

Inverse Problems Related with Non-symmetric Operators and
Inverse Problem for One-dimensional Fractional Partial Differential
Equation

(非対称作用素に関する逆問題と1次元非整数階偏微分方程式
に関する逆問題)

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January 6, 2009

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Chapter 1

Inverse Scattering Problem for Non-symmetrical Operator

1.1 Introduction and the main result

We consider the following equation :

$$A_P u = B \frac{du}{dx} + P(x)u = \lambda u \quad x \in \mathbf{R}, \quad (1.1.1)$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{pmatrix}.$$

Here we assume that $P_{jk} \in C_0^1(\mathbf{R})$. Furthermore, the operator A_P in $\{L^2(\mathbf{R})\}$ is defined by

$$\begin{cases} D(A_P) = \{H_1(\mathbf{R})\}^2, \\ A_P u = B \frac{du}{dx} + P(x)u. \end{cases} \quad (1.1.2)$$

For $\lambda \in \mathbf{R}$ with $\text{Re } \lambda = 0$, let us define $\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda), \psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)$ as the solutions to (1.1.1)

which satisfies

$$\varphi^{(+)}(x, \lambda) \sim \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad x \rightarrow \infty, \quad (1.1.3)$$

$$\varphi^{(-)}(x, \lambda) \sim \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}, \quad x \rightarrow \infty, \quad (1.1.4)$$

$$\psi^{(+)}(x, \lambda) \sim \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad x \rightarrow -\infty, \quad (1.1.5)$$

$$\psi^{(-)}(x, \lambda) \sim \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}, \quad x \rightarrow -\infty. \quad (1.1.6)$$

We call these solutions the Jost solutions.

Then, we can prove that there exists unique coefficient $\alpha^{(j)}(\lambda), \beta^{(j)}(\lambda)$, $j = 1, 2, 3, 4$ such that

$$\varphi^{(+)}(x, \lambda) = \alpha^{(1)}(\lambda)\psi^{(+)}(x, \lambda) + \beta^{(1)}(\lambda)\psi^{(-)}(x, \lambda), \quad (1.1.7)$$

$$\varphi^{(-)}(x, \lambda) = \alpha^{(2)}(\lambda)\psi^{(+)}(x, \lambda) + \beta^{(2)}(\lambda)\psi^{(-)}(x, \lambda), \quad (1.1.8)$$

$$\psi^{(+)}(x, \lambda) = \alpha^{(3)}(\lambda)\varphi^{(+)}(x, \lambda) + \beta^{(3)}(\lambda)\varphi^{(-)}(x, \lambda), \quad (1.1.9)$$

$$\psi^{(-)}(x, \lambda) = \alpha^{(4)}(\lambda)\varphi^{(+)}(x, \lambda) + \beta^{(4)}(\lambda)\varphi^{(-)}(x, \lambda), \quad (1.1.10)$$

(See Lemma 1.2.3 below). Here we call these coefficients the scattering coefficients.

Now we will consider

Inverse scattering problem.

Determine the coefficient matrix $P(x)$ from $\alpha^{(j)}(\lambda), \beta^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$).

In this paper, we will establish the uniqueness in the inverse problem.

As for works concerning inverse spectral problems for the nonsymmetric operator (1.1.2) in a finite interval, see Ning [38], Ning and Yamamoto [39], Trooshin and Yamamoto [52],[53], and Yamamoto [55]. In Trooshin and Yamamoto [52], it is proved also that the spectrum of A_P consists only of eigenvalues and that the set of generalized eigenfunctions forms a basis. In these papers, the uniqueness and the reconstruction results are proved based on the transformation formula. In [53], the inverse problem for the hyperbolic equation $\frac{\partial}{\partial t}u(t, x) = A_P u(t, x)$ are discussed.

To my best knowledge, there were no works for the scattering problem for a nonsymmetric operator, and our uniqueness result for the inverse problem with the nonsymmetric operator is the first result. The results

for direct scattering problem for the nonsymmetric operator (Theorem 1.4.3, Theorem 1.5.6) is also the first result and may have independent interest. On the other hand, there are many results on the scattering problem for the Schrödinger operator and readers can consult Chadan and Sabatier [8], Deift and Trubowitz [9], Faddeev [11],[12], Agranovich and Marchenko [2], and Marchenko [31] as monographs.

We denote the scattering coefficients for the coefficient matrix $P(x)$ by $\alpha_P^{(j)}(\lambda)$, $\beta_P^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) respectively.

In general, the uniqueness does not hold, as the following example shows.

Example

Let

$$P(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & q(x) \\ q(x) & 0 \end{pmatrix}.$$

Here, $q \in C_0^1(\mathbf{R})$ satisfy $\int_{-\infty}^{\infty} q(s)ds = 0$. Now we denote the Jost solutions to $B \frac{du}{dx} + P(x)u = \lambda u$ by $\varphi_P^{(\pm)}, \psi_P^{(\pm)}$ and denote the Jost solutions to $B \frac{du}{dx} + Q(x)u = \lambda u$ by $\varphi_Q^{(\pm)}, \psi_Q^{(\pm)}$. Furthermore, we denote the scattering coefficients for coefficient matrix $P(x)$ as $\alpha_P^{(j)}(\lambda), \beta_P^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) and we denote the scattering coefficients for coefficient matrix $Q(x)$ as $\alpha_Q^{(j)}(\lambda), \beta_Q^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$).

Then we obtain

$$\begin{aligned} \varphi_P^{(+)}(x, \lambda) &= \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad \varphi_P^{(-)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}, \\ \psi_P^{(+)}(x, \lambda) &= \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad \psi_P^{(-)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} \\ \varphi_Q^{(+)}(x, \lambda) &= \exp\left(\int_x^{\infty} q(s)ds\right) \times \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad \varphi_Q^{(-)}(x, \lambda) = \exp\left(\int_x^{\infty} q(s)ds\right) \times \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}, \\ \psi_Q^{(+)}(x, \lambda) &= \exp\left(-\int_{-\infty}^x q(s)ds\right) \times \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad \psi_Q^{(-)}(x, \lambda) = \exp\left(-\int_{-\infty}^x q(s)ds\right) \times \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}. \end{aligned}$$

Therefore we have

$$\alpha_P^{(1)}(\lambda) = \alpha_Q^{(1)}(\lambda) = 1, \quad \beta_P^{(1)}(\lambda) = \beta_Q^{(1)}(\lambda) = 0,$$

$$\alpha_P^{(2)}(\lambda) = \alpha_Q^{(2)}(\lambda) = 0, \quad \beta_P^{(2)}(\lambda) = \beta_Q^{(2)}(\lambda) = 1,$$

$$\alpha_P^{(3)}(\lambda) = \alpha_Q^{(3)}(\lambda) = 1, \quad \beta_P^{(3)}(\lambda) = \beta_Q^{(3)}(\lambda) = 0,$$

$$\alpha_P^{(4)}(\lambda) = \alpha_Q^{(4)}(\lambda) = 0, \quad \beta_P^{(4)}(\lambda) = \beta_Q^{(4)}(\lambda) = 1.$$

Consequently the uniqueness does not hold. \square

Our main result is the following theorem.

Theorem 1.1.1 *We assume that the coefficient matrices $P(x)$ and $Q(x)$ are in $C_0^1(\mathbf{R})$. Then, for all λ with*

$\operatorname{Re} \lambda = 0$,

$$\alpha_P^{(j)}(\lambda) = \alpha_Q^{(j)}(\lambda), \quad \beta_P^{(j)}(\lambda) = \beta_Q^{(j)}(\lambda), \quad j = 1, 2, 3, 4$$

if and only if the following equations hold :

$$\begin{aligned}
& (Q_{11}(x) + Q_{12}(x) - Q_{21}(x) - Q_{22}(x) - P_{11}(x) + P_{12}(x) - P_{21}(x) + P_{22}(x)) \\
& + \exp\left(\int_{-\infty}^x (Q_{11}(s) + Q_{22}(s) - P_{11}(s) - P_{22}(s))ds\right) \\
& \times (Q_{11}(x) - Q_{12}(x) + Q_{21}(x) - Q_{22}(x) - P_{11}(x) - P_{12}(x) + P_{21}(x) + P_{22}(x)) = 0, \\
& x \in \mathbf{R}. \tag{1.1.11}
\end{aligned}$$

$$\begin{aligned}
& (Q_{11}(x) + Q_{12}(x) - Q_{21}(x) - Q_{22}(x) + P_{11}(x) - P_{12}(x) + P_{21}(x) - P_{22}(x)) \\
& + \exp\left(\int_{-\infty}^x (Q_{11}(s) + Q_{22}(s) - P_{11}(s) - P_{22}(s))ds\right) \\
& \times (-Q_{11}(x) + Q_{12}(x) - Q_{21}(x) + Q_{22}(x) - P_{11}(x) - P_{12}(x) + P_{21}(x) + P_{22}(x)) = 0, \\
& x \in \mathbf{R}. \tag{1.1.12}
\end{aligned}$$

$$\int_{-\infty}^{\infty} (Q_{12}(s) + Q_{21}(s) - P_{12}(s) - P_{21}(s))ds = 0, \tag{1.1.13}$$

$$\int_{-\infty}^{\infty} (Q_{11}(s) + Q_{22}(s) - P_{11}(s) - P_{22}(s))ds = 0. \tag{1.1.14}$$

The theorem gives the uniqueness for some components. For example, the following corollary holds.

Corollary 1.1.2 *If we restrict a class of coefficient matrices to matrices in the form*

$$\left\{ \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}; a, b \in C_0^1(\mathbf{R}) \right\},$$

then the solution to the inverse problem is unique.

1.2 Jost solution

In this section, we investigate the properties of the Jost solution, that is, the solution to (1.1.1) with (1.1.3) – (1.1.6). Here and henceforce, for \mathbf{R}^2 -valued function φ , by φ_j we denote the j -th component :

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$$

and $|\varphi(x)| = \max_{j=1,2} |\varphi_j(x)|$. Using variation of constants method, we can directly prove the following integral equations.

Lemma 1.2.1

$$\varphi^{(+)}(x, \lambda) = \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} + \int_x^\infty \begin{pmatrix} \cosh(-\lambda)(t-x) & \sinh(-\lambda)(t-x) \\ \sinh(-\lambda)(t-x) & \cosh(-\lambda)(t-x) \end{pmatrix} BP(t)\varphi^{(+)}(t, \lambda)dt, \quad (1.2.1)$$

$$\varphi^{(-)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} + \int_x^\infty \begin{pmatrix} \cosh(-\lambda)(t-x) & \sinh(-\lambda)(t-x) \\ \sinh(-\lambda)(t-x) & \cosh(-\lambda)(t-x) \end{pmatrix} BP(t)\varphi^{(-)}(t, \lambda)dt \quad (1.2.2)$$

$$\psi^{(+)}(x, \lambda) = \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} - \int_{-\infty}^x \begin{pmatrix} \cosh \lambda(x-t) & \sinh \lambda(x-t) \\ \sinh \lambda(x-t) & \cosh \lambda(x-t) \end{pmatrix} BP(t)\psi^{(+)}(t, \lambda)dt, \quad (1.2.3)$$

$$\psi^{(-)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} - \int_{-\infty}^x \begin{pmatrix} \cosh \lambda(x-t) & \sinh \lambda(x-t) \\ \sinh \lambda(x-t) & \cosh \lambda(x-t) \end{pmatrix} BP(t)\psi^{(-)}(t, \lambda)dt. \quad (1.2.4)$$

Lemma 1.2.2 *For $\operatorname{Re} \lambda \geq 0$, each of (1.2.2) and (1.2.3) possesses a unique solution. Moreover, there exists a constant $C > 0$ which is independent of x and λ , such that*

$$|\varphi^{(-)}(x, \lambda)| \leq Ce^{-\operatorname{Re} \lambda x}, \quad x \in \mathbf{R}, \quad (1.2.5)$$

$$|\psi^{(+)}(x, \lambda)| \leq Ce^{\operatorname{Re} \lambda x}, \quad x \in \mathbf{R}. \quad (1.2.6)$$

Similarly, for $\operatorname{Re} \lambda \leq 0$, each of (1.2.1) and (1.2.4) possesses a unique solution and

$$|\varphi^{(+)}(x, \lambda)| \leq Ce^{\operatorname{Re} \lambda x}, \quad x \in \mathbf{R}, \quad (1.2.7)$$

$$|\psi^{(-)}(x, \lambda)| \leq Ce^{-\operatorname{Re} \lambda x}, \quad x \in \mathbf{R}. \quad (1.2.8)$$

Furthermore, $\frac{d}{d\lambda}\varphi^{(-)}(x, \lambda)$ and $\frac{d}{d\lambda}\psi^{(+)}(x, \lambda)$ exist for all fixed $x \in \mathbf{R}$ and $\operatorname{Re} \lambda > 0$. Similarly, $\frac{d}{d\lambda}\varphi^{(+)}(x, \lambda)$ and $\frac{d}{d\lambda}\psi^{(-)}(x, \lambda)$ exist for all fixed $x \in \mathbf{R}$ and $\operatorname{Re} \lambda < 0$.

Proof :

We only prove the lemma only for $\varphi^{(-)}$. We use the iteration method to solve the integral equation (1.2.2).

Now we set

$$g^{(0)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix},$$

$$g^{(n)}(x, \lambda) = \int_x^\infty \begin{pmatrix} \cosh(-\lambda)(t-x) & \sinh(-\lambda)(t-x) \\ \sinh(-\lambda)(t-x) & \cosh(-\lambda)(t-x) \end{pmatrix} BP(t)g^{(n-1)}(t, \lambda)dt. \quad n = 1, 2, \dots$$

Then we have

$$|g^{(n)}(x, \lambda)| \leq \int_x^\infty \begin{pmatrix} e^{\operatorname{Re} \lambda(t-x)} & e^{\operatorname{Re} \lambda(t-x)} \\ e^{\operatorname{Re} \lambda(t-x)} & e^{\operatorname{Re} \lambda(t-x)} \end{pmatrix} \begin{pmatrix} |P_{11}(t)| + |P_{12}(t)| \\ |P_{21}(t)| + |P_{22}(t)| \end{pmatrix} |g^{(n-1)}(t, \lambda)| dt$$

$$\leq \int_x^\infty e^{\operatorname{Re} \lambda(t-x)} (|P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)|) |g^{(n-1)}(t, \lambda)| dt.$$

Putting $P^{(0)}(t) = |P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)|$, we obtain

$$|g^{(n)}(x, \lambda)|$$

$$\leq \int_x^\infty dt_1 e^{\operatorname{Re} \lambda(t_1-x)} P^{(0)}(t_1) \int_{t_1}^\infty dt_2 e^{\operatorname{Re} \lambda(t_2-t_1)} P^{(0)}(t_2)$$

$$\times \int_{t_2}^\infty dt_3 e^{\operatorname{Re} \lambda(t_3-t_2)} P^{(0)}(t_3) \dots \int_{t_{n-1}}^\infty dt_n e^{\operatorname{Re} \lambda(t_n-t_{n-1})} P^{(0)}(t_n) |g^{(0)}(t_n, \lambda)|$$

$$\leq e^{-\operatorname{Re} \lambda x} \int_x^\infty dt_1 P^{(0)}(t_1) \int_{t_1}^\infty dt_2 P^{(0)}(t_2) \dots \int_{t_{n-1}}^\infty dt_n P^{(0)}(t_n)$$

$$= e^{-\operatorname{Re} \lambda x} \frac{1}{n!} \left(\int_x^\infty P^{(0)}(t) dt \right)^n \leq e^{-\operatorname{Re} \lambda x} \frac{1}{n!} \left(\int_{-\infty}^\infty P^{(0)}(t) dt \right)^n$$

because of $|g^{(0)}(t_n, \lambda)| \leq e^{-\operatorname{Re} \lambda t_n}$ Therefore, the series

$$g(x, \lambda) = \sum_{n=0}^{\infty} g^{(n)}(x, \lambda)$$

is absolutely and uniformly convergent for any compact set of x . Furthermore we obtain the estimate

$$|g(x, \lambda)| \leq e^{-\operatorname{Re} \lambda x} \exp \left\{ \int_x^\infty P^{(0)}(t) dt \right\} \leq e^{-\operatorname{Re} \lambda x} \exp \left\{ \int_{-\infty}^\infty P^{(0)}(t) dt \right\}. \quad (1.2.9)$$

Therefore, (1.2.5) is obtained. It is clear that $g(x, \lambda)$ is a solution to (1.2.2). The uniqueness for the solution can be proved by successive approximation method. The existence of $\frac{d}{d\lambda}\varphi^{(-)}(x, \lambda)$ for $\operatorname{Re} \lambda > 0$ follows from the differential equation (1.1.1). \square

By $W[f, g]$ we denote the Wronskian :

$$W[f(x), g(x)] = f(x)g'(x) - f'(x)g(x).$$

Lemma 1.2.3 *We assume that $\operatorname{Re} \lambda = 0$. $\{\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)\}$ is linearly independent for all x . Furthermore, $\{\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)\}$ is also linearly independent for all x .*

Proof :

Calculating the Wronskian $W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]$, we have

$$W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)] \rightarrow -2 \quad x \rightarrow \infty$$

by (1.2.1), (1.2.2) and $P \in C_0^1(\mathbf{R})$. Then the Wronskian is not 0 for sufficiently large x .

Then $\{\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)\}$ is linearly independent.

We can prove the linearly independence for $\{\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)\}$ in the similar way. \square

1.3 Scattering data

Let $\text{Re } \lambda = 0$. From lemma 1.2.2, there exists $\varphi^{(\pm)}(x, \lambda), \psi^{(\pm)}(x, \lambda)$ which satisfies (1.2.1) – (1.2.4). Furthermore, from lemma 1.2.3, $\{\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)\}$ and $\{\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)\}$ are linearly independent for all x .

Therefore, we can take the scattering coefficients $\alpha^{(j)}(\lambda), \beta^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) such as

$$\varphi^{(+)}(x, \lambda) = \alpha^{(1)}(\lambda)\psi^{(+)}(x, \lambda) + \beta^{(1)}(\lambda)\psi^{(-)}(x, \lambda), \quad (1.3.1)$$

$$\varphi^{(-)}(x, \lambda) = \alpha^{(2)}(\lambda)\psi^{(+)}(x, \lambda) + \beta^{(2)}(\lambda)\psi^{(-)}(x, \lambda), \quad (1.3.2)$$

$$\psi^{(+)}(x, \lambda) = \alpha^{(3)}(\lambda)\varphi^{(+)}(x, \lambda) + \beta^{(3)}(\lambda)\varphi^{(-)}(x, \lambda), \quad (1.3.3)$$

$$\psi^{(-)}(x, \lambda) = \alpha^{(4)}(\lambda)\varphi^{(+)}(x, \lambda) + \beta^{(4)}(\lambda)\varphi^{(-)}(x, \lambda). \quad (1.3.4)$$

Note that these $\alpha^{(j)}(\lambda), \beta^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) are uniquely determined.

Substituting (1.3.3), (1.3.4) into (1.3.1), we obtain

$$\varphi^{(+)}(x, \lambda) = (\alpha^{(1)}\alpha^{(3)} + \beta^{(1)}\alpha^{(4)})\varphi^{(+)}(x, \lambda) + (\alpha^{(1)}\beta^{(3)} + \beta^{(1)}\beta^{(4)})\varphi^{(-)}(x, \lambda).$$

Therefore we have

$$\alpha^{(1)}\alpha^{(3)} + \beta^{(1)}\alpha^{(4)} = 1, \quad \alpha^{(1)}\beta^{(3)} + \beta^{(1)}\beta^{(4)} = 0.$$

Similarly, substituting (1.3.3), (1.3.4) into (1.3.2), substituting (1.3.1), (1.3.2) into (1.3.3), and substituting (1.3.1), (1.3.2) into (1.3.4), we obtain the following equations :

$$\begin{cases} \alpha^{(1)}\alpha^{(3)} + \beta^{(1)}\alpha^{(4)} = 1, & \alpha^{(1)}\beta^{(3)} + \beta^{(1)}\beta^{(4)} = 0, \\ \alpha^{(2)}\alpha^{(3)} + \beta^{(2)}\alpha^{(4)} = 0, & \alpha^{(2)}\beta^{(3)} + \beta^{(2)}\beta^{(4)} = 1, \\ \alpha^{(1)}\alpha^{(3)} + \alpha^{(2)}\beta^{(3)} = 1, & \alpha^{(3)}\beta^{(1)} + \beta^{(2)}\beta^{(3)} = 0, \\ \alpha^{(1)}\alpha^{(4)} + \alpha^{(2)}\beta^{(4)} = 0, & \alpha^{(4)}\beta^{(1)} + \beta^{(2)}\beta^{(4)} = 1. \end{cases} \quad (1.3.5)$$

Taking the Wronskian of (1.3.1) and $\psi^{(+)}(x, \lambda)$, and taking the Wronskian of (1.3.3) and $\varphi^{(+)}(x, \lambda)$, we

have

$$\begin{cases} W[\varphi^{(+)}(x, \lambda), \psi^{(+)}(x, \lambda)] = -\beta^{(1)}(\lambda)W[\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)], \\ W[\varphi^{(+)}(x, \lambda), \psi^{(+)}(x, \lambda)] = \beta^{(3)}(\lambda)W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]. \end{cases}$$

Therefore we obtain

$$\beta^{(1)}(\lambda) = -\gamma(\lambda)\beta^{(3)}(\lambda)$$

where

$$\gamma(\lambda) = \frac{W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]}{W[\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)]}.$$

Taking the Wronskian of (1.3.1) and $\psi^{(-)}(x, \lambda)$, and taking the Wronskian of (1.3.4) and $\varphi^{(+)}(x, \lambda)$, we have

$$\begin{cases} W[\varphi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)] = \alpha^{(1)}(\lambda)W[\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)], \\ W[\varphi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)] = \beta^{(4)}(\lambda)W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]. \end{cases}$$

Therefore we obtain

$$\alpha^{(1)}(\lambda) = \gamma(\lambda)\beta^{(4)}(\lambda).$$

Taking the Wronskian of (1.3.2) and $\psi^{(+)}(x, \lambda)$, and taking the Wronskian of (1.3.3) and $\varphi^{(-)}(x, \lambda)$, we have

$$\begin{cases} W[\varphi^{(-)}(x, \lambda), \psi^{(+)}(x, \lambda)] = -\beta^{(2)}(\lambda)W[\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)], \\ W[\varphi^{(-)}(x, \lambda), \psi^{(+)}(x, \lambda)] = -\alpha^{(3)}(\lambda)W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]. \end{cases}$$

Therefore we obtain

$$\beta^{(2)}(\lambda) = \gamma(\lambda)\alpha^{(3)}(\lambda).$$

Taking the Wronskian of (1.3.2) and $\psi^{(-)}(x, \lambda)$, and taking the Wronskian of (1.3.4) and $\varphi^{(-)}(x, \lambda)$, we have

$$\begin{cases} W[\varphi^{(-)}(x, \lambda), \psi^{(-)}(x, \lambda)] = \alpha^{(2)}(\lambda)W[\psi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)], \\ W[\varphi^{(-)}(x, \lambda), \psi^{(-)}(x, \lambda)] = -\alpha^{(4)}(\lambda)W[\varphi^{(+)}(x, \lambda), \varphi^{(-)}(x, \lambda)]. \end{cases}$$

Therefore we obtain

$$\alpha^{(2)}(\lambda) = -\gamma(\lambda)\alpha^{(4)}(\lambda).$$

Thus

$$\begin{cases} \beta^{(1)}(\lambda) = -\gamma(\lambda)\beta^{(3)}(\lambda), \\ \alpha^{(1)}(\lambda) = \gamma(\lambda)\beta^{(4)}(\lambda), \\ \beta^{(2)}(\lambda) = \gamma(\lambda)\alpha^{(3)}(\lambda), \\ \alpha^{(2)}(\lambda) = -\gamma(\lambda)\alpha^{(4)}(\lambda). \end{cases} \quad (1.3.6)$$

Substituting (1.3.6) into (1.3.5), we have

$$\alpha^{(3)}(\lambda)\beta^{(4)}(\lambda) - \alpha^{(4)}(\lambda)\beta^{(3)}(\lambda) = \frac{1}{\gamma(\lambda)}. \quad (1.3.7)$$

From (1.3.6) and (1.3.7), we can represent $\alpha^{(1)}, \alpha^{(4)}, \beta^{(1)}, \beta^{(4)}, \gamma$ in terms of $\alpha^{(2)}, \alpha^{(3)}, \beta^{(2)}, \beta^{(3)}$. In fact,

by direct calculation, we obtain

$$\begin{cases} \gamma(\lambda) = \frac{\beta^{(2)}(\lambda)}{\alpha^{(3)}(\lambda)}, \\ \alpha^{(1)}(\lambda) = \frac{1 - \alpha^{(2)}(\lambda)\beta^{(3)}(\lambda)}{\alpha^{(3)}(\lambda)}, \\ \alpha^{(4)}(\lambda) = -\frac{\alpha^{(2)}(\lambda)\alpha^{(3)}(\lambda)}{\beta^{(2)}(\lambda)}, \\ \beta^{(1)}(\lambda) = -\frac{\beta^{(2)}(\lambda)\beta^{(3)}(\lambda)}{\alpha^{(3)}(\lambda)}, \\ \beta^{(4)}(\lambda) = \frac{1 - \alpha^{(2)}(\lambda)\beta^{(3)}(\lambda)}{\beta^{(2)}(\lambda)}. \end{cases} \quad (1.3.8)$$

Therefore, the information of the data $\alpha^{(j)}(\lambda), \beta^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) is essentially included in the data $\alpha^{(2)}(\lambda), \alpha^{(3)}(\lambda), \beta^{(2)}(\lambda), \beta^{(3)}(\lambda)$. Therefore we can only consider about (1.3.2) and (1.3.3). Now we rewrite (1.3.2), (1.3.3) as

$$\varphi^{(-)}(x, \lambda) = \frac{R_2(\lambda)}{T_2(\lambda)}\psi^{(+)}(x, \lambda) + \frac{1}{T_2(\lambda)}\psi^{(-)}(x, \lambda), \quad (1.3.9)$$

$$\psi^{(+)}(x, \lambda) = \frac{1}{T_1(\lambda)}\varphi^{(+)}(x, \lambda) + \frac{R_1(\lambda)}{T_1(\lambda)}\varphi^{(-)}(x, \lambda). \quad (1.3.10)$$

We call $T_j(\lambda), j = 1, 2$ and $R_j(\lambda), j = 1, 2$, the transmission coefficients and the reflection coefficients.

Note that we defined these coefficients only for $\text{Re } \lambda = 0$.

For $\text{Re } \lambda = 0$, using (1.2.2), we have

$$\begin{aligned} & \varphi^{(-)}(x, \lambda) \\ &= \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} + \int_{-\infty}^{\infty} \begin{pmatrix} \cosh(-\lambda)(t-x) & \sinh(-\lambda)(t-x) \\ \sinh(-\lambda)(t-x) & \cosh(-\lambda)(t-x) \end{pmatrix} BP(t)\varphi^{(-)}(t, \lambda)dt + o(1) \quad (x \rightarrow -\infty) \\ &= \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} + \frac{1}{2}e^{\lambda x} \int_{-\infty}^{\infty} e^{-\lambda t} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} BP(t)\varphi^{(-)}(t, \lambda)dt \\ & \quad + \frac{1}{2}e^{-\lambda x} \int_{-\infty}^{\infty} e^{\lambda t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} BP(t)\varphi^{(-)}(t, \lambda)dt + o(1) \\ &= \left\{ 1 - \frac{1}{2} \int_{-\infty}^{\infty} e^{\lambda t} \begin{pmatrix} 1 & -1 \end{pmatrix} P(t)\varphi^{(-)}(t, \lambda)dt \right\} \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} \\ & \quad + \left\{ \frac{1}{2} \int_{-\infty}^{\infty} e^{-\lambda t} \begin{pmatrix} 1 & 1 \end{pmatrix} P(t)\varphi^{(-)}(t, \lambda)dt \right\} \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} + o(1). \end{aligned}$$

Comparing this equation and (1.3.9), and taking the limit $x \rightarrow -\infty$, we obtain

$$\frac{1}{T_2(\lambda)} = \beta^{(2)}(\lambda) = 1 - \frac{1}{2} \int_{-\infty}^{\infty} e^{\lambda t} \begin{pmatrix} 1 & -1 \end{pmatrix} P(t) \varphi^{(-)}(t, \lambda) dt, \quad (1.3.11)$$

$$\frac{R_2(\lambda)}{T_2(\lambda)} = \alpha^{(2)}(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\lambda t} \begin{pmatrix} 1 & 1 \end{pmatrix} P(t) \varphi^{(-)}(t, \lambda) dt. \quad (1.3.12)$$

Similarly we obtain

$$\frac{1}{T_1(\lambda)} = \alpha^{(3)}(\lambda) = 1 - \frac{1}{2} \int_{-\infty}^{\infty} e^{-\lambda t} \begin{pmatrix} 1 & 1 \end{pmatrix} P(t) \psi^{(+)}(t, \lambda) dt, \quad (1.3.13)$$

$$\frac{R_1(\lambda)}{T_1(\lambda)} = \beta^{(3)}(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\lambda t} \begin{pmatrix} 1 & -1 \end{pmatrix} P(t) \psi^{(+)}(t, \lambda) dt. \quad (1.3.14)$$

1.4 Properties of the Jost Solutions

We will deduce the asymptotic behavior of the Jost solutions. At first, we consider the case of $\operatorname{Re} \lambda \geq 0$, and

we will deduce the asymptotic behavior for $\varphi^{(-)}(x, \lambda), \psi^{(+)}(x, \lambda)$.

From (1.1.4) and (1.1.5), we set

$$m^{(-)}(x, \lambda) = e^{\lambda x} \varphi^{(-)}(x, \lambda), \quad n^{(+)}(x, \lambda) = e^{-\lambda x} \psi^{(+)}(x, \lambda).$$

Substituting $\varphi^{(-)}(x, \lambda) = e^{-\lambda x} m^{(-)}(x, \lambda)$, $\psi^{(+)}(x, \lambda) = e^{\lambda x} n^{(+)}(x, \lambda)$ into (1.1), we obtain

$$B \frac{d}{dx} m^{(-)}(x, \lambda) + P(x) m^{(-)}(x, \lambda) = \lambda \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} m^{(-)}(x, \lambda), \quad (1.4.1)$$

$$B \frac{d}{dx} n^{(+)}(x, \lambda) + P(x) n^{(+)}(x, \lambda) = \lambda \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} n^{(+)}(x, \lambda). \quad (1.4.2)$$

From (1.2.5) and (1.2.6), we also have

$$|m^{(-)}(x, \lambda)| \leq C, \quad (1.4.3)$$

$$|n^{(+)}(x, \lambda)| \leq C. \quad (1.4.4)$$

Using (1.2.2) and (1.2.3), we obtain the integral equations for $m^{(-)}, n^{(+)}$:

$$m^{(-)}(x, \lambda) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} \int_x^\infty \begin{pmatrix} e^{-2\lambda(t-x)} - 1 & e^{-2\lambda(t-x)} + 1 \\ e^{-2\lambda(t-x)} + 1 & e^{-2\lambda(t-x)} - 1 \end{pmatrix} P(t) m^{(-)}(t, \lambda) dt, \quad (1.4.5)$$

$$n^{(+)}(x, \lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \int_{-\infty}^x \begin{pmatrix} -e^{-2\lambda(x-t)} + 1 & e^{-2\lambda(x-t)} + 1 \\ e^{-2\lambda(x-t)} + 1 & -e^{-2\lambda(x-t)} + 1 \end{pmatrix} P(t) n^{(+)}(t, \lambda) dt. \quad (1.4.6)$$

Let us investigate $m^{(-)}$. Remark that $\operatorname{Re} \lambda \geq 0$. When we set

$$g^{(0)}(x, \lambda) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$g^{(k)}(x, \lambda) = \int_x^\infty \frac{1}{2} \begin{pmatrix} -1 + e^{-2\lambda(t-x)} & 1 + e^{-2\lambda(t-x)} \\ 1 + e^{-2\lambda(t-x)} & -1 + e^{-2\lambda(t-x)} \end{pmatrix} P(t) g^{(k-1)}(t, \lambda) dt,$$

we have the following estimate :

$$|g^{(k)}(x, \lambda)| \leq \int_x^\infty (|P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)|) |g^{(k-1)}(t, \lambda)| dt.$$

Setting $P^{(0)}(t) = |P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)|$, we have

$$\begin{aligned} |g^{(k)}(x, \lambda)| &\leq \int_x^\infty dt_1 P^{(0)}(t_1) \int_{t_1}^\infty dt_2 P^{(0)}(t_2) \cdots \int_{t_{n-1}}^\infty dt_n P^{(0)}(t_n) \\ &= \frac{1}{n!} \left(\int_x^\infty P^{(0)}(t) dt \right)^n. \end{aligned}$$

Then the series

$$m^{(-)}(x, \lambda) = \sum_{k=0}^{\infty} g^{(k)}(x, \lambda) \quad (1.4.7)$$

are absolutely and uniformly convergent for $x \in \mathbf{R}$. Furthermore, we have the estimation :

$$|m^{(-)}(x, \lambda)| \leq \exp \left(\int_{-\infty}^{\infty} P^{(0)}(t) dt \right)$$

and (1.4.7) is the solution to (1.4.5).

Now we will deduce the asymptotic behavior of $m^{(-)}$ from (1.4.5). We set

$$X = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then for $k \geq 1$,

$$\begin{aligned} g^{(k)}(x, \lambda) &= \int_x^\infty \left(X + Y e^{-2\lambda(t_1-x)} \right) P(t_1) g^{(k-1)}(t_1, \lambda) dt_1 \\ &= \int_x^\infty dt_1 \left(X + Y e^{-2\lambda(t_1-x)} \right) P(t_1) \int_{t_1}^\infty dt_2 \left(X + Y e^{-2\lambda(t_2-t_1)} \right) P(t_2) \times \cdots \\ &\quad \times \int_{t_{k-1}}^\infty dt_k \left(X + Y e^{-2\lambda(t_k-t_{k-1})} \right) P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \int_x^\infty dt_1 X P(t_1) \int_{t_1}^\infty dt_2 X P(t_2) \times \cdots \times \int_{t_{k-1}}^\infty dt_k X P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (\text{remainder}). \end{aligned}$$

Now we define the operator $W_{+\infty}$ in the following way :

$$\begin{cases} W_{+\infty} : \{C(\mathbf{R})\}^2 \rightarrow \{C(\mathbf{R})\}^2 \\ D(W_{+\infty}) = \{C(\mathbf{R})\}^2 \\ (W_{+\infty}u)(x) = \int_x^\infty X P(t) u(t) dt. \end{cases}$$

Then we have

$$g^{(k)}(x, \lambda) = W_{+\infty}^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (\text{remainder}).$$

Let us consider the remainder term. Putting

$$Z^{(1)}(\lambda; t, x) = X, \quad Z^{(2)}(\lambda; t, x) = Ye^{-2\lambda(t-x)},$$

and setting $\tau_i = 1$ or $2 (i = 1, \dots, k)$, the remainder term is represented as the summation of the following terms :

$$\begin{aligned} & \int_x^\infty dt_1 Z^{(\tau_1)}(\lambda; t_1, x) P(t_1) \int_{t_1}^\infty dt_2 Z^{(\tau_2)}(\lambda; t_2, t_1) P(t_2) \times \\ & \dots \times \int_{t_{k-1}}^\infty dt_k Z^{(\tau_k)}(\lambda; t_k, t_{k-1}) P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \quad (1.4.8)$$

excepting the term for $(\tau_1, \dots, \tau_k) = (1, 1, \dots, 1)$.

Let us consider the term (1.4.8) with $(\tau_1, \dots, \tau_k) \neq (1, 1, \dots, 1)$. Taking the maximum i such that $\tau_i = 2$.

We denote this maximum i as i_0 . Then (1.4.8) is written as

$$\begin{aligned} & \int_x^\infty dt_1 Z^{(\tau_1)}(\lambda; t_1, x) P(t_1) \int_{t_1}^\infty dt_2 Z^{(\tau_2)}(\lambda; t_2, t_1) P(t_2) \times \\ & \dots \times \int_{t_{i_0-2}}^\infty dt_{i_0-1} Z^{(\tau_{i_0-1})}(\lambda; t_{i_0-1}, t_{i_0-2}) P(t_{i_0-1}) \int_{t_{i_0-1}}^\infty dt_{i_0} Y e^{-2\lambda(t_{i_0}-t_{i_0-1})} P(t_{i_0}) \times \\ & \dots \times \int_{t_{i_0}}^\infty dt_{i_0+1} X P(t_{i_0+1}) \int_{t_{k-1}}^\infty dt_k X P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \quad (1.4.9)$$

For $i_0 \neq k$, by partial integration, we have

$$\begin{aligned} & \int_{t_{i_0-1}}^\infty dt_{i_0} Y e^{-2\lambda(t_{i_0}-t_{i_0-1})} P(t_{i_0}) \int_{t_{i_0}}^\infty dt_{i_0+1} X P(t_{i_0+1}) \times \dots \times \int_{t_{k-1}}^\infty dt_k X P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & = \int_{t_{i_0-1}}^\infty dt_{i_0} Y e^{-2\lambda(t_{i_0}-t_{i_0-1})} P(t_{i_0}) \int_{t_{i_0}}^\infty dt_{i_0+1} X P(t_{i_0+1}) W_{+\infty}^{k-i_0-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t_{i_0+1}) \\ & = \int_{t_{i_0-1}}^\infty dt_{i_0} \frac{1}{-2\lambda} \left(\frac{d}{dt_{i_0}} e^{-2\lambda(t_{i_0}-t_{i_0-1})} \right) Y P(t_{i_0}) \int_{t_{i_0}}^\infty dt_{i_0+1} X P(t_{i_0+1}) W_{+\infty}^{k-i_0-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t_{i_0+1}) \\ & = \frac{1}{2\lambda} Y P(t_{i_0-1}) \int_{t_{i_0-1}}^\infty dt_{i_0+1} X P(t_{i_0+1}) W_{+\infty}^{k-i_0-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t_{i_0+1}) \\ & + \frac{1}{2\lambda} \int_{t_{i_0-1}}^\infty dt_{i_0} e^{-2\lambda(t_{i_0}-t_{i_0-1})} Y P'(t_{i_0}) \int_{t_{i_0}}^\infty dt_{i_0+1} X P(t_{i_0+1}) W_{+\infty}^{k-i_0-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t_{i_0+1}) \\ & - \frac{1}{2\lambda} \int_{t_{i_0-1}}^\infty dt_{i_0} e^{-2\lambda(t_{i_0}-t_{i_0-1})} Y P(t_{i_0}) X P(t_{i_0}) W_{+\infty}^{k-i_0-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t_{i_0}). \end{aligned}$$

For $i_0 = k$, we have

$$\int_{t_{k-1}}^{\infty} dt_k Y e^{-2\lambda(t_k - t_{k-1})} P(t_k) = \frac{1}{2\lambda} Y P(t_{k-1}) + \frac{1}{2\lambda} e^{-2\lambda(t_k - t_{k-1})} Y P'(t_k) dt_k.$$

Therefore, from (1.4.9), the absolute value of each component of remainder term is smaller than

$$C \cdot (2^k - 1) \cdot \left[\frac{1}{|2\lambda|} \frac{1}{(k-1)!} \left(\int_x^{\infty} P^{(1)}(t) dt \right)^{k-1} + \frac{1}{|2\lambda|} \frac{1}{k!} \left(\int_x^{\infty} P^{(1)}(t) dt \right)^k + \frac{1}{|2\lambda|} \frac{1}{(k-1)!} \left(\int_x^{\infty} P^{(1)}(t) dt \right)^{k-1} \right] \quad (1.4.10)$$

where

$$\begin{aligned} P^{(1)}(t) &= |P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)| \\ &+ |P'_{11}(t)| + |P'_{12}(t)| + |P'_{21}(t)| + |P'_{22}(t)| \\ &+ (|P_{11}(t)| + |P_{12}(t)| + |P_{21}(t)| + |P_{22}(t)|)^2 \end{aligned}$$

and C is a constant which depends on P . From (1.4.10), the remainder term is bounded by

$$C' \frac{1}{|\lambda|} \left\{ \frac{1}{(k-1)!} \left(\int_x^{\infty} 2P^{(1)}(t) dt \right)^{k-1} + \frac{1}{k!} \left(\int_x^{\infty} 2P^{(1)}(t) dt \right)^k \right\}$$

where C' is the positive constant.

Therefore, we obtain

$$m^{(-)}(x, \lambda) = \sum_{k=0}^{\infty} g^{(k)}(x, \lambda) = \sum_{k=0}^{\infty} W_{+\infty}^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} (x) + O\left(\frac{1}{|\lambda|}\right), \quad |\lambda| \rightarrow \infty. \quad (1.4.11)$$

Similarly we obtain

$$n^{(+)}(x, \lambda) = \sum_{k=0}^{\infty} (-V_{-\infty})^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} (x) + O\left(\frac{1}{|\lambda|}\right), \quad |\lambda| \rightarrow \infty, \quad (1.4.12)$$

where

$$\begin{cases} V_{-\infty} : \{C(\mathbf{R})\}^2 \rightarrow \{C(\mathbf{R})\}^2, \\ D(V_{-\infty}) = \{C(\mathbf{R})\}^2, \\ V_{-\infty} u(x) = \int_{-\infty}^x Y P(t) u(t) dt. \end{cases}$$

Next, we consider the case of $\operatorname{Re} \lambda \leq 0$. We will deduce the asymptotic behavior of $\varphi^{(+)}(x, \lambda), \psi^{(-)}(x, \lambda)$.

We can use the same idea for the case of $\operatorname{Re} \lambda \geq 0$. Setting

$$m^{(+)}(x, \lambda) = e^{-\lambda x} \varphi^{(+)}(x, \lambda), \quad n^{(-)}(x, \lambda) = e^{\lambda x} \psi^{(-)}(x, \lambda),$$

we have

$$B \frac{d}{dx} m^{(+)}(x, \lambda) + P(x) m^{(+)}(x, \lambda) = \lambda \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} m^{(+)}(x, \lambda), \quad (1.4.13)$$

$$B \frac{d}{dx} n^{(-)}(x, \lambda) + P(x) n^{(-)}(x, \lambda) = \lambda \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} n^{(-)}(x, \lambda), \quad (1.4.14)$$

$$m^{(+)}(x, \lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 + o(1)) \quad x \rightarrow \infty, \quad (1.4.15)$$

$$n^{(-)}(x, \lambda) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1 + o(1)) \quad x \rightarrow -\infty. \quad (1.4.16)$$

We can deduce the integral equation for $m^{(+)}, n^{(-)}$:

$$m^{(+)}(x, \lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \int_x^\infty \begin{pmatrix} -e^{2\lambda(t-x)} + 1 & e^{2\lambda(t-x)} + 1 \\ e^{2\lambda(t-x)} + 1 & -e^{2\lambda(t-x)} + 1 \end{pmatrix} P(t) m^{(+)}(t, \lambda) dt, \quad (1.4.17)$$

$$n^{(-)}(x, \lambda) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{2} \int_{-\infty}^x \begin{pmatrix} e^{2\lambda(x-t)} - 1 & e^{2\lambda(x-t)} + 1 \\ e^{2\lambda(x-t)} + 1 & e^{2\lambda(x-t)} - 1 \end{pmatrix} P(t) n^{(-)}(t, \lambda) dt. \quad (1.4.18)$$

The asymptotic behavior of $m^{(+)}, n^{(-)}$ is given by

$$m^{(+)}(x, \lambda) = \sum_{k=0}^{\infty} V_{+\infty}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} (x) + O\left(\frac{1}{|\lambda|}\right), \quad (1.4.19)$$

$$n^{(-)}(x, \lambda) = \sum_{k=0}^{\infty} (-W_{-\infty})^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} (x) + O\left(\frac{1}{|\lambda|}\right), \quad (1.4.20)$$

where

$$\begin{cases} V_{+\infty} : \{C(\mathbf{R})\}^2 \rightarrow \{C(\mathbf{R})\}^2, \\ D(V_{+\infty}) = \{C(\mathbf{R})\}^2, \\ V_{+\infty} u(x) = \int_x^\infty YP(t)u(t)dt, \\ \\ W_{-\infty} : \{C(\mathbf{R})\}^2 \rightarrow \{C(\mathbf{R})\}^2, \\ D(W_{-\infty}) = \{C(\mathbf{R})\}^2, \\ W_{-\infty} u(x) = \int_{-\infty}^x XP(t)u(t)dt. \end{cases}$$

Now we calculate $\sum_{k=0}^{\infty} W_{+\infty}^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} (x)$. By direct calculation, we obtain

$$\begin{aligned}
& \sum_{k=0}^{\infty} W_{+\infty}^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} (x) \\
&= \sum_{k=0}^{\infty} \int_x^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{k-1}}^{\infty} dt_k X P(t_1) X P(t_2) \cdots X P(t_k) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \int_x^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{k-1}}^{\infty} dt_k \begin{pmatrix} -P_{11}(t_1) + P_{21}(t_1) & -P_{12}(t_1) + P_{22}(t_1) \\ P_{11}(t_1) - P_{21}(t_1) & P_{12}(t_1) - P_{22}(t_1) \end{pmatrix} \times \cdots \\
&\quad \cdots \times \begin{pmatrix} -P_{11}(t_k) + P_{21}(t_k) & -P_{12}(t_k) + P_{22}(t_k) \\ P_{11}(t_k) - P_{21}(t_k) & P_{12}(t_k) - P_{22}(t_k) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \int_x^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{k-1}}^{\infty} dt_k \begin{pmatrix} -P_{11}(t_1) + P_{21}(t_1) & -P_{12}(t_1) + P_{22}(t_1) \\ P_{11}(t_1) - P_{21}(t_1) & P_{12}(t_1) - P_{22}(t_1) \end{pmatrix} \times \cdots \\
&\quad \cdots \times \begin{pmatrix} -P_{11}(t_{k-1}) + P_{21}(t_{k-1}) & -P_{12}(t_{k-1}) + P_{22}(t_{k-1}) \\ P_{11}(t_{k-1}) - P_{21}(t_{k-1}) & P_{12}(t_{k-1}) - P_{22}(t_{k-1}) \end{pmatrix} \\
&\quad \times (-P_{11}(t_k) + P_{12}(t_k) + P_{21}(t_k) - P_{22}(t_k)) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \int_x^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{k-1}}^{\infty} dt_k (-P_{11}(t_1) + P_{12}(t_1) + P_{21}(t_1) - P_{22}(t_1)) \times \cdots \\
&\quad \cdots \times (-P_{11}(t_k) + P_{12}(t_k) + P_{21}(t_k) - P_{22}(t_k)) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \exp \left\{ \int_x^{\infty} \frac{(-P_{11}(t) + P_{12}(t) + P_{21}(t) - P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

Similarly, we have

$$\sum_{k=0}^{\infty} (-V_{-\infty}^k) \begin{pmatrix} 1 \\ 1 \end{pmatrix} (x) = \exp \left\{ \int_{-\infty}^x \frac{(-P_{11}(t) - P_{12}(t) - P_{21}(t) - P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, we obtain the following lemma.

Lemma 1.4.1 For $\operatorname{Re} \lambda \geq 0$, we have

$$m^{(-)}(x, \lambda) = \exp \left\{ \int_x^{\infty} \frac{(-P_{11}(t) + P_{12}(t) + P_{21}(t) - P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right), \quad (1.4.21)$$

$$n^{(+)}(x, \lambda) = \exp \left\{ \int_{-\infty}^x \frac{(-P_{11}(t) - P_{12}(t) - P_{21}(t) - P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right). \quad (1.4.22)$$

For $\operatorname{Re} \lambda \leq 0$, we can obtain the similar results.

Lemma 1.4.2 For $\text{Re } \lambda \leq 0$, we have

$$m^{(+)}(x, \lambda) = \exp \left\{ \int_x^\infty \frac{(P_{11}(t) + P_{12}(t) + P_{21}(t) + P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right), \quad (1.4.23)$$

$$n^{(-)}(x, \lambda) = \exp \left\{ \int_{-\infty}^x \frac{(P_{11}(t) - P_{12}(t) - P_{21}(t) + P_{22}(t))}{2} dt \right\} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O\left(\frac{1}{|\lambda|}\right). \quad (1.4.24)$$

From (1.4.21) and (1.3.11), we have

$$\begin{aligned} \beta^{(2)}(\lambda) &= \frac{1}{T_2(\lambda)} = 1 - \frac{1}{2} \int_{-\infty}^\infty e^{\lambda t} \begin{pmatrix} 1 & -1 \end{pmatrix} P(t) \varphi^{(-)}(t, \lambda) dt \\ &= 1 - \frac{1}{2} \int_{-\infty}^\infty \begin{pmatrix} 1 & -1 \end{pmatrix} P(t) m^{(-)}(t, \lambda) dt \\ &= 1 - \frac{1}{2} \int_{-\infty}^\infty \begin{pmatrix} 1 & -1 \end{pmatrix} P(t) \exp \left\{ \int_t^\infty \frac{-P_{11}(t_1) + P_{12}(t_1) + P_{21}(t_1) - P_{22}(t_1)}{2} dt_1 \right\} \begin{pmatrix} 1 \\ -1 \end{pmatrix} dt \\ &\quad + O\left(\frac{1}{|\lambda|}\right) \end{aligned}$$

Therefore, we have

$$\beta^{(2)}(\lambda) = \frac{1}{T_2(\lambda)} = \exp \left\{ \int_{-\infty}^\infty \frac{-P_{11}(t) + P_{12}(t) + P_{21}(t) - P_{22}(t)}{2} dt \right\} + O\left(\frac{1}{|\lambda|}\right).$$

Similarly, we have

$$\alpha^{(3)}(\lambda) = \frac{1}{T_1(\lambda)} = \exp \left\{ \int_{-\infty}^\infty \frac{-P_{11}(t) - P_{12}(t) - P_{21}(t) - P_{22}(t)}{2} dt \right\} + O\left(\frac{1}{|\lambda|}\right).$$

From (1.3.12) and (1.3.14), we can see

$$\begin{aligned} \alpha^{(2)}(\lambda) &= \frac{R_2(\lambda)}{T_2(\lambda)} = O\left(\frac{1}{|\lambda|}\right), \\ \beta^{(3)}(\lambda) &= \frac{R_1(\lambda)}{T_1(\lambda)} = O\left(\frac{1}{|\lambda|}\right) \end{aligned}$$

by partial integration. From (1.3.8), we have

$$\begin{aligned} \alpha^{(1)}(\lambda) &= \exp \left\{ \int_{-\infty}^\infty \frac{P_{11}(t) + P_{12}(t) + P_{21}(t) + P_{22}(t)}{2} dt \right\} + O\left(\frac{1}{|\lambda|}\right), \\ \alpha^{(4)}(\lambda) &= O\left(\frac{1}{|\lambda|}\right), \\ \beta^{(1)}(\lambda) &= O\left(\frac{1}{|\lambda|}\right), \\ \beta^{(4)}(\lambda) &= \exp \left\{ \int_{-\infty}^\infty \frac{P_{11}(t) - P_{12}(t) - P_{21}(t) + P_{22}(t)}{2} dt \right\} + O\left(\frac{1}{|\lambda|}\right). \end{aligned}$$

Therefore we have obtained the following.

Theorem 1.4.3 *The following asymptotics hold :*

$$\alpha^{(1)}(\lambda) = \exp \left\{ \int_{-\infty}^{\infty} \frac{P_{11}(t) + P_{12}(t) + P_{21}(t) + P_{22}(t)}{2} dt \right\} + O \left(\frac{1}{|\lambda|} \right), \quad (1.4.25)$$

$$\alpha^{(2)}(\lambda) = \frac{R_2(\lambda)}{T_2(\lambda)} = O \left(\frac{1}{|\lambda|} \right), \quad (1.4.26)$$

$$\alpha^{(3)}(\lambda) = \frac{1}{T_1(\lambda)} = \exp \left\{ \int_{-\infty}^{\infty} \frac{-P_{11}(t) - P_{12}(t) - P_{21}(t) - P_{22}(t)}{2} dt \right\} + O \left(\frac{1}{|\lambda|} \right), \quad (1.4.27)$$

$$\alpha^{(4)}(\lambda) = O \left(\frac{1}{|\lambda|} \right), \quad (1.4.28)$$

$$\beta^{(1)}(\lambda) = O \left(\frac{1}{|\lambda|} \right), \quad (1.4.29)$$

$$\beta^{(2)}(\lambda) = \frac{1}{T_2(\lambda)} = \exp \left\{ \int_{-\infty}^{\infty} \frac{-P_{11}(t) + P_{12}(t) + P_{21}(t) - P_{22}(t)}{2} dt \right\} + O \left(\frac{1}{|\lambda|} \right), \quad (1.4.30)$$

$$\beta^{(3)}(\lambda) = \frac{R_1(\lambda)}{T_1(\lambda)} = O \left(\frac{1}{|\lambda|} \right), \quad (1.4.31)$$

$$\beta^{(4)}(\lambda) = \exp \left\{ \int_{-\infty}^{\infty} \frac{P_{11}(t) - P_{12}(t) - P_{21}(t) + P_{22}(t)}{2} dt \right\} + O \left(\frac{1}{|\lambda|} \right). \quad (1.4.32)$$

1.5 The spectrum of A_P

Let us define $A_P : \{L^2(\mathbf{R})\}^2 \rightarrow \{L^2(\mathbf{R})\}^2$ as

$$\begin{cases} A_P u = B \frac{d}{dx} u + P(x)u, \\ D(A_P) = \{H^1(\mathbf{R})\}^2. \end{cases} \quad (1.5.1)$$

Now we prove that the operator A_P does not have eigenvalues. Remember that $P(x) \in \{C_0^1(\mathbf{R})\}^4$. We assume that the support of $P(x)$ is included in $(-b, b)$. ($b > 0$).

Let us consider

$$A_P u = \lambda u. \quad (1.5.2)$$

At first, we consider the case of $\text{Re } \lambda > 0$. Since $P(x) = 0$ for $x < -b, b < x$, the solution to (1.4.2) is written as the linear combination of $\begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}$, $\begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}$ for $x < -b, b < x$. Therefore, the non-zero solution to (1.5.2) which is included in $\{L^2(\mathbf{R})\}$ satisfy

$$u(x, \lambda) = \begin{cases} C_1 \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix} & x > b, \\ C_2 \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} & x < -b \end{cases} \quad (1.5.3)$$

where C_1 and C_2 are non-zero constants. Conversely, the functions written in the above form are included in $\{L^2(\mathbf{R})\}$.

We can connect (1.5.3) to $-b < x < b$, if and only if

$$\begin{cases} A_P u(x, \lambda) = \lambda u(x, \lambda) \\ u(-b, \lambda) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \\ u(b, \lambda) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \end{cases} \quad (1.5.4)$$

has non-zero solution. Let us set $A_{P,b}^{(+)} : \{L^2(-b, b)\}^2 \rightarrow \{L^2(-b, b)\}^2$ as

$$\begin{cases} D(A_{P,b}^{(+)}) = \left\{ u \in \{H^1(-b, b)\}^2 ; u(-b) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0, u(b) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \right\}, \\ A_{P,b}^{(+)} u = B \frac{d}{dx} u + P(x)u \quad (-b < x < b). \end{cases} \quad (1.5.5)$$

Then we easily obtain the following lemma.

Lemma 1.5.1 *The eigenvalue λ for A_P with $\operatorname{Re} \lambda > 0$ is the eigenvalue for $A_{P,b}^{(+)}$. Conversely, the eigenvalue for $A_{P,b}^{(+)}$ with $\operatorname{Re} \lambda > 0$ is the eigenvalue for A_P .*

Similarly, for $\operatorname{Re} \lambda < 0$, we define $A_{P,b}^{(-)} : \{L^2(-b, b)\}^2 \rightarrow \{L^2(-b, b)\}^2$ as

$$\left\{ \begin{array}{l} D(A_{P,b}^{(-)}) = \left\{ u \in \{H^1(-b, b)\}^2 ; u(-b) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0, \quad u(b) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \right\}, \\ A_{P,b}^{(-)} u = B \frac{d}{dx} u + P(x)u \quad (-b < x < b). \end{array} \right. \quad (1.5.6)$$

Then we easily have the following lemma.

Lemma 1.5.2 *The eigenvalue λ for A_P with $\operatorname{Re} \lambda < 0$ is the eigenvalue for $A_{P,b}^{(-)}$. Conversely, the eigenvalue for $A_{P,b}^{(-)}$ with $\operatorname{Re} \lambda < 0$ is the eigenvalue for A_P .*

It is known that $A_{P,b}^{(\pm)}$ does not have eigenvalues. Therefore, by lemma 1.5.1 and lemma 1.5.2, A_P does not have eigenvalues in $\operatorname{Re} \lambda \neq 0$.

For $\operatorname{Re} \lambda = 0$, λ is not eigenvalue, too. In fact, since $P = 0$ for $x > b$, the solution to $A_P u = \lambda u$ is written in the following form :

$$\left\{ \begin{array}{l} u(x, \lambda) = C_1 \begin{pmatrix} \cosh \lambda x \\ \sinh \lambda x \end{pmatrix} + C_2 \begin{pmatrix} \sinh \lambda x \\ \cosh \lambda x \end{pmatrix}, \quad \operatorname{Re} \lambda = 0, \lambda \neq 0, \\ u(x, \lambda) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad \lambda = 0. \end{array} \right.$$

This is included in $\{L^2(\mathbf{R})\}^2$ only for $C_1 = C_2 = 0$. Therefore, the solution to $A_P u = \lambda u$ is only 0. Then λ is not eigenvalue for $\operatorname{Re} \lambda = 0$.

Proposition 1.5.3 *A_P does not have eigenvalues.*

Now we can prove that $\alpha^{(2)}(\lambda) \neq 0, \beta^{(3)}(\lambda) \neq 0$ for all $\lambda \in \mathbf{C}$. If not, there exists λ such that $\varphi^{(-)}(x, \lambda)$ and $\psi^{(+)}(x, \lambda)$ are linearly dependent. Thus this λ is the eigenvalue for $A_{P,b}^{(+)}$. This is contradiction.

Lemma 1.5.4 *For all $\lambda \in \mathbf{C}$, we have*

$$\alpha^{(2)}(\lambda) \neq 0, \quad \beta^{(3)}(\lambda) \neq 0.$$

Now we prove that λ for $\operatorname{Re} \lambda \neq 0$ is included in the resolvent of A_P . Let us consider

$$B \frac{d}{dx} u(x, \lambda) + P(x)u(x, \lambda) - \lambda u(x, \lambda) = g(x) \quad x \in \mathbf{R} \quad (1.5.7)$$

where $g \in \{C_0^\infty(\mathbf{R})\}^2$.

Let us solve (1.5.7) by the method of variation of constants. At first, we consider the case of $\operatorname{Re} \lambda > 0$.

We set $u(x, \lambda)$ as

$$u(x, \lambda) = C_1(x, \lambda)\varphi^{(-)}(x, \lambda) + C_2(x, \lambda)\psi^{(+)}(x, \lambda) = \begin{pmatrix} \varphi^{(-)}(x, \lambda) & \psi^{(+)}(x, \lambda) \end{pmatrix} \begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix}. \quad (1.5.8)$$

Taking Wronskian for (1.3.2) and $\psi^{(+)}(x, \lambda)$, we can see $\{\varphi^{(-)}, \psi^{(+)}\}$ is linearly independent because of Lemma 1.5.4. Since $\varphi^{(-)}, \psi^{(+)}$ are the solution to (1.1.1), we have

$$\begin{aligned} g(x) &= B \frac{d}{dx} u(x, \lambda) + P(x)u(x, \lambda) - \lambda u(x, \lambda) \\ &= B \begin{pmatrix} \varphi^{(-)}(x, \lambda) & \psi^{(+)}(x, \lambda) \end{pmatrix} \frac{d}{dx} \begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix}. \end{aligned}$$

Then we have

$$\begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} \varphi^{(-)}(t, \lambda) & \psi^{(+)}(t, \lambda) \end{pmatrix}^{-1} Bg(t) dt.$$

Therefore, by (1.5.8) we have

$$\begin{aligned} u(x, \lambda) &= c_1(\lambda)\varphi^{(-)}(x, \lambda) + c_2(\lambda)\psi^{(+)}(x, \lambda) \\ &+ \int_{-\infty}^x \begin{pmatrix} \varphi^{(-)}(x, \lambda) & \psi^{(+)}(x, \lambda) \end{pmatrix} \begin{pmatrix} \varphi^{(-)}(t, \lambda) & \psi^{(+)}(t, \lambda) \end{pmatrix}^{-1} Bg(t) dt. \end{aligned} \quad (1.5.9)$$

Because of $g \in \{C_0^\infty(\mathbf{R})\}^2$, there are x_0 such that the integral term of (1.5.9) is 0 for $x < x_0$. $\psi^{(+)}(x, \lambda)$ decreases exponentially for $x \rightarrow -\infty$. Furthermore, $\varphi^{(-)}(x, \lambda)$ is not in $\{L^2(-\infty, x_0)\}^2$. In fact, if so, $\varphi^{(-)}(x, \lambda)$ is eigenfunction for A_P , this is contradiction.

Therefore, if $u(x, \lambda)$ is in $\{L^2(\mathbf{R})\}^2$, $c_1(\lambda) = 0$. Then we have

$$\begin{aligned}
u(x, \lambda) &= c_2(\lambda)\psi^{(+)}(x, \lambda) \\
&+ \int_{-\infty}^x \begin{pmatrix} \varphi^{(-)}(x, \lambda) & \psi^{(+)}(x, \lambda) \end{pmatrix} \begin{pmatrix} \varphi^{(-)}(t, \lambda) & \psi^{(+)}(t, \lambda) \end{pmatrix}^{-1} Bg(t)dt \\
&= c_2(\lambda)\psi^{(+)}(x, \lambda) \\
&+ \int_{-\infty}^x \frac{1}{W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]} \begin{pmatrix} \varphi_1^{(-)}(x, \lambda) & \psi_1^{(+)}(x, \lambda) \\ \varphi_2^{(-)}(x, \lambda) & \psi_2^{(+)}(x, \lambda) \end{pmatrix} \begin{pmatrix} \psi_2^{(+)}(t, \lambda) & -\psi_1^{(+)}(t, \lambda) \\ -\varphi_2^{(-)}(t, \lambda) & \varphi_1^{(-)}(t, \lambda) \end{pmatrix} Bg(t)dt \\
&= c_2(\lambda)\psi^{(+)}(x, \lambda) \\
&+ \int_{-\infty}^x \frac{1}{W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]} \begin{pmatrix} \varphi_1^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & -\varphi_1^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \\ \varphi_2^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & -\varphi_2^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \end{pmatrix} Bg(t)dt \\
&+ \int_{-\infty}^x \frac{1}{W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]} \begin{pmatrix} -\varphi_2^{(-)}(t, \lambda)\psi_1^{(+)}(x, \lambda) & \varphi_1^{(-)}(t, \lambda)\psi_1^{(+)}(x, \lambda) \\ -\varphi_2^{(-)}(t, \lambda)\psi_2^{(+)}(x, \lambda) & \varphi_1^{(-)}(t, \lambda)\psi_2^{(+)}(x, \lambda) \end{pmatrix} Bg(t)dt.
\end{aligned}$$

Because of $g \in \{C_0^\infty(\mathbf{R})\}^2$, the first integral term decreases exponentially for $x \rightarrow \infty$. We can write

$$\begin{aligned}
u(x, \lambda) &= \int_{-\infty}^x \frac{1}{W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]} \begin{pmatrix} \varphi_1^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & -\varphi_1^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \\ \varphi_2^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & -\varphi_2^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \end{pmatrix} Bg(t)dt \\
&+ \left\{ c_2(\lambda) + \int_{-\infty}^x \frac{1}{W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]} \left(g_1(t)\varphi_1^{(-)}(t, \lambda) - g_2(t)\varphi_2^{(-)}(t, \lambda) \right) dt \right\} \psi^{(+)}(x, \lambda). \quad (1.5.10)
\end{aligned}$$

Therefore, to make $u \in \{L^2(\mathbf{R})\}^2$, c_2 must satisfy

$$c_2(\lambda) = - \int_{-\infty}^{\infty} \frac{g_1(t)\varphi_1^{(-)}(t, \lambda) - g_2(t)\varphi_2^{(-)}(t, \lambda)}{W[\varphi^{(-)}(t, \lambda), \psi_2^{(+)}(t, \lambda)]} dt. \quad (1.5.11)$$

Substituting (1.5.11) to (1.5.10), we have

$$\begin{aligned}
u(x, \lambda) &= - \int_{-\infty}^x \frac{g_1(t)\psi_1^{(+)}(t, \lambda) - g_2(t)\psi_2^{(+)}(t, \lambda)}{W[\varphi^{(-)}(t, \lambda), \psi_2^{(+)}(t, \lambda)]} dt \varphi^{(-)}(x, \lambda) \\
&- \int_x^{\infty} \frac{g_1(t)\varphi_1^{(-)}(t, \lambda) - g_2(t)\varphi_2^{(-)}(t, \lambda)}{W[\varphi^{(-)}(t, \lambda), \psi_2^{(+)}(t, \lambda)]} dt \psi^{(+)}(x, \lambda). \quad (1.5.12)
\end{aligned}$$

Now we have got the following representation :

$$u(x, \lambda) = \int_{-\infty}^{\infty} G(x, t; \lambda) g(t) dt, \quad (1.5.13)$$

$$G(x, t; \lambda) = -\frac{1}{W[\varphi^{(-)}(t, \lambda), \psi_2^{(+)}(t, \lambda)]} \times \begin{cases} \begin{pmatrix} \varphi_1^{(-)}(x, \lambda) \psi_1^{(+)}(t, \lambda) & -\varphi_1^{(-)}(x, \lambda) \psi_2^{(+)}(t, \lambda) \\ \varphi_2^{(-)}(x, \lambda) \psi_1^{(+)}(t, \lambda) & -\varphi_2^{(-)}(x, \lambda) \psi_2^{(+)}(t, \lambda) \end{pmatrix} & t < x \\ \begin{pmatrix} \varphi_1^{(-)}(t, \lambda) \psi_1^{(+)}(x, \lambda) & -\varphi_2^{(-)}(t, \lambda) \psi_1^{(+)}(x, \lambda) \\ \varphi_1^{(-)}(t, \lambda) \psi_2^{(+)}(x, \lambda) & -\varphi_2^{(-)}(t, \lambda) \psi_2^{(+)}(x, \lambda) \end{pmatrix} & x < t. \end{cases} \quad (1.5.14)$$

We call $G(x, t; \lambda)$ the Green's function.

Similarly, for the case of $\text{Re } \lambda < 0$, we have

$$u(x, \lambda) = -\int_{-\infty}^x \frac{g_1(t) \psi_1^{(-)}(t, \lambda) - g_2(t) \psi_2^{(-)}(t, \lambda)}{W[\varphi^{(+)}(t, \lambda), \psi_2^{(-)}(t, \lambda)]} dt \varphi^{(+)}(x, \lambda) - \int_x^{\infty} \frac{g_1(t) \varphi_1^{(+)}(t, \lambda) - g_2(t) \varphi_2^{(+)}(t, \lambda)}{W[\varphi^{(+)}(t, \lambda), \psi_2^{(-)}(t, \lambda)]} dt \psi^{(-)}(x, \lambda), \quad (1.5.15)$$

or

$$u(x, \lambda) = \int_{-\infty}^{\infty} \tilde{G}(x, t; \lambda) g(t) dt, \quad (1.5.16)$$

$$\tilde{G}(x, t; \lambda) = -\frac{1}{W[\varphi^{(+)}(t, \lambda), \psi_2^{(-)}(t, \lambda)]} \times \begin{cases} \begin{pmatrix} \varphi_1^{(+)}(x, \lambda) \psi_1^{(-)}(t, \lambda) & -\varphi_1^{(+)}(x, \lambda) \psi_2^{(-)}(t, \lambda) \\ \varphi_2^{(+)}(x, \lambda) \psi_1^{(-)}(t, \lambda) & -\varphi_2^{(+)}(x, \lambda) \psi_2^{(-)}(t, \lambda) \end{pmatrix} & t < x, \\ \begin{pmatrix} \varphi_1^{(+)}(t, \lambda) \psi_1^{(-)}(x, \lambda) & -\varphi_2^{(+)}(t, \lambda) \psi_1^{(-)}(x, \lambda) \\ \varphi_1^{(+)}(t, \lambda) \psi_2^{(-)}(x, \lambda) & -\varphi_2^{(+)}(t, \lambda) \psi_2^{(-)}(x, \lambda) \end{pmatrix} & x < t. \end{cases} \quad (1.5.17)$$

For the case of $\text{Re } \lambda > 0$, $(A_P - \lambda)^{-1}g$ is given by the right hand sides of (1.5.13) or (1.5.14). Now we prove the boundedness for $(A_P - \lambda)^{-1}$. For this purpose, we use the equation (1.5.13).

Now we consider $W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]$. If $\text{supp } P(x) \subset [-b, b]$, for $t < -b$, we have

$$\psi^{(+)}(t, \lambda) = \begin{pmatrix} e^{\lambda t} \\ e^{\lambda t} \end{pmatrix}.$$

For $t < -b$, we can write $\varphi^{(-)}$ as

$$\varphi^{(-)}(t, \lambda) = d_1(\lambda) \begin{pmatrix} e^{\lambda t} \\ e^{\lambda t} \end{pmatrix} + d_2(\lambda) \begin{pmatrix} e^{-\lambda t} \\ -e^{-\lambda t} \end{pmatrix},$$

where $d_1(\lambda), d_2(\lambda)$ are constants depending on λ . Since A_P does not have eigenvalues, $d_2(\lambda)$ is not equal to 0. Therefore, for $t < -b$, we have

$$W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)] = 2d_2(\lambda) \neq 0.$$

With similar argument, the Wronskian is constant non-zero value for $t > b$.

Therefore, there is some positive constant $c > 0$ depending on λ such that

$$|W[\varphi^{(-)}(t, \lambda), \psi^{(+)}(t, \lambda)]| \geq c > 0, \quad t \in \mathbf{R}$$

because the Wronskian takes non-zero value everywhere by the linearly independence of $\{\varphi^{(-)}, \psi^{(+)}\}$.

Then from (1.5.13), we have the following estimate for the first component u_1 of u . Here, the constants

C are general positive constants depending on λ .

$$\begin{aligned} |u_1(x, \lambda)| &\leq C \int_{-\infty}^x |\psi_1^{(+)}(t, \lambda)g_1(t) - \psi_2^{(+)}(t, \lambda)g_2(t)| dt |\varphi_1^{(-)}(x, \lambda)| \\ &\quad + C \int_x^{\infty} |\varphi_1^{(-)}(t, \lambda)g_1(t) - \varphi_2^{(-)}(t, \lambda)g_2(t)| dt |\psi_1^{(+)}(x, \lambda)| \\ &\leq C \int_{-\infty}^x (|\psi_1^{(+)}(t, \lambda)| + |\psi_2^{(+)}(t, \lambda)|)(|g_1(t)| + |g_2(t)|) dt |\varphi_1^{(-)}(x, \lambda)| \\ &\quad + C \int_x^{\infty} (|\varphi_1^{(-)}(t, \lambda)| + |\varphi_2^{(-)}(t, \lambda)|)(|g_1(t)| + |g_2(t)|) dt |\psi_1^{(+)}(x, \lambda)| \end{aligned}$$

Now using the estimates (1.2.5) and (1.2.6) in Lemma 2.2, we have

$$|u_1(x, \lambda)| \leq C \int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t-x|} (|g_1(t)| + |g_2(t)|) dt$$

By Hölder inequality, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |u_1(x, \lambda)|^2 dx &\leq C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t-x|} (|g_1(t)| + |g_2(t)|) dt \right)^2 dx \\
&\leq C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t_1-x|} dt_1 \right) \left(\int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t-x|} (|g_1(t)| + |g_2(t)|)^2 dt \right) dx \\
&\leq C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t-x|} (|g_1(t)|^2 + |g_2(t)|^2) dt dx \\
&= C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\operatorname{Re} \lambda |t-x|} dx \right) (|g_1(t)|^2 + |g_2(t)|^2) dt \\
&\leq C \int_{-\infty}^{\infty} (|g_1(t)|^2 + |g_2(t)|^2) dt = C \|g\|_{L^2(-\infty, \infty)}^2
\end{aligned}$$

Such estimate is obtained similarly for the second component u_2 . Therefore we have proved the boundedness $(A_P - \lambda)^{-1}$ for $\operatorname{Re} \lambda > 0$. Then λ with $\operatorname{Re} \lambda > 0$ is included in the resolvent of A_P . We can prove similarly that λ with $\operatorname{Re} \lambda < 0$ is also included in the resolvent of A_P .

Lemma 1.5.5 A_P does not have eigenvalues. Furthermore, λ satisfying $\operatorname{Re} \lambda \neq 0$ is in the resolvent of A_P .

Now we consider the case of $\operatorname{Re} \lambda = 0$. For $\operatorname{Re} \lambda = 0$, let us solve (1.5.7). In this case, we use the linearly independent system $\{\psi^{(+)}, \psi^{(-)}\}$ for the variation of constants method. We set

$$u(x, \lambda) = \begin{pmatrix} \psi^{(+)}(x, \lambda) & \psi^{(-)}(x, \lambda) \end{pmatrix} \begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix}.$$

Then we have

$$g(x) = B \frac{d}{dx} u + P(x)u - \lambda u = B \begin{pmatrix} \psi^{(+)}(x, \lambda) & \psi^{(-)}(x, \lambda) \end{pmatrix} \frac{d}{dx} \begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix}.$$

Therefore, we obtain

$$\begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} \psi^{(+)}(t, \lambda) & \psi^{(-)}(t, \lambda) \end{pmatrix}^{-1} B g(t) dt,$$

and we have

$$u(x, \lambda) = c_1(\lambda)\psi^{(+)}(x, \lambda) + c_2(\lambda)\psi^{(-)}(x, \lambda) \\ + \int_{-\infty}^x \begin{pmatrix} \psi^{(+)}(x, \lambda) & \psi^{(-)}(x, \lambda) \end{pmatrix} \begin{pmatrix} \psi^{(+)}(t, \lambda) & \psi^{(-)}(t, \lambda) \end{pmatrix}^{-1} Bg(t)dt.$$

Let us assume $\text{supp } g \subset (-b, b)$ and $\text{supp } P \subset (-b, b)$. If $x < -b$, the integral term vanishes. Furthermore, if

$x < -b$, we have

$$\psi^{(+)}(x, \lambda) = \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix}, \quad \psi^{(-)}(x, \lambda) = \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}.$$

Then for $x < -b$, $u(x, \lambda)$ is written as

$$u(x, \lambda) = c_1(\lambda) \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} + c_2(\lambda) \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}$$

Then if $u \in \{L^2(\mathbf{R})\}^2$, $c_1(\lambda)$ and $c_2(\lambda)$ must be 0. Therefore, we have

$$u(x, \lambda) = \int_{-\infty}^x \begin{pmatrix} \psi^{(+)}(x, \lambda) & \psi^{(-)}(x, \lambda) \end{pmatrix} \begin{pmatrix} \psi^{(+)}(t, \lambda) & \psi^{(-)}(t, \lambda) \end{pmatrix}^{-1} Bg(t)dt \\ = \int_{-\infty}^x \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \begin{pmatrix} \psi_1^{(+)}(x, \lambda) & \psi_1^{(-)}(x, \lambda) \\ \psi_2^{(+)}(x, \lambda) & \psi_2^{(-)}(x, \lambda) \end{pmatrix} \\ \times \begin{pmatrix} \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \\ -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \end{pmatrix} Bg(t)dt \\ = \int_{-\infty}^x \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \begin{pmatrix} \psi_1^{(+)}(x, \lambda)\psi_2^{(-)}(t, \lambda) & -\psi_1^{(+)}(x, \lambda)\psi_1^{(-)}(t, \lambda) \\ \psi_2^{(+)}(x, \lambda)\psi_2^{(-)}(t, \lambda) & -\psi_2^{(+)}(x, \lambda)\psi_1^{(-)}(t, \lambda) \end{pmatrix} Bg(t)dt \\ + \int_{-\infty}^x \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \begin{pmatrix} -\psi_1^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & \psi_1^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \\ -\psi_2^{(-)}(x, \lambda)\psi_2^{(+)}(t, \lambda) & \psi_2^{(-)}(x, \lambda)\psi_1^{(+)}(t, \lambda) \end{pmatrix} Bg(t)dt$$

Now, for $x > b$, we can set

$$\psi^{(+)}(x, \lambda) = a_1(\lambda) \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} + a_2(\lambda) \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}, \quad \psi^{(-)}(x, \lambda) = a_3(\lambda) \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} + a_4(\lambda) \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}.$$

Then for $x > b$, we have

$$\begin{aligned}
u(x, \lambda) &= e^{\lambda x} \int_{-b}^b \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \\
&\quad \times \left[a_1(\lambda) \begin{pmatrix} \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \\ \psi_2^{(+)}(t, \lambda) & -\psi_1^{(+)}(t, \lambda) \end{pmatrix} + a_3(\lambda) \begin{pmatrix} -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \\ -\psi_2^{(-)}(t, \lambda) & \psi_1^{(-)}(t, \lambda) \end{pmatrix} \right] Bg(t) dt \\
&+ e^{-\lambda x} \int_{-b}^b \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \\
&\quad \times \left[a_2(\lambda) \begin{pmatrix} \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \\ -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \end{pmatrix} + a_4(\lambda) \begin{pmatrix} -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \\ \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \end{pmatrix} \right] Bg(t) dt
\end{aligned}$$

Note that $\operatorname{Re} \lambda = 0$. If $u(x, \lambda) \in \{L^2(\mathbf{R})\}^2$, the two integral terms must be 0.

Therefore, for g with

$$\begin{aligned}
&\int_{-b}^b \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \\
&\quad \times \left[a_1(\lambda) \begin{pmatrix} \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \\ \psi_2^{(+)}(t, \lambda) & -\psi_1^{(+)}(t, \lambda) \end{pmatrix} + a_3(\lambda) \begin{pmatrix} -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \\ -\psi_2^{(-)}(t, \lambda) & \psi_1^{(-)}(t, \lambda) \end{pmatrix} \right] Bg(t) dt \neq 0
\end{aligned}$$

or

$$\begin{aligned}
&\int_{-b}^b \frac{1}{W[\psi^{(+)}(t, \lambda), \psi^{(-)}(t, \lambda)]} \\
&\quad \times \left[a_2(\lambda) \begin{pmatrix} \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \\ -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \end{pmatrix} + a_4(\lambda) \begin{pmatrix} -\psi_2^{(+)}(t, \lambda) & \psi_1^{(+)}(t, \lambda) \\ \psi_2^{(-)}(t, \lambda) & -\psi_1^{(-)}(t, \lambda) \end{pmatrix} \right] Bg(t) dt \neq 0,
\end{aligned}$$

$(A_P - \lambda)u = g$ does not have the solution. Then λ with $\operatorname{Re} \lambda = 0$ is in residual spectrum.

Theorem 1.5.6 We assume that $P \in \{C_0^1(\mathbf{R})\}^4$. The operator A_P such that

$$\begin{cases} A_P : \{L^2(\mathbf{R})\}^2 \rightarrow \{L^2(\mathbf{R})\}^2 \\ D(A_P) = \{H^1(\mathbf{R})\}^2 \\ A_P u = B \frac{d}{dx} u + P(x)u \end{cases}$$

satisfies

$$\rho(A_P) = \{\lambda \in \mathbf{C} ; \operatorname{Re} \lambda \neq 0\}, \quad (1.5.18)$$

$$\sigma_R(A_P) = \{\lambda \in \mathbf{C} ; \operatorname{Re} \lambda = 0\} \quad (1.5.19)$$

where $\rho(A_P)$ and $\sigma_R(A_P)$ are respectively the resolvent and the residual spectrum of A_P .

1.6 The proof of Theorem 1.1.1

1.6.1 Preliminaries

In this subsection we will show spectral properties for A_P restricted to a finite interval. To investigate the uniqueness for the inverse problem, we use the following results. Now we let b be positive number and we denote Ω_b as

$$\Omega_b = \{(x, y); -b < y < x < b\}.$$

Similarly to [55], we can prove the following properties.

Propositon 1.6.1 *For given $b > 0$, 2×2 -matrix functions $P, Q \in \{C_0^1(\mathbf{R})\}^4$ and $h \in \mathbf{R}$ satisfying $|h| \neq 1$, there exists a unique $K = K(x, y) = (K_{k\ell}(x, y))_{k,\ell=1,2} \in \{C^1(\bar{\Omega}_b)\}^4$ satisfying the following equations (1.6.1) – (1.6.4).*

$$B \frac{\partial K(x, y)}{\partial x} + Q(x)K(x, y) - K(x, y)P(y) = -\frac{\partial K(x, y)}{\partial y}B, \quad (x, y) \in \Omega_b. \quad (1.6.1)$$

$$\begin{cases} K_{12}(x, -b) = hK_{11}(x, -b) \\ K_{22}(x, -b) = hK_{21}(x, -b) \end{cases}, \quad (-b \leq x \leq b). \quad (1.6.2)$$

$$\begin{aligned} & K_{12}(x, x) - K_{21}(x, x) \\ &= \frac{1}{4} \exp(-\theta_1(x) - \theta_2(x)) \\ & \times (Q_{11}(x) + Q_{12}(x) - Q_{21}(x) - Q_{22}(x) - P_{11}(x) + P_{12}(x) - P_{21}(x) + P_{22}(x)) \\ &+ \frac{1}{4} \exp(-\theta_1(x) + \theta_2(x)) \\ & \times (Q_{11}(x) - Q_{12}(x) + Q_{21}(x) - Q_{22}(x) - P_{11}(x) - P_{12}(x) + P_{21}(x) + P_{22}(x)), \\ & \hspace{20em} (-b \leq x \leq b). \end{aligned} \quad (1.6.3)$$

$$\begin{aligned}
& K_{11}(x, x) - K_{22}(x, x) \\
&= \frac{1}{4} \exp(-\theta_1(x) - \theta_2(x)) \\
&\quad \times (Q_{11}(x) + Q_{12}(x) - Q_{21}(x) - Q_{22}(x) + P_{11}(x) - P_{12}(x) + P_{21}(x) - P_{22}(x)) \\
&\quad + \frac{1}{4} \exp(-\theta_1(x) + \theta_2(x)) \\
&\quad \times (-Q_{11}(x) + Q_{12}(x) - Q_{21}(x) + Q_{22}(x) - P_{11}(x) - P_{12}(x) + P_{21}(x) + P_{22}(x)), \\
&\hspace{25em} (-b \leq x \leq b). \tag{1.6.4}
\end{aligned}$$

Here, we set

$$P(x) = \begin{pmatrix} P_{11}(x) & P_{12}(x) \\ P_{21}(x) & P_{22}(x) \end{pmatrix}, \quad Q(x) = \begin{pmatrix} Q_{11}(x) & Q_{12}(x) \\ Q_{21}(x) & Q_{22}(x) \end{pmatrix},$$

and

$$\theta_1(x) = \frac{1}{2} \int_{-b}^x (Q_{12}(s) + Q_{21}(s) - P_{12}(s) - P_{21}(s)) ds, \quad -b \leq x \leq b, \tag{1.6.5}$$

$$\theta_2(x) = \frac{1}{2} \int_{-b}^x (Q_{11}(s) + Q_{22}(s) - P_{11}(s) - P_{22}(s)) ds, \quad -b \leq x \leq b. \tag{1.6.6}$$

Propositon 1.6.2 (Transformation formula)

We use the same notations as Proposition 1.6.1. For $\lambda \in \mathbf{C}$, if $u(x, \lambda) \in \{C^1[-b, b]\}^2$ satisfies

$$\begin{cases} B \frac{du(x, \lambda)}{dx} + P(x)u(x, \lambda) = \lambda u(x, \lambda), & -b \leq x \leq b. \\ u(-b, \lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix}, \end{cases}$$

then $\tilde{u}(x, \lambda) \in \{C^1[-b, b]\}$ defined by

$$\tilde{u}(x, \lambda) = R(x)u(x, \lambda) + \int_{-b}^x K(x, y)u(y, \lambda)dy, \quad -b \leq x \leq b \tag{1.6.7}$$

satisfies

$$\begin{cases} B \frac{d\tilde{u}(x, \lambda)}{dx} + Q(x)\tilde{u}(x, \lambda) = \lambda \tilde{u}(x, \lambda), & -b \leq x \leq b. \\ \tilde{u}(-b, \lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix}. \end{cases}$$

Here $R(x)$ is defined by

$$R(x) = \exp(-\theta_1(x)) \begin{pmatrix} \cosh(-\theta_2(x)) & \sinh(-\theta_2(x)) \\ \sinh(-\theta_2(x)) & \cosh(-\theta_2(x)) \end{pmatrix}. \tag{1.6.8}$$

We define the operator $A_{P,h,H}^b$ by

$$\left\{ \begin{array}{l} A_{P,h,H}^b : L_2(-b, b) \rightarrow L_2(-b, b) \\ D(A_{P,h,H}^b) = \left\{ u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \in \{H_1(\mathbf{R})\}^2 ; u_2(-b) = hu_1(-b), u_2(b) = Hu_1(b) \right\} , \\ A_{P,h,H}^b u(x) = B \frac{du}{dx}(x) + P(x)u(x), \quad -b < x < b \end{array} \right\} ,$$

where $h, H \in \mathbf{R}$ satisfies $|h| \neq 1$, $|H| \neq 1$ and $b > 0$, $P \in \{C_0^1(\mathbf{R})\}^4$.

Next by [52], we show the results on Riesz basis in $\{L_2(-b, b)\}^2$ and root vectors of $A_{P,h,H}^b$.

Definition 1.6.3 $\{u_n\}_{n \in \mathbf{Z}}$ is a Riesz basis in $\{L_2(-b, b)\}^2$ if and only if each $u \in \{L_2(-b, b)\}^2$ has a unique expansion

$$u = \sum_{n=-\infty}^{\infty} c_n u_n$$

with $c_n \in \mathbf{C}$, $n \in \mathbf{Z}$ and

$$M^{-1} \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \|u\|_{\{L_2(-b, b)\}^2}^2 \leq M \sum_{n=-\infty}^{\infty} |c_n|^2$$

where a constant $M > 0$ is independent of u .

Definition 1.6.4 $u \neq 0$ is a root vector of $A_{P,h,H}^b$ if and only if

$$(A - \lambda)^m u = 0$$

for some $\lambda \in \mathbf{C}$ and $m \in \mathbf{N}$.

Now we state the property of the spectrum $\sigma(A_{P,h,H}^b)$ of $A_{P,h,H}^b$.

Proposition 1.6.5 Let $h, H \in \mathbf{R}$ satisfy $|h| \neq 1$, $|H| \neq 1$. And we assume that $P(x) \in \{C_0^1(\mathbf{R})\}$. Then the following (i) and (ii) hold.

(i) There exist $N \in \mathbf{N}$ and $\Sigma_1, \Sigma_2 \subset \sigma(A_{P,h,H}^b)$ such that

$$\sigma(A_{P,h,H}^b) = \Sigma_1 \cup \Sigma_2, \quad \Sigma_1 \cap \Sigma_2 = \emptyset$$

and the following properties hold.

(1) Σ_1 consists of $2N - 1$ eigenvalues including algebraic multiplicities in

$$\left\{ \lambda ; |\operatorname{Im}(\lambda - C_{P,h,H}^b)| < \left(N - \frac{1}{2}\right) \pi \right\},$$

where $C_{P,h,H}^b$ is a constant depending on P, h, H, b .

(2) All the elements of Σ_2 are eigenvalues whose algebraic multiplicities are one, and

$$\Sigma_2 \subset \left\{ \lambda ; |\operatorname{Im}(\lambda - C_{P,h,H}^b)| > \left(N - \frac{1}{2}\right) \pi \right\}.$$

Furthermore, with suitable numbering $\{\lambda_n\}_{n \in \mathbb{Z}}$ of $\sigma(A_{P,h,H}^b)$ the eigenvalues have an asymptotic behavior

$$\lambda_n = C_{P,h,H}^b + \frac{n\pi i}{2b} + O\left(\frac{1}{|n|}\right) \quad (1.6.9)$$

as $|n| \rightarrow \infty$.

(ii) The set of all the root vectors of $A_{P,h,H}^b$ is a Riesz basis in $\{L_2(-b, b)\}^2$.

1.6.2 Completion of the proof of Theorem 1.1

Proof of Theorem 1.1

Let us assume that (1.1.11) – (1.1.14) is satisfied, and that $\operatorname{supp} P, \operatorname{supp} Q \subset (-b, b)$, for some $b > 0$.

From (1.1.11), (1.1.12), the kernel $K(x, y)$ in proposition 1.6.1 is equal to 0 in Ω_b . Furthermore, from (1.1.13), (1.1.14), $R(x)$ in Proposition 1.6.2 satisfies $R(b) = E_2$. Thus from (1.6.7), we have $\tilde{u}(b, \lambda) = u(b, \lambda)$ for all $h \in \mathbf{R}$ with $|h| \neq 1$. Therefore $\alpha_P^{(j)}(\lambda) = \alpha_Q^{(j)}(\lambda)$, $\beta_P^{(j)}(\lambda) = \beta_Q^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) hold for all λ satisfying $\operatorname{Re} \lambda = 0$.

Conversely, we assume that $\alpha_P^{(j)}(\lambda) = \alpha_Q^{(j)}(\lambda)$, $\beta_P^{(j)}(\lambda) = \beta_Q^{(j)}(\lambda)$ ($j = 1, 2, 3, 4$) hold for all λ satisfying

$\operatorname{Re} \lambda = 0$. We also assume that $\operatorname{supp} P, \operatorname{supp} Q \subset (-b, b)$.

From (1.3.11) – (1.3.14), we see that $\alpha_P^{(2)}, \alpha_Q^{(2)}, \alpha_P^{(3)}, \alpha_Q^{(3)}, \beta_P^{(2)}, \beta_Q^{(2)}, \beta_P^{(3)}$ and $\beta_Q^{(3)}$ are analytic functions.

Therefore, $\alpha_P^{(1)}, \alpha_Q^{(1)}, \alpha_P^{(4)}, \alpha_Q^{(4)}, \beta_P^{(1)}, \beta_Q^{(1)}, \beta_P^{(4)}$ and $\beta_Q^{(4)}$ are also analytic functions since $\alpha_P^{(3)}, \alpha_Q^{(3)}, \beta_P^{(2)}$ and $\beta_Q^{(2)}$ is not equal to 0 for all $\lambda \in \mathbf{C}$ because of Lemma 1.5.4. Therefore, we have $\alpha_P^{(j)}(\lambda) = \alpha_Q^{(j)}(\lambda)$, $\beta_P^{(j)}(\lambda) = \beta_Q^{(j)}(\lambda)$, $j = 1, 2, 3, 4$ for all $\lambda \in \mathbf{C}$.

Now we assume that $h \in \mathbf{R}$ satisfies $|h| \neq 1$. By $u_P(x, \lambda)$ let us denote the solution to

$$\begin{cases} B \frac{d}{dx} u_P(x, \lambda) + P(x) u_P(x, \lambda) = \lambda u_P(x, \lambda), & x \in \mathbf{R}, \\ u_P(-b, \lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix}. \end{cases}$$

Since $\alpha_P^{(j)}(\lambda) = \alpha_Q^{(j)}(\lambda)$, $\beta_P^{(j)}(\lambda) = \beta_Q^{(j)}(\lambda)$, $j = 1, 2, 3, 4$ for all $\lambda \in \mathbf{C}$, we have $u_P(b, \lambda) = u_Q(b, \lambda)$ for all $\lambda \in \mathbf{C}$. Then from Proposition 1.6.2,

$$u_P(b, \lambda) = u_Q(b, \lambda) = R(b) u_P(b, \lambda) + \int_{-b}^b K(y, b) u_P(y, \lambda) dy, \quad \lambda \in \mathbf{C}. \quad (1.6.10)$$

Let $H_1, H_2 \in \mathbf{R}$ satisfy $H_1 \neq H_2$, $|H_1| \neq 1$ and $|H_2| \neq 1$. We denote the set of eigenvalues of A_{P,h,H_1}^b by $\{\lambda_n^{(H_1)}\}_{n \in \mathbf{Z}}$. Similarly, we denote the set of eigenvalues of A_{P,h,H_2}^b by $\{\lambda_n^{(H_2)}\}_{n \in \mathbf{Z}}$. From (1.6.10), we have

$$\begin{pmatrix} u_{P,1}(b, \lambda_n^{(H_j)}) \\ H_j u_{P,1}(b, \lambda_n^{(H_j)}) \end{pmatrix} = R(b) \begin{pmatrix} u_{P,1}(b, \lambda_n^{(H_j)}) \\ H_j u_{P,1}(b, \lambda_n^{(H_j)}) \end{pmatrix} + \int_{-b}^b K(y, b) u_P(y, \lambda_n^{(H_j)}) dy, \quad j = 1, 2,$$

where $u_{P,1}(b, \lambda_n^{(H_j)})$ is the first component of $u_P(b, \lambda_n^{(H_j)})$. Taking the limit $n \rightarrow \infty$, we have

$$\begin{pmatrix} 1 \\ H_j \end{pmatrix} = R(b) \begin{pmatrix} 1 \\ H_j \end{pmatrix}$$

from Riemann-Lebesgue theorem. Then we have $R(b) = E_2$. Therefore

$$\int_{-b}^b K(y, b) u_P(y, \lambda_n^{(H_1)}) dy = 0$$

hold for all $n \in \mathbf{N}$. Because $u_P(y, \lambda_n^{(H_1)})$ forms Riesz basis, we have

$$K(y, b) = 0, \quad -b < y < b.$$

From this equation, using characteristic method, we obtain $K(y, x) = 0$ in Ω_b (See [55], Appendix III).

$R(b) = E_2$, $K(y, x) \equiv 0$ means (1.1.11) – (1.1.14). The proof is finished.

Chapter 2

Inverse Problems for Vibrating Systems of First Order

2.1 Introduction and the main result

We will consider the following initial value / boundary value problem:

$$\frac{\partial u}{\partial t}(t, x) = B_{2n} \frac{\partial u}{\partial x}(t, x) + P(x)u(t, x) \quad -T < t < T, \quad 0 < x < 1 \quad (2.1.1)$$

with boundary conditions

$$u_{\ell+n}(t, 0) = h_{\ell}u_{\ell}(t, 0), \quad \ell = 1, 2, \dots, n, \quad -T \leq t \leq T \quad (2.1.2)$$

$$u_{\ell+n}(t, 1) = H_{\ell}u_{\ell}(t, 1) \quad \ell = 1, 2, \dots, n, \quad -T \leq t \leq T \quad (2.1.3)$$

and with initial conditions

$$u(0, x) = a(x), \quad 0 \leq x \leq 1. \quad (2.1.4)$$

Here, let $n \in \mathbf{N}$, $h_{\ell}, H_{\ell} \in \mathbf{R} \setminus \{-1, 1\}$, $\ell = 1, 2, \dots, n$, and let

$$u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \\ \vdots \\ u_{2n}(t, x) \end{pmatrix}, \quad B_{2n} = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}, \quad E_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$
$$P(x) = \begin{pmatrix} p_{1,1}(x) & p_{1,2}(x) & \dots & p_{1,2n}(x) \\ p_{2,1}(x) & p_{2,2}(x) & \dots & p_{2,2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{2n,1}(x) & p_{2n,2}(x) & \dots & p_{2n,2n}(x) \end{pmatrix}, \quad a(x) = \begin{pmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_{2n}(x) \end{pmatrix}$$

and $u_\ell, p_{k,\ell}, 1 \leq k, \ell \leq 2n$ be real-valued. Henceforth 0 denotes zero matrices whose sizes may change line by line, and $(M)_{k,\ell}$ denotes the (k, ℓ) -component of a matrix M . Moreover we assume also the compatibility condition:

$$\begin{cases} a_{\ell+n}(0) = h_\ell a_\ell(0) \\ a_{\ell+n}(1) = H_\ell a_\ell(1), \end{cases} \quad \ell = 1, 2, \dots, n, \quad -T \leq t \leq T. \quad (2.1.5)$$

System (2.1.1) describes some vibrating system. For example, we consider a governing equation of an electric oscillation in parallel n transmission lines :

$$\begin{pmatrix} L(x) & 0 \\ 0 & C(x) \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} R(x) & 0 \\ 0 & G(x) \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix} = 0. \quad (2.1.6)$$

Here $I = I(t, x)$ and $V = V(t, x)$ are vector-valued functions whose j -th components are respectively the current and the voltage of the j -th transmission line. Moreover we assume that the electromagnetic properties of the n lines are not homogeneous in x and the coefficients R, L, C, G depend on $x \in (0, 1)$.

The parameters R, L, C, G are called a resistance matrix, an inductance matrix, a capacity matrix and a conductance matrix respectively. If there exists a scalar function $r(x) > 0$ such that

$$L(x)C(x) = r(x)E_n, \quad (2.1.7)$$

we can reduce system (2.1.6) to (2.1.1). In fact, in terms of (2.1.7), we can reduce (2.1.6) to a equation in the form

$$\frac{\partial}{\partial t} \begin{pmatrix} I \\ V \end{pmatrix} = -\frac{1}{r(x)} \begin{pmatrix} 0 & C(x) \\ L(x) & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} I \\ V \end{pmatrix} + \begin{pmatrix} \tilde{R}(x) & 0 \\ 0 & \tilde{G}(x) \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}. \quad (2.1.8)$$

Changing variables as

$$z \equiv \int_0^x \sqrt{r(y)} dy, \quad s \equiv t,$$

and

$$u(s, z) \equiv V(t, x(z))$$

$$v(s, z) \equiv -\frac{1}{\sqrt{r(x(z))}} L(x(z)) I(t, x(z)),$$

we obtain the following system:

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = B_{2n} \frac{\partial}{\partial z} \begin{pmatrix} u \\ v \end{pmatrix} + \tilde{P}(z) \begin{pmatrix} u \\ v \end{pmatrix}.$$

We will investigate

Inverse Problem

Determine a coefficient matrix $P(x)$ and an initial value $a(x)$ from the boundary values $u(t, 0), u(t, 1), -T \leq t \leq T$.

For inverse problems for one-dimensional first-order system such as (2.1.1), the method of characteristics is applicable (e.g. Chapter 5 in Romanov [45]). However such a method cannot characterize coefficients and initial values yielding the same boundary values, although boundary data $u(t, 0), u(t, 1), -T \leq t \leq T$, can simultaneously identify a coefficient matrix and an initial value. For inverse problems for first-order systems, see also Blagoveshchenskii [6]. For the corresponding inverse spectral problems with $n = 1$, see Ning [38], Ning and Yamamoto [39], Trooshin and Yamamoto [53], Yamamoto [55].

In this paper, we will study the uniqueness in our inverse problem. Here, we will only consider the case of $n = 2$. The basic properties for $n = 2$ such as the asymptotic behaviour of eigenvalues, are very different from $n = 1$, and already the case $n = 2$ needs essentially different treatments.

In general, the uniqueness does not hold, as the following example shows.

Example

Let

$$P(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a(x) = \begin{pmatrix} e^{-x} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Q(x) = \begin{pmatrix} 0 & 0 & 2x & 0 \\ 0 & 0 & 0 & 0 \\ 2x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b(x) = \begin{pmatrix} e^{-x^2} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $h_\ell = H_\ell = 0$, $\ell = 1, 2$. Then we can verify that the solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = B_4 \frac{\partial u}{\partial x}(t, x) + P(x)u(t, x) & -T < t < T, \quad 0 < x < 1, \\ u_3(t, 0) = u_4(t, 0) = 0, & -T \leq t \leq T, \\ u_3(t, 1) = u_4(t, 1) = 0, & -T \leq t \leq T, \\ u(0, x) = a(x) \end{cases}$$

is

$$u(t, x) = \begin{pmatrix} e^{-x} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

while the solution to

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t, x) = B_4 \frac{\partial \tilde{u}}{\partial x}(t, x) + Q(x)\tilde{u}(t, x) & -T < t < T, \quad 0 < x < 1 \\ \tilde{u}_3(t, 0) = \tilde{u}_4(t, 0) = 0, & -T \leq t \leq T \\ \tilde{u}_3(t, 1) = \tilde{u}_4(t, 1) = 0; & -T \leq t \leq T \\ \tilde{u}(0, x) = b(x) \end{cases}$$

is

$$\tilde{u}(t, x) = \begin{pmatrix} e^{-x^2} \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore we obtain the same boundary value:

$$u(t, 0) = \tilde{u}(t, 0) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u(t, 1) = \tilde{u}(t, 1) = \begin{pmatrix} e^{-1} \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the uniqueness does not hold, even though we restrict the coefficient matrices $P(x)$ in (2.1.1)

to a form $\begin{pmatrix} 0 & P_1(x) \\ P_1(x) & 0 \end{pmatrix}$ with 2×2 matrix $P_1(x)$. \square

We will find a condition for the uniqueness to our inverse problem, and the condition should be sufficiently general. Here and henceforth, by $u = u_{P,a}(t, x)$ we denote the solution to

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = B_4 \frac{\partial u}{\partial x}(t, x) + P(x)u(t, x), & -T < t < T, 0 < x < 1, \\ u_{\ell+2}(t, 0) = h_\ell u_\ell(t, 0), & \ell = 1, 2, -T \leq t \leq T, \\ u_{\ell+2}(t, 1) = H_\ell u_\ell(t, 1), & \ell = 1, 2, -T \leq t \leq T, \\ u(0, x) = a(x), \end{cases} \quad (2.1.9)$$

provided that $h_\ell, H_\ell \in \mathbf{R} \setminus \{-1, 1\}$ are fixed.

Throughout this paper, we assume that the solution $u_{P,a}(t, x)$ is sufficiently smooth. By using an energy estimate we can prove that there exists at most one solution. Moreover the existence of the solution can be proved, and the sufficient smoothness can be proved by compatibility conditions of a and P . We will omit details of the unique existence of $u_{P,a}$ in order to concentrate on the inverse problem.

Henceforth $L^2(0, 1)$ and $H^1(0, 1)$ are the usual Lebesgue space and Sobolev space of complex-valued functions.

We set

$$M_T(P, a) \equiv \{(Q, b) \in \{C^1[0, 1]\}^{20}; u_{Q,b}(t, 0) = u_{P,a}(t, 0), u_{Q,b}(t, 1) = u_{P,a}(t, 1) \quad -T < t < T\}$$

for arbitrarily fixed (P, a) guaranteeing the unique existence of smooth $u_{P,a}$. We can immediately see that $(P, a) \in M_T(P, a)$. If $M_T(P, a)$ has only one element (P, a) , then uniqueness in our inverse problem would be true. Thus it is sufficient to characterize the set $M_T(P, a)$.

Definition 2.1.1 We define an operator A_P acting from $\{L_2(0, 1)\}^4$ to $\{L_2(0, 1)\}^4$, by

$$\begin{cases} (A_P u)(x) = B_4 \frac{du}{dx}(x) + P(x)u(x), & 0 < x < 1 \\ D(A_P) = \{u \in \{H^1(0, 1)\}^4; u_{\ell+2}(0) - h_\ell u_\ell(0) = 0, u_{\ell+2}(1) - H_\ell u_\ell(1) = 0, \ell = 1, 2\}. \end{cases} \quad (2.1.10)$$

Definition 2.1.2 For an eigenvalue λ of A_P , we call $\phi \neq 0$ a root vector of an operator A_P for λ if $(A_P - \lambda)^k \phi = 0$ for some $k \in \mathbf{N}$. We call $\dim\{\phi; (A_P - \lambda)^k \phi = 0 \text{ for some } k \in \mathbf{N}\}$ and $\dim\{\phi; (A_P - \lambda)\phi = 0\}$ the algebraic multiplicity and the geometric multiplicity of λ , respectively.

In order to state the main result, we assume the following three conditions:

(I): For each root vector f^* of the adjoint operator A_P^* for A_P , the fixed initial value $a(x)$ satisfies

$$(a, f^*)_{\{L_2(0,1)\}^4} \neq 0. \quad (2.1.11)$$

(II): The following quadratic equation in α has two distinct roots:

$$\det \left\{ \alpha E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = 0 \quad (2.1.12)$$

where $h_\ell = \tanh \mu_\ell$, $H_\ell = -\tanh \nu_\ell$, and

$$\theta^P(x) = (\theta_{k,\ell}^P(x))_{k,\ell=1,2} = \left(\frac{1}{2}(p_{k,\ell}(x) + p_{k,\ell+2}(x) + p_{k+2,\ell}(x) + p_{k+2,\ell+2}(x)) \right)_{k,\ell=1,2}, \quad (2.1.13)$$

$$\tilde{\theta}^P(x) = (\tilde{\theta}_{k,\ell}^P(x))_{k,\ell=1,2} = \left(\frac{1}{2}(-p_{k,\ell}(x) + p_{k,\ell+2}(x) + p_{k+2,\ell}(x) - p_{k+2,\ell+2}(x)) \right)_{k,\ell=1,2}, \quad (2.1.14)$$

and by $G(\Theta)(x)$ for a 2×2 -matrix $\Theta = \Theta(x)$, we denote the solution to

$$\frac{d}{dx}(G(\Theta)(x)) + \Theta(x)G(\Theta)(x) = 0, \quad 0 < x < 1 \quad (2.1.15)$$

with the condition $G(\Theta)(0) = E_2$.

(III): For an arbitrary eigenvalue λ of A_P , we assume that the geometric multiplicity of λ is 1.

Remark 2.1.3 *Since Condition (II) holds if the determinant of quadratic equation (2.1.12) in α , is not zero, we can assert that the condition holds generically. Condition (III) is always true for $n = 1$. By Theorem 2.2.1 stated in Section 2.2, if Condition (II) holds, then the geometric multiplicities of the eigenvalues is one except for a finite number of eigenvalues. Moreover we can assert that Condition (III) holds also generically.*

In fact, let φ and ψ be the solutions to $(A_P - \lambda)u = 0$ with conditions at $x = 0$

$$\varphi(0) = \begin{pmatrix} 1 \\ 0 \\ h_1 \\ 0 \end{pmatrix}, \quad \psi(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ h_2 \end{pmatrix},$$

respectively. Let $u \neq 0$ satisfy $(A_P - \lambda)u = 0$. Then $u = \alpha\varphi + \beta\psi$ with some $\alpha, \beta \in \mathbb{C}$. Since $u \in D(A_P)$, we have

$$(\varphi_3(1) - H_1\varphi_1(1))\alpha + (\psi_3(1) - H_1\psi_1(1))\beta = 0$$

and

$$(\varphi_4(1) - H_2\varphi_2(1))\alpha + (\psi_4(1) - H_2\psi_2(1))\beta = 0.$$

If either of $\varphi_3(1) - H_1\varphi_1(1)$, $\psi_3(1) - H_1\psi_1(1)$, $\varphi_4(1) - H_2\varphi_2(1)$ and $\psi_4(1) - H_2\psi_2(1)$ is not zero, then $\alpha = \gamma\beta$ or $\beta = \gamma\alpha$ where γ is independent of α and β . Hence $u = \beta(\gamma\varphi + \psi)$ or $u = \alpha(\varphi + \gamma\psi)$. That is, $\{u; (A_P - \lambda)u = 0\}$ is spanned by one vector, which means that the geometric multiplicity of λ is one. Therefore Condition (III) breaks only if $\varphi_3(1) - H_1\varphi_1(1) = \psi_3(1) - H_1\psi_1(1) = \varphi_4(1) - H_2\varphi_2(1) = \psi_4(1) - H_2\psi_2(1) = 0$. Thanks to the transformation formula (2.2.10) (Theorem 2.2.5) with $P = 0$, the condition $\varphi(1)$ can be described by

$$\varphi(1) = R(1)\varphi_0(1, \lambda) + \int_0^1 K(y, 1)\varphi_0(y, \lambda)dy$$

where

$$\varphi_0(x, \lambda) = \begin{pmatrix} \frac{h_1+1}{2}e^{\lambda x} - \frac{h_1-1}{2}e^{-\lambda x} \\ 0 \\ \frac{h_1+1}{2}e^{\lambda x} + \frac{h_1-1}{2}e^{-\lambda x} \\ 0 \end{pmatrix}.$$

Thus $\varphi_3(1) - H_1\varphi_1(1) = \varphi_4(1) - H_2\varphi_2(1) = 0$ are given by two equations involving λ and K, R . We note that K and R are determined by h_1, h_2 and P . From $\psi_3(1) - H_1\psi_1(1) = \psi_4(1) - H_2\psi_2(1) = 0$, we can obtain similar equations. Hence for given h_1, h_2, H_1, H_2 , if (λ, P) does not satisfy those four equations, then Condition (III) holds true. In this sense, Condition (III) holds generically.

Let $P(x)$ and $Q(x)$ be 4×4 -matrix functions. Here let 4×4 -matrix function

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix} \quad (2.1.16)$$

with 2×2 -matrix functions $R^j(x)$, $j = 1, 2$, satisfy the following system of eight ordinary differential equations

$$\begin{cases} (B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} + (B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k+2,\ell+2} = 0 \\ (B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+2} + (B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k+2,\ell} = 0, \end{cases} \quad (2.1.17)$$

$$0 < x < 1, \quad k, \ell = 1, 2$$

and $R(0) = E_4$. Here and henceforth we set $R'(x) = \frac{dR}{dx}(x)$. By the theory of ordinary differential equations, we can prove that such an $R(x)$ exists uniquely.

Now we are ready to state our main result characterizing $M_T(P, a)$.

Theorem 2.1.4 *Let (P, a) satisfy Conditions (I), (II) and (III) and let $a \in \{C^3[0, 1]\}^4 \cap D(A^2)$. We assume that $T \geq 2$. Then*

$$(Q, b) \in M_T(P, a)$$

if and only if the following conditions hold:

$$R(1) = E_4 \quad (2.1.18)$$

$$(B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} = 0, \quad k, \ell = 1, 2 \quad 0 < x < 1 \quad (2.1.19)$$

$$(B_4 R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+2} = 0, \quad k, \ell = 1, 2 \quad 0 < x < 1 \quad (2.1.20)$$

$$b(x) = R(x)a(x). \quad (2.1.21)$$

The theorem gives the uniqueness for some components. For example, we can prove obtain the following result by verifying that (2.1.19) and (2.1.20) yield $p_{1,\ell} = q_{1,\ell}$, $1 \leq \ell \leq 4$ when $p_{k,\ell} = q_{k,\ell} = 0$ for $2 \leq k \leq 4$ and $1 \leq \ell \leq 4$.

Corollary 2.1.5 *If we restrict a class of coefficient matrices to the matrix with the form*

$$\left\{ \begin{pmatrix} a(x) & b(x) & c(x) & d(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad a, b, c, d \in C^1[0, 1] \right\}$$

and the initial value is known, then the solution to the inverse problem is unique under Conditions (I) - (III).

In Section 2.2, we will state the spectral properties of the operator A_P and in Sections 2.3,2.4, we prove them. Section 2.5 is devoted to the proof of Theorem 2.1.4.

2.2 Spectral property of A_P and transformation formulae

In this section, we will first present the spectral property of A_P defined by (2.1.10), and such properties are necessary for the proof of Theorem 2.1.4. There are very few works concerning spectral properties for a nonsymmetric operator of ordinary differential equations and Theorem 2.2.1 and Theorem 2.2.3 may be independent interests. On the other hand, there are many results on the spectral properties for the classical Sturm-Liouville problem and readers can consult Levitan and Sargsjan [26], Naimark [36] as monographs. For our system with $n = 1$, see Trooshin and Yamamoto [52].

Let $\sigma(A_P)$ denote the spectrum of the operator A_P and let $i = \sqrt{-1}$.

We present the asymptotic behaviour of $\sigma(A_P)$.

Theorem 2.2.1 *There exist $N \in \mathbf{N}$ and $\Sigma_1, \Sigma_2 \subset \sigma(A_P)$ such that*

$$\sigma(A_P) = \Sigma_1 \cup \Sigma_2, \quad \Sigma_1 \cap \Sigma_2 = \emptyset.$$

(1) *Let equation (2.1.12) possess distinct roots α_1 and α_2 . Then the following (a) and (b) hold.*

(a): Σ_1 *consists of $2(2N - 1)$ eigenvalues by taking the algebraic multiplicities into consideration, and is included in*

$$\left\{ \lambda ; |\operatorname{Im} \lambda - \tilde{\alpha}| < N\pi - \frac{\pi}{2} \right\}.$$

Here and henceforth we set

$$\tilde{\alpha} = \frac{1}{4} \operatorname{Im} \log \alpha_1 + \frac{1}{4} \operatorname{Im} \log \alpha_2$$

and we take the principal value of the logarithm : $-\pi < \text{Im} \log \alpha_j \leq \pi$, $j = 1, 2$.

(b) : All the elements of Σ_2 are eigenvalues whose algebraic multiplicities are one, and

$$\Sigma_2 \subset \left\{ \lambda \mid |\text{Im} \lambda - \tilde{\alpha}| > N\pi - \frac{\pi}{2} \right\}$$

and with suitable numbering $\{\lambda_{j,m}\}_{j=1,2,|m|\geq N, m \in \mathbf{Z}}$ of $\sigma(A)$, the eigenvalues have an asymptotic behaviour

$$\lambda_{j,m} = \frac{1}{2} \log \alpha_j + m\pi i + O\left(\frac{1}{|m|}\right) \quad (2.2.1)$$

as $|m| \rightarrow \infty$.

(2) Let (2.1.12) possess the multiple root $\alpha_1 = \alpha_2 \equiv \alpha$. Then Σ_1 has the same property as in Case (1) and

we can number all the eigenvalues of Σ_2 by $\{\lambda_{j,m}\}_{j=1,2,|m|\geq N, m \in \mathbf{Z}}$ such that $\lambda_{1,m} = \lambda_{2,m}$ may happen, but

$\lambda_{j,m} \neq \lambda_{j',m'}$ for $j, j' = 1, 2$ if $m \neq m'$, and

$$\lambda_{j,m} = \frac{1}{2} \log \alpha + m\pi i + O\left(\frac{1}{\sqrt{|m|}}\right) \quad (2.2.2)$$

as $|m| \rightarrow \infty$. Moreover for sufficiently large $|m|$, the algebraic multiplicities of $\lambda_{1,m}$ and $\lambda_{2,m}$ are one if

$\lambda_{1,m} \neq \lambda_{2,m}$ and are two if $\lambda_{1,m} = \lambda_{2,m}$.

The asymptotic behaviour in the case of $\alpha_1 \neq \alpha_2$ has two branches whose real parts are close to $\frac{1}{2} \text{Re} \log \alpha_1$ and $\frac{1}{2} \text{Re} \log \alpha_2$, and is very different from the case of $n = 1$.

Next we discuss the completeness of eigenvectors.

Definition 2.2.2 We call $\{b_m\}_{m \in \mathbf{Z}}$ a Riesz basis in $\{L_2(0, 1)\}^4$ if each $u \in \{L_2(0, 1)\}^4$ has a unique expansion

$$u = \sum_{m \in \mathbf{Z}} c_m b_m, \quad c_m \in \mathbf{C}$$

and there exists a positive number M , which is independent of the choice of u , such that

$$M^{-1} \sum_{m \in \mathbf{Z}} |c_m|^2 \leq \|u\|_{\{L_2(0,1)\}^4}^2 \leq M \sum_{m \in \mathbf{Z}} |c_m|^2.$$

We state the completeness of the root vectors.

Theorem 2.2.3 *Let (2.1.12) have two distinct roots. Then the set of all the root vectors of A_P is a Riesz basis in $\{L_2(0, 1)\}^4$.*

In Theorem 2.2.3, we note that we need not assume Condition (III).

In order to state transformation formulae, which are basic tools for our inverse problem, we prove the following lemma. Until the end of section 2.2, we will consider general $n \in \mathbf{N}$, not necessarily $n = 2$.

Lemma 2.2.4 *Assume that $P(x)$ and $Q(x)$ are $2n \times 2n$ -matrix functions whose elements are in $C^1[0, 1]$. Let $a_{k,\ell}(x), b_{k,\ell}(x), 1 \leq k, \ell \leq n$ be real valued functions. Let $h_k, 1 \leq k \leq n$ be constants and $|h_k| \neq 1, 1 \leq k \leq n$.*

Moreover we set

$$\Omega = \{(y, x) \in \mathbf{R}^2 ; 0 < y < x < 1\}$$

Then there exists a unique solution $K(y, x) \in \{C^1(\bar{\Omega})\}^{2n \times 2n}$ to

$$B_{2n} \frac{\partial K}{\partial x}(y, x) + Q(x)K(y, x) = K(y, x)P(y) - \frac{\partial K}{\partial y}(y, x)B_{2n} \quad \text{in } \Omega \quad (2.2.3)$$

$$\begin{cases} K_{k,\ell+n}(0, x) = -h_k K_{k,\ell}(0, x) \\ K_{k+n,\ell+n}(0, x) = -h_k K_{k+n,\ell}(0, x) \end{cases}, \quad k, \ell = 1, 2, \dots, n, \quad 0 \leq x \leq 1 \quad (2.2.4)$$

$$\begin{cases} K_{k,\ell+n}(x, x) - K_{k+n,\ell}(x, x) = a_{k,\ell}(x) \\ K_{k,\ell}(x, x) - K_{k+n,\ell+n}(x, x) = b_{k,\ell}(x) \end{cases} \quad k, \ell = 1, 2, \dots, n, \quad 0 \leq x \leq 1. \quad (2.2.5)$$

The proof is given in section 2.6.

Transformation formulae are given as follows. Let $P(x), Q(x)$ be fixed $2n \times 2n$ -matrix functions with $C^1[0, 1]$ -elements. Here, let $2n \times 2n$ -matrix function $R(x)$

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix}$$

with $n \times n$ matrix functions $R^j(x)$, $j = 1, 2$, satisfy system of $2n^2$ ordinary differential equations:

$$\begin{cases} (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} + (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k+n,\ell+n} = 0 \\ (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+n} + (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k+n,\ell} = 0, \end{cases} \quad (2.2.6)$$

$$0 < x < 1, \quad k, \ell = 1, 2, \dots, n$$

and

$$R(0) = E_{2n}.$$

By a classical theory of ordinary differential equations, we can prove that there exists a unique solution $R = R(x)$ to this system of ordinary differential equations.

Theorem 2.2.5 (Transformation formula in the stationary case)

Let $\tau_1, \tau_2, \dots, \tau_n \in \mathbf{R}$ and let $K = K(y, x)$ be the solution to (2.2.3), (2.2.4) and

$$\begin{cases} K_{k,\ell+n}(x, x) - K_{k+n,\ell}(x, x) = (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell} \\ K_{k,\ell}(x, x) - K_{k+n,\ell+n}(x, x) = (B_{2n}R'(x) + Q(x)R(x) - R(x)P(x))_{k,\ell+n} \end{cases}, \quad k, \ell = 1, 2, \dots, n. \quad (2.2.7)$$

Assume that $\phi(x, \lambda) = \begin{pmatrix} \phi_1(x, \lambda) \\ \vdots \\ \phi_{2n}(x, \lambda) \end{pmatrix}$ and $\psi(x, \lambda) = \begin{pmatrix} \psi_1(x, \lambda) \\ \vdots \\ \psi_{2n}(x, \lambda) \end{pmatrix}$ are \mathbf{R}^{2n} -valued functions and satisfy

$$\begin{cases} B_{2n} \frac{d\phi}{dx} + P(x)\phi = \lambda\phi, & 0 < x < 1 \\ \phi_1(0, \lambda) = \tau_1, \quad \dots, \quad \phi_n(0, \lambda) = \tau_n \\ \phi_{n+1}(0, \lambda) = h_1\tau_1, \quad \dots, \quad \phi_{2n}(0, \lambda) = h_n\tau_n \end{cases} \quad (2.2.8)$$

$$\begin{cases} B_{2n} \frac{d\psi}{dx} + Q(x)\psi = \lambda\psi, & 0 < x < 1 \\ \psi_1(0, \lambda) = \tau_1, \quad \dots, \quad \psi_n(0, \lambda) = \tau_n \\ \psi_{n+1}(0, \lambda) = h_1\tau_1, \quad \dots, \quad \psi_{2n}(0, \lambda) = h_n\tau_n. \end{cases} \quad (2.2.9)$$

Then,

$$\psi(x, \lambda) = R(x)\phi(x, \lambda) + \int_0^x K(y, x)\phi(y, \lambda)dy, \quad 0 < x < 1. \quad (2.2.10)$$

The proof of Theorem 2.2.5 is given in section 2.2.7.

Next we consider the following Cauchy problems:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = B_{2n} \frac{\partial u}{\partial x}(t, x) + P(x)u(t, x), & x > 0, \quad -T + x < t < T - x \\ u_\ell(t, 0) = \omega_\ell(t), \quad u_{n+\ell}(t, 0) = h_\ell \omega_\ell(t), & \ell = 1, 2, \dots, n \end{cases} \quad (2.2.11)$$

and

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(t, x) = B_{2n} \frac{\partial \tilde{u}}{\partial x}(t, x) + Q(x)\tilde{u}(t, x), & x > 0, \quad -T + x < t < T - x \\ \tilde{u}_\ell(t, 0) = \omega_\ell(t), \quad \tilde{u}_{n+\ell}(t, 0) = h_\ell \omega_\ell(t), & \ell = 1, 2, \dots, n \end{cases} \quad (2.2.12)$$

for given $\omega_\ell \in C^1[-T, T]$.

We can prove the transformation formula for these Cauchy problems.

Theorem 2.2.6 *Between the solution to (2.2.11) and the solution to (2.2.12), the following relation holds :*

$$\tilde{u}(t, x) = R(x)u(t, x) + \int_0^x K(y, x)u(t, y)dy, \quad x > 0, \quad -T + x < t < T - x,$$

where $R(x)$ and $K(y, x)$ are defined in Theorem 2.2.5.

Theorem 2.2.6 can be proved similarly to Theorem 2.2.5, by verifying that the right hand side satisfies (2.2.12) and using the uniqueness for the Cauchy problem (2.2.12). We omit the proof.

2.3 The proof of Theorem 2.2.1

Step 1. We shall prove the following lemma.

Lemma 2.3.1 *The spectrum $\sigma(A_P)$ consists entirely of countable isolated eigenvalues with finite algebraic multiplicities.*

Proof of Lemma 2.3.1. Let $U(x, \lambda) = (U_{k,\ell}(x, \lambda))_{k,\ell=1,2,3,4}$ be the solution to

$$\begin{cases} B_4 \frac{dU}{dx} + P(x)U = \lambda U, & 0 < x < 1 \\ U(0, \lambda) = E_4. \end{cases} \quad (2.3.1)$$

We set

$$\tilde{h} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

$$B_4^{(0)} = \begin{pmatrix} -\tilde{h} & E_2 \\ 0 & 0 \end{pmatrix}, \quad B_4^{(1)} = \begin{pmatrix} 0 & 0 \\ -\tilde{H} & E_2 \end{pmatrix}.$$

We note that 0 means a zero matrix whose sizes may change line by line and for example, in the above, 0 means the 2×2 zero matrix. Then we have

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \in D(A_P)$$

if and only if

$$\gamma \in \{H^1(0, 1)\}^4, \quad B_4^{(0)}\gamma(0) + B_4^{(1)}\gamma(1) = 0.$$

For given $f \in \{L^2(0, 1)\}^4$, let us consider the following equation:

$$\left(B_4 \frac{d}{dx} + P(x) - \lambda \right) \gamma = f.$$

By the variation of constants, a general solution to this equation is

$$\gamma(x, \lambda) = U(x, \lambda)\eta + U(x, \lambda) \int_0^x U(y, \lambda)^{-1} B_4 f(y) dy,$$

where $U(x, \lambda)$ is the fundamental solution and $\eta \in \mathbf{C}^4$ is arbitrary. In order to satisfy the condition

$\gamma \in D(A_P)$ for fixed λ , we choose η such that

$$B_4^{(0)}\gamma(0) + B_4^{(1)}\gamma(1) = 0,$$

that is to say,

$$(B_4^{(0)} + B_4^{(1)}U(1, \lambda))\eta + B_4^{(1)}U(1, \lambda) \int_0^1 U(y, \lambda)^{-1} B_4 f(y) dy = 0.$$

If $\det(B_4^{(0)} + B_4^{(1)}U(1, \lambda)) \neq 0$, then

$$\eta = -(B_4^{(0)} + B_4^{(1)}U(1, \lambda))^{-1}B_4^{(1)}U(1, \lambda) \int_0^1 U(y, \lambda)^{-1}B_4f(y)dy$$

satisfies this condition. Moreover we can write

$$\begin{aligned} \gamma(x, \lambda) &= -U(x, \lambda)(B_4^{(0)} + B_4^{(1)}U(1, \lambda))^{-1}B_4^{(1)}U(1, \lambda) \int_0^1 U(y, \lambda)^{-1}B_4f(y)dy \\ &\quad + U(x, \lambda) \int_0^x U(y, \lambda)^{-1}B_4f(y)dy. \end{aligned}$$

Therefore, if $\det(B_4^{(0)} + B_4^{(1)}U(1, \lambda_0)) \neq 0$ for some $\lambda_0 \in \mathbf{C}$, then $(A_P - \lambda_0)^{-1}$ is a compact operator from $\{L^2(0, 1)\}^4$ to itself. By Kato [22], this implies that $\sigma(A_P)$ consists of isolated eigenvalues with finite algebraic multiplicities. Hence it is sufficient to show that there exists $\lambda_0 \in \mathbf{C}$ such that $\det(B_4^{(0)} + B_4^{(1)}U(1, \lambda_0)) \neq 0$.

Since

$$U_0(x, \lambda) = \begin{pmatrix} E_2 \cosh \lambda x & E_2 \sinh \lambda x \\ E_2 \sinh \lambda x & E_2 \cosh \lambda x \end{pmatrix}$$

is the solution to

$$\begin{cases} B_4 \frac{d}{dx} U_0(x, \lambda) = \lambda U_0(x, \lambda), & 0 < x < 1 \\ U_0(0, \lambda) = E_4, \end{cases}$$

by the transformation formula, we can write

$$\begin{aligned} U(x, \lambda) &= R(x)U_0(x, \lambda) + \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy \\ &\quad + \int_0^x K^{(2)}(y, x) \begin{pmatrix} 0 & E_2 \sinh \lambda y \\ 0 & E_2 \cosh \lambda y \end{pmatrix} dy. \end{aligned} \tag{2.3.2}$$

Here, we recall that the 4×4 -matrix

$$R(x) = \begin{pmatrix} R^1(x) & R^2(x) \\ R^2(x) & R^1(x) \end{pmatrix}$$

with 2×2 -matrix functions R^j , $j = 1, 2$, satisfies

$$\begin{cases} (B_4 R'(x) + P(x)R(x))_{k,\ell} + (B_4 R'(x) + P(x)R(x))_{k+2,\ell+2} = 0, \\ (B_4 R'(x) + P(x)R(x))_{k,\ell+2} + (B_4 R'(x) + P(x)R(x))_{k+2,\ell} = 0, \end{cases} \quad k, \ell = 1, 2, \quad 0 \leq x \leq 1$$

and $R(0) = E_4$. Let $K^{(1)}$ be the solution to

$$\begin{cases} B_4 \frac{\partial K^{(1)}}{\partial x}(y, x) + P(x)K^{(1)}(y, x) = -\frac{\partial K^{(1)}}{\partial y}(y, x)B_4 & \text{in } \Omega \\ K_{k, \ell+2}^{(1)}(0, x) = 0, & k = 1, 2, 3, 4, \ell = 1, 2 \\ K_{k, \ell+2}^{(1)}(x, x) - K_{k+2, \ell}^{(1)}(x, x) = [B_4 R'(x) + P(x)R(x)]_{k, \ell}, & 0 \leq x \leq 1 \quad k, \ell = 1, 2 \\ K_{k, \ell}^{(1)}(x, x) - K_{k+2, \ell+2}^{(1)}(x, x) = [B_4 R'(x) + P(x)R(x)]_{k, \ell+2}, & 0 \leq x \leq 1 \quad k, \ell = 1, 2, \end{cases} \quad (2.3.3)$$

and $K^{(2)}$ be the solution to

$$\begin{cases} B_4 \frac{\partial K^{(2)}}{\partial x}(y, x) + P(x)K^{(2)}(y, x) = -\frac{\partial K^{(2)}}{\partial y}(y, x)B_4 & \text{in } \Omega \\ K_{k, \ell}^{(2)}(0, x) = 0, & k = 1, 2, 3, 4, \ell = 1, 2 \\ K_{k, \ell+2}^{(2)}(x, x) - K_{k+2, \ell}^{(2)}(x, x) = [B_4 R'(x) + P(x)R(x)]_{k, \ell}, & 0 \leq x \leq 1 \quad k, \ell = 1, 2 \\ K_{k, \ell}^{(2)}(x, x) - K_{k+2, \ell+2}^{(2)}(x, x) = [B_4 R'(x) + P(x)R(x)]_{k, \ell+2}, & 0 \leq x \leq 1 \quad k, \ell = 1, 2. \end{cases} \quad (2.3.4)$$

We can prove by a usual method of characteristics that $K^{(1)}$ and $K^{(2)}$ exist uniquely.

Let us consider the second term on the right hand side of (2.3.2). By integration by parts, we obtain

$$\begin{aligned} & \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy = \frac{1}{\lambda} \int_0^x K^{(1)}(y, x) \frac{d}{dy} \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} dy \\ & = \frac{1}{\lambda} \left\{ \left[K^{(1)}(y, x) \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} \right]_{y=0}^{y=x} - \int_0^x \frac{\partial}{\partial y} K^{(1)}(y, x) \begin{pmatrix} E_2 \sinh \lambda y & 0 \\ E_2 \cosh \lambda y & 0 \end{pmatrix} dy \right\}. \end{aligned}$$

Therefore, for any $C > 0$, there exists a constant $C_0 > 0$, which is dependent on C and is independent of λ ,

such that

$$\sup_{0 \leq x \leq 1} \left| \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy \right| \leq \frac{C_0}{|\lambda|} \quad \text{if } |\operatorname{Re} \lambda| \leq C.$$

Here, for a 4×4 -matrix M , we define a matrix norm $|M|$ by

$$|M| = \max_{k, \ell=1, 2, 3, 4} |M_{k, \ell}|.$$

Similarly, we can verify that there exists a constant $C_0 = C_0(C) > 0$ such that

$$\sup_{0 \leq x \leq 1} \left| \int_0^x K^{(1)}(y, x) \begin{pmatrix} E_2 \cosh \lambda y & 0 \\ E_2 \sinh \lambda y & 0 \end{pmatrix} dy + \int_0^x K^{(2)}(y, x) \begin{pmatrix} 0 & E_2 \sinh \lambda y \\ 0 & E_2 \cosh \lambda y \end{pmatrix} dy \right| \leq \frac{C_0}{|\lambda|} \quad (2.3.5)$$

if $|\operatorname{Re} \lambda| \leq C$.

Setting $\lambda = \beta + 2m\pi i$ with $\beta \in \mathbf{C}$ and $m \in \mathbf{Z}$, we can write

$$\det(B_4^{(0)} + B_4^{(1)}U(1, \lambda)) = \det \left(B_4^{(0)} + B_4^{(1)}R(1) \begin{pmatrix} E_2 \cosh \beta & E_2 \sinh \beta \\ E_2 \sinh \beta & E_2 \cosh \beta \end{pmatrix} \right) + O \left(\frac{1}{|m|} \right).$$

Let us calculate

$$B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh \beta & E_2 \sinh \beta \\ E_2 \sinh \beta & E_2 \cosh \beta \end{pmatrix}.$$

By the definition of $R(x)$, we can write

$$\begin{aligned} R'_{k,\ell}(x) &= -\frac{1}{2} \sum_{m=1}^2 (P_{k,m+2}(x) + P_{k+2,m}(x)) R_{m,\ell}(x) \\ &\quad - \frac{1}{2} \sum_{m=1}^2 (P_{k,m}(x) + P_{k+2,m+2}(x)) R_{m,\ell+2}(x), \quad k, \ell = 1, 2 \\ R'_{k,\ell+2}(x) &= -\frac{1}{2} \sum_{m=1}^2 (P_{k,m}(x) + P_{k+2,m+2}(x)) R_{m,\ell}(x) \\ &\quad - \frac{1}{2} \sum_{m=1}^2 (P_{k,m+2}(x) + P_{k+2,m}(x)) R_{m,\ell+2}(x), \quad k, \ell = 1, 2 \\ R_{k,\ell}(0) &= \delta_{k\ell}, \quad k, \ell = 1, 2, \quad R_{k,\ell+2}(0) = 0, \quad k, \ell = 1, 2. \end{aligned}$$

Here and henceforth we set $\delta_{kk} = 1$ and $\delta_{k\ell} = 0$ if $k \neq \ell$. Setting

$$\begin{cases} r_{k,\ell}(x) = R_{k,\ell}(x) + R_{k,\ell+2}(x) \\ \tilde{r}_{k,\ell}(x) = R_{k,\ell}(x) - R_{k,\ell+2}(x), \end{cases} \quad k, \ell = 1, 2$$

we can reduce the preceding differential equation into

$$\begin{cases} r'_{k,\ell}(x) + \sum_{m=1}^2 \theta_{k,m}^P(x) r_{m,\ell}(x) = 0 \\ \tilde{r}'_{k,\ell}(x) + \sum_{m=1}^2 \tilde{\theta}_{k,m}^P(x) \tilde{r}_{m,\ell}(x) = 0 \\ r_{k,\ell}(0) = \tilde{r}_{k,\ell}(0) = \delta_{k\ell}, \end{cases} \quad k, \ell = 1, 2.$$

Recalling the definition of $G(\theta^P)(x)$ and $G(\tilde{\theta}^P)(x)$, we can write

$$R(x) = \frac{1}{2} \begin{pmatrix} G(\theta^P)(x) + G(\tilde{\theta}^P)(x) & G(\theta^P)(x) - G(\tilde{\theta}^P)(x) \\ G(\theta^P)(x) - G(\tilde{\theta}^P)(x) & G(\theta^P)(x) + G(\tilde{\theta}^P)(x) \end{pmatrix}. \quad (2.3.6)$$

Hence

$$\begin{aligned} & B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh \beta & E_2 \sinh \beta \\ E_2 \sinh \beta & E_2 \cosh \beta \end{pmatrix} \\ &= \begin{pmatrix} -\tilde{h} & E_2 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -\tilde{H} & E_2 \end{pmatrix} \begin{pmatrix} e^\beta G(\theta^P)(1) + e^{-\beta} G(\tilde{\theta}^P)(1) & e^\beta G(\theta^P)(1) - e^{-\beta} G(\tilde{\theta}^P)(1) \\ e^\beta G(\theta^P)(1) - e^{-\beta} G(\tilde{\theta}^P)(1) & e^\beta G(\theta^P)(1) + e^{-\beta} G(\tilde{\theta}^P)(1) \end{pmatrix} \\ &= \begin{pmatrix} -\tilde{h} & E_2 \\ \frac{e^\beta(E_2 - \tilde{H})G(\theta^P)(1) - e^{-\beta}(E_2 + \tilde{H})G(\tilde{\theta}^P)(1)}{2} & \frac{e^\beta(E_2 - \tilde{H})G(\theta^P)(1) + e^{-\beta}(E_2 + \tilde{H})G(\tilde{\theta}^P)(1)}{2} \end{pmatrix}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \det \left(B_4^{(0)} + B_4^{(1)} R(1) \begin{pmatrix} E_2 \cosh \beta & E_2 \sinh \beta \\ E_2 \sinh \beta & E_2 \cosh \beta \end{pmatrix} \right) \\ &= \frac{1}{2} \det \begin{pmatrix} 0 & E_2 \\ e^\beta \left\{ (E_2 - \tilde{H})G(\theta^P)(1)(E_2 + \tilde{h}) - e^{-2\beta}(E_2 + \tilde{H})G(\tilde{\theta}^P)(1)(E_2 - \tilde{h}) \right\} & * \end{pmatrix}. \end{aligned}$$

This determinant is not zero if and only if

$$\det \left(e^\beta \left\{ (E_2 - \tilde{H})G(\theta^P)(1)(E_2 + \tilde{h}) - e^{-2\beta}(E_2 + \tilde{H})G(\tilde{\theta}^P)(1)(E_2 - \tilde{h}) \right\} \right) \neq 0.$$

By $h_j \neq \pm 1$ and $H_j \neq \pm 1$ for $j = 1, 2$, $\det G(\theta^P)(1) \neq 0$ and the continuity of the determinant, for sufficiently large $\operatorname{Re} \beta > 0$, the preceding determinant is not equal to zero. Here we used that $\det G(\theta^P)(1) \neq 0$. In fact, for $y \in (0, 1)$, by $G(\theta)(x; y)$ we denote the solution to (2.1.15) such that $G(\theta)(y; y) = E_2$. Then the uniqueness of the initial value problem for (2.1.15) yields $G(\theta)(x; y)G(\theta)(y; x) = E_2$, which implies $\det G(\theta)(x; y) \neq 0$ for any $x, y \in (0, 1)$. Since $G(\theta^P)(1) = G(\theta^P)(1; 0)$ by the definition, we have $\det G(\theta^P)(1) \neq 0$.

Consequently we can choose sufficiently large $|m|$ and sufficiently large $\operatorname{Re} \beta > 0$ such that

$$\det(B_4^{(0)} + B_4^{(1)}U(1, \lambda)) \neq 0$$

for $\lambda \neq \beta + 2m\pi i$.

Therefore, the proof of Lemma 2.3.1 is completed. \square

Step 2. Let a 4×2 -matrix function $\phi(x, \lambda)$ be the solution to the following equations:

$$\begin{cases} B_4 \frac{d}{dx} \phi(x, \lambda) + P(x)\phi(x, \lambda) = \lambda \phi(x, \lambda), & 0 < x < 1 \\ \phi(0, \lambda) = \begin{pmatrix} E_2 \\ \tilde{h} \end{pmatrix}. \end{cases}$$

Then, $\lambda \in \mathbf{C}$ is an eigenvalue of A_P if and only if the determinant of

$$\Phi(\lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi(1, \lambda)$$

is equal to zero. Henceforth we call $\det \Phi(\lambda)$ the characteristic function for A_P . In fact, if ψ is an eigenfunction of A_P , then we can choose $(c_1, c_2) \neq (0, 0)$ such that

$$\psi(x, \lambda) = c_1 \phi_1(x, \lambda) + c_2 \phi_2(x, \lambda),$$

where ϕ_ℓ is the ℓ -th column vector of $\phi(x, \lambda)$, $\ell = 1, 2$. Since

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \psi(1, \lambda) = 0,$$

we have

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \psi(1, \lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} c_1 \phi_1(1, \lambda) + \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} c_2 \phi_2(1, \lambda) = 0.$$

Hence

$$\left\{ \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi_1(1, \lambda), \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi_2(1, \lambda) \right\}$$

is linearly dependent, so that $\det \Phi(\lambda) = 0$ follows.

Conversely, if

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi_\ell(1, \lambda), \quad \ell = 1, 2$$

are linearly dependent, then there exists $(c_1, c_2) \neq (0, 0)$ such that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} (c_1 \phi_1(1, \lambda) + c_2 \phi_2(1, \lambda)) = 0.$$

Then $\psi(x, \lambda) = c_1 \phi_1(x, \lambda) + c_2 \phi_2(x, \lambda)$ is an eigenfunction of A_P , that is, λ is an eigenvalue of A_P . Thus we have proved that λ is an eigenvalue of A_P if and only if $\det \Phi(\lambda) = 0$.

Moreover we can prove

Lemma 2.3.2 *The algebraic multiplicity of an eigenvalue λ_0 is equal to the multiplicity of λ_0 as zero of $\det \Phi(\lambda)$.*

The proof is given in section 2.8.

Let us calculate $\Phi(\lambda)$. Using the transformation formula, we have

$$\begin{aligned}
\phi(x, \lambda) &= R(x) \begin{pmatrix} \cosh \lambda x + h_1 \sinh \lambda x & 0 \\ 0 & \cosh \lambda x + h_2 \sinh \lambda x \\ \sinh \lambda x + h_1 \cosh \lambda x & 0 \\ 0 & \sinh \lambda x + h_2 \cosh \lambda x \end{pmatrix} \\
&+ \int_0^x K^{(1)}(y, x) \begin{pmatrix} \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \\ \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \end{pmatrix} dy \\
&+ \int_0^x K^{(2)}(y, x) \begin{pmatrix} h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \\ h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \end{pmatrix} dy. \tag{2.3.7}
\end{aligned}$$

Here $K^{(1)}(y, x)$ and $K^{(2)}(y, x)$ are defined by (2.3.3) and (2.3.4).

For simplicity, by $\tilde{\phi}(\lambda)$ we denote the integral terms on the right hand side of (2.3.7) with $x = 1$. Setting

$h_\ell = \tanh \mu_\ell$, we can write

$$\phi(1, \lambda) = R(1) \begin{pmatrix} \cosh(\lambda + \mu_1) & 0 \\ 0 & \cosh(\lambda + \mu_2) \\ \sinh(\lambda + \mu_1) & 0 \\ 0 & \sinh(\lambda + \mu_2) \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \tilde{\phi}(\lambda) \tag{2.3.8}$$

By (2.3.6), we have

$$\phi(1, \lambda) = \frac{1}{2} \begin{pmatrix} (A_{k,\ell} e^{\lambda+\mu_\ell} + B_{k,\ell} e^{-\lambda-\mu_\ell})_{k,\ell=1,2} \\ (A_{k,\ell} e^{\lambda+\mu_\ell} - B_{k,\ell} e^{-\lambda-\mu_\ell})_{k,\ell=1,2} \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \tilde{\phi}(\lambda),$$

where $G(\theta^P)(1) = (A_{k,\ell})_{k,\ell=1,2}$, $G(\tilde{\theta}^P)(1) = (B_{k,\ell})_{k,\ell=1,2}$.

Multiplying $\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix}$ from the left, we obtain

$$\begin{aligned}
\Phi(\lambda) &= \frac{1}{2} \begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell} e^{\lambda+\mu_\ell+\nu_k} - B_{k,\ell} e^{-(\lambda+\mu_\ell+\nu_k)})_{k,\ell=1,2} \\
&\times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \tilde{\Phi}(\lambda) \equiv \Phi_0(\lambda) + \tilde{\Phi}(\lambda) \tag{2.3.9}
\end{aligned}$$

where $H_\ell = -\tanh \nu_\ell$ and $\tilde{\Phi}(\lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \tilde{\phi}(\lambda)$.

Let us calculate $\tilde{\Phi}(\lambda)$. By integration by parts,

$$\begin{aligned} \tilde{\Phi}(\lambda) &= \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda} \\ &\times \left[K^{(1)}(y, 1) \begin{pmatrix} \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \\ \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \end{pmatrix} + K^{(2)}(y, 1) \begin{pmatrix} h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \\ h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \end{pmatrix} \right]_{y=0}^{y=1} \\ &- \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda} \int_0^1 \frac{\partial}{\partial y} K^{(1)}(y, 1) \begin{pmatrix} \sinh \lambda y & 0 \\ 0 & \sinh \lambda y \\ \cosh \lambda y & 0 \\ 0 & \cosh \lambda y \end{pmatrix} dy \\ &- \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \frac{1}{\lambda} \int_0^1 \frac{\partial}{\partial y} K^{(2)}(y, 1) \begin{pmatrix} h_1 \cosh \lambda y & 0 \\ 0 & h_2 \cosh \lambda y \\ h_1 \sinh \lambda y & 0 \\ 0 & h_2 \sinh \lambda y \end{pmatrix} dy. \end{aligned}$$

Hence

$$|\tilde{\Phi}(\lambda)| \leq \frac{C}{|\lambda|} e^{|\operatorname{Re} \lambda|}, \quad (2.3.10)$$

where we recall that $|\tilde{\Phi}(\lambda)|$ denotes the matrix norm of $\tilde{\Phi}(\lambda)$ and $C > 0$ is a positive constant which is independent of λ .

We show that there exists a positive constant K satisfying

$$|\operatorname{Re} \lambda| \leq K \quad \text{for any } \lambda \in \sigma(A_P). \quad (2.3.11)$$

If not, then there exists a sequence $\{\lambda_m\}_{m \in \mathbb{N}} \subset \sigma(A_P)$ such that $\lim_{m \rightarrow \infty} |\operatorname{Re} \lambda_m| = \infty$. Without loss of generality, we suppose that there exists a subsequence $\{\lambda_{j_m}\}_{m \in \mathbb{N}} \subset \{\lambda_m\}_{m \in \mathbb{N}}$ satisfying $\lim_{m \rightarrow \infty} \operatorname{Re} \lambda_{j_m} = \infty$. Since λ_{j_m} are eigenvalues, by (2.3.9) we have

$$\begin{aligned} 0 &= |\det \Phi(\lambda_{j_m})| \\ &= \left| \det \left(\frac{1}{2} \begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell} e^{\lambda_{j_m} + \mu_\ell + \nu_k} - B_{k,\ell} e^{-(\lambda_{j_m} + \mu_\ell + \nu_k)})_{k,\ell=1,2} \right. \right. \\ &\quad \left. \left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \tilde{\Phi}(\lambda_{j_m}) \right) \right|. \end{aligned}$$

Here, using (2.3.10), we have

$$0 = |\det \Phi(\lambda_{j_m})| = \left| \det \left(\frac{1}{2} \begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell} e^{i \operatorname{Im} \lambda_{j_m} + \mu_\ell + \nu_k} - B_{k,\ell} e^{-2 \operatorname{Re} \lambda_{j_m} - i \operatorname{Im} \lambda_{j_m} - \mu_\ell - \nu_k})_{k,\ell=1,2} \right. \right. \\ \left. \left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} + \varepsilon_m \right) \right| e^{2 \operatorname{Re} \lambda_{j_m}}$$

where $\lim_{|m| \rightarrow \infty} |\varepsilon_m| = 0$. Here we have $\det G(\theta^P)(1) \neq 0$, which is derived at the end of Step 1. Hence, since $\det(A_{k,\ell} e^{\mu_\ell + \nu_k})_{k,\ell=1,2} = e^{\mu_1 + \mu_2 + \nu_1 + \nu_2} \det G(\theta^P)(1) \neq 0$. Then taking $|m| \rightarrow \infty$, the right hand side tends to ∞ , because of the continuity of the determinant. Thus this yields a contradiction and the proof of (2.3.11) is completed.

Step 3. We choose sufficiently large $K > 0$ satisfying (2.3.11) and

$$\left| \frac{1}{2} \operatorname{Re} \log \alpha_j \right| < K, \quad j = 1, 2.$$

We further choose $K > 0$ large enough, so that

$$|\det \Phi(\lambda) - \det \Phi_0(\lambda)| < |\det \Phi_0(\lambda)|$$

for all λ with $|\operatorname{Re} \lambda| = K$. Here we recall that $\Phi_0 = \Phi - \tilde{\Phi}$. It is possible because (2.3.10) holds and $\det(A_{k,\ell} e^{\mu_\ell + \nu_k})_{k,\ell=1,2} \neq 0$, $\det(B_{k,\ell} e^{-\mu_\ell - \nu_k})_{k,\ell=1,2} \neq 0$ in (2.3.9).

Then we set

$$K_m = \left\{ \lambda ; -K - 1 < \operatorname{Re} \lambda < K + 1, \tilde{\alpha} + m\pi - \frac{\pi}{2} < \operatorname{Im} \lambda < \tilde{\alpha} + m\pi + \frac{\pi}{2} \right\}, \quad m \in \mathbf{Z},$$

using the constant $\tilde{\alpha}$ defined in the statement of Theorem 2.2.1.

Now we will prove the following assertion :

There exists $N \in \mathbf{N}$ such that in K_m there are exactly 2 zeros of $\det \Phi$ by taking the algebraic multiplicities into consideration for $|m| \geq N$.

Noting

$$K_m = \{\lambda + m\pi i ; \lambda \in K_0\},$$

and

$$\Phi_0(\lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda + \mu_1) & 0 \\ 0 & \cosh(\lambda + \mu_2) \\ \sinh(\lambda + \mu_1) & 0 \\ 0 & \sinh(\lambda + \mu_2) \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix},$$

by definition (2.3.9) of Φ_0 , we have

$$\min_{\lambda \in \partial K_m} |\det \Phi_0(\lambda)| = \min_{\lambda \in \partial K_0} |\det \Phi_0(\lambda)| \equiv L.$$

For sufficiently large $N \in \mathbb{N}$, we have

$$\sup_{\lambda \in \partial K_m} |\det \Phi(\lambda) - \det \Phi_0(\lambda)| < L, \quad N \leq |m|$$

by (2.3.10) and the linearity of the determinant in each column. Therefore

$$|\det \Phi(\lambda) - \det \Phi_0(\lambda)| < |\det \Phi_0(\lambda)|, \quad \text{on } \lambda \in \partial K_m. \quad (2.3.12)$$

On the other hand,

$$\begin{aligned} & \det \Phi_0(\lambda) = 0 \\ \Leftrightarrow & \det \left(\frac{1}{2} \begin{pmatrix} \frac{1}{\cosh \nu_1} & 0 \\ 0 & \frac{1}{\cosh \nu_2} \end{pmatrix} (A_{k,\ell} e^{\lambda + \mu_\ell + \nu_k} - B_{k,\ell} e^{-(\lambda + \mu_\ell + \nu_k)})_{k,\ell=1,2} \right. \\ & \quad \left. \times \begin{pmatrix} \frac{1}{\cosh \mu_1} & 0 \\ 0 & \frac{1}{\cosh \mu_2} \end{pmatrix} \right) = 0 \\ \Leftrightarrow & \det \left((A_{k,\ell} e^{\lambda + \mu_\ell + \nu_k} - B_{k,\ell} e^{-(\lambda + \mu_\ell + \nu_k)})_{k,\ell=1,2} \right) = 0 \\ \Leftrightarrow & \det \left(e^{2\lambda} \begin{pmatrix} e^{\nu_1} & 0 \\ 0 & e^{\nu_2} \end{pmatrix} G(\theta^P)(1) \begin{pmatrix} e^{\mu_1} & 0 \\ 0 & e^{\mu_2} \end{pmatrix} \right. \\ & \quad \left. - \begin{pmatrix} e^{-\nu_1} & 0 \\ 0 & e^{-\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-\mu_1} & 0 \\ 0 & e^{-\mu_2} \end{pmatrix} \right) = 0 \\ \Leftrightarrow & \det \left(e^{2\lambda} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right) = 0. \end{aligned}$$

Therefore, from the definition of α_1 and α_2 , the zeros of $\det \Phi_0$ are

$$\frac{1}{2} \log \alpha_j + m\pi i, \quad m \in \mathbf{Z}.$$

By the Rouché theorem, all K_m contains exactly 2 zeros of $\det \Phi$ by taking into consideration the multiplicities. Thus the proof of Assertion is completed.

Setting

$$K^{(0)} \equiv \left\{ \lambda ; -K - 1 < \operatorname{Re} \lambda < K + 1, \quad \tilde{\alpha} - N\pi + \frac{\pi}{2} < \operatorname{Im} \lambda < \tilde{\alpha} + N\pi - \frac{\pi}{2} \right\},$$

we have

$$|\det \Phi(\lambda) - \det \Phi_0(\lambda)| < |\det \Phi_0(\lambda)| \quad \text{on } \partial K^{(0)},$$

by (2.3.12). Hence, since $\det \Phi_0(\lambda) = 0$ possesses $2(2N - 1)$ zeros in $K^{(0)}$, the Rouché theorem yields that $K^{(0)}$ contains exactly $2(2N - 1)$ zeros of $\det \Phi$ by taking into consideration the multiplicities.

According to the argument of this step, in terms of Lemma 2.3.2, we see :

There exists $N \in \mathbf{N}$ such that K_m contains exactly 2 eigenvalues of A_P for all $|m| \geq N$ and $K^{(0)}$ contains $2(2N - 1)$ eigenvalues of A_P by taking into consideration the algebraic multiplicities.

Step 4. We will show the asymptotic behaviour of the eigenvalues. Here let $N \leq |m|$. We note that two zeros of $\det \Phi$ are included in K_m with the multiplicities. Now we consider $\det \Phi(\lambda) = 0$ in K_m . By (2.3.9) and (2.3.10), using the linearity of the determinant in each column, we see that $\det \Phi(\lambda) = 0$, $\lambda \in K_m$, is rewritten as

$$\det \left((A_{k,\ell} e^{\lambda + \mu_\ell + \nu_k} - B_{k,\ell} e^{-(\lambda + \mu_\ell + \nu_k)})_{k,\ell=1,2} \right) = O \left(\frac{1}{|m|} \right), \quad \lambda \in K_m.$$

By $|\operatorname{Re} \lambda| < K + 1$, we can rewrite the left hand side to obtain

$$\det \left\{ e^{2\lambda} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = O\left(\frac{1}{|m|}\right), \quad \lambda \in K_m.$$

We rewrite this equation as

$$(e^{2\lambda})^2 + a_1 e^{2\lambda} + a_0 = O\left(\frac{1}{|m|}\right), \quad \lambda \in K_m \quad (2.3.13)$$

where a_1 and a_0 are constants. That is, λ is a root of

$$e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0, \quad \kappa_m = O\left(\frac{1}{|m|}\right).$$

We set $\zeta_m = \frac{1}{2} \log \alpha_1 + m\pi i \in K_m$. Then $\alpha_1 = e^{2\zeta_m}$ and by the definition of α_1 , we have

$$\det \left\{ e^{2\zeta_m} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = 0,$$

that is, ζ_m is a root of the equation in λ :

$$e^{4\lambda} + a_1 e^{2\lambda} + a_0 = 0.$$

Using the Rouché theorem, we will estimate the difference between ζ_m and a root of (2.3.13). First, for sufficiently large $|m|$, we consider a circle S_{ζ_m, r_m} centred at ζ_m with radius r_m . For large $|m|$, we will find r_m such that

$$|\kappa_m| < |e^{4\lambda} + a_1 e^{2\lambda} + a_0| \quad \text{on } S_{\zeta_m, r_m}. \quad (2.3.14)$$

We set $\rho(\lambda) = e^{4\lambda} + a_1 e^{2\lambda} + a_0$ and $\eta = a_1 + 2\alpha_1$. Let us calculate $|\rho(\lambda)|$ under $|\lambda - \zeta_m| = r_m$. By $\rho(\zeta_m) = 0$,

we have

$$\begin{aligned} |e^{4\lambda} + a_1 e^{2\lambda} + a_0| &= | \{(e^{2\lambda} - \alpha_1) + \alpha_1\}^2 + a_1 \{(e^{2\lambda} - \alpha_1) + \alpha_1\} + a_0 | \\ &= |(e^{2\lambda} - \alpha_1)^2 + 2(e^{2\lambda} - \alpha_1)\alpha_1 + a_1(e^{2\lambda} - \alpha_1)| = |(e^{2\lambda} - \alpha_1)^2 + \eta(e^{2\lambda} - \alpha_1)|. \end{aligned}$$

Case 1 : (2.1.12) possesses distinct roots α_1 and α_2 .

Then $\eta \neq 0$ and we have

$$|\rho(\lambda)| = |e^{2\lambda} - \alpha_1| |(e^{2\lambda} - \alpha_1) + \eta| \geq C_0 r |(e^{2\lambda} - \alpha_1) + \eta| \quad \text{on } S_{\zeta_m, r}.$$

At the last inequality, we used $\zeta_m = \frac{1}{2} \log \alpha_1 + m\pi i$ and $|e^{2\lambda} - \alpha_1| = |\alpha_1| |e^{2(\lambda - \zeta_m)} - 1|$. Taking sufficiently small $d < 1$, by $|\eta| > 0$ we can estimate

$$|(e^{2\lambda} - \alpha_1) + \eta| \geq |\eta| - C_0 r \geq C > 0 \quad \text{on } S_{\zeta_m, r}, \text{ for all } r < d,$$

where d and C are dependent on a_j, α_1 , and independent of m . Hence

$$|\rho(\lambda)| \geq Cr.$$

Therefore, since $|\kappa_m| = O\left(\frac{1}{|m|}\right)$, for sufficiently large $C' > 0$, we set $r_m = \frac{C'}{|m|}$, so that $Cr_m \geq |\kappa_m|$, that is,

(2.3.14) holds on $S_{\lambda_m^{(1)}, r_m}$.

Moreover $\rho(\lambda)$ possesses a unique zero in $\{\lambda; |\lambda - \zeta_m| < r_m\}$ for sufficiently large $|m|$. Applying the Rouché theorem, in terms of (2.3.14), we see that $e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0$ possesses a unique zero denoted by $\lambda_{1,m}$ in $\{\lambda; |\lambda - \zeta_m| < r_m\}$ and

$$\lambda_{1,m} = \frac{1}{2} \log \alpha_1 + m\pi i + O\left(\frac{1}{|m|}\right).$$

For α_2 , we can argue similarly. Thus the proof of (2.2.1) is completed in Case 1.

Case 2 : (2.1.12) possesses the multiple root $\alpha_1 = \alpha_2$. Then $\eta = 0$, and

$$|\rho(\lambda)| = |e^{2\lambda} - e^{2\zeta_m}|^2 \geq C_0^2 r^2 \quad \text{on } S_{\zeta_m, r}$$

and for sufficiently large $|m|$, the function $\rho(\lambda)$ possesses exactly two zeros in $\{\lambda; |\lambda - \zeta_m| < r_m\}$ including the multiplicity. Choosing $r_m = \frac{C'}{\sqrt{|m|}}$ with large $C' > 0$, we can argue similarly to Case 1, in terms of the

Rouché theorem to see that $e^{4\lambda} + a_1 e^{2\lambda} + a_0 + \kappa_m = 0$ possesses two zeros $\lambda_{1,m}$ and $\lambda_{2,m}$ in $\{\lambda; |\lambda - \zeta_m| < r_m\}$

by taking into consideration the multiplicities, and

$$|\lambda_{j,m} - \zeta_m| = O\left(\frac{1}{\sqrt{|m|}}\right), \quad j = 1, 2$$

as $|m| \rightarrow \infty$. Thus the proof of Theorem 2.2.1 is completed. \square

2.4 The proof of Theorem 2.2.3

In this section, we prove Theorem 2.2.3. For this, we apply the Bari theorem (e.g., Gohberg and Kreĭn [16]).

Let α_1, α_2 be the solutions to (2.1.12). Because of the assumption $\alpha_1 \neq \alpha_2$, for sufficiently large $|m|$, we see that

$$K_m = \left\{ \lambda; -K - 1 < \operatorname{Re} \lambda < K + 1, \tilde{\alpha} + m\pi - \frac{\pi}{2} < \operatorname{Im} \lambda < \tilde{\alpha} + m\pi + \frac{\pi}{2} \right\}$$

contains two eigenvalues each of whose algebraic multiplicity is one.

Let us set $\beta_j = \frac{1}{2} \log \alpha_j$, $j = 1, 2$. Now we prove that

$$\operatorname{rank} \left(\begin{array}{cc} -\tilde{H} & E_2 \end{array} \right) R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} = 1.$$

By (2.3.6), this rank is equal to

$$\operatorname{rank} \left\{ e^{2\beta_1} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\}.$$

By the assumption that $\alpha_1 = e^{2\beta_1}$ is the solution to (2.1.12), the rank is not equal to 2. We assume

$$\operatorname{rank} \left\{ e^{2\beta_1} E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} = 0.$$

Then because each column of this matrix is equal to 0, we have

$$\frac{d}{d\alpha} \left[\det \left\{ \alpha E_2 - \begin{pmatrix} e^{-2\nu_1} & 0 \\ 0 & e^{-2\nu_2} \end{pmatrix} G(\tilde{\theta}^P)(1) \begin{pmatrix} e^{-2\mu_1} & 0 \\ 0 & e^{-2\mu_2} \end{pmatrix} G(\theta^P)(1)^{-1} \right\} \right] \Big|_{\alpha=e^{2\beta_1}} = 0.$$

This contradicts the assumption that quadratic equation (2.1.12) has distinct roots. Therefore we obtain

$$\text{rank} \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} = 1.$$

Then there exists $(c_1, c_2) \neq (0, 0)$ such that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Without loss of generality, we can assume that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_1 + \mu_2) \\ 0 \\ \sinh(\beta_1 + \mu_2) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.4.1)$$

Then we have $c_1 \neq 0$.

Similarly, we can take $(d_1, d_2) \neq (0, 0)$ such that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_2 + \mu_1) & 0 \\ 0 & \cosh(\beta_2 + \mu_2) \\ \sinh(\beta_2 + \mu_1) & 0 \\ 0 & \sinh(\beta_2 + \mu_2) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we can assume that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_2 + \mu_2) \\ 0 \\ \sinh(\beta_2 + \mu_2) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

without loss of generality. Then we can directly verify that $d_1 \neq 0$.

By $S(x)$, we denote a 4×4 matrix

$$S(x) = R(x) \begin{pmatrix} c_1 \cosh(\beta_1 x + \mu_1) & d_1 \cosh(\beta_2 x + \mu_1) & c_1 \sinh(\beta_1 x + \mu_1) & d_1 \sinh(\beta_2 x + \mu_1) \\ c_2 \cosh(\beta_1 x + \mu_2) & d_2 \cosh(\beta_2 x + \mu_2) & c_2 \sinh(\beta_1 x + \mu_2) & d_2 \sinh(\beta_2 x + \mu_2) \\ c_1 \sinh(\beta_1 x + \mu_1) & d_1 \sinh(\beta_2 x + \mu_1) & c_1 \cosh(\beta_1 x + \mu_1) & d_1 \cosh(\beta_2 x + \mu_1) \\ c_2 \sinh(\beta_1 x + \mu_2) & d_2 \sinh(\beta_2 x + \mu_2) & c_2 \cosh(\beta_1 x + \mu_2) & d_2 \cosh(\beta_2 x + \mu_2) \end{pmatrix}.$$

Since the property of the determinant yields

$$\begin{aligned} \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} &= \det \begin{pmatrix} A-B & B-A \\ B & A \end{pmatrix} \\ &= \det \begin{pmatrix} A-B & 0 \\ B & A+B \end{pmatrix} = \det(A-B) \det(A+B) \end{aligned}$$

for 2×2 -matrices A, B , we have

$$\begin{aligned} \det S(x) &= \det R(x) \det \begin{pmatrix} c_1 \exp(\beta_1 x + \mu_1) & d_1 \exp(\beta_2 x + \mu_1) \\ c_2 \exp(\beta_1 x + \mu_2) & d_2 \exp(\beta_2 x + \mu_2) \end{pmatrix} \\ &\times \det \begin{pmatrix} c_1 \exp(-\beta_1 x - \mu_1) & d_1 \exp(-\beta_2 x - \mu_1) \\ c_2 \exp(-\beta_1 x - \mu_2) & d_2 \exp(-\beta_2 x - \mu_2) \end{pmatrix}. \end{aligned}$$

If $c_1 d_2 - c_2 d_1 \neq 0$, then the inverse matrix $S^{-1}(x)$ exists. We will prove $c_1 d_2 - c_2 d_1 \neq 0$. If not, then we can take a constant γ such that

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \gamma \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hence β_1 and β_2 are the solution to the following equation in λ :

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda + \mu_1) & 0 \\ 0 & \cosh(\lambda + \mu_2) \\ \sinh(\lambda + \mu_1) & 0 \\ 0 & \sinh(\lambda + \mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \quad (2.4.2)$$

which implies

$$\begin{cases} d_{1,1} e^\lambda + d_{1,2} e^{-\lambda} = 0 \\ d_{2,1} e^\lambda + d_{2,2} e^{-\lambda} = 0 \end{cases}$$

with some $d_{k,\ell} \in \mathbf{C}$, $k, \ell = 1, 2$. Then there exists $d_{k,\ell} \neq 0$. Otherwise all $\lambda \in \mathbf{C}$ is the solution to (2.4.2),

which means that $\Phi_0(\lambda) = 0$ for all $\lambda \in \mathbf{C}$. This is a contradiction.

Therefore dividing some $d_{k,\ell} \neq 0$, we obtain $e^{2\beta_1} = e^{2\beta_2}$. Hence $2\beta_1 - 2\beta_2 = 2k\pi i$ with some $k \in \mathbf{Z}$, that

is,

$$\log \alpha_1 = \log \alpha_2 + 2k\pi i.$$

This contradicts that $\alpha_1 \neq \alpha_2$. Thus we proved that $c_1 d_2 - c_2 d_1 \neq 0$.

By the definition of S , we have

$$S(x) \begin{pmatrix} \cos m\pi x \\ 0 \\ i \sin m\pi x \\ 0 \end{pmatrix} = R(x) \begin{pmatrix} c_1 \cosh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \cosh(\beta_1 x + m\pi i x + \mu_2) \\ c_1 \sinh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \sinh(\beta_1 x + m\pi i x + \mu_2) \end{pmatrix},$$

$$S(x) \begin{pmatrix} 0 \\ \cos m\pi x \\ 0 \\ i \sin m\pi x \end{pmatrix} = R(x) \begin{pmatrix} d_1 \cosh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \cosh(\beta_2 x + m\pi i x + \mu_2) \\ d_1 \sinh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \sinh(\beta_2 x + m\pi i x + \mu_2) \end{pmatrix}.$$

We set

$$e_{1,m} := R(x) \begin{pmatrix} c_1 \cosh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \cosh(\beta_1 x + m\pi i x + \mu_2) \\ c_1 \sinh(\beta_1 x + m\pi i x + \mu_1) \\ c_2 \sinh(\beta_1 x + m\pi i x + \mu_2) \end{pmatrix}, \quad e_{2,m} := R(x) \begin{pmatrix} d_1 \cosh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \cosh(\beta_2 x + m\pi i x + \mu_2) \\ d_1 \sinh(\beta_2 x + m\pi i x + \mu_1) \\ d_2 \sinh(\beta_2 x + m\pi i x + \mu_2) \end{pmatrix}.$$

Since $S(x)$ is invertible and

$$\left\{ \begin{pmatrix} \cos m\pi x \\ 0 \\ i \sin m\pi x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cos m\pi x \\ 0 \\ i \sin m\pi x \end{pmatrix} \right\}_{m \in \mathbf{Z}}$$

is a Riesz basis in $\{L^2(0,1)\}^4$, we see that $\{e_{1,m}, e_{2,m}\}_{m \in \mathbf{Z}}$ is a Riesz basis in $\{L_2(0,1)\}^4$ (e.g., Gohberg and Kreĭn [16]).

We can write an eigenfunction corresponding to $\lambda_{1,m}$ as

$$R(x) \begin{pmatrix} \cosh(\lambda_{1,m} x + \mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m} x + \mu_2) \\ \sinh(\lambda_{1,m} x + \mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m} x + \mu_2) \end{pmatrix} \begin{pmatrix} \tilde{c}_1^{(m)} \\ \tilde{c}_2^{(m)} \end{pmatrix} + \begin{pmatrix} \tilde{\phi}_1(\lambda_{1,m}, x) & \tilde{\phi}_2(\lambda_{1,m}, x) \end{pmatrix} \begin{pmatrix} \tilde{c}_1^{(m)} \\ \tilde{c}_2^{(m)} \end{pmatrix}. \quad (2.4.3)$$

from (2.3.7). Here, $\tilde{\phi}_1(\lambda_{1,m}, x), \tilde{\phi}_2(\lambda_{1,m}, x)$, $m \in \mathbf{Z}$ correspond to the integral terms on (2.3.7) and $\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}, m \in \mathbf{Z}$ are constants such that

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda_{1,m} + \mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m} + \mu_2) \\ \sinh(\lambda_{1,m} + \mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m} + \mu_2) \end{pmatrix} \begin{pmatrix} \tilde{c}_1^{(m)} \\ \tilde{c}_2^{(m)} \end{pmatrix} + \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1(\lambda_{1,m}, 1) & \tilde{\phi}_2(\lambda_{1,m}, 1) \end{pmatrix} \begin{pmatrix} \tilde{c}_1^{(m)} \\ \tilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.4.4)$$

Such $\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}$, $m \in \mathbf{Z}$ exist because $\lambda_{1,m}$ are eigenvalues. By (2.2.1) and (2.3.10), we choose $C > 0$ such that

$$|\tilde{\phi}_k(\lambda_{1,m}, x)| \leq \frac{C}{|m|}, \quad m \in \mathbf{Z}, \quad 0 < x < 1. \quad (2.4.5)$$

Now we prove that for sufficiently large $|m|$, we can take $\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}$ such that $\tilde{c}_1^{(m)} = c_1$ and $\tilde{c}_2^{(m)} - c_2 = O\left(\frac{1}{|m|}\right)$.

Because we assume (2.4.1), we have

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\lambda_{1,m} + \mu_2) \\ 0 \\ \sinh(\lambda_{1,m} + \mu_2) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.4.6)$$

for sufficiently large $|m|$. Then, by (2.4.4) and (2.4.5), we have $\tilde{c}_1^{(m)} \neq 0$ for sufficiently large $|m|$. Multiplying $\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)}$ with $\frac{c_1}{\tilde{c}_1^{(m)}}$, we can take $(c_1, \tilde{c}_2^{(m)})$ as $(\tilde{c}_1^{(m)}, \tilde{c}_2^{(m)})$.

Now let us prove $c_2 - \tilde{c}_2^{(m)} = O\left(\frac{1}{|m|}\right)$. For this purpose, we will first prove that $\tilde{c}_2^{(m)} = O(1)$. Equation

(2.4.4) yields

$$\begin{aligned} & \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda_{1,m} + \mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m} + \mu_2) \\ \sinh(\lambda_{1,m} + \mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m} + \mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ \tilde{c}_2^{(m)} \end{pmatrix} \\ & + \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1(\lambda_{1,m}, 1) & \tilde{\phi}_2(\lambda_{1,m}, 1) \end{pmatrix} \begin{pmatrix} c_1 \\ \tilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (2.4.7)$$

that is,

$$\begin{aligned} & c_1 \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\lambda_{1,m} + \mu_1) \\ 0 \\ \sinh(\lambda_{1,m} + \mu_1) \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \tilde{\phi}_1(\lambda_{1,m}, 1) \\ & + \tilde{c}_2^{(m)} \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\lambda_{1,m} + \mu_2) \\ 0 \\ \sinh(\lambda_{1,m} + \mu_2) \end{pmatrix} + \tilde{c}_2^{(m)} \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \tilde{\phi}_2(\lambda_{1,m}, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

By using (2.4.6) and $\tilde{\phi}_k(\lambda_{1,m}, 1) = O\left(\frac{1}{|m|}\right)$, we obtain $\tilde{c}_2^{(m)} = O(1)$.

We will estimate $c_2 - \tilde{c}_2^{(m)}$. Because of (2.4.7) and $\lambda_{1,m} = \beta_1 + m\pi i + \delta_m$ with $\delta_m = O\left(\frac{1}{|m|}\right)$, we have

$$\begin{aligned} & \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} \cosh \delta_m \begin{pmatrix} c_1 \\ \tilde{c}_2^{(m)} \end{pmatrix} \\ & + \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix}^* R(1) \begin{pmatrix} \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \\ \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \end{pmatrix} \sinh \delta_m \begin{pmatrix} c_1 \\ \tilde{c}_2^{(m)} \end{pmatrix} \\ & + \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1(\lambda_{1,m}, 1) & \tilde{\phi}_2(\lambda_{1,m}, 1) \end{pmatrix} \begin{pmatrix} c_1 \\ \tilde{c}_2^{(m)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

From this equation, we subtract the following:

$$\begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} \cosh(\beta_1 + \mu_1) & 0 \\ 0 & \cosh(\beta_1 + \mu_2) \\ \sinh(\beta_1 + \mu_1) & 0 \\ 0 & \sinh(\beta_1 + \mu_2) \end{pmatrix} \cosh \delta_m \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which follows from (2.4.2). Then, since the second and the third terms on the left hand side are bounded by

$O\left(\frac{1}{|m|}\right)$ in terms of $\delta_m = O\left(\frac{1}{|m|}\right)$ and (2.4.5), we obtain

$$\cosh \delta_m \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} R(1) \begin{pmatrix} 0 \\ \cosh(\beta_1 + \mu_2) \\ 0 \\ \sinh(\beta_1 + \mu_2) \end{pmatrix} (c_2 - \tilde{c}_2^{(m)}) = O\left(\frac{1}{|m|}\right).$$

Therefore, by (2.4.1) we have

$$c_2 - \tilde{c}_2^{(m)} = O\left(\frac{1}{|m|}\right).$$

Thus for sufficiently large $|m|$, we can choose an eigenfunction $f_{1,m}$ corresponding to $\lambda_{1,m}$ such that

$$f_{1,m}(x) = R(x) \begin{pmatrix} \cosh(\lambda_{1,m}x + \mu_1) & 0 \\ 0 & \cosh(\lambda_{1,m}x + \mu_2) \\ \sinh(\lambda_{1,m}x + \mu_1) & 0 \\ 0 & \sinh(\lambda_{1,m}x + \mu_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 + O\left(\frac{1}{|m|}\right) \end{pmatrix} + O\left(\frac{1}{|m|}\right). \quad (2.4.8)$$

For the eigenvalue $\lambda_{2,m}$ for sufficiently large $|m|$, we can argue similarly and can choose an eigenfunction

$f_{2,m}$ corresponding to $\lambda_{2,m}$ such that

$$f_{2,m}(x) = R(x) \begin{pmatrix} \cosh(\lambda_{2,m}x + \mu_1) & 0 \\ 0 & \cosh(\lambda_{2,m}x + \mu_2) \\ \sinh(\lambda_{2,m}x + \mu_1) & 0 \\ 0 & \sinh(\lambda_{2,m}x + \mu_2) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 + O\left(\frac{1}{|m|}\right) \end{pmatrix} + O\left(\frac{1}{|m|}\right). \quad (2.4.9)$$

Supplementing root vectors to $f_{j,m}$, $j = 1, 2$ for sufficiently large $|m|$, we can obtain the totality of all the root vectors which can be denoted by $\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$ without fear of confusion such that

$$|e_{j,m}(x) - f_{j,m}(x)| = O\left(\frac{1}{|m|}\right), \quad 0 < x < 1.$$

Therefore we have

$$\sum_{j=1,2} \sum_{m \in \mathbf{Z}} \|e_{j,m} - f_{j,m}\|_{\{L_2(0,1)\}^4}^2 < \infty.$$

If $\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$ is linearly independent, then we can complete the proof of the theorem by the Bari theorem (e.g., [16]). Let us prove the linear independence of $\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$. For this purpose, we renumber the eigenvalues of A_P and the root vectors $\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$ as follows. In terms of Theorem 2.2.1, we number the eigenvalues $\{\lambda_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$ as

$$\sigma(A_P) = \{\mu_k\}_{k \in \mathbf{Z}} \cup \{\nu_\ell\}_{1 \leq \ell \leq N},$$

where $\mu_k, k \in \mathbf{Z}$ are the eigenvalues with algebraic multiplicity one, $\nu_\ell, 1 \leq \ell \leq N$ are the eigenvalues with algebraic multiplicity $\chi_\ell \geq 2$ and

$$\mu_{k_1} \neq \mu_{k_2}, \quad \nu_{\ell_1} \neq \nu_{\ell_2}, \quad k_1 \neq k_2, \ell_1 \neq \ell_2.$$

We renumber the root vectors $\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}}$ as

$$\{f_{j,m}\}_{j=1,2, m \in \mathbf{Z}} = \{g_k\}_{k \in \mathbf{Z}} \cup \{h_{\ell,j}\}_{1 \leq \ell \leq N, 1 \leq j \leq \chi_\ell},$$

where g_k is an eigenfunction corresponding to the eigenvalue μ_k , and $\{h_{\ell,j}\}_{1 \leq j \leq \chi_\ell}$ is a basis of $\{\phi; (A_P - \nu_\ell)^k \phi = 0 \text{ for some } k \in \mathbf{N}\}$.

Now we verify that

$$\sum_{\ell=1,2,\dots,N, j=1,2,\dots,\chi_\ell} \alpha_{\ell,j} h_{\ell,j} + \sum_{k \in \mathbf{Z}} \beta_k g_k = 0, \quad \alpha_{\ell,j}, \beta_k \in \mathbf{C} \quad (2.4.10)$$

implies $\alpha_{\ell,j} = 0$, $1 \leq \ell \leq N$, $1 \leq j \leq \chi_\ell$ and $\beta_k = 0$, $k \in \mathbf{Z}$. We define

$$P_k = \frac{1}{2\pi i} \int_{\Gamma_k} (\mu - A_P)^{-1} d\mu, \quad k \in \mathbf{Z}$$

where Γ_k , $k \in \mathbf{Z}$ is a sufficiently small circle centred at μ_k including no other points of $\sigma(A_P)$. By Theorem 2.2.1, such Γ_k exists. Then

$$P_k g_k = g_k, \quad P_k g_{k_1} = 0, \quad P_k h_{\ell,j} = 0 \quad \text{if } k \neq k_1, \quad 1 \leq \ell \leq N, \quad 1 \leq j \leq \chi_\ell$$

hold (e.g., Kato [22]). Applying P_k to (2.4.10), we have $\beta_k = 0$, $k \in \mathbf{Z}$. Since $\{h_{\ell,j}\}_{1 \leq \ell \leq N, 1 \leq j \leq \chi_\ell}$ is a linearly independent system, we obtain $\alpha_{\ell,j} = 0$, $1 \leq \ell \leq N$, $1 \leq j \leq \chi_\ell$. Thus the proof of Theorem 2.2.3 is completed.

2.5 The proof of Theorem 2.1.4

We denote the adjoint operator of A_P by A_P^* . We can easily see that

$$(A_P^* u)(x) = -B_4 \frac{du}{dx}(x) + P^t(x)u(x) \quad 0 < x < 1$$

$$D(A_P^*) = \{u \in \{H^1(0,1)\}^4; u_{\ell+n}(0) + h_\ell u_\ell(0) = 0, \quad u_{\ell+n}(1) + H_\ell u_\ell(1) = 0, \quad \ell = 1, 2\}.$$

Here P^t denotes the transpose matrix of P . By Theorem 2.2.1, we can number all the eigenvalues of A_P as $\{\lambda_m\}_{|m| \leq N-1} \cup \{\lambda_{j,m}\}_{|m| \geq N, j=1,2}$ such that the algebraic multiplicity of $\lambda_{j,m}$ is one for $|m| \geq N$ and $j = 1, 2$, and the value λ_m , $|m| \leq N - 1$ appears as many times as its algebraic multiplicity. According to the numbering of the eigenvalues, we number the eigenvectors and the associated root vectors. That is, in the case $|m| \geq N$, for $j = 1, 2$ we choose an eigenvector $f_{j,m}$ of A_P for $\lambda_{j,m}$ satisfying (2.4.8) and (2.4.9). We note that by Condition (III) an eigenvector is determined uniquely up to multiples. Furthermore we know (e.g., [22]) that $\overline{\sigma(A_P)} = \sigma(A_P^*)$ and the algebraic multiplicity of $\bar{\lambda} \in \sigma(A_P^*)$ is equal to the one of

$\lambda \in \sigma(A_P)$. By $g_{j,m}$, $j = 1, 2, |m| \geq N$, we denote an eigenvector of A_P^* for $\overline{\lambda_{j,m}}$ such that

$$(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4} \neq 0.$$

In fact, $g_{j,m}$ is orthogonal to $\{\phi \in \{L^2(0,1)\}^4; (A_P - \lambda)^k \phi = 0 \text{ for some } k \in \mathbf{N}\}$ for any eigenvalue λ of A_P which is different from $\lambda_{j,m}$ (e.g., [22]). Therefore if $(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4} = 0$, then Theorem 2.2.3 implies that $g_{j,m} = 0$, which is impossible. Hence, for any $a \in \{L^2(0,1)\}^4$, we can set

$$\alpha_{j,m} = \frac{(a, g_{j,m})_{\{L^2(0,1)\}^4}}{(f_{j,m}, g_{j,m})_{\{L^2(0,1)\}^4}}.$$

Moreover we put

$$\theta_{j,m}(t) = \alpha_{j,m} e^{\lambda_{j,m} t}, \quad |m| \geq N, \quad j = 1, 2.$$

In the case $|m| \leq N - 1$, the eigenvalue λ_m appears χ_m -times according to its algebraic multiplicity χ_m : $\lambda_q = \dots = \lambda_{q+\chi_m-1}$. Then by f_q we denote a corresponding eigenvector, and by $f_{q+\ell}(x)$, $1 \leq \ell \leq \chi_m - 1$, a Jordan chain of the associated root vectors. That is, $f_{q+\ell}$, $1 \leq \ell \leq \chi_m - 1$, satisfy $(A_P - \lambda_q) f_{q+\ell} = f_{q+\ell-1}$. We denote by $g_{q+\chi_m-1}$ an eigenvector of the adjoint operator A_P^* for the eigenvalue $\overline{\lambda_q}$, and by $g_{q+\chi_m-\ell}$, $2 \leq \ell \leq \chi_m$ we denote a Jordan chain of associated root vectors. Here, $(A_P - \lambda_q) g_{q+\chi_m-\ell} = g_{q+\chi_m-\ell+1}$ for $\ell = 2, 3, \dots, \chi_m$. Then we can prove (e.g., Propositions 2.2 and 2.3 in [39]) that $(f_{q+\ell}, g_{q+\ell})_{\{L^2(0,1)\}^4} \neq 0$, $0 \leq \ell \leq \chi_m - 1$. Thus for any $a \in \{L^2(0,1)\}^4$, we can set

$$\gamma_{q+\ell} = \frac{(a, g_{q+\ell})_{\{L^2(0,1)\}^4}}{(f_{q+\ell}, g_{q+\ell})_{\{L^2(0,1)\}^4}}, \quad 0 \leq \ell \leq \chi_m - 1$$

and

$$\theta_{q+\ell}(t) = e^{\lambda_q t} \left(\sum_{k=0}^{\chi_m-\ell-1} \frac{t^k}{k!} \gamma_{q+\ell+k} \right), \quad 0 \leq \ell \leq \chi_m - 1.$$

Then we renumber $\{f_{j,m}\}_{|m| \geq N, j=1,2}$, $f_{q+\ell}$, $\{\theta_{j,m}\}_{|m| \geq N, j=1,2}$, $\theta_{q+\ell}$, $\{g_{j,m}\}_{|m| \geq N, j=1,2}$, $g_{q+\ell}$ with $0 \leq j \leq \chi_m - 1$ as $\{f_m\}_{m \in \mathbf{Z}}$, $\{\theta_m\}_{m \in \mathbf{Z}}$ and $\{g_m\}_{m \in \mathbf{Z}}$.

In terms of θ_m and f_m , we can prove an expansion of the solution to the initial value/boundary value problem (2.1.9). The proof is done by arguments similar to Appendix in [39] and Proposition 2.2 in [53], and is omitted.

Proposition 2.5.1 *Let $a \in \{C^3[0, 1]\}^4 \cap D(A^2)$ and $u_{P,a}$ satisfy (2.1.9). Then*

$$u(t, x) = \sum_{m \in \mathbf{Z}} \theta_m(t) f_m(x),$$

where the series converges absolutely and uniformly in $-T \leq t \leq T$ and $0 \leq x \leq 1$.

Proof of Theorem 2.1.4. The "if" part is directly proved. In fact, by (2.1.19) and (2.1.20), we see that $K = 0$ satisfies (2.2.3), (2.2.4) and (2.2.7), so that $\tilde{u}(t, x) = R(x)u(t, x)$ satisfies (2.2.12) with some $\omega_1(t)$ and $\omega_2(t)$ by Theorem 2.2.6. In terms of (2.1.18) and (2.1.21), we can conclude that $(Q, b) \in M_T(P, a)$.

Proof of "only if" part. Let us recall that $u_{P,a}$ is the solution to (2.1.9) with coefficient matrix P and initial value a . Let us suppose that $u_{P,a}(t, 0) = u_{Q,b}(t, 0)$ and $u_{P,a}(t, 1) = u_{Q,b}(t, 1)$ for $-T \leq t \leq T$. Then it follows from Theorem 2.2.6 that for $-T + 1 \leq t \leq T - 1$

$$u_{Q,b}(t, 1) = u_{P,a}(t, 1) = R(1)u_{P,a}(t, 1) + \int_0^1 K(y, 1)u_{P,a}(t, y)dy.$$

We recall that

$$R(1) = \begin{pmatrix} R^1(1) & R^2(1) \\ R^2(1) & R^1(1) \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where R^1, R^2 are 2×2 matrices. For simplicity, we set

$$u(t, x) = u_{P,a}(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \\ u_3(t, x) \\ u_4(t, x) \end{pmatrix}, \quad K(y, 1) = \begin{pmatrix} K_1(y, 1) \\ K_2(y, 1) \end{pmatrix}$$

where $K_1(y, 1)$ and $K_2(y, 1)$ are 2×4 -matrices. Then it follows from $u_{Q,b}(t, 1) = u(t, 1)$ and $u_{\ell+2}(t, 1) =$

$H_{\ell}u_{\ell}(t, 1)$, $\ell = 1, 2$ that

$$\begin{aligned} (E_2 - R^1(1) - R^2(1)\tilde{H}) \begin{pmatrix} u_1(t, 1) \\ u_2(t, 1) \end{pmatrix} &= \int_0^1 K_1(y, 1)u(t, y)dy, \\ (\tilde{H} - R^2(1) - R^1(1)\tilde{H}) \begin{pmatrix} u_1(t, 1) \\ u_2(t, 1) \end{pmatrix} &= \int_0^1 K_2(y, 1)u(t, y)dy. \end{aligned}$$

By Proposition 2.5.1, we have

$$\begin{aligned} &(E_2 - R^1(1) - R^2(1)\tilde{H}) \sum_{m \in \mathbf{Z}} \theta_m(t) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} \\ &= \int_0^1 K_1(y, 1) \sum_{m \in \mathbf{Z}} \theta_m(t) f_m(y) dy, \\ &(\tilde{H} - R^2(1) - R^1(1)\tilde{H}) \sum_{m \in \mathbf{Z}} \theta_m(t) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} \\ &= \int_0^1 K_2(y, 1) \sum_{m \in \mathbf{Z}} \theta_m(t) f_m(y) dy \end{aligned}$$

for $-T + 1 \leq t \leq T - 1$. Here f_m^{ℓ} , $\ell = 1, 2$ is the ℓ -th component of f_m . Since the series on the right hand

converge uniformly by Proposition 2.5.1, we can change orders of summation and integration :

$$\begin{aligned} \sum_{m \in \mathbf{Z}} \theta_m(t) \left\{ (E_2 - R^1(1) - R^2(1)\tilde{H}) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} - \int_0^1 K_1(y, 1) f_m(y) dy \right\} &= 0 \\ \sum_{m \in \mathbf{Z}} \theta_m(t) \left\{ (\tilde{H} - R^2(1) - R^1(1)\tilde{H}) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} - \int_0^1 K_2(y, 1) f_m(y) dy \right\} &= 0 \end{aligned}$$

for $-1 \leq t \leq 1$. Here we used that $-T + 1 \leq t \leq T - 1$ and $T \geq 2$ implies $-1 \leq t \leq 1$.

We can prove that for the system $\mathcal{S} = \{\theta_m\}_{m \in \mathbf{Z}}$, there exists another system $\tilde{\mathcal{S}} \subset L^2(-1, 1)$ such that for any $\varphi \in \mathcal{S}$, we can choose a unique $\tilde{\varphi} \in \tilde{\mathcal{S}}$ satisfying $(\varphi, \psi)_{L^2(-1, 1)} = 0$ if and only if $\psi \in \tilde{\mathcal{S}} \setminus \{\tilde{\varphi}\}$. The proof is based on Theorem 1.1.1 in Sedletskii [47], and see Appendix C in [53] for the proof. Taking the

scalar products in $L^2(-1, 1)$ with all $\psi \in \tilde{\mathcal{S}}$, we can obtain

$$\begin{aligned} (E_2 - R^1(1) - R^2(1)\tilde{H}) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} - \int_0^1 K_1(y, 1) f_m(y) dy &= 0, \quad m \in \mathbf{Z}, \\ (\tilde{H} - R^2(1) - R^1(1)\tilde{H}) \begin{pmatrix} f_m^1(1) \\ f_m^2(1) \end{pmatrix} - \int_0^1 K_2(y, 1) f_m(y) dy &= 0, \quad m \in \mathbf{Z}. \end{aligned}$$

Here for sufficiently large $|m|$, as f_m , we see that $f_{j,m} = \begin{pmatrix} f_{j,m}^1 \\ f_{j,m}^2 \\ f_{j,m}^3 \\ f_{j,m}^4 \end{pmatrix}$, $j = 1, 2$, are two linearly independent eigenvectors corresponding to the eigenvalue $\lambda_{j,m}$. We will prove that

$$\left\{ \lim_{m \rightarrow \infty} \begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \end{pmatrix}, \quad \lim_{m \rightarrow \infty} \begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \end{pmatrix} \right\}$$

is linearly independent. In order to prove this, it is sufficient to prove that

$$\left\{ \lim_{m \rightarrow \infty} \begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \\ f_{1,m}^3(1) \\ f_{1,m}^4(1) \end{pmatrix}, \quad \lim_{m \rightarrow \infty} \begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \\ f_{2,m}^3(1) \\ f_{2,m}^4(1) \end{pmatrix} \right\}$$

is linearly independent because of

$$f_{j,m}^3(1) = H_1 f_{j,m}^1(1), \quad f_{j,m}^4(1) = H_2 f_{j,m}^2(1), \quad j = 1, 2.$$

By (2.4.8) and (2.4.9), we have

$$\begin{aligned} \lim_{m \rightarrow +\infty} \begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \\ f_{1,m}^3(1) \\ f_{1,m}^4(1) \end{pmatrix} &= R(1) \begin{pmatrix} c_1 \cosh(\beta_1 + \mu_1) \\ c_2 \cosh(\beta_1 + \mu_2) \\ c_1 \sinh(\beta_1 + \mu_1) \\ c_2 \sinh(\beta_1 + \mu_2) \end{pmatrix}, \\ \lim_{m \rightarrow +\infty} \begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \\ f_{2,m}^3(1) \\ f_{2,m}^4(1) \end{pmatrix} &= R(1) \begin{pmatrix} d_1 \cosh(\beta_2 + \mu_1) \\ d_2 \cosh(\beta_2 + \mu_2) \\ d_1 \sinh(\beta_2 + \mu_1) \\ d_2 \sinh(\beta_2 + \mu_2) \end{pmatrix}. \end{aligned}$$

Since $R^{-1}(1)$ exists and $c_1 d_2 - c_2 d_1 \neq 0$ which is proved for (2.4.2), we can verify that

$$\left\{ R(1) \begin{pmatrix} c_1 \cosh(\beta_1 + \mu_1) \\ c_2 \cosh(\beta_1 + \mu_2) \\ c_1 \sinh(\beta_1 + \mu_1) \\ c_2 \sinh(\beta_1 + \mu_2) \end{pmatrix}, \quad R(1) \begin{pmatrix} d_1 \cosh(\beta_2 + \mu_1) \\ d_2 \cosh(\beta_2 + \mu_2) \\ d_1 \sinh(\beta_2 + \mu_1) \\ d_2 \sinh(\beta_2 + \mu_2) \end{pmatrix} \right\}$$

is linearly independent. Thus

$$\left\{ \lim_{m \rightarrow \infty} \begin{pmatrix} f_{1,m}^1(1) \\ f_{1,m}^2(1) \end{pmatrix}, \lim_{m \rightarrow \infty} \begin{pmatrix} f_{2,m}^1(1) \\ f_{2,m}^2(1) \end{pmatrix} \right\}$$

is linearly independent.

Furthermore, from Riemann-Lebesgue lemma, we have

$$\lim_{m \rightarrow \infty} \int_0^1 K_\ell(y, 1) f_m(y) dy = 0, \quad \ell = 1, 2.$$

Therefore, we obtain

$$E_2 - R^1(1) - R^2(1)\tilde{H} = 0, \quad \tilde{H} - R^2(1) - R^1(1)\tilde{H} = 0 \quad (2.5.1)$$

and

$$\int_0^1 K_\ell(y, 1) f_m(y) dy = 0, \quad \ell = 1, 2, \quad m \in \mathbf{Z}.$$

Since $\{f_m\}_{m \in \mathbf{Z}}$ forms a Riesz basis, it follows that

$$K_1(y, 1) = K_2(y, 1) = 0, \quad 0 \leq y \leq 1. \quad (2.5.2)$$

Therefore, using a characteristic method, we can prove the uniqueness in the problem (2.2.3) - (2.2.4) with

(2.5.2) (e.g., [52], [55]), and obtain

$$K(y, x) = 0, \quad 0 \leq y \leq x \leq 1.$$

Consequently, we obtain (2.1.19), (2.1.20) and (2.1.21). Since $H_\ell \neq \pm 1$, we can directly derive $R^1(1) = E_2$

and $R^2(1) = 0$ from (2.5.1). Thus we obtain (2.1.18), and the proof of Theorem 2.1.4 is completed.

2.6 Proof of Lemma 2.2.4

We set

$$\begin{cases} L_{k,\ell}^{(1)}(y,x) = K_{k,\ell}(y,x) - K_{k+n,\ell+n}(y,x) \\ L_{k,\ell+n}^{(1)}(y,x) = K_{k,\ell+n}(y,x) - K_{k+n,\ell}(y,x) \\ L_{k,\ell}^{(2)}(y,x) = K_{k,\ell}(y,x) + K_{k+n,\ell+n}(y,x) \\ L_{k,\ell+n}^{(2)}(y,x) = K_{k,\ell+n}(y,x) + K_{k+n,\ell}(y,x), \end{cases} \quad k, \ell = 1, 2, \dots, n$$

and

$$f_{k,\ell}(y,x) = (K(y,x)P(x) - Q(x)K(y,x))_{k,\ell}, \quad k, \ell = 1, 2, \dots, 2n.$$

From (2.2.3), we obtain

$$\begin{cases} \frac{\partial}{\partial x} K_{k+n,\ell} + \frac{\partial}{\partial y} K_{k,\ell+n} = f_{k,\ell} \\ \frac{\partial}{\partial x} K_{k+n,\ell+n} + \frac{\partial}{\partial y} K_{k,\ell} = f_{k,\ell+n} \\ \frac{\partial}{\partial x} K_{k,\ell} + \frac{\partial}{\partial y} K_{k+n,\ell+n} = f_{k+n,\ell} \\ \frac{\partial}{\partial x} K_{k,\ell+n} + \frac{\partial}{\partial y} K_{k+n,\ell} = f_{k+n,\ell+n}, \end{cases} \quad \text{in } \Omega, \quad k, \ell = 1, 2, \dots, n.$$

Hence we obtain the following system for $k, \ell = 1, 2, \dots, n$:

$$\frac{\partial}{\partial x} L_{k,\ell}^{(1)} - \frac{\partial}{\partial y} L_{k,\ell}^{(1)} = \tilde{f}_{k,\ell} \equiv f_{k+n,\ell} - f_{k,\ell+n} \quad (2.6.1)$$

$$\frac{\partial}{\partial x} L_{k,\ell+n}^{(1)} - \frac{\partial}{\partial y} L_{k,\ell+n}^{(1)} = \tilde{f}_{k,\ell+n} \equiv f_{k+n,\ell+n} - f_{k,\ell} \quad (2.6.2)$$

$$\frac{\partial}{\partial x} L_{k,\ell}^{(2)} + \frac{\partial}{\partial y} L_{k,\ell}^{(2)} = \tilde{f}_{k+n,\ell} \equiv f_{k,\ell+n} + f_{k+n,\ell} \quad (2.6.3)$$

$$\frac{\partial}{\partial x} L_{k,\ell+n}^{(2)} + \frac{\partial}{\partial y} L_{k,\ell+n}^{(2)} = \tilde{f}_{k+n,\ell+n} \equiv f_{k+n,\ell+n} + f_{k,\ell}. \quad (2.6.4)$$

By (2.2.5), we have

$$L_{k,\ell}^{(1)}(x,x) = b_{k,\ell}(x), \quad 0 < x < 1, \quad k, \ell = 1, 2, \dots, n. \quad (2.6.5)$$

$$L_{k,\ell+n}^{(1)}(x,x) = a_{k,\ell}(x), \quad 0 < x < 1, \quad k, \ell = 1, 2, \dots, n. \quad (2.6.6)$$

Moreover from (2.2.4), we have

$$\begin{cases} L_{k,\ell}^{(2)}(0,x) = K_{k,\ell}(0,x) + K_{k+n,\ell+n}(0,x) = K_{k,\ell}(0,x) - h_k K_{k+n,\ell}(0,x) \\ L_{k,\ell+n}^{(2)}(0,x) = K_{k,\ell+n}(0,x) + K_{k+n,\ell}(0,x) = -h_k K_{k,\ell}(0,x) + K_{k+n,\ell}(0,x). \end{cases}$$

Since

$$\begin{cases} L_{k,\ell}^{(1)}(0, x) = K_{k,\ell}(0, x) - K_{k+n,\ell+n}(0, x) = K_{k,\ell}(0, x) + h_k K_{k+n,\ell}(0, x) \\ L_{k,\ell+n}^{(1)}(0, x) = K_{k,\ell+n}(0, x) - K_{k+n,\ell}(0, x) = -h_k K_{k,\ell}(0, x) - K_{k+n,\ell}(0, x), \end{cases}$$

we have

$$\begin{cases} K_{k,\ell}(0, x) = \frac{1}{1-h_k^2} L_{k,\ell}^{(1)}(0, x) + \frac{h_k}{1-h_k^2} L_{k,\ell+n}^{(1)}(0, x) \\ K_{k+n,\ell}(0, x) = -\frac{h_k}{1-h_k^2} L_{k,\ell}^{(1)}(0, x) - \frac{1}{1-h_k^2} L_{k,\ell+n}^{(1)}(0, x). \end{cases}$$

Consequently we have

$$L_{k,\ell}^{(2)}(0, x) = \frac{1+h_k^2}{1-h_k^2} L_{k,\ell}^{(1)}(0, x) + \frac{2h_k}{1-h_k^2} L_{k,\ell+n}^{(1)}(0, x) \quad (2.6.7)$$

$$L_{k,\ell+n}^{(2)}(0, x) = -\frac{2h_k}{1-h_k^2} L_{k,\ell}^{(1)}(0, x) - \frac{1+h_k^2}{1-h_k^2} L_{k,\ell+n}^{(1)}(0, x). \quad (2.6.8)$$

Here, we introduce the other variables

$$\begin{cases} u = \frac{x+y}{2} \\ v = \frac{x-y}{2}. \end{cases}$$

Then, we integrate (2.6.1) and (2.6.2) for v with (2.6.5) and (2.6.6) and we have

$$L_{k,\ell}^{(1)}(y, x) = \int_{\frac{x+y}{2}}^x \tilde{f}_{k,\ell}(-\xi + x + y, \xi) d\xi + b_{k,\ell} \left(\frac{x+y}{2} \right), \quad (y, x) \in \Omega, \quad 1 \leq k, \ell \leq n \quad (2.6.9)$$

$$L_{k,\ell+n}^{(1)}(y, x) = \int_{\frac{x+y}{2}}^x \tilde{f}_{k,\ell+n}(-\xi + x + y, \xi) d\xi + a_{k,\ell} \left(\frac{x+y}{2} \right), \quad (y, x) \in \Omega, \quad 1 \leq k, \ell \leq n \quad (2.6.10)$$

Integrating (2.6.3) and (2.6.4) for u , we have

$$L_{k,\ell}^{(2)}(y, x) = \int_{x-y}^x \tilde{f}_{k+n,\ell}(\xi - x + y, \xi) d\xi + L_{k,\ell}^{(2)}(0, x - y), \quad (y, x) \in \Omega, \quad 1 \leq k, \ell \leq n, \quad (2.6.11)$$

$$L_{k,\ell+n}^{(2)}(y, x) = \int_{x-y}^x \tilde{f}_{k,\ell+n}(\xi - x + y, \xi) d\xi + L_{k,\ell+n}^{(2)}(0, x - y), \quad (y, x) \in \Omega, \quad 1 \leq k, \ell \leq n. \quad (2.6.12)$$

By (2.6.7) - (2.6.10), we have

$$\begin{aligned}
L_{k,\ell}^{(2)}(y,x) &= \int_{x-y}^x \tilde{f}_{k+n,\ell}(\xi-x+y,\xi)d\xi \\
&+ \int_{\frac{x-y}{2}}^{x-y} \left\{ g_k \tilde{f}_{k,\ell}(-\xi+x-y,\xi) + \tilde{g}_k \tilde{f}_{k,\ell+n}(-\xi+x-y,\xi) \right\} d\xi \\
&+ g_k b_{k,\ell} \left(\frac{x-y}{2} \right) + \tilde{g}_k a_{k,\ell} \left(\frac{x-y}{2} \right), \tag{2.6.13}
\end{aligned}$$

$$\begin{aligned}
L_{k,\ell+n}^{(2)}(y,x) &= \int_{x-y}^x \tilde{f}_{k,\ell+n}(\xi-x+y,\xi)d\xi \\
&+ \int_{\frac{x-y}{2}}^{x-y} \left\{ -\tilde{g}_k \tilde{f}_{k,\ell}(-\xi+x-y,\xi) - g_k \tilde{f}_{k,\ell+n}(-\xi+x-y,\xi) \right\} d\xi \\
&- \tilde{g}_k b_{k,\ell} \left(\frac{x-y}{2} \right) - g_k a_{k,\ell} \left(\frac{x-y}{2} \right) \tag{2.6.14}
\end{aligned}$$

for $(y,x) \in \Omega$ and $k,\ell = 1,2,\dots,n$. Here we set

$$g_k = \frac{1+h_k^2}{1-h_k^2}, \quad \tilde{g}_k = \frac{2h_k}{1-h_k^2}.$$

Therefore we obtain Volterra integral equations (2.6.9), (2.6.10), (2.6.13) and (2.6.14) of the second kind, which are equivalent to (2.2.3) - (2.2.5). Using the iteration method, we can complete the proof. \square

2.7 Proof of Theorem 2.2.5

According to the general theory of the ordinary differential equation, equation (2.2.9) possesses a unique solution in $\{C^1[0,1]\}^{2n}$. Let us denote the right hand side of (2.2.10) by $\tilde{\psi}(x,\lambda)$. Hence it suffices to verify that $\tilde{\psi}$ satisfies (2.2.9). Clearly, initial conditions of (2.2.9) are satisfied.

We have

$$\begin{aligned}
&B_{2n} \frac{d\tilde{\psi}}{dx}(x,\lambda) + Q(x)\tilde{\psi}(x,\lambda) - \lambda\tilde{\psi}(x,\lambda) \\
&= B_{2n}R(x) \frac{d\phi}{dx}(x,\lambda) + \{B_{2n}R'(x) + B_{2n}K(x,x) + Q(x)R(x)\}\phi(x,\lambda) - \lambda R(x)\phi(x,\lambda) \\
&+ \int_0^x B_{2n} \frac{\partial K}{\partial x}(y,x)\phi(y,\lambda)dy + (Q(x) - \lambda) \int_0^x K(y,x)\phi(y,\lambda)dy.
\end{aligned}$$

Using (2.2.3) in Lemma 2.2.4 and (2.2.8), we obtain by integration by parts,

$$\begin{aligned} & B_{2n} \frac{d\tilde{\psi}}{dx}(x, \lambda) + Q(x)\tilde{\psi}(x, \lambda) - \lambda\tilde{\psi}(x, \lambda) \\ &= B_{2n}R(x) \frac{d\phi}{dx}(x, \lambda) + \{B_{2n}R'(x) + B_{2n}K(x, x) + Q(x)R(x)\}\phi(x, \lambda) - \lambda R(x)\phi(x, \lambda) \\ &+ K(0, x)B_{2n}\phi(0, \lambda) - K(x, x)B_{2n}\phi(x, \lambda). \end{aligned}$$

Here (2.2.4) and the condition in (2.2.8) at $x = 0$, yield

$$K(0, x)B_{2n}\phi(0, \lambda) = 0.$$

Hence

$$\begin{aligned} & B_{2n} \frac{d\tilde{\psi}}{dx}(x, \lambda) + Q(x)\tilde{\psi}(x, \lambda) - \lambda\tilde{\psi}(x, \lambda) \\ &= B_{2n}R(x) \frac{d\phi}{dx}(x, \lambda) - \lambda R(x)\phi(x, \lambda) + \{B_{2n}R'(x) + Q(x)R(x) - (K(x, x)B_{2n} - B_{2n}K(x, x))\}\phi(x, \lambda). \end{aligned}$$

By the differential equation in (2.2.8) and $R = \begin{pmatrix} R^1 & R^2 \\ R^2 & R^1 \end{pmatrix}$, we have

$$B_{2n}R(x) \frac{d\phi}{dx}(x, \lambda) = R(x)B_{2n} \frac{d\phi}{dx}(x, \lambda) = R(x)(-P(x) + \lambda)\phi(x, \lambda),$$

so that

$$\begin{aligned} & B_{2n} \frac{d\tilde{\psi}}{dx}(x, \lambda) + Q(x)\tilde{\psi}(x, \lambda) - \lambda\tilde{\psi}(x, \lambda) \\ &= \{B_{2n}R'(x) + Q(x)R(x) - R(x)P(x) - (K(x, x)B_{2n} - B_{2n}K(x, x))\}\phi(x, \lambda). \end{aligned}$$

By (2.2.6) and (2.2.7), we can directly verify that the right hand side of this equation is zero. Then

$\tilde{\psi}(x, \lambda) = \psi(x, \lambda)$. Thus the proof is completed. \square

2.8 Proof of Lemma 2.3.2

We set

$$f_j(\lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi_j(1, \lambda), \quad j = 1, 2.$$

Here, $\phi_j(x, \lambda)$, $j = 1, 2$ satisfies

$$\begin{cases} B_4 \frac{d\phi_j}{dx}(x, \lambda) + P(x)\phi_j(x, \lambda) = \lambda\phi_j(x, \lambda), \\ \phi_1(0, \lambda) = \begin{pmatrix} 1 \\ 0 \\ h_1 \\ 0 \end{pmatrix}, \quad \phi_2(0, \lambda) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ h_2 \end{pmatrix}. \end{cases}$$

Then we have

$$\det \Phi(\lambda) = \det \begin{pmatrix} f_1(\lambda) & f_2(\lambda) \end{pmatrix}.$$

Let $\ell_0 \in \mathbf{N} \cup \{0\}$ be the smallest number in

$$\left\{ \ell \in \mathbf{N} \cup \{0\}; \text{rank} \begin{pmatrix} \frac{d^\ell f_1}{d\lambda^\ell}(\lambda_0) & \frac{d^\ell f_2}{d\lambda^\ell}(\lambda_0) \end{pmatrix} \neq 0 \right\}.$$

We consider two cases separately ; **Case I:** $\ell_0 \geq 1$ and **Case II:** $\ell_0 = 0$.

If $\ell_0 = 0$, then we can argue similarly to the Case I-B stated below. Thus we argue only for the Case I.

Case I: Let $\ell_0 \geq 1$. Then

$$\frac{d^\ell f_1}{d\lambda^\ell}(\lambda_0) = \frac{d^\ell f_2}{d\lambda^\ell}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \leq \ell \leq \ell_0 - 1. \quad (2.8.1)$$

In particular, we have $f_1(\lambda_0) = f_2(\lambda_0) = 0$, that is, $(-\tilde{H} \ E_2)\phi_j(1, \lambda_0) = 0$, $j = 1, 2$, which means that ϕ_j , $j = 1, 2$ satisfies the boundary condition at $x = 1$ in (2.1.10). Therefore $\phi_1, \phi_2 \in D(A_P)$.

Let us define $\{\phi^{(j,\ell)}(x)\}_{j=1,2,\ell=1,2,\dots,\ell_0}$ as follows:

$$\begin{cases} \phi^{(1,1)}(x) = \phi_1(x, \lambda_0) \\ \phi^{(1,2)}(x) = \frac{1}{1!} \frac{\partial \phi_1}{\partial \lambda}(x, \lambda_0) \\ \phi^{(1,3)}(x) = \frac{1}{2!} \frac{\partial^2 \phi_1}{\partial \lambda^2}(x, \lambda_0) \\ \vdots \\ \phi^{(1,\ell_0)}(x) = \frac{1}{(\ell_0-1)!} \frac{\partial^{\ell_0-1} \phi_1}{\partial \lambda^{\ell_0-1}}(x, \lambda_0), \end{cases} \quad \begin{cases} \phi^{(2,1)}(x) = \phi_2(x, \lambda_0) \\ \phi^{(2,2)}(x) = \frac{1}{1!} \frac{\partial \phi_2}{\partial \lambda}(x, \lambda_0) \\ \phi^{(2,3)}(x) = \frac{1}{2!} \frac{\partial^2 \phi_2}{\partial \lambda^2}(x, \lambda_0) \\ \vdots \\ \phi^{(2,\ell_0)}(x) = \frac{1}{(\ell_0-1)!} \frac{\partial^{\ell_0-1} \phi_2}{\partial \lambda^{\ell_0-1}}(x, \lambda_0). \end{cases} \quad (2.8.2)$$

Now by (2.8.1) we can easily check that $\phi^{(j,\ell)} \in D(A_P)$ for all $j = 1, 2$ and $\ell = 1, 2, \dots, \ell_0$. Moreover

$$(A_P - \lambda_0)\phi^{(j,\ell)} = \phi^{(j,\ell-1)}$$

holds for all $j = 1, 2$ and $\ell = 1, 2, \dots, \ell_0$ where $\phi^{(j,0)} = 0$, $j = 1, 2$. This fact is checked by differentiating the equation

$$A_P \phi_j(x, \lambda) = \lambda \phi_j(x, \lambda)$$

with respect to λ successively.

By (2.8.2), we can check that $\{\phi^{(j,\ell)}(x)\}_{j=1,2,\ell=1,2,\dots,\ell_0}$ is a linearly independent system. In fact, let $\sum_{\ell=1}^{\ell_0} \sum_{j=1,2} a_{j,\ell} \phi^{(j,\ell)} = 0$. Applying $(A_P - \lambda_0)$ successively and using $(A_P - \lambda_0)\phi^{(j,\ell)} = \phi^{(j,\ell-1)}$ for $2 \leq \ell \leq \ell_0$ and $(A_P - \lambda_0)\phi^{(j,1)} = 0$, we see by the linear independence of $\phi^{(1,1)}$ and $\phi^{(2,1)}$ that $a_{j,\ell} = 0$.

Therefore, the algebraic multiplicity of the eigenvalue λ_0 is at least $2\ell_0$.

Moreover, by (2.8.1) and the linearity of the determinant, we have

$$\left. \frac{d^\ell}{d\lambda^\ell} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} = 0, \quad 0 \leq \ell \leq 2\ell_0 - 1.$$

for $\ell_0 \geq 1$. Hence, for $\ell_0 \geq 1$, the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is at least $2\ell_0$.

Therefore, we proved that the algebraic multiplicity of λ_0 and the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ are at least $2\ell_0$.

We separately discuss the following two cases :

Case I-A: The case of

$$\text{rank} \left(\begin{array}{cc} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) = 2.$$

Case I-B: The case of

$$\text{rank} \left(\begin{array}{cc} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) = 1.$$

Case I-A: Let

$$\text{rank} \left(\begin{array}{cc} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) = 2.$$

We will prove that the algebraic multiplicity of the eigenvalue λ_0 is $2\ell_0$.

The set of all the solutions to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{j=1,2,\ell=1,2,\dots,\ell_0} a_{j,\ell} \phi^{(j,\ell)}(x) \\ \left(\begin{array}{cc} \tilde{h} & E_2 \end{array} \right) \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

with given $a_{j,\ell} \in \mathbf{C}$, is written as

$$\left\{ \begin{aligned} & \sum_{j=1,2, \ell=1,2, \dots, \ell_0-1} a_{j,\ell} \phi^{(j,\ell+1)}(x) + a_{1,\ell_0} \frac{1}{\ell_0!} \frac{\partial^{\ell_0} \phi_1}{\partial \lambda^{\ell_0}}(x, \lambda_0) \\ & + a_{2,\ell_0} \frac{1}{\ell_0!} \frac{\partial^{\ell_0} \phi_2}{\partial \lambda^{\ell_0}}(x, \lambda_0) + b_1 \phi_1(x, \lambda_0) + b_2 \phi_2(x, \lambda_0) ; b_1, b_2 \in \mathbf{C} \end{aligned} \right\}. \quad (2.8.3)$$

Then there exists a solution to

$$\left\{ \begin{aligned} (A_P - \lambda_0)\phi(x) &= \sum_{j=1,2, \ell=1,2, \dots, \ell_0} a_{j,\ell} \phi^{(j,\ell)}(x) \\ \begin{pmatrix} -\tilde{h} & E_2 \end{pmatrix} \phi(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi(1) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \right. \quad (2.8.4)$$

if and only if

$$a_{1,\ell_0} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) + a_{2,\ell_0} \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because of

$$\text{rank} \begin{pmatrix} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 2,$$

this condition holds if and only if $a_{1,\ell_0} = a_{2,\ell_0} = 0$. Therefore for $j = 1, 2$, there exist no solutions to $(A_P - \lambda_0)\phi = \phi^{(j,\ell_0)}$. Hence the Jordan block corresponding $\phi^{(j,1)}$ is of size $\ell_0 \times \ell_0$, and the algebraic multiplicities of λ_0 is $2\ell_0$.

Because of (2.8.1) and the linearity of the determinant, we have

$$\frac{d^{2\ell_0}}{d\lambda^{2\ell_0}} \det \Phi(\lambda) \Big|_{\lambda=\lambda_0} = \frac{(2\ell_0)!}{(\ell_0!)^2} \det \begin{pmatrix} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} \neq 0,$$

that is, the multiplicity of the zero λ_0 of $\det \Phi(\lambda)$ is $2\ell_0$. Therefore, the algebraic multiplicity of λ_0 is equal to the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$.

Case I-B: Let

$$\text{rank} \begin{pmatrix} \frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) & \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 1.$$

Without the loss of generality, we assume that there exists some $c \in \mathbf{C}$ such that

$$\frac{d^{\ell_0} f_1}{d\lambda^{\ell_0}}(\lambda_0) = c \frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \quad (2.8.5)$$

and we assume that

$$\frac{d^{\ell_0} f_2}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.8.6)$$

Now we define $\{\tilde{\phi}^{(j,\ell)}(x)\}_{j=1,2, \ell=1,2,\dots,\ell_0}$ as follows:

$$\begin{cases} \tilde{\phi}^{(1,1)}(x) = \phi^{(1,1)}(x) - c\phi^{(2,1)}(x) \\ \tilde{\phi}^{(1,2)}(x) = \phi^{(1,2)}(x) - c\phi^{(2,2)}(x) \\ \vdots \\ \tilde{\phi}^{(1,\ell_0)}(x) = \phi^{(1,\ell_0)}(x) - c\phi^{(2,\ell_0)}(x), \end{cases} \quad \begin{cases} \tilde{\phi}^{(2,1)}(x) = \phi^{(2,1)}(x) \\ \tilde{\phi}^{(2,2)}(x) = \phi^{(2,2)}(x) \\ \vdots \\ \tilde{\phi}^{(2,\ell_0)}(x) = \phi^{(2,\ell_0)}(x). \end{cases}$$

We can easily check that $(A_P - \lambda_0)\tilde{\phi}^{(j,\ell)} = \tilde{\phi}^{(j,\ell-1)}$ for $j = 1, 2$ and $\ell \in \{1, 2, \dots, \ell_0\}$ and that $\tilde{\phi}^{(j,\ell)} \in D(A_P)$

for all $j = 1, 2$ and $\ell \in \{1, 2, \dots, \ell_0\}$. Here we set $\tilde{\phi}^{(j,0)} = 0$, $j = 1, 2$.

We set

$$\tilde{f}_j(\lambda) = \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \tilde{\phi}^{(j,1)}(1, \lambda), \quad j = 1, 2.$$

Then

$$\det \begin{pmatrix} \tilde{f}_1(\lambda) & \tilde{f}_2(\lambda) \end{pmatrix} = \det \begin{pmatrix} f_1(\lambda) & f_2(\lambda) \end{pmatrix} = \det \Phi(\lambda).$$

Because

$$\frac{d^\ell f_j}{d\lambda^\ell}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1, 2, \ell = 1, 2, \dots, \ell_0 - 1, \quad (2.8.7)$$

we obtain

$$\frac{d^\ell \tilde{f}_j}{d\lambda^\ell}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 1, 2, \ell = 1, 2, \dots, \ell_0 - 1. \quad (2.8.8)$$

By the definition of $\tilde{\phi}^{(1,1)}$ and $\tilde{\phi}^{(2,1)}$, and from (2.8.5) and (2.8.6), we have

$$\frac{d^{\ell_0} \tilde{f}_1}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.8.9)$$

Therefore, we obtain

$$\frac{d^{2\ell_0}}{d\lambda^{2\ell_0}} \det \Phi(\lambda) \Big|_{\lambda=\lambda_0} = 0,$$

that is, the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is at least $2\ell_0 + 1$. Now we define $\tilde{\phi}^{(1, \ell_0+1)}(x)$ by

$$\tilde{\phi}^{(1, \ell_0+1)}(x) = \frac{1}{\ell_0!} \left(\frac{\partial^{\ell_0} \phi_1}{\partial \lambda^{\ell_0}}(x, \lambda_0) - c \frac{\partial^{\ell_0} \phi_2}{\partial \lambda^{\ell_0}}(x, \lambda_0) \right).$$

Then $\tilde{\phi}^{(1, \ell_0+1)} \in D(A_P)$ and $(A_P - \lambda_0) \tilde{\phi}^{(1, \ell_0+1)} = \tilde{\phi}^{(1, \ell_0)}$.

According to the following respective cases, we proceed:

Case I-B-a: the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is $2\ell_0 + 1$.

Case I-B-b: the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ is $2\ell_0 + \ell_1$ with $\ell_1 \geq 2$.

Case I-B-a: Let the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ be $2\ell_0 + 1$, that is,

$$\frac{d^{2\ell_0+1} \Phi}{d\lambda^{2\ell_0+1}}(\lambda_0) \neq 0.$$

Then by (2.8.8) and (2.8.9), we have

$$\det \left(\begin{array}{cc} \frac{d^{\ell_0+1} \tilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) & \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) \neq 0. \quad (2.8.10)$$

The set of all the solutions to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\dots,\ell_0+1} a_{1,\ell} \tilde{\phi}^{(1,\ell)}(x) + \sum_{\ell=1,2,\dots,\ell_0} a_{2,\ell} \tilde{\phi}^{(2,\ell)}(x) \\ \begin{pmatrix} -\tilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

with $a_{1,\ell} \in \mathbf{C}$, $\ell \in \{1, 2, \dots, \ell_0 + 1\}$ and $a_{2,\ell} \in \mathbf{C}$, $\ell \in \{1, 2, \dots, \ell_0\}$, is written as

$$\begin{aligned} & \left\{ \sum_{\ell=1,2,\dots,\ell_0} a_{1,\ell} \phi^{(1,\ell+1)}(x) + \sum_{\ell=1,2,\dots,\ell_0-1} a_{2,\ell} \phi^{(2,\ell+1)}(x) \right. \\ & + a_{1,\ell_0+1} \frac{1}{(\ell_0+1)!} \left(\frac{\partial^{\ell_0+1} \phi_1}{\partial \lambda^{\ell_0+1}}(x, \lambda_0) - c \frac{\partial^{\ell_0+1} \phi_2}{\partial \lambda^{\ell_0+1}}(x, \lambda_0) \right) \\ & \left. + a_{2,\ell_0} \frac{1}{\ell_0!} \frac{\partial^{\ell_0} \phi_2}{\partial \lambda^{\ell_0}}(x, \lambda_0) + b_1 \phi_1(x, \lambda_0) + b_2 \phi_2(x, \lambda_0) ; b_1, b_2 \in \mathbf{C} \right\}. \quad (2.8.11) \end{aligned}$$

Then there exists a solution to

$$\begin{cases} (A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\dots,\ell_0+1} a_{1,\ell}\tilde{\phi}^{(1,\ell)}(x) + \sum_{\ell=1,2,\dots,\ell_0} a_{2,\ell}\tilde{\phi}^{(2,\ell)}(x) \\ \begin{pmatrix} -\tilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases} \quad (2.8.12)$$

if and only if

$$a_{1,\ell_0+1} \frac{1}{(\ell_0+1)!} \frac{d^{\ell_0+1} \tilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) + a_{2,\ell_0} \frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By (2.8.10), this condition holds if and only if $a_{1,\ell_0+1} = a_{2,\ell_0} = 0$. Therefore, by an argument similar to Case I-A, the algebraic multiplicities of λ_0 is $2\ell_0 + 1$. Hence we see that the algebraic multiplicity of λ_0 is $2\ell_0 + 1$ which is equal to the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$.

Case I-B-b: Let the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$ be $2\ell_0 + \ell_1$ with $\ell_1 \geq 2$. Let us define $c_1, c_2, \dots, c_{\ell_1-1}$ as follows.

(a): The definition of c_1 .

We define c_1 by

$$\frac{1}{\ell_0!} c_1 \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0+1)!} \frac{d^{\ell_0+1} \tilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0). \quad (2.8.13)$$

Such c_1 exists. In fact, since $\left. \frac{d^{2\ell_0+1}}{d\lambda^{2\ell_0+1}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} = 0$, by means of (C.8) and (C.9), we have

$$\det \begin{pmatrix} \frac{d^{\ell_0+1} \tilde{f}_1}{d\lambda^{\ell_0+1}}(\lambda_0) & \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{pmatrix} = 0.$$

Recalling

$$\frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from (2.8.9) we can obtain c_1 such that (2.8.13) holds.

(b): The definition of $c_2, c_3, \dots, c_{\ell_1-1}$.

We define $c_2, c_3, \dots, c_{\ell_1-1}$ in an inductive way as follows. For $k = 2, 3, \dots, \ell_1 - 2$, assume that c_1, c_2, \dots, c_{k-1}

are already defined. Then we will define c_k such that

$$\frac{1}{\ell_0!} c_k \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0 + k)!} \frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) - \sum_{q=1}^{k-1} \frac{c_q}{(\ell_0 + k - q)!} \frac{d^{\ell_0+k-q} \tilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0). \quad (2.8.14)$$

holds. Now we prove that such c_k exists and that there does not exist c_{ℓ_1} such that

$$\frac{1}{\ell_0!} c_{\ell_1} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \frac{1}{(\ell_0 + \ell_1)!} \frac{d^{\ell_0+\ell_1} \tilde{f}_1}{d\lambda^{\ell_0+\ell_1}}(\lambda_0) - \sum_{q=1}^{\ell_1-1} \frac{c_q}{(\ell_0 + \ell_1 - q)!} \frac{d^{\ell_0+\ell_1-q} \tilde{f}_2}{d\lambda^{\ell_0+\ell_1-q}}(\lambda_0). \quad (2.8.15)$$

Let us calculate

$$\begin{aligned} & \left. \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} \\ &= \sum_{q=0}^{2\ell_0+k} \frac{(2\ell_0+k)!}{q!(2\ell_0+k-q)!} \det \left(\begin{array}{cc} \frac{d^q \tilde{f}_1}{d\lambda^q}(\lambda_0) & \frac{d^{2\ell_0+k-q} \tilde{f}_2}{d\lambda^{2\ell_0+k-q}}(\lambda_0) \end{array} \right) \\ &= \sum_{q=\ell_0+1}^{\ell_0+k} \frac{(2\ell_0+k)!}{q!(2\ell_0+k-q)!} \det \left(\begin{array}{cc} \frac{d^q \tilde{f}_1}{d\lambda^q}(\lambda_0) & \frac{d^{2\ell_0+k-q} \tilde{f}_2}{d\lambda^{2\ell_0+k-q}}(\lambda_0) \end{array} \right) \\ &= \sum_{q=1}^k \frac{(2\ell_0+k)!}{(\ell_0+q)!(\ell_0+k-q)!} \det \left(\begin{array}{cc} \frac{d^{\ell_0+q} \tilde{f}_1}{d\lambda^{\ell_0+q}}(\lambda_0) & \frac{d^{\ell_0+k-q} \tilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{array} \right). \end{aligned}$$

Here we used (2.8.8) and (2.8.9).

Now we eliminate $\frac{d^{\ell_0+q} \tilde{f}_1}{d\lambda^{\ell_0+q}}(\lambda_0)$ for $q \in \{1, 2, \dots, k-1\}$ by (2.8.14). Using (2.8.14), we have

$$\begin{aligned} & \left. \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} \\ &= \sum_{q=1}^{k-1} \frac{(2\ell_0+k)!}{(\ell_0+q)!(\ell_0+k-q)!} (\ell_0+q)! \det \left(\begin{array}{cc} \sum_{p=1}^q \frac{c_p}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p} \tilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) & \frac{d^{\ell_0+k-q} \tilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{array} \right) \\ & \quad + \frac{(2\ell_0+k)!}{(\ell_0+k)! \ell_0!} \det \left(\begin{array}{cc} \frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) \\ &= (2\ell_0+k)! \sum_{q=1}^{k-1} \sum_{p=1}^q c_p \det \left(\begin{array}{cc} \frac{1}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p} \tilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) & \frac{1}{(\ell_0+k-q)!} \frac{d^{\ell_0+k-q} \tilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \end{array} \right) \\ & \quad + \frac{(2\ell_0+k)!}{(\ell_0+k)! \ell_0!} \det \left(\begin{array}{cc} \frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right). \end{aligned}$$

Then changing orders of the summations, we have

$$\begin{aligned}
& \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \Big|_{\lambda=\lambda_0} \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \sum_{q=p}^{k-1} \det \left(\frac{1}{(\ell_0+q-p)!} \frac{d^{\ell_0+q-p} \tilde{f}_2}{d\lambda^{\ell_0+q-p}}(\lambda_0) \quad \frac{1}{(\ell_0+k-q)!} \frac{d^{\ell_0+k-q} \tilde{f}_2}{d\lambda^{\ell_0+k-q}}(\lambda_0) \right) \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)! \ell_0!} \det \left(\frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) \quad \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \right) \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \sum_{q=0}^{k-p-1} \det \left(\frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q} \tilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0) \quad \frac{1}{(\ell_0+k-p-q)!} \frac{d^{\ell_0+k-p-q} \tilde{f}_2}{d\lambda^{\ell_0+k-p-q}}(\lambda_0) \right) \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)! \ell_0!} \det \left(\frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) \quad \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \right).
\end{aligned}$$

Now we prove that

$$\begin{aligned}
& \sum_{q=0}^{k-p-1} \det \left(\frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q} \tilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0) \quad \frac{1}{(\ell_0+k-p-q)!} \frac{d^{\ell_0+k-p-q} \tilde{f}_2}{d\lambda^{\ell_0+k-p-q}}(\lambda_0) \right) \\
&= \det \left(\frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \quad \frac{1}{(\ell_0+k-p)!} \frac{d^{\ell_0+k-p} \tilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \right).
\end{aligned}$$

In fact, let $b_q = \frac{1}{(\ell_0+q)!} \frac{d^{\ell_0+q} \tilde{f}_2}{d\lambda^{\ell_0+q}}(\lambda_0)$ for $q = 1, 2, \dots, k-p-1$. Let $k-p-1$ be odd. Then the left hand side of the above equation is

$$\begin{aligned}
& \sum_{q=0}^{k-p-1} \det(b_q \quad b_{k-p-q}) \\
&= \det(b_0 \quad b_{k-p}) + \det(b_1 \quad b_{k-p-1}) + \det(b_2 \quad b_{k-p-2}) + \dots + \det(b_{k-p-1} \quad b_1) \\
&= \det(b_0 \quad b_{k-p}) + \left\{ \det(b_1 \quad b_{k-p-1}) + \det(b_{k-p-1} \quad b_1) \right\} \\
&\quad + \left\{ \det(b_2 \quad b_{k-p-2}) + \det(b_{k-p-2} \quad b_2) \right\} + \dots \\
&\quad + \left\{ \det(b_{\frac{k-p}{2}-1} \quad b_{\frac{k-p}{2}+1}) + \det(b_{\frac{k-p}{2}+1} \quad b_{\frac{k-p}{2}-1}) \right\} + \det(b_{\frac{k-p}{2}} \quad b_{\frac{k-p}{2}}) \\
&= \det(b_0 \quad b_{k-p}) = \det \left(\frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \quad \frac{1}{(\ell_0+k-p)!} \frac{d^{\ell_0+k-p} \tilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \right).
\end{aligned}$$

For even $k-p-1$, the argument is similar.

Therefore, we obtain

$$\begin{aligned}
& \left. \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} \\
&= (2\ell_0+k)! \sum_{p=1}^{k-1} c_p \det \left(\begin{array}{cc} \frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) & \frac{1}{(\ell_0+k-p)!} \frac{d^{\ell_0+k-p} \tilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) \end{array} \right) \\
&\quad + \frac{(2\ell_0+k)!}{(\ell_0+k)! \ell_0!} \det \left(\begin{array}{cc} \frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) & \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right) \\
&= (2\ell_0+k)! \det \left(\begin{array}{cc} \frac{1}{(\ell_0+k)!} \frac{d^{\ell_0+k} \tilde{f}_1}{d\lambda^{\ell_0+k}}(\lambda_0) - \sum_{p=1}^{k-1} \frac{c_p}{(\ell_0+k-p)!} \frac{d^{\ell_0+k-p} \tilde{f}_2}{d\lambda^{\ell_0+k-p}}(\lambda_0) & \frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) \end{array} \right). \quad (2.8.16)
\end{aligned}$$

Now, since λ_0 is a zero of $\det \Phi(\lambda)$ with multiplicity $2\ell_0 + \ell_1$, we have

$$\begin{cases} \left. \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} = 0 & \text{if } 1 \leq k \leq \ell_1 - 1 \\ \left. \frac{d^{2\ell_0+k}}{d\lambda^{2\ell_0+k}} \det \Phi(\lambda) \right|_{\lambda=\lambda_0} \neq 0 & \text{if } k = \ell_1. \end{cases}$$

Then there exists c_k satisfying (2.8.14) for $k = 2, 3, \dots, \ell_1 - 1$, and there is no c_{ℓ_1} satisfying (2.8.15).

Now we define $\tilde{\phi}^{(1, \ell_0+2)}, \tilde{\phi}^{(1, \ell_0+3)}, \dots, \tilde{\phi}^{(1, \ell_0+\ell_1)}$ as follows:

$$\begin{aligned}
\tilde{\phi}^{(1, \ell_0+k)}(x) &= \frac{1}{(\ell_0+k-1)!} \left(\frac{d^{\ell_0+k-1} \phi_1}{d\lambda^{\ell_0+k-1}}(x, \lambda_0) - c \frac{d^{\ell_0+k-1} \phi_2}{d\lambda^{\ell_0+k-1}}(x, \lambda_0) \right) \\
&\quad - \sum_{q=1}^{k-1} \frac{c_q}{(\ell_0+k-1-q)!} \frac{d^{\ell_0+k-1-q} \phi_2}{d\lambda^{\ell_0+k-1-q}}(x, \lambda_0)
\end{aligned}$$

for $k \in \{2, 3, \dots, \ell_1\}$. Then, for all $k \in \{2, 3, \dots, \ell_1\}$, we have $\tilde{\phi}^{(1, \ell_0+k)} \in D(A_P)$ and

$$\begin{aligned}
& (A_P - \lambda_0) \tilde{\phi}^{(1, \ell_0+k)}(x) \\
&= \tilde{\phi}^{(1, \ell_0+k-1)}(x) + \left(\text{a linear combination of } \phi_2(x, \lambda_0), \frac{\partial \phi_2}{\partial \lambda}(x, \lambda_0), \dots, \frac{\partial^{\ell_0-1} \phi_2}{\partial \lambda^{\ell_0-1}}(x, \lambda_0) \right).
\end{aligned}$$

We define $\zeta^{(1,1)}(x), \zeta^{(1,2)}(x), \dots, \zeta^{(1,\ell_0+\ell_1)}(x), \zeta^{(2,1)}, \zeta^{(2,2)}, \dots, \zeta^{(2,\ell_0)}$ by

$$\left\{ \begin{array}{l} \zeta^{(1,\ell_0+\ell_1)}(x) = \tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \zeta^{(1,\ell_0+\ell_1-1)}(x) = (A_P - \lambda_0)\tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \zeta^{(1,\ell_0+\ell_1-2)}(x) = (A_P - \lambda_0)^2\tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \vdots \\ \zeta^{(1,\ell_0+1)}(x) = (A_P - \lambda_0)^{\ell_1-1}\tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \zeta^{(1,\ell_0)}(x) = (A_P - \lambda_0)^{\ell_1}\tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \zeta^{(1,\ell_0-1)}(x) = (A_P - \lambda_0)^{\ell_0+1}\tilde{\phi}^{(1,\ell_0+\ell_1)}(x) \\ \vdots \\ \zeta^{(1,1)}(x) = (A_P - \lambda_0)^{\ell_0+\ell_1-1}\tilde{\phi}^{(1,\ell_0+\ell_1)}(x), \\ \zeta^{(2,\ell_0)}(x) = \tilde{\phi}^{(2,\ell_0)}(x) \\ \zeta^{(2,\ell_0-1)}(x) = \tilde{\phi}^{(2,\ell_0-1)}(x) \\ \vdots \\ \zeta^{(2,1)}(x) = \tilde{\phi}^{(2,1)}(x). \end{array} \right.$$

Then we can see that $\{\zeta^{(1,\ell)}(x)\}_{\ell=1,2,\dots,\ell_0+\ell_1} \cup \{\zeta^{(2,\ell)}\}_{\ell=1,2,\dots,\ell_0}$ is a linearly independent system.

For fixed $a_{1,\ell} \in \mathbf{C}$, $j \in \{1, 2, \dots, \ell_0 + \ell_1\}$ and $a_{2,\ell} \in \mathbf{C}$, $\ell \in \{1, 2, \dots, \ell_0\}$, the set of all the solutions to

$$\left\{ \begin{array}{l} (A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\dots,\ell_0+\ell_1} a_{1,\ell}\zeta^{(1,\ell)}(x) + \sum_{\ell=1,2,\dots,\ell_0} a_{2,\ell}\zeta^{(2,\ell)}(x) \\ \begin{pmatrix} -\tilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array} \right.$$

is written as

$$\left\{ \begin{array}{l} \sum_{\ell=1,2,\dots,\ell_0+\ell_1-1} a_{1,\ell}\zeta^{(1,\ell+1)}(x) + \sum_{\ell=1,2,\dots,\ell_0-1} a_{2,\ell}\zeta^{(2,\ell+1)}(x) \\ + a_{1,\ell_0+\ell_1} \left[\frac{1}{(\ell_0 + \ell_1)!} \left(\frac{\partial^{\ell_0+\ell_1}\phi_1}{\partial\lambda^{\ell_0+\ell_1}}(x, \lambda_0) - c \frac{\partial^{\ell_0+\ell_1}\phi_2}{\partial\lambda^{\ell_0+\ell_1}}(x, \lambda_0) \right) \right. \\ \left. - \sum_{q=1}^{j_1-1} \frac{c_q}{(\ell_0 + \ell_1 - q)!} \frac{\partial^{\ell_0+\ell_1-q}\phi_2}{\partial\lambda^{\ell_0+\ell_1-q}}(x, \lambda_0) \right] \\ + a_{2,\ell_0} \frac{1}{\ell_0!} \frac{\partial^{\ell_0}\phi_2}{\partial\lambda^{\ell_0}}(x, \lambda_0) + b_1\phi_1(x, \lambda_0) + b_2\phi_2(x, \lambda_0); b_1, b_2 \in \mathbf{C} \end{array} \right\}. \quad (2.8.17)$$

Then there exists a solution to

$$\left\{ \begin{array}{l} (A_P - \lambda_0)\phi(x) = \sum_{\ell=1,2,\dots,\ell_0+\ell_1} a_{1,\ell}\tilde{\phi}^{(1,\ell)}(x) + \sum_{\ell=1,2,\dots,\ell_0} a_{2,\ell}\tilde{\phi}^{(2,\ell)}(x) \\ \begin{pmatrix} -\tilde{h} & E_2 \end{pmatrix} \phi(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\tilde{H} & E_2 \end{pmatrix} \phi(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array} \right. \quad (2.8.18)$$

if and only if

$$a_{1,\ell_0+\ell_1} \left(\frac{1}{(\ell_0 + \ell_1)!} \frac{d^{\ell_0+\ell_1} \tilde{f}_1}{d\lambda^{\ell_0+\ell_1}}(\lambda_0) - \sum_{q=1}^{\ell_1-1} \frac{c_q}{(\ell_0 + \ell_1 - q)!} \frac{d^{\ell_0+\ell_1-q} \tilde{f}_2}{d\lambda^{\ell_0+\ell_1-q}}(\lambda_0) \right) + a_{2,\ell_0} \frac{1}{\ell_0!} \frac{d^{\ell_0} \tilde{f}_2}{d\lambda^{\ell_0}}(\lambda_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because there is no c_{ℓ_1} satisfying (2.8.15), this condition holds if and only if $a_{1,\ell_0+\ell_1} = a_{2,\ell_0} = 0$.

Therefore, if the solution to (2.8.18) exists, then it is in the space spanned by $\{\zeta^{(1,\ell)}(x)\}_{\ell=1,2,\dots,\ell_0+\ell_1} \cup \{\zeta^{(2,\ell)}\}_{\ell=1,2,\dots,\ell_0}$. Hence, by an argument similar to Case I-A, the algebraic multiplicity of an eigenvalue λ_0 of A_P is $2\ell_0 + \ell_1$. This is equal to the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$.

Thus in all the cases, we have seen that the algebraic multiplicity of λ_0 is equal to the multiplicity of a zero λ_0 of $\det \Phi(\lambda)$, that is, the proof is completed. \square

Chapter 3

Inverse Problem for One-Dimensional Fractional Diffusion Equation

3.1 Introduction

We consider a one-dimensional fractional diffusion equation: $\partial_t^\alpha u(x, t) = \frac{\partial}{\partial x} (p(x) \frac{\partial u}{\partial x}(x, t))$, $0 < x < \ell$, where $0 < \alpha < 1$ and ∂_t^α denotes the Caputo derivative in time of order α . We attach the homogeneous Neumann boundary condition at $x = 0, \ell$ and the initial value given by the Dirac delta function. We prove that α and $p(x)$, $0 < x < \ell$, are uniquely determined by data $u(0, t)$, $0 < t < T$. The uniqueness result is a theoretical background in experimentally determining the order α of many anomalous diffusion phenomena which are important for example in the environmental engineering. The proof is based on the eigenfunction expansion of the weak solution to the initial value/boundary value problem and the Gel'fand-Levitan theory.

Recently there are many anomalous diffusion phenomena observed which show different aspects from the classical diffusion. For example, Admas and Gelhar [1] pointed that field data in the saturated zone of a highly heterogeneous aquifer are not well simulated by the classical advection-diffusion equation which is based on the random walk, and the data indicate "slower" diffusion than the classical one. The slow diffusion is characterized by the long-tailed profile in spatial distribution of densities as the time passes. Also see Zhou and Selim [57]. Such slow diffusion is called the anomalous diffusion. Since [1], there are many studies for

better models, because from the practical viewpoint, the anomalous diffusion is seriously concerned e.g., with the quantitative environmental problems such as evaluation of underground contaminants. In particular, Berkowitz, Scher and Silliman [5], Y. Hatano and N. Hatano [18] have applied the continuous-time random walk to the underground environmental problem.

For applying the continuous-time random walk, we have to determine some parameters in the continuous-time random walk, and one important parameter is the power in the large-time behaviour of a waiting-time distribution function. We can refer to Y. Hatano and N. Hatano [18] where the authors fit the parameters by data of column experiments at laboratory. See also Xiong, G. Huang and Q. Huang [54], and Berkowitz, Cortis, Dentz and Scher [4] as a survey. Although there have been many works which are concerned more experimentally with the continuous-time random walk, there are very few mathematical analyses for the parameter identification. The continuous-time random walk is a microscopic model for the anomalous diffusion, while from it, we can derive a macroscopic model equation, e.g., Metzler and Klafter [33] (pp.14-18), Roman and Alemany [44], Sokolov, Klafter and Blumen [48]. The derivation corresponds to the way with which the classical diffusion equation is derived from the random walk, and as a macroscopic model from the continuous-time random walk, we have a fractional diffusion equation:

$$\partial_t^\alpha u(x, t) = \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x}(x, t) \right), \quad 0 < x < \ell, \quad t > 0, \quad (3.1.1)$$

where the diffusion coefficient $p(x)$ describes the heterogeneity of the medium, $\alpha > 0$, and $\partial_t^\alpha u(x, t)$ means the Caputo derivative :

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{\partial u}{\partial s}(x, s) ds. \quad (3.1.2)$$

In the slow diffusion, we can take $0 < \alpha < 1$. The fractional order α is related with the power parameter in the waiting-time distribution function. As related papers, see Giona, Gerbelli and Roman [14], Giona and Roman [15], Mainardi [27] - [29], Metzler, Glöckle and Nonnenmacher [32], Metzler and Klafter [34],

Nigmatullin [37], Roman [43] and see section 10.10 in Podlubny [41].

The main purpose of this paper is to establish the uniqueness in determining α and $p(x)$ by means of observation data $u(0, t)$, $0 < t < T$ at one end point. By our uniqueness result, we expect that by experiments, we can identify an important parameter α and $p(x)$ characterizing the anomalous diffusion.

There are many works on the forward problem for fractional diffusion equations such as an initial value/ boundary value problem and we refer to Bazhlekova [3], Eidelman and Kochubei [10], Metzler and Klafter [34], Gorenflo, Luchko and Zabrejko [17], Hanyga [19] and the references therein. Also see Prüss [42] (e.g., Section 2 of Chapter I) as a monograph. However, to my best knowledge, there are very few works on inverse problems for fractional diffusion equations in spite of the physical and practical importance, and our uniqueness is the first mathematical result for the coefficient inverse problem for a fractional differential equation.

The chapter is composed of 4 sections. In section 2, we formulate our inverse problem and state the uniqueness in the inverse problem as main result. In section 3, we prove the unique existence of weak solution, and in section 4, we complete the proof of the main result.

3.2 Formulation and the main result.

We consider the following fractional partial differential equation.

$$\partial_t^\alpha u(x, t) = \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x}(x, t) \right), \quad 0 < x < \ell, \quad 0 < t < T, \quad (3.2.1)$$

$$u(x, 0) = \delta(x), \quad (3.2.2)$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\ell, t) = 0, \quad 0 < t \leq T. \quad (3.2.3)$$

Here $T > 0, \ell > 0$ are fixed and $\delta(x)$ is the Dirac delta function,

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{\partial u}{\partial s}(x, s) ds$$

(e.g., Kilbas, Srivastava and Trujillo [23], Podlubny [41]). We assume that $p \in C^2[0, \ell]$ and $0 < \alpha < 1$. The initial condition (3.2.2) means that we start experiments by setting up a density profile concentrating at $x = 0$, and the boundary condition (3.2.3) requires no fluxes at the both end points.

We discuss

Inverse problem. Determine the order $\alpha \in (0, 1)$ of the time derivative and the diffusion coefficient $p(x)$ from boundary data $u(0, t)$, $0 < t \leq T$.

Due to the irregular initial value in (3.2.2), we have to consider a weak solution to (3.2.1) - (3.2.3) which is defined below. In terms of the weak solution, we can state our main result.

Theorem 3.2.1 *Let us assume $p, q \in C^2[0, \ell]$, $p, q > 0$ on $[0, \ell]$, $\alpha, \beta \in (0, 1)$. Let u be the weak solution to (3.2.1) - (3.2.3), and let v be the weak solution to (3.2.4) with the same initial and boundary conditions as (3.2.2) and (3.2.3) :*

$$\partial_t^\beta v(x, t) = \frac{\partial}{\partial x} \left(q(x) \frac{\partial v}{\partial x}(x, t) \right), \quad 0 < x < \ell, \quad 0 < t < T. \quad (3.2.4)$$

Then $u(0, t) = v(0, t)$, $0 < t \leq T$ with some $T > 0$, implies $\alpha = \beta$ and $p(x) = q(x)$, $0 \leq x \leq \ell$.

In the case of $\alpha = \beta = 1$, our inverse problem is for the one-dimensional diffusion equation and we can refer to Isakov and Kindermann [21], Murayama [35], Pierce [40], Suzuki [49], [50], Suzuki and Murayama [51]. As source books for inverse problems for partial differential equations without fractional order derivatives, see for example, Isakov [20], Klibanov and Timonov [24] and Lavrent'ev, Romanov and Shishat-skiĭ[25], Romanov [45].

Now we define the weak solution to (3.2.1) - (3.2.3). First we define an operator A_p in $L^2(0, \ell)$ by

$$(A_p \psi)(x) = -\frac{d}{dx} \left(p(x) \frac{d}{dx} \psi(x) \right), \quad 0 < x < \ell, \quad \mathcal{D}(A_p) = \left\{ \psi \in H^2(0, \ell); \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(\ell) = 0 \right\}.$$

It is known that the operator A_p has only real and simple eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$, and with suitable numbering,

we have

$$0 = \lambda_1 < \lambda_2 < \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Moreover by means of the Liouville transform (e.g., Yosida [56]) and Levitan and Sargsjan [26], we see the following asymptotic:

$$\lambda_n = \left(\int_0^\ell \frac{1}{\sqrt{p(x)}} dx \right)^{-2} n^2 \pi^2 + O(1), \quad n \rightarrow \infty. \quad (3.2.5)$$

By φ_n we denote the eigenfunction corresponding to λ_n which satisfies $\varphi_n(0) = 1$. Henceforth (\cdot, \cdot) denotes the scalar product in $L^2(0, \ell)$ and we set $\|\varphi\|_{L^2(0, \ell)} = \|\varphi\| = (\varphi, \varphi)^{\frac{1}{2}}$. We define

$$\rho_n = \|\varphi_n\|^{-2}.$$

Then, for each $v \in L^2(0, \ell)$, we have the eigenfunction expansion :

$$\psi = \sum_{n=1}^{\infty} \rho_n (\psi, \varphi_n) \varphi_n.$$

Moreover $\{\rho_n\}_{n \in \mathbb{N}}$ satisfies the asymptotic behaviour: there exists a constant $c_0 > 0$ such that

$$\rho_n = c_0 + o(1), \quad n \rightarrow \infty, \quad (3.2.6)$$

which is derived by the Liouville transform (e.g., Yosida [56]) and Levitan and Sargsjan [26].

Now we arbitrarily choose a constant $M > 0$ and define the operator $A_{p, M}$ in $L^2(0, \ell)$ as follows :

$$\left\{ \begin{array}{l} (A_{p, M} \psi)(x) = -\frac{d}{dx} \left(p(x) \frac{d}{dx} \psi(x) \right) + M\psi, \quad 0 < x < \ell, \\ \mathcal{D}(A_{p, M}) = \left\{ \psi \in H^2(0, \ell); \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(\ell) = 0 \right\}. \end{array} \right.$$

Then the set of all the eigenvalues of $A_{p, M}$ is $\{\lambda_n + M\}_{n \in \mathbb{N}}$, and we set $\lambda_n^{(M)} = \lambda_n + M$. Then we have

$$\lambda_n^{(M)} > 0, \quad n \in \mathbb{N}.$$

We define the function space $\mathcal{D}(A_{p, M}^\kappa)$ for $\kappa > 0$ by

$$\mathcal{D}(A_{p, M}^\kappa) = \left\{ \psi \in L^2(0, \ell); \sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi, \varphi_n)|^2 < \infty \right\}.$$

Then we see that $\mathcal{D}(A_{p,M}^\kappa)$ is a Banach space with the norm :

$$\|\psi\|_{\mathcal{D}(A_{p,M}^\kappa)} = \left\{ \sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{2\kappa} |(\psi, \varphi_n)|^2 \right\}^{\frac{1}{2}}.$$

We have $\mathcal{D}(A_{p,M}^\kappa) = H^{2\kappa}(0, \ell)$ if $0 \leq \kappa < \frac{3}{4}$. Since $\mathcal{D}(A_{p,M}^\kappa) \subset L^2(0, \ell)$, identifying the dual $L^2(0, \ell)'$ with itself, we have $\mathcal{D}(A_{p,M}^\kappa) \subset L^2(0, \ell) \subset (\mathcal{D}(A_{p,M}^\kappa))'$. Henceforth we set $\mathcal{D}(A_{p,M}^{-\kappa}) = (\mathcal{D}(A_{p,M}^\kappa))'$, which consists of bounded linear functionals on $\mathcal{D}(A_{p,M})$. For $f \in \mathcal{D}(A_{p,M}^{-\kappa})$ and $\psi \in \mathcal{D}(A_{p,M}^\kappa)$, by $-\kappa < f, \psi >_\kappa$ we denote the value which is obtained by operating f to ψ . We note that $\mathcal{D}(A_{p,M}^{-\kappa})$ is a Banach space with the norm :

$$\|f\|_{\mathcal{D}(A_{p,M}^{-\kappa})} = \left\{ \sum_{n=1}^{\infty} \rho_n |\lambda_n^{(M)}|^{-2\kappa} |-\kappa < f, u >_\kappa|^2 \right\}^{\frac{1}{2}}.$$

Now we fix $0 < \epsilon < \frac{1}{2}$. By the Sobolev embedding theorem, we have $\delta \in \mathcal{D}(A^{-\frac{1}{4}-\epsilon})$ and $\delta = \sum_{n=1}^{\infty} \rho_n \varphi_n$ in $\mathcal{D}(A^{-\frac{1}{4}-\epsilon})$. We set $\langle \cdot, \cdot \rangle = {}_{-\frac{1}{4}-\epsilon} \langle \cdot, \cdot \rangle_{\frac{1}{4}+\epsilon}$. We note

$$\langle f, \psi \rangle = (f, \psi) \quad \text{if } f \in L^2(0, \ell) \text{ and } \psi \in \mathcal{D}(A_{p,M}^{\frac{1}{4}+\epsilon})$$

(e.g., Chapter V in Brezis [7]).

Let us define the weak solution to system (3.2.1) - (3.2.3) as follows.

Definition 3.2.2 We call that u is a weak solution to (2.1) - (2.3) if the following conditions hold :

$$\begin{cases} u(\cdot, t) \in L^2(0, \ell) \text{ for } 0 < t \leq T, \\ u \in C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})), \\ \frac{\partial}{\partial t} u, \partial_t^\alpha u, A_{p,M} u \in C((0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})), \end{cases} \quad (3.2.7)$$

$$\lim_{t \rightarrow 0} \|u(\cdot, t) - \delta\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = 0, \quad (3.2.8)$$

$$\langle \partial_t^\alpha u(\cdot, t), \psi \rangle + (u(\cdot, t), A_p \psi) = 0 \quad \text{for } t \in (0, T], \psi \in \mathcal{D}(A_p). \quad (3.2.9)$$

Remark. Let u be a sufficiently smooth weak solution. Then, integrating (3.2.9) by parts, we have

$$\begin{aligned} 0 &= \langle \partial_t^\alpha u(\cdot, t), \psi \rangle + (u(\cdot, t), A_p \psi) \\ &= \left(\partial_t^\alpha u - \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x} \right), \psi \right) + \left[\psi(x) p(x) \frac{\partial u}{\partial x}(x, t) \right]_{x=0}^{x=\ell} \end{aligned}$$

for $\psi \in \mathcal{D}(A_p)$. Taking $\psi \in C_0^\infty(0, \ell)$, we see that $\partial_t^\alpha u(x, t) = \frac{\partial}{\partial x} (p(x) \frac{\partial u}{\partial x})$ for $x \in (0, \ell)$ and $t \in (0, T]$.

Since we arbitrarily choose $\psi(0)$ and $\psi(\ell)$ within $\psi \in \mathcal{D}(A_p)$, we obtain $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\ell, t) = 0$ for $t \in (0, T]$.

Therefore the smooth weak solution satisfies (3.2.1) and (3.2.3) in a usual sense.

Proposition 3.2.3 *There exists a unique weak solution to (3.2.1) - (3.2.3) and*

$$u(x, t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x) \quad \text{in } C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})). \quad (3.2.10)$$

Here for $\alpha > 0$ and $\beta \in \mathbf{R}$, the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (3.2.11)$$

(e.g., [23], [41]) and Γ is the gamma function. We note that $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbf{C}$ (e.g., [23]).

3.3 Proof of Proposition 3.2.3.

First Step. We will prove the uniqueness of the weak solutions to system (3.2.1)-(3.2.3).

Let u be a weak solution with $u(\cdot, 0) = 0$. We set

$$v_n(t) = \langle u(\cdot, t), \varphi_n \rangle, \quad 0 < t \leq T.$$

By $u \in C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$, we see that $v_n \in C[0, T]$ and $v_n(0) = 0$. By $u(\cdot, t) \in L^2(0, \ell)$, $t \in (0, T]$, we have $v_n(t) = \langle u(\cdot, t), \varphi_n \rangle$ for $t \in (0, T]$. Therefore (3.2.9) implies

$$\langle \partial_t^\alpha u(\cdot, t), \varphi_n \rangle + \langle u(\cdot, t), A_p \varphi_n \rangle = 0, \quad 0 < t \leq T,$$

that is,

$$\langle \partial_t^\alpha u(\cdot, t), \varphi_n \rangle + \lambda_n v_n(t) = 0, \quad 0 < t \leq T. \quad (3.3.1)$$

Now we prove the following.

Lemma 3.3.1 *We have*

$$\langle \partial_t^\alpha u(\cdot, t), \varphi_n \rangle = \partial_t^\alpha (u(\cdot, t), \varphi_n), \quad 0 < t \leq T.$$

Proof. Since the third condition in (3.2.7) yields

$$\partial_t^\alpha u(\cdot, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(\cdot, s) ds \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}), \quad 0 < t \leq T,$$

setting

$$J_{\varepsilon_1, \varepsilon_2}(\cdot, t) = \frac{1}{\Gamma(1-\alpha)} \int_{\varepsilon_1}^{t-\varepsilon_2} (t-s)^{-\alpha} \frac{\partial u}{\partial s}(\cdot, s) ds,$$

we have $\lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0, \varepsilon_1, \varepsilon_2 > 0} J_{\varepsilon_1, \varepsilon_2}(\cdot, t) = \partial_t^\alpha u(\cdot, t)$ in $\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $0 < t \leq T$. Approximating $J_{\varepsilon_1, \varepsilon_2}(\cdot, t)$ by the

Riemann sum, in terms of $\frac{\partial u}{\partial s} \in C([\varepsilon_1, T - \varepsilon_2]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$, we can see

$$\langle J_{\varepsilon_1, \varepsilon_2}(\cdot, t), \varphi_n \rangle = \frac{1}{\Gamma(1-\alpha)} \int_{\varepsilon_1}^{t-\varepsilon_2} (t-s)^{-\alpha} \left\langle \frac{\partial u}{\partial s}(\cdot, s), \varphi_n \right\rangle ds, \quad 0 < t \leq T.$$

Hence letting $\varepsilon_1, \varepsilon_2 \rightarrow 0$, by (3.2.7) we have

$$\langle \partial_t^\alpha u(\cdot, t), \varphi_n \rangle = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left\langle \frac{\partial u}{\partial s}(\cdot, s), \varphi_n \right\rangle ds, \quad 0 < t \leq T.$$

Moreover (3.2.7) yields

$$\left\langle \frac{\partial u}{\partial s} u(\cdot, s), \varphi_n \right\rangle = \frac{\partial}{\partial s} \langle u(\cdot, s), \varphi_n \rangle = \frac{\partial}{\partial s} (u(\cdot, s), \varphi_n), \quad 0 < s \leq T.$$

Then we have

$$\langle \partial_t^\alpha u(\cdot, t), \varphi_n \rangle = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} (u(\cdot, s), \varphi_n) ds = \partial_t^\alpha (u(\cdot, s), \varphi_n), \quad 0 < t \leq T.$$

Thus the proof of the lemma is completed.

Applying Lemma 3.3.1 in (3.3.1), we have

$$\partial_t^\alpha v_n(t) + \lambda_n v_n(t) = 0, \quad 0 < t \leq T, \quad v_n(0) = 0.$$

The uniqueness of the initial value problem for the fractional ordinary differential equation (e.g., Kilbas, Srivastava and Trujillo [23], Chapter 3 in Podlubny [41]) implies $v_n(t) = 0$ for $0 \leq t \leq T$ and $n \in \mathbf{N}$. Since

$\{\varphi_n\}_{n \in \mathbb{N}}$ is complete in $L^2(0, \ell)$, we see that $u(\cdot, t) = 0$ for $0 \leq t \leq T$. The proof of the uniqueness is completed.

Second Step Next, we will verify that representation (3.2.10) gives the weak solution to system (3.2.1) - (3.2.3). In the following, we set

$$\tilde{u}(x, t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x).$$

We will use the following results on the Mittag-Leffler function.

Lemma 3.3.2 *If $\alpha < 2$, β is an arbitrary real number and μ satisfies $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there exists a constant $C_1 > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C_1}{1 + |z|}, \quad z \in \mathbf{C}, \quad \mu \leq |\arg(z)| \leq \pi.$$

For the proof, we refer to Theorem 1.6 (p.35) in Podlubny [41] for example.

Lemma 3.3.3 *Let $\lambda > 0$.*

(i) :

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0, \quad \alpha > 0.$$

(ii) :

$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0, \quad 0 < \alpha < 1.$$

By noting that $E_{\alpha,1}(z)$ is an entire function in $z \in \mathbf{C}$, the proof of the lemma follows directly by the termwise differentiation of (3.2.11) and

$$(\partial_\eta^\alpha \eta^{\alpha k})(t) = \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + 1 - \alpha)} t^{-\alpha + \alpha k}, \quad 0 < \alpha < 1, \quad k \in \mathbb{N}.$$

Now we prove (3.2.7).

(i) **Verification of $\tilde{u} \in C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:**

Let us fix $t \in [0, T]$. It follows from (3.2.5), (3.2.6) and Lemma 3.3.2 that

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 < \infty.$$

Thus for fixed $t \in [0, T]$, we have $\tilde{u}(\cdot, t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$. For $t, t+h \in [0, T]$, we have

$$\begin{aligned} & \|\tilde{u}(\cdot, t+h) - \tilde{u}(\cdot, t)\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)|^2. \end{aligned} \quad (3.3.2)$$

Here it follows from Lemma 3.3.2 that $|E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)|^2$ is uniformly bounded for $n \in \mathbf{N}$.

Thus using the Lebesgue convergence theorem, in terms of (3.2.5) we have

$$\lim_{h \rightarrow 0} \|\tilde{u}(\cdot, t+h) - \tilde{u}(\cdot, t)\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} = 0.$$

Therefore $\tilde{u} \in C([0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$.

(ii) Verification of $\tilde{u}(\cdot, t) \in L^2(0, \ell)$ for $t \in (0, T]$:

For fixed $t \in (0, T]$, Lemma 3.3.2, (3.2.5) and (3.2.6) yield

$$\|\tilde{u}(\cdot, t)\|_{L^2(0, \ell)}^2 = \sum_{n=1}^{\infty} \rho_n |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \leq \sum_{n=1}^{\infty} \rho_n \left(\frac{C}{1 + |\lambda_n t^\alpha|} \right)^2 < \infty,$$

which means that $\tilde{u}(\cdot, t) \in L^2(0, \ell)$ for $t \in (0, T]$.

(iii) Verification of $\frac{\partial \tilde{u}}{\partial t} \in C((0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:

First, we consider

$$U(x, t) = \sum_{n=1}^{\infty} \rho_n \frac{d}{dt} (E_{\alpha,1}(-\lambda_n t^\alpha)) \varphi_n(x)$$

for $t \in (0, T]$. By Lemma 3.3.3 (i), we have

$$U(x, t) = \sum_{n=1}^{\infty} \rho_n (-\lambda_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \varphi_n(x).$$

By Lemma 3.3.2, (3.2.5) and (3.2.6), we have

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |(-\lambda_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 < \infty, \quad 0 < t \leq T.$$

Thus $U(\cdot, t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $t \in (0, T]$.

Next we have

$$\begin{aligned} & \left\| \frac{\tilde{u}(\cdot, t+h) - \tilde{u}(\cdot, t)}{h} - U(\cdot, t) \right\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n \left| \frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)}{h} - \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha)) \right|^2. \end{aligned} \quad (3.3.3)$$

Since the mean value theorem implies that

$$\left| \frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)}{h} \right| = \left| \frac{dE_{\alpha,1}(-\lambda_n \eta^\alpha)}{d\eta} \Big|_{\eta=t+\theta h} \right|$$

with some $\theta \in [0, 1]$,

$$\left| \frac{E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)}{h} - \frac{d}{dt}(E_{\alpha,1}(-\lambda_n t^\alpha)) \right|^2$$

is uniformly bounded for $n \in \mathbb{N}$ from Lemma 3.3.2 and Lemma 3.3.3 (i). Therefore the left-hand side of

(3.3.3) tends to 0 for $h \rightarrow 0$. Hence $\frac{\partial \tilde{u}}{\partial t}(\cdot, t)$ exists and is equal to $U(\cdot, t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ for $0 < t \leq T$:

$$\frac{\partial \tilde{u}}{\partial t}(\cdot, t) = \sum_{n=1}^{\infty} \rho_n (-\lambda_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \varphi_n, \quad 0 < t \leq T. \quad (3.3.4)$$

The continuity of $\frac{\partial \tilde{u}}{\partial t}(\cdot, t)$ in $t \in (0, T]$ is proved similarly to (3.3.3). Therefore $\frac{\partial \tilde{u}}{\partial t} \in C((0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$ is

proved.

(iv) Verification of $\partial_t^\alpha \tilde{u} \in C((0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$:

Let us fix $t \in (0, T]$. For $0 < s < t$, by Lemmata 3.2 and 3.3 (i), the following estimation hold :

$$\begin{aligned} & \left\| (t-s)^{-\alpha} \frac{\partial \tilde{u}}{\partial s}(\cdot, s) \right\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}^2 \\ &= (t-s)^{-2\alpha} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |\lambda_n|^2 s^{2\alpha-2} |E_{\alpha,\alpha}(-\lambda_n s^\alpha)|^2 \\ &\leq C_2 s^{2\alpha-2} (t-s)^{-2\alpha} \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \frac{|\lambda_n|^2}{(1+|\lambda_n|s^\alpha)^2}, \end{aligned}$$

where $C_2 > 0$ is some constant. On the other hand,

$$\begin{aligned} & \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \frac{|\lambda_n|^2}{(1+|\lambda_n|s^\alpha)^2} = \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}} \frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon} \frac{|\lambda_n|^{2-\epsilon}}{(1+|\lambda_n|s^\alpha)^{2-\epsilon}} \frac{1}{(1+|\lambda_n|s^\alpha)^\epsilon} \\ & \leq \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}} \frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon} \frac{(|\lambda_n|s^\alpha)^{2-\epsilon}}{(1+|\lambda_n|s^\alpha)^{2-\epsilon}} \frac{1}{s^{(2-\epsilon)\alpha}} \leq \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+\epsilon}} \frac{|\lambda_n|^\epsilon}{|\lambda_n^{(M)}|^\epsilon} s^{-(2-\epsilon)\alpha}, \end{aligned}$$

so that

$$\left\| (t-s)^{-\alpha} \frac{\partial \tilde{u}}{\partial s}(\cdot, s) \right\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})} \leq C_3 s^{-1+\frac{1}{2}\epsilon\alpha} (t-s)^{-\alpha}$$

with some constant $C_3 > 0$. Therefore $\|(t-s)^{-\alpha} \frac{\partial \tilde{u}}{\partial s}(\cdot, s)\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}$ is integrable over the interval $s \in (0, t)$.

Then $\partial_t^\alpha \tilde{u}(\cdot, t) \in \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})$ exists. From (3.3.4) and Lemma 3.3.3 (i), in terms of the Lebesgue convergence theorem, we can prove

$$\partial_t^\alpha \tilde{u}(\cdot, t) = \sum_{n=1}^{\infty} \rho_n(-\lambda_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n, \quad 0 < t \leq T. \quad (3.3.5)$$

The continuity of $\partial_t^\alpha \tilde{u}(\cdot, t)$ in $t \in (0, T]$ is proved similarly to (3.3.3).

Therefore, $\partial_t^\alpha \tilde{u} \in C((0, T]; \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}))$ is verified.

(v) Verification of (3.2.8):

Since

$$\delta = \sum_{n=1}^{\infty} \rho_n \varphi_n \quad \text{in } \mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon}),$$

we have

$$\|\tilde{u}(\cdot, t) - \delta\|_{\mathcal{D}(A_{p,M}^{-\frac{1}{4}-\epsilon})}^2 = \sum_{n=1}^{\infty} \frac{1}{|\lambda_n^{(M)}|^{\frac{1}{2}+2\epsilon}} \rho_n |E_{\alpha,1}(-\lambda_n t^\alpha) - 1|^2.$$

Taking $t \rightarrow 0$, by Lemma 3.3.2 and the Lebesgue convergence theorem, we verify (3.2.8).

(vi) Verification of (3.2.9):

Let us take $\psi \in \mathcal{D}(A_p)$ arbitrarily. Then we have $\psi = \sum_{n=1}^{\infty} \rho_n(\psi, \varphi_n) \varphi_n$ in $\mathcal{D}(A_p)$. Then by (3.3.5), we

have

$$\begin{aligned} \langle \partial_t^\alpha \tilde{u}(\cdot, t), \psi \rangle &= \left\langle \sum_{n=1}^{\infty} \rho_n (-\lambda_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n, \sum_{n=1}^{\infty} \rho_n (\psi, \varphi_n) \varphi_n \right\rangle \\ &= \sum_{n=1}^{\infty} \rho_n (-\lambda_n) E_{\alpha,1}(-\lambda_n t^\alpha) (\psi, \varphi_n), \quad 0 < t \leq T. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\tilde{u}(\cdot, t), A_p \psi) &= \left(\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n, \sum_{m=1}^{\infty} \lambda_m \rho_m (\psi, \varphi_m) \varphi_m \right) \\ &= \sum_{n=1}^{\infty} \rho_n \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) (\psi, \varphi_n), \end{aligned}$$

which means (3.2.9).

From (i)-(vi), the eigenfunction expansion (3.2.10) gives the weak solution.

3.4 Proof of Theorem 3.2.1.

By Proposition 3.2.1, the weak solutions u and v are given by

$$u(x, t) = \sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x) \quad (3.4.1)$$

and

$$v(x, t) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta) \psi_n(x). \quad (3.4.2)$$

Here $0 = \lambda_1 < \lambda_2 < \dots$, $n \in \mathbf{N}$ are all the eigenvalues of A_p and φ_n is the eigenfunction corresponding to λ_n with $\varphi_n(0) = 1$ and we set $\rho_n = \|\varphi_n\|_{L^2(0,\ell)}^{-2}$, while $0 = \mu_1 < \mu_2 < \dots$ are all the eigenvalues of A_q , ψ_n is the eigenfunction corresponding to μ_n with $\psi_n(0) = 1$, and we set $\sigma_n = \|\psi_n\|_{L^2(0,\ell)}^{-2}$. Let $t_0 > 0$ be arbitrarily fixed. By the Sobolev embedding theorem, we have

$$\|\varphi_n\|_{C[0,\ell]} \leq C'_0 \|\varphi_n\|_{H^{\frac{1}{2}+2\varepsilon}(0,\ell)}$$

with sufficiently small $\varepsilon > 0$. Moreover we see that

$$\|\varphi_n\|_{H^{\frac{1}{2}+2\varepsilon}(0,\ell)} \leq C'_0 \|A_{p,M}^{\frac{1}{4}+\varepsilon} \varphi_n\|_{L^2(0,\ell)} = C'_0 |\lambda_n^{(M)}|^{\frac{1}{4}+\varepsilon} \frac{1}{\sqrt{\rho_n}}.$$

Hence by Lemma 3.3.2, (3.2.5) and (3.2.6), we have

$$\sum_{n=1}^{\infty} \max_{0 \leq x \leq \ell} |\rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x)| \leq C_0 \sum_{n=1}^{\infty} \sqrt{\rho_n} |\lambda_n^{(M)}|^{\frac{1}{4}+\varepsilon} \frac{1}{1+|\lambda_n t^\alpha|} < \infty$$

for $t_0 \leq t \leq T$. Therefore we see that the series on the right-hand sides of (3.4.1) and (3.4.2) are convergent uniformly in $x \in [0, \ell]$.

Consequently, assuming that $u(0, t) = v(0, t)$ for $0 < t \leq T$, we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta), \quad 0 < t \leq T. \quad (3.4.3)$$

Since the both sides of this equation are analytic in $\operatorname{Re} t > 0$, we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta), \quad t > 0.$$

For $E_{\alpha,1}(z)$, we have the following asymptotic behaviour

$$E_{\alpha,1}(-t) = \frac{1}{t\Gamma(1-\alpha)} + O(|t|^{-2}), \quad \text{as } t \rightarrow \infty. \quad (3.4.4)$$

(e.g., Theorem 1.4 (pp.33-34) in [41]).

First Step. First we will deduce $\alpha = \beta$ and

$$\int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx.$$

Since $\lambda_1 = 0$ and $\lambda_n > 0$ for $n = 2, 3, 4, \dots$, we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \rho_0 + \sum_{n=2}^{\infty} \rho_n \left[\frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} + \left\{ E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} \right\} \right].$$

By (3.4.4) and $\lambda_n > 0$ for $n \geq 2$, there exists a constant $C_1 > 0$ such that

$$\left| E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} \right| \leq \frac{C_1}{\lambda_n^2 t^{2\alpha}}, \quad n \geq 2$$

for sufficiently large t . Taking the summation for $n = 1, 2, \dots$, by (3.2.5) and (3.2.6) we have

$$\sum_{n=1}^{\infty} \rho_n \left| E_{\alpha,1}(-\lambda_n t^\alpha) - \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\lambda_n t^\alpha} \right| \leq \frac{C_2}{t^{2\alpha}}$$

with some $C_2 > 0$. Then we have

$$\sum_{n=1}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \rho_0 + \sum_{n=2}^{\infty} \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n t^\alpha} + O\left(\frac{1}{t^{2\alpha}}\right). \quad (3.4.5)$$

Similarly arguing for $\sum_{n=1}^{\infty} \sigma_n E_{\beta,1}(-\mu_n t^\beta)$, we have

$$\rho_0 + \frac{1}{t^\alpha} \sum_{n=2}^{\infty} \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n} + O\left(\frac{1}{t^{2\alpha}}\right) = \sigma_0 + \frac{1}{t^\beta} \sum_{n=2}^{\infty} \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n} + O\left(\frac{1}{t^{2\beta}}\right)$$

as $t \rightarrow \infty$. This means that $\alpha = \beta$ and $\rho_0 = \sigma_0$. In fact, letting $t \rightarrow \infty$, we see that $\rho_0 = \sigma_0$. Let $\alpha > \beta$.

Then

$$\rho_0 t^\beta = -\frac{t^\beta}{t^\alpha} \left(\sum_{n=2}^{\infty} \rho_n \frac{1}{\Gamma(1-\alpha)} \frac{1}{\lambda_n} \right) + O\left(\frac{t^\beta}{t^{2\alpha}}\right) + \sigma_0 + \sum_{n=2}^{\infty} \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n} + O\left(\frac{1}{t^\beta}\right).$$

Then, letting $t \rightarrow \infty$, we have

$$\infty = \rho_0 \lim_{t \rightarrow \infty} t^\beta = \sigma_0 + \sum_{n=2}^{\infty} \sigma_n \frac{1}{\Gamma(1-\beta)} \frac{1}{\mu_n},$$

because $\rho_0 > 0$. This is a contradiction. Similarly $\beta > \alpha$ is impossible. Therefore $\alpha = \beta$ follows.

Hence we have

$$\sum_{n=2}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=2}^{\infty} \sigma_n E_{\alpha,1}(-\mu_n t^\alpha), \quad t > 0. \quad (3.4.6)$$

Second Step. We will prove $\lambda_n = \mu_n$, $n \in \mathbf{N}$. We take the Laplace transform and we can obtain

$$\int_0^{\infty} e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt = \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \operatorname{Re} z > 0. \quad (3.4.7)$$

In fact, we can take the Laplace transforms termwise in (3.2.11) to obtain

$$\int_0^{\infty} e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt = \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \operatorname{Re} z > \lambda_n^{\frac{1}{\alpha}}$$

(cf. formula (1.80) on p.21 in [41]). Since $\sup_{t \geq 0} |E_{\alpha,1}(-\lambda_n t^\alpha)| < \infty$ (e.g., Theorem 1.6 on p.35 in [41]), we

see that $\int_0^{\infty} e^{-zt} E_{\alpha,1}(-\lambda_n t^\alpha) dt$ is analytic with respect to z in $\operatorname{Re} z > 0$. Therefore the analytic continuation

yields (3.4.7) for $\operatorname{Re} z > 0$.

By Lemma 3.3.2, (3.2.5), (3.2.6) and the Lebesgue convergence theorem, noting that

$$\left| e^{-t \operatorname{Re} z} \sum_{n=2}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) \right| \leq C_1' e^{-t \operatorname{Re} z} \left(\sum_{n=2}^{\infty} \frac{1}{|\lambda_n|} \right) \frac{1}{t^\alpha} \leq \frac{C_1''}{t^\alpha} e^{-t \operatorname{Re} z}, \quad t > 0,$$

and $e^{-t \operatorname{Re} z} t^{-\alpha}$ is integrable in $t \in (0, \infty)$ for fixed z satisfying $\operatorname{Re} z > 0$, we have

$$\int_0^{\infty} e^{-zt} \sum_{n=2}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha) dt = \sum_{n=2}^{\infty} \rho_n \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \quad \operatorname{Re} z > 0.$$

Similarly

$$\int_0^{\infty} e^{-zt} \sum_{n=2}^{\infty} \sigma_n E_{\alpha,1}(-\mu_n t^\alpha) dt = \sum_{n=2}^{\infty} \sigma_n \frac{z^{\alpha-1}}{z^\alpha + \mu_n}, \quad \operatorname{Re} z > 0.$$

Hence (3.4.6) yields

$$\sum_{n=2}^{\infty} \frac{\rho_n}{z^\alpha + \lambda_n} = \sum_{n=2}^{\infty} \frac{\sigma_n}{z^\alpha + \mu_n}, \quad \operatorname{Re} z > 0.$$

That is,

$$\sum_{n=2}^{\infty} \frac{\rho_n}{\eta + \lambda_n} = \sum_{n=2}^{\infty} \frac{\sigma_n}{\eta + \mu_n}, \quad \operatorname{Re} \eta > 0. \quad (3.4.8)$$

By (3.2.5) and (3.2.6), we can analytically continue the both sides of (3.4.8) in η , so that (3.4.8) holds for $\eta \in \mathbf{C} \setminus (\{-\lambda_n\}_{n \geq 2} \cup \{-\mu_n\}_{n \geq 2})$.

Now we deduce $\lambda_2 = \mu_2$ from (3.4.8). Let us assume $\lambda_2 \neq \mu_2$. Without loss of generality, we can assume that $\lambda_2 < \mu_2$. Then we can take a suitable disk which includes $-\lambda_2$ and does not include $\{-\lambda_n\}_{n \geq 3} \cup \{-\mu_n\}_{n \geq 3}$. Integrating (3.4.8) in a disk, we have

$$2\pi i \rho_2 = 0.$$

This is contradiction because of $\rho_2 \neq 0$. Then we obtain $\lambda_2 = \mu_2$. Repeating this argument, we can obtain

$$\lambda_n = \mu_n, \quad n = 2, 3, 4, \dots$$

Moreover by (3.2.5) we see that

$$\int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx. \quad (3.4.9)$$

Third Step. In order to prove that $p = q$ on $[0, \ell]$, we apply the Gel'fand-Levitan theory. For it, we have to transform (3.2.1) to the canonical form by means of the Liouville transform (e.g., Yosida [56]). The argument in this step is a modification of Murayama [35].

By (3.4.9), we set

$$\ell_0 = \int_0^\ell \frac{1}{\sqrt{p(x)}} dx = \int_0^\ell \frac{1}{\sqrt{q(x)}} dx.$$

By the Liouville transform, we have

$$z = z(x) = \int_0^x \frac{1}{\sqrt{p(\xi)}} d\xi$$

and

$$\tilde{u}(z, t) = u(x, t)p(x)^{1/4},$$

system (3.2.1) - (3.2.3) is transformed to

$$\left\{ \begin{array}{l} \frac{\partial \tilde{u}}{\partial t} + \left(a - \frac{\partial^2}{\partial z^2} \right) \tilde{u} = 0, \quad 0 < z < \ell_0, \quad 0 < t < T, \\ \frac{\partial \tilde{u}}{\partial z}(0, t) - h\tilde{u}(0, t) = 0, \quad 0 < t < T, \\ \frac{\partial \tilde{u}}{\partial z}(\ell_0, t) + H\tilde{u}(\ell_0, t) = 0, \quad 0 < t < T, \\ \tilde{u}(z, 0) = \delta(z)f(z), \quad 0 < z < \ell_0, \end{array} \right.$$

where

$$a(z) = \frac{1}{f(z)} \frac{d^2}{dz^2} f(z), \quad f(z) = p(x)^{1/4} \tag{3.4.10}$$

and

$$h = \frac{1}{f(0)} \frac{df}{dx}(0), \quad H = -\frac{1}{f(\ell_0)} \frac{df}{dx}(\ell_0). \tag{3.4.11}$$

Similarly, by

$$w = w(y) = \int_0^y \frac{1}{\sqrt{q(\xi)}} d\xi$$

and

$$\tilde{v}(z, t) = v(y, t)q(y)^{1/4},$$

system (3.2.2) - (3.2.4) is transformed to

$$\begin{cases} \frac{\partial \tilde{v}}{\partial t} + \left(b - \frac{\partial^2}{\partial w^2} \right) \tilde{v} = 0, & 0 < w < \ell_0, \quad 0 < t < T, \\ \frac{\partial \tilde{v}}{\partial w}(0, t) - j\tilde{v}(0, t) = 0, & 0 < t < T, \\ \frac{\partial \tilde{v}}{\partial w}(\ell_0, t) + J\tilde{v}(\ell_0, t) = 0, & 0 < t < T, \\ \tilde{v}(w, 0) = \delta(w)g(w), & 0 < w < \ell_0, \end{cases}$$

where

$$b(w) = \frac{1}{g(w)} \frac{d^2}{dw^2} g(w), \quad g(w) = q(y)^{1/4} \quad (3.4.12)$$

and

$$j = \frac{1}{g(0)} \frac{dg}{dw}(0), \quad J = -\frac{1}{g(\ell_0)} \frac{dg}{dw}(\ell_0). \quad (3.4.13)$$

Then $u(0, t) = v(0, t)$, $0 < t < T$ is equivalent to

$$p(0)^{-1/4} \tilde{u}(0, t) = q(0)^{-1/4} \tilde{v}(0, t), \quad 0 < t < T. \quad (3.4.14)$$

We will define an operator $A_{a,h,H}$ in $L^2(0, \ell_0)$ by

$$\begin{cases} (A_{a,h,H}\psi)(z) = -\frac{\partial^2}{\partial z^2} \psi + a(z)\psi(z), & 0 < z < \ell_0, \\ \mathcal{D}(A_{a,h,H}) = \left\{ \psi \in H^2(0, \ell_0); \frac{d\psi}{dz}(0) - h\psi(0) = \frac{d\psi}{dz}(\ell_0) + H\psi(\ell_0) = 0 \right\} \end{cases}$$

and we define an operator $A_{b,j,J}$ similarly. By $\sigma(A_{a,h,H})$, we denote the set of all the eigenvalues of $A_{a,h,H}$.

Since the Liouville transform does not change the eigenvalues, by $\sigma(A_p) = \sigma(A_q)$ we obtain

$$\sigma(A_{a,h,H}) = \sigma(A_{b,j,J}) = \{\lambda_n\}_{n \in \mathbf{N}}. \quad (3.4.15)$$

Let $\tilde{\varphi}_n$ and $\tilde{\psi}_n$, $n \in \mathbf{N}$ be the corresponding eigenfunctions of $A_{a,h,H}$ and $A_{b,j,J}$ for λ_n respectively such

that $\tilde{\varphi}_n(0) = \tilde{\psi}_n(0) = 1$. We set

$$\tilde{\rho}_n = \frac{1}{\|\tilde{\varphi}_n\|_{L^2(0, \ell_0)}^2}, \quad \tilde{\sigma}_n = \frac{1}{\|\tilde{\psi}_n\|_{L^2(0, \ell_0)}^2}.$$

Similarly to Proposition 3.2.1, noting that $\tilde{u}(z, 0) = \delta(z)p(x)^{\frac{1}{4}}$ and $\tilde{v}(w, 0) = \delta(w)q(y)^{\frac{1}{4}}$, we obtain

$$\begin{cases} \tilde{u}(z, t) = p(0)^{1/4} \sum_{n=1}^{\infty} \tilde{\rho}_n E_{\alpha,1}(-\lambda_n t^\alpha) \tilde{\varphi}_n(z), \\ \tilde{v}(z, t) = q(0)^{1/4} \sum_{n=1}^{\infty} \tilde{\sigma}_n E_{\alpha,1}(-\lambda_n t^\alpha) \tilde{\psi}_n(z), \end{cases} \quad (3.4.16)$$

where the convergences are understood in a corresponding space to (3.2.7). Moreover it is known (e.g., [26]) that $\sup_{n \in \mathbf{N}} \tilde{\rho}_n, \sup_{n \in \mathbf{N}} \tilde{\sigma}_n < \infty$. Therefore by (3.2.5) and Lemma 3.3.2, similarly to (3.4.3), we can prove that the series on the right-hand sides of (3.4.16) are convergent in $C((0, T]; C[0, \ell_0])$.

Hence (3.4.14) yields

$$\sum_{n=1}^{\infty} \tilde{\rho}_n E_{\alpha,1}(-\lambda_n t^\alpha) = \sum_{n=1}^{\infty} \tilde{\sigma}_n E_{\alpha,1}(-\lambda_n t^\alpha), \quad 0 < t \leq T.$$

Similarly to (3.4.8), we can argue to obtain

$$\sum_{n=1}^{\infty} \frac{\tilde{\rho}_n}{\eta + \lambda_n} = \sum_{n=1}^{\infty} \frac{\tilde{\sigma}_n}{\eta + \lambda_n}, \quad \eta \in \mathbf{C} \setminus \{-\lambda_n\}_{n \in \mathbf{N}}.$$

Integrating the both sides in a sufficiently small disk centred at $-\lambda_n$, we see that

$$\tilde{\rho}_n = \tilde{\sigma}_n, \quad n \in \mathbf{N}. \quad (3.4.17)$$

By (3.4.15) and (3.4.17), we apply the Gel'fand-Levitan theory (e.g., Theorem 1.4.2 (p.21) in Freiling and Yurko [13], Marchenko [30]) to have

$$a(z) = b(z), \quad 0 \leq z \leq \ell_0, \quad h = j, \quad H = J. \quad (3.4.18)$$

Finally we have to derive $p(x) = q(x)$, $0 \leq x \leq \ell_0$ from (3.4.18). The argument is same as in Murayama [35] and we repeat it for the completeness. We first have

$$\ell = \int_0^{\ell_0} \frac{dx}{dz} dz = \int_0^{\ell_0} \sqrt{p(x)} dz = \int_0^{\ell_0} f(z)^2 dz \quad (3.4.19)$$

and similarly

$$\ell = \int_0^{\ell_0} g(z)^2 dz.$$

On the other hand, we can prove that the positive solution $e = e(z)$ to

$$\begin{cases} \frac{d^2 e}{dz^2}(z) = a(z)e(z), & 0 < z < \ell_0, \\ \frac{1}{e(0)} \frac{de}{dz}(0) = h, & \int_0^{\ell_0} e(z)^2 dz = \ell, \end{cases}$$

is unique. Consequently we have

$$g(z) = f(z), \quad 0 \leq z \leq \ell_0$$

by (3.4.10) - (3.4.13) and (3.4.18). Therefore, since

$$\frac{dz}{dx} = \frac{1}{f(z)^2}, \quad 0 \leq x \leq \ell, \quad z(0) = 0$$

and

$$\frac{dw}{dx} = \frac{1}{g(w)^2}, \quad 0 \leq x \leq \ell, \quad w(0) = 0,$$

we obtain $w(x) = z(x)$, $0 \leq x \leq \ell$. Therefore

$$q(x) = \left(\frac{dw}{dx}(x) \right)^{-2} = \left(\frac{dz}{dx}(x) \right)^{-2} = p(x), \quad 0 \leq x \leq \ell.$$

Thus the proof of Theorem 2.1 is completed.

Acknowledgements.

First I thank Professor Masahiro Yamamoto of Graduate School of Mathematical Sciences of The University of Tokyo for his kind supports, constant encouragement and supervision for all these studies.

I also thank Professor Jin Cheng of School of Mathematical Sciences of Fudan University in Shanghai, China for inspiring me to work on the inverse problem for the fractional diffusion equation.

I am very grateful for discussions with Dr Igor Trooshin. He gave me the valuable information of the Gel'fand-Levitan theory and nonsymmetric ordinary differential operators.

Thanks also goes to everyone who has helped me in this study. In particular, to Mr. Ken-ichi Sakamoto of Graduate School of Mathematical Sciences of The University of Tokyo for his suggestions about eigenfunction expansion for the fractional diffusion equation, and his friendship.

I was supported partly by the Global COE Program "The Research and Training Center for New Development in Mathematics" at The Graduate School of Mathematical Sciences of The University of Tokyo.

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