf + \Phi, \tag{2.23}$$

where

$$\begin{aligned} (Kf)(x) = \frac{1}{h(x,\gamma(x))}\{&u_t(x,\gamma(x)) + 2\nabla u_t(x,\gamma(x)) \cdot \nabla\gamma(x) \\ &+ u_{tt}(x,\gamma(x))|\nabla\gamma(x)|^2 + u_t(x,\gamma(x))\Delta\gamma(x)\}, \end{aligned} \tag{2.24}$$

$$\Phi(x) = -\frac{\Delta\varphi(x)}{h(x,\gamma(x))}. \tag{2.25}$$

(ii) For $\varphi \in C^{2+\lambda}(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$, there exists a unique solution $\{u(f), f\} \in$

$C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]) \times C^{\lambda}(\overline{\Omega})$ *to* (2.1) - (2.4) *if and only if* (2.23) *possesses a unique solution* $f \in C^{\lambda}(\overline{\Omega})$.

Proof: (i) Let us assume that $\{u(f), f\} \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T]) \times C^{\lambda}(\overline{\Omega})$ satisfies (2.1) - (2.4). By (2.4), we obtain

$$\begin{aligned}\Delta\varphi(x) =&\Delta u(x,\gamma(x)) + 2\nabla u_t(x,\gamma(x)) \cdot \nabla\gamma(x) \\ &+ u_{tt}(x,\gamma(x))|\nabla\gamma(x)|^2 + u_t(x,\gamma(x))\Delta\gamma(x). \end{aligned} \tag{2.26}$$

Therefore the equation (2.1) is reduced to

$$\begin{aligned}u_t(x,\gamma(x)) = &\Delta\varphi(x) - 2\nabla u_t(x,\gamma(x)) \cdot \nabla\gamma(x) \\ &- u_{tt}(x,\gamma(x))|\nabla\gamma(x)|^2 - u_t(x,\gamma(x))\Delta\gamma(x) + f(x)h(x,\gamma(x)). \end{aligned} \tag{2.27}$$

By (2.7), we can divide by $h(x,\gamma(x))$ both sides of (2.27). From this, it follows that $f \in C^{\lambda}(\overline{\Omega})$ is the solution to (2.23).

Next let us assume that there exists a solution $f \in C^{\lambda}(\overline{\Omega})$ to (2.23). We substitute the solution f into (2.1) and obtain the unique solution $u(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$ to (2.1) - (2.3). We have to prove that the solution $u(f)$ satisfies (2.4). We set $u(f)(x,\gamma(x)) = \varphi_1(x), x \in \overline{\Omega}$. By $u(f), u_t(f), u_{tt}(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$, we can see $\varphi_1 \in C^{2+\lambda}(\overline{\Omega})$ with $\varphi_1|_{\partial\Omega} = 0$. The equation (2.27) is valid for $u(f)$ and φ_1 by similar computations. From this and (2.23), we can see that φ_1 and φ satisfy

$$\Delta(\varphi_1 - \varphi)(x) = 0, \quad x \in \Omega, \qquad (\varphi_1 - \varphi)(x) = 0, \quad x \in \partial\Omega.$$

Accordingly, we derive

$$\varphi_1(x) = \varphi(x), \quad x \in \overline{\Omega},$$

which implies $\{u(f), f\}$ is the solution to (2.1) - (2.4).

(ii) We assume that the solution $\{u(f_1), f_1\}$ to (2.1) - (2.4) exists uniquely. Then by the part (i), the function f_1 gives the solution to (2.23). Seeking a contradiction, we assume that f_2 ($\neq f_1$) gives the solution to (2.23). Then there exists a solution $\{u(f_2), f_2\}$ to (2.1) - (2.4). However, this contradicts the assumption. Conversely, let us assume that (2.23) possesses a unique solution f_1. Then by the part (i), there exists a solution $\{u(f_1), f_1\}$ to (2.1) - (2.4). We suppose that both $\{u(f_1), f_1\}$ and $\{u(f_2), f_2\}$ are distinct solution to (2.1) - (2.4). Let $f_1 = f_2$. Then $u(f_1) = u(f_2)$ because of the unique solvability of the direct problem (2.1) - (2.3). If $f_1 \neq f_2$, then f_1 and f_2 are solutions to (2.23) from the part (i), which contradicts the unique solvability of (2.23). Thus the proof of Lemma 2.3.2 is completed.

Lemma 2.3.3 *$K : C^\lambda(\overline{\Omega}) \to C^\lambda(\overline{\Omega})$ is compact.*

Proof: By Lemma 2.3.1, for an arbitrarily fixed $f \in C^\lambda(\overline{\Omega})$, the unique solution $u(f) \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$ to (2.1) - (2.3) has $u_t(f)$ and $u_{tt}(f)$, which belong to $C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0,T])$.

Let us set

$$\begin{aligned}(\tilde{K}f)(x) &= h(x,\gamma(x))(Kf)(x) \\ &= u_t(x,\gamma(x))(1+\Delta\gamma(x)) + 2\nabla u_t(x,\gamma(x)) \cdot \nabla\gamma(x) \\ &\quad + u_{tt}(x,\gamma(x))|\nabla\gamma(x)|^2. \end{aligned} \tag{2.28}$$

Since $\gamma \in C^3(\overline{\Omega})$, both $u_t(\cdot,\gamma(\cdot))$ and $u_{tt}(\cdot,\gamma(\cdot))$ are belong to $C^{1+\frac{\lambda}{2}}(\overline{\Omega})$. Therefore by (2.10), we have

$$\|(\tilde{K}f)_x\|_{C^{\frac{\lambda}{2}}(\overline{\Omega})} \leq C_4 \|f\|_{C^\lambda(\overline{\Omega})}.$$

Namely,

$$\|\tilde{K}f\|_{C^{1+\frac{\lambda}{2}}(\overline{\Omega})} \leq C_5 \|f\|_{C^\lambda(\overline{\Omega})}. \tag{2.29}$$

By Sobolev embedding theorem and the choice of $\lambda' > \lambda$ such that $1 - \lambda' > n/p$ (e.g., Adams [1]), we obtain

$$\|\tilde{K}f\|_{C^{\lambda'}(\overline{\Omega})} \leq C_6 \|\tilde{K}f\|_{W^{1,p}(\Omega)} \leq C_7 \|\tilde{K}f\|_{C^{1+\frac{\lambda}{2}}(\overline{\Omega})} \leq C_8 \|f\|_{C^\lambda(\overline{\Omega})}. \tag{2.30}$$

Since the embedding $C^{\lambda'}(\overline{\Omega}) \longrightarrow C^\lambda(\overline{\Omega})$ is compact for $\lambda' > \lambda$, the operator $\tilde{K}$ is compact from $C^\lambda(\overline{\Omega})$ to $C^\lambda(\overline{\Omega})$. Furthermore by (2.7), the division operator by $h(\cdot,\gamma(\cdot)) \in C^\lambda(\overline{\Omega})$ is bounded from $C^\lambda(\overline{\Omega})$ to $C^\lambda(\overline{\Omega})$. As a result, we see that K is compact. Thus the proof of Lemma 2.3.3 is completed.

Now we complete the proof of Theorem 2.2.1. By Lemma 2.3.3, we have the following alternative:

(i) For an arbitrary $\Phi \in C^\lambda(\overline{\Omega})$, there exists a unique solution $f \in C^\lambda(\overline{\Omega})$ to (2.23).

(ii) There exists a nontrivial solution to the following equation:

$$Kf = f. \tag{2.31}$$

By Lemma 2.3.2, (i) implies that the inverse problem (2.1) - (2.4) has a unique solution $\{u(f), f\}$. In case that (ii) holds, Lemma 2.3.2 implies that for $\Phi(x) \equiv 0, x \in \overline{\Omega}$ (which is equivalent to $\varphi(x) \equiv 0, x \in \overline{\Omega}$), the inverse problem (2.1) - (2.4) has a solution $\{u(f), f\}$ such that $f \not\equiv 0$. We can see that $u \not\equiv 0$ is also valid. In fact, when we assume that $u \equiv 0$, since $Kf \equiv 0$ by (2.23), we obtain $f \equiv 0$ by (2.31). However, this contradict the assumption $f \not\equiv 0$. Thus the proof of Theorem 2.2.1 is completed.

Part II

Inverse Source Problems for Fractional Diffusion Equations

Chapter 3

A fractional diffusion equation and a fractional diffusion-wave equation in a bounded domain and applications to inverse problems

3.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We consider a time fractional equation:

$$^cD_t^\alpha u(x,t) = (Lu)(x,t) + F(x,t), \quad x \in \Omega,\ t \in (0,T),\ 0 < \alpha \leq 2, \tag{3.1}$$

where $^cD_t^\alpha$ denotes the Caputo fractional derivative with respect to t. The Caputo fractional derivative of order α is defined as

$$^cD_t^\alpha g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} g(\tau) d\tau, & n-1 < \alpha < n,\ n \in \mathbb{N}, \\ \dfrac{d^n}{dt^n} g(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

where Γ is the Gamma function. Here the operator L is symmetric, F is a given function on $\Omega \times [0,T]$ and $T > 0$ is a fixed value. Note that if $\alpha = 1$ and $\alpha = 2$, the equation (3.1) represent the parabolic equation and the hyperbolic equation respectively. Since our concern is the fractional cases in this chapter, we restrict the order α to the two cases $0 < \alpha < 1$ and $1 < \alpha < 2$. We will solve the equation (3.1) satisfying the following initial-boundary value conditions:

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in (0,T), \tag{3.2}$$

$$u(x,0) = a(x), \qquad x \in \Omega. \tag{3.3}$$

For $1 < \alpha < 2$, we add in the condition

$$u_t(x,0) = b(x), \qquad x \in \Omega. \tag{3.4}$$

In the case such that $L = \Delta$, the equation (3.1) is called a fractional diffusion equation in the case $0 < \alpha < 1$, while the equation is called a fractional diffusion-wave equation or a fractional wave equation in the case $1 < \alpha < 2$. The fractional diffusion equation has been explicitly introduced in physics by Nigmatullin [36] to describe diffusion in media with fractal geometry. Metzler and Klafter [33] have demonstrated that fractional diffusion equation describes a non-Markovian diffusion process with a memory. Roman and Alemany [43] have investigated a continuous time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically. Ginoa, Cerbelli and Roman [13] have presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. Mainardi [28] has pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media.

As for the mathematical treatments for the equation (3.1), we can refer, for example, Wyss [47], Schneider and Wyss [45], Kochubei [22], [23]. Wyss [47] and Schneider and Wyss [45] have used Mellin transforms and Fox H-functions for an integrodifferential equation which is equivalent to the fractional diffusion equation (3.1). Kochubei [22], [23] has used semigroup theory in Banach spaces. However, these mathematical treatments are made in the unbounded domain. Additionally, these mathematical results seem not easy for application because of their high level of generality. Mainardi [30], [31] has solved a fractional diffusion-wave equation using the method of the Laplace trnsform in a 1-dimensional bounded domain, which seems easy for application. Gejji and Jafari [11] has also solved a nonhomogeneous fractional diffusin-wave equation in a 1-dimensional bounded domain. Fujita [9], [10] has solved an Integrodifferential equation which interpolates the heat equation and the wave equation in an unbounded domain. Agrawal [2] has solved a fractional diffusion equation using finite sine transform technique and presented numerical results in a 1-dimensional bounded domain.

In this chapter, we solve the equation (3.1) using the separation of variables and prove the unique solvability of the solution to (3.1) in a n-dimensional bounded domain. The asymptotic behaviors of the solution are also derived. Moreover, we apply the solution described in the eigenfunction expansion to inverse problems.

As for the other references, we can refer the following. An encyclopedic treatment of fractional calculus can be found in [44]. Additional background, survey, and application of this field in science, engineering, and mathematics can be found in [27], [37], [34], [14], [39]. In [21], a lot of surveys of a variety of applications of fractional differential equations are treated briefly and the reviews of some important applications involving fractional models is presented systematically. This also included a large number of and up-to-date bibliography.

The remainder of this chapter is composed of three sections. In Section 2, we state our main result. In Section 3, we complete the proof of the main result. In Section 4, we apply the results to inverse problems.

3.2 Main results

We denote the Sobolev spaces by $H^l(\Omega)$ with $l > 0$ (e.g., Adams [1]). In what follow, the operator L is symmetric so that it can be written as

$$Lu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} A_{ij}(x)\frac{\partial}{\partial x_j}u\right) + C(x)u,$$

where $A_{ij} = A_{ji}$ $(1 \le i, j \le n)$. Moreover, we assume that the operator L is uniformly elliptic in Ω and that its coefficients have some smoothness; that is, there exists $\nu > 0$ such that

$$\nu \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j,$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^d$, and the coefficients satisfy

$$A_{ij} \in C^1(\overline{\Omega}), \quad C \in C^0(\overline{\Omega}), \quad C(x) \leq 0,$$

for all $x \in \overline{\Omega}$.

Here and henceforth C_j denotes positive constants which are independent of F in (3.1), but may depend on α and the coefficients of the operator L.

We are ready to state our main theorems and corollaries.

Theorem 3.2.1 *Let $0 < \alpha < 1$, $a \in L^2(\Omega)$ and $F \in C^1([0,T]; L^2(\Omega))$. Then there exists a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (3.1) - (3.3), and the Caputo fractional derivative $D_t^\alpha u$ belongs to $C((0,T]; L^2(\Omega))$. Moreover, there exists a constant $C_1 > 0$ such that*

$$\|u\|_{C([0,T];L^2(\Omega))} \leq C_1(\|a\|_{L^2(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))}). \tag{3.5}$$

Theorem 3.2.2 *In addition to the assumptions of Theorem 3.2.1, we assume that $a \in H_0^1(\Omega)$. Then the unique solution u belongs to $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, the Caputo fractional derivative ${}^cD_t^\alpha u$ belongs to $L^2(\Omega \times (0,T))$ and there exists a constant $C_2 > 0$ satisfying the following inequality:*

$$\|u\|_{L^2((0,T);H^2(\Omega))} + \|{}^cD_t^\alpha u\|_{L^2(\Omega\times(0,T))} \leq C_2(\|a\|_{H^1(\Omega)} + \|F\|_{L^2(\Omega\times(0,T))}). \tag{3.6}$$

Moreover, we assume that $a \in H^2(\Omega) \cap H_0^1(\Omega)$, then the unique solution u belongs to $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, the Caputo fractional derivative ${}^cD_t^\alpha u$ belongs to $C([0,T]; L^2(\Omega)) \cap C((0,T]; H_0^1(\Omega))$ and the following inequality holds:

$$\|u\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u\|_{C([0,T];L^2(\Omega))} \leq C_3(\|a\|_{H^2(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))}) \tag{3.7}$$

Theorem 3.2.3 *Let $1 < \alpha < 2$, $a \in H^2(\Omega) \cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$ and $F \in C^1([0,T]; L^2(\Omega))$, then there exists a unique solution $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap$*

$C^1([0,T]:L^2(\Omega))$ *to* (3.1) - (3.4) *and the Caputo fractional derivative* ${}^cD_t^\alpha u$ *belongs to* $C([0,T];L^2(\Omega))$. *Moreover, there exists a constant* $C_4 > 0$ *satisfying the inequality:*

$$\|u\|_{C^1([0,T];L^2(\Omega))} + \|u\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u\|_{C([0,T];L^2(\Omega))}$$
$$\le C_4\{\|a\|_{H^2(\Omega)} + \|b\|_{H^1(\Omega)} + \|F\|_{H^1(0,T;L^2(\Omega))}\}. \tag{3.8}$$

Corollary 3.2.4 *Let* $0 < \alpha < 1$, $a \in L^2(\Omega)$ *and* $F = 0$. *Then for the unique solution* $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ *to* (3.1) - (3.3), *the following asymptotic behavior holds. There exists constants* $C_5 > 0$ *and* $\lambda_1 > 0$ *satisfying the inequality:*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_5}{1+\lambda_1 t^\alpha}\|a\|_{L^2(\Omega)} \qquad \text{for} \quad t \ge 0. \tag{3.9}$$

Corollary 3.2.5 *Under the assumptions of Corollary 3.2.4,*
we have $u \in C^\infty((0,T);L^2(\Omega))$.

Corollary 3.2.6 *Let* $1 < \alpha < 2$, $a \in H^2(\Omega)\cap H_0^1(\Omega)$, $b \in H_0^1(\Omega)$ *and* $F = 0$. *Then for the unique solution* $u \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T]:L^2(\Omega))$ *to* (3.1) - (3.4), *the following asymptotic behaviors hold. There exist constants* $C_5 > 0$ *and* $C_6 > 0$ *satisfying the inequalities:*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_6}{1+\lambda_1 t^\alpha}\{\|a\|_{L^2(\Omega)} + t\|b\|_{L^2(\Omega)}\} \qquad \text{for} \quad t \ge 0, \tag{3.10}$$
$$\|u_t(\cdot,t)\|_{L^2(\Omega)} \le \frac{C_7}{1+\lambda_1 t^\alpha}\{t^{\alpha-1}\|a\|_{H^2(\Omega)} + \|b\|_{L^2(\Omega)}\} \qquad \text{for} \quad t \ge 0. \tag{3.11}$$

Corollary 3.2.7 *Under the assumptions of Corollary 3.2.6,*
we have $u \in C^\infty((0,T);L^2(\Omega))$.

Corollary 3.2.4 show that the solution to (3.1) - (3.3) in the case $0 < \alpha < 1$ is likely to decay slowly compared with the solution of the standard diffusion equaton ($\alpha = 1$). In Section 4, we will see that the slow decay certainly happen. Corollaries 3.2.6 and 3.2.7 indicate that the solution to (3.1) - (3.4) in the case $1 < \alpha < 2$ has also asymptotic decay and smoothing which differ from the standard wave equation ($\alpha = 2$).

3.3 Proof of the main results

The two-parameter function of the Mittag-Leffler type is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}, \tag{3.12}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$. In particular, it follows from the definition that $E_{1,1}(z) = e^z$. In detail, we can refer in Kilbas, Srivastava and Trujillo [21], Podlubny [39] and Mainardi [27].

Lemma 3.3.1 *Let* $0 < \alpha < 2$ *and* $\beta \in \mathbb{R}$ *be arbitrary. We suppose that* μ *is such that* $\pi\alpha/2 < \mu < \min(\pi, \pi\alpha)$. *Then there exists a constant* $C_8 = C_8(\alpha, \beta, \mu) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C_8}{1 + |z|}, \qquad \mu \le |\arg(z)| \le \pi. \tag{3.13}$$

Proof: The proof can be found in Podlubny [39].

Lemma 3.3.2 *Let* $\lambda > 0$. *For* $\alpha > 0$ *and positive integer* $m \in \mathbb{N}$, *we have*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha) \qquad \text{for} \quad t > 0, \tag{3.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(t E_{\alpha,2}(-\lambda t^\alpha)) = E_{\alpha,1}(-\lambda t^\alpha) \qquad \text{for} \quad t \ge 0. \tag{3.15}$$

Proof: $E_{\alpha,\beta}(z)$ is an entire function of z and a polynomial series. Therefore when we restrict the domain in $\mathbb{R}$, $E_{\alpha,\beta}(x)$ is real analytic and termwise differentiable in $\mathbb{R}$. Since t^α is also real analytic in $t > 0$, so is $E_{\alpha,\beta}(-\lambda t^\alpha)$ in $t > 0$. Therefore the equations above obtained by differentiating termwisely are valid.

We formaly solve the time fractional diffusion equation (3.1) - (3.3) for $0 < \alpha < 1$ and the time fractional diffusion-wave equation (3.1) - (3.4) for $1 < \alpha < 2$ respectively.

Since L is symmetric, The spectrum of L is discrete and the each eigenvalue of L is real and has finite multiplicity such that $0 > -\lambda_1 \ge -\lambda_2 \ge \cdots, -\lambda_n \to -\infty$

as $n \to \infty$. We set the sequence $\{-\lambda_n\}$ of eigenvalues of L, counted according to multiplicity and by $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $-\lambda_n$. Then the sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ is orthonormal basis of $L^2(\Omega)$.

Let us multiply both sides of (3.1) by the function $\varphi_n(x)$ and integrate the equation with respect to x. Using Green's formula, Fubini's theorem and boundary condition of $\varphi_n(x)$, we obtain

$$ {}^cD_t^{\alpha} u_n(t) = -\lambda_n u_n(t) + F_n(t), \quad t > 0, \tag{3.16} $$

where $u_n(t) = (u(\cdot,t), \varphi_n)$ and $F_n(t) = (F(\cdot,t), \varphi_n)$. Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. We set $a_n = (a, \varphi_n)$ and $b_n = (b, \varphi_n)$. From the theory of the ordinary fractional differential equation (e.g., Podlubny [39], Kilbas, Srivastava, Trujillo [21]), we have

$$ u_n(x,t) = a_n E_{\alpha,1}(-\lambda_n t^{\alpha}) + \int_0^t F_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \mathrm{d}\tau, \tag{3.17} $$

$$ u_n(t) = \begin{cases} a_n E_{\alpha,1}(-\lambda_n t^{\alpha}) + \int_0^t F_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \mathrm{d}\tau, & 0 < \alpha < 1, \\ a_n E_{\alpha,1}(-\lambda_n t^{\alpha}) + b_n t E_{\alpha,2}(-\lambda_n t^{\alpha})) & \\ + \int_0^t F_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \mathrm{d}\tau, \quad 1 < \alpha < 2. & \end{cases} \tag{3.18} $$

Then the solutions to (3.1) - (3.3) and (3.1) - (3.4) have the form respectively:

$$ u(x,t) = \sum_{n=1}^{\infty} \Big\{ (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^{\alpha}) $$
$$ + \int_0^t (F(\cdot,\tau), \varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \mathrm{d}\tau \Big\} \varphi_n(x), \quad 0 < \alpha < 1, \tag{3.19} $$

$$ u(x,t) = \sum_{n=1}^{\infty} \Big\{ (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^{\alpha}) + (b, \varphi_n) t E_{\alpha,2}(-\lambda_n t^{\alpha})) $$
$$ + \int_0^t (F(\cdot,\tau), \varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \mathrm{d}\tau \Big\} \varphi_n(x), \quad 1 < \alpha < 2. \tag{3.20} $$

Proof: (Proof of Theorem 3.2.1.) First, we will show that the formally derived solution (3.19) certainly give the solution to (3.1) - (3.3).

We introduce a new scalar product into $H_0^1(\Omega)$:

$$[u,v] \equiv \int_\Omega \left\{ \sum_{i,j=1}^{n} A_{ij}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} v - C(x)uv \right\} \mathrm{d}x. \tag{3.21}$$

Then the norm $[u,u]^{\frac{1}{2}}$ is equivalent to $\|u\|_{H^1(\Omega)}$ in $H_0^1(\Omega)$. Likewise, another scalar product into $H^2(\Omega) \cap H_0^1(\Omega)$ is given by

$$\{u,v\} \equiv (Lu, Lv).$$

Then the norm $\{u,u\}^{\frac{1}{2}}$ is equivalent to $\|u\|_{H^2(\Omega)}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ (e.g., Ladyzhenskaya [24]). Note that the eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ are mutually orthogonal in the following ways:

$$[\varphi_k, \varphi_l] = \lambda_k \delta_{lk}, \qquad \{\varphi_k, \varphi_l\} = \lambda_k^2 \delta_{lk},$$

where δ_{lk} is Kronecker's delta. Since

$$\begin{aligned} &\int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \\ &= \int_0^t (F(\cdot,\tau),\varphi_n) \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\lambda_n} E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha) \right) \mathrm{d}\tau \\ &= -\frac{1}{\lambda_n} \Big\{ (F(\cdot,0),\varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) - (F(\cdot,t),\varphi_n) \\ &+ \int_0^t (F_\tau(\cdot,\tau),\varphi_n) E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \Big\}, \end{aligned} \tag{3.22}$$

by Lemma 3.3.1, we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \Big\{ (a,\varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \\ &+ \int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \Big\}^2 \\ &\le 4 \sum_{n=1}^{\infty} \Big\{ C_8^2 (a,\varphi_n)^2 + \frac{1}{\lambda_1^2} \Big\{ C_8^2 (F(\cdot,0),\varphi_n)^2 + (F(\cdot,t),\varphi_n)^2 \\ &+ \left(\int_0^t (F_\tau(\cdot,\tau),\varphi_n) E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \right)^2 \Big\} \Big\} \\ &\le C_9 \{ \|a\|_{L^2(\Omega)}^2 + \|F\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|F_t\|_{L^2(\Omega\times(0,T))}^2 \}. \end{aligned}$$

Therefore the inequality (3.5) is valid. Moreover, we have

$$\{u(\cdot,t),u(\cdot,t)\} \le C_{10}\{\|a\|^2_{L^2(\Omega)}t^{-2\alpha} + \|F\|^2_{L^\infty(0,T;L^2(\Omega))} + \|F_t\|^2_{L^2(\Omega\times(0,T))}\} \quad \text{for} \quad t>0. \tag{3.23}$$

The right side of (3.23) is uniform convergence in the wider sense with respect to $t>0$. Therefore we get $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H^1_0(\Omega))$.

Next, we will show that $D_t^\alpha u \in C((0,T] : L^2(\Omega))$. By $D_t^\alpha E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)$ and (3.22), we have

$$\begin{aligned} &D_t^\alpha\Big(\int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big) \\ &= (F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) + \int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau. \end{aligned} \tag{3.24}$$

Therefore we get

$$\begin{aligned} \|{}^cD_t^\alpha u(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^\infty\Big\{(a,\varphi_n)(-\lambda_n)E_{\alpha,1}(-\lambda_n t^\alpha) + (F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) \\ &\quad + \int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \\ &\le C_{11}\{\|a\|^2_{L^2(\Omega)}t^{-2\alpha} + \|F(\cdot,0)\|^2_{L^2(\Omega)} + \|F_t\|^2_{L^2(\Omega\times(0,T))}\}, \quad \text{for} \quad t>0, \end{aligned} \tag{3.25}$$

which implies $D_t^\alpha u \in C((0,T];L^2(\Omega))$.

Next, we prove the uniqueness of the solution to (3.1) - (3.3). Under the conditions $a=0$ and $F=0$, we prove the system (3.1) - (3.3) has a trivial solution only. Since $\varphi_n(x)$ is the eigenfunctions to the following eigenvalue problem:

$$(L\varphi_n)(x) = -\lambda_n\varphi_n(x), \quad x\in\Omega, \qquad \varphi_n(x)=0, \quad x\in\partial\Omega, \tag{3.26}$$

multiplying both sides of (3.1) by $\varphi_n(x)$, we obtain

$$D_t^\alpha u_n(t) = -\lambda_n u_n(t), \qquad t>0. \tag{3.27}$$

By (3.2), we have $u_n(0)=0$. Due to the existence and uniqueness of the ordinary fractional differential equation (Kilbas, Srivastava, Trujillo [21]), we get $u_n(t) = 0$, $n=1,2,3\cdots$. Since $\{\varphi_n(x)\}$ are complete orthonormal system of $L^2(\Omega)$, the solution $u(x,t)$ is given by $u(x,t)=\sum_{n=1}^\infty u_n(t)\varphi_n(x)$. So we have $u=0$.

Thus the proof of Theorem 3.2.1 is completed.

Proof: (Proof of Theorem 3.2.2.) Since $\int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\mathrm{d}t = \frac{1}{\lambda_n}(1-E_{\alpha,1}(-\lambda_n T^\alpha))$, by Lemma 3.3.1 and the Young's inequality for convolution, we have

$$\begin{aligned}
\int_0^T \|u(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t &= \int_0^T \sum_{n=1}^\infty \Big\{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) \\
&\quad + \int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \mathrm{d}t \\
&\le 2\sum_{n=1}^\infty \int_0^T (a,\varphi_n)^2 E_{\alpha,1}(-\lambda_n t^\alpha)^2 \mathrm{d}t \\
&\quad + 2\sum_{n=1}^\infty \Big(\int_0^T (F(\cdot,t),\varphi_n)^2\mathrm{d}t\Big)\Big(\int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\mathrm{d}t\Big)^2 \\
&\le 2C_8^2 T\|a\|^2_{L^2(\Omega)} + \frac{2(1+C_8)^2}{\lambda_1^2}\|F\|^2_{L^2(\Omega\times(0,T))},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \{u(\cdot,t),u(\cdot,t)\}\mathrm{d}t &= \int_0^T \sum_{n=1}^\infty \Big\{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) \\
&\quad + \int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \lambda_n^2 \mathrm{d}t \\
&\le \frac{2C_8^2 T^{1-\alpha}}{1-\alpha}\sum_{n=1}^\infty (a,\varphi_n)^2\lambda_n + 2(1+C_8)^2\|F\|^2_{L^2(\Omega\times(0,T))} \\
&\le C_{12}(\|a\|^2_{H^1(\Omega)} + \|F\|^2_{L^2(\Omega\times(0,T))}).
\end{aligned}$$

Therefore we have $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$.

Next we will show that ${}^cD_t^\alpha u \in L^2(\Omega\times(0,T))$. Since ${}^cD_t^\alpha E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)$ and

$$\begin{aligned}
&{}^cD_t^\alpha\Big(\int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big) \\
&= -\lambda_n \int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau + (F(\cdot,t),\varphi_n),
\end{aligned}$$

we have

$$\int_0^T \|{}^cD_t^\alpha u(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}t \le 3\Big\{\frac{C_8^2T^{1-\alpha}}{1-\alpha}\sum_{n=1}^\infty (a,\varphi_n)^2\lambda_n$$
$$+\sum_{n=1}^\infty (1-E_{\alpha,1}(-\lambda_nT^\alpha))^2\int_0^T (F(\cdot,t),\varphi_n)^2\mathrm{d}t + \|F\|^2_{L^2(\Omega\times(0,T))}\Big\}$$
$$\le C_{13}(\|a\|^2_{H^1(\Omega)}+\|F\|^2_{L^2(\Omega\times(0,T))}).$$

Therefore we have ${}^cD_t^\alpha u \in L^2(\Omega\times(0,T))$ and the inequality (3.6) is valid.

In the additional case such that $a\in H^2(\Omega)\cap H_0^1(\Omega)$, By (3.22), we have

$$\{u(\cdot,t),u(\cdot,t)\} \le 4\sum_{n=1}^\infty\Big\{(a,\varphi_n)^2E_{\alpha,1}(-\lambda_nt^\alpha)^2+\frac{1}{\lambda_n^2}\Big\{(F(\cdot,t),\varphi_n)^2$$
$$+(F(\cdot,0),\varphi_n)^2E_{\alpha,1}(-\lambda_nt^\alpha)^2 + C_8^2T\int_0^T (F_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\Big\}\Big\}\lambda_n^2$$
$$\le C_{14}\{\|a\|^2_{H^2(\Omega)}+\|F\|^2_{L^\infty(0,T;L^2(\Omega))}+\|F_t\|^2_{L^2(\Omega\times(0,T))}\},\quad \text{for}\quad t\ge 0.$$

Moreover, by (3.24), we have

$$\|{}^cD_t^\alpha u(\cdot,t)\|^2_{L^2(\Omega)} \le \sum_{n=1}^\infty\Big\{(a,\varphi_n)(-\lambda_n)E_{\alpha,1}(-\lambda_nt^\alpha)+(F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_nt^\alpha)$$
$$+\int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2$$
$$\le C_{15}\{\|a\|^2_{H^2(\Omega)}+\|F\|^2_{L^\infty(0,T;L^2(\Omega))}+\|F_t\|^2_{L^2(\Omega\times(0,T))}\},\quad \text{for}\quad t\ge 0. \tag{3.28}$$

Therefore we get (3.7). Furthermore, we have

$$[{}^cD_t^\alpha u(\cdot,t),{}^cD_t^\alpha u(\cdot,t)] \le \sum_{n=1}^\infty\Big\{(a,\varphi_n)(-\lambda_n)E_{\alpha,1}(-\lambda_nt^\alpha)$$
$$+(F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_nt^\alpha)+\int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2\lambda_n$$
$$\le C_{16}(\|a\|^2_{H^1(\Omega)}t^{-2\alpha}+\|F\|^2_{L^\infty(0,T;L^2(\Omega))}t^{-\alpha}+\|F_t\|^2_{L^2(\Omega\times(0,T))}),\quad \text{for}\quad t>0. \tag{3.29}$$

Therefore we have $u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ and ${}^cD_t^\alpha u\in C([0,T];L^2(\Omega))\cap C((0,T];H_0^1(\Omega))$.

Proof: (Proof of Theorem 3.2.3.) Similar to the proof of Theorem 3.2.1, we have

$$\begin{aligned}
\|u(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\Big\{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) + (b,\varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha) \\
&\quad + \int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \\
&\le 3\Big\{\sum_{n=1}^{\infty}(a,\varphi_n)^2\Big(\frac{C_8}{1+\lambda_n t^\alpha}\Big)^2 + \sum_{n=1}^{\infty}(b,\varphi_n)^2\Big(\frac{C_8 t}{1+\lambda_n t^\alpha}\Big)^2 \\
&\quad + \frac{C_8^2 T^{2\alpha-1}}{\lambda_1(2\alpha-1)}\sum_{n=1}^{\infty}\int_0^t (F(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\Big\} \\
&\le C_{17}\{\|a\|^2_{L^2(\Omega)} + \|b\|^2_{L^2(\Omega)} + \|F\|^2_{L^2(\Omega\times(0,T))}\}. \qquad (3.30)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (F(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \\
&\quad = \int_0^t (F(\cdot,\tau),\varphi_n)\frac{\mathrm{d}}{\mathrm{d}t}\Big((t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\Big)\mathrm{d}\tau \\
&\quad = (F(\cdot,0),\varphi_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha) \\
&\quad + \int_0^t (F_\tau(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau.
\end{aligned}$$

By Lemma 3.3.2, we have

$$\begin{aligned}
\|u_t(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\Big\{(a,\varphi_n)(-\lambda_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha) + (b,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha) \\
&\quad + (F(\cdot,0),\varphi_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha) \\
&\quad + \int_0^t (F_\tau(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \\
&\le 4\sum_{n=1}^{\infty}\Big\{(a,\varphi_n)^2\lambda_n^2 t^{2(\alpha-1)}\Big(\frac{C_8}{1+\lambda_n t^\alpha}\Big)^2 + (b,\varphi_n)^2\Big(\frac{C_8}{1+\lambda_n t^\alpha}\Big)^2 \\
&\quad + C_8^2 T^{2(\alpha-1)}(F(\cdot,0),\varphi_n)^2 + \int_0^t (F_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\int_0^t C_8^2(t-\tau)^{2(\alpha-1)}\mathrm{d}\tau\Big\} \\
&\le 4\Big\{C_8^2 T^{2(\alpha-1)}\|a\|^2_{H^2(\Omega)} + C_8^2\|b\|^2_{L^2(\Omega)} \\
&\quad + C_8^2 T^{2(\alpha-1)}\|F(\cdot,0)\|_{L^2(\Omega)} + \frac{C_8^2 T^{2\alpha-1}}{2\alpha-1}\|F_t\|^2_{L^2(\Omega\times(0,T))}\Big\}. \qquad (3.31)
\end{aligned}$$

Therefore we obtain $u \in C^1([0,T];L^2(\Omega))$.

Furthermore, by (3.22), we have

$$\begin{aligned}\{u(\cdot,t),u(\cdot,t)\} &= \sum_{n=1}^{\infty}\Big\{(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)+(b,\varphi_n)tE_{\alpha,2}(-\lambda_n t^\alpha)\\ &- \frac{1}{\lambda_n}\Big((F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)-(F(\cdot,t),\varphi_n)\\ &+\int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big)\Big\}^2\lambda_n^2\\ &\le C_8\{\|a\|^2_{H^2(\Omega)}+T^{2-\alpha}\|b\|^2_{H^1(\Omega)}+\|F(\cdot,0)\|^2_{L^2(\Omega)}\\ &+\|F(\cdot,t)\|^2_{L^2(\Omega)}+T\|F_t\|^2_{L^2(\Omega\times(0,T))}\}\\ &\le C_{18}\{\|a\|^2_{H^2(\Omega)}+\|b\|^2_{H^1(\Omega)}+\|F\|^2_{H^1(0,T;L^2(\Omega))}\}. \end{aligned} \tag{3.32}$$

Therefore we have $u \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$.

Likewise, by (3.24), we have

$$\begin{aligned}\|D_t^\alpha u\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\Big\{(a,\varphi_n)(-\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha))+(b,\varphi_n)(-\lambda_n tE_{\alpha,2}(-\lambda_n t^\alpha))\\ &+(F(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)+\int_0^t (F_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2\\ &\le 4C_8\{\|a\|^2_{H^2(\Omega)}+T^{2-\alpha}\|b\|^2_{H^1(\Omega)}+\|F(\cdot,0)\|^2_{L^2(\Omega)}+T\|F_t\|^2_{L^2(\Omega\times(0,T))}\}. \end{aligned} \tag{3.33}$$

Therefore we obtain $D_t^\alpha u \in C([0,T];L^2(\Omega))$ and the inequality (3.8) is satisfied. The proof of the uniqueness is similar to that of Theorem 3.2.1.

Proof: (Proof of Corollary 3.2.4.) By Lemma 3.3.1, we have

$$\begin{aligned}\|u(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}(a,\varphi_n)^2E_{\alpha,1}(-\lambda_n t^\alpha)^2\\ &\le \sum_{n=1}^{\infty}(a,\varphi_n)^2\Big(\frac{C_8}{1+\lambda_n t^\alpha}\Big)^2\\ &\le \Big(\frac{C_8}{1+\lambda_1 t^\alpha}\|a\|_{L^2(\Omega)}\Big)^2, \quad \text{for} \quad t\ge 0. \end{aligned} \tag{3.34}$$

Proof: (Proof of Corollary 3.2.5.) Since the Mittag-Leffler function $E_{\alpha,1}(-\lambda t^\alpha)$ is infinitely differentiable with respect to $t > 0$ and given by Lemma 3.3.2., we can estimate with an arbitrary positive integer $m \in \mathbb{N}$,

$$\|\frac{\partial^m}{\partial t^m}u(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}(a,\varphi_n)^2\left(\frac{\mathrm{d}^m}{\mathrm{d}t^m}E_{\alpha,1}(-\lambda_n t^\alpha)\right)^2$$
$$\le C_8^2 t^{-2m}\|a\|^2_{L^2(\Omega)},$$

which implies $u \in C^\infty((0,T);L^2(\Omega))$.

Proof: (Proof of Corollary 3.2.6.)

By the estimations of (3.30) and (3.31), we can see (3.10) and (3.11).

Proof: (Proof of Corollary 3.2.7.)

By Lemma 3.3.2, for $m \ge 2$, we have

$$\|\frac{\partial^m}{\partial t^m}u(\cdot,t)\|_{L^2(\Omega)} = \sum_{n=1}^{\infty}\{-\lambda_n(a,\varphi_n)t^{\alpha-m}E_{\alpha,\alpha-m+1}(-\lambda_n t^\alpha)$$
$$- \lambda_n(b,\varphi_n)t^{\alpha-(m-1)}E_{\alpha,\alpha-(m-1)+1}(-\lambda_n t^\alpha)\}^2$$
$$\le C_8^2\{\|a\|^2_{L^2(\Omega)}t^{-2m} + \|b\|^2_{L^2(\Omega)}t^{-2(m-1)}\},$$

which implies $u \in C^\infty((0,T);L^2(\Omega))$.

3.4 Applications of the eigenfunction expansion

We apply the eigenfunction expansion of the solution only in the case of $0 < \alpha < 1$. The arguments in the case of $1 < \alpha < 2$ are similar. Let L be the same ellitpic operator defined in Section 2.

3.4.1 Backward problem in time

Theorem 3.4.1 *Let $T > 0$ be arbitrarily fixed. For any given $a_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega) \cap H_0^1(\Omega))$ such that $u(\cdot,T) = a_1$ to (3.1) with $F = 0$ and (3.2). Moreover there exist constants $C_{19}, C_{20} > 0$ such that*

$$C_{19}\|u(\cdot,0)\|_{L^2(\Omega)} \le \|u(\cdot,T)\|_{H^2(\Omega)} \le C_{20}\|u(\cdot,0)\|_{L^2(\Omega)}.$$

Proof: The proof is seen from (3.9) and the asymptotic formula of the Mittag-Leffler function (e.g., Theorem 1.4 (pp. 33-34) in [39]).

3.4.2 Uniqueness of solution to a boundary-value problem

We note that $-L$ defines the fractional power $(-L)^\beta$ with $\beta \in \mathbb{R}$ and

$$\|u\|_{H^{2\beta}(\Omega)} \le C'_{21}\|(-L)^\beta u\|_{L^2(\Omega)}$$

(e.g., Section 15 of Chapter 1 in [46]).

Theorem 3.4.2 *Let $a \in \mathcal{D}((-L)^{2\beta})$ with $\beta > \frac{d}{4}$. Let $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ satisfy (3.1) with $F = 0$ and (3.2). Let $\omega \subset \Omega$ be an arbitraily chosen subdomain and let $T > 0$. Then $u(x,t) = 0$, $x \in \omega$, $0 < t < T$, implies $u = 0$ in $\Omega \times (0,T)$.*

Remark 3.4.3 We do not know if the uniqueness holds without (3.2).

Proof: By $\lambda_n = O\left(n^{\frac{2}{d}}\right)$ and $a \in \mathcal{D}((-L)^{2\beta})$ and the Sobolev embedding theorem, we have

$$\|\varphi_n\|_{L^\infty(\Omega)} \le C''_{21}\|\varphi_n\|_{H^{2\beta}(\Omega)} \le C'''_{21}\|(-L)^\beta\varphi_n\|_{L^2(\Omega)} \le C_{21}|\lambda_n|^\beta$$

and

$$\sum_{n=1}^\infty |(a,\varphi_n)|\|\varphi_n\|_{L^\infty(\Omega)} \le C_{21}\sum_{n=1}^\infty |(a,\varphi_n)||\lambda_n|^\beta = C_{21}\sum_{n=1}^\infty |(a,\varphi_n)||\lambda_n|^{2\beta}|\lambda_n|^{-\beta}$$
$$\le C_{21}\left(\sum_{n=1}^\infty |(a,\varphi_n)|^2|\lambda_n|^{4\beta}\right)^{\frac{1}{2}}\left(\sum_{n=1}^\infty \frac{1}{\lambda_n^{2\beta}}\right)^{\frac{1}{2}}$$
$$\le C_{22}\left(\sum_{n=1}^\infty \frac{1}{n^{\frac{4\beta}{d}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^\infty |(a,\varphi_n)|^2(|\lambda_n|^{2\beta})^2\right)^{\frac{1}{2}} < \infty. \tag{3.35}$$

Then, by Lemma 3.3.1, $\sum_{n=1}^\infty (a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x)$ can be analytically in t to $\{z \in \mathbb{C};\ z \ne 0,\ |\arg z| \le \mu_0\}$ with some $\mu_0 > 0$. Therefore, since

$$u(x,t) = \sum_{n=1}^\infty (a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) = 0, \quad x \in \omega,\ 0 < t < T,$$

we have

$$\sum_{n=1}^\infty (a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x) = 0, \quad x \in \omega,\ t > 0. \tag{3.36}$$

We set $\sigma(L) = \{-\mu_k\}_{k\in\mathbb{N}}$ and by $\{\varphi_{kj}\}_{1\le j\le m_k}$ we denote an orthonormal basis of Ker $(-\mu_k - L)$. Therefore we can rewrite (3.36) by

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x)\right)E_{\alpha,1}(-\mu_k t^\alpha) = 0, \quad x\in\omega,\ t>0. \tag{3.37}$$

By (3.35) and Lemma 3.3.1, we have

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}|(a,\varphi_{kj})\varphi_{kj}(x)||E_{\alpha,1}(-\mu_k t^\alpha)| \le C_{23}\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}|(a,\varphi_{kj})|\|\varphi_{kj}\|_{L^\infty(\Omega)} < \infty.$$

Hence the Lebesgue convergence theorem yields that

$$\int_0^\infty e^{-zt}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x)E_{\alpha,1}(-\mu_k t^\alpha)\right)dt$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\int_0^\infty e^{-zt}E_{\alpha,1}(-\mu_k t^\alpha)dt\varphi_{kj}(x), \qquad x\in\omega,\ \mathrm{Re}\ z>0. \tag{3.38}$$

We take the Laplace transform to have

$$\int_0^\infty e^{-zt}E_{\alpha,1}(-\mu_k t^\alpha)dt = \frac{z^{\alpha-1}}{z^\alpha+\mu_k}, \quad \mathrm{Re}\ z>0. \tag{3.39}$$

In fact, we can take the Laplace transforms termwise in (3.12) to obtain

$$\int_0^\infty e^{-zt}E_{\alpha,1}(-\mu_k t^\alpha)dt = \frac{z^{\alpha-1}}{z^\alpha+\mu_k}, \quad \mathrm{Re}\ z>\mu_k^{\frac{1}{\alpha}}$$

(cf. formula (1.80) on p.21 in [39]). Since $\sup_{t\ge0}|E_{\alpha,1}(-\mu_k t^\alpha)|<\infty$ by Lemma 3.3.1, we see that $\int_0^\infty e^{-zt}E_{\alpha,1}(-\mu_k t^\alpha)dt$ is analytic with respect to z in Re $z>0$. Therefore the analytic continuation yields (3.39) for Re $z>0$.

Hence (3.38) and (3.39) yield

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\frac{z^{\alpha-1}}{z^\alpha+\mu_k}\varphi_{kj}(x) = 0, \quad x\in\omega,\ \mathrm{Re}\ z>0,$$

that is,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a,\varphi_{kj})\frac{1}{\eta+\mu_k}\varphi_{kj}(x) = 0, \quad x\in\omega,\ \mathrm{Re}\ \eta>0. \tag{3.40}$$

By (3.35), we can analytically continue the both sides of (3.40) in η, so that (3.40) holds for $\eta \in \mathbb{C} \setminus \{-\mu_k\}_{k\in\mathbb{N}}$. We can take a suitable disk which includes $-\mu_\ell$ and does not include $\{-\mu_k\}_{k\neq\ell}$. Integrating (3.40) in a disk, we have

$$u_\ell(x) \equiv \sum_{j=1}^{m_\ell}(a, \varphi_{\ell j})\varphi_{\ell j}(x) = 0, \quad x \in \omega.$$

Since $(L + \mu_\ell)u_\ell = 0$ in Ω, and $u_\ell = 0$ in ω, the unique continuation (e.g., Isakov [18]) implies $u_\ell = 0$ in Ω for each $\ell \in \mathbb{N}$. Since $\{\varphi_{\ell j}\}_{1\le j\le m_\ell}$ is linearly independent in Ω, we see that $(a, \varphi_{\ell j}) = 0$ for $1 \le j \le m_\ell$, $\ell \in \mathbb{N}$. Thus $u = 0$ in $\Omega \times (0, T)$. Thus the proof of Theorem 3.4.2 is completed.

3.4.3 Decay rate at $t = \infty$

We state a different version of Corollary 3.2.4. In fact, the following theorem asserts that the solution can not decay faster than $\frac{1}{t^m}$ with any $m \in \mathbb{N}$ if the solution does not vanish identically. It is a remarkable property of the fractional diffusion equation because the classical diffusion equation with $\alpha = 1$ admits non-zero solutions decaying exponentially. This is one description of the slower diffusion than the classical one.

Theorem 3.4.4 *Let $a \in \mathcal{D}((-L)^{2\beta})$ with $\beta > \frac{d}{4}$ and let $\omega \subset \Omega$ be an arbitrary subdomain. Let $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ satisfy (3.1) with $F = 0$ and (3.2). We assume that for any $m \in \mathbb{N}$, there exists a constant $C(m) > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\omega)} \le \frac{C(m)}{t^m} \quad as\ t \to \infty. \tag{3.41}$$

Then $u = 0$ in $\Omega \times (0, \infty)$.

Proof: By (3.35), we have

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}(a, \varphi_{kj})E_{\alpha,1}(-\mu_k t^\alpha)\varphi_{kj}(x)$$

converges uniformly for $x \in \overline{\Omega}$ and $\delta \le t \le T$ with any $\delta, T > 0$. By Theorem 1.4 (pp. 33-34) in [39], for any $p \in \mathbb{N}$, we have

$$u(x,t) = -\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\sum_{\ell=1}^{p} \frac{(-1)^\ell}{\Gamma(1-\alpha\ell)\mu_k^\ell t^{\alpha\ell}}(a,\varphi_{kj})\varphi_{kj}(x)$$
$$+\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} O\left(\frac{1}{\mu_k^{p+1}t^{\alpha(p+1)}}\right)(a,\varphi_{kj})\varphi_{kj}(x) \quad \text{as } t \to \infty.$$

Setting $m = 1$ in (3.41) and $p = 1$, multiplying t^α and letting $t \to \infty$, we have

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} \frac{1}{\Gamma(1-\alpha)\mu_k}(a,\varphi_{kj})\varphi_{kj}(x) = 0, \quad x \in \omega.$$

Setting $p = 2, 3, ...$, applying (3.41) and repeating the above argument, we can find $\ell_i \in \mathbb{N}$ satisfying $\lim_{i\to\infty} \ell_i = \infty$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k^{\ell_i}}\left(\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x)\right) = 0, \quad x \in \omega,\ \ell_i \in \mathbb{N}.$$

Hence

$$\sum_{j=1}^{m_1}(a,\varphi_{1j})\varphi_{1j}(x) + \sum_{k=2}^{\infty}\left(\frac{\mu_1}{\mu_k}\right)^{\ell_i}\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x) = 0, \quad x \in \omega,\ \ell_i \in \mathbb{N}.$$

By (3.35) and $0 < \mu_1 < \mu_2 < ....$, we have

$$\left\|\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}\left(\frac{\mu_1}{\mu_k}\right)^{\ell_i}(a,\varphi_{kj})\varphi_{kj}(x)\right\|_{L^\infty(\Omega)} \le \sum_{k=2}^{\infty}\sum_{j=1}^{m_k}\left|\frac{\mu_1}{\mu_k}\right|^{\ell_i}|(a,\varphi_{kj})|\|\varphi_{kj}\|_{L^\infty(\Omega)}$$
$$\le \left|\frac{\mu_1}{\mu_2}\right|^{\ell_i}\sum_{k=2}^{\infty}\sum_{j=1}^{m_k}|(a,\varphi_{kj})|\|\varphi_{kj}\|_{L^\infty(\Omega)} \le C_{23}\left|\frac{\mu_1}{\mu_2}\right|^{\ell_i}.$$

Letting $\ell_i \to \infty$ and $\left|\frac{\mu_1}{\mu_2}\right| < 1$, we see that

$$\sum_{j=1}^{m_1}(a,\varphi_{1j})\varphi_{1j}(x) = 0, \quad x \in \omega.$$

Similarly we obtain

$$\sum_{j=1}^{m_k}(a,\varphi_{kj})\varphi_{kj}(x) = 0, \quad x \in \omega,\ k \in \mathbb{N}.$$

Similarly to the end of the proof of Theorem 3.4.2, we can conclude that $u = 0$ in $\Omega \times (0,\infty)$. Thus the proof of Theorem 3.4.4 is completed.

Chapter 4

Inverse source problem with a final overdetermination for a fractional diffusion equation

4.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We consider an initial-boundary problem for a time fractional parabolic equation:

$$^cD_t^\alpha u(x,t) = r^\alpha (Lu)(x,t) + f(x)h(x,t), \quad x \in \Omega,\ t \in (0,T),\ 0 < \alpha \leq 1, \tag{4.1}$$

$$u(x,0) = 0, \qquad x \in \Omega \tag{4.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in (0,T). \tag{4.3}$$

where $^cD_t^\alpha$ denotes the Caputo fractional derivative with respect to t. The operator L is symmetric, $r > 0$ is the parameter, h is a given function on $\overline{Q_T} \equiv \overline{\Omega} \times [0,T]$ and $T > 0$ is a fixed value. Note that if $\alpha = 1$, the equation (4.1) represents the parabolic equation. Since our concern is the fractional cases in this chapter, we restrict the order α to $0 < \alpha < 1$.

We discuss the following inverse problem:

Inverse Problem. Let $r > 0$ be fixed. Determine $u(x,t) = u(r,f)(x,t)$ and $f(x)$; $x \in \Omega$, $t \in (0,T)$ satisfying (4.1) - (4.3) and

$$u(x,T) = \varphi(x), \qquad x \in \overline{\Omega}. \tag{4.4}$$

For $\alpha = 1$ and $r = 1$, similar kinds of inverse problem has been studied by different author. As for inverse problems with final overdetermining observations, see Choulli and Yamamoto [4], [5], [6], Hoffmann and Yamamoto [15], Isakov [17], [18], Prilepko, Orlovsky and Vasin [42] and the references therein.

We prove that the inverse problem is well-posed in the sense of Hadamard except for a finite set of $r > 0$. This idea of parametalizing is based on Choulli and Yamamoto [5], [6] and Hoffmann and Yamamoto [15], which treat inverse problems of determination of a coefficient of lower-order term.

Our main tools are the unique solvability of the fractonal diffusion equation, the theory of analytic perturbation of linear operators and uniqueness in the inverse problem (4.1) - (4.4) for large $r > 0$.

The reminder of this paper is composed of two sections. In Section 2, we state our main result. In Section 3, we give a proof of our main result.

4.2 Main results

We denote the Sobolev spaces by $H^l(\Omega)$ with $l > 0$ (e.g., Adams [1]). In what follow, the operator L is the same elliptic one defined in Section 2 of Chapter 3. Here and henceforth C_j denotes positive constants which are independent of f in (4.1), but may depend on α, h and the coefficients of the operator L.

Throughtout this paper, we assume:

(*i*) $h \in C^1([0,T]; L^\infty(\Omega))$.

(*ii*) There exists a constant $\delta > 0$ satisfying

$$|h(x,T)| \geq \delta > 0, \quad \text{for} \quad x \in \overline{\Omega}. \tag{4.5}$$

We set an arbitrarily fixed open interval $I \subset (0,\infty)$. Then for arbitrarily fixed $r \in I$ and $f \in L^2(\Omega)$, by Theorems 3.2.2 in Chapter 3, there exists a unique solution $u = u(r,f) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ which satisfies ${}^cD_t^\alpha u \in C([0,T]; L^2(\Omega) \cap C((0,T]; H_0^1(\Omega))$ to (4.1)-(4.3) and by (3.19) in Chapter 3, this solution is given by

$$u(r,f)(x,t) = \sum_{n=1}^{\infty}\left\{\int_0^t (f(\cdot)h(\cdot,\tau),\varphi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n r^\alpha (t-\tau)^\alpha)\mathrm{d}\tau\right\}\varphi_n(x). \tag{4.6}$$

Here the sequence $\{-\lambda_n\}$ are eigenvalues of L and by $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $-\lambda_n$. $\{\varphi_n\}_{n=1}^{\infty}$ is orthonormal basis of $L^2(\Omega)$. Moreover, there exists a constant $C_1 > 0$ such that

$$\|u(r,f)\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u(r,f)\|_{C([0,T];L^2(\Omega))} \le C_1\|fh\|_{H^1(0,T;L^2(\Omega))}. \quad (4.7)$$

We are ready to state our main theorem.

Theorem 4.2.1 *There exists a finite set* $E = E(\alpha, h, I) \subset I$ *satisfying: For* $r \in I \setminus E$ *and* $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, *there exists a unique solution* $\{u(r,f), f\} \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega)$ *to* (4.1) - (4.4). *Moreover there exists a constant* $C_2 > 0$ *satisfying*

$$\|f\|_{L^2(\Omega)} + \|u(r,f)\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u(r,f)\|_{C([0,T];L^2(\Omega))} \le C_2\|\varphi\|_{H^2(\Omega)}. \quad (4.8)$$

4.3 Proof of Theorem 4.2.1

We set

$$(A_r f)(x) = \frac{{}^cD_t^\alpha u(x,t)|_{t=T}}{h(x,T)}, \qquad \Phi(x) = -\frac{r^\alpha (L\varphi)(x)}{h(x,T)}. \quad (4.9)$$

and

$$A_r f + \Phi = f. \quad (4.10)$$

Lemma 4.3.1 *Let* $r \in I$ *be arbitrarily fixed.*

(i)Equation (4.10) *possesses a solution* $f \in L^2(\Omega)$ *if and only if* $\{u(r,f), f\} \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ *satisfies* (4.1) - (4.4).

(ii)Equation (4.10) *possesses a unique solution if and only if there exists a unique solution* $\{u(r,f), f\}$ *to* (4.1) - (4.4).

Proof: (i) First, we assume that (4.10) possesses a solution $f \in L^2(\Omega)$. Substituting the solution f into (4.1), then by Theorems 3.2.2 in Chapter 3, we have a unique solution $u(r,f) \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (4.1) - (4.3). It is sufficient to prove that the solution $u(r,f)$ satisfies (4.4). We set

$$u(x,T) = \varphi_1(x), \qquad x \in \Omega.$$

Then we can see $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. From (4.10), we have $(L(\varphi_1 - \varphi))(x) = 0,\ x \in \Omega$. Since 0 is not the eigenvalue of L and $(\varphi_1 - \varphi)|_{\partial\Omega} = 0$, we get

$$\varphi(x) = \varphi_1(x), \qquad x \in \overline{\Omega}.$$

Next converse assertion of (i) is directly seen in transforming (4.1) into (4.10).

(ii) Next, we prove part (ii). Let us assume that (4.10) possesses a unique solution $f \in L^2(\Omega)$. Seeking a contradiction, we suppose that both $\{u(r, f_1), f_1\}$ and $\{u(r, f_2), f_2\}$ are distinct solutions to (4.1) - (4.4). Let $f_1 = f_2$. Then we have $u(r, f_1) = u(r, f_2)$ because of the unique solvability of the direct problem to (4.1) - (4.3). If $f_1 \neq f_2$, then both f_1 and f_2 are solutions to (4.10) from the part (i), which contradicts the unique solvability of (4.10). Conversely, we assume that the solution to (4.1) - (4.4) exists uniquely, say $\{u(r, f), f\}$. If f_1 and f_2 are distinct solution to (4.10), then by the part (i), there exist the solutions $\{u(r, f_1), f_1\}$ and $\{u(r, f_2), f_2\}$ to (4.1) - (4.4), which contradicts the unique solvability to (4.1) - (4.4). Thus the proof of Lemma 4.3.1 is completed.

Lemma 4.3.2 *For an arbitrarily fixed $f \in L^2(\Omega)$,*

$$A_r f : I \longrightarrow L^2(\Omega)$$

is real analytic in $r \in I$.

Proof: For simplicity we set $u(r, f) = u(r)$. For the real analyticity in $r \in I$, it is sufficient to prove the following two things (e.g., pp.65-66 in John [19]):

(i) ${}^cD_t^\alpha u(r)(\cdot, t)|_{t=T} \in C^\infty((0, \infty); L^2(\Omega))$.

(ii) For every closed interval $J = [r_0, R_0] \subset I$ where $0 < r_0 < R_0$,

there exist positive constants $M = M(f, h, J)$ and $\eta = \eta(f, h, J)$ satisfying

$$\|{}^cD_t^\alpha u^{(m)}(r)(\cdot, t)|_{t=T}\|_{L^2(\Omega)} \leq M\eta^m m! \qquad m \in N,\ r \in J. \tag{4.11}$$

We set $M = \sqrt{2C_3^2\{\|h(\cdot, 0)\|^2_{L^\infty(\Omega)} + T\|h_t\|^2_{L^\infty(\Omega\times(0,T))}\}}\|f\|_{L^2(\Omega)}$, where the coefficient C_3 is equal to C_8 in Chapter 3.

For an arbitrary fixed $f \in L^2(\Omega)$, by (4.6) and Lemma 3.3.1 in Chapter 3, we have

$$
\begin{aligned}
\|{}^cD_t^\alpha u(r)(\cdot,t)|_{t=T}\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha) \\
&\quad + \int_0^t (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2 \\
&\le 2C_3^2\{\|f(\cdot)h(\cdot,0)\|^2_{L^2(\Omega)} + T\|fh_\tau\|^2_{L^2(\Omega\times(0,T))}\} \\
&\le 2C_3^2\{\|h(\cdot,0)\|^2_{L^\infty(\Omega)} + T\|h_t\|^2_{L^\infty(\Omega\times(0,T))}\}\|f\|^2_{L^2(\Omega)},
\end{aligned}
$$

hence

$$
\|{}^cD_t^\alpha u(\cdot,t)|_{t=T}\|_{L^2(\Omega)} \le M. \tag{4.12}
$$

Since $\frac{\mathrm{d}^m}{\mathrm{d}r^m}E_{\alpha,1}(-\lambda_n r^\alpha T^\alpha) = -\lambda_n r^{\alpha-m}T^\alpha E_{\alpha,\alpha-m+1}(-\lambda_n r^\alpha T^\alpha)$ by Lemma 3.3.2 in Chapter 3, we have

$$
\begin{aligned}
\frac{\mathrm{d}^m}{\mathrm{d}r^m}{}^cD_t^\alpha u(r)(\cdot,t)|_{t=T} &= \sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)(-\lambda_n)r^{\alpha-m}T^\alpha E_{\alpha,\alpha-m+1}(-\lambda_n r^\alpha T^\alpha) \\
&+ \int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)(-\lambda_n)r^{\alpha-m}(T-\tau)^\alpha E_{\alpha,\alpha-m+1}(-\lambda_n r^\alpha(T-\tau)^\alpha)\mathrm{d}\tau\Big\}\varphi_n(x).
\end{aligned} \tag{4.13}
$$

Therefore we get

$$
\begin{aligned}
\|\frac{\mathrm{d}^m}{\mathrm{d}r^m}{}^cD_t^\alpha u(r)(\cdot,t)|_{t=T}\|^2_{L^2(\Omega)} &\le 2\sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)^2\lambda_n^2 r^{2(\alpha-m)}T^{2\alpha}\Big(\frac{C_3}{1+\lambda_n r^\alpha T^\alpha}\Big)^2 \\
&+ \int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)^2\lambda_n^2 r^{2(\alpha-m)}\mathrm{d}\tau\int_0^T (T-\tau)^{2\alpha}\Big(\frac{C_3}{1+\lambda_n r^\alpha(T-\tau)^\alpha}\Big)^2\mathrm{d}\tau\Big\} \\
&\le \frac{2C_3^2}{r^{2m}}\{\|f(\cdot)h(\cdot,0)\|^2_{L^2(\Omega)} + T\|fh_t\|^2_{L^2(\Omega\times(0,T))}\} \\
&\le M^2 r^{-2m}.
\end{aligned} \tag{4.14}
$$

As a result, we have

$$
\|\frac{\mathrm{d}^m}{\mathrm{d}r^m}(A_r f)\|^2_{L^2(\Omega)} \le M\delta^{-1}r_0^{-m}. \tag{4.15}
$$

Thus the proof of Lemma 4.3.2 is completed.

Lemma 4.3.3 *There exists a large $R^* > 0$ such that for $R^* < r$, there exists a constant $0 < \theta(r) < 1$ satisfying*

$$\|A_r f\|_{L^2(\Omega)} \le \theta(r)\|f\|_{L^2(\Omega)}, \qquad f \in L^2(\Omega).$$

Proof:

$$\|A_r f\|^2_{L^2(\Omega)} \tag{4.16}$$
$$\le \frac{2}{\delta^2}\sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)^2\Big(\frac{C_3}{1+\lambda_n r^\alpha T^\alpha}\Big)^2$$
$$+\Big\{\int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha (T-\tau)^\alpha)\mathrm{d}\tau\Big\}^2\Big\}$$
$$\le \frac{2}{\delta^2}\sum_{n=1}^{\infty}\Big\{\frac{C_3^2}{\lambda_n^2 r^{2\alpha}T^{2\alpha}}(f(\cdot)h(\cdot,0),\varphi_n)^2 + \frac{C_3^2 T^{1-\alpha}}{\lambda_n(1-\alpha)r^\alpha}\int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\Big\}$$
$$\le \frac{2C_3^2}{\delta^2}\Big\{\frac{1}{\lambda_1^2 r^{2\alpha}T^{2\alpha}}\|h(\cdot,0)\|^2_{L^\infty(\Omega)} + \frac{T^{1-\alpha}}{\lambda_1(1-\alpha)r^\alpha}\|h_t\|^2_{L^\infty(\Omega\times(0,T))}\Big\}\|f\|^2_{L^2(\Omega)} \tag{4.17}$$

The coefficient of $\|f\|^2_{L^2(\Omega)}$ can be less than 1 for large $r > 0$. Thus the proof of Lemma 4.3.3 is completed.

Lemma 4.3.4 $A_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ *is compact for* $r \in I$.

Proof: For arbitrarily fixed $r \in I$ and $f \in L^2(\Omega)$, we can solve (4.1) - (4.3) and the solution $u(r,f) \in C([0,T]; H^2(\Omega)\cap H_0^1(\Omega))$ is unique. Moreover this solution satisfies ${}^cD_t^\alpha u(\cdot,t)|_{t=T} \in H_0^1(\Omega)$. In fact, by using the equivalent norm to $\|\cdot\|_{H^1(\Omega)}$ in Chapter 3, we have

$$[{}^cD_t^\alpha u(\cdot,t)|_{t=T}, {}^cD_t^\alpha u(\cdot,t)|_{t=T}] \le 2\sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)^2 E_{\alpha,1}(-\lambda_n r^\alpha T^\alpha)^2$$
$$+\Big(\int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha(T-\tau)^\alpha)\mathrm{d}\tau\Big)^2\Big\}\lambda_n$$
$$\le 2\sum_{n=1}^{\infty}\lambda_n\Big\{(\frac{C_3}{\lambda_n r^\alpha T^\alpha})^2(f(\cdot)h(\cdot,0),\varphi_n)^2$$
$$+\int_0^T (\frac{C_3}{1+\lambda_n r^\alpha(T-\tau)^\alpha})^2\mathrm{d}\tau\int_0^T (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\Big\}$$
$$\le 2C_3^2\Big\{\frac{1}{\lambda_1 r^{2\alpha}T^{2\alpha}}\|f(\cdot)h(\cdot,0)\|^2_{L^2(\Omega)} + \frac{T^{1-\alpha}}{r^\alpha(1-\alpha)}\|fh_t\|^2_{L^2(\Omega\times(0,T))}\Big\}$$
$$\le C_4\|f\|^2_{L^2(\Omega)}. \tag{4.18}$$

Since the enbedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact, $h(\cdot,T)A_r$ is a compact operator. Furthermore since $|h(x,T)| \ge \delta > 0$ for $x \in \overline{\Omega}$, the division operator by $h(\cdot,T)$ is bounded from $L^2(\Omega)$ to $L^2(\Omega)$. As a result, $A_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact for $r \in I$. Thus the proof of Lemma 4.3.4 is completed.

Now we complete the proof of Theorem 4.2.1. By Lemma 4.3.1, it is sufficient to prove that (4.10) is uniquely solvable. By Lemmata 4.3.2 and 4.3.4, we can apply the result on analytic perturbation to the operator $A_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ (e.g., Theorem 1.9 on p.370 in Kato [20]). Then the following alternative holds.

(i) There exists a finite set $E = E(\alpha, h, I) \subset I$ such that $1 \notin \sigma(A_r)$ for all $r \in I \setminus E$.

or

(ii)$1 \in \sigma(A_r)$ for all $r \in I$.

Lemma 4.3.3 implies that 1 can not be an eigenvalue of A_r for large $r > 0$. Consequently the second alternative (ii) can not occur. We see that E is the set described in Theorem 4.2.1. Let $r \in I \setminus E$. By Lemma 4.3.4., we can apply the Fredholm alternative in $L^2(\Omega)$ to (4.10), and obtain

$$\|f\|_{L^2(\Omega)} \le C_7' \left\| \frac{r^\alpha(L\varphi)}{h(\cdot,T)} \right\|_{L^2(\Omega)} \le C_7 \|\varphi\|_{H^2(\Omega)}.$$

Applying the solution f to (4.1), by (4.7) we obtain the inequality of Theorem 4.2.1. Thus the proof of Theorem 4.2.1 is completed.

Chapter 5

Inverse source problem from time distributing overdetermination for a fractional diffusion equation

5.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We consider an initial-boundary problem for a time fractional parabolic equation:

$$^{c}D_t^{\alpha}u(x,t) = r^{\alpha}\Delta u(x,t) + f(x)h(x,t), \quad (x,t) \in Q_T \equiv \Omega \times (0,T),\ 0 < \alpha < 1, \tag{5.1}$$

$$u(x,0) = 0, \qquad x \in \Omega \tag{5.2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in (0,T). \tag{5.3}$$

$r > 0$ is the parameter, h is a given function on $\overline{\Omega} \times [0,T]$ and $T > 0$ is fixed value. We note that r^{α} is a diffusion coefficient.

We discuss the following inverse problem:

Inverse Problem. Let $r > 0$ be fixed and let $\rho(x,t)$ be given. Determine $u(x,t) = u(r,f)(x,t)$ and $f(x)$, $x \in \Omega$, $t \in (0,T)$ satisfying (5.1) - (5.3) and

$$\int_0^T \rho(x,t)u(x,t)dt = \varphi(x), \qquad x \in \overline{\Omega}. \tag{5.4}$$

This observation is same in Chapter 1. We will show that the inverse problem in Chapter 1 is valid for the above fractional diffusion equation.

The remainder of this paper is composed of two sections. In Section 2, we state our main result. In Section 3, we complete the proof of the main result.

5.2 Main results

In what follow, the operator A in $L^2(\Omega)$ is same in Chapter 1 so that A^{-1} exists and $A^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega))$ is valid.

Throughout this chapter, we assume:

$$\begin{aligned} &\text{(i)} \quad h \in C^1([0,T]; L^\infty(\Omega)), \quad \rho \in L^2(0,T; W^{2,\infty}(\Omega)), \\ &\text{(ii)} \quad M^{-1} \in \mathcal{L}(H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega)). \end{aligned} \tag{5.5}$$

Here we set

$$Mf = \int_0^T \rho(\cdot,t) A^{-1}(h(\cdot,t)f)dt.$$

We note again that $M \in \mathcal{L}(L^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega))$. We arbitrarily choose $r_0 > 0$ such that $0 < r_0 < R$ and set $I = (r_0, R)$. Here and henceforth C_j denotes positive constants which are independent of choices of $r \in I$ and f in (5.1), but may depend on α, h, ρ and I.

Then for every arbitrary fixed $r \in I$ and $f \in L^2(\Omega)$, by means of $h \in C^1([0,T]; L^\infty(\Omega))$ and Theorem 3.2.2 in Chapter 3, there exists a unique solution $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (5.1) - (5.3) and this solution is given by

$$u(r,f)(x,t) = \sum_{n=1}^{\infty} \left\{ \int_0^t (f(\cdot)h(\cdot,\tau), \varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n r^\alpha (t-\tau)^\alpha) \mathrm{d}\tau \right\} \varphi_n(x). \tag{5.6}$$

Here the sequence $\{-\lambda_n\}$ are eigenvalues of the Laplacian Δ and by $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ we denote the orthonormal eigenfunction corresponding to $-\lambda_n$. $\{\varphi_n\}_{n=1}^\infty$ is orthonormal basis of $L^2(\Omega)$. Moreover, by Theorem 3.2.2 in Chapter 3, there exists a constant $C_1 > 0$ satisfying

$$\|u\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C_1 \|fh\|_{H^1(0,T;L^2(\Omega))} \tag{5.7}$$

We are ready to state our main theorem.

Theorem 5.2.1 *There exists a finite set $E = E(\alpha, h, \rho, I) \subset I$ satisfying: For $r \in I \setminus E$ and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $\{u(r,f), f\} \in$*

$C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$ *to* (5.1) - (5.4). *Moreover there exists a constant* $C_2 > 0$ *satisfying*

$$\|f\|_{L^2(\Omega)} + \|u\|_{C([0,T];H^2(\Omega))} + \|{}^cD_t^\alpha u\|_{C([0,T];L^2(\Omega))} \le C_2\|\varphi\|_{H^2(\Omega)}. \tag{5.8}$$

5.3 Proof of Theorem 5.2.1

Similar to the computation of section 3 in Chapter 1, we reduce the inverse problem to a Fredholm equation of second kind:

$$f = K_r f + \Phi_r. \tag{5.9}$$

where we set

$$\begin{aligned} K_r f =& M^{-1}\int_0^T \rho(t)A^{-1}\Bigg\{\sum_{n=1}^{\infty}\Bigg(\big(f(\cdot)h(\cdot,0),\varphi_n\big)E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha) \\ &+\int_0^t \big(f(\cdot)h_\tau(\cdot,\tau),\varphi_n\big)E_{\alpha,1}(-\lambda_n r^\alpha (t-\tau)^\alpha)\mathrm{d}\tau dt\Bigg)\varphi_n(x)\Bigg\}, \end{aligned} \tag{5.10}$$

and

$$\Phi_r = r^\alpha M^{-1}\varphi. \tag{5.11}$$

Lemma 5.3.1 *Let* $r \in I$ *be arbitrarily fixed.*
(i) Equation (5.9) *possesses a solution* $f \in L^2(\Omega)$ *if and only if* $\{u(r,f),f\} \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega)$ *satisfies* (5.1) - (5.4).
(ii) Equation (5.9) *possesses a unique solution if and only if there exists a unique solution* $\{u(r,f),f\}$ *to* (5.1) - (5.4).

Proof: The proof is similar to that of Lemma 1.3.1 in Chapter 1.

Lemma 5.3.2 *For an arbitrary fixed* $f \in L^2(\Omega)$,

$$K_r f : I \longrightarrow L^2(\Omega)$$

is real analytic in $r \in I$.

It is sufficient to prove that ${}^cD_t^\alpha u(r)$ is real analytic in $r \in I$ with the norm in $L^2(\Omega \times (0,T))$, that is, the following two things hold:

(i) ${}^cD_t^\alpha u(r) \in C^\infty(I; L^2(\Omega \times (0,T)))$.

(ii) For every closed interval $J \subset I$, there exist positive constants $\tilde{M} = \tilde{M}(f, h, J)$ and $\eta = \eta(f, h, J)$ satisfying

$$\|D_t^\alpha u^{(m)}(r)\|_{L^2(\Omega\times(0,T))} \le \tilde{M}\eta^m m!, \quad m \in \mathbb{N},\ r \in J, \quad (5.12)$$

For an arbitrary fixed $f \in L^2(\Omega)$, by (3.24) and Lemma 3.3.1 in Chapter 3, we have

$$\begin{aligned}
&\|{}^cD_t^\alpha u(r)(t)\|_{L^2(\Omega)}^2\\
&= \sum_{n=1}^\infty \left\{ (f(\cdot)h(\cdot,0), \varphi_n) E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha) + \int_0^t (f(\cdot)h_\tau(\cdot,\tau), \varphi_n) E_{\alpha,1}(-\lambda_n r^\alpha (t-\tau)^\alpha) \mathrm{d}\tau \right\}^2\\
&\le 2C_3^2 \{\|h(\cdot,0)\|_{L^\infty(\Omega)}^2 + T\|h_t\|_{L^\infty(\Omega\times(0,T))}^2\} \|f\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore we have

$$\|{}^cD_t^\alpha u(r)\|_{L^2(\Omega\times(0,T))}^2 \le \kappa,$$

where we set

$$\kappa = \sqrt{2C_3^2 T\{\|h(\cdot,0)\|_{L^\infty(\Omega)}^2 + T\|h_t\|_{L^\infty(\Omega\times(0,T))}^2\}} \|f\|_{L^2(\Omega)}.$$

By Lemmata 3.3.1 and 3.3.2 in Chapter 3, We have

$$\begin{aligned}
\|{}^cD_t^\alpha u^{(m)}(r)(t)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^\infty \Big\{ (f(\cdot)h(\cdot,0), \varphi_n)(-\lambda_n t^\alpha r^{\alpha-m}) E_{\alpha,\alpha-m+1}(-\lambda_n r^\alpha t^\alpha)\\
&+ \int_0^t (f(\cdot)h_\tau(\cdot,\tau), \varphi_n)(-\lambda_n (t-\tau)^\alpha r^{\alpha-m}) E_{\alpha,\alpha-m+1}(-\lambda_n r^\alpha (t-\tau)^\alpha) \mathrm{d}\tau \Big\}^2\\
&\le 2C_3^2 r^{-2m} \{\|h(\cdot,0)\|_{L^\infty(\Omega)}^2 + T\|h_t\|_{L^\infty(\Omega\times(0,T))}^2\} \|f\|_{L^2(\Omega)}^2,
\end{aligned}$$

hence

$$\|{}^cD_t^\alpha u^{(m)}\|_{L^2(\Omega\times(0,T))}^2 \le \kappa^2 r^{-2m}.$$

Thus the proof of Lemma 5.3.2 is completed.

Lemma 5.3.3 *$K_r : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact for $r \in I$.*

Proof: Let $f_m \to f_0$ weakly in $L^2(\Omega)$. For every $f_m \in L^2(\Omega)$, there exists a unique solution $u = u(r, f_m)$ to (5.1) - (5.3) and this solution satisfies

$$
\begin{aligned}
{}^cD_t^\alpha u(f_m)(\cdot,t) = \sum_{n=1}^{\infty}\Big(& (f_m(\cdot)h(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha) \\
& + \int_0^t (f_m(\cdot)h_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha (t-\tau)^\alpha)\mathrm{d}\tau dt\Big)\varphi_n(x).
\end{aligned}
$$

By (5.10), we have

$$
K_r f_m = M^{-1}\int_0^T \rho(t)A^{-1c}D_t^\alpha u(f_m)(\cdot,t)dt.
$$

By Theorem 3.2.2 in Chapter 3, we have $\|{}^cD_t^\alpha u(f_m)(\cdot,t)\|_{L^2(\Omega)} \le C_4\|f_m\|_{L^2(\Omega)}$. Since $\sup_{m\in\mathbb{N}}\|f_m\|_{L^2(\Omega)} < \infty$, the Lebesgue theorem yields that

$$
\lim_{m\to\infty}\|{}^cD_t^\alpha u(f_m)(\cdot,t)\|_{L^2(\Omega)} = \|{}^cD_t^\alpha u(f_0)(\cdot,t)\|_{L^2(\Omega)}, \quad t\in[0,T].
$$

Since ${}^cD_t^\alpha u(f_m)(\cdot,t) \to {}^cD_t^\alpha u(f_0)(\cdot,t)$ weakly in $L^2(\Omega)$, $t\in[0,T]$, we get

$$
\begin{aligned}
&\|{}^cD_t^\alpha u(f_m)(\cdot,t) - {}^cD_t^\alpha u(f_0)(\cdot,t)\|^2_{L^2(\Omega)} \\
&= \|{}^cD_t^\alpha u(f_m)(\cdot,t)\|^2_{L^2(\Omega)} + \|{}^cD_t^\alpha u(f_0)(\cdot,t)\|^2_{L^2(\Omega)} - 2({}^cD_t^\alpha u(f_m)(\cdot,t), {}^cD_t^\alpha u(f_0)(\cdot,t)) \\
&\to 0, \quad \text{as} \quad m\to\infty.
\end{aligned}
$$

On the othe hand, since $A^{-1}: L^2(\Omega) \to H^2(\Omega)$ is bounded and $\rho \in L^2(0,T;W^{2,\infty}(\Omega))$, we have $\rho(t)A^{-1c}D_t^\alpha u(f_m)(\cdot,t) \to \rho(t)A^{-1c}D_t^\alpha u(f_0)(\cdot,t)$ in $H^2(\Omega)\cap H_0^1(\Omega)$, $t\in(0,T)$. Moreover we have

$$
\|\rho(t)A^{-1c}D_t^\alpha u(f_m)(\cdot,t)\|_{H^2(\Omega)} \le C_5'\|{}^cD_t^\alpha u(f_m)(\cdot,t)\|_{L^2(\Omega)} \le C_5
$$

by $\sup_{m\in\mathbb{N}}\|f_m\|_{L^2(\Omega)} < \infty$. By the Lebesgue theorem, we have

$$
\int_0^T \rho(t)A^{-1c}D_t^\alpha u(f_m)(\cdot,t)\mathrm{d}t \to \int_0^T \rho(t)A^{-1c}D_t^\alpha u(f_0)(\cdot,t)\mathrm{d}t
$$

in $H^2(\Omega)\cap H_0^1(\Omega)$. Furthermore, since $M^{-1}: H^2(\Omega)\cap H_0^1(\Omega) \to L^2(\Omega)$ is bounded, we have

$$
M^{-1}\int_0^T \rho(t)A^{-1c}D_t^\alpha u(f_m)(\cdot,t)\mathrm{d}t \to M^{-1}\int_0^T \rho(t)A^{-1c}D_t^\alpha u(f_0)(\cdot,t)\mathrm{d}t
$$

in $L^2(\Omega)$. Thus the proof of Lemma 5.3.3 is completed.

Lemma 5.3.4 *There exists a large $R^* > 0$ such that for $R^* < r$, there exists a constant $0 < \theta(r) < 1$ satisfying*

$$\|K_r f\|_{L^2(\Omega)} \le \theta(r)\|f\|_{L^2(\Omega)}, \qquad f \in L^2(\Omega).$$

Proof: We have

$$\|K_r f\|_{L^2(\Omega)} \le C_6\|\rho\|_{L^2(0,T;W^{2,\infty}(\Omega))}\|{}^cD_t^\alpha u\|_{L^2(\Omega\times(0,T))}. \tag{5.13}$$

On the othe hand, by Lemmata 3.3.1 and 3.3.2 in Chapter 3, we have

$$\begin{aligned}
&\|{}^cD_t^\alpha u(\cdot,t)\|^2_{L^2(\Omega)}\\
&= \sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha) + \int_0^t (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)E_{\alpha,1}(-\lambda_n r^\alpha(t-\tau)^\alpha)\mathrm{d}\tau\Big\}^2\\
&\le 2\sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)^2E_{\alpha,1}(-\lambda_n r^\alpha t^\alpha)^2\\
&+ \int_0^t (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\int_0^t E_{\alpha,1}(-\lambda_n r^\alpha(t-\tau)^\alpha)^2\mathrm{d}\tau\Big\}\\
&\le 2C_3^2\sum_{n=1}^{\infty}\Big\{(f(\cdot)h(\cdot,0),\varphi_n)^2\frac{1}{\lambda_1 r^\alpha t^\alpha}\\
&+ \frac{t^{1-\alpha}}{\lambda_1 r^\alpha(1-\alpha)}\int_0^t (f(\cdot)h_\tau(\cdot,\tau),\varphi_n)^2\mathrm{d}\tau\mathrm{d}\tau\Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|{}^cD_t^\alpha u\|^2_{L^2(\Omega\times(0,T))}\\
&\le 2C_3^2\Big\{\frac{T^{1-\alpha}}{\lambda_1 r^\alpha(1-\alpha)}\|fh(\cdot,0)\|^2_{L^2(\Omega)} + \frac{T^{2-\alpha}}{\lambda_1 r^\alpha(2-\alpha)}\|fh_t\|^2_{L^2(\Omega\times(0,T))}\Big\}\\
&\le \frac{2C_3^2}{\lambda_1 r^\alpha}\Big\{\frac{T^{1-\alpha}}{(1-\alpha)}\|h(\cdot,0)\|^2_{L^\infty(\Omega)} + \frac{T^{2-\alpha}}{(2-\alpha)}\|h_t\|^2_{L^\infty(\Omega\times(0,T))}\Big\}\|f\|^2_{L^2(\Omega)}.
\end{aligned}$$

For large $r > 0$, the operator K_r is contractible operator. Thus the proof of Lemma 5.3.4 is completed.

Now we complete the proof of Theorem 5.2.1. By Lemma 5.3.1, it is sufficient to prove that (5.9) is uniquely solvable. Similar to Chapter 1, by Lemmata 5.3.2 and 5.3.3, we have the following alternative holds.

(i)There exists a finite set $E = E(\alpha, h, \rho, I) \subset I$ such that $1 \notin \sigma(K_r)$ for all $r \in I \setminus E$.

or

(ii)$1 \in \sigma(K_r)$ for all $r \in I$.

Lemma 5.3.4 implies that 1 can not be an eigenvalue of K_r for large $r > 0$. As a result, the second alternative (ii) can not occur. We see that E is the set described in Theorem 5.2.1. Let $r \in I \setminus E$. By Lemma 5.3.3, we can apply the Fredholm alternative in $L^2(\Omega)$, and obtain

$$\|f\|_{L^2(\Omega)} \leq C_6' \left\|r^\alpha M^{-1}\varphi\right\|_{L^2(\Omega)} \leq C_6\|\varphi\|_{H^2(\Omega)}.$$

We apply (5.7) and obtain (5.8). Thus the proof of Theorem 5.2.1 is completed.

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