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Hydrodynamic Limit for the Ginzburg-Landau $\nabla\phi$ Interface Model with a Conservation Law

By Takao NISHIKAWA

Abstract. Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model was established in [5] under the periodic boundary conditions. This paper extends their results to the modified dynamics which preserve the total volume of each microscopic phase. Nonlinear partial differential equation of fourth order

$$\frac{\partial h}{\partial t} = -\Delta\left[\mathrm{div}\left\{(\nabla\sigma)(\nabla h(t,\theta))\right\}\right], \quad \theta \in \mathbb{T}^d \equiv [0,1)^d,\, t > 0$$

is derived as the macroscopic equation, where $\sigma = \sigma(u)$ is the surface tension of the surface with tilt $u \in \mathbb{R}^d$. The main tool is H^{-2}-method, which is a modification of H^{-1}-method used in [5]. The Gibbs measures associated with the dynamics are characterized.

1. Introduction

The Ginzburg-Landau $\nabla\phi$ interface model determines stochastic dynamics for a discretized hypersurface separating two microscopic phases embedded in the $d+1$ dimensional space. The position of the hypersurface is described by height variables $\phi = \{\phi(x); x \in \Gamma\}$ measured from a fixed d dimensional hyperplane Γ. The hyperplane Γ is discretized and considered as $\Gamma = \mathbb{Z}^d$ or $\Gamma = \Gamma_N := (\mathbb{Z}/N\mathbb{Z})^d$ when we introduce periodic boundary conditions.

Once an interface energy (Hamiltonian) is admitted to ϕ by the formula

$$H(\phi) = \frac{1}{2} \sum_{\substack{x,y\in\Gamma,\\ |x-y|=1}} V(\phi(x) - \phi(y)), \tag{1.1}$$

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the dynamics of the interface can be introduced by means of the Langevin equation

$$(1.2)\qquad d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)\,dt + \sqrt{2}dw_t(x), \quad x \in \Gamma,$$

where $\{w_t(x); x \in \Gamma\}$ is the family of independent one dimensional Brownian motions. Funaki and Spohn [5] discussed the large scale hydrodynamic behavior of the dynamics governed by (1.2) with periodic boundary conditions, namely, by taking $\Gamma = \Gamma_N$.

Another dynamics can be associated with the Hamiltonian H by considering the equation

$$(1.3)\qquad d\phi_t(x) = \Delta\frac{\partial H}{\partial \phi(x)}(\phi_t)\,dt + \sqrt{-2\Delta}dw_t(x), \quad x \in \Gamma,$$

where Δ is the discrete Laplacian, see below. The dynamics determined by (1.3) have a different feature from those governed by (1.2). Indeed, (1.3) preserves the total volume $\sum_{x\in\Gamma}\phi_t(x)$ of the phase under the interface, although (1.2) doesn't have such conservation law. Hohenberg and Halperin [7] called the equation (1.2) model A and (1.3) model B, and studied qualitative difference between these two models. The models corresponding to (1.2) and (1.3) in the particles' systems are the Glauber dynamics and the Kawasaki dynamics, respectively. The aim of this paper is to investigate the hydrodynamic limit of the dynamics determined by (1.3).

From the point of view of the theory of the interfaces, the equation (1.3) serves as a model of the so-called surface diffusion, which is a model of the interface for binary arroys, see Spohn [10]. The mass of each arroy is conserved so that the particles (atoms of each arroy) move around on the surface separating two phases. To formulate such phenomena microscopically, the integer-valued height variables are introduced by counting the number of atoms of one type piled up over the fixed hyperplane Γ. The model studied in this paper is a continuous analog of such SOS type model, although the spatial structure is kept to be discrete.

Now, it is the position to formulate our problem more precisely. We take $\Gamma = \Gamma_N$ and consider the dynamics of the interface ϕ which are governed by the stochastic differential equation (SDE)

$$d\phi_t(x) = \sum_{\substack{y\in\Gamma_N,\\ |x-y|=1}} \{U_y(\phi_t) - U_x(\phi_t)\}\, dt + \sqrt{2}d\tilde{w}_t(x), \quad x \in \Gamma_N, \tag{1.4}$$

where $\{\tilde{w}_t(x); x \in \Gamma_N\}$ is the family of Gaussian processes with mean 0 and covariance structure

$$E[\tilde{w}_t(x)\tilde{w}_s(y)] = -\Delta_{\Gamma_N}(x,y)t \wedge s. \tag{1.5}$$

The function $U_x(\phi)$ is defined by

$$U_x(\phi) := \sum_{\substack{y\in\Gamma_N,\\ |x-y|=1}} V'(\phi(x) - \phi(y)) \tag{1.6}$$

from a potential $V : \mathbb{R} \to \mathbb{R}$ satisfying the conditions (V1)-(V3) stated below and $\Delta_{\Gamma_N}(x,y)$ is the kernel of the discrete Laplacian Δ_{Γ_N} on Γ_N determined by

$$\begin{aligned}(\Delta_{\Gamma_N}\psi)(x) &\equiv \sum_{y\in\Gamma_N} \Delta_{\Gamma_N}(x,y)\psi(y) \\ &= \sum_{y\in\Gamma_N, |x-y|=1} \{\psi(y) - \psi(x)\}, \quad \psi \in \mathbb{R}^{\Gamma_N},\, x \in \Gamma_N.\end{aligned}$$

The equation (1.4) is the same as (1.3), but written in more accurate manner. Under the dynamics (1.4), the total volume $\sum_{x\in\Gamma_N} \phi_t(x)$ is conserved in t as we have already pointed out. From the microscopic dynamics determined by (1.4) by changing scales in space and time properly, we shall derive a fourth order nonlinear partial differential equation (PDE) of parabolic type which describes macroscopic phenomena.

We assume the following conditions on the potential V:

(V1) (smoothness) $V \in C^2(\mathbb{R})$.

(V2) (symmetry) $V(\eta) = V(-\eta), \quad \eta \in \mathbb{R}$.

(V3) (strict convexity) There exist two constants $0 < c_+, c_- < \infty$ such that $c_- \le V''(\eta) \le c_+$ for all $\eta \in \mathbb{R}$.

These assumptions guarantee that the equation (1.4) on Γ_N and also on an infinite lattice $\mathbb{Z}^d$ have strong solutions (see Section 2.2) and the method of energy estimates works (see Section 5.1).

We introduce the macroscopic height processes from microscopic ones as follows:

$$h^N(t,\theta) = \sum_{x\in\Gamma_N} N^{-1}\phi_{N^4t}(x)1_{B(x/N,1/N)}(\theta), \quad \theta\in\mathbb{T}^d, \tag{1.7}$$

where $\mathbb{T}^d \equiv [0,1)^d$ stands for the d-dimensional unit torus and $B(\theta,a) = \prod_{\alpha=1}^d[\theta_\alpha - a/2, \theta_\alpha + a/2)$ denotes a box in $\mathbb{T}^d$ with center $\theta = (\theta_\alpha)_{\alpha=1}^d$ and side length $a > 0$. We note that the diffusive scaling is not appropriate for (1.4) and, indeed, another scaling should be introduced: the factor N for the spatial scaling, while N^4 for the time scaling.

Moreover, we assume the following conditions on the initial data ϕ_0 of (1.4):

(I1) There exists $h_0 \in L^2(\mathbb{T}^d)$ such that $\lim_{N\to\infty} E\|h^N(0)-h_0\|^2_{H^{-1}(\mathbb{T}^d)} = 0$, see Section 4.2 for the definition of the H^{-1}-norm $\|\cdot\|_{H^{-1}(\mathbb{T}^d)}$.

(I2) The sequence $\{h^N(0)\}$ satisfies $\sup_{N\geq 1} E\|h^N(0)\|^2_{L^2(\mathbb{T}^d)} < \infty$.

Now, we state the main theorem of this article:

THEOREM 1.1. *Assume (V1)-(V3),(I1) and (I2). Then, for every $t > 0$, $h^N(t)$ converges as $N\to\infty$ to $h(t)$ which is the unique weak solution of the PDE*

$$\begin{aligned} \frac{\partial}{\partial t}h(t,\theta) &= -\Delta\Big[\mathrm{div}\Big\{(\nabla\sigma)(\nabla h(t,\theta))\Big\}\Big] \\ &\equiv -\sum_{\alpha=1}^d \frac{\partial^2}{\partial\theta_\alpha^2}\sum_{\beta=1}^d\frac{\partial}{\partial\theta_\beta}\left\{\frac{\partial\sigma}{\partial u_\beta}(\nabla h(t,\theta))\right\}, \quad \theta\in\mathbb{T}^d,\ t>0 \end{aligned} \tag{1.8}$$

with initial data h_0, where $\nabla h = (\partial h/\partial\theta_\alpha)_{\alpha=1}^d$. The function $\sigma = \sigma(u)$ is the surface tension of the surface with tilt $u\in\mathbb{R}^d$, see Section 3.3. More precisely, for every $t > 0$,

$$\lim_{N\to\infty} E\|h^N(t) - h(t)\|^2_{H^{-1}(\mathbb{T}^d)} = 0. \tag{1.9}$$

We shall treat not only the height variables ϕ but also the gradient fields to study the associated Gibbs measures, and to establish the local equilibrium similarly to [5]. The gradient fields on Γ_N and $\mathbb{Z}^d$ together with their time evolutions will be introduced in Section 2. In Section 3, we characterize the family of all shift-invariant Gibbs measures for the gradient fields. The method of energy estimates used by [5] doesn't seem to work straightforwardly for this purpose. We therefore use another method: comparison between our dynamics and non-conservative dynamics studied in [5]. The local equilibrium is shown in Section 5 for the coupling of the stochastic dynamics with the discretization of the PDE (1.8). To do this, we study PDE (1.8) and its discretization in Section 4. After these preparations, we prove Theorem 1.1 in Section 6. Finally in Section 7, we give some remarks to our results.

2. Dynamics

In this section, we introduce the dynamics of the height variables on the infinite lattice $\mathbb{Z}^d$ and those for the corresponding gradient (height difference) fields.

2.1. Basic Notations

Let $(\mathbb{Z}^d)^*$ be the set of all directed bonds $b = (x, y)$, $x, y \in \mathbb{Z}^d$, $|x-y| = 1$ in $\mathbb{Z}^d$, and let Γ_N^* be the set of those consisting of sites of Γ_N. The bond $b = (x, y)$ is directed from y to x. We write $x_b = x$ and $y_b = y$ for $b = (x, y)$. The bond $-b$ means the bond b reversely directed, that is $-b = (y_b, x_b)$. The bond b is called positively directed if $(x_b - y_b) \cdot e_\alpha \geq 0$ holds for every $1 \leq \alpha \leq d$, where e_α are unit vectors of direction α in $\mathbb{Z}^d$. We denote the bond $(e_\alpha, 0)$ by e_α again if it doesn't cause any confusion. For every subset Λ of $\mathbb{Z}^d$, we denote the set of all directed bonds touching Λ by $\overline{\Lambda^*}$, that is

$$\overline{\Lambda^*} = \{b \in (\mathbb{Z}^d)^*;\, x_b \in \Lambda \text{ or } y_b \in \Lambda\}.$$

For $\psi = \{\psi(x); x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$, the gradient ∇ and the discrete Laplacian Δ are defined by

$$\nabla\psi(b) = \psi(x) - \psi(y), \quad b = (x, y) \in (\mathbb{Z}^d)^*,$$

$$\Delta\psi(x) = \sum_{y\in\mathbb{Z}^d, |x-y|=1} \{\psi(y) - \psi(x)\}, \quad x \in \mathbb{Z}^d,$$

respectively. We denote the kernel of Δ by $\Delta(x, y)$. We can define operators ∇_{Γ_N} on the torus Γ_N and ∇_Λ on the domains $\Lambda \subset \mathbb{Z}^d$ with Γ_N^* and $\overline{\Lambda^*}$ in place of $(\mathbb{Z}^d)^*$, respectively. And the operators Δ_{Γ_N} and Δ_Λ are defined by

$$\Delta_{\Gamma_N}\psi_{\Gamma_N}(x) = \sum_{y\in\Gamma_N, |x-y|=1} \{\psi_{\Gamma_N}(y) - \psi_{\Gamma_N}(x)\}, \quad x \in \Gamma_N,$$

$$\Delta_\Lambda\psi_\Lambda(x) = \sum_{y\in\Lambda, |x-y|=1} \{\psi_\Lambda(y) - \psi_\Lambda(x)\}, \quad x \in \Lambda$$

for $\psi_{\Gamma_N} = \{\psi_{\Gamma_N}(x); x \in \Gamma_N\} \in \mathbb{R}^{\Gamma_N}$ and $\psi_\Lambda = \{\psi_\Lambda(x); x \in \Lambda\} \in \mathbb{R}^\Lambda$, respectively. In the case without confusion, we omit noting the domains Γ_N and Λ.

A sequence of bonds $\mathcal{C} = \{b_1, \dots, b_n\}$ is called a chain connecting from y to x, $x, y \in \mathbb{Z}^d$, if $y_{b_1} = y$, $x_{b_i} = y_{b_{i+1}}$, $1 \le i \le n-1$ and $x_{b_n} = x$. The chain $\mathcal{C}$ is called a closed loop if $x_{b_n} = y_1$. A plaquette is a closed loop $\mathcal{P} = \{b_1, \dots, b_4\}$ such that $\{x_{b_1}, \dots, x_{b_4}\}$ consists of four distinct points.

Now, let $\mathcal{X}$ be the family of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ which satisfy the plaquette condition:

(P1) $\eta(b) = -\eta(-b), \quad b \in (\mathbb{Z}^d)^*.$

(P2) For any plaquette $\mathcal{P}$, $\sum_{b\in\mathcal{P}} \eta(b) = 0$ holds.

We note that the gradient field $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ defined by $\eta(b) = \nabla\phi(b)$, $b \in (\mathbb{Z}^d)^*$ from the height variable $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ always satisfies the plaquette condition. Similarly, let $\mathcal{X}_{\Gamma_N^*}$ be the set of all $\eta \in \mathbb{R}^{\Gamma_N^*}$ which satisfy the plaquette condition. Let $\mathbb{L}_r^2$ be the set of all $\eta \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ such that

$$|\eta|_r^2 := \sum_{b\in(\mathbb{Z}^d)^*} |\eta(b)|^2 e^{-2r|x_b|} < \infty.$$

We denote $\mathcal{X}_r = \mathcal{X} \cap \mathbb{L}_r^2$ equipped with the norm $|\cdot|_r$.

2.2. Dynamics on the Infinite Lattice

The SDE (1.4) determines dynamics on the periodic cubic lattice Γ_N. The corresponding dynamics for $\phi_t = \{\phi_t(x); x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$ on the infinite lattice $\mathbb{Z}^d$ can be introduced by the SDE

$$(2.1) \qquad d\phi_t(x) = \sum_{\substack{y \in \mathbb{Z}^d, \\ |x-y|=1}} \{U_y(\phi_t) - U_x(\phi_t)\}\, dt + \sqrt{2} d\tilde{w}_t(x), \quad x \in \mathbb{Z}^d,$$

where the process $\{\tilde{w}_t(x); x \in \mathbb{Z}^d\}$ and the function $U_x(\phi)$ are defined similarly as before, i.e., $\{\tilde{w}_t(x)\}$ is the family of Gaussian processes with mean 0 and covariance structure (1.5) with $\Delta(x, y)$ in place of $\Delta_{\Gamma_N}(x, y)$ and

$$U_x(\phi) := \sum_{y \in \mathbb{Z}^d, |x-y|=1} V'(\phi(x) - \phi(y)).$$

Note that $U_x(\phi)$ can be regarded as a function of $\nabla\phi = \{\nabla\phi(b); b \in (\mathbb{Z}^d)^*\}$ and we denote it by $\tilde{U}_x(\nabla\phi)$, namely,

$$\tilde{U}_x(\nabla\phi) := \sum_{b \in (\mathbb{Z}^d)^*, \, x_b = x} V'(\nabla\phi(b)).$$

Then, the first term in the right hand side of (2.1) can be rewritten as $\{\Delta\tilde{U}.(\nabla\phi_t)\}(x)$. The corresponding dynamics for gradient fields $\eta_t = \{\eta_t(b); b \in (\mathbb{Z}^d)^*\} \in \mathbb{R}^{(\mathbb{Z}^d)^*}$ are determined by the SDE

$$(2.2) \qquad d\eta_t(b) = \nabla\Delta\tilde{U}.(\eta_t)(b)\, dt + \sqrt{2} d\left(\nabla\tilde{w}_t\right)(b), \quad b \in (\mathbb{Z}^d)^*.$$

We summarize the relationship between the two dynamics for the height field ϕ and the gradient field η. The potential V satisfies the conditions (V1)-(V3).

PROPOSITION 2.1.

(i) *If ϕ_t is the solution of* (2.1), $\eta_t := \nabla\phi_t$ *satisfies the equation* (2.2).

(ii) *Assume that η_t is the solution of* (2.2) *and ϕ_t is constructed from η_t and $\phi_0(0)$ by*

$$\phi_t(x) = \phi_t(0) + \sum_{b \in \mathcal{C}_{0,x}} \eta_t(b),$$

$$\phi_t(0) = \phi_0(0) + \int_0^t \Delta\tilde{U}.(\eta_s)(0)\, ds + \sqrt{2}\tilde{w}_t(0),$$

where $\mathcal{C}_{0,x}$ is an arbitrary chain connecting 0 *and* x. *Then ϕ_t is well-defined and satisfies the equation* (2.1).

Since the proof is straightforward, it is omitted.

PROPOSITION 2.2. *For every* $\eta \in \mathcal{X}_r$, $r > 0$, *the SDE* (2.2) *has a unique* $\mathcal{X}_r$*-valued continuous solution* η_t *starting at* $\eta_0 = \eta$.

PROOF. We can easily see that the drift term of the SDE is globally Lipschitz continuous on $\mathcal{X}_r$. Hence, the conclusion can be shown by using standard arguments. □

3. Gibbs Measures

In this section, we focus on Gibbs measures associated with the dynamics (2.2) of the gradient field η_t. At first, we discuss the identification of them. [5] treated similar problem for non-conservative system and characterized all equilibrium states of the dynamics due to the method of energy estimates. Our method is different. We compare our dynamics with those studied by [5]. After that, in Section 3.3, we summarize known results on the surface tension $\sigma = \sigma(u)$ introduced by Gibbs measures, which will be useful in the subsequent sections.

3.1. Definition and Notation

Let $\mathcal{P}(\mathcal{X})$ be the set of all probability measures on $\mathcal{X}$ and let $\mathcal{P}_2(\mathcal{X})$ be those $\mu \in \mathcal{P}(\mathcal{X})$ satisfying $E^\mu[|\eta(b)|^2] < \infty$ for each $b \in (\mathbb{Z}^d)^*$. The measure $\mu \in \mathcal{P}_2(\mathcal{X})$ is sometimes called tempered.

We introduce the canonical Gibbs measure associated with our model. Recall that the dynamics (2.1), for instance considered on the torus Γ_N (i.e. the dynamics (1.4)), conserve the total volume $\sum_{x\in\Gamma_N} \phi_t(x)$. For every $\xi \in \mathcal{X}$ and $\Lambda \Subset \mathbb{Z}^d$ (i.e. finite subset of $\mathbb{Z}^d$), the space of gradient fields on $\overline{\Lambda^*}$ with given boundary condition ξ is defined by

$$\mathcal{X}_{\overline{\Lambda^*},\xi} = \left\{ (\eta(b))_{b\in\overline{\Lambda^*}};\ \eta \vee \xi \in \mathcal{X}, \sum_{b\in\overline{\Lambda^*}}{}' x_b \eta(b) = \sum_{b\in\overline{\Lambda^*}}{}' x_b \xi(b) \right\},$$

where $\sum'_{b\in\overline{\Lambda^*}}$ means the sum over all positively directed bonds b in $\overline{\Lambda^*}$ and the configuration $\eta \vee \xi$ is defined by

$$(\eta \vee \xi)(b) = \begin{cases} \eta(b), & b \in \overline{\Lambda^*}, \\ \xi(b), & b \in \left(\overline{\Lambda^*}\right)^c. \end{cases}$$

The quantity $\sum'_{b\in\overline{\Lambda^*}} x_b\eta(b)$ corresponds to $\sum_{x\in\Lambda}\phi(x)$ for the associated height field ϕ. The uniform measure on the affine space $\mathcal{X}_{\overline{\Lambda^*},\xi}$ is denoted by $\nu_{\Lambda,\xi}$. Then, the finite volume canonical Gibbs measure $\mu_{\Lambda,\xi}\in\mathcal{P}(\mathcal{X}_{\overline{\Lambda^*},\xi})$ is defined by

$$\mu_{\Lambda,\xi}(d\eta) = Z_{\Lambda,\xi}^{-1}\exp\left(-H_\Lambda(\eta)\right)\nu_{\Lambda,\xi}(d\eta),$$

where H_Λ is the restriction of the Hamiltonian H to the finite set Λ, that is,

$$H_\Lambda(\eta) := \sum_{b\in\overline{\Lambda^*}} V(\eta(b)) \tag{3.1}$$

for $\eta\in\mathbb{R}^{\overline{\Lambda^*}}$ and $Z_{\Lambda,\xi}$ is the normalizing constant. The probability measure $\mu\in\mathcal{P}(\mathcal{X})$ is called a canonical Gibbs measure if it satisfies the DLR equations

$$\mu(\cdot|\mathcal{F}_\Lambda)(\xi) = \mu_{\Lambda,\xi}(\cdot), \quad \mu\text{-a.e.}\,\xi,$$

where $\mathcal{F}_\Lambda$ is the σ-field generated by $\left\{\eta(b); b\in\left(\overline{\Lambda^*}\right)^c\right\}$ and $\sum'_{b\in\overline{\Lambda^*}} x_b\eta(b)$. We denote by $\mathcal{G}$ the family of all shift-invariant canonical Gibbs measures $\mu\in\mathcal{P}_2(\mathcal{X})$ and by ext$\mathcal{G}$ those $\mu\in\mathcal{G}$ which are ergodic with respect to shifts. Moreover, for each $u=(u_\alpha)_{\alpha=1}^d\in\mathbb{R}^d$ we denote by $(\mathrm{ext}\mathcal{G})_u$ the family of all $\mu\in\mathrm{ext}\mathcal{G}$ having mean u, i.e., $E^\mu[\eta(e_\alpha)] = u_\alpha, \alpha=1,\dots,d$.

Similarly, we define Gibbs measures without conservation law, see [5]. We call them grand-canonical Gibbs measures. Namely, first define the finite volume grand-canonical Gibbs measure $\mu^0_{\Lambda,\xi}\in\mathcal{P}(\mathcal{X}^0_{\overline{\Lambda^*},\xi})$ by

$$\mu^0_{\Lambda,\xi}(d\eta) = (Z^0_{\Lambda,\xi})^{-1}\exp\left(-H_\Lambda(\eta)\right)\nu^0_{\Lambda,\xi}(d\eta), \tag{3.2}$$

where $\nu^0_{\Lambda,\xi}$ is the uniform measure on the affine space

$$\mathcal{X}^0_{\Lambda,\xi} = \left\{(\eta(b))_{b\in\overline{\Lambda^*}};\, \eta\vee\xi\in\mathcal{X}\right\}$$

and $Z^0_{\Lambda,\xi}$ is the normalizing constant. Then, $\mu^0\in\mathcal{P}(\mathcal{X})$ is called a grand-canonical Gibbs measure if it satisfies another DLR equations

$$\mu^0(\cdot|\mathcal{F}^0_\Lambda)(\xi) = \mu^0_{\Lambda,\xi}(\cdot), \quad \mu^0\text{-a.e.}\,\xi, \tag{3.3}$$

where $\mathcal{F}_\Lambda^0$ is the σ-field generated by $\left\{\eta(b); b \in \left(\overline{\Lambda^*}\right)^c\right\}$. We denote by $\mathcal{G}^0$ the family of all shift-invariant grand-canonical Gibbs measures $\mu^0 \in \mathcal{P}_2(\mathcal{X})$ and by ext$\mathcal{G}^0$ those $\mu^0 \in \mathcal{G}^0$ which are ergodic with respect to shifts. Moreover, for each $u \in \mathbb{R}$ we denote by $(\mathrm{ext}\mathcal{G}^0)_u$ the family of all $\mu^0 \in \mathrm{ext}\mathcal{G}^0$ having mean u.

The main result in this section is the following.

THEOREM 3.1. *For every $u \in \mathbb{R}^d$, $(\mathrm{ext}\mathcal{G})_u = (\mathrm{ext}\mathcal{G}^0)_u$ holds. In particular, $\mu_u \in (\mathrm{ext}\mathcal{G})_u$ exists uniquely.*

The theorem tells that the factor of the conserved quantity $\sum'_{b\in\Lambda^*} x_b\eta(b)$ doesn't make an apparent contribution. This is because we only discuss the class of shift-invariant canonical Gibbs measures.

3.2. Comparison with Non-Conservative Dynamics

To prove the uniqueness of tempered shift-invariant and ergodic canonical Gibbs measure with assigned mean u, it is helpful to compare the dynamics determined by (2.2) with non-conservative one studied in [5]. Such idea was used by [9]. We shall prove that the tempered shift-invariant canonical Gibbs measure is reversible under the non-conservative dynamics and then apply the result of [5].

Let $\mathcal{R}^0$ be the family of all shift-invariant $\mu \in \mathcal{P}_2(\mathcal{X})$ which are reversible under the dynamics governed by the SDE

$$d\eta_t^0(b) = -\nabla\tilde{U}.(\eta_t^0)(b)\,dt + \sqrt{2}d\left(\nabla w_t\right)(b), \quad b \in (\mathbb{Z}^d)^*, \tag{3.4}$$

where $\{w_t(x);\, x \in \mathbb{Z}^d\}$ is the family of independent one dimensional Brownian motions. We can define ext$\mathcal{R}^0$ and $(\mathrm{ext}\mathcal{R}^0)_u$ in a similar way to ext$\mathcal{G}^0$ and $(\mathrm{ext}\mathcal{G}^0)_u$ respectively. Then, we have the following proposition.

PROPOSITION 3.2. *$\mathcal{G} \subset \mathcal{R}^0$ holds.*

Assuming Proposition 3.2, we prove Theorem 3.1.

PROOF OF THEOREM 3.1. From Theorems 2.1, 3.1 and 3.2 of [5], $(\mathrm{ext}\mathcal{R}^0)_u = (\mathrm{ext}\mathcal{G}^0)_u$ holds and these sets consist of a single element μ_u^0. On the other hand, one can easily show that $\mu_u^0 \in (\mathrm{ext}\mathcal{G})_u$ because a grand-canonical Gibbs measure is also a canonical Gibbs measure by definition. We therefore get the conclusion from Proposition 3.2. □

We are going to prove Proposition 3.2.

PROOF OF PROPOSITION 3.2. We introduce the local version of the SDE (3.4): For $\xi \in \mathcal{X}$ and $\Lambda \Subset \mathbb{Z}^d$, let $\eta^{0,\Lambda,\xi}$ be the solution of the SDE

$$
\begin{cases}
d\eta_t^{0,\Lambda,\xi}(b) = -\tilde{U}_{x_b}(\eta_t^{0,\Lambda,\xi})(x_b)1_\Lambda(x_b)\,dt \\
\qquad\qquad + \tilde{U}_{y_b}(\eta_t^{0,\Lambda,\xi})(y_b)1_\Lambda(y_b)\,dt & b \in \overline{\Lambda^*}, \\
\qquad\qquad + \sqrt{2}d(\nabla w_t^\Lambda)(b), \\
\eta_t^{0,\Lambda,\xi}(b) = \xi(b), & b \notin \overline{\Lambda^*},\ t \geq 0 \\
\eta_0^{0,\Lambda,\xi}(b) = \xi(b), & b \in (\mathbb{Z}^d)^*,
\end{cases}
\tag{3.5}
$$

where $w_t^\Lambda(x) = 1_{x\in\Lambda} w_t(x)$. We denote the generators (acting on the class of "nice" functions) for the processes determined by (3.5) and (3.4) by L_Λ^0 and L^0, respectively. For the details, see Section 4 of [5].

For $l \in \mathbb{N}$, let $\Lambda_l = [-l, l]^d \cap \mathbb{Z}^d$ be a cube of side length $2l+1$ with (outer) boundary $\partial\Lambda_l = \{x = (x_1, \ldots, x_d);\ |x_\alpha| = l+1 \text{ for some } 1 \leq \alpha \leq d\}$. Let $C_{\mathrm{loc},0}^\infty(\mathcal{X})$ be the family of all functions F on $\mathcal{X}$ of the forms $F(\eta) = \bar{F}\left(\{\eta(b)\}_{b\in\overline{\Lambda^*}}\right)$ for some $\Lambda \Subset \mathbb{Z}^d$ and $\bar{F} \in C_0^\infty\left(\mathbb{R}^{\overline{\Lambda^*}}\right)$. The (minimal) set Λ is called the support of F for $F \in C_{\mathrm{loc},0}^\infty(\mathcal{X})$.

Choose Λ_l as Λ. Using the Dirichlet form given in Section 4.2 of [5], it holds that every $F, G \in C_{\mathrm{loc},0}^\infty(\mathcal{X})$ whose supports are included by Λ_{l-1} and $\sigma(\sum'_{b\in\overline{\Lambda^*}} x_b\eta(b))$-measurable function J

$$
\int_{\mathcal{X}^0_{\overline{\Lambda^*},\xi}} JFL_\Lambda^0 G\, d\mu_{\Lambda,\xi}^0 = \int_{\mathcal{X}^0_{\overline{\Lambda^*},\xi}} JGL_\Lambda^0 F\, d\mu_{\Lambda,\xi}^0.
$$

We therefore obtain that

$$
\int_{\mathcal{X}_{\overline{\Lambda^*},\xi}} FL_\Lambda^0 G\, d\mu_{\Lambda,\xi} = \int_{\mathcal{X}_{\overline{\Lambda^*},\xi}} GL_\Lambda^0 F\, d\mu_{\Lambda,\xi}
$$

holds. Noting that $L^0 F = L_\Lambda^0 F$ holds, we get for $\mu \in \mathcal{G}$

$$
\begin{aligned}
E^\mu\left[FL^0G\right] &= E^\mu\left[E^\mu[FL^0G|\mathcal{F}_\Lambda](\xi)\right] \\
&= E^\mu\left[E^\mu[FL_\Lambda^0G|\mathcal{F}_\Lambda](\xi)\right] \\
&= E^\mu\left[E^{\mu_{\Lambda,\xi}}[FL_\Lambda^0G]\right] \\
&= E^\mu\left[E^{\mu_{\Lambda,\xi}}[GL_\Lambda^0F]\right]
\end{aligned}
$$

$$= E^{\mu}\left[GL^0 F\right].$$

This shows the reversibility of the generator L^0 under μ. Hence we conclude the proof of Proposition 3.2. □

REMARK 3.1. It seems difficult to extend the coupling argument used for the proof of Proposition 2.1 of [5] directly to our model. Such argument could characterize all stationary measures of the dynamics (3.4). The statement of Theorem 3.1 is weaker, since it only characterizes the class of associated Gibbs measures. However, for the proof of the hydrodynamic limit (i.e. Theorem 1.1), the result of Theorem 3.1 is sufficient if one uses the method of [6].

3.3. Surface Tension

Here, we summarize the known results on the surface tension $\sigma = \sigma(u)$ determined from the potential V. Recall that the finite volume surface tension $\sigma_{\Lambda_l} = \sigma_{\Lambda_l}(u)$, $u \in \mathbb{R}^d$ is defined by

$$\sigma_{\Lambda_l}(u) := -\left|\Lambda_l\right|^{-1} \log Z^0_{\Lambda_l,\xi_u}, \tag{3.6}$$

where $Z^0_{\Lambda,\xi}$ is the normalizing constant appearing in (3.2) and ξ_u is determined as follows:

$$\xi_u(b) = \begin{cases} u_\alpha, & x_b - y_b = e_\alpha, \\ -u_\alpha, & x_b - y_b = -e_\alpha \end{cases}$$

for $1 \le \alpha \le d$.

PROPOSITION 3.3 ([4] and [5]).

(i) *The following limit exists:*

$$\sigma(u) := \lim_{l\to\infty} \sigma_{\Lambda_l}(u).$$

(ii) $\sigma \in C^1(\mathbb{R}^d)$ *and there exists a constant* $C_1 > 0$ *such that for all* $u, v \in \mathbb{R}^d$

$$|\sigma(u)| \le C_1(1 + |u|),$$
$$|\nabla\sigma(u) - \nabla\sigma(v)| \le C_1|u - v|,$$
$$u \cdot \nabla\sigma(u) \ge c_-|u|^2 - 1,$$

where c_- *is the constant appearing in the assumption (V3).*

(iii) *The function σ is strictly convex in the sense that there exist constants $C_2, C_3 > 0$ such that*

$$C_2|u-v|^2 \le (\nabla\sigma(u) - \nabla\sigma(v)) \cdot (u-v) \le C_3|u-v|^2.$$

(iv) *The following identities hold for the Gibbs measure μ_u appearing in Theorem 3.1:*

(a) $E^{\mu_u}[V'(\eta(e_\alpha))] = \nabla_\alpha \sigma(u) \, (= \partial\sigma/\partial u_\alpha)$.

(b) $E^{\mu_u}\left[\sum_{\alpha=1}^{d} \eta(e_\alpha) V'(\eta(e_\alpha))\right] = u \cdot \nabla\sigma(u) + 1.$

4. Estimates for Partial Differential Equation

In this section, we introduce the discretization for the PDE (1.8) and derive various uniform estimates.

4.1. Discretization Scheme

We define the finite difference operators by

$$\begin{aligned}
\nabla_\alpha^N f(\theta) &= N(f(\theta + e_\alpha/N) - f(\theta)), \\
\nabla_\alpha^{N,*} f(\theta) &= -N(f(\theta) - f(\theta - e_\alpha/N)), \\
\nabla^N f(\theta) &= (\nabla_1^N f(\theta), \ldots, \nabla_d^N f(\theta)), \\
\operatorname{div}_N g(\theta) &= -\sum_{\alpha=1}^{d} \nabla_\alpha^{N,*} g_\alpha(\theta), \\
\Delta_N f(\theta) &= \operatorname{div}_N \nabla^N f(\theta),
\end{aligned}$$

for $f : \mathbb{T}^d \to \mathbb{R}$, $g = (g_\alpha)_{1\le\alpha\le d} : \mathbb{T}^d \to \mathbb{R}^d, 1 \le \alpha \le d$ and $\theta \in \mathbb{T}^d$. With these notations the discretized PDE of (1.8) reads

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{h}^N(t,\theta) &= A^N(\bar{h}^N(t))(\theta) \\
&:= -\Delta_N\Big[\operatorname{div}_N\big\{(\nabla\sigma)(\nabla^N \bar{h}^N(t))\big\}\Big](\theta), \quad \theta \in \mathbb{T}^d.
\end{aligned} \tag{4.1}$$

The equation (4.1) will be solved with the initial data given by

$$\bar{h}_0^N(\theta) = N^d \int_{[\![\theta]\!]_N} h_0(\theta')\, d\theta',$$

where $[\![\theta]\!]_N$ denotes the box with center in $\mathbb{T}^d_N := \{\theta \in \mathbb{T}^d;\, N\theta \in \Gamma_N\}$ and side length $1/N$ containing $\theta \in \mathbb{T}^d$ and $h_0 \in L^2(\mathbb{T}^d)$ as in the assumption (I1). We note that the assumption (I1) implies

$$\sup_{N\geq 1} \|\bar{h}^N_0\|_{L^2(\mathbb{T}^d)} < \infty. \tag{4.2}$$

Since the initial data $\bar{h}^N_0$ is a step function, the solution $\bar{h}^N(t,\theta)$ is also a step function over $\mathbb{T}^d$, that is,

$$\bar{h}^N(t,\theta) = \bar{h}^N(t,[\theta]_N) \tag{4.3}$$

holds, where $[\theta]_N$ denotes the center of $[\![\theta]\!]_N$.

4.2. Notations and Definition

We will consider the PDE (1.8) and its discretization (4.1) in the Sobolev space $H^{-1}(\mathbb{T}^d)$. Let $\mathscr{D}$ be the space $C^\infty(\mathbb{T}^d)$ with the usual topology and let $\mathscr{D}'$ be the dual space of $\mathscr{D}$. We denote the duality relation between $\mathscr{D}'$ and $\mathscr{D}$ by ${}_{\mathscr{D}'}\langle\cdot,\,\cdot\rangle_{\mathscr{D}}$. For $m \in \mathbb{N}$, the Sobolev space $H^{-m}(\mathbb{T}^d)$ is defined by

$$H^{-m}(\mathbb{T}^d) := \left\{ h \in \mathscr{D}';\, \|h\|_{H^{-m}(\mathbb{T}^d)} := \sup_{\|J\|_{H^m(\mathbb{T}^d)}=1} |{}_{\mathscr{D}'}\langle h,\, J\rangle_{\mathscr{D}}| < \infty \right\}.$$

Here, $\|\cdot\|_{H^m(\mathbb{T}^d)}$ is the usual H^m-norm, i.e.,

$$\|f\|^2_{H^m(\mathbb{T}^d)} := \sum_{|\gamma|\leq m} \int_{\mathbb{T}^d} |\partial^\gamma f(\theta)|^2\, d\theta,$$

where the summation is taken over all multi-indices $\gamma = (\gamma_1, \ldots, \gamma_d) \in (\mathbb{N}\cup\{0\})^d$ satisfying $|\gamma| := \sum_{i=1}^d \gamma_\alpha \leq m$ and ∂^γ is a differential operator defined by

$$\partial^\gamma := \frac{\partial^{\gamma_1}}{\partial\theta^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial\theta^{\gamma_d}}.$$

We easily see that $H^{-1}(\mathbb{T}^d)$ is the Hilbert space with the inner product

$$\langle h_1, h_2\rangle_{H^{-1}(\mathbb{T}^d)} = \sum_{i=1}^\infty \frac{1}{1+\lambda_i} {}_{\mathscr{D}'}\langle h_1,\, \Psi_i\rangle_{\mathscr{D}\mathscr{D}'}\langle h_2,\, \Psi_i\rangle_{\mathscr{D}},\ h_1, h_2 \in H^{-1}(\mathbb{T}^d),$$

where $\{\lambda_i > 0;\, i \in \mathbb{N}\}$ is an increasing sequence of eigenvalues of continuum Laplacian $-\Delta$ on $\mathbb{T}^d$ and $\{\Psi_i;\, i \in \mathbb{N}\}$ is a family of the corresponding eigenfunctions which consist of complete orthonormal system of $L^2(\mathbb{T}^d)$. We note that $\lambda_1 = 0$, $\Psi_1 \equiv 1$ and $\lambda_i > 0$ for $i \geq 2$. Moreover, we can take another inner product:

$$\langle\!\langle h_1, h_2 \rangle\!\rangle_{H^{-1}(\mathbb{T}^d)} = \sum_{i=2}^{\infty} \frac{1}{\lambda_i} {}_{\mathscr{D}'}\langle h_1,\, \Psi_i \rangle_{\mathscr{D}\mathscr{D}'} \langle h_2,\, \Psi_i \rangle_{\mathscr{D}} + {}_{\mathscr{D}'}\langle h_1, \Psi_1 \rangle_{\mathscr{D}\mathscr{D}'} \langle h_2, \Psi_1 \rangle_{\mathscr{D}},\ h_1, h_2 \in H^{-1}(\mathbb{T}^d).$$

We can obtain $\langle \cdot, \cdot \rangle_{H^{-1}(\mathbb{T}^d)}$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{H^{-1}(\mathbb{T}^d)}$ are equivalent. For $h \in H^{-1}(\mathbb{T}^d)$, we denote $\langle h, h \rangle_{H^{-1}(\mathbb{T}^d)}$ and $\langle\!\langle h, h \rangle\!\rangle_{H^{-1}(\mathbb{T}^d)}$ by $\|h\|_{H^{-1}(\mathbb{T}^d)}$ and $|\!|\!|h|\!|\!|_{H^{-1}(\mathbb{T}^d)}$, respectively. Formally, $\|\cdot\|_{H^{-1}(\mathbb{T}^d)}$ and $|\!|\!|\cdot|\!|\!|_{H^{-1}(\mathbb{T}^d)}$ are written as follows:

$$\begin{aligned} ||h||^2_{H^{-1}(\mathbb{T}^d)} &= \int_{\mathbb{T}^d} h(\theta)\, (I - \Delta)^{-1} h(\theta)\, d\theta, \\ |\!|\!|h|\!|\!|^2_{H^{-1}(\mathbb{T}^d)} &= \int_{\mathbb{T}^d} (h(\theta) - \langle h \rangle)\, (-\Delta)^{-1}\, (h - \langle h \rangle)\, (\theta)\, d\theta + \langle h \rangle^2, \end{aligned}$$

where $\langle h \rangle (= {}_{\mathscr{D}'}\langle h, \Psi_1 \rangle_{\mathscr{D}})$ is the average of h over $\mathbb{T}^d$, that is, $\langle h \rangle = \int_{\mathbb{T}^d} h(\theta)\, d\theta$.

We need to state the discretized H^{-1}-norm. For step functions h_1^N, h_2^N with mesh size $1/N$ (i.e. h_i^N satisfies $h_i^N(\theta) = h^N([\theta]_N)$ for $i = 1, 2$), we define $\langle h_1^N, h_2^N \rangle_{-1,N}$ and $\langle\!\langle h_1^N, h_2^N \rangle\!\rangle_{-1,N}$ respectively by

$$\begin{aligned} \langle h_1^N, h_2^N \rangle_{-1,N} &:= N^{-d-2} \sum_{x \in \Gamma_N} \phi_1(x)\, (I - N^2 \Delta_{\Gamma_N})^{-1} \phi_2(x), \\ \langle\!\langle h_1^N, h_2^N \rangle\!\rangle_{-1,N} &:= N^{-d-2} \sum_{x \in \Gamma_N} (\phi_1(x) - \langle \phi_1 \rangle_N)\, (-N^2 \Delta_{\Gamma_N})^{-1} \\ &\qquad \times (\phi_2 - \langle \phi_2 \rangle_N)\, (x) \\ &\qquad + N^{-2d-2} \langle \phi_1 \rangle_N \langle \phi_2 \rangle_N, \end{aligned}$$

where ϕ_1 and ϕ_2 are corresponding microscopic height variables, that is, $\phi_i(x) = h_i^N(x/N)$ for $x \in \Gamma_N$, $i = 1, 2$ and $\langle \phi \rangle_N = \sum_{x \in \Gamma_N} \phi(x)$ is the average of $\phi \in \mathbb{R}^{\Gamma_N}$. Here, we note that the inverse operator $(-\Delta_{\Gamma_N})^{-1}$ acting on $\mathbb{R}^{\Gamma_N}$ is defined only for height variables with average 0. For a step function h^N with mesh size $1/N$ we denote $\langle h^N, h^N\rangle_{-1,N}$ and $\langle\!\langle h^N, h^N\rangle\!\rangle_{-1,N}$ by $\|h^N\|_{-1,N}$ and $|\!|\!|h^N|\!|\!|_{-1,N}$, respectively. We can see that these norms as well as $\|h^N\|_{H^{-1}(\mathbb{T}^d)}$ are mutually equivalent uniformly in N, i.e.,

$$c_1\|h^N\|_{-1,N} \le |\!|\!|h^N|\!|\!|_{-1,N} \le c_2\|h^N\|_{-1,N},$$
$$c_3\|h^N\|_{-1,N} \le \|h^N\|_{H^{-1}(\mathbb{T}^d)} \le c_4\|h^N\|_{-1,N},$$

hold for every $N > 0$ and every step function h^N with the mesh size $1/N$ with constants $c_3, c_4 > 0$ independent of N.

4.3. Uniform L^p bound for $\{\nabla \bar{h}^N\}$

To guarantee the uniform integrability of the function $u \cdot \nabla\sigma(u)$ with respect to coupled measure p^N in (5.8), we need several uniform moment estimates for the solution $\bar{h}^N(t)$ of (4.1).

We introduce several notations. For $1 \le p < \infty$ and step functions h^N with mesh size $1/N$, define $\|\nabla^N h^N\|_{L^p(\mathbb{T}^d)}$ and $\|\nabla^N\nabla^N h^N\|_{L^p(\mathbb{T}^d)}$ by

$$\|\nabla^N h^N\|^p_{L^p(\mathbb{T}^d)} := \sum_{\alpha=1}^d \|\nabla^N_\alpha h^N\|^p_{L^p(\mathbb{T}^d)},$$
$$\|\nabla^N\nabla^N h^N\|^p_{L^p(\mathbb{T}^d)} := \sum_{\alpha,\beta=1}^d \|\nabla^N_\alpha\nabla^N_\beta h^N\|^p_{L^p(\mathbb{T}^d)},$$

respectively. Similarly as above, we define $|\!|\!|\nabla^N h^N|\!|\!|_{-1,N}$ by

$$|\!|\!|\nabla^N h^N|\!|\!|^2_{-1,N} := \sum_{\alpha=1}^d |\!|\!|\nabla^N_\alpha h^N|\!|\!|^2_{-1,N}.$$

To simplify notations, we sometimes omit the domain $\mathbb{T}^d$.

THEOREM 4.1. *The following estimates hold for the solution $\bar{h}^N(t)$ of (4.1):*

(i) $\displaystyle\sup_{N\ge1}\sup_{0\le t\le T}\|\bar{h}^N(t)\|_{L^2} < \infty.$

(ii) $\displaystyle\sup_{N\ge1}\int_0^T \|\nabla^N\nabla^N\bar{h}^N(t)\|^2_{L^2}\,dt < \infty.$

(iii) *For some* $p > 2$, $\quad \sup_{N\ge 1} \int_0^T \|\nabla^N \bar h^N(t)\|_{L^p}^p \, dt < \infty.$

Proof. Differentiating $\left\| \bar h^N(t) \right\|_{-1,N}^2$ in t and noting that $\langle h^N(t)\rangle$ is constant in t, we get from (4.1)

$$\begin{aligned}
\frac{d}{dt} \left\| \bar h^N(t) \right\|_{-1,N}^2 &= 2N^{-d} \sum_{x\in\Gamma_N} \bar h^N(t,x/N) \operatorname{div}_N \nabla\sigma(\nabla^N \bar h^N(t))(x/N) \\
&= -2N^{-d} \sum_{x\in\Gamma_N} \nabla^N \bar h^N(t,x/N) \cdot \nabla\sigma(\nabla^N \bar h^N(t,x/N)) \\
&\le -2C_2 \left\| \nabla^N \bar h^N(t) \right\|_{L^2}^2 .
\end{aligned}$$

We have used (iii) of Proposition 3.3 and $\nabla\sigma(0) = 0$ in the last line. Integrating in t, we obtain

$$\left\| \bar h^N(t) \right\|_{-1,N}^2 + 2C_2 \int_0^t \left\| \nabla^N \bar h^N(s) \right\|_{L^2}^2 \, ds \le \left\| \bar h^N(0) \right\|_{-1,N}^2 . \tag{4.4}$$

It therefore follows from (I1) that

$$\sup_{N\ge 1} \sup_{0\le t\le T} \left\| \bar h^N(t) \right\|_{-1,N}^2 < \infty \tag{4.5}$$

and

$$\sup_{N\ge 1} \int_0^T \left\| \nabla^N \bar h^N(t) \right\|_{L^2}^2 \, dt < \infty. \tag{4.6}$$

Similarly as above, differentiating $\left\| \nabla^N_\alpha \bar h^N(t) \right\|_{-1,N}^2$ in t,

$$\begin{aligned}
&\frac{d}{dt} \left\| \nabla^N_\alpha \bar h^N(t) \right\|_{-1,N}^2 \\
&\quad = 2N^{-d} \sum_{x\in\Gamma_N} \nabla^N_\alpha \bar h^N(t,x/N) \nabla^N_\alpha \operatorname{div}_N \nabla\sigma(\nabla^N \bar h^N(t))(x/N) \\
&\quad = -2N^{-d} \sum_{x\in\Gamma_N} \sum_{\beta=1}^d \nabla^N_\beta \nabla^N_\alpha \bar h^N(t,x/N) \nabla^N_\alpha \nabla_\beta \sigma(\nabla^N \bar h^N(t))(x/N) \\
&\quad = -2N^{-d+2} \sum_{x\in\Gamma_N} \left\{ \nabla^N \bar h^N(t,(x+e_\alpha)/N) - \nabla^N \bar h^N(t,x/N) \right\}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left\{\nabla\sigma(\nabla^N \bar{h}^N(t,(x+e_\alpha)/N)) - \nabla\sigma(\nabla^N \bar{h}^N(t,x/N))\right\}\\
&\le -2C_2 N^{-d+2} \sum_{x\in\Gamma_N} \left|\nabla^N \bar{h}^N(t,(x+e_\alpha)/N) - \nabla^N \bar{h}^N(t,x/N)\right|^2\\
&= -2C_2 \sum_{\beta=1}^{d} \left\|\nabla_\alpha^N \nabla_\beta^N \bar{h}^N(t)\right\|_{L^2}^2 .
\end{aligned}
$$

Integrating both sides in t and summing up in α, we get

$$
\left\|\nabla^N \bar{h}^N(t)\right\|_{-1,N}^2 + 2C_2 \int_0^t \|\nabla^N \nabla^N \bar{h}^N(s)\|_{L^2}^2 \, ds \le \left\|\nabla^N \bar{h}^N(0)\right\|_{-1,N}^2 .
$$

By $\left\|\nabla^N h^N\right\|_{-1,N}^2 = \|h^N\|_{L^2}^2$ and (4.2), we obtain (i) and (ii).

For (iii), applying discretized version of Theorem 4.17 of [1],

$$
\left\|\nabla^N h^N\right\|_{L^2} \le K \left(\left\|\nabla^N \nabla^N h^N\right\|_{L^2}^2 + \left\|\nabla^N h^N\right\|_{L^2}^2 + \left\|h^N\right\|_{L^2}^2\right)^{1/4} \left\|h^N\right\|_{L^2}^{1/2}
$$

and $ab \le a^p/p + b^q/q$ if $a, b \ge 0$ and $1/p + 1/q = 1$, we get

$$
\begin{aligned}
\left\|\nabla^N h^N\right\|_{L^2}^r \le K^r p^{-1} &\left(\left\|\nabla^N \nabla^N h^N\right\|_{L^2}^2 + \left\|\nabla^N h^N\right\|_{L^2}^2 + \left\|h^N\right\|_{L^2}^2\right)^{rp/4}\\
&+ K^r q^{-1} \left\|h^N\right\|_{L^2}^{rq/2} .
\end{aligned}
$$

for $r > 0$. For every $2 < r < 4$, choose $1 < p < 2$ such that $rp = 4$ and take $h^N = \bar{h}^N(t)$ in the above estimate. Then, integrating its both sides in t, we obtain

$$
\begin{aligned}
&\int_0^T \left\|\nabla^N \bar{h}^N(t)\right\|_{L^2}^r \, dt\\
&\quad \le K^r p^{-1} \int_0^T \left(\left\|\nabla^N \nabla^N \bar{h}^N(t)\right\|_{L^2}^2 + \left\|\nabla^N \bar{h}^N(t)\right\|_{L^2}^2 + \left\|\bar{h}^N(t)\right\|_{L^2}^2\right) dt\\
&\quad\quad + K^r q^{-1} T \left(\sup_{0\le t\le T} \left\|\bar{h}^N(t)\right\|_{L^2}\right)^{rq/2} .
\end{aligned}
$$

Using (i), (ii) and (4.5), we get

$$
\sup_{N\ge 1} \int_0^T \left\|\nabla^N \bar{h}^N(t)\right\|_{L^2}^r \, dt < \infty \tag{4.7}
$$

for every $2 < r < 4$. Now,we use the discretized version of Sobolev's lemma stated in the proof of Proposition I.4 of [5], that is,

$$\|f^N\|_{L^{2^*}}^2 \le C(\|\nabla^N f^N\|_{L^2}^2 + \|f^N\|_{L^2}^2) \tag{4.8}$$

holds for every step function f^N with mesh size $1/N$ and for some constant $C > 0$ which is independent of N. Here, 2^* is the Sobolev's conjugate of 2 which is defined by $2^* = 2d/(d-2)$ if $d \ge 3$, 2^* is an arbitrary number larger than 1 if $d = 2$ and $2^* = \infty$ if $d = 1$. If $d = 1$, choose $p = 5/2$. Then, since

$$\|f^N\|_{L^p}^p \le \|f^N\|_{L^\infty} \|f^N\|_{L^{p-1}}^{p-1}$$

holds, by Hölder's inequality and (4.8) we get

$$\begin{aligned}\int_0^T \|\nabla^N \bar{h}^N(t)\|_{L^p}^p \, dt &\le \frac{1}{2}\int_0^T \|\nabla^N \bar{h}^N(t)\|_{L^\infty}^2 \, dt + \frac{1}{2}\int_0^T \|\nabla^N \bar{h}^N(t)\|_{L^{p-1}}^{2p-2} \, dt \\ &\le \frac{1}{2} C \int_0^T \left(\|\nabla^N \bar{h}^N(t)\|_{L^2}^2 + \|\bar{h}^N(t)\|_{L^2}^2\right) dt + \frac{1}{2}\int_0^T \|\nabla^N \bar{h}^N(t)\|_{L^2}^3 \, dt.\end{aligned}$$

Combining (ii), (4.2) and (4.7), the case of $d = 1$ is shown. Next, we consider the case $d \ge 2$. By Hölder's inequality, we get

$$\|f^N\|_{L^p} \le \|f^N\|_{L^2}^{1-\tau} \|f^N\|_{L^{2^*}}^{\tau} \tag{4.9}$$

for $2 < p < 2^*$ and $\tau \in (0, 1)$ such that $1/p = (1-\tau)/2 + \tau/2^*$. Choosing $p = 3 - 2/2^* > 2$, $\tau = 1/p$, $r = 2p - 2 < 4$ and combining (4.8) and (4.9) for $f^N = \nabla^N h^N$, we get

$$\begin{aligned}\|\nabla^N h^N\|_{L^p}^p &\le \|\nabla^N h^N\|_{L^2}^{p(1-\tau)} \|\nabla^N h^N\|_{L^{2^*}}^{p\tau} \\ &\le \|\nabla^N h^N\|_{L^2}^r + C(\|\nabla^N \nabla^N h^N\|_{L^2}^2 + \|\nabla^N h^N\|_{L^2}^2)\end{aligned}$$

and therefore, by (ii), (4.2) and (4.7), we have shown (iii) in the case of $d \ge 2$. □

4.4. Existence and Uniqueness for Partial Differential Equation

In this section, we establish existence and uniqueness for the PDE (1.8). We introduce a triple of separable Hilbert spaces $V \subset H = H^* \subset V^*$ by $H = H^{-1}(\mathbb{T}^d)$ with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{H^{-1}(\mathbb{T}^d)}$, $V = H^1(\mathbb{T}^d)$ and $V^* = H^{-3}(\mathbb{T}^d)$.

These spaces are equipped with their norms denoted by $\|\!|\cdot|\!\|_{H^{-1}(\mathbb{T}^d)}$, $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively. We denote the duality relation between V and V^* by ${}_V\langle\cdot,\cdot\rangle_{V^*}$ such that ${}_V\langle f,g\rangle_{V^*} = \langle\langle f,g\rangle\rangle_{H^{-1}(\mathbb{T}^d)}$ if $g \in H^{-1}(\mathbb{T}^d)$.

The nonlinear fourth order differential operator $A : V \to V^*$ is defined by

$$A(h) := -\Delta\left[\operatorname{div}\{(\nabla\sigma)(\nabla h)\}\right]. \tag{4.10}$$

for $h \in V^*(\mathbb{T}^d)$.

We call $h(t)$ a (weak) solution of (1.8) with initial data $h_0 \in L^2(\mathbb{T}^d)$ if $h(t) \in \mathbb{D}$ and

$$h(t) = h(0) + \int_0^t A(h(s))\,ds, \quad \text{in } V^*$$

holds for a.e.$t \in [0,T]$, where

$$\mathbb{D} = C([0,T], H^{-1}(\mathbb{T}^d)) \cap L^2([0,T], H^1(\mathbb{T}^d)).$$

Let us first discuss the uniqueness of the solution.

THEOREM 4.2. *The solution of the PDE* (1.8) *with the initial data* $h_0 \in H^{-1}(\mathbb{T}^d)$ *is unique if it exists. Moreover, there are constants* $K_1, K_2 > 0$ *such that the solution satisfies the following inequality:*

$$\sup_{0\le t\le T} \|\!|h(t)|\!\|^2_{H^{-1}(\mathbb{T}^d)} + K_1 \int_0^T \|h(t)\|^2_{H^1(\mathbb{T}^d)}\,dt \le K_2 \|\!|h_0|\!\|^2_{H^{-1}(\mathbb{T}^d)}. \tag{4.11}$$

PROOF. Assume that $h(t)$ and $\hat{h}(t)$ are two solutions of (1.8) with initial data h_0 and $\hat{h}_0$ respectively. Noting $\langle h(t)\rangle$ and $\langle\check{h}(t)\rangle$ are constants in t, we obtain

$$\begin{aligned}
\left|\!\left|\!\left| h(t) - \hat{h}(t) \right|\!\right|\!\right|^2_{H^{-1}} &= \left|\!\left|\!\left| h(0) - \hat{h}(0) \right|\!\right|\!\right|^2_{H^{-1}} \\
&\quad - 2\int_0^t \int_{\mathbb{T}^d} \{\nabla h(s,\theta) - \nabla\hat{h}(s,\theta)\} \\
&\qquad\quad \cdot \{\nabla\sigma(\nabla h(s,\theta)) - \nabla\sigma(\nabla\hat{h}(s,\theta))\}\,d\theta\,ds \\
&\le \left|\!\left|\!\left| h(0) - \hat{h}(0) \right|\!\right|\!\right|^2_{H^{-1}}
\end{aligned}$$

$$- 2C_2 \int_0^t \int_{\mathbb{T}^d} \left| \nabla h(s,\theta) - \nabla \hat{h}(s,\theta) \right|^2 d\theta \, ds,$$

where C_2 is the constant appearing in (iii) of Proposition 3.3. In the third inequality, we have used (iii) of Proposition 3.3. Now, we note that the Laplacian Δ on $\mathbb{T}^d$ has a spectral gap $c > 0$, that is, for every $f \in H^1(\mathbb{T}^d)$

$$c\|f - \langle f \rangle\|_{L^2}^2 \le \|\nabla(f - \langle f \rangle)\|_{L^2}^2 = \|\nabla f\|_{L^2}^2 \tag{4.12}$$

and therefore

$$c\|f\|_{L^2}^2 \le \|\nabla f\|_{L^2}^2 + c\langle f \rangle^2. \tag{4.13}$$

Letting $f = h(s) - \hat{h}(s)$, we conclude

$$\begin{aligned} \sup_{0 \le t \le T} \left\| h(t) - \hat{h}(t) \right\|_{H^{-1}}^2 &+ \min\{c, 1\} C_2 \int_0^T \|h(t) - \hat{h}(t)\|_{H^{-1}}^2 \, dt \\ &\le (1 + cC_2) \left\| h_0 - \hat{h}_0 \right\|_{H^{-1}}^2 . \end{aligned} \tag{4.14}$$

This shows the uniqueness of the solution of the PDE (1.8). Since $\hat{h}(t) \equiv 0$ is the solution of the PDE (1.8), we obtain (4.11) by letting $\hat{h}(t) \equiv 0$. □

We discuss the convergence of solutions $\bar{h}^N(t)$ of the discretized PDE (4.1) to the solution $h(t)$ of the PDE (1.8). The following theorem guarantees for the existence of the solution of (1.8), too.

THEOREM 4.3. *The sequence of solutions $\{\bar{h}^N(t)\}$ of the discretized PDE* (4.1) *with initial data $\bar{h}_0^N$ converges to the unique solution of PDE* (1.8) *with initial data h_0 in $H^{-1}(\mathbb{T}^d)$ strongly. Namely,*

$$\lim_{N \to \infty} \|\bar{h}^N(t) - h(t)\|_{H^{-1}} = 0 \tag{4.15}$$

holds for every $t > 0$.

PROOF. Step 1: We recall the statement of Theorem 4.1:

$$\sup_{0 \le t \le T} \|\bar{h}^N(t)\|_{L^2}^2 + \int_0^T \|\nabla^N \bar{h}^N(t)\|_{L^2}^2 \, dt \le C,$$

where C is a constant independent of N.
Step 2: Let $\{N\} \subset \mathbb{N}$ be an arbitrary sequence such that

$$\begin{aligned} \bar{h}^N(t) \to \bar{h}(t) &\quad \text{weakly in } L^2([0,T], L^2(\mathbb{T}^d)), \\ \nabla^N \bar{h}^N(t) \to g(t) &\quad \text{weakly in } L^2([0,T], (L^2(\mathbb{T}^d))^d), \\ A^N(\bar{h}^N(t)) \to \bar{A}(t) &\quad \text{weakly in } L^2([0,T], V^*), \\ \bar{h}^N(T) \to \tilde{h}(T) &\quad \text{weakly in } L^2(\mathbb{T}^d), \end{aligned}$$

for some $\bar{h}(t)$, $g(t)$, $\bar{A}(t)$ and $\tilde{h}(T)$ as $N \to \infty$, where A^N is same as in (4.1). However, we easily see that $\bar{g}(t) = \nabla \bar{h}(t)$ holds for a.e.t. Hence $\bar{h}(t) \in H^1(\mathbb{T}^d)$ a.e.t and $\bar{h} \in L^2([0,T], H^1(\mathbb{T}^d))$. Moreover, using the integral form of the discretized PDE (4.1) and letting $N \to \infty$, we obtain

$$\tilde{h}(T) = h_0 + \int_0^T \bar{A}(s)\, ds, \quad \text{in } V^*$$

and for a.e.t,

$$\bar{h}(t) = h_0 + \int_0^t \bar{A}(s)\, ds, \quad \text{in } V^*.$$

We can therefore take the continuous modification of $\bar{h}$ and denote it by $\bar{h}$ again. Then, from

$$\text{(4.16)} \qquad \left\| \bar{h}^N(T) \right\|_{-1,N}^2 - \left\| \bar{h}_0^N \right\|_{-1,N}^2 = 2 \int_0^T \langle\langle \bar{h}^N(t), A^N(\bar{h}^N(t)) \rangle\rangle_{-1,N}\, dt$$

we get

$$\text{(4.17)} \qquad \left\| \bar{h}(T) \right\|_{H^{-1}}^2 = \| h_0 \|_{H^{-1}}^2 + 2 \int_0^T {}_V\langle \bar{h}(t), \bar{A}(t) \rangle_{V^*}\, dt.$$

Step 3: Let $y(t,\theta) \in C^\infty([0,T] \times \mathbb{T}^d)$ and $y^N(t,\theta) := N^d \int_{[\![\theta]\!]_N} y(t,\theta')\, d\theta'$. Then, by convexity of σ,

$$\begin{aligned} 0 &\geq \int_0^T \langle\langle \bar{h}^N(t) - y^N(t), A^N(\bar{h}^N(t)) - A^N(y^N(t)) \rangle\rangle_{-1,N}\, dt \\ &= \int_0^T \langle\langle \bar{h}^N(t), A^N(\bar{h}^N(t)) \rangle\rangle_{-1,N}\, dt - \int_0^T \langle\langle y^N(t), A^N(\bar{h}^N(t)) \rangle\rangle_{-1,N}\, dt \end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \langle\!\langle \bar{h}^N(t), A^N(y^N(t)) \rangle\!\rangle_{-1,N}\, dt + \int_0^T \langle\!\langle y^N(t), A^N(y^N(t)) \rangle\!\rangle_{-1,N}\, dt \\
&=: I_1^N - I_2^N - I_3^N + I_4^N.
\end{aligned}
$$

For I_1^N, from (4.16) and (4.17) we obtain

$$
\liminf_{N\to\infty} I_1^N \geq \frac{1}{2}\{\|h(T)\|_{H^{-1}}^2 - \|h_0\|_{H^{-1}}^2\} = \int_0^T {}_V\langle \bar{h}(t), \bar{A}(t) \rangle_{V^*}\, dt. \tag{4.18}
$$

For I_2^N, since $y \in C^\infty([0,T]\times\mathbb{T}^d)$ and therefore

$$
\left| \langle\!\langle y^N(t), A^N(\bar{h}^N(t)) \rangle\!\rangle_{-1,N} - {}_V\langle y(t), A^N(\bar{h}^N(t)) \rangle_{V^*} \right| \leq \epsilon(N)
$$

holds with $\epsilon(N) > 0$ which goes to 0 as $N \to \infty$, we obtain

$$
\lim_{N\to\infty} I_2^N = \int_0^T {}_V\langle y(t), \bar{A}(t) \rangle_{V^*}\, dt. \tag{4.19}
$$

Next, we note that I_3^N and I_4^N are represented as follows:

$$
\begin{aligned}
I_3^N &= \sum_{\alpha=1}^d \int_0^T \left(\nabla_\alpha^N \bar{h}^N(t), \nabla_\alpha \sigma(\nabla^N y^N(t)) \right)_{L^2} dt, \\
I_4^N &= \sum_{\alpha=1}^d \int_0^T \left(\nabla_\alpha^N y^N(t), \nabla_\alpha \sigma(\nabla^N y^N(t)) \right)_{L^2} dt,
\end{aligned}
$$

where $(\cdot,\cdot)_{L^2}$ is the inner product of the space $L^2(\mathbb{T}^d)$. Then, since for every $1 \leq \alpha \leq d$, $\nabla_\alpha \sigma(\nabla^N y^N(t))$ conveges to $\nabla_\alpha \sigma(\nabla y(t))$ strongly in $L^2(\mathbb{T}^d)$ as $N \to \infty$, we get

$$
\begin{aligned}
\lim_{N\to\infty} I_3^N &= \sum_{\alpha=1}^d \int_0^T \left(\nabla_\alpha \bar{h}(t), \nabla_\alpha \sigma(\nabla y(t)) \right)_{L^2} dt \\
&= \int_0^T {}_V\langle \bar{h}(t), A(y(t)) \rangle_{V^*}\, dt
\end{aligned} \tag{4.20}
$$

and

$$
\lim_{N\to\infty} I_4^N = \sum_{\alpha=1}^d \int_0^T \left(\nabla_\alpha y(t), \nabla_\alpha \sigma(\nabla y(t)) \right)_{L^2} dt \tag{4.21}
$$

$$= \int_0^T {}_V\langle y(t), A(y(t))\rangle_{V^*}\, dt.$$

From (4.18)-(4.21), we conclude

$$0 \geq \int_0^T {}_V\langle \bar{h}(t) - y(t), \bar{A}(t) - A(y(t))\rangle_{V^*}\, dt. \tag{4.22}$$

Now, since $C^\infty([0,T]\times\mathbb{T}^d)$ is densely embedded in $L^2([0,T], H^1(\mathbb{T}^d))$, we get (4.22) for every $y(t) \in L^2([0,T], H^1(\mathbb{T}^d))$ also. Letting $y(t) = \bar{h}(t) - \lambda x(t)$ for $x \in L^2([0,T], H^1(\mathbb{T}^d))$ and $\lambda > 0$,

$$0 \geq \lambda \int_0^T {}_V\langle x(t), \bar{A}(t) - A(\bar{h}(t) - \lambda x(t))\rangle_{V^*}\, dt.$$

Dividing λ and letting $\lambda \to 0$, we get

$$0 \geq \int_0^T {}_V\langle x(t), \bar{A}(t) - A(\bar{h}(t))\rangle_{V^*}\, dt.$$

Since $x(t)$ is arbitrary, we conclude $\bar{A}(t) = A(\bar{h}(t))$ a.e.t. This shows that $\bar{h}(t)$ is the solution of (1.8).

Step 4: From (i) of Theorem 4.1, the sequence $\{\bar{h}^N(t)\}$ is strongly relative compact in $H^{-1}(\mathbb{T}^d)$. Using this fact, we can conclude that $\{\bar{h}^N(t)\}$ converges to $h(t)$ strongly in $H^{-1}(\mathbb{T}^d)$ as $N \to \infty$. □

5. Local Equilibrium

In this section, we establish the local equilibrium (Proposition 5.3) for the dynamics of the gradient field associated with the height process $\phi(t) = \{\phi_t(x);\, x \in \Gamma_N\}$ determined by (1.4).

5.1. Uniform L^2-bounds

Let μ_t^N be the distribution of $\eta_t \equiv \nabla\phi_t$ on $\mathcal{X}_{\Gamma_N^*}$ and let $\mathrm{Av}_T(\mu^N)$ be its space-time average over $[0, N^4T] \times \Gamma_N$:

$$\mathrm{Av}_T(\mu^N)(d\eta) = N^{-d} \sum_{x\in\Gamma_N} (N^4T)^{-1} \int_0^{N^4T} \mu_t^N \circ \tau_x(d\eta)\, dt,$$

where τ_x is the spatial shift by x on Γ_N. We define the probability measure $\mu_N \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})$ by

$$\mu_N(d\eta) = Z_N^{-1} \exp(-H_{\Gamma_N}(\eta))\, \nu_N(d\eta),$$

where ν_N is the uniform measure on $\mathcal{X}_{\Gamma_N^*}$ and Z_N is the normalizing constant.

To obtain the uniform L^2-bounds on the measures $\mathrm{Av}_T(\mu^N) \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})$, we use the coupling method as in Section 4 of [5]. We assume that two initial data ϕ_0 and $\hat{\phi}_0$ are given and let ϕ_t and $\hat{\phi}_t$ be the corresponding two solutions of SDE (1.4) on Γ_N with common Gaussian process. We denote the macroscopic fields which come from ϕ_t and $\hat{\phi}_t$ by scaling in space and time by $h^N(t,\theta)$ and $\hat{h}^N(t,\theta)$, respectively.

Proposition 5.1.

(i) *We have*

$$E \left\| h^N(t) - \hat{h}^N(t) \right\|_{-1,N}^2 \le E \left\| h^N(0) - \hat{h}^N(0) \right\|_{-1,N}^2 .$$

(ii) *Assume the condition (I1) on the distribution μ_0^N of ϕ_0. Then,*

$$\sup_{N\ge 1} E^{\mathrm{Av}_T(\mu^N)}[\eta(b)^2] < \infty, \quad b \in (\mathbb{Z}^d)^*.$$

(iii) *Moreover, assume*

$$\sup_{N\ge 1} E^{\mu_0^N} \left[N^{-d} \sum_{b\in\Gamma_N^*} \eta(b)^2 \right] < \infty. \tag{5.1}$$

Then,

$$\sup_{N\ge 1,\, 0\le t\le T} E^{\mu_t^N} \left[N^{-d} \sum_{b\in\Gamma_N^*} \eta(b)^2 \right] < \infty, \quad T > 0, \tag{5.2}$$

$$\sup_{N\ge 1} N^4 E \left\| h^N(N^{-4}t) - h^N(0) \right\|_{-1,N}^2 < \infty, \quad t > 0. \tag{5.3}$$

PROOF. Let $\tilde{h}^N(t,\theta) = h^N(t,\theta) - \hat{h}^N(t,\theta)$, $\tilde{\phi}_t = \phi_t - \hat{\phi}_t$ and $\tilde{\eta}_t = \nabla\tilde{\phi}_t$. Then,

$$\frac{d}{ds}\left\|\tilde{h}^N(s)\right\|_{-1,N}^2 = -N^{-d}\sum_{b\in\Gamma_N^*}\tilde{\eta}_{N^4s}(b)\{V'(\eta_{N^4s}(b)) - V'(\hat{\eta}_{N^4s}(b))\}$$

By integrating both sides in s, we get

$$\begin{aligned}&\left\|\tilde{h}^N(t)\right\|_{-1,N}^2 - \left\|\tilde{h}^N(0)\right\|_{-1,N}^2 \\ &\qquad = -N^{-d-4}\int_0^{N^4t}\sum_{b\in\Gamma_N^*}\tilde{\eta}_s(b)\{V'(\eta_s(b)) - V'(\hat{\eta}_s(b))\}\,ds.\end{aligned} \tag{5.4}$$

Now, we take a special initial data for $\hat{\phi}_0$: Let $\hat{\eta}_0$ be an $\mathcal{X}_{\Gamma_N^*}$-valued random variable distributed under μ_N and let $\hat{\phi}_0$ be defined by

$$\hat{\phi}_0(x) = \sum_{b\in\mathcal{C}_{0,x}}\hat{\eta}(b),$$

where $\mathcal{C}_{0,x}$ is a chain in Γ_N^* connecting from 0 to x. Then, for every $b\in\Gamma_N^*$

$$\begin{aligned}E^{\mathrm{Av}_T(\mu^N)}[\eta(b)^2] &\le (2d)^{-1}N^{-d}\sum_{b'\in\Gamma_N^*}(N^4T)^{-1}\int_0^{N^4T}E[(\nabla\phi_t(b'))^2]\,dt \\ &\le d^{-1}N^{-d}\sum_{b'\in\Gamma_N^*}(N^4T)^{-1}\int_0^{N^4T}E[(\nabla\tilde{\phi}_t(b'))^2]\,dt \\ &\qquad + 2d^{-1}\sum_{\alpha=1}^d E^{\mu_N}[\eta(e_\alpha)^2].\end{aligned}$$

In the second inequality, we have used that μ_N is the stationary measure of $\nabla\phi_t$ which is shift-invariant. However, (5.4) implies that

$$N^{-d-4}\int_0^{N^4T}\sum_{b\in\Gamma_N^*}(\nabla\tilde{\phi}_t(b))^2\,dt \le c_-^{-1}\left\|\tilde{h}^N(0)\right\|_{-1,N}^2,$$

where c_- is the constant appearing in the assumption (V3). Therefore, we get

$$E^{\mathrm{Av}_T(\mu^N)}[\eta(b)^2] \le 2c_-^{-1}T^{-1}d^{-1}E\left[\left\|h^N(0)\right\|_{-1,N}^2 + \left\|\hat{h}^N(0)\right\|_{-1,N}^2\right]$$

$$+ 2d^{-1}\sum_{\alpha=1}^{d} E^{\mu_N}[\eta(e_\alpha)^2].$$

By the assumption (I2), we have $\sup_{N\ge1} E\|h^N(0)\|^2_{-1,N} < \infty$. Therefore, the statement (ii) is shown, once one proves $\sup_{N\ge1} E\|\hat{h}^N(0)\|^2_{-1,N} < \infty$. We choose the chain $\mathcal{C}_{0,x}$ as follows: First we connect 0 and $(x_1, 0, \ldots, 0)$ through changing only the first coordinate one by one. Next, we connect $(x_1, 0, 0, \ldots, 0)$ and $(x_1, x_2, 0, \ldots, 0)$ through changing the second coordinate, and so on. With this choice, we obtain

$$\begin{aligned}
E\left\|\hat{h}^N(0)\right\|^2_{-1,N} &\le E\left\|\hat{h}^N(0)\right\|^2_{L^2} = N^{-d-2}\sum_{x\in\Gamma_N} E\left[\left(\hat{\phi}_0(x)\right)^2\right]\\
&\le N^{-d-2}\sum_{x\in\Gamma_N} E^{\mu_N}\left[\left(\sum_{b\in\mathcal{C}_{0,x}}\eta(b)\right)^2\right]\\
&\le dN^{-d-1}\sum_{x\in\Gamma_N}\sum_{b\in\mathcal{C}_{0,x}} E^{\mu_N}\left[\eta(b)^2\right]\\
&\le C\sum_{\alpha=1}^{d} E^{\mu_N}\left[\eta(e_\alpha)^2\right].
\end{aligned}$$

Since the right hand side is bounded in N, we have shown (ii).

Applying Itô's formula to $\sum_{b\in\Gamma_N^*}\eta_t(b)^2$, we get

$$E\left[\sum_{b\in\Gamma_N^*}\eta_t(b)^2\right] \le E\left[\sum_{b\in\Gamma_N^*}\eta_0(b)^2\right] + C\int_0^t E\left[\sum_{b\in\Gamma_N^*}\left\{\eta_s(b)^2+1\right\}\right] ds$$

for some $C > 0$ independent of N. Now, we have used that V' is linearly growing. Multiplying both sides with N^{-d} and using Gronwall's lemma, we obtain (5.2). Applying Itô's formula also to $\|h^N(N^{-4}t) - h^N(0)\|^2_{-1,N}$, we get

$$\begin{aligned}
E\left\|h^N(N^{-4}t) - h^N(0)\right\|^2_{-1,N} \le C'\int_0^t &\big\{E\|h^N(N^{-4}s) - h^N(0)\|^2_{-1,N}\\
&+ N^{-d-4}E\sum_{b\in\Gamma_N^*}\left\{\eta_s(b)^2+1\right\}\big\}\, ds
\end{aligned}$$

for some $C' > 0$ independent of N. Multiplying both sides with N^4 and using Gronwall's lemma again, we obtain (5.3). □

5.2. The Generator and Dirichlet Form

We define the differential operators ∂_x for $x \in \mathbb{Z}^d$ acting on $C^2_{\mathrm{loc}}(\mathcal{X})$ by

$$\partial_x := \sum_{b\in(\mathbb{Z}^d)^*, x_b = x} \frac{\partial}{\partial\eta(b)}.$$

For $x \in \Gamma_N$, we can regard ∂_x as an operator acting on $C^2_b(\mathcal{X}_{\Gamma^*_N})$. We define the differential operator L_N acting on $C^2_b(\mathcal{X}_{\Gamma^*_N})$ by

$$L_N = -4\sum_{x\in\Gamma_N} \partial_x \left(\Delta_{\Gamma_N}\partial.\right)(x) + 2\sum_{x\in\Gamma_N}\left(\Delta_{\Gamma_N}\tilde{U}.(\eta)\right)(x)\partial_x.$$

Then, L_N is the generator of the dynamics governed by SDE

$$d\eta_t(b) = \nabla\Delta_{\Gamma_N}\tilde{U}.(\eta)(b)\,dt + \sqrt{2}d(\nabla\tilde{w}_t)(b), \quad b \in \Gamma^*_N. \tag{5.5}$$

Note that η_t is the gradient fields associated with ϕ_t on Γ_N determined by the SDE (1.4). For $\Lambda \Subset \mathbb{Z}^d$, we define the differential operator L_Λ by

$$L_\Lambda = -4\sum_{x\in\Lambda} \partial_x \left(\Delta_\Lambda\partial.\right)(x) + 2\sum_{x\in\Lambda}\left(\Delta_\Lambda\tilde{U}.(\eta)\right)(x)\partial_x.$$

Then, L_Λ is the generator corresponding to the SDE

$$\begin{cases} d\eta^{\Lambda,\xi}_t(b) = \Delta_\Lambda\tilde{U}.(\eta^{\Lambda,\xi}_t)(x_b)1_\Lambda(x_b)\,dt & \\ \qquad\qquad - \Delta_\Lambda\tilde{U}.(\eta^{\Lambda,\xi}_t)(y_b)1_\Lambda(y_b)\,dt & b \in \overline{\Lambda^*}, \\ \qquad\qquad + \sqrt{2}d(\nabla\tilde{w}^\Lambda_t)(b), & \\ \eta^{\Lambda,\xi}_t(b) = \xi(b), & b \notin \overline{\Lambda^*},\ t \ge 0 \\ \eta^{\Lambda,\xi}_0(b) = \xi(b), & b \in (\mathbb{Z}^d)^*, \end{cases} \tag{5.6}$$

where the processes $\{\tilde{w}^\Lambda_t(x);\, x \in \mathbb{Z}^d\}$ is defined as follows: $\{\tilde{w}^\Lambda_t(x);\, x \in \Lambda\}$ is the family of Gaussian processes with mean 0 and covariance structure (1.5) with $\Delta_\Lambda(x,y)$ in place of $\Delta_{\Gamma_N}(x,y)$ and $w^\Lambda_t(x) \equiv 0$ if $x \notin \Lambda$. The dynamics $\eta^{\Lambda,\xi}_t$ determined by (5.6) is the gradient fields of the solution $\phi^{\Lambda,\bar{\phi}}_t$

of the following SDE, which is the local version of (2.1):

$$\begin{cases} d\phi_t^{\Lambda,\bar{\phi}}(x) = \Delta_\Lambda U.(\phi_t^{\Lambda,\bar{\phi}})(x)\,dt + \sqrt{2}d\tilde{w}_t^\Lambda(x), & x \in \Lambda, \\ \phi_t^{\Lambda,\bar{\phi}}(x) = \bar{\phi}(x), & x \notin \Lambda,\ t \geq 0 \\ \phi_0^{\Lambda,\bar{\phi}}(b) = \bar{\phi}(x), & x \in \mathbb{Z}^d \end{cases} \tag{5.7}$$

with $\nabla\bar{\phi} = \xi$. Here, we note that the boundary condition ξ is contained in the space $\mathcal{X}_{\overline{\Lambda^*},\xi}$ and ξ does not appear in the generator L_Λ.

Noting that the dynamics governed by (5.7) is ergodic on the affine space

$$\left\{\phi \in \mathbb{R}^\Lambda;\ \sum_{x\in\Lambda}\phi(x) = \sum_{x\in\Lambda}\bar{\phi}(x)\right\},$$

we can see that the dynamics governed by (5.6) on $\mathcal{X}_{\overline{\Lambda^*},\xi}$ is also ergodic. Moreover, we can see that its unique stationary measure is $\mu_{\Lambda,\xi}$. Similarly, the dynamics governed by (2.2) on Γ_N^* is ergodic and its unique stationary measure is μ_N. Performing integration by parts, Dirichlet forms of these dynamics are given by

$$\begin{aligned} &\int_{\mathcal{X}_{\Gamma_N^*}} F\, L_N G\, d\mu_N \\ &\qquad = 4\sum_{x\in\Gamma_N}\int_{\mathcal{X}_{\Gamma_N^*}} \{\partial_x F\}\{(\Delta_{\Gamma_N}\partial.)(x)G\}\,d\mu_N,\ F,G \in C_b^2(\mathcal{X}_{\Gamma_N^*}), \\ &\int_{\mathcal{X}_{\overline{\Lambda^*},\xi}} F\, L_\Lambda G\, d\mu_{\Lambda,\xi} \\ &\qquad = 4\sum_{x\in\Lambda}\int_{\mathcal{X}_{\overline{\Lambda^*},\xi}} \{\partial_x F\}\{(\Delta_\Lambda\partial.)(x)G\}\,d\mu_{\Lambda,\xi},\ F,G \in C_b^2(\mathcal{X}_{\overline{\Lambda^*},\xi}), \end{aligned}$$

respectively. For $\nu \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})$, let $I_N(\nu)$ be the entropy production defined by

$$I_N(\nu) = -4\int_{\mathcal{X}_{\Gamma_N^*}} \sqrt{f_N}L_N\sqrt{f_N}d\mu_N,$$

where $f_N(\eta) = d\nu/d\mu_N$.

5.3. Local Equilibrium

Here, we shall prove the following lemma. Note that we can regard $\tilde{\mu}^N \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})$ as an element of $\mathcal{P}(\mathcal{X})$ by extending periodically.

LEMMA 5.2. *Assume that the sequence of measures $\{\tilde{\mu}^N \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})\}$ is tight in $\mathcal{P}(\mathcal{X})$ and satisfies*

$$\lim_{N\to\infty} N^{-d} I_N(\tilde{\mu}^N) = 0.$$

Then, every limit point μ of $\{\tilde{\mu}^N \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})\}$ is a canonical Gibbs measure.

PROOF. We introduce the entropy production on infinite lattice. For $\mu \in \mathcal{P}(\mathcal{X})$ and $\Lambda \Subset \mathbb{Z}^d$, we define $I_\Lambda(\nu)$ by

$$I_\Lambda(\nu) = -4 \int_{\mathcal{X}} \sqrt{f_\Lambda} L_\Lambda \sqrt{f_\Lambda}\, d\mu,$$

where $f_\Lambda = d\nu/d\mu|_{\mathcal{F}_{\overline{\Lambda^*}}}$ and $\mu = \mu_0$. We obtain

$$\begin{aligned} I_\Lambda(\tilde{\mu}^N) &= \sup\left\{ -\int_{\mathcal{X}} \frac{L_\Lambda u}{u}\, d\tilde{\mu}^N;\ u \text{ is positive, } \mathcal{F}_{\overline{\Lambda^*}}\text{-measurable} \right\} \\ &\le \sup\left\{ -\int_{\mathcal{X}_{\Gamma_N^*}} \frac{L_\Lambda u}{u}\, d\tilde{\mu}^N;\ u \text{ is positive on } \mathcal{X}_{\Gamma_N^*} \right\} \\ &= \frac{|\Lambda|}{|\Gamma_N|} I_N(\tilde{\mu}^N). \end{aligned}$$

Therefore, by assumption, $\lim_{N\to\infty} I_\Lambda(\tilde{\mu}^N) = 0$. This implies, using lower semicontinuity of I_Λ, that

$$I_\Lambda(\nu) \le \liminf_{N\to\infty} I_\Lambda(\tilde{\mu}^N) = 0$$

for every limit point ν of $\{\tilde{\mu}^N\}$. Now, for $\mathcal{F}_{\overline{\Lambda^*}}$-measurable $\psi \in C^2_{loc,b}(\mathcal{X})$, we get

$$\left| \int_{\mathcal{X}} L_\Lambda \psi\, d\nu \right| = \left| \int_{\mathcal{X}} L_\Lambda \psi f_\Lambda\, d\mu \right|$$

$$= 4 \left| \sum_{x\in\Lambda} \int_{\mathcal{X}} \{\partial_x \psi\} \{(\Delta_\Lambda \partial.)(x) f_\Lambda\}\, d\mu \right|$$
$$\leq 2 \sqrt{\sum_{x\in\Lambda} \int_{\mathcal{X}} \{\partial_x \psi\} \{(\Delta_\Lambda \partial.)(x) \psi\}\, d\nu} \times \sqrt{I_\Lambda(\nu)} = 0.$$

This shows that $\nu|_{\mathcal{F}_{\overline{\Lambda^*}}}$ is stationary measure of L_Λ which is the generator for the SDE (5.6) when the boundary condition ξ is fixed. Since the unique stationary measure of this dynamics is the finite volume Gibbs measure $\mu_{\Lambda,\xi}$, decomposing ν with respect to boundary conditions and assigning conserved quantity, we get DLR equations for ν,

$$\nu(\cdot|\mathcal{F}_\Lambda) = \mu_{\Lambda,\xi}(\cdot), \quad \nu\text{-a.e.}\xi.$$

This shows that ν is a Gibbs measure. □

5.4. Coupled Local Equilibrium

We define the probability measure $p^N(d\eta\, du)$ on $\mathcal{X}_{\Gamma_N^*} \times \mathbb{R}^d$ by

$$p^N(d\eta\, du) = t^{-1} \int_0^t N^{-d} \sum_{x\in\Gamma_N} \delta_{u^N(s,x)}(du) \mu^N_{N^4 s} \circ \tau_x(d\eta)\, ds, \tag{5.8}$$

where $u^N(s,x) = \nabla^N \bar{h}^N(s, x/N)$. This means that we have coupled the distribution of stochastic dynamics and the solution of the discretized PDE. Theorem 4.1 and Proposition 5.1 show that there exists $p > 2$ such that

$$\sup_{N\geq 1, b\in(\mathbb{Z}^d)^*} \int \{\eta(b)^2 + |u|^p\} p^N(d\eta\, du) < \infty. \tag{5.9}$$

Therefore, the sequence $\{p^N(d\eta\, du)\}$ is tight and we can choose a subsequence $N'' \to \infty$ from an arbitrary sequence $N' \to \infty$ such that $p^{N''}$ converges weakly to some $\bar{p}$ on $\mathcal{X} \times \mathbb{R}^d$ as $N'' \to \infty$.

To characterize the limit $\bar{p}$, we impose the following entropy bound on the initial distribution μ_0^N of ϕ_0:

$$\lim_{N\to\infty} N^{-(d+4)} H_N(\mu_0^N) = 0. \tag{5.10}$$

We first prove the main theorem assuming (5.10) and remove it later. Here, $H_N(\nu)$ denotes the relative entropy of $\nu \in \mathcal{P}(\mathcal{X}_{\Gamma_N^*})$ with respect to μ_N, that is,

$$H_N(\nu) = \int f_N \log f_N \, d\mu_N,$$

where $f_N = d\nu/d\mu_N$.

Proposition 5.3. *Under the condition* (5.10), *for every limit point $\bar{p}(d\eta\, du)$ of $\{p^N(d\eta\, du)\}$, there exists $\lambda(dv\, du) \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\bar{p}$ is represented as*

$$\bar{p}(d\eta\, du) = \int_{\mathbb{R}} \mu_v(d\eta) \bar{\lambda}(dv\, du).$$

We can obtain Proposition 5.3 in a quite parallel manner to the proof of Theorem 4.1 of [5] under the conditions (5.9) and (5.10). Hence, we omit the proof.

6. The Proof of Main Theorem

We shall prove Theorem 1.1 first under the condition (5.10) on the entropy. Then, we remove it. The assumption (I2) is necessary only for the second step.

6.1. Derivation of PDE (1.8)

We assume under the condition (5.10). From

$$\begin{aligned}\|h^N(t) - h(t)\|^2_{H^{-1}(\mathbb{T}^d)} \le 2\|h^N(t) - \bar{h}^N(t)\|^2_{H^{-1}(\mathbb{T}^d)} \\ + 2\|\bar{h}^N(t) - h(t)\|^2_{H^{-1}(\mathbb{T}^d)}\end{aligned}$$

and Theorem 4.3, the proof of Theorem 1.1 is completed once we can prove that the first term of the right hand side tends to 0 as $N \to \infty$. Moreover, by the uniform equivalence of $\|\cdot\|_{H^{-1}(\mathbb{T}^d)}$ and $|\!|\!|\cdot|\!|\!|_{-1,N}$, this follows from

$$\lim_{N\to\infty} E \left|\!\left|\!\left| h^N(t) - \bar{h}^N(t) \right|\!\right|\!\right|^2_{-1,N} = 0. \tag{6.1}$$

Using Itô's formula, we get

$$
\begin{aligned}
&E \left\| h^N(t) - \bar{h}^N(t) \right\|_{-1,N}^2 \\
&\quad = E \left\| h^N(0) - \bar{h}^N(0) \right\|_{-1,N}^2 \\
&\qquad + 2E \int_0^t \sum_{x\in\Gamma_N} \left(N^{-1}\phi_s^N(x) - \bar{h}^N(s,x/N) \right) \\
&\qquad\qquad \times \left\{ -NU_x(\phi_s^N) - \nabla^{N,*}\nabla\sigma(\nabla^N \bar{h}^N(s,x/N))(x) \right\} ds + 2t \\
&\quad = E \left\| h^N(0) - \bar{h}^N(0) \right\|_{-1,N}^2 \\
&\qquad + 2 \int_0^t \left(I_1^N(s) + I_2^N(s) + I_3^N(s) + I_4^N(s) \right) ds,
\end{aligned}
$$

where $I_i^N(s)$, $1 \le i \le 4$ are given by

$$
\begin{aligned}
I_1^N(s) &= -\sum_{x\in\Gamma_N} \sum_{\alpha=1}^d \nabla_\alpha \phi_s^N(x) V'(\nabla_\alpha \phi_s^N(x)) + 1, \\
I_2^N(s) &= \sum_{x\in\Gamma_N} \sum_{\alpha=1}^d \nabla_\alpha^N \bar{h}^N(s,x/N) V'(\nabla_\alpha \phi_s^N(x)), \\
I_3^N(s) &= \sum_{x\in\Gamma_N} \sum_{\alpha=1}^d \nabla_\alpha \phi_s^N(x) \nabla_\alpha \sigma(\nabla^N \bar{h}^N(s,x/N)), \\
I_4^N(s) &= -\sum_{x\in\Gamma_N} \sum_{\alpha=1}^d \nabla_\alpha^N \bar{h}^N(s,x/N) \nabla_\alpha \sigma(\nabla^N \bar{h}^N(s,x/N)),
\end{aligned}
$$

respectively. Using $p^N(d\eta\, du)$ introduced in Section 5.4, we can rewrite these terms as follows:

$$
\begin{aligned}
\int_0^t I_1^N(s)\, ds &= -t \sum_{\alpha=1}^d \int_{\mathcal{X}\times\mathbb{R}^d} \left\{ \eta(e_\alpha) V'(\eta(e_\alpha)) - 1 \right\} p^N(d\eta\, du), \\
\int_0^t I_2^N(s)\, ds &= t \sum_{\alpha=1}^d \int_{\mathcal{X}\times\mathbb{R}^d} u_\alpha V'(\eta(e_\alpha))\, p^N(d\eta\, du), \\
\int_0^t I_3^N(s)\, ds &= t \sum_{\alpha=1}^d \int_{\mathcal{X}\times\mathbb{R}^d} \eta(e_\alpha) \nabla_\alpha \sigma(u)\, p^N(d\eta\, du),
\end{aligned}
$$

$$\int_0^t I_4^N(s)\,ds = -t\sum_{\alpha=1}^d \int_{\mathcal{X}\times\mathbb{R}^d} u_\alpha \nabla_\alpha \sigma(u)\, p^N(d\eta\, du).$$

Recall that we chose the subsequence $N'' \to \infty$ from an arbitrary sequence $N' \to \infty$ such that $p^{N''}$ converges to some $\bar{p}$. Now, since

$$\begin{aligned} |u\cdot\nabla\sigma(u)|^{p/2} &\le C\left(|u|^p+1\right), \\ |\eta(e_\alpha)\nabla_\alpha\sigma(u)|^q &\le C\left(1+|u|^p+|\eta(e_\alpha)|^2\right) \\ \left|V'(\eta(e_\alpha))u_\alpha\right|^{p/2} &\le C\left(1+|u|^p+|\eta(e_\alpha)|^2\right) \end{aligned}$$

for $p > 2$, $q = 2p/(2+p) > 1$ and some constant $C > 0$, the integrands of I_2^N, I_3^N, I_4^N are uniformly integrable with respect to the probability measures $\{p^N\}$ because of (5.9). Moreover, Proposition 5.3 gives the representation of the limit $\bar{p}$ in term of $\bar{\lambda}(dv\,du)$. Hence, by Proposition 3.3, we obtain

$$\begin{aligned} \lim_{N''\to\infty}\int_0^t I_2^{N''}(s)\,ds &= 2t\int_{\mathbb{R}^{2d}} u\cdot\nabla\sigma(v)\,\bar{\lambda}(dv\,du), \\ \lim_{N''\to\infty}\int_0^t I_3^{N''}(s)\,ds &= 2t\int_{\mathbb{R}^{2d}} v\cdot\nabla\sigma(u)\,\bar{\lambda}(dv\,du), \\ \lim_{N''\to\infty}\int_0^t I_4^{N''}(s)\,ds &= -2t\int_{\mathbb{R}^{2d}} u\cdot\nabla\sigma(u)\,\bar{\lambda}(dv\,du). \end{aligned}$$

For I_1^N, since $\eta V'(\eta) \ge c_-\eta^2 \ge 0$, we get

$$\limsup_{N''\to\infty}\int_0^t I_1^N(s)\,ds \le -2t\sum_{\alpha=1}^d\int_{\mathcal{X}\times\mathbb{R}^d} u\cdot\nabla\sigma(u)\,\bar{\lambda}(dv\,du)$$

by applying Fatou's lemma. Summarizing these results and from the assumption (I1), we get

$$\begin{aligned} &\limsup_{N''\to\infty} E\left\|h^N(t)-\bar{h}^N(t)\right\|_{-1,N''}^2 \\ &\quad\le \limsup_{N''\to\infty} E\left\|h^N(0)-\bar{h}^N(0)\right\|_{-1,N''}^2 \\ &\qquad\qquad - 2t\int_{\mathbb{R}^{2d}}(u-v)\cdot(\nabla\sigma(u)-\nabla\sigma(v))\,\bar{\lambda}(dv\,du) \\ &\quad\le 0. \end{aligned}$$

We have used the convexity of the surface tension σ. Since the subsequence N' is arbitrary, we obtain without choosing subsequences

$$\lim_{N\to\infty} E \left\| h^N(t) - \bar{h}^N(t) \right\|_{-1,N}^2 = 0.$$

This shows (6.1) and therefore the conclusion of Theorem 1.1.

6.2. Removal of the Entropy Bound

We have proved Theorem 1.1 under the entropy bound (5.10). Here, we are going to remove it.

We take $0 < a < 1$ as an approximation parameter. Let

$$\phi_0^a(x) := \frac{1}{|\Lambda_{[Na]}|} \sum_{y \in x + \Lambda_{[Na]}} \phi_0(y), \quad x \in \Gamma_N,$$

and ϕ_t^a be the solution of the SDE (1.4) with the initial data ϕ_0^a. We define the corresponding macroscopic field $h^{N,a}$ by

$$h^{N,a}(t,\theta) = \sum_{x \in \Gamma_N} N^{-1} \phi_{N^4 t}^a(x) 1_{B(x/N,1/N)}(\theta), \quad \theta \in \mathbb{T}^d. \tag{6.2}$$

Here, note that $h^{N,a}(0,\theta) = h^N(0,\cdot) * \psi^a(\theta)$ with $\psi^a(u) = a^{-d} 1_{[-1,1]}(u/a)$, $u \in \mathbb{R}^d$. We denote by $\mu_t^{N,a}$ the distribution of $\nabla \phi_t^a$.

Lemma 6.1. *Assume (I1) and (I2). Then, for fixed* $0 < a < 1$*,*

(i) $\mu_0^{N,a}$ *satisfies* (5.1).

(ii) *For* $t \geq 0$*, (I1) holds with* $\mu_t^{N,a}$ *and* $h_0 * \psi^a$ *in place of* μ_t^N *and* h_0*, respectively.*

We can prove this lemma quite similarly to Lemma 5.1 of [5] by replacing L^2-norm with H^{-1}-norm and using Proposition 5.1. Hence, we omit the proof.

Lemma 6.2. *Suppose that the sequence* $\{\mu_0^N\}$ *satisfies (I1) and* (5.1)*. Then, for* $t > 0$ *there exists a constant* $C > 0$ *such that*

$$H_N(\mu_t^N) \leq C N^d. \tag{6.3}$$

Since the proof runs quite parallel to the argument in [8], we omit the proof. In the proof, we have used the bound (5.1).

Lemmas 6.1 and 6.2 imply the entropy bound (5.10) and (I1) for $\{\mu_1^{N,a}; N \geq 1\}$ with $h_0 * \psi^a$ in place of h_0. Therefore, we can apply the results obtained in the last section and we conclude

$$\lim_{N\to\infty} E\|h^{N,a}(t+N^{-4}) - h^a(t)\|_{H^{-1}}^2 = 0,$$

where h^a is the solution of PDE (1.8) with the initial data $h_0 * \psi^a$. Let $\hat{h}^N(t)$ be the macroscopic field obtained from the solution of the SDE (1.4) on Γ_N with Gaussian process $\{\hat{w}_t(x) := \tilde{w}_{t+N^{-4}}(x) - \tilde{w}_{N^{-4}}(x)\}$ and initial data ϕ_0. Here, we note that h^N and $\hat{h}^N$ have the same distributions. Then, using (i) of Proposition 5.1 we obtain

$$\begin{aligned} &E\|h^{N,a}(t+N^{-4}) - \hat{h}^N(t)\|_{H^{-1}}^2 \\ &\leq E\|h^{N,a}(N^{-4}) - h^N(0)\|_{H^{-1}}^2 \\ &\leq 2E\|h^{N,a}(N^{-4}) - h^{N,a}(0)\|_{H^{-1}}^2 + 2E\|h^{N,a}(0) - h^N(0)\|_{H^{-1}}^2. \end{aligned} \tag{6.4}$$

The first term on the right hand side of (6.4) goes to 0 as $N \to \infty$ for fixed $a > 0$ by (iii) of Proposition 5.1. The second term tends to 0 as $N \to \infty$ and then $a \to 0$ by (ii) of Lemma 6.1. By (4.14) of Theorem 4.2, we get the conclusion.

7. Concluding Remarks

(i) The total surface tension of the macroscopic hypersurface $h \in C^1(\mathbb{T}^d)$ is defined by

$$\Sigma(h) := \int_{\mathbb{T}^d} \sigma(\nabla h(\theta))\, d\theta.$$

Its (formal) functional derivative is then given by

$$\frac{\delta\Sigma}{\delta h(\theta)} = -\operatorname{div}\{(\nabla\sigma)(\nabla h(\theta))\},$$

in the sense that

$$\frac{d}{d\epsilon}\Sigma(h+\epsilon g)\bigg|_{\epsilon=0} = \int_{\mathbb{T}^d} \frac{\delta\Sigma}{\delta h(\theta)} g(\theta)\, d\theta \left(= \left(\frac{\delta\Sigma}{\delta h}, g\right)_{L^2}\right)$$

holds for every $g \in C^1(\mathbb{T}^d)$. As explained in [5], the macroscopic equation derived from the microscopic dynamics determined by (1.2) is the gradient flow which relaxes the total energy Σ:

$$\frac{\partial h}{\partial t} = -\frac{\delta \Sigma}{\delta h(\theta)}.$$

In our case, the basic (Riemannian) structure should be introduced to the space of heights based on the H^{-1}-inner product. Accordingly, the functional derivative of Σ would be changed into

$$\frac{\tilde{\delta}\Sigma}{\tilde{\delta}h(\theta)} = \Delta[\mathrm{div}\{(\nabla\sigma)(\nabla h(\theta))\}],$$

since we have

$$\frac{d}{d\epsilon}\Sigma(h+\epsilon g)\bigg|_{\epsilon=0} = \left(\frac{\tilde{\delta}\Sigma}{\tilde{\delta}h}, g\right)_{H^{-1}},$$

where $(f,g)_{H^{-1}} = ((-\Delta)^{-1}f, g)_{L^2}$. In terms of the functional derivative in this sense, the PDE (1.8) can be written as

$$\frac{\partial h}{\partial t} = -\frac{\tilde{\delta}\Sigma}{\tilde{\delta}h(\theta)}.$$

(ii) The microscopic dynamics determined by the SDE (1.4) preserve the total volume so that the macroscopic equation (1.8) also has the same property. In particular, as the time t goes to ∞, the solution $h(t)$ of the PDE (1.8) tends to the minimizer of the total surface tension $\Sigma(h)$ in the class of all h's satisfying the condition $\int_{\mathbb{T}^d} h(\theta)\,d\theta = c(= \int_{\mathbb{T}^d} h_0(\theta)\,d\theta)$. Under the periodic boundary conditions, however, the minimizer h is simply a constant function. Moreover, although the microscopic dynamics studied by [5] have no conservation law, the corresponding macroscopic PDE preserves the total volume if one imposes the periodic boundary conditions. To observe the apparent differences between these two PDEs, it is required to discuss the problems in a bounded domain in $\mathbb{R}^d$ imposing proper boundary conditions. The corresponding static problem was studied by [4], which gave the mathematical foundation to the derivation of the Wulff shapes from the $\nabla\phi$-interface model.

(iii) [2] investigated the problem of the hydrodynamic limit and derived the nonlinear fourth order PDE as the macroscopic equation. In the case of $d = 1$ the microscopic dynamics studied by that paper are the same as the gradient fields associated with ϕ_t on Γ_N determined by SDE (1.4). However, if $d \geq 2$, our model is quite different from [2]. Indeed, the Gibbs measures of their model are product measures, while the Gibbs measures of our model have long-range correlations.

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References

[1] Adams, R. A., *Sobolev spaces*, Academic Press, 1975.

[2] Bertini, L., Landim, C. and S. Olla, Derivation of Cahn-Hilliard equations from Ginzburg-Landau models, J. Stat. Phys. **88** (1997), 365–381.

[3] Chang, C. C. and H.-T. Yau, Fluctuations of one dimensional Ginzburg-Landau models in nonequilibrium, Commun. Math. Phys. **145** (1992), 209–239.

[4] Deuschel, J.-D., Giacomin, G. and D. Ioffe, Large deviations and concentration properties for $\nabla\varphi$ interface models, Probab. Theory Relat. Fields **117** (2000), 49–111.

[5] Funaki, T. and H. Spohn, Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model, Commun. Math. Phys. **185** (1997), 1–36.

[6] Guo, M. Z., Papanicolaou, G. C. and S. R. S. Varadhan, Nonlinear diffusion limit for a system with nearest neighbor interactions, Commun. Math. Phys. **118** (1988), 31–59.

[7] Hohenberg, P. C. and B. I. Halperin, Theory of dynamic critical phenomena, Rev. Mod. Phys. **49** (1977), 435–475.

[8] Lu, S., Hydrodynamic scaling limits with deterministic initial configurations, Ann. Probab. **23** (1995), 1831–1852.

[9] Sakagawa, H., The reversible measures of interacting diffusion system with plural conservation laws, Markov Proc. Relat. Fields **7** (2001), 289–300.

[10] Spohn, H., Interface motion in models with stochastic dynamics, J. Stat. Phys. **71** (1993), 1081–1132.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, JAPAN
E-mail: nisikawa@ms.u-tokyo.ac.jp