J. Math. Sci. Univ. Tokyo
9 (2002), 1–163.

Crystalline Fundamental Groups II — Log Convergent Cohomology and Rigid Cohomology

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Abstract. In this paper, we investigate the log convergent cohomology in detail. In particular, we prove the log convergent Poincaré lemma and the comparison theorem between log convergent cohomology and rigid cohomology in the case that the coefficient is an F^a-isocrystal. We also give applications to finiteness of rigid cohomology with coefficient, Berthelot-Ogus theorem for crystalline fundamental groups and independence of compactification for crystalline fundamental groups.

Contents

2000 *Mathematics Subject Classification.* Primary 14F30; Secondary 14F35.

The title of the previous version of this paper was *Crystalline Fundamental Groups II — Overconvergent Isocrystals*.

Introduction

This paper is the continuation of the previous paper [Shi]. In the previous paper, we gave a definition of crystalline fundamental groups for certain fine log schemes over a perfect field of positive characteristic and proved some fundamental properties of them.

Let us briefly recall what we have done in the previous paper. First, let K be a field and let $f : (X,M) \longrightarrow (\operatorname{Spec} K, N)$ be a morphism of fine log schemes such that the 0-th log de Rham cohomology $H^0_{\mathrm{dR}}((X,M)/(\operatorname{Spec} K,N))$ is equal to K and let x be a K-valued point of $X_{f\text{-triv}} := \{x \in X \mid (f^*N)_{\bar{x}} \xrightarrow{\sim} M_{\bar{x}}\}$. Then we defined the de Rham fundamental group $\pi_1^{\mathrm{dR}}((X,M)/(\operatorname{Spec} K,N),x)$ of (X,M) over $(\operatorname{Spec} K,N)$ with base point x as the Tannaka dual of the category $\mathcal{N}C((X,M)/(\operatorname{Spec} K,N))$ of nilpotent integrable log connections on (X,M) over $(\operatorname{Spec} K,N)$. It is a pro-unipotent algebraic group over K. (For precise definition, see Section 3.1 in [Shi].) Second, let k be a perfect field of characteristic $p>0$, let W be the Witt ring of k and let us assume given the diagram

$$(X,M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} W, N),$$

where f is a morphism of fine log schemes of finite type, N is a fine log structure on $\operatorname{Spf} W$ and ι is the canonical exact closed immersion. Assume moreover that the 0-th log crystalline cohomology $H^0((X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W})$ is equal to W. Let x be a k-valued point of $X_{f\text{-triv}}$. Then we defined the crystalline fundamental group $\pi_1^{\mathrm{crys}}((X,M)/(\operatorname{Spf} W,N),x)$ of (X,M) over $(\operatorname{Spf} W,N)$ with base point x as the Tannaka dual of the category $\mathcal{N}I_{\mathrm{crys}}((X,M)/(\operatorname{Spf} W,N))$ of nilpotent isocrystals on the log crystalline site $((X,M)/(\operatorname{Spf} W,N))_{\mathrm{crys}}$. It is a pro-unipotent algebraic group over $K_0 := \mathbb{Q} \otimes_{\mathbb{Z}} W$. (For precise definition, see Section 4.1 in [Shi].) The abelianization of it is isomorphic to the dual of the first log crystalline cohomology (Hurewicz isomorphism), and we have the action of Frobenius on it which is isomorphic when f is log smooth integral and of Cartier type. In the case that N is trivial, X is proper smooth over k and M is the log structure associated to a normal crossing divisor D on X, the crystalline fundamental group $\pi_1^{\mathrm{crys}}((X,M)/\mathrm{Spf}\,W,x)$ should be regarded as the crystalline realization of the (conjectual) motivic fundamental group of $U := X - D$. (Note that it is not good to consider the crystalline fundamental group of U, for the crystalline cohomology of U is not finitely generated in general.) So it is natural to ask whether $\pi_1^{\mathrm{crys}}((X,M)/\mathrm{Spf}\,W,x)$ depends only on U and x and is independent of the choice of the compactification (X,M) of U as above. We asked it in Problem 4.2.1 of [Shi] and we gave the affirmative answer in the case $\dim X \leq 2$, by using resolution of singularities due to Abhyankar [A] and the structure theorem of a proper birational morphism between surfaces due to Shafarevich ([Sha]).

Thirdly, we proved the comparison theorem between de Rham fundamental groups and crystalline fundamental groups (which we call the Berthelot-Ogus theorem for fundamental groups), whose statement is as follows: Let k be a perfect field of characterictic $p > 0$, W the Witt ring of k and V a totally ramified finite extension of W. Denote the fraction fields of W, V by K_0, K, respectively. Assume we are given the following commutative diagram of fine log schemes

$$\begin{array}{ccccc}
(X_k, M) & \hookrightarrow & (X, M) & \hookleftarrow & (X_K, M) \\
\downarrow & & f\downarrow & & \downarrow \\
(\mathrm{Spec}\,k, N) & \hookrightarrow & (\mathrm{Spec}\,V, N) & \hookleftarrow & (\mathrm{Spec}\,K, N) \\
& \searrow & \downarrow & & \\
& & (\mathrm{Spec}\,W, N), & &
\end{array}$$

where the two squares are Cartesian, f is proper log smooth integral and X_k is reduced. Assume moreover that $H^0_{\mathrm{dR}}((X,M)/(\mathrm{Spf}\,V,N)) = V$ holds, and that we are given a V-valued point x of $X_{f\text{-triv}}$. Denote the special fiber (resp. generic fiber) of x by x_k (resp. x_K). Then there exists a canonical isomorphism of pro-algebraic groups

$$\pi_1^{\mathrm{crys}}((X_k,M)/(\mathrm{Spf}\,W,N),x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K,M)/(\mathrm{Spec}\,K,N),x_K).$$

To prove this theorem, we introduced, in Section 5.1 in [Shi], the notion of log convergent site and the isocrystals on it. Then we defined the convergent fundamental groups as follows. Let k, V, K be as above and let us assume given the diagram

$$(Y, M_Y) \xrightarrow{g} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N),$$

where g is a morphism of fine log schemes of finite type, N is a fine log structure on $\operatorname{Spf} V$ and ι is the canonical exact closed immersion. Assume moreover that the 0-th log convergent cohomology $H^0((Y/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{Y/V})$ is equal to K. Let x be a k-valued point of $Y_{g\text{-triv}}$. Then we defined the convergent fundamental group $\pi_1^{\mathrm{conv}}((Y, M_Y)/(\operatorname{Spf} V, N), x)$ of (Y, M_Y) over $(\operatorname{Spf} V, N)$ with base point x as the Tannaka dual of the category $\mathcal{N}I_{\mathrm{conv}}((Y, M_Y)/(\operatorname{Spf} V, N))$ of nilpotent isocrystals on the log convergent site $((Y, M_Y)/(\operatorname{Spf} V, N))_{\mathrm{conv}}$. Then the Berthelot-Ogus theorem is the consequence of the following three isomorphisms

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K \cong \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k), \tag{0.0.1}$$

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k) \cong \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K), \tag{0.0.2}$$

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \cong \pi_1^{\mathrm{crys}}((X_k, M)/(\operatorname{Spf} W, N), x_k), \tag{0.0.3}$$

which were proved in Chapter 5 of [Shi].

In this paper, we investigate the log convergent cohomology (the cohomology on log convergent site) of isocrystals in detail. Let k, V, K be as above and let us assume given the diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N),$$

where f is a morphism of fine log schemes of finite type, N is a fine log structure on $\operatorname{Spf} V$ and ι is the canonical exact closed immersion. First, for a locally free isocrystal $\mathcal{E}$ on log convergent site $((X, M)/(\operatorname{Spf} V, N))_{\mathrm{conv}}$, we introduce the analytic cohomology (in rigid analytic sense) $H^i_{\mathrm{an}}((X, M)/(\operatorname{Spf} V, N), \mathcal{E})$ as the cohomology of log de Rham complex associated to $\mathcal{E}$ on 'tubular neighborhood' of (X, M) (it is a rigid analytic space). Then we prove that the log convergent cohomology $H^i(((X,M)/(\mathrm{Spf}\, V, N))_{\mathrm{conv}}, \mathcal{E})$ is isomorphic to the analytic cohomology. This theorem says that the log convergent cohomology can be calculated by certain de Rham complex. This type of theorem is sometimes called Poincaré lemma. So we call this theorem log convergent Poincaré lemma. This is a log version of convergent Poincaré lemma proved by Ogus ([Og2]). Second, let us consider the case where N is trivial, X is proper smooth and M is the log structure associated to a simple normal crossing divisor D on X. Put $U := X - D$ and denote the open immersion $U \hookrightarrow X$ by j. Then we define the restriction $j^\dagger\mathcal{E}$ of $\mathcal{E}$ to an overconvergent isocrystal (see [Be3] or Section 1.4 in this paper for the definition of an overconvergent isocrystal on U) and we construct a canonical homomorphism

$$H^i_{\mathrm{an}}((X,M)/\mathrm{Spf}\, V, \mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E}). \tag{0.0.4}$$

Roughly speaking, both sides are cohomologies of certain de Rham complexes. So we define this homomorphism by constructing a homomorphism between these de Rham complexes. Then we prove that the homomorphism (0.0.4) is an isomorphism if $\mathcal{E}$ is trivial or an F^a-isocrystal (for definition, see Definition 2.4.2). Combining with log convergent Poincaré lemma, we obtain the comparison

$$H^i(((X,M)/\mathrm{Spf}\, V)_{\mathrm{conv}}, \mathcal{E}) \overset{\sim}{=} H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E}) \tag{0.0.5}$$

between log convergent cohomology and rigid cohomology. This isomorphism is quite natural from the motivic point of view, because both should be the p-adic realization of the motivic cohomology groups (with certain 'motivic coefficients').

The isomorphism (0.0.5) has the following importance: First, one can relate the left hand side to the log crystalline cohomology of (X,M). (Indeed, if $V = W$ holds, it is isomorphic to the log crystalline cohomology of (X,M) with certain coefficients.) Since X is proper smooth, this allows us to prove the finiteness of the left hand side: Hence we obtain the finiteness of the right hand side. That is, we can prove the finiteness of rigid cohomology with certain coefficients. The finiteness of rigid cohomology is proved by Berthelot ([Be4]) and Tsuzuki ([Ts2]) in the case that the coefficient is trivial or a unit-root overconvergent F^a-isocrystal. (There is also a result of Crew [Cr2] in the case of curves.) If we admit a version of quasi-unipotent conjecture for overconvergent F^a-isocrystals, our finiteness implies the finiteness of rigid cohomology when the coefficient is an overconvergent F^a-isocrystal. Second, it is known ([Be2]) that the right hand side of the isomorphism (0.0.5) depends only on U, although one uses a compactification of U to define it. So the left hand side also depends only on U. By using this fact in the case of trivial coefficient, we can prove that $\pi_1^{\mathrm{crys}}((X,M)/\operatorname{Spf} W, x)$ depends only on U and x and it is independent of the choice of the compactification (X,M) of U as above. That is, we can give the affirmative answer to Problem 4.2.1 in [Shi] in general case. (We need to work a little more since the irreducible components of D need not be smooth in Problem 4.2.1 in [Shi].)

Moreover, we note that the log convergent Poincaré lemma allows us to give an alternative proof of the isomorphisms (0.0.2) and (0.0.3), hence gives an alternative proof of Berthelot-Ogus theorem for fundamental groups. We can slightly weaken the hypothesis of the theorem in this new proof: the conditions 'f is integral' and 'X_k is reduced' are not necessary in the new proof.

Now let us explain the content of each chapter briefly. In Chapter 1, we give some preliminary results which we need in later chapters. In Section 1.1, we give a result concerning log schemes which we did not proved in the previous paper. We introduce the notion 'of Zariki type' (for definition, see Definition 1.1.1), which plays an important role in later chapters. In Section 1.2, we axiomize the relation between stratifications and integrable log connections which was proved in the case of certain (formal) log schemes in [Shi, §3.2]. It is a variant of [Be1, Chap. II]. We use the result in this section to define the log de Rham complex associated to an isocrystal on log convergent site on certain rigid analytic space. In Section 1.3, we review the results on rigid geometry which is due mainly to Berthelot ([Be2], [Be3]).

In Chapter 2, we investigate the log convergent cohomology of isocrystals in detail. In Section 2.1, we give basic definitions concerning log convergent site. We slightly change the definition of enlargement and log convergent site from those in the previous paper, but we prove that these changes cause no problem. We prove basic descent properties of log convegent site. We also introduce some new notions such as (pre-)widenings, which is a log version of widenings in [Og2]. In section 2.2, we introduce the notion of the tubular neighborhood for certain closed immersion of a fine log scheme into a fine log formal scheme, and define the analytic cohomology (in rigid analytic sense) of log schemes. In Section 2.3, we prove log convergent Poincaré lemma. That is, we prove that the log convergent cohomology is isomorphic to the analytic cohomology which is defined above. In Section 2.4, we prove the comparison theorem between the log convergent cohomology and the rigid cohomology.

In Chapter 3, we give some applications of the results in the previous chapter. In Section 3.1, we prove the application of the results in the previous chapter to the finiteness of rigid cohomology with certain coefficients: We prove the finiteness of the rigid cohomology in the case that the coefficient is an F^a-isocryatsl on a log compactification. Moreover, under a version of a quasi-unipotent conjecture (for overconvergent F^a-isocrystals), we prove the finiteness of the rigid cohomology in the case that the coefficient is an overconvergent F^a-isocrystal. In Section 3.2, we give an alternative proof of the Berthelot-Ogus theorem for fundamental groups, which is proved in the previous paper, under a slightly weaker assumption. In Section 3.3, we give the affirmative answer to Problem 4.2.1 in [Shi] in general case. That is, we prove that, when X is a scheme which is proper smooth over k and M is the log structure associated to a normal crossing divisor D on X, $\pi_1^{\text{crys}}((X,M)/\operatorname{Spf} W, x)$ depends only on $U := X - D$ and x and it is independent of the choice of the compactification (X,M) of U.

After writing the first version of this paper, the author learned that Chiarellotto, Le Stum and Trihan ([Ch], [Ch-LS], [Ch-LS2], [LS-T]) also studied on closely related subjects independently. In particular, the definition of rigid fundamental group which we introduce in Section 3.3 is due to Chiarellotto and Le Stum. There is also a related work of Mokrane ([Mo]).

This series of papers is a revised version of the author's thesis in Tokyo University. The author would like to express his profound gratitude to his thesis advisor Professor Takeshi Saito for valuable advices and encouragements. He also would like to express his thanks to Professors Yukiyoshi Nakkajima, Nobuo Tsuzuki and Doctor Kenichi Bannai for useful advices and conversations. He would like to express his thanks to Doctor Kiran Kedlaya for pointing out some mistakes in the earlier version of this paper. The author would like to thank to the referee for reading the first version of this series of papers carefully and patiently, and for giving him many advices. Without his advices, it would be impossible for the author to make this paper understandable. The author would like to express his thanks to those who encouraged him during the revision of this paper. Without their encouragement, he could not finish the revision of this paper. The author revised this paper during his stay at Université de Paris-Sud. The author would like to thank to the members there for the hospitality. Finally, the author would like to apologize to the editors and staffs of Journal of Mathematical Sciences, University of Tokyo, especially to Mrs. Ikuko Takagi, for the long delay of the revision of this paper, and express his thanks for their patience.

The author was supported by JSPS Research Fellowships for Young Scientists in 1996-97, while the main part of this work was done. As for the author's stay in Univerisité de Paris-Sud, he was supported by JSPS Postdoctoral Fellowships for Research Abroad.

Conventions

(1) Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$. A formal scheme T is called a formal V-scheme if T is a p-adic Noetherian formal scheme over $\mathrm{Spf}\, V$ and $\Gamma(U, \mathcal{O}_T)$ is topologically of finite type over V for any open affine $U \subset T$.

(2) For a scheme or a formal scheme T, we denote the category of coherent sheaves of $\mathcal{O}_T$-modules by $\mathrm{Coh}(\mathcal{O}_T)$ and for a formal scheme T over $\mathrm{Spf}\, V$ (where V is as in (1)), we denote by $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ the category of sheaves of $K \otimes \mathcal{O}_T$-modules on T which is isomorphic to $K \otimes F$ for some $F \in \mathrm{Coh}(\mathcal{O}_T)$, where K denotes the fraction field of V. For elementary properties of $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ for a formal V-scheme T, see [Og1, §1]. We call an object of $\mathrm{Coh}(K \otimes \mathcal{O}_T)$ an isocoherent sheaf on T.

(3) In this paper, we use freely the terminologies concerning the log structure on schemes or formal schemes in the sense of Fontaine, Illusie and Kato. Basic facts about log structures are written in [Kk]. See also [Shi, Chap. 2].

(4) Let $X \longrightarrow Y$ be a morphism of formal schemes and let N be a log structure on Y. Then the log structure on X defined by the pullback of the log structure N is also denoted by N, if there will be no confusions.

(5) Contrary to the convention of the previous paper [Shi], we denote the completed tensor product (resp. completed fiber product) of topological modules (resp. formal log schemes) by $\hat{\otimes}$ (resp. $\hat{\times}$).
(6) For a site $\mathcal{S}$, we will denote the topos associated to $\mathcal{S}$ by $\mathcal{S}^{\sim}$.

Chapter 1. Preliminaries

In this chapter, we give some preliminary results which we need in later chapters. In Section 1.1, we give a result concerning log schemes which we did not proved in the previous paper. We introduce the notion 'of Zariki type' (for definition, see Definition 1.1.1), which plays an important role in later chapters. In Section 1.2, we axiomize the relation between stratifications and integrable log connections, which was proved in the case of certain (formal) log schemes in [Shi, §3.2]. It is a variant of [Be1, Chap. II]. We use the result in this section to define the log de Rham complex associated to an isocrystal on log convergent site on certain rigid analytic space. In Section 1.3, we review the results on rigid geometry which is due mainly to Berthelot ([Be2], [Be3]). In particular, we recall the definition of overconvergent (F^a-)isocrystals and the rigid cohomology with coefficient.

1.1. A remark on log schemes

In this section, we prove a property on log schemes which we did not stated in the prevous paper [Shi].

Recall that a log structure on a (formal) scheme X is a pair (M, α), where M is a sheaf of monoid on *etale site* of X and α is a homomorphism $M \longrightarrow \mathcal{O}_X$ of sheaves of monoids which induces the isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$. (In the previous paper, we defined the notion of log structures only for schemes and p-adic formal schemes, but we can define it for any formal schemes.) We call the triple (X, M, α) a log (formal) scheme. In the following, we often denote (X, M, α) simply as (X, M), by abuse of notation. As for the definition of 'fineness' of a log (formal) scheme, see [Kk] or [Shi, §2.1].

We introduce a new terminology which plays an important role in later chapters:

Definition 1.1.1. A fine log scheme (resp. a fine log formal scheme) (X, M) is said to be of Zariski type if there exists an open covering $X =$

$\bigcup_i X_i$ with respect to Zariski topology such that $(X_i, M|_{X_i})$ admits a chart for any i.

The main result of this section is the following:

PROPOSITION 1.1.2. *Let (X, M) be a fine log scheme (resp. a fine log formal scheme) of Zariski type and let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes (resp. fine log formal schemes). Assume that (Y, N) admits a chart $\varphi : Q \longrightarrow N$. Then, Zariski locally on X, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ extending φ. If $f^*N \longrightarrow M$ is surjective, we may assume that the homomorphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is surjective.*

COROLLARY 1.1.3. *Any morphism between fine log schemes (resp. fine log formal schemes) of Zariski type admits a chart Zariski locally.*

In the following, we give a proof of the above proposition in the case of log schemes. (The case of log formal schemes can be proved in the same way.) To prove the proposition, we need to introduce the notion of log scheme with respect to Zariski topology:

DEFINITION 1.1.4.

(1) Let X be a scheme. A pre-log structure with respect to Zariski topology on X is a pair (M, α), where M is a sheaf of monoids on X_{Zar} and $\alpha : M \longrightarrow \mathcal{O}_X$ is a homomorphism of sheaves of monoids.
(2) A pre-log structure with respect to Zariski topology (M, α) is called a log structure with respect to Zariski topology if α induces the isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$.
(3) A log scheme with respect to Zariski topology is a triple (X, M, α), where X is a scheme and (M, α) is a log structure with respect to Zariski topology on X. In the following, we denote the log scheme with respect to Zariski topology (X, M, α) by (X, M), by abuse of notation.

DEFINITION 1.1.5. Let X be a scheme and let (M, α) be a pre-log structure with respect to Zariski topology on X. Then we define the log structure with respect to Zariski topology (M^a, α^a) associated to (M, α) as follows: M^a is defined to be the push-out of the diagram

$$\mathcal{O}_X^\times \xleftarrow{\alpha} \alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow M$$

in the category of sheaves of monoids on X_{Zar} and α^a is defined to be the morphism

$$M^a \longrightarrow \mathcal{O}_X\,;\ (a,b) \mapsto \alpha(a)b\ (a \in M, b \in \mathcal{O}_X).$$

DEFINITION 1.1.6. Let (X, M) be a log scheme with respect to Zariski topology. Then X is said to be fine (resp. fs) if Zariski locally on X, there exists a fine (resp. fs) monoid P and a homomorphism $P_X \longrightarrow \mathcal{O}_X$ on X_{Zar} whose associated log structure with respect to Zariski topology is isomorphic to M.

DEFINITION 1.1.7.

(1) For a fine log scheme with respect to Zariski topology (X, M), a chart of (X, M) is a homomorphism $P_X \longrightarrow M$ for a fine monoid P which induces the isomorphism $(P_X)^a \overset{\sim}{=} M$.
(2) For a morphism $f : (X, M) \longrightarrow (Y, N)$ of fine log schemes with respect to Zariski topology, a chart of f is a triple $(P_X \to M, Q_Y \to N, Q \to P)$, where $P_X \to M$ and $Q_Y \to N$ are charts of M and N respectively and $Q \to P$ is a homomorphism such that the diagram

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \big\downarrow & & \big\downarrow \\ f^{-1}N & \longrightarrow & M \end{array}$$

is commutative.

Then one can prove the following proposition in the similar way to the case of usual log schemes ([Kk, §2], [Shi, (2.1.10)]). (We omit the proof.)

PROPOSITION 1.1.8. *Let $f : (X, M) \longrightarrow (Y, N)$ be a morphism of fine log schemes with respect to Zariski topology. Assume (Y, N) admits a chart $\varphi : Q_Y \longrightarrow N$. Then, Zariski locally on X, there exists a chart $(P_X \to M, Q_Y \overset{\varphi}{\to} N, Q \to P)$ of f extending φ. If the homomorphism $f^*N \longrightarrow M$ is surjective, we may assume that the homomorphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is surjective.*

For a scheme X, let ϵ be the canonical morphism of sites $X_{\mathrm{et}} \longrightarrow X_{\mathrm{Zar}}$. For a log structure (M, α) on X, we define the log structure with respect to Zariski topology $(\epsilon_* M, \epsilon_* \alpha)$ on X by

$$\epsilon_* \alpha : \epsilon_* M \longrightarrow \epsilon_* \mathcal{O}_{X_{\mathrm{et}}} = \mathcal{O}_{X_{\mathrm{Zar}}}.$$

Conversely, for a log structure with respect to Zariski topology (M, α), we define the log structure $(\epsilon^* M, \epsilon^* \alpha)$ as the associated log structure to the pre-log structure

$$\epsilon^{-1} M \xrightarrow{\epsilon^{-1}\alpha} \epsilon^{-1} \mathcal{O}_{X_{\mathrm{Zar}}} \longrightarrow \mathcal{O}_{X_{\mathrm{et}}}.$$

Then we have the following proposition, which is the key to the proof of Proposition 1.1.2.

PROPOSITION 1.1.9.

(1) *Let (X, M) be a fine log scheme with respect to Zariski topology and let $\varphi : P_X \longrightarrow M$ be a chart. Then the homomorphism*

$$\epsilon^* \varphi : P_X \xrightarrow{\epsilon^{-1}\varphi} \epsilon^{-1} M \longrightarrow \epsilon^* M$$

induces the isomorphism $P_X^a \overset{\sim}{=} \epsilon^* M$.

(2) *Let (X, M) be a fine log scheme and let $\varphi : P_X \longrightarrow M$ be a chart. Then the homomorphism*

$$\epsilon_* \varphi : P_X \longrightarrow \epsilon_* M$$

induces the isomorphism $P_X^a \overset{\sim}{=} \epsilon_* M$.

Before the proof of the proposition, we prepare a lemma. For a monoid Q, log structures (resp. log structures with respect to Zariski topology) $(M_i, \alpha_i)\ (i = 1, 2)$ and homomorphisms of sheaves of monoids $\varphi_i : Q_X \longrightarrow M_i$, we denote the set

$$\{f : M_1 \longrightarrow M_2 \,|\, \varphi_2 = f \circ \varphi_1, \alpha_1 = \alpha_2 \circ f\}$$

by $\mathrm{Hom}^{\mathrm{et}}_{Q,\varphi_1,\varphi_2}(M_1, M_2)$ (resp. $\mathrm{Hom}^{\mathrm{Zar}}_{Q,\varphi_1,\varphi_2}(M_1, M_2)$.) Then one has the following elementary lemma:

LEMMA 1.1.10. *Let (M, α_M) be a log structure with respect to Zariski topology on a scheme X and let (N, α_N) be a log structure on X. Let P be a monoid and let $\varphi : P_{X_{\mathrm{Zar}}} \longrightarrow M$, $\psi : P_{X_{\mathrm{et}}} \longrightarrow N$ be homomorphisms of sheaves of monoids. Let $\epsilon^*\varphi : P_{X_{\mathrm{et}}} \longrightarrow \epsilon^* M$, $\epsilon_*\psi : P_{X_{\mathrm{Zar}}} \longrightarrow \epsilon_* N$ be the morphisms naturally induced by φ, ψ, respectively. Then we have the canonical bijection of sets*

$$\mathrm{Hom}^{\mathrm{Zar}}_{P,\varphi,\epsilon_*\psi}(M, \epsilon_* N) = \mathrm{Hom}^{\mathrm{et}}_{P,\epsilon^*\varphi,\psi}(\epsilon^* M, N). \tag{1.1.1}$$

PROOF. Let f be an element of $\mathrm{Hom}^{\mathrm{Zar}}_{P,\varphi,\epsilon_*\psi}(M, \epsilon_* N)$. Let g be the element of $\mathrm{Hom}(\epsilon^{-1}M, N)$ corresponding to f by the canonical bijection $\mathrm{Hom}(M, \epsilon_* N) \overset{\sim}{=} \mathrm{Hom}(\epsilon^{-1}M, N)$. Then, by the functoriality of the bijection

$$\mathrm{Hom}(-, \epsilon_* -) \overset{\sim}{=} \mathrm{Hom}(\epsilon^{-1} -, -),$$

we have $\epsilon^{-1}\alpha_M = \alpha_N \circ g$ and $g \circ \epsilon^{-1}\varphi = \psi$. Since N is a log structure, the homomorphism g factors uniquely as

$$\epsilon^{-1}M \longrightarrow \epsilon^* M \xrightarrow{h} N,$$

and we have the commutativities $\epsilon^*\alpha_M = \alpha_N \circ h, h \circ \epsilon^*\varphi = \psi$. Hence h is an element in $\mathrm{Hom}^{\mathrm{et}}_{P,\epsilon^*\varphi,\psi}(\epsilon^* M, N)$. One can check easily that the correspondence $f \leftrightarrow h$ gives the desired bijection. $\square$

PROOF OF PROPOSITION 1.1.9. To prove the assertion (1), we have only to prove the following: For any log structure N endowed with a homomorphism $\psi : P_{X_{\mathrm{et}}} \longrightarrow N$, there exists uniquely a homomorphism $g : \epsilon^* M \longrightarrow N$ of log structures such that $g \circ \epsilon^*\varphi = \psi$ holds.

Since $\varphi : P_{X_{\mathrm{Zar}}} \longrightarrow M$ is a chart, there exists uniquely a homomorphism $f : M \longrightarrow \epsilon_* N$ of log structures with respect to Zariski topology such that $\epsilon_*\psi = f \circ \varphi$ holds. So f is the unique element in $\mathrm{Hom}^{\mathrm{Zar}}_{P,\varphi,\epsilon_*\psi}(M, \epsilon_* N)$. Then the unique element g in $\mathrm{Hom}^{\mathrm{et}}_{P,\epsilon^*\varphi,\psi}(\epsilon^* M, N)$ corresponding to f by the above lemma satisfies the desired condition. So the assertion (1) is proved.

Let us prove the assertion (2). Denote the structure morphism $M \longrightarrow \mathcal{O}_{X_{\mathrm{et}}}$ by α_M and denote the composite $P_X \xrightarrow{\varphi} M \xrightarrow{\alpha_M} \mathcal{O}_{X_{\mathrm{et}}}$ by α_P. To prove the assertion (2), it suffices to prove that the homomorphism $\epsilon_*\varphi : P_X \longrightarrow \epsilon_* M$ induces the isomorphism

$$P_X \oplus_{(\epsilon_*\alpha_P)^{-1}(\mathcal{O}^\times_{X_{\mathrm{Zar}}})} 1 \xrightarrow{\sim} \epsilon_*M/\mathcal{O}^\times_{X_{\mathrm{Zar}}} = \epsilon_*M/\epsilon_*\mathcal{O}^\times_{X_{\mathrm{et}}}.$$

First let us note the following:

CLAIM. We have the isomorphism $\epsilon_*M/\epsilon_*\mathcal{O}^\times_{X_{\mathrm{et}}} \overset{\sim}{=} \epsilon_*(M/\mathcal{O}^\times_{X_{\mathrm{et}}})$.

PROOF OF CLAIM. Let us apply the functor ϵ_* to the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}^\times_{X_{\mathrm{et}}} & \longrightarrow & M & \longrightarrow & M/\mathcal{O}^\times_{X_{\mathrm{et}}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}^\times_{X_{\mathrm{et}}} & \longrightarrow & M^{\mathrm{gp}} & \longrightarrow & M^{\mathrm{gp}}/\mathcal{O}^\times_{X_{\mathrm{et}}} & \longrightarrow & 1. \end{array}$$

By Hilbert 90, we have $R^1\epsilon_*\mathcal{O}^\times_{X_{\mathrm{et}}} = 0$. So we get the following:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}^\times_{X_{\mathrm{Zar}}} & \longrightarrow & \epsilon_*M & \xrightarrow{\pi} & \epsilon_*(M/\mathcal{O}^\times_{X_{\mathrm{et}}}) & & \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}^\times_{X_{\mathrm{Zar}}} & \longrightarrow & \epsilon_*M^{\mathrm{gp}} & \xrightarrow{\pi^{\mathrm{gp}}} & \epsilon_*(M^{\mathrm{gp}}/\mathcal{O}^\times_{X_{\mathrm{et}}}) & \longrightarrow & 1. \end{array}$$

It suffices to prove the surjectivity of π. Let $U \subset X$ be an open set and let $a \in \Gamma(U, \epsilon_*(M/\mathcal{O}^\times_{X_{\mathrm{et}}})) = \Gamma(U, M/\mathcal{O}^\times_{X_{\mathrm{et}}})$. Then there exists an open covering $U' \longrightarrow U$ and an element $b \in \Gamma(U', \epsilon_*M^{\mathrm{gp}}) = \Gamma(U', M^{\mathrm{gp}})$ satisfying $\pi^{\mathrm{gp}}(b) = a|_{U'}$. To prove the claim, it suffices to prove that $b \in \Gamma(U', \epsilon_*M)$ holds. Since $a \in \Gamma(U, M/\mathcal{O}^\times_{X_{\mathrm{et}}})$, there exists an etale covering $V \longrightarrow U$ and an element $c \in \Gamma(V, M)$ satisfying $\pi(c) = a|_V$. We may assume that the morphism $V \longrightarrow U$ factors as the composite of surjective etale morphisms $V \longrightarrow U' \longrightarrow U$. Then we have $\pi(c) = \pi^{\mathrm{gp}}(b|_V)$. So there exists an element $u \in \Gamma(V, \mathcal{O}^\times_V)$ such that $b|_V = cu$ holds. Hence we have $b|_V \in \Gamma(V, M)$. So b is in $\Gamma(U', M) = \Gamma(U', \epsilon_*M)$. So we have shown the surjectivity of π and the proof of the claim is finished. $\square$

By the above claim, it suffices to show the isomorphism

$$P_X \oplus_{(\epsilon_*\alpha_P)^{-1}(\mathcal{O}^\times_{X_{\mathrm{Zar}}})} 1 \xrightarrow{\sim} \epsilon_*(M/\mathcal{O}^\times_{X_{\mathrm{et}}}).$$

For etale morphism $U \longrightarrow X$, let α_U be the composite

$$P \overset{\Gamma(X,\alpha_P)}{\longrightarrow} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{O}_U).$$

Then, the sheaf $P_X \oplus_{(\epsilon_* \alpha_P)^{-1}(\mathcal{O}^\times_{X_{\mathrm{Zar}}})} 1$ is the sheaf associated to the presheaf on X_{Zar}

$$(U \subset X) \mapsto P \oplus_{\alpha_U^{-1}(\Gamma(U, \mathcal{O}_U^\times))} 1.$$

Let N_0 be the presheaf on X_{Zar} defined by

$$(U \subset X) \mapsto P \oplus_{\alpha_U^{-1}(\Gamma(U, \mathcal{O}_U^\times))} 1/ \sim_{\mathrm{Zar}},$$

where $\sim_{\mathrm{Zar}}$ is the equivalence relation defined as follows: $a \sim_{\mathrm{Zar}} b$ holds if there exists a Zariski covering $\mathcal{U} := \{U_i \longrightarrow U\}_i$ such that $\mathrm{Im}(a) = \mathrm{Im}(b)$ holds in $P \oplus_{\alpha_{U_i}^{-1}(\Gamma(U_i, \mathcal{O}_{U_i}^\times))} 1$ for any i. Let N_1 be the presheaf defined by

$$N_1(U) := \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, N_0),$$

where $\mathcal{U}$ runs through Zariski open coverings of U. Then, by definition, N_1 is a sheaf and we have $P_X \oplus_{(\epsilon_* \alpha_P)^{-1}(\mathcal{O}^\times_{X_{\mathrm{Zar}}})} 1 = N_1$.

On the other hand, since $\alpha_P : P_X \longrightarrow M$ is a chart of M, the sheaf $M/\mathcal{O}^\times_{X_{\mathrm{et}}}$ is the sheaf on X_{et} associated to the presheaf

$$(U \longrightarrow X) \mapsto P \oplus_{\alpha_U^{-1}(\Gamma(U, \mathcal{O}_U^\times))} 1.$$

Let N_0' be the presheaf on X_{et} defined by

$$(U \longrightarrow X) \mapsto P \oplus_{\alpha_U^{-1}(\Gamma(U, \mathcal{O}_U^\times))} 1/ \sim_{\mathrm{et}},$$

where $\sim_{\mathrm{et}}$ is the equivalence relation defined as follows: $a \sim_{\mathrm{et}} b$ holds if there exists an etale covering $\mathcal{U} := \{U_i \longrightarrow U\}_i$ such that $\mathrm{Im}(a) = \mathrm{Im}(b)$ holds in $P \oplus_{\alpha_{U_i}^{-1}(\Gamma(U_i, \mathcal{O}_{U_i}^\times))} 1$ for any i. Let N_1' be the presheaf on X_{et} defined by

$$N_1'(U) := \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, N_0'),$$

where $\mathcal{U}$ runs through etale coverings of U. Then, by definition, N_1' is a sheaf and we have $M/\mathcal{O}^\times_{X_{\mathrm{et}}} = N_1'$. Hence it suffices to prove the following claim.

Claim. For any open $U \subset X$, we have $N_1(U) = N_1'(U)$.

To prove the claim, first we prove the following assertion: For any open $U \subset X$ and any etale surjective morphism $V \longrightarrow U$, we have the isomorphism $N_0(U) \overset{\sim}{=} N_0'(V)$. Since $V \longrightarrow U$ is surjective, we have $\alpha_U^{-1}(\Gamma(U, \mathcal{O}_U^\times)) = \alpha_V^{-1}(\Gamma(V, \mathcal{O}_V^\times))$. Hence we have the isomorphism

$$P \oplus_{\alpha_U^{-1}(\Gamma(U,\mathcal{O}_U^\times))} 1 \overset{\sim}{=} P \oplus_{\alpha_V^{-1}(\Gamma(V,\mathcal{O}_V^\times))} 1.$$

Therefore, to prove the eqality $N_0(U) \overset{\sim}{=} N_0'(V)$, it suffices to prove the equivalence of the relations $\sim_{\text{Zar}}$ (for $N_0(U)$) and $\sim_{\text{et}}$ (for $N_0'(V)$) via the above isomorphism. If $a \sim_{\text{Zar}} b$, then it is easy to see that $a \sim_{\text{et}} b$ holds. Let us prove the converse. Assume $a \sim_{\text{et}} b$ holds and let $\{V_i \longrightarrow V\}_i$ be an etale covering such that $\text{Im}(a) = \text{Im}(b)$ holds in $P \oplus_{\alpha_{V_i}^{-1}(\Gamma(V_i,\mathcal{O}_{V_i}^\times))} 1$ for any i. Let U_i be the image of the morphism $V_i \longrightarrow V \longrightarrow U$. Then $\{U_i \hookrightarrow U\}_i$ is a Zariski covering and we have the isomorphism

$$P \oplus_{\alpha_{V_i}^{-1}(\Gamma(V_i,\mathcal{O}_{V_i}^\times))} 1 \overset{\sim}{=} P \oplus_{\alpha_{U_i}^{-1}(\Gamma(U_i,\mathcal{O}_{U_i}^\times))} 1.$$

So $\text{Im}(a) = \text{Im}(b)$ holds in $P \oplus_{\alpha_{U_i}^{-1}(\Gamma(U_i,\mathcal{O}_{U_i}^\times))} 1$, that is, we have $a \sim_{\text{Zar}} b$, as desired. Hence we have $N_0(U) = N_0'(V)$.

Now we prove the claim. For an etale covering $\mathcal{U} := \{U_i \longrightarrow U\}$ of U, let us denote the Zariski covering $\{\text{Im}(U_i) \subset U\}$ by $\mathcal{U}^{\text{Zar}}$. Then the assertion in the previous paragraph implies the isomorphism

$$\check{H}^0(\mathcal{U}^{\text{Zar}}, N_0) \overset{\sim}{=} \check{H}^0(\mathcal{U}, N_0').$$

Note that, if $\mathcal{U}$ runs through etale coverings of U, $\mathcal{U}^{\text{Zar}}$ runs through Zariski coverings of U. Hence we have

$$N_1(U) = \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}^{\text{Zar}}, N_0) \overset{\sim}{=} \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, N_0') = N_1'(U).$$

Hence the claim is finished and the proof of proposition is now completed. □

As an immediate corollary of Proposition 1.1.9, we have the following:

Corollary 1.1.11. *There exists a canonical equivalence of categories*

$$\begin{pmatrix}\text{fine log schemes}\\ \text{of Zariski type}\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}\text{fine log schemes with}\\ \text{respect to Zariski topology}\end{pmatrix}$$

given by $(X, M) \mapsto (X, \epsilon_* M)$. *The quasi-inverse is given by* $(X, M) \mapsto (X, \epsilon^* M)$.

Now we give a proof of Proposition 1.1.2.

Proof of Proposition 1.1.2. Let $f : (X, M) \longrightarrow (Y, N)$ be as in the statement of the proposition and let $\epsilon_* f : (X, \epsilon_* M) \longrightarrow (Y, \epsilon_* N)$ be the associated morphism between fine log schemes with respect to Zariski topology. Then $\epsilon_* \varphi : Q_Y \longrightarrow \epsilon_* N$ is a chart of $(Y, \epsilon_* N)$. Then, by Proposition 1.1.8, there exists a chart $(P_X \to \epsilon_* M, Q_Y \overset{\epsilon_*\varphi}{\to} \epsilon_* N, Q \to P)$ of $\epsilon_* f$ extending $\epsilon_* \varphi$. Then, by pulling back the chart by ϵ^*, we get a chart $(P_X \to M, Q_Y \overset{\varphi}{\to} N, Q \to P)$ of f extending φ, by Proposition 1.1.9 and Corollary 1.1.11. □

1.2. Stratifications and integrable connections on formal groupoids

Let $f : (X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$ (resp. a formally log smooth morphism of fine log formal V-schemes, where V is a complete discrete valuation ring of mixed characteristic). In the previous paper ([Shi, §3.2]), we proved the equivalence of categories between the following two categories:

(1) The category $C((X, M)/(S, N))$ of coherent sheaves with integrable log connections (resp. The category $\hat{C}((X, M)/(S, N))$ of isocoherent sheaves with integrable formal log connections).
(2) The category $\mathrm{Str}((X, M)/(S, N))$ of coherent sheaves with log stratifications (resp. The category $\widehat{\mathrm{Str}}((X, M)/(S, N))$ of isocoherent sheaves with formal log stratifications).

In this section, we remark that we can axiomize this result by using the notion of a formal groupoid in a topos ([Be1, Chap. II]). So the result in this section is a variant of a result in [Be1, Chap. II]. We will use this result to construct the log de Rham complex associated to an isocrystal on log convergent site on certain rigid analytic space in the next chapter.

First we recall the notion of formal groupoid, which is defined in [Be1, Chap. II.1.1.3].

Definition 1.2.1 (Berthelot). Let T be a topos. A formal groupoid in T is the data

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}}),$$

where O is a ring in T, $\{P^n\}_n$ is a projective system of rings in T whose transition morphisms are surjective, $p_{i,n}, \pi_n, \delta_{n,m}, \tau_n$ are the homomorphisms of rings over O

$$\begin{aligned} p_{i,n} &: O \longrightarrow P^n, \\ \pi_n &: P^n \longrightarrow O, \\ \delta_{m,n} &: P^{m+n} \longrightarrow P^m \otimes_O P^n, \\ \tau_n &: P^n \longrightarrow P^n, \end{aligned}$$

which commute with transition maps of the projective system $\{P_n\}$ and which are subject to the following conditions:

(1) $\pi_n \circ p_{1,n} = \pi_n \circ p_{2,n} = \mathrm{id}$.
(2) We have

$$\delta_{m,n} \circ p_{1,m+n} = q_{1,m,n} \circ p_{1,m+n}, \quad \delta_{m,n} \circ p_{2,m+n} = q_{2,m,n} \circ p_{2,m+n},$$

where $q_{1,m,n}, q_{2,m,n}$ are homomorphisms $P^{m+n} \longrightarrow P^m \otimes_O P^n$ obtained by composing $P^{m+n} \longrightarrow P^m, P^{m+n} \longrightarrow P^n$ with the canonical homomorphisms $P^m \longrightarrow P^m \otimes_O P^n, P^n \longrightarrow P^m \otimes_O P^n$.
(3) $(\pi_m \otimes \mathrm{id}) \circ \delta_{m,n}$ and $(\mathrm{id} \otimes \pi_n) \circ \delta_{m,n}$ coincide with transition maps.
(4) $(\delta_{m,n} \otimes \mathrm{id}) \circ \delta_{m+n,p} = (\mathrm{id} \otimes \delta_{n,p}) \circ \delta_{m,n+p}$.
(5) $\tau_n \circ p_{1,n} = p_{2,n}$, $\tau_n \circ p_{2,n} = p_{1,n}$.
(6) $\pi_n \circ \tau_n = \pi_n$.
(7) The following diagrams are commutative:

$$\begin{array}{ccc} P^n \otimes_O P^n & \xrightarrow{\mathrm{id}\otimes\tau_n} & P^n \\ {\scriptstyle \delta_{n,n}}\uparrow & & \uparrow{\scriptstyle p_{1,n}} \\ P^{2n} & \xrightarrow{\pi_{2n}} & O, \end{array} \qquad \begin{array}{ccc} P^n \otimes_O P^n & \xrightarrow{\tau_n\otimes\mathrm{id}} & P^n \\ {\scriptstyle \delta_{n,n}}\uparrow & & \uparrow{\scriptstyle p_{2,n}} \\ P^{2n} & \xrightarrow{\pi_{2n}} & O. \end{array}$$

(In the above definition, we regard P^n as a bi-(O,O)-module via the left (resp. right) O-module struture defined by $p_{1,n}$ (resp. $p_{2,n}$)). We say that the formal groupoid $\mathcal{X}$ is of characteristic zero if O and P_n's are characteristic zero as rings in T.

We introduce the notion of 'differential log smoothness' as follows:

Definition 1.2.2. Let T be a topos. A formal groupoid in T

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

is said to be differentially log smooth if there exists locally an integer m and the elements $\{\xi_{j,n}\}_{j=1}^m$ of P^n for $n \in \mathbb{N}$ which satisfy the following conditions:

(1) For $n' > n$, the transition map $P^{n'} \longrightarrow P^n$ sends $\xi_{j,n'}$ to $\xi_{j,n}$.
(2) There exists the canonical isomorphism of left O-algebras

$$P^n \overset{\sim}{=} O[\xi_{j,n}\ (1 \leq j \leq m)]/(I_n)^{n+1},$$

where $I_n := (\xi_{1,n}, \cdots, \xi_{m,n}) \subset O[\xi_{j,n}\ (1 \leq j \leq m)]$.
(3) $\delta_{m,n}(\xi_{j,m+n} + 1) = (\xi_{j,m} + 1) \otimes (\xi_{j,n} + 1)$.

Remark 1.2.3. A differentially log smooth formal groupoid in a topos T is adic of finite type in the sense of [Be1, II.4.2.1].

Remark 1.2.4. A formal groupoid of characteristic zero in T

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

is differentially log smooth if and only if there exists locally an integer m and the elements $\{t_{j,n}\}_{j=1}^m$ of P^n for $n \in \mathbb{N}$ which satisfy the following conditions:

(1) For $n' > n$, the transition map $P^{n'} \longrightarrow P^n$ sends $t_{j,n'}$ to $t_{j,n}$.
(2) There exists the canonical isomorphism of left O-algebras

$$P^n \overset{\sim}{=} O[t_{j,n}\,(1 \leq j \leq m)]/(I'_n)^{n+1},$$

where $I'_n := (t_{1,n}, \cdots, t_{m,n}) \subset O[t_{j,n}\,(1 \leq j \leq m)]$.
(3) $\delta_{m,n}(t_{j,m+n}) = t_{j,m} \otimes 1 + 1 \otimes t_{j,n}$.

Indeed, assume we are given the elements $\{t_{j,n}\}_{j,n}$. Then if we put $\xi_{j,n} := \sum_{k=1}^{n} \frac{1}{k!} t_{j,n}^k$, the elements $\{\xi_{j,n}\}$ satisfy the conditions in Definition 1.2.2. Conversely, assume we are given the elements $\{\xi_{j,n}\}_{j,n}$ as in Definition 1.2.2. Then if we put $t_{j,n} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \xi_{j,n}^k$, the elements $\{t_{j,n}\}$ satisfy the conditions in this remark.

In particular, if $\mathcal{X}$ is an adic differentially smooth formal groupoid of finite type (in the sense of [Be1, II.4.2.3]) of characteristic zero, it is differentially log smooth.

Let T be a topos and let

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

be a differentially log smooth formal groupoid. We put $\omega^1 := \mathrm{Ker}(\pi_1 : P^1 \longrightarrow O)$. Then the left action and the right action of O to ω^1 coincide. So we regard ω^1 as O-module by this action, and let ω^q ($q \in \mathbb{N}$) be the q-th exterior power of ω^1 over O.

We define the differentials $d^0 : O \longrightarrow \omega^1$ and $d^1 : \omega^1 \longrightarrow \omega^2$ as follows: First, d^0 is defined by $d^0(a) = p_{2,1}(a) - p_{1,1}(a)$. Next let us consider the morphism

$$\partial : P^2 \longrightarrow P^1 \otimes P^1$$

defined by $\partial = q_{1,1,1} + q_{2,1,1} - \delta_{1,1}$, where $q_{i,m,n}$ is as in Definition 1.2.1. Then, by [Be1, p.117], one can see that the morphism ∂ induces the morphism

$$\mathrm{Ker}(\pi_2 : P^2 \longrightarrow O) \longrightarrow \omega^1 \otimes \omega^1,$$

which we denote also by ∂.

Now we check that the composite

$$\mathrm{Ker}(\pi_2 : P^2 \longrightarrow O) \xrightarrow{\partial} \omega^1 \otimes \omega^1 \longrightarrow \omega^2 \tag{1.2.1}$$

kills $\mathrm{Ker}(P^2 \longrightarrow P^1)$. Indeed, one can check it locally, so it suffices to prove the image of $p_{1,2}(a)\xi_{j,2}\xi_{j',2}$ by the above map is zero. One can see, by definition, that $\partial(p_{1,2}(a)\xi_{j,2}\xi_{j',2}) = -(p_{1,1}(a) \otimes 1)(\xi_{j,1} \otimes \xi_{j',1} + \xi_{j',1} \otimes \xi_{j,1})$ holds. So it is zero in ω^2. Hence the composite (1.2.1) induces the morphism $\omega^1 \longrightarrow \omega^2$. We denote this morphism by d^1.

One can check that the composite $d^1 \circ d^0$ is equal to zero. Locally, the morphism d^1 is characterized by the equation

$$d^1(\sum_{j=1}^{m} a_j \xi_{j,1}) = \sum_{j=1}^{m} d^0(a_j) \wedge \xi_{j,1}.$$

We recall the notion of integrable connections and stratifications.

DEFINITION 1.2.5. Let T be a topos and let

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

be a differentially log smooth formal groupoid in T. Define ω^q $(q = 1, 2)$, $d^0 : O \longrightarrow \omega^1$ and $d^1 : \omega^1 \longrightarrow \omega^2$ as above.

(1) For an O-module E, a connection on E with respect to $\mathcal{X}$ is a homomorphism

$$\nabla : E \longrightarrow E \otimes_O \omega^1$$

satisfying $\nabla(ae) = a\nabla(e) + e \otimes d^0(a)$.

(2) A connection ∇ on an O-module E is said to be integrable if we have $\nabla \circ \nabla = 0$, where we extend ∇ to the morphism

$$E \otimes \omega^1 \longrightarrow E \otimes \omega^2$$

by $\nabla(e \otimes \eta) = \nabla(e) \wedge \eta + e \otimes d^1(\eta)$.

For a subcategory $\mathcal{C}$ of the category of O-modules, let us denote the category of objects in $\mathcal{C}$ endowed with integrable connections by $C(\mathcal{C})$.

DEFINITION 1.2.6. Let T be a topos and let

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

be a formal groupoid in T. For an O-module E, a stratification on E with respect to $\mathcal{X}$ is a family of isomorphisms

$$\epsilon_n : P^n \otimes_O E \longrightarrow E \otimes_O P^n$$

satisfying the following conditions:

(1) Each ϵ_n is P^n-linear and $\epsilon_0 = \mathrm{id}$ holds.
(2) For any $n' > n$, $\epsilon_{n'}$ modulo $\mathrm{Ker}(P^{n'} \longrightarrow P^n)$ coincides with ϵ_n.
(3) (Cocycle condition) For any n and n',

$$(\mathrm{id} \otimes \delta_{n,n'}) \circ \epsilon_{n+n'} = (\epsilon_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon_{n'}) \circ (\delta_{n,n'} \otimes \mathrm{id}) : P^{n+n'} \otimes E \longrightarrow E \otimes P^n \otimes P^{n'}$$

holds.

For a subcategory $\mathcal{C}$ of the category of O-modules, let us denote the category of objects in $\mathcal{C}$ endowed with stratifications by $\mathrm{Str}(\mathcal{C})$.

Then the main result in this section is as follows:

Proposition 1.2.7. *Let T be a topos and let*

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

be a differentially log smooth formal groupoid of characteristic zero. Then we have the canonical equivalence of categories

$$C(\mathcal{C}) \overset{\sim}{=} \mathrm{Str}(\mathcal{C}).$$

Proof. One can prove the assertion exactly in the same way as Section 3.2 of [Shi] (see Definition 3.2.6, Lemma 3.2.7 (2), Definition 3.2.8, Proposition 3.2.9, Definition 3.2.10 and Proposition 3.2.11). □

Corollary 1.2.8. *Let the notations be as above and let $(E, \{\epsilon_n\})$ be an object in $\mathrm{Str}(\mathcal{C})$. Denote the composite*

$$E = O \otimes_O E \overset{p_{2,1}\otimes\mathrm{id}}{\longrightarrow} P^1 \otimes_O E \overset{\epsilon_1}{\longrightarrow} E \otimes_O P^1$$

by θ. Then the object in $C(\mathcal{C})$ corresponging to $(E, \{\epsilon_n\})$ via the equivalence in the above proposition is given by (E, ∇), where $\nabla : E \longrightarrow E \otimes_O \omega^1$ is defined by $\nabla(e) = \theta(e) - e \otimes 1$.

PROOF. It is immediate from the construction of the equivalence of categories. See Section 3.2 in [Shi], especially the proof of [Shi, (3.2.9)]. □

Example 1.2.9. Let $f : (X, M) \longrightarrow (S, N)$ be a log smooth morphism of fine log schemes over $\mathbb{Q}$. Let T be a topos associated to X_{et}. Then, one can define a differentially log smooth formal groupoid of characteristic zero

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

as follows: Put $O := \mathcal{O}_X$ and put $P^n := \mathcal{O}_{X^n}$, where (X^n, M^n) is the n-th log infinitesimal neighborhood of (X, M) in $(X, M) \times_{(S,N)} (X, M)$ ([Shi, §3.2]). Let $p_{i,n}$ be the morphism $\mathcal{O}_X \longrightarrow \mathcal{O}_{X^n}$ induced by the morphism

$$(X^n, M^n) \longrightarrow (X, M) \times_{(S,N)} (X, M) \overset{i\text{-th proj.}}{\longrightarrow} (X, M)$$

and let $\pi_n : \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_X$ be the morphism induced by the exact closed immersion $(X, M) \hookrightarrow (X^n, M^n)$. Finally, let $\delta_{m,n}$, τ_n be the morphisms $\delta^*_{m,n}, \tau^*_n$ in [Shi, §3.2].

If we apply Proposition 1.2.7 to $\mathcal{C} = \mathrm{Coh}(\mathcal{O}_X)$, we obtain the equivalence of categories

$$C((X, M)/(S, N)) \simeq \mathrm{Str}((X, M)/(S, N))$$

of Propositions 3.2.9 and 3.2.11 (in the case of fine log schemes) in [Shi].

Example 1.2.10. Let $f : (X, M) \longrightarrow (S, N)$ be a formally log smooth morphism of fine log formal V-schemes, where V is a complete discrete valuation ring of mixed characteristic. Let K be the fraction field of V and let T be a topos associated to X_{et}. Then, one can define a differentially log smooth formal groupoid of characteristic zero

$$\mathcal{X} := (O, \{P^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau_n\}_{n\in\mathbb{N}})$$

as follows: Put $O := K \otimes_V \mathcal{O}_X$ and put $P^n := K \otimes_V \mathcal{O}_{X^n}$, where (X^n, M^n) is the n-th log infinitesimal neighborhood of (X, M) in $(X, M) \hat{\times}_{(S,N)} (X, M)$ ([Shi, §3.2]). Let $p_{i,n}, \pi_n$ be as in the previous example and let $\delta_{m,n}$, τ_n be the morphisms $\mathrm{id} \otimes \delta^*_{m,n}, \mathrm{id} \otimes \tau^*_n$, where $\delta^*_{m,n}, \tau^*_n$ is as in [Shi, §3.2].

If we apply Proposition 1.2.7 to $\mathcal{C} = \mathrm{Coh}(K \otimes \mathcal{O}_X)$, we obtain the equivalence of categories

$$\hat{C}((X,M)/(S,N)) \simeq \widehat{\mathrm{Str}}((X,M)/(S,N))$$

of Propositions 3.2.9 and 3.2.11 (in the case of fine log formal schemes) in [Shi].

REMARK 1.2.11. We can give a proof of Proposition 1.2.7 by using the system of elements $\{t_{j,n}\}$ in Remark 1.2.4 and applying the argument in [Be-Og, §2]. In particular, this proof gives another proof of Propositions 3.2.9, 3.2.11 in [Shi] which uses only the usual differential calculus and which does not use 'log differential calculus' in [Shi, §3.2]. Details are left to the reader.

1.3. Review of rigid analytic geometry

In this section, we review some basic definitions and known results concerning rigid analytic geometry. The basic references are [Be2] and [Be3]. See also [Ta], [Be4] and so on.

First we fix some notations. Let k be a perfect field of characteristic $p > 0$ and let V be a complete discrete valuation ring of mixed characteristic with residue field k. Let π be a uniformizer of V. Denote the fraction field of V by K and the algebraic closure of K by $\overline{K}$. Let $|\cdot| : \overline{K} \longrightarrow \mathbb{R}_{\geq 0}$ be the valuation satisfying $|p| = p^{-1}$ and put $\Gamma_0 := \{|x| \,|\, x \in K^\times\} \subset \mathbb{R}_{>0}$, $\Gamma := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_0 \subset \mathbb{R}_{>0}$.

For $n \in \mathbb{N}$, let $K\{t_1, \cdots, t_n\}$ be the ring

$$\{\sum_{k \in \mathbb{N}^n} a_k t^k \,|\, a_k \in K, |a_k| \to 0 \,(|k| \to \infty)\}.$$

(That is, $K\{t_1, \cdots, t_n\}$ is the ring of power series which are convergent on the closed disc of radius 1.) A topological K-algegra A is called a Tate algebra if there exists an integer $n \in \mathbb{N}$ and an ideal $I \subset K\{t_1, \cdots, t_n\}$ (which is necessarily closed ([Ta])) such that A is isomorphic to $K\{t_1, \cdots, t_n\}/I$ as topological K-algebras.

Let A be a Tate algebra and put $X := \mathrm{Spm}\, A (:=$ the set of maximal ideals of A). Then it is known that one can naturally endow a Grothendieck topology in the sense of [Be3, (0.1.1)] (which we denote by θ_X) and a structure sheaf of rings $\mathcal{O}_X$ with respect to the topology θ_X ([Be3, (0.1.2)]). We call the triple $(X, \theta_X, \mathcal{O}_X)$ the affinoid rigid analytic space associated to A. We denote the triple $(X, \theta_X, \mathcal{O}_X)$ simply by X or $\mathrm{Spm}\, A$, by abuse of notation.

In the following, for a set X endowed with a Grothedieck topology θ_X, we call an open set (resp. an open covering) with respect to the Grothendieck topology θ_X an admissible open set (resp. an admissible open covering). A triple $X := (X, \theta_X, \mathcal{O}_X)$, where X is a set endowed with a Grothendieck topology θ_X and $\mathcal{O}_X$ is a sheaf of rings on (X, θ_X), is called a rigid analytic space if there exists an admissible open covering $X := \bigcup_i X_i$ of X such that the triple $(X_i, \theta_X|_{X_i}, \mathcal{O}_X|_{X_i})$ is an affinoid rigid analytic space for each i.

Let P be a formal V-scheme and let P_K be the set of closed sub formal schemes $Z \subset P$ which are integral and finite flat over $\mathrm{Spf}\, V$. Then, for any open affine formal scheme $P \supset U := \mathrm{Spf}\, A$, we have the isomorphism of sets $U_K \overset{\sim}{=} \mathrm{Spm}\,(K \otimes A)$. So, one can introduce a structure of an affinoid rigid analytic space on U_K for any affine open $U \subset P$. Then, by [Be3, (0.2.3)], one can glue these structures and so one can define the structure of a rigid analytic space on P_K. We call this rigid analytic space the rigid analytic space associated to P. We define the specialization map $\mathrm{sp} : P_K \longrightarrow P$ by $Z \mapsto$ (support of Z). Then sp is a morphism of sites ([Be3, (0.2.3)]).

Let $i : X \hookrightarrow P$ be a locally closed immersion of a k-scheme X into a formal V-scheme P. Then we define the tubular neighborhood $]X[_P$ of X in P by $]X[_P := \mathrm{sp}^{-1}(X)$. Let us assume that P is affine and i is a closed immersion. Suppose that the defining ideal of X in P is generated by π and $f_1, f_2, \cdots, f_n \in \Gamma(P, \mathcal{O}_P)$. Then one can check the equality

$$]X[_P = \{ x \in P_K \mid |f_i(x)| < 1 \ (1 \le i \le n) \},$$

where $f_i(x) := f_i \bmod x \in \kappa(x)$(:= the residue field of $x \in P_K$).

Let $i : X \hookrightarrow P$ be a closed immersion of a k-scheme X into an affine formal scheme P. Let $f_1, \cdots, f_n$ be as above. For $|\pi| < \lambda \le 1$, we define the (open) tubular neighborhood $]X[_{P,\lambda}$ of X in P of radius λ by

$$]X[_{P,\lambda} = \{ x \in P_K \,|\, |f_i(x)| < \lambda \ (1 \le i \le n) \}$$

and for $|\pi| < \lambda \le 1, \lambda \in \Gamma$, we define the closed tubular neighborhood $[X]_{P,\lambda}$ of X in P of radius λ by

$$[X]_{P,\lambda} = \{ x \in P_K \,|\, |f_i(x)| \le \lambda \ (1 \le i \le n) \}.$$

This definition is independent of the choice of f_i's by the assumption $|\pi| < \lambda$ ([Be3, (1.1.8)]). Hence we can define $]X[_{P,\lambda}$ and $[X]_{P,\lambda}$ even when P is not necessarily affine. It is known that the tubular neighborhoods $]X[_{P,\lambda}, [X]_{P,\lambda}$ are admissible open sets of P_K.

Let $i : X \hookrightarrow P := \mathrm{Spf}\, A$ be a closed immersion of a k-scheme X into an affine formal scheme P and let $f_1, \cdots, f_n$ be as above. Let $\lambda := |\pi|^{a/b}$, where $a, b \in \mathbb{N}, a < b$. Then $[X]_{P,\lambda}$ is naturally isomorphic to the affinoid rigid analytic space $\mathrm{Spm}\, B$, where

$$B := (K \otimes_V A)\{t_1, \cdots, t_n\}/(\pi^a t_1 - f_1^b, \cdots, \pi^a t_n - f_n^b).$$

Let A be a topological V-algebra which is topologically of finite type over V, and let $X \hookrightarrow P$ be the canonical closed immersion $\mathrm{Spec}\, A/(\pi) \hookrightarrow \mathrm{Spf}\, A\{t_1, \cdots, t_n\}$. Then we call the rigid analytic space

$$]X[_P := \{x \in P_K \,|\, |t_i(x)| < 1 \ (1 \le i \le n)\}$$

the n-dimensional unit open disc over A and denote it by D^n_A. For $|\pi| < \lambda < 1, \lambda \in \Gamma$, we call the rigid analytic space

$$]X[_P - [X]_{P,\lambda} := \{x \in P_K \,|\, \lambda < |t_i(x)| < 1 \ (1 \le i \le n)\}$$

the n-dimensional open annulus of radius between λ and 1 over A and denote it by $C^n_{A,\lambda}$. We call the functions $t_1, \cdots, t_n$ the coordinates of $D^n_{A,\lambda}$, $C^n_{A,\lambda}$.

A rigid analytic space X is called quasi-Stein if there exists an admissible covering $X = \bigcup_{n=1}^{\infty} X_i$ by increasing family of affinoid rigid analytic spaces $\{X_i\}_{i \in \mathbb{N}}$ such that the image of the map $\Gamma(X_{i+1}, \mathcal{O}_{X_{i+1}}) \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$ is dense for each $i \in \mathbb{N}$. For example, $D^n_A, C^n_{A,\lambda}$ are quasi-Stein. The following theorem of Kiehl is important:

THEOREM 1.3.1 (Theorem B of Kiehl). *For a quasi-Stein rigid analytic space X and a coherent $\mathcal{O}_X$-module E, we have the vanishing $H^i(X, E) = 0 \, (i > 0)$.*

COROLLARY 1.3.2. *Let $f : X \longrightarrow Y$ be a morphism of sites from a rigid analytic space X to a rigid analytic space Y (resp. a scheme Y) such* *that $f^{-1}(U)$ is quasi-Stein for any sufficiently small admissible open affinoid rigid analytic space (resp. any sufficiently small affine open) $U \subset Y$. Then, for any coherent $\mathcal{O}_X$-module E, we have the vanishing $R^i f_* E = 0$ $(i > 0)$.*

We recall here an important theorem on the structure of tubular neighborhoods ([Be3, (1.3.2)]):

THEOREM 1.3.3 (Weak fibration theorem). *Suppose we are given the following diagram*

$$\begin{array}{ccc} X & \longrightarrow & P' \\ \| & & \downarrow u \\ X & \longrightarrow & P, \end{array}$$

where X is a k-scheme, P, P' are formal V-schemes, u is a formally smooth V-morphism of relative dimension n and horizontal arrows are closed immersions. Denote the morphism $P'_K \longrightarrow P_K$ of rigid analytic spaces associated to u by u_K. Then there exists an open covering $P := \bigcup_\alpha P_\alpha$ and isomorphisms $f_\alpha :]X[_{P'} \cap u_K^{-1}(P_{\alpha,K}) \xrightarrow{\sim} (]X[_P \cap P_{\alpha,K}) \times D_V^n$ such that the following diagram is commutative:

$$\begin{array}{ccc}]X[_{P'} \cap u_K^{-1}(P_{\alpha,K}) & \xrightarrow{f_\alpha} & (]X[_P \cap P_{\alpha,K}) \times D_V^n \\ {\scriptstyle u_K}\downarrow & & \downarrow{\scriptstyle \text{1-st proj.}} \\ ]X[_P \cap P_{\alpha,K} & = &]X[_P \cap P_{\alpha,K}. \end{array}$$

Now we recall some notions concerning strict neighborhood. We recall also a part of strong fibration theorem, which we need later.

Let $j : X \hookrightarrow Y$ be an open immersion between k-schemes and let $Y \hookrightarrow P$ be a closed immersion of Y into a formal V-scheme P. Put $Z := Y - X$. Then, an admissible open set $U \subset]Y[_P$ containing $]X[_P$ is called a strict neighborhood of $]X[_P$ in $]Y[_P$ if $(U,]Z[_P)$ is an admissible covering of $]Y[_P$. It is known that the intersection of two strict neighborhoods is again a strict neighborhood. Hence the set of strict neighborhoods forms a filtered category.

For strict neighborhoods $U' \subset U$ of $]X[_P$ in $]Y[_P$, denote the inclusion map by $\alpha_{UU'}$ and denote the map $\alpha_{]Y[_P U}$ simply by α_U. Then, for a strict neighborhood U and an $\mathcal{O}_U$-module E, we define the sheaf $j^\dagger E$ on $]Y[_P$ by

$$j^\dagger E := \varinjlim_{U'} \alpha_{U'*}\alpha_{UU'}^* E,$$

where U' runs through strict neighborhoods of $]X[_P$ in $]Y[_P$ which are contained in U.

We give an example of strict neighborhoods which we use later. Let $X \subset Y \hookrightarrow P$ be as above. Then, for $|\pi| < \lambda < 1, \lambda \in \Gamma$, $U_\lambda :=]Y[_P - [X]_{P,\lambda}$ is a strict neighborhood of $]X[_P$ in $]Y[_P$. Denote the open immersion $U_\lambda \hookrightarrow]Y[_P$ by j_λ. Then, [Be3, (2.1.1.5)] implies that we have the isomorphism

$$j^\dagger E \overset{\sim}{=} \varinjlim_{\lambda \to 1} j_{\lambda,*} j_\lambda^* E$$

for an $\mathcal{O}_{]Y[_P}$-module E.

We recall basic functorialities for strict neighborhoods and the functor $j^\dagger$ ([Be3, (1.2.7),(2.1.4)]):

PROPOSITION 1.3.4. *Let us consider the following diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{j'} & Y' & \xrightarrow{i'} & P' \\ \downarrow & & v\downarrow & & u\downarrow \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & P, \end{array}$$

where X, X', Y, Y' are k-schemes, P and P' are formal V-schemes, j and j' are open immersions and i and i' are closed immersions. Denote the morphism of rigid analytic spaces $]Y'[_{P'} \longrightarrow]Y[_P$ induced by u by u_K. Then:

(1) *For a strict neighborhood U of $]X[_P$ in $]Y[_P$, $(u_K)^{-1}(U)$ is a strict neighborhood of $]X'[_{P'}$ in $]Y'[_{P'}$.*

(2) *For any $\mathcal{O}_{]Y[_P}$-module E, there exists a canonical homomorphism*

$$u_K^* j^\dagger E \longrightarrow (j')^\dagger u_K^* E$$

and it is an isomorphism if $v^{-1}(X) = X'$ holds.

We also recall the following proposition of Berthelot ([Be3, (1.3.5)]), which is a part of strong fibration theorem.

THEOREM 1.3.5. *Assume we are given the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{j'} & Y' & \xrightarrow{i'} & P' \\ \| & & {\scriptstyle v}\downarrow & & {\scriptstyle u}\downarrow \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & P, \end{array}$$

where X, Y, Y' are k-schemes, P and P' are formal V-schemes, j and j' are open immersions and i and i' are closed immersions. Denote the closure of X in $P' \times_P Y$ by $\overline{X}$. Assume moreover that u is formally etale on a neighborhood of X and that the restriction of v to $\overline{X}$ is proper. Then the morphism $u_K : P'_K \longrightarrow P_K$ induces an isomorphism between a strict neighborhood of $]X[_{P'}$ in $]Y'[_{P'}$ and a strict neighborhood of $]X[_P$ in $]Y[_P$.

Next we recall the definition of (over)convergent (F^a-)isocrystals, the de Rham complex associated to (over)convergent isocrystals and rigid cohomology (and analytic cohomology) with coefficient.

Let $X \subset Y$ be an open immersion of k-schemes of finite type and let $Y \hookrightarrow P$ be a closed immersion into a formal V-scheme which is formally smooth over $\mathrm{Spf}\, V$ on a neighborhood of X. For $n \in \mathbb{N}$, let $P(n)$ be the $(n+1)$-fold fiber product of P over $\mathrm{Spf}\, V$. Then the projections

$$p'_i : P(1) \longrightarrow P \quad (i = 1, 2), \qquad p'_{ij} : P(2) \longrightarrow P(1) \quad (1 \leq i < j \leq 3)$$

and the diagonal morphism $\Delta' : P \hookrightarrow P(1)$ induce the morphisms

$$p_i :]Y[_{P(1)} \longrightarrow]Y[_P \quad (i = 1, 2), \qquad p_{ij} :]Y[_{P(2)} \longrightarrow]Y[_{P(1)} \quad (1 \leq i < j \leq 3)$$

and the diagonal morphism $\Delta :]Y[_P \hookrightarrow]Y[_{P(1)}$, respectively. By Proposition 1.3.4 (2), one can see that the above morphisms induce the functors

$$\begin{aligned} p_i^* &: (j^\dagger \mathcal{O}_{]Y[_P}\text{-modules}) \longrightarrow (j^\dagger \mathcal{O}_{]Y[_{P(1)}}\text{-modules}), \\ p_{ij}^* &: (j^\dagger \mathcal{O}_{]Y[_{P(1)}}\text{-modules}) \longrightarrow (j^\dagger \mathcal{O}_{]Y[_{P(2)}}\text{-modules}), \\ \Delta^* &: (j^\dagger \mathcal{O}_{]Y[_{P(1)}}\text{-modules}) \longrightarrow (j^\dagger \mathcal{O}_{]Y[_P}\text{-modules}), \end{aligned}$$

respectively. Now we define an overconvergent isocrystal on the triple (X, Y, P) as follows:

Definition 1.3.6. Let the notations be as above. Then an overconvergent isocrystal on the triple (X, Y, P) is a pair (E, ϵ), where E is a locally free $j^\dagger \mathcal{O}_{]Y[_P}$-module of finite type and ϵ is a $j^\dagger \mathcal{O}_{]Y[_{P(1)}}$-linear isomorphism $p_2^* E \xrightarrow{\sim} p_1^* E$ satisfying $\Delta^*(\epsilon) = \mathrm{id}$ and $p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon)$. We denote the category of overconvergent isocrystals on the triple (X, Y, P) by $I^\dagger(X, Y, P)$.

Then the following proposition is known ([Be3, (2.3.1)]):

Proposition 1.3.7. *The category $I^\dagger(X, Y, P)$ is independent of the choice of $Y \hookrightarrow P$ up to canonical equivalence. That is, if we have another closed immersion $Y \hookrightarrow Q$ which satisfies the same condition as $Y \hookrightarrow P$ and a morphism $f : Q \longrightarrow P$ compatible with the closed immersions $Y \hookrightarrow P, Y \hookrightarrow Q$ which is formally smooth on a neighborhood of X, then the functor*

$$I^\dagger(X, Y, P) \longrightarrow I^\dagger(X, Y, Q)$$

induced by the morphism of rigid analytic spaces

$$]Y[_{Q(n)} \longrightarrow]Y[_{P(n)} \quad (n = 0, 1, 2)$$

is an equivalence of categories.

Next, let $X \hookrightarrow Y$ be an open immersion of k-schemes. Take a diagram

$$Y \xleftarrow{f} Y^{(\bullet)} \overset{i}{\hookrightarrow} P^{(\bullet)}, \tag{1.3.1}$$

where $Y^{(\bullet)}$ is a simplicial k-scheme, $P^{(\bullet)}$ is a simplicial formal V-scheme such that $P^{(n)}$ is formally smooth over $\mathrm{Spf}\, V$ on a neighborhood of the image of $X^{(n)} := X \times_Y Y^{(n)}$, f is a Zariski hypercovering and i is a morphism of simplicial formal schemes which induces the closed immersions $Y^{(n)} \hookrightarrow P^{(n)}$

for each n. (Note that there exists such a diagram.) Let us denote the functor

$$I^\dagger(X^{(0)}, Y^{(0)}, P^{(0)}) \longrightarrow I^\dagger(X^{(1)}, Y^{(1)}, P^{(1)})$$
$$(\text{resp.} \quad I^\dagger(X^{(0)}, Y^{(0)}, P^{(0)}) \longrightarrow I^\dagger(X^{(1)}, Y^{(1)}, P^{(1)}) \quad)$$

induced by the projection of triples

$$q_i : (X^{(1)}, Y^{(1)}, P^{(1)}) \longrightarrow (X^{(0)}, Y^{(0)}, P^{(0)}) \quad (i = 1, 2)$$
$$(\text{resp.} \quad q_{ij} : (X^{(2)}, Y^{(2)}, P^{(2)}) \longrightarrow (X^{(1)}, Y^{(1)}, P^{(1)}) \quad (1 \leq i < j \leq 3) \quad)$$

by q_i^* (resp. q_{ij}^*), and denote the functor

$$I^\dagger(X^{(1)}, Y^{(1)}, P^{(1)}) \longrightarrow I^\dagger(X^{(0)}, Y^{(0)}, P^{(0)})$$

induced by the diagonal

$$\Delta : (X^{(0)}, Y^{(0)}, P^{(0)}) \longrightarrow (X^{(1)}, Y^{(1)}, P^{(1)})$$

by Δ^*. Then we define the category of overconvergent isocrystals on (X, Y) as the category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is an object in $I^\dagger(X^{(0)}, Y^{(0)}, P^{(0)})$ and φ is an isomorphism $q_2^*\mathcal{E} \xrightarrow{\sim} q_1^*\mathcal{E}$ in $I^\dagger(X^{(1)}, Y^{(1)}, P^{(1)})$ satisfying $\Delta^*(\varphi) = \mathrm{id}$ and $q_{12}^*(\varphi) \circ q_{23}^*(\varphi) = q_{13}^*(\varphi)$. One can see, by using Proposition 1.3.7, that this definition is independent of the choice of the diagram (1.3.1). We denote the category of overconvergent isocrystals on (X, Y) by $I^\dagger(X, Y)$.

In the case where $X = Y$ holds, we call an object in $I^\dagger(X, X)$ a convergent isocrystal on X. It is known ([Be3, (2.3.4)]) that the category $I^\dagger(X, X)$ is equivalent to the category of convergent isocrystals on X over $\mathrm{Spf}\, V$ in the sense of Ogus [Og1], which we denoted by $I_{\mathrm{conv}}(X/V)$ in the previous paper [Shi].

For a separated k-scheme of finite type, the following proposition is known ([Be3, (2.3.5)]):

Proposition 1.3.8. *Let X be a separated k-scheme of finite type, and let $X \subset \overline{X}$ be a k-compactification. Then the category $I^\dagger(X, \overline{X})$ is independent of the choice of the k-compactification $X \subset \overline{X}$, up to canonical equivalence.*

Under the situation of Proposition 1.3.8, we call an object in the category $I^\dagger(X,\overline{X})$ an overconvergent isocrystal on X and we denote the category $I^\dagger(X,\overline{X})$ simply by $I^\dagger(X)$.

We define the notion of an overconvergent F^a-isocrystal as follows:

DEFINITION 1.3.9. Let X be a separated scheme of finite type over k and let $F_X : X \longrightarrow X$, $F_k : \mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\, k$ be the absolute Frobenius endomorphisms. Let $a \in \mathbb{N}, a > 0$ and assume there exists a morphism $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ which coincides with F_k^a modulo the maximal ideal of V. Then we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{F_X^a} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \xrightarrow{F_k^a} & \mathrm{Spec}\, k \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, V & \xrightarrow{\sigma} & \mathrm{Spf}\, V. \end{array}$$

For an overconvergent isocrystal $\mathcal{E}$ on X, denote the pull-back of E by (F_X^a, F_k^a, σ) in the above diagram by $F^{a,*}\mathcal{E}$. An overconvergent F^a-isocrystal on X with respect to σ is a pair $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is an overconvergent isocrystal on X and Φ is an isomorphism $F^{a,*}E \xrightarrow{\sim} E$.

Now we recall the definition of the de Rham complex associated to an overconvergent isocrystal.

First, for a rigid analytic space X, we put $\mathcal{I}_X := \mathrm{Ker}(\mathcal{O}_X \hat{\otimes} \mathcal{O}_X \longrightarrow \mathcal{O}_X)$ and let $\Omega_X^1 := \mathcal{I}_X/\mathcal{I}_X^2$. For $q \in \mathbb{N}$, we define Ω_X^q as the q-th exterior power of Ω_X^1 over $\mathcal{O}_X$. We call the sheaf Ω_X^q as the sheaf of q-th differential forms on X, as in the usual case.

Now let $j : X \hookrightarrow Y$ be an open immersion of k-schemes and let $Y \hookrightarrow P$ be a closed immersion into a formal V-scheme which is formally smooth on a neighborhood of X. Let $P(n)\,(n \in \mathbb{N})$ be the $(n+1)$-fold fiber product of P over $\mathrm{Spf}\, V$. For $n \in \mathbb{N}$, let P^n be the n-th infinitesimal neighborhood of P in $P(1)$. Let

$$\tau_n : P^n \longrightarrow P^n \quad (n \in \mathbb{N}),$$
$$\delta_{m,n} : P^m \times_P P^n \longrightarrow P^{m+n} \quad (m, n \in \mathbb{N}),$$

be the morphisms induced by the morphisms

$$\begin{aligned} P(1) &\longrightarrow P(1); \quad (x,y) \mapsto (y,x), \\ P(1)\hat{\times}_P P(1) &\longrightarrow P(1); \quad ((x,y),(y,z)) \mapsto (x,z), \end{aligned}$$

respectively. Let us define the data

$$\mathcal{X} := (\mathcal{O}, \{\mathcal{P}^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta'_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau'_n\}_{n\in\mathbb{N}})$$

as follows: Let $\mathcal{O} := j^\dagger\mathcal{O}_{]Y[_P}$ and $\mathcal{P}^n := j^\dagger\mathcal{O}_{]Y[_{P^n}}$. (Since $]Y[_{P^n}$ is homeomorphic to $]Y[_P$, we can regard $\mathcal{P}^n$ as a sheaf on $]Y[_P$.) Let $p_{i,n}\,(i = 1,2, n \in \mathbb{N})$ be the homomorphism $\mathcal{O} \longrightarrow \mathcal{P}^n$ corresponding to the morphism $]Y[_{P^n}\longrightarrow]Y[_P$ induced by the i-th projection, let π_n be the homomorphism $\mathcal{P}^n \longrightarrow \mathcal{O}$ corresponding to the morphism $]Y[_P\longrightarrow]Y[_{P^n}$ induced by the closed immersion $P \hookrightarrow P^n$, let τ'_n be the homomorphism $\mathcal{P}^n \longrightarrow \mathcal{P}^n$ corresponding to the morphism $]Y[_{P^n}\longrightarrow]Y[_{P^n}$ induced by the morphism τ_n and let $\delta'_{m,n}$ be the homomorphism $\mathcal{P}^{m+n} \longrightarrow \mathcal{P}^m \otimes_{\mathcal{O}} \mathcal{P}^n$ corresponding to the morphism $]Y[_{P^m\times_P P^n}\longrightarrow]Y[_{P^{m+n}}$ induced by $\delta_{m,n}$. Then it is known ([Be3, (2.2.2)]) that the data $\mathcal{X}$ is an adic differentially smooth formal groupoid of finite type of characteristic zero in the topos associated to $]Y[_P$.

Note that the canonical closed immersion $P^n \hookrightarrow P(1)$ induces the morphism of rigid analytic spaces $]Y[_{P^n}\longrightarrow]Y[_{P(1)}$, which we denote by Δ_n. Denote the homomorphism

$$\Delta_n^{-1}j^\dagger\mathcal{O}_{]Y[_{P(1)}} = j^\dagger\Delta_n^{-1}\mathcal{O}_{]Y[_{P(1)}} \longrightarrow j^\dagger\mathcal{O}_{]Y[_{P^n}}$$

by θ_n.

Now let (E,ϵ) be an object in $I^\dagger(X,Y) = I^\dagger(X,Y,P)$. Then we define the $\mathcal{P}^n$-linear isomorphism $\epsilon_n : \mathcal{P}^n \otimes E \xrightarrow{\sim} E \otimes \mathcal{P}^n$ as the composite

$$\begin{aligned} \mathcal{P}^n \otimes E &= \mathcal{P}^n \otimes_{\theta_n,\Delta_n^{-1}j^\dagger\mathcal{O}_{]Y[_{P(1)}}} \Delta_n^{-1}p_2^*E \\ &\xrightarrow{\mathrm{id}\otimes\Delta_n^{-1}\epsilon} \mathcal{P}^n \otimes_{\theta_n,\Delta_n^{-1}j^\dagger\mathcal{O}_{]Y[_{P(1)}}} \Delta_n^{-1}p_1^*E = E \otimes \mathcal{P}^n. \end{aligned}$$

Then $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a stratification on E with respect to the formal groupoid $\mathcal{X}$. Hence, by Proposition 1.2.7, $\{\epsilon_n\}_n$ defines an integrable connection

$$\nabla : E \longrightarrow E \otimes_{j^\dagger\mathcal{O}_{]Y[_P}} j^\dagger\Omega^1_{]Y[_P} = E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^1_{]Y[_P}.$$

Then we define the de Rham complex on $]Y[_P$ associated to the overconvergent isocrystal $\mathcal{E} := (E, \epsilon)$ by the complex

$$\mathrm{DR}(]Y[_P, \mathcal{E}) := [0 \to E \overset{\nabla}{\to} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^1_{]Y[_P} \overset{\nabla}{\to} \cdots \overset{\nabla}{\to} E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^q_{]Y[_P} \overset{\nabla}{\to} \cdots],$$

where we extend ∇ to

$$E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^q_{]Y[_P} \longrightarrow E \otimes_{\mathcal{O}_{]Y[_P}} \Omega^{q+1}_{]Y[_P}$$

by setting

$$x \otimes \eta \mapsto \nabla(x) \wedge \eta + x \otimes d\eta.$$

Now we give the definition of rigid cohomology with coefficient. First, let $X \subset Y$ be an open immersion of k-schemes of finite type and let $\mathcal{E}$ be an overconvergent isocrystal on (X, Y). Take a diagram as (1.3.1) and let $\mathcal{E}^{(n)}$ be the pull-back of $\mathcal{E}$ to $I^\dagger(X^{(n)}, Y^{(n)}) = I^\dagger(X^{(n)}, Y^{(n)}, P^{(n)})$. Then we define the rigid cohomology of the pair (X, Y) with coefficient $\mathcal{E}$ by

$$H^i_{\mathrm{rig}}(X \subset Y/K, \mathcal{E}) := H^i(Y, Rf_*R\mathrm{sp}^{(\bullet)}_*\mathrm{DR}(]Y^{(\bullet)}[_{P^{(\bullet)}}, \mathcal{E})),$$

where $\mathrm{sp}^{(\bullet)}$ denotes the specialization map $]Y^{(\bullet)}[_{P^{(\bullet)}} \longrightarrow Y^{(\bullet)}$.

It is known that the following proposition holds ([Be2]):

Proposition 1.3.10 (Berthelot). *Let us assume given the following commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xrightarrow{i'} & P' \\ \| & & \| & & \downarrow u \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & P, \end{array}$$

where $j : X \hookrightarrow Y$ is an open immersion of k-schemes of finite type and i, i' are closed immersions into formal V-schemes which are formally smooth *on a neighborhood of X. Assume moreover that u is formally smooth on a neighborhood of X. Let $u_K :]Y[_{P'} \longrightarrow]Y[_P$ be the morphism induced by u. Then, for an object $\mathcal{E}$ in $I^\dagger(X,Y) (= I^\dagger(X,Y,P) = I^\dagger(X,Y,P'))$, the canonical homomorphism*

$$\mathrm{DR}(]Y[_P, \mathcal{E}) \longrightarrow Ru_{K,*}\mathrm{DR}(]Y[_{P'}, \mathcal{E})$$

is a quasi-isomorphism.

As a consequence, one can check that the above definition of rigid cohomology $H^i_{\mathrm{rig}}(X \subset Y/K, \mathcal{E})$ is independent of the choice of the diagram (1.3.1). (Details are left to the reader.)

In the case where $X = Y$ holds, we denote the rigid cohomology $H^i_{\mathrm{rig}}(X \subset X/K, \mathcal{E})$ of the pair $X \subset X$ with coefficient $\mathcal{E} \in I^\dagger(X,X)$ by $H^i_{\mathrm{an}}(X/K, \mathcal{E})$ and call it the analytic cohomology (in rigid analytic sense) of X with coefficient $\mathcal{E}$. Note that the definition in this case is much simpler than the general case, since $j^\dagger = \mathrm{id}$ holds in this case. (In particular, we do not need the notion of strict neighborhoods in this case.) Convergent Poincaré lemma of Ogus ([Og2, (0.5.4), (0.6.6)]) implies that the analytic cohomology $H^i_{\mathrm{an}}(X/K, \mathcal{E})$ is isomorphic to the convergent cohomology $H^i((X/V)_{\mathrm{conv}}, \mathcal{E})$.

For a separated k-scheme X, the following proposition is known ([Be2]):

Proposition 1.3.11 (Berthelot). *Let X be a separated k-scheme of finite type, and let $X \subset \overline{X}$ be a k-compactification of X. Let $\mathcal{E}$ be an object in $I^\dagger(X) = I^\dagger(X, \overline{X})$. Then the rigid cohomology $H^i_{\mathrm{rig}}(X \subset \overline{X}/K, \mathcal{E})$ of $(X, \overline{X})$ with coefficient $\mathcal{E}$ is independent of the choice of the k-compactification $X \subset \overline{X}$.*

Under the situation of Proposition 1.3.11, we denote the group $H^i_{\mathrm{rig}}(X \subset \overline{X}/K, \mathcal{E})$ simply by $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ and call it the rigid cohomology of X with coefficient $\mathcal{E}$.

Remark 1.3.12. Let X be a smooth scheme over k. Then we have the rigid cohomology $H^i_{\mathrm{rig}}(X/K, -)$ and the analytic cohomology $H^i_{\mathrm{an}}(X/K, -)$. In the case where X is proper over k, the two cohomologies are the same, but they are not isomorphic in general.

The analytic cohomology is not a good p-adic cohomology theory, for it is not finite-dimensional in general even in the case that the coefficient is trivial. On the other hand, the rigid cohomology is expected to be a good p-adic cohomology theory in the case where the coefficient is an overconvergent F^a-isocrystal. In fact, in the case where the coefficient is trivial or a unit-root overconvergent F^a-isocrystal, it is finite-dimensional and satisfies several nice properties ([Be4], [Be5], [Ts2]).

Finally, we recall the definition of the rigid analytic space associated to certain formal schemes over $\mathrm{Spf}\, V$ which are not necessarily p-adic ([Be3, (0.2.6), (0.2.7)]), which we need in later chapters.

Let P be a Noetherian formal scheme over $\mathrm{Spf}\, V$ and let $\mathcal{I}$ be an ideal of definition. Let $P_0 \subset P$ be the scheme defined by $\mathcal{I}$, and suppose that it is locally of finite type over $\mathrm{Spf}\, V$. We define the rigid analytic space P_K associated to P. (In the case that P is a formal V-scheme, the rigid analytic space P_K coincides with the previous one.) Suppose first that $P := \mathrm{Spf}\, A$ is affine, and let $f_1, \cdots, f_r$ be a generator of the ideal $I := \Gamma(P, \mathcal{I})$ of A. For $n \in \mathbb{N}, n > 1$, let B_n be the ring

$$(A[t_1, \cdots, t_r]/(f_1^n - \pi t_1, \cdots, f_r^n - \pi t_r))^{\wedge},$$

where $^{\wedge}$ denotes the p-adic completion. For $n < n' \in \mathbb{N}$, let $B_{n'} \longrightarrow B_n$ be the continuous ring homomorphism over A which sends t_i to $f^{n'-n} t_i$. Then, via the morphism of rigid analytic spaces

$$\mathrm{Spm}\,(K \otimes_V B_n) \longrightarrow \mathrm{Spm}\,(K \otimes_V B_{n'})$$

associated to the above ring homomorphism, $\mathrm{Spm}\,(K \otimes_V B_n)$ is identified with the admissible open set $\{x \,|\, |f_i(x)| \leq |\pi|^{1/n} \ (1 \leq i \leq r)\}$ of $\mathrm{Spm}\,(K \otimes_V B_{n'})$. We define the rigid analytic space P_K as the union of $\mathrm{Spm}\,(K \otimes_V B_n)$'s. It is known that this definition is independent of the choice of the system of generators $(f_1, \cdots, f_r)$ of I. In the case that P is not necessarily affine, we take an affine open covering $P = \bigcup P_i$ of P and define P_K as the union of $P_{i,K}$'s. It is known that this definition is well-defined. We can define the specialization map $\mathrm{sp} : P_K \longrightarrow P$ as the union of the maps $\mathrm{Spm}\,(K \otimes_V B_n) \xrightarrow{\mathrm{sp}} \mathrm{Spf}\, B_n \longrightarrow P$.

As for the relation of the above construction and the tubular neighborhood, the following proposition is known ([Be3, (0.2.7)]):

PROPOSITION 1.3.13. *Let P, P_0 be as above and let $X \subset P_0$ be a closed subscheme. Denote the completion of P along X by $\hat{P}$. Then the canonical morphism $\hat{P}_K \longrightarrow P_K$ induces the isomorphism of rigid analytic spaces $\hat{P}_K \xrightarrow{\sim}]X[_P$.*

Chapter 2. Log Convergent Site Revisited

The purpose of this chapter is to develop the theory of cohomologies of isocrystals on log convergent site in detail. First, In Section 2.1, we recall the definition of log convergent site (which are introduced in Chapter 5 in [Shi]), and prove basic properties of them. We also introduce some new notions, such as pre-widenings and widenings, which we use later. They are the log version of widenings introduced by Ogus ([Og2]). Note that we will slightly change the definition of log convergent site, but this change causes no problem. (See Proposition 2.1.7.)

In Section 2.2, we extend the notion of tubular neighborhood to the case of closed immersion of certain log formal schemes, and we define the analytic cohomology (in rigid analytic sense) of log schemes with coefficients. In Section 2.3, we prove that the cohomology of a locally free isocrystal on log convergent site is isomorphic to the analytic cohomology defined in Section 2.2. The theorem of this type is often called as Poincaré lemma. So we call this theorem the log convergent Poincaré lemma. This is a generalization of convergent Poincaré lemma of Ogus ([Og2]). Finally, in Section 2.4, we prove that the analytic cohomology of certain log schemes $X := (X, M)$ is isomorphic to the rigid cohomology of X_{triv} in the case where the coefficient is an F^a-isocrystal on $((X, M)/V)_{\mathrm{conv}}$. We use the results of Baldassarri and Chiarellotto developed in [Ba-Ch] and [Ba-Ch2] in local situation.

Throughout this chapter, k denotes a perfect field of characteristic $p > 0$ and V denotes a complete discrete valuation ring of mixed characteristic with residue field k. Let π be a uniformizer of V and denote the fraction field of V by K. Let $|\cdot|$ be the normalized valuation of $\overline{K}$(:= the algebraic closure of K), and let $\Gamma_0, \Gamma \subset \mathbb{R}_{>0}$ be $|K^\times|$, $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_0$, respectively. For a formal V-scheme T, denote the closed sub formal scheme defined by the ideal $\{x \in \mathcal{O}_T \,|\, p^n x = 0 \text{ for some } n\}$ by T_{fl}. T_{fl} is the largest closed sub formal scheme of T which is flat over $\mathrm{Spf}\, V$.

2.1. Log convergent site

In this section, first we give the definition of enlargement and log convergent site. We slightly change the definitions of them from those in [Shi]. But we remark that the category of isocrystals on log convergent site is unchanged. After that, we define the notion of pre-widening and widening, which are generalizations of the notion of enlargement. They are the log version of the corresponding notions introduced in [Og2]. Finally, we prove some basic properties of log convergent site and the category of isocrystals on it which we need later. We also recall the definition of direct limit site and an acyclicity property of it, which are due to Ogus ([Og2]).

Throughout this section, let (X, M) be a fine log scheme of finite type over k and let us fix a diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N) \tag{2.1.1}$$

of fine log formal V-schemes, where f is of finite type and ι is the canonical exact closed immersion.

First, we define the notion of enlargement. Note that we change the definition from that in [Shi]. The definition here is a log version of that in [Og2], while the definition in [Shi] is a log version of that in [Og1].

Definition 2.1.1.

(1) An enlargement of (X, M) over $(\operatorname{Spf} V, N)$ is a 4-tuple $((T, M_T), (Z, M_Z), i, z)$, where (T, M_T) is a fine log formal V-scheme over $(\operatorname{Spf} V, N)$ such that T is flat over $\operatorname{Spf} V$, (Z, M_Z) is a fine log scheme over $(\operatorname{Spec}\ k, N)$, i is an exact closed immersion $(Z, M_Z) \longrightarrow (T, M_T)$ over $(\operatorname{Spf} V, N)$ such that Z contains $\underline{\operatorname{Spec}}\,(\mathcal{O}_T/p\mathcal{O}_T)_{\mathrm{red}}$ and z is a morphism $(Z, M_Z) \longrightarrow (X, M)$ over $(\operatorname{Spec} k, N)$. We often denote an enlargement $((T, M_T), (Z, M_Z), i, z)$ simply by T.

(2) Let $T := ((T, M_T), (Z, M_Z), i, z)$ and $T' := ((T', M_{T'}), (Z', M_{Z'}), i', z')$ be enlargements. Then we define a morphism $g : T \longrightarrow T'$ of enlargements as a pair of morphisms

$$\begin{aligned} g_1 &: (T, M_T) \longrightarrow (T', M_{T'}), \\ g_2 &: (Z, M_Z) \longrightarrow (Z', M_{Z'}), \end{aligned}$$

such that $g_1 \circ i = i' \circ g_2$ and $z = z' \circ g_2$ hold.

We denote the category of enlargements of (X, M) over $(\operatorname{Spf} V, N)$ by $\operatorname{Enl}((X, M)/(\operatorname{Spf} V, N))$, or simply by $\operatorname{Enl}((X/V)^{\log})$.

REMARK 2.1.2. In this remark, for a formal V-scheme S, denote the closed subscheme $\underline{\operatorname{Spec}}\,(\mathcal{O}_S/p\mathcal{O}_S)_{\mathrm{red}}$ by S_0. In the previous paper [Shi], we defined an enlargement of (X, M) over $(\operatorname{Spf} V, N)$ as the triple (T, M_T, z), where (T, M_T) is a fine log formal V-scheme over $(\operatorname{Spf} V, N)$ (where T is not necessarily flat over $\operatorname{Spf} V$) and z is a morphism $(T_0, M_T) \longrightarrow (X, M)$ over $(\operatorname{Spec} k, N)$. Let us denote the category of enlargements of (X, M) over $(\operatorname{Spf} V, N)$ in this sense by $\operatorname{Enl}'((X, M)/(\operatorname{Spf} V, N))$ (or $\operatorname{Enl}'((X/V)^{\log})$, for short). Then we have the canonical functor

$$\Phi : \operatorname{Enl}((X/V)^{\log}) \longrightarrow \operatorname{Enl}'((X/V)^{\log})$$

which is defined by

$$((T, M_T), (Z, M_Z), i, z) \mapsto (T, M_T, (T_0, M_T|_{T_0}) \hookrightarrow (Z, M_Z) \xrightarrow{z} (X, M)),$$

and the canonical functor

$$\Phi' : \operatorname{Enl}'((X/V)^{\log}) \longrightarrow \operatorname{Enl}((X/V)^{\log})$$

which is defined by

$$\begin{aligned} &(T, M_T, z) \mapsto ((T_{\mathrm{fl}}, M_T|_{T_{\mathrm{fl}}}), ((T_{\mathrm{fl}})_0, M_T|_{(T_{\mathrm{fl}})_0}), \\ &((T_{\mathrm{fl}})_0, M_T|_{(T_{\mathrm{fl}})_0}) \hookrightarrow (T_{\mathrm{fl}}, M_T|_{T_{\mathrm{fl}}}), \\ &\qquad ((T_{\mathrm{fl}})_0, M_T|_{(T_{\mathrm{fl}})_0}) \hookrightarrow (T_0, M_T|_{T_0}) \xrightarrow{z} (X, M)). \end{aligned}$$

Note that the functors Φ, Φ' are neither full nor faithful. (In particular, they are not quasi-inverses of each other.) Indeed, one can check easily that the composite $\Phi' \circ \Phi$ sends $((T, M_T), (Z, M_Z), i, z)$ to $((T, M_T), (T_0, M_T|_{T_0}),$ $(T_0, M_T|_{T_0}) \hookrightarrow (T, M_T), (T_0, M_T|_{T_0}) \hookrightarrow (Z, M_Z) \to (X, M))$, and that the composite $\Phi' \circ \Phi$ sends (T, M_T, z) to $(T_{\mathrm{fl}}, M_T|_{T_{\mathrm{fl}}}, (T_{\mathrm{fl}}, M_T|_{T_{\mathrm{fl}}}) \hookrightarrow (T_0, M_T|_{T_0}) \to (X, M))$.

Next we define the notion of log convergent site and isocrystals on it:

Definition 2.1.3. Let τ be one of the words $\{\mathrm{Zar}(=\text{Zariski}), \mathrm{et}(=\text{etale})\}$. Then we define the log convergent site $((X,M)/(\mathrm{Spf}\,V,N))_{\mathrm{conv},\tau}$ of (X,M) over $(\mathrm{Spf}\,V,N)$ with respect to τ-topology as follows: The objects of this category are the enlargements T of (X,M) over $(\mathrm{Spf}\,V,N)$ and the morphisms are the morphism of enlargements. A family of morphisms

$$\{g_\lambda := (g_{1,\lambda}, g_{2,\lambda}) : ((T_\lambda, M_{T_\lambda}), (Z_\lambda, M_{Z_\lambda}), i_\lambda, z_\lambda) \longrightarrow ((T, M_T), (Z, M_Z), i, z)\}_{\lambda\in\Lambda}$$

is a covering if the following conditions are satisfied:

(1) $g_{1,\lambda}^* M_T \overset{\sim}{=} M_{T_\lambda}$ holds for any $\lambda \in \Lambda$.
(2) The family of morphisms $\{g_{1,\lambda} : T_\lambda \longrightarrow T\}$ is a covering with respect to τ-topology on T.
(3) $(Z_\lambda, M_{Z_\lambda})$ is isomorphic to $(T_\lambda, M_{T_\lambda}) \times_{g_{1,\lambda},(T,M_T),i} (Z, M_Z)$ via the morphism induced by i_λ and $g_{2,\lambda}$.

We often denote the site $((X,M)/(\mathrm{Spf}\,V,N))_{\mathrm{conv},\tau}$ simply by $(X/V)^{\log}_{\mathrm{conv},\tau}$, when there will be no confusion on log structures. When the log structures are trivial, we omit the superscript $^{\log}$.

Remark 2.1.4. The definition of the log convergent site here is different from that in the previous paper [Shi], since the definition of enlargement is different. Also, note that we considered only the log convergent site with respect to etale topology in the previous paper.

Definition 2.1.5. Let the notations be as above. An isocrystal on the log convergent site $(X/V)^{\log}_{\mathrm{conv},\tau}$ is a sheaf $\mathcal{E}$ on the site $(X/V)^{\log}_{\mathrm{conv},\tau}$ satisfying the following conditions:

(1) For any enlargement T, the sheaf $\mathcal{E}$ on T_τ induced by $\mathcal{E}$ is an isocoherent sheaf.
(2) For any morphism $f : T' \longrightarrow T$ of enlargements, the homomorphism $f^*\mathcal{E}_T \longrightarrow \mathcal{E}_{T'}$ of sheaves on T'_τ induced by $\mathcal{E}$ is an isomorphism.

We denote the category of isocrystals on the log convergent site $((X,M)/(\mathrm{Spf}\,V,N))_{\mathrm{conv},\tau}$ by $I_{\mathrm{conv},\tau}((X,M)/(\mathrm{Spf}\,V,N))$. When there are no confusions on log structures, we will denote it simply by $I_{\mathrm{conv},\tau}((X/V)^{\log})$. When the log structures are trivial, we omit the superscript $^{\log}$. We denote the isocrystal on $(X/V)^{\log}_{\mathrm{conv},\tau}$ defined by $T \mapsto K \otimes_V \Gamma(T, \mathcal{O}_T)$ by $\mathcal{K}_{X/V}$.

Definition 2.1.6. Let the notations be as above. Then an isocrystal $\mathcal{E}$ is said to be locally free if, for any enlargement T, the sheaf $\mathcal{E}$ on T_τ

induced by $\mathcal{E}$ is a locally free $K \otimes_V \mathcal{O}_T$-module. We denote the category of locally free isocrystals on the log convergent site $((X, M)/(\mathrm{Spf}\, V, N))_{\mathrm{conv},\tau}$ by $I^{\mathrm{lf}}_{\mathrm{conv},\tau}((X, M)/(\mathrm{Spf}\, V, N))$. When there are no confusions on log structures, we will denote it simply by $I^{\mathrm{lf}}_{\mathrm{conv},\tau}((X/V)^{\log})$.

The definition of the category of isocrystals above is *a priori* different from that in the previous paper, because the definition of the log convergent site is different. But we can prove the following proposition:

PROPOSITION 2.1.7. *Let the notations be as above. Let us denote the log convergent site of* (X, M) *over* $(\mathrm{Spf}\, V, N)$ *defined in* [Shi] *by* $(X/V)^{\log}_{\mathrm{conv}}$ *and the category of isocrystals on it by* $I_{\mathrm{conv}}((X/V)^{\log})$. *Then we have the canonical equivalence of categories*

$$I_{\mathrm{conv}}((X/V)^{\log}) \simeq I_{\mathrm{conv,et}}((X/V)^{\log}).$$

PROOF. First we define the functor $\Psi : I_{\mathrm{conv}}((X/V)^{\log}) \longrightarrow I_{\mathrm{conv,et}}((X/V)^{\log})$. Let $\mathcal{E}$ be an object in $I_{\mathrm{conv}}((X/V)^{\log})$ and let T be an object in $\mathrm{Enl}((X/V)^{\log})$. Then we define the value $\Psi(\mathcal{E})$ of $\mathcal{E}$ at T by $\Psi(\mathcal{E})(T) := \mathcal{E}(\Phi(T))$, where $\Phi : \mathrm{Enl}((X/V)^{\log}) \longrightarrow \mathrm{Enl}'((X/V)^{\log})$ is as in Remark 2.1.2. (It is easy to check that $\Psi(\mathcal{E})$ is an isocrystal.) Next we define the functor $\Psi' : I_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I_{\mathrm{conv}}((X/V)^{\log})$. Let $\mathcal{E}$ be an object in $I_{\mathrm{conv,et}}((X/V)^{\log})$ and let T be an object in $\mathrm{Enl}'((X/V)^{\log})$. Then we define the value $\Psi'(\mathcal{E})$ of $\mathcal{E}$ at T by $\Psi'(\mathcal{E})(T) := \mathcal{E}(\Phi'(T))$, where $\Phi' : \mathrm{Enl}((X/V)^{\log}) \longrightarrow \mathrm{Enl}'((X/V)^{\log})$ is as in Remark 2.1.2. One can check that $\Psi'(\mathcal{E})$ is an isocrystal by using the fact that the canonical functor

$$\mathrm{Coh}(K \otimes \mathcal{O}_S) \longrightarrow \mathrm{Coh}(K \otimes \mathcal{O}_{S_{\mathrm{fl}}})$$

is an equivalence of categories for any formal V-scheme S. One can also check that Ψ and Ψ' are quasi-inverses of each other, by using the above equivalence. Hence the assertion is proved. $\square$

Next, we define the notion of pre-widening and widening, which are generalized notion of enlargement. They are log versions of widening defined in [Og2].

Definition 2.1.8. Let the situation be as in the beginning of this section. Define the category $\mathcal{Q}((X,M)/(\mathrm{Spf}\, V,N))$ (or $\mathcal{Q}((X/V)^{\log})$, if there are no confusions on log structures) of 4-tuples on (X,M) over $(\mathrm{Spf}\, V,N)$ as follows: The objects are the 4-tuples $((T,M_T),(Z,M_Z),i,z)$, where (T,M_T) is a Noetherian fine log formal scheme over $(\mathrm{Spf}\, V,N)$ (which is not necessarily p-adic), (Z,M_Z) is a fine log scheme of finite type over $(\mathrm{Spec}\, k,N)$, i is a closed immersion $(Z,M_Z) \longrightarrow (T,M_T)$ over $(\mathrm{Spf}\, V,N)$ and z is a morphism $(Z,M_Z) \longrightarrow (X,M)$ over $(\mathrm{Spec}\, k,N)$. We define a morphism of 4-tuples

$$(T,M_T,Z,M_Z,i,z) \xrightarrow{g} (T',M_{T'},Z',M_{Z'},i',z')$$

as a pair of V-morphisms $(T,M_T) \xrightarrow{g_1} (T',M_{T'})$, $(Z,M_Z) \xrightarrow{g_2} (Z',M_{Z'})$ which satisfy $g_1 \circ i = i' \circ g_2$ and $z = z' \circ g_2$.

Definition 2.1.9.

(1) A 4-tuple $((T,M_T),(Z,M_Z),i,z)$ is called a pre-widening if T is a formal V-scheme. We often denote a pre-widening $((T,M_T),(Z,M_Z),i,z)$ simply by $((T,M_T),(Z,M_Z))$ or T.
(2) A 4-tuple $((T,M_T),(Z,M_Z),i,z)$ is called a widening if Z is a scheme of definition of T via $i : Z \hookrightarrow T$. We often denote a widening $((T,M_T),(Z,M_Z),i,z)$ simply by $((T,M_T),(Z,M_Z))$ or T.
(3) Let $T := ((T,M_T),(Z,M_Z),i,z)$ be a pre-widening. Then we define the associated widening by the 4-tuple $((\hat{T},M_T|_{\hat{T}}),(Z,M_Z),i,z)$, where $\hat{T}$ is the completion of T along Z. We often denote this widening simply by $\hat{T}$. We have the canonical morphism of 4-tuples $\hat{T} \longrightarrow T$.
(4) A pre-widening or a widening $((T,M_T),(Z,M_Z),i,z)$ is said to be exact if i is exact.

We denote the full subcategory of $\mathcal{Q}((X/V)^{\log})$ consisting of the pre-widenings of (X,M) over $(\mathrm{Spf}\, V,N)$ by $\mathrm{PWide}((X,M)/(\mathrm{Spf}\, V,N))$ or $\mathrm{PWide}((X/V)^{\log})$, and the full subcategory of $\mathcal{Q}((X/V)^{\log})$ consists of widenings of (X,M) over $(\mathrm{Spf}\, V,N)$ by $\mathrm{Wide}((X,M)/(\mathrm{Spf}\, V,N))$ or $\mathrm{Wide}((X/V)^{\log})$. For a (pre-)widening T, denote the sheaf on $(X/V)^{\log}_{\mathrm{conv},\tau}$ (where $\tau = \mathrm{Zar}$ or et) defined by $T' \mapsto \mathrm{Hom}_{\mathcal{Q}((X/V)^{\log})}(T',T)$ by h_T.

Let T be a pre-widening and let $\hat{T}$ be the associated widening. Then one can check easily that the canonical morphism of sheaves $h_{\hat{T}} \longrightarrow h_T$ is an isomorphism.

REMARK 2.1.10. In the paper [Og2], the notion of pre-widenings was not defined, but it seems to the author that pre-widenings have essentially appeared in his paper as widenings by abuse of terminology.

REMARK 2.1.11. A widening $T := ((T, M_T), (Z, M_Z), i, z)$ is an enlargement in the sense of Definition 2.1.1 if $Z \supset \underline{\mathrm{Spec}}\,(\mathcal{O}_T/p\mathcal{O}_T)_{\mathrm{red}}$ holds and exact.

REMARK 2.1.12. In the category $\mathrm{PWide}((X/V)^{\log})$ or $\mathrm{Wide}((X/V)^{\log})$, there exist products. For pre-widenings $T := ((T, M_T), (Z, M_Z))$ and $T' := ((T', M_{T'}), (Z', M_{Z'}))$, the product $T \times T'$ is defined by

$$T \times T' := ((T, M_T)\hat{\times}_{(\mathrm{Spf}\,V,N)}(T', M_{T'}), (Z, M_Z) \times_{(X,M)} (Z', M_{Z'}))$$

and for widenings $T := ((T, M_T), (Z, M_Z))$ and $T' := ((T', M_{T'}), (Z', M_{Z'}))$, the product $T \times T'$ is defined by

$$\begin{aligned} T \times T' := (((T, M_T)\hat{\times}_{(\mathrm{Spf}\,V,N)}(T', M_{T'}))_{/(Z,M_Z)\times_{(X,M)}(Z',M_{Z'})}, \\ (Z, M_Z) \times_{(X,M)} (Z', M_{Z'})), \end{aligned}$$

where $((T, M_T)\hat{\times}_{(\mathrm{Spf}\,V,N)}(T', M_{T'}))_{/(Z,M_Z)\times_{(X,M)}(Z',M_{Z'})}$ denotes the completion of $(T, M_T)\hat{\times}_{(\mathrm{Spf}\,V,N)}(T', M_{T'})$ along the underlying scheme of $(Z, M_Z) \times_{(X,M)} (Z', M_{Z'})$. Hence one can check easily that the association from a pre-widening to a widening commutes with the formation of the product.

Beware of the following facts: The notion of the product as widenings and that as pre-widenings are different for objects in $\mathrm{PWide}((X/V)^{\log}) \cap \mathrm{Wide}((X/V)^{\log})$. The category $\mathrm{Enl}((X/V)^{\log})$ is not closed under the product in the category $\mathrm{PWide}((X/V)^{\log})$ or $\mathrm{Wide}((X/V)^{\log})$. The product $T \times T'$ is not necessarily exact even if T and T' are exact (pre-)widenings.

We define the notion of affinity of (pre-)widenings as follows:

DEFINITION 2.1.13. Let $(X, M) \xrightarrow{f} (\mathrm{Spec}\,k, N) \xrightarrow{\iota} (\mathrm{Spf}\,V, N)$ be as in the beginning of this section and assume that we are given a chart

$\mathcal{C}_0 := (P_X \to M, Q_V \to N, Q \to P)$ of the morphism $\iota \circ f$. (In particular, we assume the existence of such a chart.) Then a (pre-)widening $T := ((T, M_T), (Z, M_Z), i, z)$ is called affine with respect to $\mathcal{C}_0$ if T is affine, z is an affine morphism and the diagram

$$\begin{array}{ccc} (Z, M_Z) & \xrightarrow{i} & (T, M_T) \\ {\scriptstyle z}\downarrow & & \downarrow \\ (X, M) & \xrightarrow{\iota \circ f} & (\mathrm{Spf}\, V, N) \end{array} \tag{2.1.2}$$

admits a chart $\mathcal{C}$ satisfying the following conditions:

(1) The chart $\mathcal{C}$ extends the given chart $\mathcal{C}_0$ of $\iota \circ f$.

(2) If we denote the restriction of the chart $\mathcal{C}$ to i by $(R_T \to M_T, S_Z \to M_Z, R \overset{\alpha}{\to} S)$, then α^{gp} is surjective.

We call a pair $(T, \mathcal{C})$ of a (pre-)widening and a chart of the diagram (2.1.2) as above a charted affine (pre-)widening.

We define the exactification of a charted affine (pre-)widening as follows.

Definition 2.1.14. Let the situation be as in the above definition.

(1) Let $T := (((T, M_T), (Z, M_Z), i, z), \mathcal{C})$ be a charted affine pre-widening of (X, M) over $(\mathrm{Spf}\, V, N)$ with respect to $\mathcal{C}_0$. Let $T = \mathrm{Spf}\, A$, $Z = \mathrm{Spec}\, B$ and let $(R_T \to M_T, S_Z \to M_Z, R \overset{\alpha}{\to} S)$ be the restriction of $\mathcal{C}$ to i. Then we put $R' := (\alpha^{\mathrm{gp}})^{-1}(R)$, $A' := A \otimes_{\mathbb{Z}[R]} \mathbb{Z}[R']$, $T^{\mathrm{ex}} := \varinjlim_n \mathrm{Spec}\, (A'/(p^n))$ and let $M_{T^{\mathrm{ex}}}$ be the log structure associated to the homomorphism $R' \longrightarrow A' \longrightarrow \varprojlim_n A'/(p^n)$. Then the morphism i factors through the exact closed immersion $i' : (Z, M_Z) \hookrightarrow (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ and the 4-tuple

$$((T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}}), (Z, M_Z), i', z)$$

is an exact affine pre-widening. We call this pre-widening the exactification of T and we often denote it simply by T^{ex}.

(2) Let $T := ((T, M_T), (Z, M_Z), i, z)$ be a charted affine widening of (X, M) over $(\mathrm{Spf}\, V, N)$ with respect to $\mathcal{C}_0$. Let $T = \mathrm{Spf}\, A$, $Z =$

Spec B and let $(R_T \to M_T, S_Z \to M_Z, R \stackrel{\alpha}{\to} S)$ be the restriction of $\mathcal{C}$ to i. Then we put $R' := (\alpha^{\mathrm{gp}})^{-1}(R)$, $A' := A \otimes_{\mathbb{Z}[R]} \mathbb{Z}[R']$, $I := \mathrm{Ker}(A' \longrightarrow B)$, $T^{\mathrm{ex}} := \varinjlim_n \mathrm{Spec}\,(A'/I^n)$ and let $M_{T^{\mathrm{ex}}}$ be the log structure associated to the homomorphism $P' \longrightarrow A' \longrightarrow \varprojlim_n A'/I^n$. Then the morphism i factors through the exact closed immersion $i' : (Z, M_Z) \hookrightarrow (T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}})$ and the 4-tuple $((T^{\mathrm{ex}}, M_{T^{\mathrm{ex}}}), (Z, M_Z), i', z)$ is an exact affine widening. We call this widening the exactification of T and we often denote it simply by T^{ex}.

In each case, we have the canonical morphism $T^{\mathrm{ex}} \longrightarrow T$.

Remark 2.1.15.

(1) The exactification T^{ex} depends on the chart $\mathcal{C}$.
(2) For a charted affine pre-widening, the associated widening has a chart naturally. Then one can check easily that the association from a charted affine pre-widening to a charted affine widening commutes with the formation of the exactification.

Then one can check easily the following proposition. We omit the proof, since this is the immediate consequence of [Kk, (5.8)].

Proposition 2.1.16. *Let the notations be as in Definitions 2.1.13 and 2.1.14. For a charted affine widening T, the canonical morphism $T^{\mathrm{ex}} \longrightarrow T$ induces the isomorphism $h_{T^{\mathrm{ex}}} \stackrel{\sim}{\longrightarrow} h_T$ of sheaves on $(X/V)^{\log}_{\mathrm{conv},\tau}$.*

Now we define the restricted log convergent site.

Definition 2.1.17. Let $(X, M) \stackrel{f}{\longrightarrow} (\mathrm{Spec}\,k, N) \stackrel{\iota}{\hookrightarrow} (\mathrm{Spf}\,V, N)$ be as in the beginning of this section and let τ be one of the words $\{\mathrm{Zar}(= \mathrm{Zariski}), \mathrm{et}(= \mathrm{etale})\}$. For a (pre-)widening T, we define the restricted log convergent site $((X, M)/(\mathrm{Spf}\,V, N))_{\mathrm{conv},\tau}|_T$ (or $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ for short) as follows: The objects are the enlargements T' endowed with a morphism $T' \longrightarrow T$ of (pre-)widenings. The morphisms are the morphisms of enlargements over T. A family of morphisms over T

$$\{g_\lambda := (g_{1,\lambda}, g_{2,\lambda}) : ((T'_\lambda, M_{T'_\lambda}), (Z'_\lambda, M_{Z'_\lambda}), i'_\lambda, z'_\lambda) \longrightarrow ((T', M_{T'}), (Z', M_{Z'}), i', z')\}_{\lambda \in \Lambda}$$

is a covering if the following conditions are satisfied:

(1) $g^*_{1,\lambda} M_{T'} \stackrel{\sim}{=} M_{T'_\lambda}$ holds for any $\lambda \in \Lambda$.

(2) The family of morphisms $\{g_{1,\lambda} : T'_\lambda \longrightarrow T'\}$ is a covering with respect to τ-topology on T'.
(3) $(Z'_\lambda, M_{Z'_\lambda})$ is isomorphic to $(T'_\lambda, M_{T'_\lambda}) \times_{g_{1,\lambda},(T',M_{T'}),i'} (Z', M_{Z'})$ via the morphism induced by i'_λ and $g_{2,\lambda}$.

REMARK 2.1.18. One can check easily that the topos $(X/V)^{\log,\sim}_{\mathrm{conv},\tau}|_T$ associated to the site $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ is equivalent to the category of sheaves E on $(X/V)^{\log}_{\mathrm{conv},\tau}$ endowed with a morphism of sheaves $E \longrightarrow h_T$. In particular, for a pre-widening T, the site $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ is canonically equivalent to $(X/V)^{\log}_{\mathrm{conv},\tau}|_{\hat{T}}$, where $\hat{T}$ is the associated widening of T.

We can define the notion of isocrystals on restricted log convergent site in the same way as in the case of usual log convergent site:

DEFINITION 2.1.19. Let the notations be as above. Then, an isocrystal on the restricted log convergent site $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ is a sheaf $\mathcal{E}$ on the site $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ satisfying the following conditions:

(1) For any enlargement T' over T, the sheaf $\mathcal{E}$ on T'_τ induced by $\mathcal{E}$ is an isocoherent sheaf.
(2) For any morphism $f : T'' \longrightarrow T'$ of enlargements over T, the homomorphism $f^*\mathcal{E}_{T'} \longrightarrow \mathcal{E}_{T''}$ of sheaves on T''_τ induced by $\mathcal{E}$ is an isomorphism.

We denote the category of isocrystals on the restricted log convergent site $((X,M)/(\mathrm{Spf}\, V,N))_{\mathrm{conv},\tau}|_T$ by $I_{\mathrm{conv},\tau}((X,M)/(\mathrm{Spf}\, V,N)|_T)$. When there are no confusions on log structures, we will denote it simply by $I_{\mathrm{conv},\tau}((X/V)^{\log}|_T)$.

Now we remark two basic properties on the cohomology of sheaves on log convergent site which we need later.

The first one is the cohomological descent. Let $\iota : (\mathrm{Spec}\, k, N) \hookrightarrow (\mathrm{Spf}\, V, N)$ be the canonical exact closed immersion in the beginning of this section and let $(X^{(\bullet)}, M^{(\bullet)})$ be a simplicial fine log scheme over $(\mathrm{Spec}\, k, N)$ such that $X^{(n)}$ is of finite type over k for each n. Then, by Saint-Donat [SD], we can define the log convergent topos of the simplicial log scheme $(X^{(\bullet)}, M^{(\bullet)})$ over $(\mathrm{Spf}\, V, N)$ as the category of sections of the bifibered topos

$$\coprod_{n\in\mathbb{N}}((X^{(n)},M^{(n)})/(\mathrm{Spf}\,V,N))^{\sim}_{\mathrm{conv},\tau}\longrightarrow\Delta^{\mathrm{op}},$$

where Δ^{op} denotes the opposite category of the category of simplices. We denote it by $(X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau}$. Then we have the following proposition:

PROPOSITION 2.1.20. *Let $(X,M)\longrightarrow(\mathrm{Spec}\,k,N)\hookrightarrow(\mathrm{Spf}\,V,N)$ be as in the beginning of this section and let τ be one of the words* $\{\mathrm{Zar}(=\mathrm{Zariski}),\mathrm{et}(=\mathrm{etale})\}$. *Let $g:X^{(\bullet)}\longrightarrow X$ be a hypercovering of X with respect to τ-topology and put $M^{(\bullet)}:=M|_{X^{(\bullet)}}$. Let*

$$\theta:=(\theta_*,\theta^{-1}):(X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau}\longrightarrow(X/V)^{\log,\sim}_{\mathrm{conv},\tau}$$

be the morphism of topoi defined by $\theta_*(E^{(\bullet)}):=\mathrm{Ker}(g^{(0)}_*E^{(0)}\overset{\rightarrow}{\rightarrow}g^{(1)}_*E^{(1)})$, $\theta^{-1}(E)^{(i)}:=g^{(i),-1}(E)$ $(i=0,1)$ (*where $g^{(i)}$ is the homomorphism $X^{(i)}\longrightarrow X$*), $\theta^{-1}(E)^{(i)}:=0$, $(i>1)$. *Then, for any abelian sheaf E on $(X/V)^{\log}_{\mathrm{conv},\tau}$, the canonical homomorphism*

$$E\longrightarrow R\theta_*\theta^{-1}E$$

is an isomorphism in the derived category of the category of abelian sheaves on $(X/V)^{\log}_{\mathrm{conv},\tau}$.

PROOF. Let us take an enlargement $T:=((T,M_T),(Z,M_Z),i,z)$ of (X,M) over $(\mathrm{Spf}\,V,N)$ and for $n\in\mathbb{N}$, define an enlargement $T^{(n)}:=((T^{(n)},M_{T^{(n)}}),(Z^{(n)},M_{Z^{(n)}}),i^{(n)},z^{(n)})$ of $(X^{(n)},M^{(n)})$ over $(\mathrm{Spf}\,V,N)$ as follows: $Z^{(n)}$ is the scheme $Z\times_X X^{(n)}$, $M_{Z^{(n)}}$ is the pull-back of M_Z to $Z^{(n)}$, $z^{(n)}$ is the projection $(Z^{(n)},M_{Z^{(n)}})\longrightarrow(X^{(n)},M^{(n)})$, $T^{(n)}$ is the unique formal V-scheme which is formally etale over T satisfying $T^{(n)}\times_T Z\overset{\sim}{=}Z^{(n)}$, $M_{T^{(n)}}$ is the pull-back of M_T to $T^{(n)}$ and $i^{(n)}$ is the pull-back of i by the morphism $(T^{(n)},M_{T^{(n)}})\longrightarrow(T,M_T)$. Then, as formal schemes, we have a hypercovering $h:T^{(\bullet)}\longrightarrow T$ with respect to τ-topology.

Now let $\varphi:=(\varphi_*,\varphi^{-1}):T^{(\bullet),\sim}_\tau\longrightarrow T^\sim_\tau$ be the morphism of topoi induced by h and let

$$u^{(\bullet)}_*:(X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau}\longrightarrow T^{(\bullet),\sim}_\tau,$$
$$u_*:(X/V)^{\log,\sim}_{\mathrm{conv},\tau}\longrightarrow T^\sim_\tau$$

be ‘the functors of the evaluation’, which are exact by definition. Then one can check the equalities

$$\varphi_* \circ u_*^{(\bullet)} = u_* \circ \theta_*, \tag{2.1.3}$$

$$u_*^{(\bullet)} \circ \theta^{-1} = \varphi^{-1} \circ u_*. \tag{2.1.4}$$

Note that $u_*^{(\bullet)}$ sends injectives to injectives: Indeed, $u_*^{(\bullet)}$ is the composite of the functors $j^{-1} : (X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau} \longrightarrow (X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau}|_{T^{(\bullet)}}$ and $v_* : (X^{(\bullet)}/V)^{\log,\sim}_{\mathrm{conv},\tau}|_{T^{(\bullet)}} \longrightarrow T^{(\bullet),\sim}_{\tau}$, where j^{-1} is the canonical restriction and v_* is again the functor of evaluation. Then one can see, as in the case without log structures ([Og2]), that both j^{-1} and v_* admit exact left adjoint functors. Hence they send injectives to injectives. So the same is true also for the functor u_*. Hence, by Leray spectral sequence, the equality (2.1.3) implies the equality

$$R\varphi_* \circ u_*^{(\bullet)} = u_* \circ R\theta_*. \tag{2.1.5}$$

By the equalities (2.1.4) and (2.1.5), we have the following equality for an abelian sheaf E on $(X/V)^{\log}_{\mathrm{conv},\tau}$:

$$\begin{aligned} u_*(R\theta_*\theta^{-1}E) &= R\varphi_*(u_*^{(\bullet)} \circ \theta^{-1}E) \\ &= R\varphi_*\varphi^{-1}(u_*E) \\ &= u_*E, \end{aligned}$$

where the last equality follows from the cohomological descent for the hypercovering with respect to τ-topology. Since the above equality holds for any enlargement T, we have the isomorphism $R\theta_*\theta^{-1}E \overset{\sim}{=} E$, as desired. □

The second one is the comparison of the cohomology of isocrystals on $(X/V)^{\log}_{\mathrm{conv,et}}$ and $(X/V)^{\log}_{\mathrm{conv,Zar}}$ (cf. [Be-Br-Me, (1.1.19)]).

PROPOSITION 2.1.21. *Let* $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \xrightarrow{\iota} (\mathrm{Spf}\, V, N)$ *be as above and let* $T := ((T, M_T), (Z, M_Z), i, z)$ *be a (pre-)widening of* (X, M) *over* $(\mathrm{Spf}\, V, N)$*. Let us denote the canonical morphism of topoi*

$$(X/V)^{\log,\sim}_{\mathrm{conv,et}} \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}$$
$$(\textit{resp.} \quad (X/V)^{\log,\sim}_{\mathrm{conv,et}}|_T \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_T \quad)$$

by ϵ. Then:

(1) *for any $E \in I_{\mathrm{conv,et}}((X/V)^{\log})$ (resp. $E \in I_{\mathrm{conv,et}}((X/V)^{\log}|_T)$), we have $R\epsilon_* E = \epsilon_* E$.*

(2) *The functor $E \mapsto \epsilon_* E$ induces the equivalence of categories*

$$I_{\mathrm{conv,et}}((X/V)^{\log}) \xrightarrow{\sim} I_{\mathrm{conv,Zar}}((X/V)^{\log})$$
$$(\textit{resp.} \quad I_{\mathrm{conv,et}}((X/V)^{\log}|_T) \xrightarrow{\sim} I_{\mathrm{conv,Zar}}((X/V)^{\log}|_T) \quad).$$

Proof. $R^q\epsilon_* E$ is the sheaf associated to the presheaf $T' \mapsto H^q((X/V)^{\log}_{\mathrm{conv,et}}|_{T'}, E)$ (resp. $(T' \longrightarrow T) \mapsto H^q((X/V)^{\log}_{\mathrm{conv,et}}|_{T'}, E)$). So it suffices to prove the equations

$$H^q((X/V)^{\log}_{\mathrm{conv,et}}|_{T'}, E) = \begin{cases} E(T'), & q = 0, \\ 0, & q > 0, \end{cases}$$

for an enlargement $T' := ((T', M_{T'}), (Z', M_{Z'}), i', z')$ such that T' is affine. The case $q = 0$ is obvious. By [SGA4, V 4.3, III 4.1] (see also [Mi, III.2.12]), it suffices to prove the vanishing of the Čech cohomology $\check{H}^q((X/V)^{\log}_{\mathrm{conv,et}}|_{T'}, E) = 0\,(q > 0)$ for any enlargement T' as above. Let $\mathcal{U} := \{S_i \longrightarrow T'\}_{i\in I}$ be a covering in $(X/V)^{\log}_{\mathrm{conv,et}}$ such that each S_i is an affine formal V-scheme and $|I|$ is finite. Since any covering of T' has a refinement by a covering of this type, it suffices to prove the vanishing $\check{H}^q(\mathcal{U}, E) = 0\,(q > 0)$. Put $T' := \mathrm{Spf}\, A, \coprod_{i\in I} S := \mathrm{Spf}\, B$ and $E(T') := M = K \otimes_V N$, where N is a finitely generated A-module. Since E is an isocrystal, $\check{H}^q(\mathcal{U}, E)$ is the q-th cohomology of the complex

$$C^\bullet := [0 \to M \to M\hat{\otimes}_A B \to M\hat{\otimes}_A B\hat{\otimes}_A B \to \cdots].$$

(Here $\hat{\otimes}$ means the p-adically completed tensor product.) Put $N_n := N/p^nN, A_n := A/p^nA, B := B/p^nB$ and let $C^\bullet_n$ be the complex

$$0 \to N_n \to N_n \otimes_{A_n} B_n \to N_n \otimes_{A_n} B_n \otimes_{A_n} B_n \to \cdots .$$

Then we have $H^q(C^\bullet_n) = K \otimes_V H^q(\varprojlim_n C^\bullet_n)$. Since $\{C^q_n\}_{n=1}^\infty$ satisfies the Mittag-Leffler condition, we have the exact sequence

$$0 \to \varprojlim_n{}^1 H^{q-1}(C^\bullet_n) \to H^q(\varprojlim C^\bullet_n) \to \varprojlim_n H^q(C^\bullet_n) \to 0.$$

Moreover, we have $H^0(C_n^\bullet) = N_n, H^q(C_n^\bullet) = 0\,(q>0)$, since B_n is faithfully flat over A_n. Hence $H^q(\varprojlim_n C_n^\bullet) = 0$ holds for $q>0$. So we have $H^q(\mathcal{U}, E) = 0\,(q>0)$ and the proof of the assertion (1) is finished.

Next we prove the assertion (2). Let E be an object in $(X/V)^{\log}_{\mathrm{conv,Zar}}$ (resp. $(X/V)^{\log}_{\mathrm{conv,Zar}}|_T$). Then, by rigid analytic faithfully flat descent of Gabber ([Og1, (1.9)]), the presheaf on $(X/V)^{\log}_{\mathrm{conv,et}}$ (resp. $(X/V)^{\log}_{\mathrm{conv,et}}|_T$) defined by $T' \mapsto E(T')$ (resp. $(T' \to T) \mapsto E(T' \to T)$) is an isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$ (resp. $(X/V)^{\log}_{\mathrm{conv,et}}|_T$). Let us denote it by $\delta(E)$. Then it is obvious that the functor δ is the quasi-inverse of ϵ_*. □

Next, we recall the notion of the system of universal enlargements of exact (pre-)widenings, which is a slight modification of that defined in [Shi, (5.2.3)].

Definition 2.1.22. Let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \xrightarrow{\iota} (\mathrm{Spf}\, V, N)$ be as in the beginning of this section and let $((T, M_T), (Z, M_Z), i, z)$ be an exact (pre-)widening. Put $Z' := T_{\mathrm{fl}} \times_T Z$ and let $\mathcal{I}$ be the ideal $\mathrm{Ker}(\mathcal{O}_{T_{\mathrm{fl}}} \longrightarrow \mathcal{O}_{Z'})$. For $n \in \mathbb{N}$, let $B_{Z,n}(T)$ be the p-adically completed formal blow-up of T_{fl} with respect to the ideal $\pi\mathcal{O}_{T_{\mathrm{fl}}} + \mathcal{I}^n$ and let $T_{Z,n}(T)$ be the open set

$$\{x \in B_{Z,n}(T) \,|\, (\pi\mathcal{O}_{T_{\mathrm{fl}}} + \mathcal{I}^n)\cdot \mathcal{O}_{B_{Z,n}(T),x} = \pi\mathcal{O}_{B_{Z,n}(T),x}\}$$

of $B_{Z,n}(T)$. Let $\lambda_n : T_{Z,n}(T) \longrightarrow T$ be the canonical morphism and let $Z_n := \lambda_n^{-1}(Z)$. Then the 4-tuple

$$T_{Z,n}(T) := ((T_{Z,n}(T), M_T), (Z_n, M_Z), Z_n \hookrightarrow T_{Z,n}(T), (Z_n, M_Z) \xrightarrow{\lambda_n} (Z, M_Z) \xrightarrow{z} (X, M))$$

is an enlargement for each n and the family $\{T_{Z,n}(T)\}_{n\in\mathbb{N}}$ forms an inductive system of enlargements. The morphisms λ_n's define the morphisms of (pre-)widenings $T_{Z,n}(T) \longrightarrow T\,(n \in \mathbb{N})$ which is compatible with transition morphisms of the inductive system $\{T_{Z,n}(T)\}_{n\in\mathbb{N}}$. We call this inductive system the system of universal enlargements of T.

Then we have the following (cf. [Shi, (5.2.4)]):

Lemma 2.1.23. *Let the notations be as above and let $T' := ((T', M_{T'}), (Z', M_{Z'}), i', z')$ be an enlargement. Then a morphism $g : T' \longrightarrow T$ in*

$\mathcal{Q}((X/V)^{\log})$ *factors through* $T_{Z,n}(T)$ *for some* n. *Moreover, such a factorization* $T' \longrightarrow T_{Z,n}(Z)$ *is unique as a morphism to the inductive system* $\{T_{Z,n}(T)\}_{n\in\mathbb{N}}$.

PROOF. Put $T_{\mathrm{fl}} := ((T_{\mathrm{fl}}, M_T), (Z', M_Z), i \times_T T_{\mathrm{fl}}, (Z', M_Z) \hookrightarrow (Z, M_Z) \xrightarrow{z} (X, M))$. Then, since T' is flat over $\operatorname{Spf} V$, g factors through $g' : T' \longrightarrow T_{\mathrm{fl}}$. Put $\mathcal{J} := \operatorname{Ker}(\mathcal{O}_{T'} \longrightarrow \mathcal{O}_{Z'})$. Then $\mathcal{J}^n \subset \pi\mathcal{O}_{T'}$ holds for some n. Hence we have $(g')^*(\pi\mathcal{O}_{T_{\mathrm{fl}}} + \mathcal{I}^n) = \pi\mathcal{O}_{T'}$. So, by the universality of blow-up, $g' : T' \longrightarrow T_{\mathrm{fl}}$ factors through $T' \longrightarrow T_{Z,n}(T)$ for some n, and one can easily check that this defines a morphism of enlargements. The uniqueness of the factorization also follows from the universality of blow-up. □

The above lemma implies that the canonical morphism of sheaves $\varinjlim_n h_{T_{Z,n}(T)} \longrightarrow h_T$ is an isomorphism. Moreover, the following stronger result is known ([Og2, (0.2.2)]).

LEMMA 2.1.24. *With the notation above, the morphism of sheaves* $h_{T_{Z,n}(T)} \longrightarrow h_T$ *is injective.*

PROOF. Since the proof is the same as that in [Og2, (0.2.2)], we omit it. (Note that we have assumed that the formal V-scheme T' which appears in an enlargement $T' := ((T', M_{T'}), (Z', M_{Z'}), i', z')$ is assumed to be flat over $\operatorname{Spf} V$ in this paper. We need this assumption in the proof of [Og2, (0.2.2)]. This is the reason why we imposed this condition.) □

We recall the explicit description of $T_{Z,n}(T)$ in affine case, following [Og1, (2.3), (2.6.2)]. (The proof is easy and left to the reader.) Let $T := ((T, M_T), (Z, M_Z), i, z)$ be an exact (pre-)widening and assume that $T := \operatorname{Spf} A$ is affine. Put $I := \Gamma(T, \mathcal{I})$ and take a generator $g_1, \cdots, g_r$ of I. For $m := (m_1, \cdots, m_r) \in \mathbb{N}^r$, put $|m| := (m_1, \cdots, m_r)$ and $g^m := g_1^{m_1} \cdots g_r^{m_r}$. Then we have

$$T_{Z,n}(T) = \operatorname{Spf}(A[t_m \ (m \in \mathbb{N}^r, |m| = n)]/(\pi t_m - g^m \ (m \in \mathbb{N}^r, |m| = n)) + (p\text{-torsion}))^\wedge,$$

where $^\wedge$ denotes the p-adic completion. Note that one has

$$K \otimes_V \Gamma(T_{Z,n}(T), \mathcal{O}_{T_{Z,n}(T)}) \cong K \otimes (A[t_1, \cdots, t_r]/(\pi t_1 - g_1^n, \cdots, \pi t_r - g_r^n) + (p\text{-torsion}))^\wedge.$$

As a consequence of the above description, we can prove the following, which is also due to Ogus ([Og1, (2.6.1)]):

Lemma 2.1.25. *Let* $(X,M) \longrightarrow (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} V, N)$ *be as above and let* T *be a pre-widening. Denote the widening associated to* T *by* $\hat{T}$. *Then we have the canonical isomorphism of enlargements* $T_{Z,n}(\hat{T}) \xrightarrow{\sim} T_{Z,n}(T)$.

Proof. We may reduce to the case that $T = \operatorname{Spf} A$ is affine. Let $I, g_i \, (1 \leq i \leq r)$ be as above and denote the I-adic completion of A by $\hat{A}$. It suffices to prove that the canonical homomorphism of rings

$$B := (A[t_m \ (m \in \mathbb{N}^r, |m| = n)]/(\pi t_m - g^m \ (m \in \mathbb{N}^r, |m| = n)) + (p\text{-torsion}))^\wedge$$
$$\longrightarrow B' := (\hat{A}[t_m \ (m \in \mathbb{N}^r, |m| = n)]/(\pi t_m - g^m \ (m \in \mathbb{N}^r, |m| = n)) + (p\text{-torsion}))^\wedge$$

is an isomorphism. Since $I^n B$ is contained in the ideal $\pi B \subset B$, the canonical homomorphism $A \longrightarrow B$ uniquely extends to the continuous homomorphism $\hat{A} \longrightarrow B$. We can extend this homomorphism to the homomorphism $B' \longrightarrow B$ by sending t_m to t_m. It is obvious that this homomorphism gives the inverse of the above homomorphism. □

We recall some basic properties of system of enlargements which we need later. The first one is essentially proved in [Og2, (0.2.4)].

Lemma 2.1.26. *Let*

$$g : ((T', M_{T'}), (Z', M_{Z'}), i', z') \longrightarrow ((T, M_T), (Z, M_Z), i, z)$$

be a morphism of (pre-)widenings and assume $g^{-1}(Z) = Z'$ *holds. Then* g *induces the natural isomorphism of enlargements*

$$T_{Z',n'}(T) \xrightarrow{\sim} T_{Z,n}(T) \times_{T_{\mathrm{fl}}} T'_{\mathrm{fl}}.$$

If $T' \longrightarrow T$ *is flat, we have the isomorphism*

$$T_{Z',n'}(T) \xrightarrow{\sim} T_{Z,n}(T) \times_T T'.$$

Proof. This is obvious from the universality of blow-ups and the fact that $T_{\mathrm{fl}} \times_T T' \overset{\sim}{=} T'_{\mathrm{fl}}$ holds when $T' \longrightarrow T$ is flat. □

The next proposition, which is also due to Ogus ([Og2, (0.2.5), (0.2.6)]), establishes the influence of the choice of Z to the construction of the system of universal enlargement.

PROPOSITION 2.1.27. *Let* $(X,M) \longrightarrow (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} V, N)$ *be as in the beginning of this section. Let* $((T,M_T),(Z,M_Z),i,z)$ *and* $((T,M_T),(Z',M_{Z'}),i',z')$ *be exact (pre-)widenings of* (X,M) *over* $(\operatorname{Spf} V,N)$ *such that* $Z \subset Z'$ *and* $z = z'|_Z$ *hold. Assume that there exists an ideal* J *of* $\mathcal{O}_T$ *and* $m \in \mathbb{N}$ *such that* $J^{m+1} \subset \pi\mathcal{O}_T$ *and* $I_Z \subset I_{Z'} + J$ *holds, where* $I_Z, I_{Z'}$ *are the defining ideals of* Z*,* Z' *in* T*, respectively. Denote the canonical morphism of enlargements*

$$((T,M_T),(Z,M_Z),i,z) \longrightarrow ((T,M_T),(Z',M_{Z'}),i',z')$$

induced by id_T *and the closed immersion* $Z \hookrightarrow Z'$ *by* g *and denote the morphism of enlargements* $T_{Z,n}(T) \longrightarrow T_{Z',n}(T)$ *induced by* g *by* g_n*. Then:*

(1) *There exists a homomorphism of formal schemes* $h_n : T_{Z',n}(T) \longrightarrow T_{Z,m+n}(T)$ *such that the composites*

$$T_{Z',n}(T) \xrightarrow{h_n} T_{Z,m+n}(T) \xrightarrow{g_{m+n}} T_{Z',m+n}(T),$$
$$T_{Z,n}(T) \xrightarrow{g_n} T_{Z',n}(T) \xrightarrow{h_n} T_{Z,m+n}(T),$$

coincide with the canonical transition morphisms. (Note that h_n *is just a morphism of formal schemes and it is not a morphism of enlargements.)*

(2) *For an isocrystal* E *on* $(X/V)^{\log}_{\mathrm{conv},\tau}$*, we have the natural isomorphism*

$$\varphi_n : h_n^* E_{T_{Z,m+n}(T)} \xrightarrow{\sim} E_{T_{Z',n}(T)}$$

such that the composites

$$h_n^* \circ g_{m+n}^* E_{T_{Z',m+n}(T)} \xrightarrow{h_n^* E(g_{m+n})} h_n^* E_{T_{Z,m+n}(T)} \xrightarrow{\varphi_n} E_{T_{Z',n}(T)},$$
$$g_n^* \circ h_n^* E_{T_{Z,m+n}(T)} \xrightarrow{g_n^*\varphi_n} g_n^* E_{T_{Z',n}(T)} \xrightarrow{E(g_n)} E_{T_{Z,n}(T)},$$

coincide with the isomorphisms induced by the canonical morphism of enlargements $T_{Z',n}(T) \longrightarrow T_{Z',m+n}(T)$ *and* $T_{Z,n}(T) \longrightarrow T_{Z,m+n}(T)$*, respectively.*

Proof. First we prove the assertion (1). Since we have inclusions of ideals

$$\pi\mathcal{O}_T + I_Z^{m+n} \subset \pi\mathcal{O}_T + (I_{Z'} + J)^{m+n} \subset \pi\mathcal{O}_T + I_{Z'}^n + J^{m+1} \subset \pi\mathcal{O}_T + I_{Z'}^n,$$

we obtain the morphism h_n by the universality of blow-up. One can check easily the properties in the statement.

Next we prove the assertion (2). Let $\tilde{Z}_n$ be the inverse image of Z in $T_{Z',n}(T)$ and let $M_{\tilde{Z}_n}$ be the pull-back of the log structure M_T to $\tilde{Z}_n$. Then we have the diagram of enlargements

$$\begin{aligned} T_{Z,m+n}(T) &:= ((T_{Z,m+n}(T), M_{T_{Z,m+n}(T)}), (Z_{m+n}, M_{Z_{m+n}})) \\ &\xleftarrow{h_n} \widetilde{T_{Z,m+n}}(T) := ((T_{Z',n}(T), M_{T_{Z',n}(T)}), (\tilde{Z}_n, M_{\tilde{Z}_n})) \\ &\xrightarrow{j} T_{Z',n}(T) := ((T_{Z',n}(T), M_{T_{Z',n}(T)}), (Z'_n, M_{Z'_n})), \end{aligned}$$

where h_n is the morphism of enlargements induced by h_n in the assertion (1) and j is the morphism of enlargements induced by $\mathrm{id}_{T_{Z',n}(T)}$ and the closed immersion $\tilde{Z}_n \hookrightarrow Z'_n$. Then, for an isocrystal E on $(X/V)^{\log}_{\mathrm{conv},\tau}$, we have the isomorphisms

$$h_n^* E_{T_{Z,m+n}(T)} \xrightarrow{\sim} E_{\widetilde{T_{Z,m+n}}(T)} \xleftarrow{\sim} j^* E_{T_{Z',n}(T)} = E_{T_{Z',n}(T)}.$$

So we obtain the isomorphism $h_n^* E_{T_{Z,m+n}(T)} \xrightarrow{\sim} E_{T_{Z',n}(T)}$. We define φ_n as this isomorphism. It is easy to check that the isomorphism φ_n satisfies the desired properties. □

Finally we recall the definition and an acyclicity property of direct limit site $\vec{T}$ for an exact widening T, which is due to Ogus [Og2, §3].

Definition 2.1.28. Let T be an exact widening and $\{T_n := T_{Z,n}(T)\}_{n\in\mathbb{N}}$ be the system of universal enlargements of T. Then we define the direct limit site $\vec{T}$ as follows: Objects are the open sets of some T_n. For open sets $U \subset T_n$ and $V \subset T_m$, $\mathrm{Hom}_{\vec{T}}(U,V)$ is empty unless $n \leq m$ and in the case $n \leq m$, $\mathrm{Hom}_{\vec{T}}(U,V)$ is defined to be the set of morphisms

$f : U \longrightarrow V$ which commutes with the transition morphism $T_n \longrightarrow T_m$. (In particular, $\mathrm{Hom}_{\vec{T}}(U,V)$ consists of at most one element.) The coverings are defined by Zarisi topology for each object.

We define the structure sheaf $\mathcal{O}_{\vec{T}}$ by $\mathcal{O}_{\vec{T}}(U) := \Gamma(U, \mathcal{O}_U)$.

REMARK 2.1.29. In the above definition, we have changed the notation slightly from that in [Og2]. In the paper [Og2], the notation $\vec{T}$ is used to denote the topos associated to the direct limit site. In this papar, we will use the notation $\vec{T}$ to express the site and the associated topos will be denoted by $\vec{T}^{\sim}$.

Note that giving a sheaf E on the site $\vec{T}$ is equivalent to giving a compatible family $\{E_n\}_n$, where E_n is a sheaf on $T_{n,\mathrm{Zar}}$.

DEFINITION 2.1.30. A sheaf of $K \otimes \mathcal{O}_{\vec{T}}$-modules E is called crystalline if the following condition is satisfied: For any transition map $\psi : T_n \longrightarrow T_m$, the transition map of sheaves $\psi^{-1}E_m \longrightarrow E_n$ induces an isomorphism

$$\mathcal{O}_{T_n} \otimes_{\psi^{-1}\mathcal{O}_{T_m}} \psi^{-1}E_m \xrightarrow{\sim} E_n,$$

where E_n, E_m are the sheaves on $T_{n,\mathrm{Zar}}$, $T_{m,\mathrm{Zar}}$ induced by E.

We define the morphism of topoi

$$\gamma : \vec{T}^{\sim} \longrightarrow T^{\sim}_{\mathrm{Zar}}$$

as follows: γ^* is the functor defined by the pull-back and γ_* is the functor of taking the inverse limit of the direct image. Then one has the following acyclicity, which is due to Ogus ([Og2, (0.3.7)]).

PROPOSITION 2.1.31. *Let E be a crystalline sheaf of $K \otimes_V \mathcal{O}_{\vec{T}}$-modules. Then $R^q\gamma_*E = 0$ holds for $q > 0$.*

PROOF. We omit the proof, since it is the same as that in [Og2, (0.3.7)]. □

2.2. Analytic cohomology of log schemes

In this section, we extend the notion of tubular neighborhood to the case of closed immersions of fine log schemes into a fine log formal schemes satisfying certain condition. Then, for an isocrystal on log convergent site, we define the associated log de Rham complex on some tubular neighborhood. Finally, we give a definition of analytic cohomology (in rigid analytic sense) of a fine log scheme which has a locally free isocrystal on log convergent site as coefficient.

We extend the notion of tubular neighborhood to the case of closed immersions of fine log schemes into a fine log formal schemes satisfying certain condition. First, let us consider the following situation: Let (X, M) be a fine log scheme over k and let $i : (X, M) \hookrightarrow (P, L)$ be a closed immersion of (X, M) into a Noetherian fine log formal scheme (P, L) over $\mathrm{Spf}\, V$ whose scheme of definition is of finite type over $\mathrm{Spf}\, V$. (Note that (P, L) is not necessarily p-adic.) Let us consider the following condition on i:

$(*)$ There exists (at least one) factorization of i of the form

$$(X, M) \xrightarrow{i'} (P', L') \xrightarrow{f'} (P, L),$$

where i' is an exact closed immersion and f' is a formally log etale morphism.

Remark 2.2.1. If i admits a chart $(R_P \to L, S_X \to M, R \xrightarrow{\alpha} S)$ such that α^{gp} is surjective, the condition $(*)$ is satisfied. Indeed, if we put $R' := (\alpha^{\mathrm{gp}})^{-1}(S)$ and define (P', L') by

$P' := P \hat{\times}_{\mathrm{Spf}\, \mathbb{Z}_p\{R\}} \mathrm{Spf}\, \{R'\}$,
$L' :=$ the pull back of the canonical log structure on $\mathrm{Spf}\, \mathbb{Z}_p\{R'\}$,

then there exists a factorization as in $(*)$.

Let us note the following lemma:

Lemma 2.2.2. *Let $i : (X, M) \hookrightarrow (P, L)$ be as above and assume that i satisfies the condition $(*)$. Let $\hat{P}'$ be the completion of P' along X. Then the rigid analytic space $\hat{P}'_K$ is independent of the choice of the factorization as in $(*)$.*

PROOF. Let

$$(X,M) \xrightarrow{i''} (P'',L'') \xrightarrow{f''} (P,L)$$

be another factorization as in $(*)$. Put $(P^0, L^0) := ((P',L')\hat{\times}_{(P,L)} (P'',L''))^{\mathrm{int}}$. Then we have the factorization

$$(X,M) \xrightarrow{j} (P^0,L^0) \xrightarrow{h} (P,L)$$

defined by $j = i' \times i'', h = f' \circ \mathrm{pr}_1 = f'' \circ \mathrm{pr}_2$. Then h is formally log etale and j is a locally closed immersion. By shrinking P^0, we may assume that j is a closed immersion.

Now we check that j is an exact closed immersion. We only have to check that j^*L^0 is isomorphic to M. By definition, j^*L^0 is the push-out of the following diagram in the category of fine log structures on X:

$$(i')^*L' \longleftarrow i^*L \longrightarrow (i'')^*L''.$$

But the diagram is completed into the following commutative diagram:

$$\begin{array}{ccc} (i')^*L' & \xrightarrow{a} & M \\ \uparrow & & \uparrow b \\ i^*L & \longrightarrow & (i'')^*L'', \end{array}$$

where a and b are isomorphisms. So one can see that j^*L^0 is isomorphic to M.

Let us consider the following commutative diagram

$$\begin{array}{ccc} (X,M) & \xrightarrow{j} & (P^0,L^0) \\ \| & & \downarrow \mathrm{pr}_1 \\ (X,M) & \xrightarrow{i'} & (P',L'). \end{array}$$

Then pr_1 is formally log etale. Now note that

$$P^0_{\mathrm{pr}_1\text{-triv}} := \{x \in P^0 \,|\, (\mathrm{pr}_1^*L')_{\bar{x}} \xrightarrow{\sim} L^0_{\bar{x}}\}$$

is open in P^0 ([Shi, (2.3.1)]) and contains X. So, by shrinking P^0, we may assume that the morphism pr_1 is formally etale (in the classical sense).

Let $\hat{P}^0$, $\hat{P}'$ be the completion of P^0, P' along X. We prove that the morphism $\hat{\mathrm{pr}}_1 : \hat{P}^0 \longrightarrow \hat{P}'$ induced by pr_1 is an isomorphism. It suffices to prove that it is formally etale. Let $\tilde{P}^0$ be the completion of P^0 along $X' := X \times_{P'} P^0$. Then $\tilde{P}^0 \longrightarrow \hat{P}'$ is formally etale. On the other hand, the morphism $X' \longrightarrow X$ is etale and has a section. Hence X' contains X as a summand. Hence $\tilde{P}^0$ contains $\hat{P}^0$ as a summand, and so $\hat{\mathrm{pr}}_1$ is formally etale.

As a consequence, we have the isomorphism $\hat{P}^0_K \overset{\simeq}{=} \hat{P}'_K$. By the same argument, we have the isomorphism $\hat{P}^0_K \overset{\simeq}{=} \hat{P}''_K$. So the assertion is proved. □

Let $i : (X, M) \hookrightarrow (P, L)$ be a closed immersion of a fine log scheme over k into a Noetherian fine log formal scheme over $\mathrm{Spf}\, V$ whose scheme of definition is of finite type over $\mathrm{Spf}\, V$ which satisfies the condition $(*)$. Then we define the tubular neighborhood $](X, M)[_{(P,L)}$ of (X, M) in (P, L) by $](X, M)[_{(P,L)} := \hat{P}'_K$, where P' is as in $(*)$. We have the specialization map

$$\mathrm{sp} :](X, M)[_{(P,L)} \longrightarrow \hat{P} (\simeq X)$$

(where $\hat{P}$ is the completion of P along X) defined as the composite

$$\hat{P}'_K \xrightarrow{\mathrm{sp}} \hat{P}' \longrightarrow \hat{P}.$$

Remark 2.2.3. The above definition of tubular neighborhood is functorial in the following case: Let us assume given a diagram

$$\begin{array}{ccc} (X_1, M_1) & \xrightarrow{i_1} & (P_1, L_1) \\ {\scriptstyle g_X}\downarrow & & {\scriptstyle g_P}\downarrow \\ (X_2, M_2) & \xrightarrow{i_2} & (P_2, L_2) \end{array}$$

(where i_1, i_2 are the closed immersion as i above) and let us assume that there exists (at least one) diagram

$$\begin{array}{ccccc} (X_1, M_1) & \xrightarrow{i'_1} & (P'_1, L'_1) & \xrightarrow{f'_1} & (P_1, L_1) \\ {\scriptstyle g_X}\downarrow & & {\scriptstyle g_{P'}}\downarrow & & {\scriptstyle g_P}\downarrow \\ (X_2, M_2) & \xrightarrow{i'_2} & (P'_2, L'_2) & \xrightarrow{f'_2} & (P_2, L_2), \end{array}$$

where the horizontal lines are factorizations of i_1, i_2 as in the factorization in $(*)$. Then we have the morphism of tubular neighborhoods

$$](X_1, M_1)[_{(P_1,L_1)} \longrightarrow](X_2, M_2)[_{(P_2,L_2)}.$$

Next we extend the definition of tubular neighborhood to more general cases.

PROPOSITION 2.2.4. *Let $\mathcal{Z}$ be the category of locally closed immersions $i : (X, M) \hookrightarrow (P, L)$ of a fine log scheme over k into a Noetherian fine log formal scheme over* $\mathrm{Spf}\, V$ *whose scheme of definition is of finite type over* $\mathrm{Spf}\, V$ *satisfying the following condition*:

$(**)$ *There exists an open covering $P = \bigcup_{\alpha \in I} P_\alpha$ such that the morphisms $(X, M) \times_{(P,L)} (P_\alpha, L) \hookrightarrow (P_\alpha, L)$ are closed immersions and they satisfy the condition $(*)$.*

Then we have the unique functor

$$\mathcal{Z} \longrightarrow \text{(Rigid analytic space with morphism of sites)};$$
$$((X, M) \overset{i}{\hookrightarrow} (P, L)) \mapsto (](X, M)[_{(P,L)}, \mathrm{sp} :](X, M)[_{(P,L)} \to \hat{P}),$$

(*where $\hat{P}$ is the completion of P along X*) *satisfying the following conditions*:

(1) *When i satisfies the condition $(*)$, then $](X, M)[_{(P,L)}$ and* sp *coincide with the definition given above. If the morphism in $\mathcal{C}$*

$$((X_1, M_1) \xrightarrow{i_1} (P_1, L_1)) \longrightarrow ((X_2, M_2) \xrightarrow{i_2} (P_2, L_2))$$

satisfies the condition in Remark 2.2.3, then the morphism $](X_1, M_1)[_{(P_1,L_1)} \longrightarrow](X_2, M_2)[_{(P_2,L_2)}$ coincides with the one given in Remark 2.2.3.

(2) *If $P' \subset P$ is an open sub formal scheme and $X' = X \times_P P'$ holds, $](X', M)[_{(P',L)}$ is canonically identical with the admissible open set $\mathrm{sp}^{-1}(\hat{P}')$, where $\hat{P}'$ is the completion of P' along X'.*

(3) *If $P = \bigcup_{\alpha \in I} P_\alpha$ is an open covering and $X_\alpha = X \times_P P_\alpha$ holds, $](X, M)[_{(P,L)} = \bigcup_{\alpha \in I}](X_\alpha, M)[_{(P_\alpha,L)}$ is an admissible covering.*

Proof. Take an open covering $P = \bigcup_{\alpha \in I} P_\alpha$ such that the closed immersion $(X_\alpha, M) := (X, M) \times_{(P,L)} (P_\alpha, L) \hookrightarrow (P_\alpha, L)$ satisfies the condition $(*)$, and for $\alpha, \alpha' \in I$, put $P_{\alpha\alpha'} := P_\alpha \cap P_{\alpha'}, X_{\alpha\alpha'} := X_\alpha \cap X_{\alpha'}$. Then the closed immersion $(X_{\alpha\alpha'}, M) \hookrightarrow (P_{\alpha\alpha'}, L)$ also satisfies the condition $(*)$. By the conditions, the rigid analytic space $](X, M)[_{(P,L)}$ should be the reunion of $](X_\alpha, M)[_{(P_\alpha,L)}$'s which are glued along $](X_{\alpha\alpha'}, M)[_{(P_{\alpha\alpha'},L)}$'s. So it is unique. Conversely, we can define the rigid analytic space $](X, M)[_{(P,L)}$ by gluing $](X_\alpha, M)[_{(P_\alpha,L)}$'s along $](X_{\alpha\alpha'}, M)[_{(P_{\alpha\alpha'},L)}$'s and we can also define the specialization map $](X, M)[_{(P,L)} \longrightarrow \hat{P}$ by gluing $](X_\alpha, M)[_{(P_\alpha,L)} \longrightarrow \hat{P}_\alpha$'s, where $\hat{P}_\alpha$ is the completion of P_α along X_α. So the assertion is proved. □

Definition 2.2.5. Let $i : (X, M) \hookrightarrow (P, L)$ be a closed immersion of a fine log scheme over k into a Noetherian fine log formal scheme over $\mathrm{Spf}\, V$ whose scheme of definition is of finite type over $\mathrm{Spf}\, V$ which belongs to the category $\mathcal{Z}$ in Proposition 2.2.4. Then we define the tubular neighborhood $](X, M)[_{(P,L)}$ of (X, M) in (P, L) as the rigid analytic space in Proposition 2.2.4. We call the morphism of sites $](X, M)[_{(P,L)} \longrightarrow \hat{P}$ in Proposition 2.2.4 the specialization map. We denote the tubular neighborhood $](X, M)[_{(P,L)}$ simply by $]X[^{\log}_P$, where there will be no confusions on log structures.

Remark 2.2.6. Let $i : (X, M) \hookrightarrow (P, L)$ be a locally closed immersion of a fine log scheme over k into a Noetherian fine log formal scheme over $\mathrm{Spf}\, V$ whose scheme of definition is of finite type over $\mathrm{Spf}\, V$ and assume that (X, M) and (P, L) are of Zariski type. Then, by Proposition 1.1.2 and Remark 2.2.1, the closed immersion i belongs to the category $\mathcal{Z}$ in Proposition 2.2.4. So we can define the tubular neighborhood $](X, M)[_{(P,L)}$.

Next, for an isocrystal on log convergent site, we define the associated log de Rham complex on tubular neighborhood. Let us consider the following situation:

$$
\begin{array}{ccc}
(X, M) & \xrightarrow{\ i\ } & (P, L) \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\
(\mathrm{Spec}\, k, N) & \xrightarrow{\ \iota\ } & (\mathrm{Spf}\, V, N),
\end{array}
\tag{2.2.1}
$$

where N is a fine log structure on $\operatorname{Spf} V$, ι is the canonical exact closed immersion, (X, M) is a fine log scheme of finite type over k, (P, L) is a fine log formal V-scheme over $(\operatorname{Spf} V, N)$, i is a closed immersion and g is a formally log smooth morphism. Assume moreover that (X, M) and (P, L) are of Zariski type and that $(\operatorname{Spf} V, N)$ admits a chart.

For $n \in \mathbb{N}$, let $(P(n), L(n))$ be the $(n+1)$-fold fiber product of (P, L) over $(\operatorname{Spf} V, N)$ and let $i(n)$ be the locally closed immersion $(X, M) \hookrightarrow (P(n), L(n))$ induced by i. Then, since $(P(n), L(n))$ is of Zariski type (we use the assumption that $(\operatorname{Spf} V, N)$ admits a chart), $i(n)$ is in the category $\mathcal{Z}$ in Proposition 2.2.4. Hence we can define the tubular neighborhood $]X[^{\log}_{P(n)} :=](X, M)[_{(P(n),L(n))}$. Moreover, the projections

$$
\begin{aligned}
p'_i &: (P(1), L(1)) \longrightarrow (P, L) \quad (i = 1, 2), \\
p'_{ij} &: (P(2), L(2)) \longrightarrow (P(1), L(1)) \quad (1 \le i < j \le 3),
\end{aligned}
$$

and the diagonal morphism

$$\Delta' : (P, L) \longrightarrow (P(1), L(1))$$

induce the morphisms of rigid analytic spaces

$$
\begin{aligned}
p_i &:]X[^{\log}_{P(1)} \longrightarrow]X[^{\log}_{P} \quad (i = 1, 2), \\
p_{ij} &:]X[^{\log}_{P(2)} \longrightarrow]X[^{\log}_{P(1)} \quad (1 \le i < j \le 3), \\
\Delta &:]X[^{\log}_{P} \longrightarrow]X[^{\log}_{P(1)},
\end{aligned}
$$

respectively. Let $\operatorname{Str}''((X, M) \hookrightarrow (P, L)/(\operatorname{Spf} V, N))$ be the category of pairs (E, ϵ), where E is a coherent $\mathcal{O}_{]X[^{\log}_P}$-module and ϵ is an $\mathcal{O}_{]X[^{\log}_{P(1)}}$-linear isomorphism $p_2^* E \xrightarrow{\sim} p_1^* E$ satisfying $\Delta^*(\epsilon) = \mathrm{id}$, $p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon)$. Then we have the following:

Proposition 2.2.7. *Let the notations be as above. Then we have the canonical, functorial equivalence of categories*

$$I_{\mathrm{conv,et}}((X, M)/(\operatorname{Spf} V, N)) \xrightarrow{\sim} \operatorname{Str}''((X, M) \hookrightarrow (P, L)/(\operatorname{Spf} V, N)).$$

PROOF. Since both sides satisfy the descent for Zariski open covering of P, it suffices to construct the canonical functorial functor

$$I_{\mathrm{conv,et}}((X,M)/(\mathrm{Spf}\,V,N)) \longrightarrow \mathrm{Str}''((X,M)\hookrightarrow(P,L)/(\mathrm{Spf}\,V,N))$$

inducing an equivalence of categories in the case where the diagram

$$(X,M)\overset{i}{\hookrightarrow}(P,L)\overset{g}{\longrightarrow}(\mathrm{Spf}\,V,N)$$

admits a chart $\mathcal{C} := (Q_V\to N, R_P\to L, S_X\to M, Q\overset{\alpha}{\to}R\overset{\beta}{\to}S)$ such that β^{gp} is surjective.

For $n\in\mathbb{N}$, let $R(n)$ be the $(n+1)$-fold push-out of R over Q in the category of fine monoids and let $\alpha(n): Q\longrightarrow R(n)$, $\beta(n): R(n)\longrightarrow S$ be the monoid homomorphism defined by $q\mapsto(\alpha(q),1,\cdots,1), (r_1,\cdots,r_n)\mapsto\beta(r_1\cdots r_n)$, respectively. Then $\beta(n)^{\mathrm{gp}}$ is surjective and the diagram

$$(X,M)\overset{i(n)}{\hookrightarrow}(P(n),L(n))\longrightarrow(\mathrm{Spf}\,V,N)$$

admits the chart $\mathcal{C}(n) := (Q_V\to N, R(n)_{P(n)}\to L, S_X\to M, Q\overset{\alpha(n)}{\to}R(n)\overset{\beta(n)}{\to}S)$.

For $n\in\mathbb{N}$, put $\tilde{R}(n) := (\beta(n)^{\mathrm{gp}})^{-1}(S)$, $\tilde{P}(n) := P(n)\hat{\times}_{\mathrm{Spf}\,\mathbb{Z}_p\{R(n)\}}\mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}(n)\}$ and let $\tilde{L}(n)$ be the log structure on $\tilde{P}(n)$ defined as the pull-back of the canonical log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}(n)\}$. Then the closed immersion $i(n)$ factors as

$$(X,M)\overset{\tilde{i}(n)}{\hookrightarrow}(\tilde{P}(n),\tilde{L}(n))\overset{f'(n)}{\longrightarrow}(P(n),L(n)),$$

where $\tilde{i}(n)$ is an exact closed immersion and $f'(n)$ is formally log etale.

Let us note that $\tilde{P}(n) := ((\tilde{P}(n),\tilde{L}(n)),(X,M),\tilde{i}(n),\mathrm{id})$ is an exact widening. Let $\{T_{X,m}(\tilde{P}(n))\}$ be the system of universal enlargements of $\tilde{P}(n)$. Then the projections $p'_i\,(i=1,2)$, $p'_{ij}\,(1\le i<j\le 3)$ and the diagonal morphism Δ' induce the morphisms of enlargements

$$\tilde{p}_{i,m}: T_{X,m}(\tilde{P}(1))\longrightarrow T_{X,m}(\tilde{P}(0))\quad(i=1,2),$$
$$\tilde{p}_{ij,m}: T_{X,m}(\tilde{P}(2))\longrightarrow T_{X,m}(\tilde{P}(1))\quad(1\le i<j\le 3),$$
$$\tilde{\Delta}_m: T_{X,m}(\tilde{P}(0))\longrightarrow T_{X,m}(\tilde{P}(1)).$$

Let $\mathrm{Str}'((X,M) \hookrightarrow (P,L)/(\mathrm{Spf}\, V, N))$ be the category of compatible family of isocoherent sheaves E_m on $T_{X,m}(\tilde{P}(0))$ endowed with compatible isomorphisms

$$\epsilon_m : \tilde{p}_{2,m}^{*} E \xrightarrow{\sim} \tilde{p}_{1,m}^{*} E$$

satisfying $\tilde{\Delta}_m^*(\epsilon_m) = \mathrm{id}$, $\tilde{p}_{12,m}^*(\epsilon_m) \circ \tilde{p}_{23,m}^*(\epsilon_m) = \tilde{p}_{13,m}^*(\epsilon_m)$. Then we have the following:

Claim. We have the canonical equivalence of categories

$$\text{(2.2.2)} \quad I_{\mathrm{conv,et}}((X,M)/(\mathrm{Spf}\, V, N)) \xrightarrow{\sim} \mathrm{Str}'((X,M) \hookrightarrow (P,L)/(\mathrm{Spf}\, V, N))$$

defined by $E \mapsto (\{E_{T_{X,m}(\tilde{P}(0))}\}, \{\tilde{p}_{2,m}^*(E_{T_{X,m}(\tilde{P}(0))}) \xrightarrow{\sim} E_{T_{X,m}(\tilde{P}(1))} \xleftarrow{\sim} \tilde{p}_{1,m}^*(E_{T_{X,m}(\tilde{P}(0))})\})$.

Proof of Claim. Note that we have the canonical equivalence

$$\begin{aligned} &\mathrm{Str}'((X,M) \hookrightarrow (P,L)/(\mathrm{Spf}\, V, N)) \\ &\quad \simeq \mathrm{Str}'((X,M) \hookrightarrow (\tilde{P}(0), \tilde{L}(0))/(\mathrm{Spf}\, V, N)), \end{aligned}$$

and the equivalence between the categories $\mathrm{Str}'((X,M) \hookrightarrow (\tilde{P}(0), \tilde{L}(0))/(\mathrm{Spf}\, V, N))$ and $I_{\mathrm{conv,et}}((X,M)/(\mathrm{Spf}\, V, N))$ (via the functor as above) is already shown in [Shi, (5.2.6)]. So we are done. □

Now we construct the equivalence of categories

$$\text{(2.2.3)} \quad \begin{aligned} &\mathrm{Str}'((X,M) \hookrightarrow (P,L)/(\mathrm{Spf}\, V, N)) \\ &\quad \xrightarrow{\sim} \mathrm{Str}''((X,M) \hookrightarrow (P,L)/(\mathrm{Spf}\, V, N)). \end{aligned}$$

To construct it, we need the following claim:

Claim. There exists a canonical and functorial equivalence of categories

$$\Phi : \begin{pmatrix} \text{compatible family of} \\ \text{isocoherent sheaves on} \\ \{T_{X,m}(\tilde{P}(n))\}_m \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{coherent} \\ \mathcal{O}_{]X[^{\log}_{P(n)}}\text{-module} \end{pmatrix}.$$

Proof of Claim. Since both sides admit the descent property for Zariski covering of $P(n)$, we may assume that $P(n)$ is affine. Then $T_{X,m}(\tilde{P}(n))$ is also affine. Put $T_{X,m}(\tilde{P}(n)) := \mathrm{Spf}\, A_m$. Then, by the explicit description of the ring A_m which we gave in the previous section, we have the isomorphism

$$\mathrm{Spm}\,(K \otimes_V A_m) \overset{\sim}{=}]X[_{\tilde{P}(n),|\pi|^{1/m}}, \tag{2.2.4}$$

and the morphism of rigid analytic spaces

$$\mathrm{Spm}\,(K \otimes_V A_m) \longrightarrow \mathrm{Spm}\,(K \otimes_V A_{m+1})$$

corresponds to the natural inclusion

$$]X[_{\tilde{P}(n),|\pi|^{1/m}} \hookrightarrow]X[_{\tilde{P}(n),|\pi|^{1/(m+1)}}$$

via the isomorphism (2.2.4). So we have the equivalences of categories

$$\begin{aligned}\begin{pmatrix}\text{compatible family of}\\ \text{isocoherent sheaves on}\\ \{T_{X,m}(\tilde{P}(n))\}_m\end{pmatrix} &\simeq \begin{pmatrix}\text{compatible family of}\\ \text{finitely generated}\\ K \otimes_V A_m\text{-modules}\end{pmatrix}\\ &\simeq \begin{pmatrix}\text{compatible family of}\\ \text{coherent } \mathcal{O}_{]X[_{\tilde{P}(n),|\pi|^{1/m}}}\text{-modules}\end{pmatrix}\\ &\simeq \begin{pmatrix}\text{coherent}\\ \mathcal{O}_{]X[_{\tilde{P}(n)}}\text{-module}\end{pmatrix}.\end{aligned}$$

By definition of tubular neighborhood, we have $]X[^{\log}_{P(n)} =]X[_{\tilde{P}(n)}$. So the proof of the claim is finished. □

We construct the functor (2.2.3) by $(\{E_m\}, \{\epsilon_m\}) \mapsto (\Phi(\{E_m\}), \Phi(\{\epsilon_m\}))$. Then, by the claim, this functor is an equivalence of categories. Combining the functors (2.2.2) and (2.2.3), we obtain the functor

$$I_{\mathrm{conv,et}}((X, M)/(\mathrm{Spf}\, V, N)) \overset{\sim}{\longrightarrow} \mathrm{Str}''((X, M) \hookrightarrow (P, L)/(\mathrm{Spf}\, V, N))$$

giving the equivalence of categories. One can check that this functor is functorial with respect to the diagram

$$(X, M) \hookrightarrow (P, L) \longrightarrow (\mathrm{Spf}\, V, N).$$

So the proof of the proposition is finished. □

Let $(X, M) \hookrightarrow (P, L) \longrightarrow (\mathrm{Spf}\, V, N)$ be as above. For $n \in \mathbb{N}$, let (P^n, L^n) be the n-th log infinitesimal neighborhood of (P, L) in $(P(1), L(1))$. Since (X, M) and $(P(1), L(1))$ are of Zariski type, one can see that (P^n, L^n) is also of Zariski type. So the closed immersion $(X, M) \hookrightarrow (P^n, L^n)$ is in the category $\mathcal{Z}$ and hence we can define the tubular neighborhood $]X[^{\log}_{P^n}$. Let

$$\tau_n : (P^n, L^n) \longrightarrow (P^n, L^n) \quad (n \in \mathbb{N}),$$
$$\delta_{m,n} : (P^m, L^m) \times_{(P,L)} (P^n, L^n) \longrightarrow (P^{m+n}, L^{m+n}) \quad (m, n \in \mathbb{N})$$

be the morphisms defined in [Shi, §3.2]. Let us define the data

$$\mathcal{X} := (\mathcal{O}, \{\mathcal{P}^n\}_{n\in\mathbb{N}}, \{p_{1,n}\}_{n\in\mathbb{N}}, \{p_{2,n}\}_{n\in\mathbb{N}}, \{\pi_n\}_{n\in\mathbb{N}}, \{\delta'_{n,m}\}_{n,m\in\mathbb{N}}, \{\tau'_n\}_{n\in\mathbb{N}})$$

as follows: Let $\mathcal{O} := \mathcal{O}_{]X[^{\log}_P}$ and $\mathcal{P}^n := \mathcal{O}_{]X[^{\log}_{P^n}}$. (Since $]X[^{\log}_{P^n}$ is homeomorphic to $]X[^{\log}_P$, we can regard $\mathcal{P}^n$ as a sheaf on $]X[^{\log}_P$.) Let $p_{i,n}\, (i = 1, 2, n \in \mathbb{N})$ be the homomorphism $\mathcal{O} \longrightarrow \mathcal{P}^n$ corresponding to the morphism $]X[^{\log}_{P^n} \longrightarrow]X[^{\log}_P$ induced by the i-th projection $(P^n, L^n) \longrightarrow (P, L)$, let π_n be the homomorphism $\mathcal{P}^n \longrightarrow \mathcal{O}$ corresponding to the morphism $]X[^{\log}_P \longrightarrow]X[^{\log}_{P^n}$ induced by the closed immersion $(P, L) \hookrightarrow (P^n, L^n)$, let τ'_n be the homomorphism $\mathcal{P}^n \longrightarrow \mathcal{P}^n$ corresponding to the morphism $]X[^{\log}_{P^n} \longrightarrow]X[^{\log}_{P^n}$ induced by τ_n above and let $\delta'_{m,n} : \mathcal{P}^{m+n} \longrightarrow \mathcal{P}^m \otimes_{\mathcal{O}} \mathcal{P}^n$ be the morphism corresponding to the morphism $]X[^{\log}_{P^m \times_P P^n} \longrightarrow]X[^{\log}_{P^{m+n}}$ induced by $\delta_{m,n}$ above. Then we have the following:

Lemma 2.2.8. *The data $\mathcal{X}$ is a differentially log smooth formal groupoid of characteristic zero on the topos associated to $]X[^{\log}_P$.*

Proof. It is easy to check that the data $\mathcal{X}$ is a formal groupoid of characteristic zero. We prove that it is differentially log smooth.

One can see easily that it suffices to prove the following claim: Zariski locally on P, there exists an integer m and elements $\{\xi_{j,n}\}_{j=1}^{m}$ of $\mathcal{O}_{P^n}$ which satisfy the following conditions:

(1) For $n' > n$, the transition map $\mathcal{O}_{P^{n'}} \longrightarrow \mathcal{O}_{P^n}$ sends $\xi_{j,n'}$ to $\xi_{j,n}$.

(2) There exists the canonical isomorphism of left $\mathcal{O}_P$-algebras

$$\mathcal{O}_{P^n} \overset{\sim}{=} \mathcal{O}_P[\xi_{j,n}\,(1 \le j \le m)]/(I_n)^{n+1},$$

where $I_n := (\xi_{1,n}, \cdots, \xi_{m,n}) \subset \mathcal{O}_P[\xi_{j,n}\,(1 \le j \le m)]$.

(3) $\delta^*_{m,n}(\xi_{j,m+n}+1) = (\xi_{j,m}+1) \otimes (\xi_{j,n}+1)$ holds.

Since (P, L) is of Zariski type, we may assume that (P, L) admits a chart $\varphi : R_P \longrightarrow L$ to prove the claim. Let $\omega^1_{P/V}$ be the formal log differential module of (P, L) over $(\mathrm{Spf}\, V, N)$.

Let us note that we can reduce to the following claim: Zarisi locally on P, there exists elements $r_1, \cdots, r_m \in R^{\mathrm{gp}}$ such that $\mathrm{dlog}\, r_1, \cdots, \mathrm{dlog}\, r_m$ form a basis of $\omega^1_{P/V}$. Indeed, we have defined, in [Shi, §3.2], the compatible family of elements $(x^{-1}, x)_n \in \mathcal{O}^{\times}_{P^n}$ $(n \in \mathbb{N})$ for $x \in L$ satisfying $(x^{-1}, x)_1 - 1 = \mathrm{dlog}\, x$ and $\delta^*_{m+n}((x^{-1}, x)_{m+n}) := (x^{-1}, x)_m \otimes (x^{-1}, x)_n$. For $s = s_1 s_2^{-1} \in R^{\mathrm{gp}}$ $(s_1, s_2 \in R)$, define $(s^{-1}, s)_n := (s_1^{-1}, s_1)_n (s_2^{-1}, s_2)_n^{-1}$ (it is well-defined) and put $\xi_{j,n} := (r_j^{-1}, r_j)_n - 1$. Then the elements $\{\xi_{j,n}\}_{j=1}^{m}$ satisfy the desired conditions: The conditions (1) and (3) are easy to see and the condition (2) can be verified in the same way as the proof of [Shi, (3.2.7) (1)].

Now we prove the claim in the previous paragraph. Since $\omega^1_{P/V}$ is locally free, we may replace $(P, L) \longrightarrow (\mathrm{Spf}\, V, N)$ by $(P \times_{\mathrm{Spf}\, V} \mathrm{Spec}\, k, L) \longrightarrow (\mathrm{Spec}\, k, N)$ to prove the claim. Put $P_1 := P \times_{\mathrm{Spf}\, V} \mathrm{Spec}\, k$. By [Kk, (1.7)], the homomorphism

$$\mathcal{O}_{P_1} \otimes_{\mathbb{Z}} (L^{\mathrm{gp}}/N^{\mathrm{gp}}) \longrightarrow \omega^1_{P_1/k}; \quad a \otimes b \mapsto a \mathrm{dlog}\, b$$

is surjective on $P_{1,\mathrm{et}}$. On the other hand, the homomorphism

$$\varphi^{\mathrm{gp}} : R^{\mathrm{gp}}_{P_1} \longrightarrow L^{\mathrm{gp}}/N^{\mathrm{gp}}$$

is also surjective on $P_{1,\mathrm{et}}$. So the composite

$$\mathcal{O}_{P_1} \otimes_{\mathbb{Z}} R^{\mathrm{gp}}_{P_1} \overset{\mathrm{id} \otimes \varphi^{\mathrm{gp}}}{\longrightarrow} \mathcal{O}_{P_1} \otimes_{\mathbb{Z}} (L^{\mathrm{gp}}/N^{\mathrm{gp}}) \longrightarrow \omega^1_{P_1/k},$$

which we will denote by h, is surjective on $P_{1,\mathrm{et}}$. Since the sheaves $\mathcal{O}_{P_1} \otimes_{\mathbb{Z}} R^{\mathrm{gp}}_{P_1}$ and $\omega^1_{P_1/k}$ are coherent, h is surjective as homomorphism of sheaves on $P_{1,\mathrm{Zar}}$. Let x be a point in P_1. Then

$$\mathrm{id} \otimes h : \kappa(x) \otimes_{\mathbb{Z}} R^{\mathrm{gp}} \longrightarrow \kappa(x) \otimes_{\mathcal{O}_{P_1}} \omega^1_{P_1/k}$$

is surjective. Let $r_1, \cdots, r_m \in R^{\mathrm{gp}}$ be elements such that the images of $1 \otimes r_i$'s by $\mathrm{id} \otimes h$ form a basis of $\kappa(x) \otimes_{\mathcal{O}_{P_1}} \omega^1_{P_1/k}$. Then, by Nakayama's lemma, there exists a Zariski neighborhood U of x such that the images of $1 \otimes r_i$'s by $h|_U$ form a basis of $\omega^1_{P_1/k}|_U$. So the claim is proved. □

Note that the canonical morphism $(P^n, L^n) \longrightarrow (P(1), L(1))$ induces the morphism of rigid analytic spaces $]X[^{\log}_{P^n} \longrightarrow]X[^{\log}_{P(1)}$, which we denote by Δ_n. Denote the homomorphism

$$\Delta_n^{-1}\mathcal{O}_{]X[^{\log}_{P(1)}} \longrightarrow \mathcal{O}_{]X[^{\log}_{P^n}}$$

by θ_n.

Now let $\mathcal{E}$ be an object in $I_{\mathrm{conv,et}}((X/V)^{\log})$ and let (E, ϵ) be the corresponding object in $\mathrm{Str}''((X, M) \hookrightarrow (P, L)/(\mathrm{Spf}\, V, N))$. Then we define the $\mathcal{P}^n$-linear isomorphism $\epsilon_n : \mathcal{P}^n \otimes E \xrightarrow{\sim} E \otimes \mathcal{P}^n$ as the composite

$$\begin{aligned}\mathcal{P}^n \otimes E = \mathcal{P}^n \otimes_{\theta_n, \Delta_n^{-1}\mathcal{O}_{]X[^{\log}_{P(1)}}} \Delta_n^{-1} p_2^* E &\overset{\mathrm{id}\otimes\Delta_n^{-1}\epsilon}{\longrightarrow} \mathcal{P}^n \otimes_{\theta_n, \Delta_n^{-1}\mathcal{O}_{]X[^{\log}_{P(1)}}} \Delta_n^{-1} p_1^* E \\ &= E \otimes \mathcal{P}^n.\end{aligned}$$

Then $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a stratification on E with respect to the formal groupoid $\mathcal{X}$. Hence, by Proposition 1.2.7, $\{\epsilon_n\}_n$ defines an integrable connection

$$\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{]X[^{\log}_P}} \omega^1_{]X[^{\log}_P},$$

where $\omega^1_{]X[^{\log}_P} := \mathrm{Ker}(\mathcal{P}^1 \longrightarrow \mathcal{O})$. (Note that $\omega^1_{]X[^{\log}_P}$ is the restriction of the coherent sheaf $\omega^1_{P_K}$ on P_K which corresponds to the isocoherent sheaf $K \otimes_V \omega^1_{P/V}$ via the equivalence of categories

$$\begin{pmatrix}\text{isocoherent sheaves} \\ \text{on } P\end{pmatrix} \simeq \begin{pmatrix}\text{coherent sheaves} \\ \text{on } P_K\end{pmatrix}.)$$

Then we define the log de Rham complex on $]X[^{\log}_P$ associated to the isocrystal $\mathcal{E}$ by the complex

$$\begin{aligned}&\mathrm{DR}(]X[^{\log}_P, \mathcal{E}) \\ &\quad := [0 \to E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_{]X[^{\log}_P}} \omega^1_{]X[^{\log}_P} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} E \otimes_{\mathcal{O}_{]X[^{\log}_P}} \omega^q_{]X[^{\log}_P} \xrightarrow{\nabla} \cdots],\end{aligned}$$

where $\omega^q_{]X[^{\log}_P}$ is the q-th exterior power of $\omega^1_{]X[^{\log}_P}$ over $\mathcal{O}_{]X[^{\log}_P}$ and we extend ∇ to

$$E \otimes_{\mathcal{O}_{]X[^{\log}_P}} \omega^q_{]X[^{\log}_P} \longrightarrow E \otimes_{\mathcal{O}_{]X[^{\log}_P}} \omega^{q+1}_{]X[^{\log}_P}$$

by setting

$$x \otimes \eta \mapsto \nabla(x) \wedge \eta + x \otimes d\eta.$$

Remark 2.2.9. Let $(X, M) \overset{i}{\hookrightarrow} (P, L) \xrightarrow{g} (\mathrm{Spf}\, V, N)$ be as above and assume that this diagram admits a chart $(Q_V \to N, R_P \to L, S_X \to M, Q \to R \to S)$ such that $R^{\mathrm{gp}} \to S^{\mathrm{gp}}$ is surjective. Let $\mathcal{E}$ be an isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$. In this remark, we give a description of the complex $\mathrm{sp}_*\mathrm{DR}(]X[^{\log}_P, \mathcal{E})$ which we need later.

Let $(\tilde{P}, \tilde{L})$ be $(\tilde{P}(0), \tilde{L}(0))$ in the proof of Proposition 2.2.7, and let $(\tilde{P}^1, \tilde{L}^1)$ be the first log infinitesimal neighborhood of $(\tilde{P}, \tilde{L})$ in $(\tilde{P}, \tilde{L}) \hat{\times}_{(\mathrm{Spf}\, V, N)} (\tilde{P}, \tilde{L})$. Let $q_i : (\tilde{P}^1, \tilde{L}^1) \longrightarrow (\tilde{P}, \tilde{L})\,(i = 1, 2)$ be the i-th projection and put $X_i := q_i^{-1}(X)$. Then we have the following commutative diagram of exact pre-widenings for $i = 1, 2$:

$$\begin{array}{ccc} ((\tilde{P}^1, \tilde{L}^1), (X, M)) & \xrightarrow{(\mathrm{id},\mathrm{incl.})} & ((\tilde{P}^1, \tilde{L}^1), (X_i, \tilde{L}^1)) \\ {\scriptstyle (q_i,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle (q_i,q_i)} \\ ((\tilde{P}, \tilde{L}), (X, M)) & = & ((\tilde{P}, \tilde{L}), (X, M)). \end{array}$$

Then, for $m \in \mathbb{N}$, we have the associated diagram of enlargements:

$$\begin{array}{ccc} T_{X,m}(\tilde{P}^1) & \xrightarrow{d_{i,m}} & T_{X_i,m}(\tilde{P}^1) \\ \downarrow & & \downarrow \\ T_{X,m}(\tilde{P}) & = & T_{X,m}(\tilde{P}). \end{array}$$

Note that, as formal V-schemes, the objects in the above diagram are all homeomorphic to X. So, from now on in this remark, we regard the sheaves on them as sheaves on X_{Zar}.

By Lemma 2.1.26, we have the isomorphism

$$T_{X,m}(\tilde{P}) \times_{\tilde{P}_{\mathrm{fl}},(q_i)_{\mathrm{fl}}} \tilde{P}^1_{\mathrm{fl}} \xrightarrow{\simeq} T_{X_i,m}(\tilde{P}_{\mathrm{fl}}).$$

So we have

$$\text{(2.2.5)} \qquad \begin{cases} \mathcal{E}_{T_{X_1,m}(\tilde{P}^1)} \xrightarrow{\simeq} \mathcal{E}_{T_{X,m}(\tilde{P})} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{O}_{\tilde{P}^1}, \\ \mathcal{E}_{T_{X_2,m}(\tilde{P}^1)} \xrightarrow{\simeq} \mathcal{O}_{\tilde{P}^1} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{E}_{T_{X,m}(\tilde{P})}. \end{cases}$$

On the other hand, by Proposition 2.1.27, we have morphisms of formal schemes

$$T_{X_1,m-1}(\tilde{P}^1) \xrightarrow{h_m} T_{X,m}(\tilde{P}^1) \xrightarrow{d_{2,m}} T_{X_2,m}(\tilde{P}^1),$$

such that $\{h_m\}_m$ and $\{d_{2,m}\}_m$ are isomorphisms as homomorphisms of inductive system of formal schemes, and we have a system of homomorphisms

$$\varphi_m : \mathcal{E}_{T_{X_2,m}(\tilde{P}^1)} \longrightarrow \mathcal{E}_{T_{X_1,m-1}(\tilde{P}^1)}$$

induced by the composite $d_{2,m} \circ h_m$ such that $\varprojlim_m \varphi_m$ is an isomorphism. Via the isomorphisms (2.2.5), φ_m induce the homomorphism

$$\theta_m : \mathcal{O}_{\tilde{P}^1} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{E}_{T_{X,m}(\tilde{P})} \longrightarrow \mathcal{E}_{T_{X,m-1}(\tilde{P})} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{O}_{\tilde{P}^1}.$$

Note that $(d_{2,m} \circ h_m) \otimes_{\mathcal{O}_{\tilde{P}^1}} \mathcal{O}_{\tilde{P}}$ coincides with the canonical transition morphism $T_{X,m-1}(\tilde{P}) \longrightarrow T_{X,m}(\tilde{P})$. So θ_m reduces to the canonical transition $\mathcal{E}_{T_{X,m}(\tilde{P})} \longrightarrow \mathcal{E}_{T_{X,m-1}(\tilde{P})}$ when we consider modulo $K \otimes_V \omega^1_{\tilde{P}/V}$. Let us define

$$\nabla_m : \mathcal{E}_{T_{X,m}(\tilde{P})} \longrightarrow \mathcal{E}_{T_{X,m-1}(\tilde{P})} \otimes_{\mathcal{O}_{\tilde{P}}} \omega^1_{\tilde{P}/V} = \mathcal{E}_{T_{X,m-1}(\tilde{P})} \otimes_{\mathcal{O}_P} \omega^1_{P/V}$$

by $\nabla_m(e) := \theta_m(1 \otimes e) - e \otimes 1$. Put $E' := \varprojlim_m \mathcal{E}_{T_{X,m}(\tilde{P})}$. Then ∇_m's define a homomorphism

$$\nabla := \varprojlim_m \nabla_m : E' \longrightarrow E' \otimes_{\mathcal{O}_P} \omega^1_{P/V}.$$

We extend it to the diagram

$$\mathrm{DR} := [0 \to E' \xrightarrow{\nabla} E' \otimes_{\mathcal{O}_P} \omega^1_{P/V} \xrightarrow{\nabla} \cdots E' \otimes_{\mathcal{O}_P} \omega^q_{P/V} \xrightarrow{\nabla} \cdots]$$

by extending ∇ to

$$E' \otimes_{\mathcal{O}_P} \omega^q_{P/V} \longrightarrow E' \otimes_{\mathcal{O}_P} \omega^{q+1}_{P/V}$$

by setting $x \otimes \eta \mapsto \nabla(x) \wedge \eta + x \otimes d\eta$. Then we have the following:

Claim The diagram DR forms a complex and it is identical with $\mathrm{sp}_*\mathrm{DR}(]X[^{\log}_P, \mathcal{E})$.

Proof of Claim. Here we only sketch the outline of proof. The details are left to the reader as an exercise.

Let (E, ϵ) be the object in $\mathrm{Str}''((X, M) \hookrightarrow (P, L)/(\mathrm{Spf}\, V, N))$ corresponding to $\mathcal{E}$ and let $\epsilon_1 : \overline{p}_2^* E \xrightarrow{\sim} \overline{p}_1^* E$ (where $\overline{p}_i :]X[^{\log}_{P^1} \longrightarrow]X[^{\log}_P$ is the i-th projection) be the pull-back of ϵ to $]X[^{\log}_{P^1}$. Then it suffices to prove that $\varprojlim_m \theta_m$ is canonically identical with $\mathrm{sp}_*\epsilon_1$. By definition, $\mathrm{sp}_*\epsilon_1$ is defined as the projective limit (with respect to m) of the compatible family of diagrams

$$\mathcal{O}_{\tilde{P}^1} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{E}_{T_{X,m}(\tilde{P})} \longrightarrow \mathcal{E}_{T_{X,m}(\tilde{P}^1)} \longleftarrow \mathcal{E}_{T_{X,m}(\tilde{P})} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{O}_{\tilde{P}^1},$$

and this diagram fits into the upper horizontal line of the following diagram:

$$\begin{array}{ccccc}
\mathcal{O}_{\tilde{P}^1} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{E}_{T_{X,m}(\tilde{P}^1)} & \longrightarrow & \mathcal{E}_{T_{X,m}(\tilde{P})} & \longleftarrow & \mathcal{E}_{T_{X,m}(\tilde{P})} \otimes_{\mathcal{O}_{\tilde{P}}} \mathcal{O}_{\tilde{P}^1} \\
\sim\downarrow & & \alpha\downarrow & & \sim\downarrow \\
\mathcal{E}_{T_{X_2,m}(\tilde{P}^1)} & \xrightarrow{\varphi_m} & \mathcal{E}_{T_{X_1,m-1}(\tilde{P}^1)} & \longleftarrow & \mathcal{E}_{T_{X_1,m}(\tilde{P}^1)},
\end{array}$$

where α is the homomorphism induced by h_m (by Proposition 2.1.27). By taking the inverse limit of this diagram with respect to m, one can see that $\varprojlim_m \theta_m$ is canonically identical with $\mathrm{sp}_*\epsilon_1$. $\square$

Now we give the definition of analytic cohomology of log schemes which has a locally free isocrystal on log convergent site as coefficient. Assume we are given the following diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N),$$

where f is a morphism of fine log schemes of finite type, N is a fine log structure on $\operatorname{Spf} V$ and ι is the canonical exact closed immersion. Assume moreover that $(\operatorname{Spf} V, N)$ admits a chart $\varphi : Q_V \to N$. First we introduce the notion of a good embedding system:

Definition 2.2.10. Let the notations be as above. A good embedding system of (X, M) over $(\operatorname{Spf} V, N)$ is a diagram

$$(X, M) \xleftarrow{g} (X^{(\bullet)}, M^{(\bullet)}) \overset{i}{\hookrightarrow} (P^{(\bullet)}, L^{(\bullet)}),$$

where $(X^{(\bullet)}, M^{(\bullet)})$ is a simplicial fine log scheme over (X, M) such that each $(X^{(n)}, M^{(n)})$ is of finite type over k and of Zariski type, $(P^{(\bullet)}, L^{(\bullet)})$ is a simplicial fine log formal V-scheme over $(SpfV, N)$ such that each $(P^{(n)}, L^{(n)})$ is formally log smooth over $(\operatorname{Spf} V, N)$ and of Zariski type, $g := \{g^{(n)}\}_n : X^{(\bullet)} \longrightarrow X$ is an etale hypercovering such that $g^{(n),*}M \longrightarrow M^{(n)}$ is isomorphic for any $n \in \mathbb{N}$ and $i := \{i^{(n)}\}_n$ is a morphism of simplicial fine log formal V-schemes such that each $i^{(n)}$ is a locally closed immersion.

As for the existence of a good embedding system, we have the following:

Proposition 2.2.11. *Let* $(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N)$ *be as above.* (*Note that we have assumed that* $(\operatorname{Spf} V, N)$ *admits a chart* $\varphi : Q_V \to N$.) *Then there exists at least one good embedding system of* (X, M) *over* $(\operatorname{Spf} V, N)$.

Proof. First, let $g^{(0)} : X^{(0)} := \coprod_{i \in I} X_i \longrightarrow X$ be an etale covering with $|I| < \infty$ such that each X_i is affine of finite type over k and that $\iota \circ f \circ g^{(0)}|_{X_i} : (X_i, M) \longrightarrow (\operatorname{Spf} V, N)$ has a chart $\mathcal{C}_i := (Q_V \xrightarrow{\varphi} N, R_{i,X_i} \to M, Q \xrightarrow{\psi} R_i)$ extending φ. Let us take surjections

$$\alpha_i : k[\mathbb{N}^{n_i}] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}),$$
$$\beta_i : \mathbb{N}^{m_i} \longrightarrow R_i,$$

and denote the composite

$$Q \xrightarrow{\Gamma(V,\varphi)} \Gamma(V, N) \longrightarrow \Gamma(V, \mathcal{O}_V) = V,$$
$$R_i \xrightarrow{\Gamma(X_i,\psi)} \Gamma(X_i, M) \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$$

by γ, δ_i respectively.

Let P_i be $\mathrm{Spf}\, V\{\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}\}$ and let L_i be the log structure on P_i which is associated to the pre-log structure

$$\epsilon_i : Q \oplus \mathbb{N}^{m_i} \longrightarrow V\{\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}\}, \quad (q, x) \mapsto \gamma(q)x.$$

Then (P_i, L_i) is a fine log formal V-scheme which is formally log smooth over $(\mathrm{Spf}\, V, N)$. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \Gamma(X_i, \mathcal{O}_{X_i}) & \longleftarrow & V\{\mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}\} \\ {\scriptstyle \delta_i}\uparrow & & {\scriptstyle \epsilon_i}\uparrow \\ R_i & \longleftarrow & Q \oplus \mathbb{N}^{m_i}, \end{array}$$

where the upper horizontal arrow is defined by

$$v \cdot (x, y) \mapsto (v \bmod \pi) \cdot \alpha(x) \cdot \delta_i \circ \beta_i(y) \quad (v \in V, (x, y) \in \mathbb{N}^{n_i} \oplus \mathbb{N}^{m_i}),$$

and the lower horizontal arrow is defined by

$$(q, x) \mapsto \psi(q)\beta_i(x) \quad (q \in Q, x \in \mathbb{N}^{m_i}).$$

This diagram induces the closed immersion

$$j_i : (X_i, M) \hookrightarrow (P_i, L_i)$$

over $(\mathrm{Spf}\, V, N)$. Now put $(X^{(0)}, M^{(0)}) := (\coprod_{i\in I} X_i, M|_{\coprod_{i\in I} X_i})$, $(P^{(0)}, L^{(0)}) := \coprod_{i\in I}(P_i, L_i)$ and let $i^{(0)} := \coprod_{i\in I} j_i$. For $n \in \mathbb{N}$, let $(X^{(n)}, M^{(n)})$ be the $(n+1)$-fold fiber product of $(X^{(0)}, M^{(0)})$ over (X, M), let $(P^{(n)}, L^{(n)})$ be the $(n+1)$-fold fiber product of $(P^{(0)}, L^{(0)})$ over $(\mathrm{Spf}\, V, N)$ and let $i^{(n)} : (X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})$ be the closed immersion defined by the $(n+1)$-fold fiber product of $i^{(0)}$. Then, for each n, $(X^{(n)}, M^{(n)})$ and

$(P^{(n)}, L^{(n)})$ admit charts. (Attention: we do not say that $i^{(n)}$ admits a chart.) So they are of Zariski type. Hence the diagram

$$(X, M) \longleftarrow (X^{(\bullet)}, M^{(\bullet)}) \overset{i^{(\bullet)}}{\hookrightarrow} (P^{(\bullet)}, L^{(\bullet)})$$

is a good embedding system. $\square$

Now we define the analytic cohomology of a log scheme which has a locally free isocrystal on log convergent site as coefficient, as follows:

Definition 2.2.12. Assume we are given the following diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N),$$

where f is a morphism of fine log formal schemes of finite type, N is a fine log structure on $\operatorname{Spf} V$ and ι is the canonical exact closed immersion. Assume moreover that $(\operatorname{Spf} V, N)$ admits a chart. Let $\mathcal{E}$ be a locally free isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$. Take a good embedding system

$$(X, M) \xleftarrow{g} (X^{(\bullet)}, M^{(\bullet)}) \overset{i}{\hookrightarrow} (P^{(\bullet)}, L^{(\bullet)})$$

of (X, M) over $(\operatorname{Spf} V, N)$ and let $\mathcal{E}^{(\bullet)}$ be the restriction of $\mathcal{E}$ to the site $(X^{(\bullet)}/V)^{\log}_{\mathrm{conv,et}}$. Denote the specialization map

$$]X^{(\bullet)}[^{\log}_{P^{(\bullet)}} \longrightarrow X^{(\bullet)}$$

by $\mathrm{sp}^{(\bullet)}$. Then we define the analytic cohomology of (X, M) over $(\operatorname{Spf} V, N)$ with coefficient $\mathcal{E}$ by

$$H^i_{\mathrm{an}}((X, M)/(\operatorname{Spf} V, N), \mathcal{E}) := H^i(X, Rg_* R\mathrm{sp}^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}^{(\bullet)})).$$

When there will be no confusions on log structures, we denote the analytic cohomology $H^i_{\mathrm{an}}((X, M)/(\operatorname{Spf} V, N), \mathcal{E})$ simply by $H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E})$.

Remark 2.2.13. By Theorem B of Kiehl, we have

$$R\mathrm{sp}^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}^{(\bullet)}) = \mathrm{sp}^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}^{(\bullet)}).$$

We should prove that the above definition is well-defined, that is, we should prove the following proposition:

Proposition 2.2.14. *Let the notations be as in Definition 2.2.12. Then the above definition of the analytic cohomology* $H^i_{\mathrm{an}}((X,M)/(\mathrm{Spf}\,V,N),\mathcal{E})$ *is independent of the choice of the good embedding system chosen above.*

First we prepare a lemma:

Lemma 2.2.15. *Let* $(X,M)\xrightarrow{f}(\mathrm{Spec}\,k,N)\overset{\iota}{\hookrightarrow}(\mathrm{Spf}\,V,N)$ *be as in Definition 2.2.12 and assume that* (X,M) *is of Zariski type. Assume moreover that we are given the commutative diagram*

$$\begin{array}{ccc} (X,M) & \xrightarrow{i_1} & (P_1,L_1) \\ \| & & \downarrow{\scriptstyle\varphi} \\ (X,M) & \xrightarrow{i_2} & (P_2,L_2), \end{array}$$

where $i_j\,(j=1,2)$ *is a closed immersion over* $(\mathrm{Spf}\,V,N)$ *into a fine log formal* V*-scheme* (P_j,L_j) *of Zariski type and* φ *is a formally log smooth morphism. Let us denote the morphism of rigid analytic spaces*

$$]X[^{\log}_{P_1}\longrightarrow]X[^{\log}_{P_2}$$

induced by φ *by* φ_K*. Then, for a locally free isocrystal* $\mathcal{E}$ *on* $(X/V)^{\log}_{\mathrm{conv,et}}$*, we have the isomorphism*

$$R\varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E})=\mathrm{DR}(]X[^{\log}_{P_2},\mathcal{E}).$$

Proof. Let x be a point of X. Then we have open neighborhoods $x\in U_x\subset X$, $x\in V_{j,x}\subset P_j\,(j=1,2)$ which satisfies the following conditions: $i_j(U_x)\subset V_{j,x}$ and $\varphi(V_{1,x})\subset V_{2,x}$ hold and the diagram

$$\begin{array}{ccc} (U_x,M) & \xrightarrow{i_1} & (V_{1,x},L_1) \\ \| & & \downarrow{\scriptstyle\varphi} \\ (U_x,M) & \xrightarrow{i_2} & (V_{2,x},L_2) \end{array}$$

admits a chart $(R_{j,V_{j,x}} \to L_j, S_{j,U_x} \to M\ (j=1,2), \mathcal{D})$, where $\mathcal{D}$ is the diagram of monoids

$$\begin{array}{ccc} S_1 & \xleftarrow{\alpha_1} & R_1 \\ \uparrow & & \uparrow \\ S_2 & \xleftarrow{\alpha_2} & R_2, \end{array}$$

such that the homomorphisms $R_j^{\mathrm{gp}} \to S_j^{\mathrm{gp}}\ (j=1,2)$ are surjective. By shrinking $V_{j,x}$, we may assume that $V_{j,x} \times_{P_j} X = U_x$ holds. We fix a triple $(U_x, V_{1,x}, V_{2,x})$ as above for each $x \in X$. Then there exist $x_1, \cdots, x_r \in X$ such that $X = \bigcup_{i=1}^r U_{x_i}$ holds. Then we can check that $]X[^{\log}_{P_j} = \bigcup_{i=1}^r]U_{x_i}[^{\log}_{V_{j,x_i}}\ (j=1,2)$ is an admissible covering and that $\varphi_K^{-1}(]U_{x_i}[^{\log}_{V_{2,x_i}}) =]U_{x_i}[^{\log}_{V_{1,x_i}}$ holds for $1 \le i \le r$. So we have

$$R\varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1}, \mathcal{E})|_{]U_{x_i}[^{\log}_{V_{2,x_i}}} = R(\varphi_K|_{]U_{x_i}[^{\log}_{V_{1,x_i}}})_*\mathrm{DR}(]U_{x_i}[^{\log}_{V_{1,x}}, \mathcal{E}).$$

By this isomorphism, we can see that we may replace X, P_1, P_2 by U_{x_i}, V_{1,x_i}, V_{2,x_i} respectively, that is, we may assume the existence of the chart $(R_{j,V_{j,x}} \to L_j, S_{j,U_x} \to M\ (j=1,2), \mathcal{D})$ (where $\mathcal{D}$ is as above) such that $R_j^{\mathrm{gp}} \to S_j^{\mathrm{gp}}\ (j=1,2)$ are surjective. So we assume it.

Put $\tilde{R}_2 := (\alpha_2^{\mathrm{gp}})^{-1}(S_2)$, $\tilde{P}_2 := P_2 \hat{\times}_{\mathrm{Spf}\,\mathbb{Z}_p\{R_2\}} \mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}_2\}$ and let $\tilde{L}_2$ be the pull-back of the canonical log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}_2\}$ to $\tilde{P}_2$. Next, put $\overline{R}_1 := (R_1 \oplus_{R_2} \tilde{R}_2)^{\mathrm{int}}$ and let $\beta : \overline{R}_1 \longrightarrow S_1$ be the homomorphism induced by $(\alpha_1 \oplus \alpha_2)^{\mathrm{gp}}$. Put $\overline{P}_1 := P_1 \hat{\times}_{\mathrm{Spf}\,\mathbb{Z}_p\{R_1\}} \mathrm{Spf}\,\mathbb{Z}_p\{\overline{R}_1\}$ and let $\overline{L}_1$ be the pull-back of the canonical log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{\overline{R}_1\}$ to $\overline{P}_1$. Finally, put $\tilde{R}_1 := (\beta^{\mathrm{gp}})^{-1}(S_1)$, $\tilde{P}_1 := \overline{P}_1 \hat{\times}_{\mathrm{Spf}\,\mathbb{Z}_p\{\overline{R}_1\}} \mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}_1\}$ and let $\tilde{L}_1$ be the pull-back of the canonical log structure on $\mathrm{Spf}\,\mathbb{Z}_p\{\tilde{R}_1\}$ to $\tilde{P}_1$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} (X,M) & \xrightarrow{i_1'} & (\tilde{P}_1, \tilde{L}_1) & \xrightarrow{h''} & (\overline{P}_1, \overline{L}_1) & \xrightarrow{h'} & (P_1, L_1) \\ \| & & \tilde{\varphi}\downarrow & & \downarrow & & \varphi\downarrow \\ (X,M) & \xrightarrow{i_2'} & (\tilde{P}_2, \tilde{L}_2) & = & (\tilde{P}_2, \tilde{L}_2) & \xrightarrow{h} & (P_2, L_2), \end{array}$$

satisfying the following conditions:

(1) The morphisms h, h', h'' are formally log etale.

(2) i'_1, i'_2 are exact closed immersions.
(3) The right square is Cartesian.

By the conditions (1) and (3), $\tilde{\varphi}$ is formally log smooth. Note that $\tilde{P}_{1,\tilde{\varphi}\text{-triv}}$ is an open sub formal scheme of $\tilde{P}_1$ which contains X. By replacing $\tilde{P}_1$ by $\tilde{P}_{1,\tilde{\varphi}\text{-triv}}$, we may assume further that $\tilde{\varphi}$ is formally smooth in classical sense. Then, we have $]X[^{\log}_{P_j}=]X[_{\tilde{P}_j}$ $(j=1,2)$ by definition of tubular neighborhood. By weak fibration theorem, we have $]X[_{\tilde{P}_1}\cong]X[_{\tilde{P}_2}\times D^m_V$ (for some m) locally on $\tilde{P}_2$. So we have the isomorphism

$$]X[^{\log}_{P_1} \cong\,]X[^{\log}_{P_2} \times D^m_V \tag{2.2.6}$$

locally. Morever, by construction of the above isomorphism (see [Be3]), one can see the following fact:

(∗) The sheaf $\omega^1_{]X[^{\log}_{P_1}}$ on $]X[^{\log}_{P_1}$ corresponds to the sheaf $\mathrm{pr}_1^*\omega^1_{]X[^{\log}_{P_2}}\oplus \mathrm{pr}_2^*\Omega^1_{D^m_V}$ on $]X[^{\log}_{P_2}\times D^m_V$ via the isomorphism (2.2.6).

Now we prove that the canonical homomorphism

$$\mathrm{DR}(]X[^{\log}_{P_2},\mathcal{E}) \longrightarrow R\varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E})$$

is a quasi-isomorphism. By the isomorphism (2.2.6) and Theorem B of Kiehl, we have

$$R\varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E}) = \varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E}).$$

Then, to prove the quasi-isomorphism

$$\mathrm{DR}(]X[^{\log}_{P_2},\mathcal{E}) \xrightarrow{\sim} \varphi_{K,*}\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E}),$$

it suffices to prove the quasi-isomorphism

$$\begin{aligned} &H^0(\mathrm{Spm}\,(K\otimes A),\mathrm{DR}(]X[^{\log}_{P_2},\mathcal{E})) \\ &\quad\xrightarrow{\sim} H^0(\varphi_K^{-1}(\mathrm{Spm}\,(K\otimes A)),\mathrm{DR}(]X[^{\log}_{P_1},\mathcal{E})) \end{aligned} \tag{2.2.7}$$

as complexes of modules for any admissible open $\mathrm{Spm}\,(K \otimes A) \subset]X[^{\log}_{P^1}$, where A is a p-adic topological ring which is topologically of finite type over V. We may assume moreover that the coherent sheaf E on $\mathrm{Spm}\,(K \otimes A)$ induced by $\mathcal{E}$ and the differential module $\Omega^q_{\mathrm{Spm}\,(K\otimes A)}$ are free and that $\varphi_K^{-1}(\mathrm{Spm}\,(K \otimes A))$ is isomorphic to D^m_A via the isomorphism (2.2.6).

Now we prove the quasi-isomorphism (2.2.7). Put $A_K := K \otimes A$ and for $n \in \mathbb{N}$, denote the ring $\Gamma(D^n_A, \mathcal{O}_{D^n_A})$ by $A(n)_K$. Put $\Omega^q_{A_K} := \Gamma(\mathrm{Spm}\, A_K, \Omega^q_{\mathrm{Spm}\, A_K})$, put

$$\begin{aligned}\Omega^1_{A(n)_K} &:= \Gamma(D^n_A, \Omega^1_{D^n_A})/A(n)_K \otimes_{A_K} \Omega^1_{A_K} \\ &= \bigoplus_{i=1}^{n} A(n)_K dt_i\end{aligned}$$

(where $t_1, \cdots, t_n$ are the coordinates of D^n_A) and let $\Omega^q_{A(n)_K}$ be the q-th exterior power of $\Omega^1_{A(n)_K}$ over $A(n)_K$. Then the left hand side of (2.2.7) has the form

$$0 \to E \xrightarrow{\nabla} E \otimes_{A_K} \Omega^1_{A_K} \xrightarrow{\nabla} E \otimes_{A_K} \Omega^2_{A_K} \xrightarrow{\nabla} \cdots,$$

where E is a free A_K-module of finite type. On the other hand, one can see, by the fact $(*)$, that the right hand side of (2.2.7) is isomorphic to the simple complex associated to the double complex $\{(E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^q_{A(m)_K}\}_{p,q}$, where the differential

$$(E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^q_{A(m)_K} \longrightarrow (E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^{q+1}_{A(m)_K}$$

is defined by $e \otimes \omega \otimes \eta \mapsto e \otimes \omega \otimes d\eta$ and the differential

$$(E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^q_{A(m)_K} \longrightarrow (E \otimes_{A_K} \Omega^{p+1}_{A_K}) \otimes \Omega^q_{A(m)_K}$$

is defined by $e \otimes \omega \otimes \eta \mapsto \nabla(e \otimes \omega) \otimes \eta$. So, to prove the quasi-isomorphism (2.2.7), it suffices to prove the complex

$$\begin{aligned}C^\bullet := [0 \to (E \otimes_{A_K} \Omega^p_{A_K}) \otimes A(m)_K \to (E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^1_{A(m)_K} \\ \to (E \otimes_{A_K} \Omega^p_{A_K}) \otimes \Omega^2_{A(m)_K} \to \cdots]\end{aligned}$$

satisfies the equations

$$H^q(C^\bullet) = \begin{cases} E \otimes_{A_K} \Omega^p_{A_K}, & q = 0, \\ 0, & q > 0. \end{cases} \tag{2.2.8}$$

Since $E \otimes \Omega^p_{A_K}$ is a free A_K-module, we may replace $E \otimes \Omega^p_{A_K}$ by A_K to prove the equation (2.2.8). In this case, the equation (2.2.8) is well-known. For reader's convenience, we prove it in the lemma below. (We use this lemma again in the proof of log convergent Poincaré lemma in the next section.) □

LEMMA 2.2.16. *Let A be a Noetherian p-adic topological ring which is topologically of finite type over V. Put $A_K := K \otimes A$ and for $n \in \mathbb{N}$, denote the ring $\Gamma(D^n_A, \mathcal{O}_{D^n_A})$ by $A(n)_K$. Put*

$$\begin{aligned} \Omega^1_{A(n)_K} &:= \Gamma(D^n_A, \Omega^1_{D^n_A})/A(n)_K \otimes_{A_K} \Omega^1_{A_K} \\ &= \bigoplus_{i=1}^n A(n)_K dt_i \end{aligned}$$

(where $t_1, \cdots, t_n$ are the coordinates of D^n_A) and let $\Omega^q_{A(n)_K}$ be the q-th exterior power of $\Omega^1_{A(n)_K}$ over $A(n)_K$. Let $C(A,n)$ be the relative de Rham complex

$$0 \to A(n)_K \to \Omega^1_{A(n)_K} \to \Omega^2_{A(n)_K} \to \cdots .$$

Then we have the equation

$$H^q(C(A,n)) = \begin{cases} A_K, & q = 0, \\ 0, & q > 0. \end{cases} \tag{2.2.9}$$

PROOF. Let us define a filtration (of Katz-Oda type)

$$F^pC(A,n) := [0 \to F^pA(n)_K \to F^p\Omega^1_{A(n)_K} \to F^p\Omega^2_{A(n)_K} \to \cdots] \subset C(A,n)$$

by

$$F^p\Omega^q_{A(n)_K} := \mathrm{Im}((A(n)_K \otimes_{A(n-1)_K} \Omega^p_{A(n-1)_K}) \otimes_{A(n)_K} \Omega^{q-p}_{A(n)_K} \longrightarrow \Omega^q_{A(n)_K}).$$

Put $\Omega^1_{A(n)_K/A(n-1)_K} := \Omega^1_{A(n)_K}/F^1\Omega^1_{A(n)_K} \cong A(n)_K dt_n$ and let $\overline{C}(A,n)$ be the complex $[A(n)_K \xrightarrow{d} \Omega^1_{A(n)_K/A(n-1)_K}]$. Then the filtration $F^pC(A,n)$ ($p \in \mathbb{N}$) induces the spectral sequence

$$E_1^{p,q} = H^q(\overline{C}(A,n)) \otimes_{A(n-1)_K} \Omega^p_{A(n-1)_K} \Longrightarrow H^{p+q}(C(A,n)).$$

So, if we prove the equation

$$\text{(2.2.10)} \qquad H^q(\overline{C}(A,n)) = \begin{cases} A(n-1)_K, & q=0, \\ 0, & q=1, \end{cases}$$

we can deduce the equation (2.2.9) by induction on n and the above spectral sequence. So the proof is reduced to the above equations. Since $\mathrm{Ker}(d) = \mathrm{Ker}(d : A(n-1)_K[[t_n]] \longrightarrow A(n-1)_K[[t_n]]dt_n) \cap A(n)_K$ holds, one can easily see that $H^0(\overline{C}(A,n)) = A(n-1)_K$ holds. On the other hand, for $\eta := (\sum_{i=1}^{\infty} f_i t_n^i)dt_n \in \Omega^1_{A(n)_K/A(n-1)_K}$ (where $f_i \in A(n-1)_K$), one can see that the element $g := \sum_{i=0}^{\infty} \frac{f_i}{i} t_n^i$ in $A(r-1)_K[[t_r]]$ is in fact contained in $A(r)_K$. Since we have $d(g) = \eta$, d is surjective, that is, we have $H^1(\overline{C}(A,n)) = 0$. So the proof of the lemma is finished. □

Now we give a proof of Proposition 2.2.14.

Proof of Proposition 2.2.14. Let

$$(X,M) \xleftarrow{g_j} (X_j^{(\bullet)}, M_j^{(\bullet)}) \overset{i_j^{(\bullet)}}{\hookrightarrow} (P_j^{(\bullet)}, L_j^{(\bullet)}) \quad (j=1,2)$$

be good embedding systems and denote the restriction of $\mathcal{E}$ to $(X_j^{(\bullet)}/V)^{\log}_{\mathrm{conv,et}}$ by $\mathcal{E}_j^{(\bullet)}$ ($j=1,2$). Denote the specialization map

$$]X_j^{(\bullet)}[^{\log}_{P_j^{(\bullet)}} \longrightarrow X_j^{(\bullet)} \quad (j=1,2)$$

by $\mathrm{sp}_j^{(\bullet)}$.

For $m,n \in \mathbb{N}$, put $X^{(m,n)} = X^{(m)} \times_X X^{(n)}$, $M^{(m,n)} := M|_{X^{(m,n)}}$ and let $(P^{(m,n)}, L^{(m,n)})$ be $(P_1^{(m)}, L_1^{(m)}) \hat{\times}_{(\mathrm{Spf}\, V,N)} (P_2^{(n)}, L_2^{(n)})$. Then $(X^{(\bullet,\bullet)}, M^{(\bullet,\bullet)})$ forms a bisimplicial fine log scheme over (X,M) and $(P^{(\bullet,\bullet)}, L^{(\bullet,\bullet)})$ forms a bisimplicial fine log formal V-scheme over $(\mathrm{Spf}\, V, N)$. Note that $(X^{(m,n)}, M^{(m,n)})$ and $(P^{(m,n)}, L^{(m,n)})$ are of Zariski type. Denote the structure morphism $(X^{(\bullet,\bullet)}, M^{(\bullet,\bullet)}) \longrightarrow (X, M)$ by g and denote the locally closed immersion $(X^{(\bullet,\bullet)}, M^{(\bullet,\bullet)}) \longrightarrow (P^{(\bullet,\bullet)}, L^{(\bullet,\bullet)})$ induced by $i_j^{(\bullet)}$'s by $i^{(\bullet,\bullet)}$. Denote the restriction of $\mathcal{E}$ to $(X^{(\bullet,\bullet)}/V)^{\log}_{\mathrm{conv,et}}$ by $\mathcal{E}^{(\bullet,\bullet)}$, and denote the specialization map

$$]X^{(\bullet,\bullet)}[^{\log}_{P^{(\bullet,\bullet)}} \longrightarrow X^{(\bullet,\bullet)}$$

by $\mathrm{sp}^{(\bullet,\bullet)}$. To prove the proposition, it suffices to show the isomorphisms

$$\begin{aligned} &Rg_{j,*}R\mathrm{sp}^{(\bullet)}_{j,*}\mathrm{DR}(]X_j^{(\bullet)}[^{\log}_{P_j^{(\bullet)}}, \mathcal{E}_j^{(\bullet)}) \\ &\quad \cong Rg_* R\mathrm{sp}^{(\bullet,\bullet)}_* \mathrm{DR}(]X^{(\bullet,\bullet)}[^{\log}_{P^{(\bullet,\bullet)}}, \mathcal{E}^{(\bullet,\bullet)}) \quad (j = 1, 2). \end{aligned}$$

It suffices to treat the case $j = 1$. For each $n \in \mathbb{N}$, $(X^{(n,\bullet)}, M^{(n,\bullet)})$ forms a simplicial scheme over $(X_1^{(n)}, M_1^{(n)})$. Let us denote the structure morphism $(X^{(n,\bullet)}, M^{(n,\bullet)}) \longrightarrow (X_1^{(n)}, M_1^{(n)})$ by g_n. To prove the above isomorphism (for $j = 1$), it suffices to prove the isomorphism

$$\begin{aligned} (2.2.11) \qquad &Rg_{n,*}R\mathrm{sp}^{(n,\bullet)}_*\mathrm{DR}(]X^{(n,\bullet)}[^{\log}_{P^{(n,\bullet)}}, \mathcal{E}^{(n,\bullet)}) \\ &\quad \cong R\mathrm{sp}^{(n)}_{1,*}\mathrm{DR}(]X_1^{(n)}[^{\log}_{P_1^{(n)}}, \mathcal{E}_1^{(n)}). \end{aligned}$$

In the following, we give a proof of the isomorphism (2.2.11). First, by shrinking $P_1^{(n)}$ and $X_1^{(n)}$, we may assume that $P_1^{(n)}$ is affine and the closed immersion $(X_1^{(n)}, M_1^{(n)}) \hookrightarrow (P_1^{(n)}, L_1^{(n)})$ has a factorization

$$(X_1^{(n)}, M_1^{(n)}) \hookrightarrow (\overline{P}_1^{(n)}, \overline{L}_1^{(n)}) \longrightarrow (P_1^{(n)}, L_1^{(n)})$$

such that the first arrow is an exact closed immersion and the second arrow is formally log etale and $\overline{P}_1^{(n)}$ is also affine. Then, by replacing $(P_1^{(n)}, L_1^{(n)})$ by $(\overline{P}_1^{(n)}, \overline{L}_1^{(n)})$ and $(P^{(n,\bullet)}, L^{(n,\bullet)})$ by $((P^{(n,\bullet)}, L^{(n,\bullet)}) \hat{\times}_{(P_1^{(n)}, L_1^{(n)})} (\overline{P}_1^{(n)}, \overline{L}_1^{(n)}))^{\mathrm{int}}$, we may assume that $(X_1^{(n)}, M_1^{(n)}) \hookrightarrow (P_1^{(n)}, L_1^{(n)})$ is an exact closed immersion and $P_1^{(n)}$ is affine.

Let $\hat{P}_1^{(n)}$ be the formal completion of $P_1^{(n)}$ along $X_1^{(n)}$. Then, since there exists the canonical equivalence of sites $X^{(n)}_{1,\mathrm{et}} \simeq \hat{P}^{(n)}_{1,\mathrm{et}}$, there exists uniquely an etale hypercovering $h_n : \hat{P}_1^{(n,\bullet)} \longrightarrow \hat{P}_1^{(n)}$ such that $\hat{P}^{(n,\bullet)} \times_{\hat{P}_1^{(n)}} X_1^{(n)} \overset{\sim}{=} X^{(n,\bullet)}$ holds. Put $\hat{L}_1^{(n,\bullet)} := L_1^{(n)}|_{\hat{P}_1^{(n,\bullet)}}$. On the other hand, let $\hat{P}^{(n,m)}$ be the formal completion of $P^{(n,m)}$ along $X^{(n,m)}$ and put $\hat{L}^{(n,m)} := L^{(n,m)}|_{\hat{P}^{(n,m)}}$. Let $\tilde{P}^{(n,m)}$ be the formal completion of $\hat{P}_1^{(n,m)} \hat{\times}_{\mathrm{Spf}\, V} \hat{P}^{(n,m)}$ along $X^{(n,m)}$ and let $\tilde{L}^{(n,m)}$ be the pull-back of the log structure on the fine log formal scheme $((\hat{P}_1^{(n,m)}, \hat{L}_1^{(n,m)}) \hat{\times}_{(\mathrm{Spf}\, V, N)} (\hat{P}^{(n,m)}, \hat{L}^{(n,m)}))^{\mathrm{int}}$ to $\tilde{P}^{(n,m)}$. Then we have the following diagram:

$$\begin{array}{ccccccc}
(X_1^{(n)}, M_1^{(n)}) & \xleftarrow{g_n} & (X^{(n,\bullet)}, M^{(n,\bullet)}) & = & (X^{(n,\bullet)}, M^{(n,\bullet)}) & = & (X^{(n,\bullet)}, M^{(n,\bullet)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\hat{P}_1^{(n)}, \hat{L}_1^{(n)}) & \xleftarrow{h_n} & (\hat{P}_1^{(n,\bullet)}, \hat{L}_1^{(n,\bullet)}) & \xleftarrow{\mathrm{pr}_1^{(\bullet)}} & (\tilde{P}^{(n,\bullet)}, \tilde{L}^{(n,\bullet)}) & \xrightarrow{\mathrm{pr}_2^{(\bullet)}} & (\hat{P}^{(n,\bullet)}, \hat{L}^{(n,\bullet)}),
\end{array}$$

where the vertical arrows are the canonical closed immersions. Let us note the following claim:

CLAIM. Zariski locally on $X^{(n,m)}$, there exists an exact closed immersion

$$(X^{(n,m)}, M^{(n,m)}) \hookrightarrow (P_1^{(n,m)}, L_1^{(n,m)})$$

of $(X^{(n,m)}, M^{(n,m)})$ into a fine log formal V-scheme $(P_1^{(n,m)}, L_1^{(n,m)})$ formally etale (in the classical sense) over $(P_1^{(n)}, L_1^{(n)})$ such that $(\hat{P}_1^{(n,m)}, \hat{L}_1^{(n,m)})$ is the completion of $(P_1^{(n,m)}, L_1^{(n,m)})$ along $(X^{(n,m)}, M^{(n,m)})$.

PROOF OF CLAIM. Since the morphism $X^{(n,m)} \longrightarrow X_1^{(n)}$ is etale, there exists an affine open sub formal scheme $U = \mathrm{Spf}\, A \subset P_1^{(n)}$ and an affine open subscheme $V_0 \subset X^{(n,m)}$ which satisfies the following condition: If we put $U_0 := \mathrm{Spec}\, A_0 = U \times_{P_1^{(n)}} X_1^{(n)}$, then $V_0 = \mathrm{Spec}\, A_0[t_1, \cdots, t_r]/(f_1, \cdots, f_r)$ holds, where $f_i\, (1 \le i \le r)$ are elements in $A_0[t_1, \cdots, t_r]$ such that $\det \left(\frac{\partial f_i}{\partial t_j}\right)_{i,j}$ is invertible on V_0. Let $\tilde{f}_i\, (1 \le i \le r)$ be any lift of f_i to $A[t_1, \cdots, t_r]$ and let V be $\mathrm{Spf}\, (A[t_1, \cdots, t_r]/(\tilde{f}_1, \cdots, \tilde{f}_r))^\wedge$, where $^\wedge$ denotes the p-adic completion. Then, since we have $V \times_U U_0 = V_0$, the natural morphism $V \longrightarrow U$ is formally etale on a neighborhood of V_0. Let $V' \subset V$ be the etale locus. Then $(V', L_1^{(n)}|_{V'}) \longrightarrow (P_1^{(n)}, L_1^{(n)})$ is formally etale, $(V_0, M^{(n,m)}|_{V_0}) \hookrightarrow (V', L_1^{(n)}|_{V'})$ is an exact closed immersion and the completion of V' along V_0 is isomorphic to some open subscheme of $\hat{P}_1^{(n,m)}$. So the assertion is proved. $\square$

Now we define the log de Rham complexes associated to $\mathcal{E}$ on three tubular neighborhoods of $X^{(n,m)}$. First, since we have $]X^{(n,m)}[^{\log}_{\hat{P}^{(n,m)}} =]X^{(n,m)}[^{\log}_{P^{(m,n)}}$, $\mathcal{E}$ induces the log de Rham complex $\mathrm{DR}(]X^{(n,m)}[^{\log}_{P^{(n,m)}}, \mathcal{E})$ on $]X^{(n,m)}[^{\log}_{\hat{P}^{(n,m)}}$. We denote it simply by $\widehat{\mathrm{DR}}^{(n,m)}$. Second, by the above claim, we have the isomorphism $]X^{(n,m)}[^{\log}_{\hat{P}_1^{(n,m)}} =]X^{(n,m)}[^{\log}_{P_1^{(n,m)}}$ locally. So $\mathcal{E}$ induces the log de Rham complex $\mathrm{DR}(]X^{(n,m)}[^{\log}_{P_1^{(n,m)}}, \mathcal{E})$ locally on $]X^{(n,m)}[^{\log}_{\hat{P}_1^{(n,m)}}$. One can check that this de Rham complex is independent of the choice of $P_1^{(n,m)}$ in the claim and so it defines the log de Rham complex globally on $]X^{(n,m)}[^{\log}_{\hat{P}_1^{(n,m)}}$. We denote it simply by $\widehat{\mathrm{DR}}_1^{(n,m)}$. Finally, let $(\overline{P}^{(n,m)}, \overline{P}^{(n,m)})$ be $((P_1^{(n,m)}, L_1^{(n,m)}) \times_{(\mathrm{Spf}\, V,N)} (P^{(n,m)}, L^{(n,m)}))^{\mathrm{int}}$ (it is defined locally). Then $\tilde{P}^{(n,m)}$ is the completion of $\overline{P}^{(n,m)}$ along $X^{(n,m)}$. So, we have the isomorphism $]X^{(n,m)}[_{\tilde{P}^{(n,m)}} =]X^{(n,m)}[_{\overline{P}^{(n,m)}}$ locally. So $\mathcal{E}$ induces the log de Rham complex $\mathrm{DR}(]X^{(n,m)}[^{\log}_{\overline{P}^{(n,m)}}, \mathcal{E})$ locally on $]X^{(n,m)}[^{\log}_{\tilde{P}^{(n,m)}}$. One can check that this de Rham complex is independent of the choice of $P_1^{(n,m)}$ in the claim and so it defines the log de Rham complex globally on $]X^{(n,m)}[^{\log}_{\tilde{P}^{(n,m)}}$. We denote it simply by $\widetilde{\mathrm{DR}}^{(n,m)}$. Let us denote the morphism of simplicial rigid analytic spaces

$$]X^{(n,\bullet)}[^{\log}_{\tilde{P}^{(n,\bullet)}} \longrightarrow]X^{(n,\bullet)}[^{\log}_{\hat{P}_1^{(n,\bullet)}},$$
$$]X^{(n,\bullet)}[^{\log}_{\tilde{P}^{(n,\bullet)}} \longrightarrow]X^{(n,\bullet)}[^{\log}_{\hat{P}^{(n,\bullet)}},$$

by $\mathrm{pr}_{1,K}^{(\bullet)}, \mathrm{pr}_{2,K}^{(\bullet)}$ respectively and denote the specialization map

$$]X^{(n,\bullet)}[^{\log}_{\hat{P}_1^{(n,\bullet)}} \longrightarrow X^{(n,\bullet)}$$

by $\widehat{\mathrm{sp}}^{(\bullet)}$. Then the proof of the isomorphism (2.2.11) is reduced to the following claim:

Claim. Let the notations be as above. Then:

(1) We have the isomorphism

$$R\mathrm{pr}_{1,K,*}^{(\bullet)} \widetilde{\mathrm{DR}}^{(n,\bullet)} = \widehat{\mathrm{DR}}_1^{(n,\bullet)}.$$

(2) We have the isomorphism

$$R\mathrm{pr}^{(\bullet)}_{2,K,*}\widetilde{\mathrm{DR}}^{(n,\bullet)} = \widehat{\mathrm{DR}}^{(n,\bullet)}.$$

(3) We have the isomorphism

$$Rg_{n,*}R\widehat{\mathrm{sp}}^{(\bullet)}_{*}\widehat{\mathrm{DR}}^{(n,\bullet)}_1 = R\mathrm{sp}^{(n)}_{1,*}\mathrm{DR}(]X^{(n)}[_{P^{(n)}_1}, \mathcal{E}).$$

We prove the above claim. First, let us prove the assertions (1) and (2). We may replace $\bullet$ by $m \in \mathbb{N}$ and we may consider locally. So, the morphisms

$$\begin{aligned} \mathrm{pr}^{(m)}_1 &: (\tilde{P}^{(n,m)}, \tilde{L}^{(n,m)}) \longrightarrow (\hat{P}^{(n,m)}_1, \hat{L}^{(n,m)}_1), \\ \mathrm{pr}^{(m)}_2 &: (\tilde{P}^{(n,m)}, \tilde{L}^{(n,m)}) \longrightarrow (\hat{P}^{(n,m)}, \hat{L}^{(n,m)}) \end{aligned}$$

are the completions of the morphisms

$$(\overline{P}^{(n,m)}, \overline{L}^{(n,m)}) \longrightarrow (P^{(n,m)}_1, L^{(n,m)}_1), \tag{2.2.12}$$

$$(\overline{P}^{(n,m)}, \overline{L}^{(n,m)}) \longrightarrow (P^{(n,m)}, L^{(n,m)}), \tag{2.2.13}$$

respectively. Since the morphisms (2.2.12) and (2.2.13) are formally log smooth, the assertions (1) and (2) follows from Lemma 2.2.15.

The assertion (3) follows from the lemma below. Hence the above claim is proved and the proof of the proposition is now finished (modulo the lemma below). □

LEMMA 2.2.17. *Let X be a scheme of finite type over k and let $X \hookrightarrow P$ be a closed immersion of X into a Noetherian formal scheme P over* Spf V *such that X is a scheme of definition of P. Let $\psi : X^{(\bullet)} \longrightarrow X$ be an etale hypercovering of X and let $\varphi : P^{(\bullet)} \longrightarrow P$ be the unique etale hypercovering of P satisfying $P^{(\bullet)} \times_P X = X^{(\bullet)}$. Let us denote the morphism of rigid analytic spaces $P^{(\bullet)}_K \longrightarrow P_K$ associated to φ by $\varphi_K = \{\varphi^{(\bullet)}_K\}$. Denote the spacialization maps*

$$\begin{aligned} P^{(\bullet)}_K &\longrightarrow P^{(\bullet)}, \\ P_K &\longrightarrow P, \end{aligned}$$

by $\mathrm{sp}^{(\bullet)}, \mathrm{sp}$*, respectively. Let* E *be a coherent sheaf on* P_K *and put* $E^{(\bullet)} := \varphi_K^* E$*. Then we have the isomorphism*

$$R\varphi_{K,*} R\mathrm{sp}_*^{(\bullet)} E^{(\bullet)} = R\mathrm{sp}_* E.$$

Proof. Note that the 4-tuples $P := ((P, \text{triv.}\log\text{str.}), (X, \text{triv.}\log\text{str.}), X \hookrightarrow P, \mathrm{id})$ and $P^{(n)} := ((P^{(n)}, \text{triv.}\log\text{str.}), (X^{(n)}, \text{triv.}\log\text{str.}), X^{(n)} \hookrightarrow P^{(n)}, \mathrm{id})$ form exact widenings of X over $\mathrm{Spf}\, V$. So one can define the system of universal enlargements $T_m := T_{X,m}(P)$, $T_m^{(n)} := T_{X^{(n)},m}(P^{(n)})$. By comparing the explicit description of the system of universal enlargements given in Section 2.2 and the definition of the rigid analytic spaces $P_K, P_K^{(n)}$ given in Section 2.1, one can see the following:

(1) We have the admissible covering $P_K = \bigcup_m T_{m,K}$.
(2) $T_{m,K}^{(n)} = (\varphi_K^{(n)})^{-1}(T_{m,K})$ holds.

Let $\vec{P}$, $\vec{P}^{(\bullet)}$ be the direct limit topos associated to P, $P^{(\bullet)}$ respectively and let

$$\gamma : \vec{P}^{\sim} \longrightarrow P_{\mathrm{Zar}},$$
$$\gamma^{(\bullet)} : \vec{P}^{(\bullet),\sim} \longrightarrow P_{\mathrm{Zar}}^{(\bullet)}$$

be the morphism of topoi defined in the end of Section 2.2. Then we have

$$\begin{aligned} R\varphi_* R\mathrm{sp}_*^{(\bullet)} E^{(\bullet)} &= R\varphi_* \mathrm{sp}_*^{(\bullet)} E^{(\bullet)} \quad (\text{Theorem B of kiehl}) \\ &= R\varphi_* \gamma_*^{(\bullet)} \{ (\mathrm{sp}^{(\bullet)}|_{T_{m,K}^{(\bullet)}})_* (E^{(\bullet)}|_{T_{m,K}^{(\bullet)}}) \}_m \\ &= R\varphi_* R\gamma_*^{(\bullet)} \{ (\mathrm{sp}^{(\bullet)}|_{T_{m,K}^{(\bullet)}})_* (E^{(\bullet)}|_{T_{m,K}^{(\bullet)}}) \}_m \\ &\qquad (\text{Proposition 2.1.31}) \\ &= R\gamma_* R\vec{\varphi}_* \{ (\mathrm{sp}^{(\bullet)}|_{T_{m,K}^{(\bullet)}})_* (E^{(\bullet)}|_{T_{m,K}^{(\bullet)}}) \}_m, \end{aligned}$$

when we denoted the morphism of topoi

$$\vec{P}^{(\bullet),\sim} \longrightarrow \vec{P}^{\sim}$$

induced by φ by $\vec{\varphi}$.

On the other hand, we have

$$\begin{aligned} R\mathrm{sp}_* E &= \mathrm{sp}_* E \quad \text{(Theorem B of kiehl)} \\ &= \gamma_*\{(\mathrm{sp}|_{T_{m,K}})_*(E|_{T_{m,K}})\}_m \\ &= R\gamma_*\{(\mathrm{sp}|_{T_{m,K}})_*(E|_{T_{m,K}})\}_m \quad \text{(Propsotion 2.1.31).} \end{aligned}$$

So we have to show the isomorphism in the derived category of the category of sheaves on $\vec{P}$

$$R\vec{\varphi}_*\{(\mathrm{sp}^{(\bullet)}|_{T^{(\bullet)}_{m,K}})_*(E^{(\bullet)}|_{T^{(\bullet)}_{m,K}})\}_m = \{(\mathrm{sp}|_{T_{m,K}})_*(E|_{T_{m,K}})\}_m.$$

Note that it suffices to show the restriction of both sides to the derived category of the category of sheaves on $T_{m,\mathrm{Zar}}$ is isomorphic. Noting the equality

$$(\mathrm{sp}^{(\bullet)}|_{T^{(\bullet)}_{m,K}})_*(E^{(\bullet)}|_{T^{(\bullet)}_{m,K}}) = (\varphi^{(\bullet)}_m)^*(\mathrm{sp}|_{T_{m,K}})_*(E|_{T_{m,K}})$$

(where $\varphi^{(n)}_m$ is the morphism $T^{(n)}_m \longrightarrow T_m$), one can see that it suffices to prove the following claim:

Claim. For an isocoherent sheaf E on T_m, one has the isomorphism

$$R\varphi^{(\bullet)}_{m,*}\varphi^{(\bullet),*}_m E = E.$$

We give a proof of the claim. If we can prove that the morphism $T^{(\bullet)}_m \longrightarrow T_m$ is an etale hypercovering, the claim follows from the fpqc descent for formal schemes. So it suffices to prove it. Moreover, to prove it, it suffices to prove that the morphism $T^{(n)}_m \longrightarrow T_m$ is formally etale. To prove this claim, we may assume that $P := \mathrm{Spf}\, A$ and $P^{(n)} := \mathrm{Spf}\, A^{(n)}$ are affine. Put $X := \mathrm{Spec}\, A/I$ and fix a system of generators $f_1, \cdots, f_r$ of I. Let us define the rings $A_m, A^{(n)}_m$ by

$$\begin{aligned} A_m &:= A[t_i \ (i \in \mathbb{N}^r, |i| = m)]/(\pi t_i - f^i \ (i \in \mathbb{N}^r, |i| = m)), \\ A^{(n)}_m &:= A^{(n)}[t_i \ (i \in \mathbb{N}^r, |i| = m)]/(\pi t_i - f^i \ (i \in \mathbb{N}^r, |i| = m)) \end{aligned}$$

(where we used the multi-index notation). Then we have

$$T_m = \mathrm{Spf}\,(A_m/(p\text{-torsion}))^\wedge, \quad T_m^{(n)} = \mathrm{Spf}\,(A_m^{(n)}/(p\text{-torsion}))^\wedge,$$

where $^\wedge$ denotes the p-adic completion. Since A_m and $A_m^{(n)}$ are Noetherian, we have

$$(A_m/(p\text{-torsion}))^\wedge \overset{\simeq}{=} A_m^\wedge/(p\text{-torsion}),$$
$$(A_m^{(n)}/(p\text{-torsion}))^\wedge \overset{\simeq}{=} (A_m^{(n)})^\wedge/(p\text{-torsion}).$$

If we can prove that the homomorphism $\alpha : A_m^\wedge \longrightarrow (A_m^{(n)})^\wedge$ is formally etale, then the homomorphism

$$A_m^\wedge/(p\text{-torsion}) \longrightarrow (A_m^{(n)})^\wedge/(p\text{-torsion})$$

coincides with the push-out of α by the projection

$$A_m^\wedge \longrightarrow A_m^\wedge/(p\text{-torsion}).$$

In particular, it is formally etale and we are done. So it suffices to prove the formal etaleness of the homomorphism α. That is, we have only to prove the etaleness of the homomorphism

$$A_m/\pi^k A_m \longrightarrow A_m^{(n)}/\pi^k A_m^{(n)}$$

for any $k \in \mathbb{N}$. Since we have $I^m A_m \subset \pi A_m$ and $I^m A_m^{(n)} \subset \pi A_m^{(n)}$, this assertion is reduced to the etaleness of the homomorphism

$$A_m/I^N A_m \longrightarrow A_m^{(n)}/I^N A_m^{(n)}$$

for any $N \in \mathbb{N}$. Since we have

$$A_m/I^N A_m = A/I^N[t_i\ (i \in \mathbb{N}^r, |i| = m)]/(\pi t_i - f^m\ (i \in \mathbb{N}^r, |i| = m)),$$

$$\begin{aligned} &A_m^{(n)}/I^N A_m^{(n)} \\ &\quad := A^{(n)}/I^N A^{(n)}[t_i\ (i \in \mathbb{N}^r, |i| = m)]/(\pi t_i - f^m\ (i \in \mathbb{N}^r, |i| = m)), \end{aligned}$$

the assertion is reduced to the etaleness of the homomorphism

$$A/I^N \longrightarrow A^{(n)}/I^N A^{(n)}$$

and it follows from the formal etaleness of the morphism $P^{(n)} \longrightarrow P$. Hence the claim is proved and so the proof of the lemma is now finished. □

REMARK 2.2.18. Note that the above definition is independent of the choice of a chart of $(\mathrm{Spf}\, V, N)$ whose existence is assumed above. Moreover, we can define the analytic cohomology even when $(\mathrm{Spf}\, V, N)$ does not admit a chart in the following way.

Let us assume given the diagram

$$(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N), \tag{2.2.14}$$

where N is a fine log structure on V, ι is the canonical exact closed immersion and f is a morphism of fine log schemes of finite type. (We do not assume the existence of a chart of $(\mathrm{Spf}\, V, N)$.) Then there exists a finite Galois extension $V \subset V_1$ such that $(\mathrm{Spf}\, V_1, N)$ admits a chart. Let G_1 be the Galois group of V_1 over V and let

$$(X_1, M) \xrightarrow{f} (\mathrm{Spec}\, k_1, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V_1, N) \tag{2.2.15}$$

be the base change of the diagram (2.2.14) by the morphism $(\mathrm{Spf}\, V_1, N) \longrightarrow (\mathrm{Spf}\, V, N)$. Let $\mathcal{E}$ be a locally free isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$ and denote the pull-back of $\mathcal{E}$ to $(X_1/V_1)^{\log}_{\mathrm{conv,et}}$ by $\mathcal{E}_1$. Then we define the analytic cohomology of (X, M) over $(\mathrm{Spf}\, V, N)$ with coefficient $\mathcal{E}$ by

$$H^i_{\mathrm{an}}((X, M)/(\mathrm{Spf}\, V, N), \mathcal{E}) := H^i_{\mathrm{an}}((X_1, M)/(\mathrm{Spf}\, V_1, N), \mathcal{E}_1)^{G_1}.$$

Let us prove the well-definedness of the above definition. Let V_2 be another finite Galois extension of V with Galois group G_2 such that $(\mathrm{Spf}\, V_2, N)$ also admits a chart. To prove the well-definedness, we may assume that $V_1 \subset V_2$ holds. Let G be the Galois group of V_2 over V_1. Let

$$(X_2, M) \xrightarrow{f} (\mathrm{Spec}\, k_2, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V_2, N)$$

be the base change of the diagram (2.2.14) by the morphism $(\mathrm{Spf}\, V_2, N) \longrightarrow (\mathrm{Spf}\, V, N)$ and denote the pull-back of $\mathcal{E}$ to $(X_2/V_2)^{\log}_{\mathrm{conv,et}}$ by $\mathcal{E}_2$. Take a good embedding system

$$(X_1, M) \longleftarrow (X_1, M_1^{(\bullet)}) \hookrightarrow (P_1^{(\bullet)}, L_1^{(\bullet)}) \tag{2.2.16}$$

of (X_1, M) over $(\mathrm{Spf}\, V_1, N)$ and let

$$(X_2, M) \longleftarrow (X_2, M_2^{(\bullet)}) \hookrightarrow (P_2^{(\bullet)}, L_2^{(\bullet)}) \tag{2.2.17}$$

be the base change of the diagram (2.2.16) by the morphism $(\mathrm{Spf}\, V_2, N) \longrightarrow (\mathrm{Spf}\, V_1, N)$. Then the diagram (2.2.17) is also a good embedding system. Denote the natural morphism of simplicial rigid analytic spaces

$$]X_2^{(\bullet)}[_{P_2^{(\bullet)}} \longrightarrow]X_1^{(\bullet)}[_{P_1^{(\bullet)}}$$

by $\varphi^{(\bullet)}$. Then it is easy to see the following:

$$\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2)^G = \mathrm{DR}(]X_1^{(\bullet)}[_{P_1^{(\bullet)}}, \mathcal{E}_1),$$
$$R\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2) = \varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2).$$

Applying the specialization map $\mathrm{sp}^{(\bullet)} :]X_1^{(\bullet)}[_{P_1^{(\bullet)}} \longrightarrow X_1^{(\bullet)}$, we obtain the following:

$$\mathrm{sp}_*^{(\bullet)}\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2)^G = \mathrm{sp}_*^{(\bullet)}\mathrm{DR}(]X_1^{(\bullet)}[_{P_1^{(\bullet)}}, \mathcal{E}_1),$$
$$R\mathrm{sp}_*^{(\bullet)}R\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2) = \mathrm{sp}_*^{(\bullet)}\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2).$$

Noting that $\mathrm{sp}_*^{(\bullet)}\varphi_*^{(\bullet)}\mathrm{DR}(]X_2^{(\bullet)}[_{P_2^{(\bullet)}}, \mathcal{E}_2)$ is a sheaf of $\mathbb{Q}$-vector spaces, the above two isomorphisms imply the isomorphism

$$H^i_{\mathrm{an}}((X_2, M)/(\mathrm{Spf}\, V_2, N), \mathcal{E}_2)^G = H^i_{\mathrm{an}}((X_1, M)/(\mathrm{Spf}\, V_1, N), \mathcal{E}_1).$$

By taking the G_1-invariant part, we obtain the isomorphism

$$H^i_{\mathrm{an}}((X_2, M)/(\mathrm{Spf}\, V_2, N), \mathcal{E}_2)^{G_2} = H^i_{\mathrm{an}}((X_1, M)/(\mathrm{Spf}\, V_1, N), \mathcal{E}_1)^{G_1}.$$

Hence we obtain the well-definedness.

REMARK 2.2.19. In this section, we have defined the analytic cohomology for log schemes. It would be natural to ask whether we can define rigid cohomology of a fine log scheme. Let $(\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N)$ be the canonical exact closed immersion such that $(\operatorname{Spf} V, N)$ admits a chart. Let us assume given an open immersion $(X, M) \subset (X', M')$ of fine log schemes of finite type over $(\operatorname{Spec} k, N)$. Then, by taking a good embedding system of (X', M'), it is possible to define the candidate of the rigid cohomology $H^i_{\mathrm{rig}}((X, M) \subset (X', M')/(\operatorname{Spf} V, N), \mathcal{E})$ of the pair $(X, M) \subset (X', M')$ over $(\operatorname{Spf} V, N)$ (where $\mathcal{E}$ is a locally free isocrystal on $(X'/V)^{\log}_{\mathrm{conv,et}}$) as in the case of rigid cohomology. Then, one should prove the following conjecture to assure the well-definedness:

CONJECTURE 2.2.20. *The above definition of the rigid cohomology* $H^i_{\mathrm{rig}}((X, M) \subset (X', M')/(\operatorname{Spf} V, N), \mathcal{E})$ *is independent of the choice of the good embedding system of* (X', M') *chosen above.*

Next, we would like to define the rigid cohomology for a fine log scheme (X, M) which is separated and of finite type over $(\operatorname{Spec} k, N)$. For simplicity, let us consider the case of trivial coefficient. To define it, we need the following conjecture:

CONJECTURE 2.2.21 (Log version of Nagata's theorem). *Let* (X, M) *be a fine log scheme over* $(\operatorname{Spec} k, N)$. *Then there exists an open immersion* $(X, M) \hookrightarrow (X', M')$ *of* (X, M) *into a fine log scheme* (X', M') *over* $(\operatorname{Spec} k, N)$ *such that* X' *is proper over* $\operatorname{Spec} k$. (*We call such an open immersion* $(X, M) \hookrightarrow (X', M')$ *as a log compactification of* (X, M)).

Under this conjecture, we can define the candidate of the rigid cohomology $H^i_{\mathrm{rig}}((X, M)/(\operatorname{Spf} V, N))$ by $H^i_{\mathrm{rig}}((X, M)/(\operatorname{Spf} V, N)) := H^i_{\mathrm{an}}((X, M) \subset (X', M')/(\operatorname{Spf} V, N))$, where $(X, M) \subset (X', M')$ is a log compactification. Then, one should prove the following conjecture to assure the well-definedness:

CONJECTURE 2.2.22. *Let* (X, M) *be as above and assume given two log compactifications* $(X, M) \hookrightarrow (X', M')$, $(X, M) \hookrightarrow (X'', M'')$. *Then the* *rigid cohomology of the pair* $(X, M) \hookrightarrow (X', M')$ *is isomorphic to that of the pair* $(X, M) \hookrightarrow (X'', M'')$.

To define the rigid cohomology of a single fine log scheme with coefficient, we need to develop a nice theory of overconvergent isocrystals on log schemes.

2.3. Log convergent Poincaré lemma

Assume given the diagram $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N)$, where f is a morphism of finite type between fine log schemes, N is a fine log structure on $\mathrm{Spf}\, V$ and ι is the canonical exact closed immersion. Let $\mathcal{E}$ be a locally free isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$. In this section, we prove that the cohomology $H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E})$ of $\mathcal{E}$ in the log convergent site $(X/V)^{\log}_{\mathrm{conv,et}}$ (which we call the log convergent cohomology of $\mathcal{E}$) is canonically isomorphic to the analytic cohomology $H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E})$ of (X, M) over $(\mathrm{Spf}\, V, N)$ with coefficient $\mathcal{E}$, which is defined in the previous section. Recall that the analytic cohomology $H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E})$ is, roughly speaking, defined as the cohomology of the log de Rham complex associated to $\mathcal{E}$ on certain rigid analytic space. The theorems which calculate cohomology groups by certain de Rham complexes are sometimes called Poincaré lemma. So we call this result as log convergent Poincaré lemma. It is a log version of convergent Poincaré lemma proved by Ogus ([Og2]).

First we prepare several morphisms of ringed topoi which we need in this section. Let $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N)$ be as above. Let $\epsilon : (X/V)^{\log,\sim}_{\mathrm{conv,et}} \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}$ be the morphism of topoi defined as follows: For a sheaf E on $(X/V)^{\log}_{\mathrm{conv,et}}$, $\epsilon_* E$ is defined by $\epsilon_* E(T) := E(T)$ $(T \in \mathrm{Enl}((X/V)^{\log}))$ and for a sheaf E on $(X/V)^{\log}_{\mathrm{conv,Zar}}$, $\epsilon^* E$ is defined as the sheafification of the presheaf $T \mapsto E(T)$ on $(X/V)^{\log}_{\mathrm{conv,et}}$. For a (pre-)widening T, the similar morphism $(X/V)^{\log,\sim}_{\mathrm{conv,et}}|_T \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_T$ will be also denoted by ϵ.

Next, define a morphism of topoi

$$u : (X/V)^{\log,\sim}_{\mathrm{conv,Zar}} \longrightarrow X^{\sim}_{\mathrm{Zar}}$$

as follows: For a sheaf E on $(X/V)^{\log}_{\mathrm{conv,Zar}}$, define $u_* E$ by

$$u_* E(U) := \Gamma((U/V)^{\log}_{\mathrm{conv,Zar}}, j^* E),$$

where $U \subset X$ is an open set and $j^* : (X/V)^{\log,\sim}_{\mathrm{conv,Zar}} \longrightarrow (U/V)^{\log,\sim}_{\mathrm{conv,Zar}}$ is the natural restriction. For a sheaf E on X_{Zar}, define u^*E by $(u^*E)(T) := \Gamma(T, z^*E)$ for each enlargement $((T, M_T), (Z, M_Z), i, z)$. Let $\tilde{u} : (X/V)^{\log,\sim}_{\mathrm{conv,et}} \longrightarrow X^{\sim}_{\mathrm{Zar}}$ be the composite $u \circ \epsilon$.

Next, for a (pre-)widening T, define a morphism of topoi

$$j_T : (X/V)^{\log,\sim}_{\mathrm{conv},\tau}|_T \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv},\tau} \quad (\tau = \mathrm{Zar}\ \text{ or }\ \mathrm{et})$$

as follows: For an object $g : E \longrightarrow h_T$ in $(X/V)^{\log,\sim}_{\mathrm{conv}}|_T$, define $j_{T,*}(g : E \longrightarrow h_T)$ to be the sheaf of sections of g, and for an object E in $(X/V)^{\log,\sim}_{\mathrm{conv}}$, define j_T^*E to be the projection $E \times h_T \longrightarrow h_T$. We put $u_T := u \circ j_T$, $\tilde{u}_T = \tilde{u} \circ j_T$.

Next, for an exact widening T, define a morphism of ringed topoi

$$\phi_T : (X/V)^{\log,\sim}_{\mathrm{conv},\tau}|_T \longrightarrow T^{\sim}_{\tau} \quad (\tau = \mathrm{Zar}\ \text{ or }\ \mathrm{et})$$

as follows: Let E be a sheaf on $(X/V)^{\log}_{\mathrm{conv},\tau}|_T$ and let $U \longrightarrow T$ be an object in T_τ. Then U has the canonical structure of exact widening defined as the pull-back of that of T to U. Let $\{U_n\}$ be the system of universal enlargements of U. Then we define $\phi_{T,*}E$ by $\phi_{T,*}E(U) := \varprojlim_n E(U_n \to T)$, where $U_n \to T$ is the composite $U_n \to U \to T$. For a sheaf E on T_τ, define ϕ_T^*E by $\phi_T^*E(g) := g^*E(T)$ for an enlargement T' and a morphism of widenings $g : T' \longrightarrow T$.

Finally, for an exact widening T, define the functor

$$\phi_{\vec{T},*} : (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_T \longrightarrow \vec{T}^{\sim}$$

by $\phi_{\vec{T},*}E(U) := E(U \hookrightarrow T_n \longrightarrow T)$, where E is an object on left hand side and U is an open set in T_n. (Note that $\phi_{\vec{T},*}$ is not a part of a morphism of topoi.)

One can check that there exists the following diagram of topoi for an exact widening T:

$$\begin{array}{ccccc}
(X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_T & \xrightarrow{\phi_{\vec{T},*}} & \vec{T}^{\sim} & \xrightarrow{\gamma_*} & T^{\sim}_{\mathrm{Zar}} \\
{\scriptstyle j_{T,*}}\downarrow & & & & \Vert \\
(X/V)^{\log,\sim}_{\mathrm{conv,Zar}} & \xrightarrow[u_*]{} & X^{\sim}_{\mathrm{Zar}} & \xleftarrow[z_*]{} & Z^{\sim}_{\mathrm{Zar}},
\end{array}$$

where γ_* is the functor of taking the inverse limit of the direct image defined in Section 2.1.

Then we have the following lemma:

LEMMA 2.3.1. *The functor $\phi_{\vec{T},*}$ sends an injective sheaf to a flasque sheaf.*

PROOF. We omit the proof, since it is the same as that in [Og2, (0.4.1)]. (Note that we need Lemma 2.1.24(=[Og2, (0.2.2)]) to prove this lemma.) □

Next we prove the following propositions:

PROPOSITION 2.3.2. *Let $(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N)$ be as above and assume that $\iota \circ f$ admits a chart $\mathcal{C}_0$. Let T be an affine exact widening of (X, M) over $(\operatorname{Spf} V, N)$ with respect to $\mathcal{C}_0$. Then the following assertion holds.*

(1) *$\phi_{\vec{T},*}$ is exact.*

(2) *Let $\mathcal{E}$ be an isocrystal on $(X/V)^{\log}_{\mathrm{conv,Zar}}|_T$. Then we have*

$$H^q((X/V)^{\log}_{\mathrm{conv,Zar}}|_T, \mathcal{E}) \overset{\sim}{=} H^q(\vec{T}, \phi_{\vec{T},*}\mathcal{E})$$

for all $q \geq 0$ and these groups vanish for $q > 0$.

PROOF. The proof is similar to that in [Og2, (0.4.2)].

One can check the assertion (1) directly, so we omit the proof. Now we will prove the assertion (2). By Lemma 2.3.1, there exists a Leray spectral sequence for $\phi_{\vec{T},*}$ and it degenerates by (1). So we get the former statement. Moreover, by definition, $\phi_{\vec{T},*}E$ is a crystalline $K \otimes_V \mathcal{O}_{\vec{T}}$-module. So the above cohomology groups vanish for $q > 0$ by Proposition 2.1.31. □

PROPOSITION 2.3.3. *Let $(X, M) \longrightarrow (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} V, N), \mathcal{C}_0$ be as in the previous proposition. Let T be an affine exact widening of (X, M) over $(\operatorname{Spf} V, N)$ with respect to $\mathcal{C}_0$ and let $\mathcal{E}$ be an isocrystal in $(X/V)^{\log,\sim}_{\mathrm{conv,et}}|_T$. Then we have $R^q\tilde{u}_{T,*}\mathcal{E} = 0$ and $R^q j_{T,*}\mathcal{E} = 0$ for $q > 0$.*

PROOF. The proof is similar to that in [Og2, (0.4.3)]. First, we may assume X is affine.

Since $R^q\epsilon_*\mathcal{E} = 0$ holds for $q > 0$ by Proposition 2.1.21, $R^q\tilde{u}_{T,*}\mathcal{E}$ is the sheaf associated to the presheaf

$$U \mapsto H^q((U/V)^{\log}_{\mathrm{conv,Zar}}|_T, \epsilon_*\mathcal{E})$$

and it vanishes for $q > 0$ by Proposition 2.3.2. So we have $R^q\tilde{u}_{T,*}\mathcal{E} = 0$ for $q > 0$.

Next we prove the vanishing $R^q j_{T,*}\mathcal{E} = 0\,(q > 0)$. To show this, it suffices to show that the sheaf $(R^q j_{T,*}\mathcal{E})_{T'}$ on T'_{Zar} induced by $R^q j_{T,*}\mathcal{E}$ vanishes for any charted affine enlargement T' and $q > 0$. Let $T \times T'$ be the direct product as a charted affine widening, and form the exactification $(T \times T')^{\mathrm{ex}}$. Now let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
(T' \times T)^{\mathrm{ex},\sim}_{\mathrm{Zar}} & \xleftarrow{\gamma} & (T' \times \vec{T})^{\mathrm{ex},\sim} & \xleftarrow{\phi_{(T' \times \vec{T})^{\mathrm{ex}},*}} & (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_{(T' \times T)^{\mathrm{ex}}} & & \\
{\scriptstyle \mathrm{pr}}\big\downarrow & & & & {\scriptstyle j_T|_{T'}}\big\downarrow & & \\
T'^{\sim}_{\mathrm{Zar}} & = & T'^{\sim}_{\mathrm{Zar}} & \xleftarrow[\phi_{T'}]{} & (X/V)^{\log,\sim}_{\mathrm{conv,Zar}}|_{T'} & & \\
& & & & & & \\
& & & \xleftarrow{\epsilon} & (X/V)^{\log,\sim}_{\mathrm{conv,et}}|_{(T' \times T)^{\mathrm{ex}}} & \xrightarrow{j_{T'}|_T} & (X/V)^{\log,\sim}_{\mathrm{conv,et}}|_T \\
& & & & {\scriptstyle j_T|_{T'}}\big\downarrow & & {\scriptstyle j_T}\big\downarrow \\
& & & \xleftarrow{\epsilon} & (X/V)^{\log,\sim}_{\mathrm{conv,et}}|_{T'} & \xrightarrow[j_{T'}]{} & (X/V)^{\log,\sim}_{\mathrm{conv,et}}.
\end{array}$$

Since there exists an exact left adjoint functor $j_{T',!}$ of the functor $j^*_{T'}$ (which can be shown as in the case without log structure [Og2]), the functor $j^*_{T'}$ sends injectives to injectives. Similarly, $(j_{T'}|_T)^*$ sends injectives to injectives. Moreover, the functors $(j_{T'}|_T)^*$, $\phi_{T',*}$, $\phi_{(T' \times \vec{T})^{\mathrm{ex}},*}$, $j^*_{T'}$ and ϵ_* are exact. So we get the following equations.

$$\begin{aligned}
(R^q j_{T,*}\mathcal{E})_{T'} &= \phi_{T',*}\epsilon_* j^*_{T'} R^q j_{T,*}\mathcal{E} \\
&= \phi_{T',*}\epsilon_* R^q (j_T|_{T'})_* (j_{T'}|_T)^*\mathcal{E} \\
&= R^q(\mathrm{pr} \circ \gamma)_* \phi_{(T' \times \vec{T})^{\mathrm{ex}},*}\epsilon_*(j_{T'}|_T)^*\mathcal{E} \\
&= R^q\mathrm{pr}_*\gamma_*\phi_{(T' \times \vec{T})^{\mathrm{ex}},*}\epsilon_*(j_{T'}|_T)^*\mathcal{E} \quad \text{(Proposition 2.1.31)},
\end{aligned}$$

and the last term is equal to zero for $q > 0$ because $\gamma_*\phi_{(T' \times \vec{T})^{\mathrm{ex}},*}\epsilon_*(j_{T'}|_T)^*\mathcal{E}$ is quasi-coherent and the morphism $(T \times T')^{\mathrm{ex}} \longrightarrow T'$ is affine. So we have the vanishing $R^q j_{T,*}\mathcal{E} = 0$ for $q > 0$. $\square$

Corollary 2.3.4. *Let T be an affine exact widening and E be an isocrystal on $(X/V)^{\log,\sim}_{\mathrm{conv,et}}|_T$. Then $j_{T,*}\mathcal{E}$ is $\tilde{u}_*$-acyclic.*

Proof. In fact,

$$\begin{aligned} R\tilde{u}_*(j_{T,*}\mathcal{E}) &= Ru_* Rj_{T,*}\epsilon_*\mathcal{E} \quad \text{(Proposition 2.3.3)} \\ &= Rz_* R\gamma_* R\phi_{\vec{T},*}\epsilon_*\mathcal{E} \\ &= Rz_*\gamma_*\phi_{\vec{T},*}\epsilon_*\mathcal{E} \quad \text{(Lemma 2.3.1, Proposition 2.1.31)} \\ &= z_*\gamma_*\phi_{\vec{T},*}\epsilon_*\mathcal{E} \quad \text{(affinity of } z\text{)} \\ &= \tilde{u}_* j_{T,*}\mathcal{E}. \ \square \end{aligned}$$

Now we begin the proof of the log convergent Poincaré lemma. We will work on the commutative diagram

$$(2.3.1) \qquad \begin{array}{ccc} (X,M) & \xrightarrow{\ i\ } & (P,L) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ (\operatorname{Spec} k, N) & \xrightarrow{\ \iota\ } & (\operatorname{Spf} V, N), \end{array}$$

where f is a morphism of finite type between fine log schemes, N is a fine log structure on $\operatorname{Spf} V$, ι is the canonical exact closed immersion, i is a closed immersion into a fine log formal V-scheme (P,L) and g is a formally log smooth morphism. Assume moreover that $(\operatorname{Spf} V, N)$ admits a chart $\varphi : Q_V \longrightarrow N$ and (X,M), (P,L) are of Zariski type.

For the moment, let us assume moreover the following conditions:

(a) P is affine.

(b) The diagram $(X,M) \overset{i}{\hookrightarrow} (P,L) \xrightarrow{g} (\operatorname{Spf} V, N)$ admits a chart $\mathcal{C} = (Q_V \overset{\varphi}{\to} N, R_P \to L, S_X \to M, Q \to R \overset{\alpha}{\to} S)$ extending φ such that α^{gp} is surjective.

Denote the chart $(Q_V \overset{\varphi}{\to} N, S_X \to M, Q \to S)$ of $g \circ i = \iota \circ f$ induced by $\mathcal{C}$ by $\mathcal{C}'_0$. Then the diagram

$$\begin{array}{ccc} (X,M) & \xrightarrow{\ i\ } & (P,L) \\ \| & & {\scriptstyle g}\downarrow \\ (X,M) & \xrightarrow{\ \iota\circ f\ } & (\operatorname{Spf} V, N) \end{array}$$

admits a chart $\mathcal{C}_P := (Q_V \xrightarrow{\varphi} N, R_P \to L, S_X \to M, S_X \to M, \mathcal{D}_P)$, where $\mathcal{D}_P$ is the diagram

$$\begin{array}{ccc} S & \xleftarrow{\alpha} & R \\ \| & & \uparrow \\ S & \longleftarrow & Q. \end{array}$$

So $P := ((P, L), (X, M), i, \mathrm{id}), \mathcal{C}_P)$ is a charted affine pre-widening of (X, M) over $(\mathrm{Spf}\, V, N)$ with respect to the chart $\mathcal{C}'_0$. Let $\underline{P} := ((\underline{P}, \underline{L}), (X, M), \underline{i}, \mathrm{id})$ be the exactification of P and let $\hat{P} := (\hat{P}, \hat{L}), (X, M), \hat{\imath}, \mathrm{id})$ be the widening associated to $\underline{P}$. Put $\underline{R} := (\alpha^{\mathrm{gp}})^{-1}(S)$. Then the diagram

$$\begin{array}{ccc} (X, M) & \xrightarrow{\underline{i}} & (\underline{P}, \underline{L}) \\ \| & & \Big\downarrow g \\ (X, M) & \xrightarrow{\iota \circ f} & (\mathrm{Spf}\, V, N) \end{array}$$

admits a chart $\mathcal{C}_{\underline{P}} := (Q_V \xrightarrow{\varphi} N, \underline{R}_{\underline{P}} \to L, \underline{R}_X \to M, \underline{R}_X \to M, \mathcal{D}_{\underline{P}})$, where $\mathcal{D}_{\underline{P}}$ is the diagram

$$\begin{array}{ccc} \underline{R} & \xleftarrow{\mathrm{id}} & \underline{R} \\ \| & & \uparrow \\ \underline{R} & \longleftarrow & Q, \end{array}$$

($Q \to \underline{R}$ is the composite $Q \to R \hookrightarrow \underline{R}$). Let $\mathcal{C}_0$ be the restriction of the chart $\mathcal{C}_{\underline{P}}$ to the morphism $\iota \circ f$. Then $(\underline{P}, \mathcal{C}_{\underline{P}})$ is an exact charted affine pre-widening of (X, M) over $(\mathrm{Spf}\, V, N)$ with respect to the chart $\mathcal{C}_0$. Let $\mathcal{C}_{\hat{P}}$ be the pull-back of the chart $\mathcal{C}_{\underline{P}}$ to the diagram

$$\begin{array}{ccc} (X, M) & \xrightarrow{\hat{\imath}} & (\hat{P}, \hat{L}) \\ \| & & \Big\downarrow g \\ (X, M) & \xrightarrow{\iota \circ f} & (\mathrm{Spf}\, V, N). \end{array}$$

Then $(\hat{P}, \mathcal{C}_{\hat{P}})$ is an exact charted affine widening of (X, M) over $(\mathrm{Spf}\, V, N)$ with respect to the chart $\mathcal{C}_0$. Now, for a locally free isocrystal $\mathcal{E}$ on $(X/V)^{\log}_{\mathrm{conv,et}}$, put

$$\omega^i_{\hat{P}}(\mathcal{E}) := j_{\hat{P},*}(j^*_{\hat{P}}\mathcal{E} \otimes_{\mathcal{O}_{X/V}} \phi^*_{\hat{P}}(\omega^i_{P/V}|_{\hat{P}})).$$

Then we have the following theorem, which is a log version of [Og2, (0.5.4)]:

Theorem 2.3.5. *Let the notations be as above. Then,*

(1) *For a charted affine enlargement* $T = ((T, M_T), (Z, M_Z), i, z), \mathcal{C}_T)$ *of* (X, M) *over* $(\mathrm{Spf}\, V, N)$ *with respect to the chart* $\mathcal{C}_0$*, we have*

$$\omega^i_{\hat{P}}(\mathcal{E})(T) = \varprojlim_n \mathcal{E}(T_{Z,n}((T \times \hat{P})^{\mathrm{ex}})) \otimes_{\mathcal{O}_P} \omega^i_{P/V}.$$

(2) *There exists a canonical structure of complex on* $\omega^\bullet_{\hat{P}}(\mathcal{E})$.

(3) (*Log convergent Poincaré lemma*) *The adjoint homomorphism* $\mathcal{E} \longrightarrow j_{\hat{P},*} j^*_{\hat{P}} \mathcal{E} = \omega^0_{\hat{P}}(\mathcal{E})$ *induces the quasi-isomorphism*

$$\mathcal{E} \xrightarrow{\sim} \omega^\bullet_{\hat{P}}(\mathcal{E}).$$

Proof. First we prove the assertion (1). Let $\psi : \mathcal{F} \longrightarrow h_{\hat{P}}$ be the object in $(X/V)^{\log}_{\mathrm{conv,et}}|_{\hat{P}}$ defined by $\psi^{-1}(S \to \hat{P}) = \mathcal{E}(S) \otimes_{\mathcal{O}_P} \omega^i_{P/V}$. Then we have

$$\begin{aligned}
\omega^i_{\hat{P}}(\mathcal{E})(T) &= \mathrm{Hom}(h_T, j_{\hat{P},*}(j^*_{\hat{P}}\mathcal{E} \otimes_{\mathcal{O}_{X/V}} \phi^*_{\hat{P}}(\omega^i_{P/V}|_{\hat{P}}))) \\
&= \mathrm{Hom}(j^*_{\hat{P}} h_T, j^*_{\hat{P}}\mathcal{E} \otimes_{\mathcal{O}_{X/V}} \phi^*_{\hat{P}}(\omega^i_{P/V}|_{\hat{P}})) \\
&= \mathrm{Hom}(h_{T \times \hat{P}} \to h_{\hat{P}}, \mathcal{F} \xrightarrow{\psi} h_{\hat{P}}) \\
&= \varprojlim_n \mathrm{Hom}(h_{T_{Z,n}((T \times \hat{P})^{\mathrm{ex}})} \to h_{\hat{P}}, \mathcal{F} \xrightarrow{\psi} h_{\hat{P}}) \\
&= \varprojlim_n \mathcal{E}(T_{Z,n}((T \times \hat{P})^{\mathrm{ex}})) \otimes_{\mathcal{O}_P} \omega^i_{P/V},
\end{aligned}$$

as desired.

Next we prove the assertions (2) and (3). Since X is affine, any enlargement is affine (with respect to the chart $\mathcal{C}_0$) etale locally. So it suffices to construct a canonical, functorial structure of complex on $\omega^\bullet_{\hat{P}}(\mathcal{E})_T$ (:= the sheaf on T_{Zar} induced by $\omega^\bullet_{\hat{P}}(\mathcal{E})$) for charted affine enlargements $T := (((T, M_T), (Z, M_Z), i, z), \mathcal{C}_T)$ with respect to $\mathcal{C}_0$. Put $\mathcal{C}_T = (Q_V \xrightarrow{\varphi} N, \underline{R}_X \to M, U_T \to M_T, U'_Z \to Z, \mathcal{D}_T)$, where $\mathcal{D}_T$ is the diagram

$$\begin{array}{ccc}
U' & \longleftarrow & U \\
\uparrow & & \uparrow \\
\underline{R} & \longleftarrow & Q
\end{array}$$

such that $U^{\mathrm{gp}} \to (U')^{\mathrm{gp}}$ is surjective. Let $(\underline{P}^m, \underline{L}^m)$ be the m-th log infinitesimal neighborhood of $(\overline{P}, \overline{L})$ in $(\overline{P}, \overline{L}) \times_{(\mathrm{Spf}\, V, N)} (\overline{P}, \overline{L})$. Let $p_{i,m}$: $(\underline{P}^m, \underline{L}^m) \longrightarrow (\underline{P}, \underline{L})$ be the i-th projection ($i = 1, 2$) and put $X_i^m := p_{i,m}^{-1}(X)$. Let $s : \underline{R} \oplus_Q \underline{R} \longrightarrow \underline{R}$ be the homomorphism induced by the summation and put $R(1) := (s^{\mathrm{gp}})^{-1}(\underline{R})$. Let β_i be the composite $\underline{R} \overset{i\text{-th incl.}}{\longrightarrow} \underline{R} \oplus_Q \underline{R} \hookrightarrow R(1)$. Then the diagram

$$\begin{array}{ccc} (X_i^m, \underline{L}^m) & \longrightarrow & (\underline{P}^m, \underline{L}^m) \\ \big\downarrow & & \big\downarrow \\ (X, M) & \xrightarrow{\iota \circ f} & (\mathrm{Spf}\, V, N) \end{array}$$

admits a chart $\mathcal{C}_i := (Q_V \to N, \underline{R}_X \to M, R(1)_{\underline{P}^m} \to \underline{L}^m, R(1)_{X_i^m} \to \underline{L}^m, \mathcal{D}_i)$, where $\mathcal{D}_i$ is the diagram

$$\begin{array}{ccc} R(1) & \xleftarrow{\mathrm{id}} & R(1) \\ {\scriptstyle \beta_i}\big\uparrow & & \big\uparrow \\ \underline{R} & \longleftarrow & Q, \end{array}$$

and the diagram

$$\begin{array}{ccc} (X, M) & \longrightarrow & (\underline{P}^m, \underline{L}^m) \\ \big\| & & \big\downarrow \\ (X, M) & \xrightarrow{\iota \circ f} & (\mathrm{Spf}\, V, N) \end{array}$$

admits two charts which are defined as the pull-backs of the charts $\mathcal{C}_i$ ($i = 1, 2$), which we denote also by $\mathcal{C}_i$ by abuse of notation. Then, for $m \in \mathbb{N}$ and $i = 1, 2$, we have exact charted affine pre-widenings

$$\underline{P}^{m,i} := (((\underline{P}^m, \underline{L}^m), (X, M)), \mathcal{C}_i),$$
$$\underline{P}_i^m := (((\underline{P}^m, \underline{L}^m), (X_i^m, \underline{L}^m)), \mathcal{C}_i),$$
$$\underline{P} := (((\underline{P}, \underline{L}), (X, M)), \mathcal{C}_i),$$

and the following commutative diagram of exact charted affine pre-widenings:

$$\begin{array}{ccc} \underline{P}^{m,i} & \longrightarrow & \underline{P}_i^m \\ \big\downarrow & & {\scriptstyle q_i''}\big\downarrow \\ \underline{P} & = & \underline{P}, \end{array}$$

where the chart of the morphism q_i'' is induced by β_i. Now we take the products with $T := (((T, M_T), (Z, M_Z), i, z), \mathcal{C}_T)$ (in the category of affine pre-widenings with respect to $\mathcal{C}_0$). Then we obtain the following diagram of charted affine pre-widenings (but not exact in general):

$$\begin{array}{ccc} \underline{P}^{m,i} \times T & \longrightarrow & \underline{P}_i^m \times T \\ \big\downarrow & & \big\downarrow{\scriptstyle q_i'} \\ \underline{P} \times T & = & \underline{P} \times T. \end{array}$$

Now we calculate the exactifications of the charted affine pre-widenings which appear in the above diagram. In the following, for a (pre-)widening $S := ((S, M_S), (Y, M_Y))$, we denote the ring of global sections of $\mathcal{O}_S$ simply by $\Gamma(S)$. Note that the restriction of the chart of $\underline{P} \times T$ to the closed immersion

$$(Z, M_Z) \hookrightarrow (\underline{P}, \underline{L}) \times_{(\mathrm{Spf}\, V, N)} (T, M_T)$$

is given by the monoid homomorphism $\delta : (\underline{R} \oplus_Q U)^{\mathrm{int}} \longrightarrow U'$, and the restriction of the chart of $\underline{P}_i^m \times T$ (resp. $\underline{P}_i^m \times T$) to the closed immersion

$$(Z, M_Z) \hookrightarrow (\underline{P}^m, \underline{L}^m) \times_{(\mathrm{Spf}\, V, N)} (T, M_T)$$
$$(\text{resp.} \quad (Z_i^m, M_Z) \hookrightarrow (\underline{P}^m, \underline{L}^m) \times_{(\mathrm{Spf}\, V, N)} (T, M_T),$$

where $Z_i^m := (q_i')^{-1}(Z))$ is given by the monoid homomorphism

$$\epsilon_i : (R(1) \oplus_{\tau_i, Q} U)^{\mathrm{int}} \longrightarrow (R(1) \oplus_{\beta_1, \underline{R}} U')^{\mathrm{int}},$$

where τ_i is the composite $Q \longrightarrow R \xrightarrow{\beta_1} R(1)$. Let $\gamma_i : R(1) \longrightarrow \underline{R} \oplus (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})\, (i = 1, 2)$ be the monoid homomorphism given by $\gamma_1((x, y)) := (xy, y), \gamma_2((x, y)) := (xy, x)$. Then one can see easily that γ_i's are isomorphisms and that the composite

$$\underline{R} \hookrightarrow \underline{R} \oplus (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\gamma_i^{-1}} R(1)$$

coincides with the homomorphism β_i. So we have the following isomorphisms:

$$\begin{aligned} (R(1) \oplus_{\beta_i, \underline{R}} U')^{\mathrm{int}} &\cong ((\underline{R} \oplus (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})) \oplus_{\underline{R}} U')^{\mathrm{int}} \quad (\text{via } \gamma_i) \\ &\cong (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}}) \oplus U', \end{aligned}$$

$$\begin{aligned}(R(1)\oplus_{\tau_i,Q}U)^{\mathrm{int}} &\cong (R(1)\oplus_{\beta_i,\underline{R}}(\underline{R}\oplus_Q U)^{\mathrm{int}})^{\mathrm{int}} \\ &\cong ((\underline{R}\oplus(\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}}))\oplus_{\underline{R}}(\underline{R}\oplus_Q U)^{\mathrm{int}})^{\mathrm{int}} \quad (\text{via } \gamma_i) \\ &\cong (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})\oplus(\underline{R}\oplus_Q U)^{\mathrm{int}}.\end{aligned}$$

One can see that, via the above isomorphisms, the homomorphism ϵ_i is compatible with

$$\mathrm{id}\oplus\delta : (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})\oplus(\underline{R}\oplus_Q U)^{\mathrm{int}} \longrightarrow (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})\oplus U'.$$

Put $W := (\delta^{\mathrm{gp}})^{-1}(U')$ and $W_i := (\epsilon_i^{\mathrm{gp}})^{-1}((R(1)\oplus_{\beta_i,\underline{R}}U')^{\mathrm{int}})\,(i=1,2)$. Then we have the isomorphism $W_i \cong (\underline{R}^{\mathrm{gp}}/Q^{\mathrm{gp}})\oplus W$. Hence we obtain the following isomorphism:

$$\begin{aligned}(2.3.2)\qquad &\Gamma((\underline{P}^{m,i}\times T)^{\mathrm{ex}}) \\ &\cong (\Gamma(\underline{P}^{m,i}\times T)\otimes_{\mathbb{Z}[(R(1)\oplus_{\tau_i,Q}U)^{\mathrm{int}}]}\mathbb{Z}[W_i])^\wedge \\ &\cong (\Gamma(\underline{P}^{m})\otimes_{\Gamma(\underline{P})}\Gamma(\underline{P}\times T)\otimes_{\mathbb{Z}[(\underline{R}\oplus_Q U)^{\mathrm{int}}]}\mathbb{Z}[W])^\wedge \\ &\cong \Gamma(\underline{P}^{m})\otimes_{\Gamma(\underline{P})}\Gamma((\underline{P}\times T)^{\mathrm{ex}}).\end{aligned}$$

By the same argument, we obtain the isomorphism

$$(2.3.3)\qquad \Gamma((\underline{P}_i^{m}\times T)^{\mathrm{ex}}) \cong \Gamma(\underline{P}^{m})\otimes_{\Gamma(\underline{P})}\Gamma((\underline{P}\times T)^{\mathrm{ex}}).$$

Now let us consider the following diagram of exact pre-widenings for $i=1,2$:

$$(2.3.4)\qquad \begin{array}{ccc}(\underline{P}^{m,i}\times T)^{\mathrm{ex}} & \xrightarrow{r_i} & (\underline{P}_i^{m}\times T)^{\mathrm{ex}} \\ \downarrow & & \downarrow{\scriptstyle q_i} \\ (\underline{P}\times T)^{\mathrm{ex}} & = & (\underline{P}\times T)^{\mathrm{ex}}.\end{array}$$

By the isomorphisms (2.3.2) and (2.3.3), the underlying morphism of formal schemes of $q_i\,(i=1,2)$ are flat and the underlying morphism of formal schemes of $r_i\,(i=1,2)$ are identities.

For $n\in\mathbb{N}$, put $E_n := \mathcal{E}_{T_{Z,n}((\underline{P}\times T)^{\mathrm{ex}})} \cong \mathcal{E}_{T_{Z,n}((\hat{P}\times T)^{\mathrm{ex}})}$, $E_{i,n}^{m} := \mathcal{E}_{T_{Z,n}((\underline{P}_i^{m}\times T)^{\mathrm{ex}})}$ and $E_n^{m,i} := \mathcal{E}_{T_{Z,n}((\underline{P}^{m,i}\times T)^{\mathrm{ex}})}$. On the other hand, let E

be the coherent sheaf on the rigid analytic space $]Z[^{\log}_{\underline{P}^m \times T}$ associated to $\mathcal{E}$. Then, by Lemma 2.1.26 and the isomorphism (2.3.3), we have

$$\begin{cases} E^m_{1,n} \overset{\sim}{=} E_n \otimes_{\mathcal{O}_{\underline{P}}} \mathcal{O}_{\underline{P}^m}, \\ E^m_{2,n} \overset{\sim}{=} \mathcal{O}_{\underline{P}^m} \otimes_{\mathcal{O}_{\underline{P}}} E_n. \end{cases} \tag{2.3.5}$$

Next, by Proposition 2.1.27, we have the isomorphism of inductive systems of formal V-schemes

$$\{T_{Z,n}((\underline{P}^{m,i} \times T)^{\mathrm{ex}})\}_n \overset{\sim}{=} \{T_{Z,n}((\underline{P}^m_i \times T)^{\mathrm{ex}})\}_n$$

and it induces the isomorphism of the projective systems of sheaves on Z

$$\{E^m_{i,n}\}_n \overset{\sim}{=} \{E^{m,i}_n\}_n. \tag{2.3.6}$$

Thirdly, let us note that we have the equivalences of categories

$$\Phi_i : \begin{pmatrix} \text{compatible family of} \\ \text{isocoherent sheaves on} \\ \{T_{Z,n}((\underline{P}^{m,i} \times T)^{\mathrm{ex}})\}_n \end{pmatrix} \overset{\sim}{\longrightarrow} \begin{pmatrix} \text{coherent sheaf on} \\ ]Z[^{\log}_{\underline{P}^m \times T} \end{pmatrix},$$

and that we have

$$\Phi_i(\{E^m_{i,n}\}_n) = E \quad (i = 1, 2), \tag{2.3.7}$$

by definition of E. (See the proof of Proposition 2.2.7.) By the equations (2.3.5), (2.3.6) and (2.3.7), we have the isomorphism

$$\Phi_2(\{\mathcal{O}_{\underline{P}^m} \otimes_{\mathcal{O}_{\underline{P}}} E_n\}_n) \overset{\sim}{=} \Phi_1(\{E_n \otimes_{\mathcal{O}_{\underline{P}}} \mathcal{O}_{\underline{P}^m}\}_n).$$

Denote the specialization map $]Z[^{\log}_{\underline{P}^m \times T} \longrightarrow Z$ by sp. Then, by applying sp_* to the above isomorphism, we get the isomorphism

$$\begin{aligned} \mathcal{O}_{\underline{P}^m} \otimes_{\mathcal{O}_{\underline{P}}} \omega^0_{\hat{P}}(\mathcal{E})_T = \mathcal{O}_{\underline{P}^m} \otimes_{\mathcal{O}_{\underline{P}}} \varprojlim_n E_n \longrightarrow \varprojlim_n E_n \otimes_{\mathcal{O}_{\underline{P}}} \mathcal{O}_{\underline{P}^m} \\ = \omega^0_{\hat{P}}(\mathcal{E})_T \otimes_{\mathcal{O}_{\underline{P}}} \mathcal{O}_{\underline{P}^m}, \end{aligned}$$

which we denote by θ_m. Then we define the homomorphism

$$d : \omega^0_{\hat{P}}(\mathcal{E})_T \longrightarrow \omega^1_{\hat{P}}(\mathcal{E})_T = \omega^0_{\hat{P}}(\mathcal{E})_T \otimes_{\mathcal{O}_{\underline{P}}} \omega^1_{P/V}$$

by $d(e) := \theta_1(1 \otimes e) - e \otimes 1$, and extend it to the diagram

$$\omega^\bullet_{\hat{P}}(\mathcal{E})_T := [\omega^0_{\hat{P}}(\mathcal{E})_T \xrightarrow{d} \omega^1_{\hat{P}}(\mathcal{E})_T \xrightarrow{d} \cdots \xrightarrow{d} \omega^q_{\hat{P}}(\mathcal{E})_T \xrightarrow{d} \cdots]$$

by extending d to

$$d : \omega^q_{\hat{P}}(\mathcal{E})_T = \omega^0_{\hat{P}}(\mathcal{E})_T \otimes_{\mathcal{O}_{\underline{P}}} \omega^q_{P/V} \longrightarrow \omega^0_{\hat{P}}(\mathcal{E})_T \otimes_{\mathcal{O}_{\underline{P}}} \omega^{q+1}_{P/V} = \omega^{q+1}_{\hat{P}}(\mathcal{E})_T$$

by setting $x \otimes \eta \mapsto d(x) \wedge \eta + x \otimes d\eta$. The construction of the diagram $\omega^\bullet_{\hat{P}}(\mathcal{E})_T$ is functorial with respect to the charted affine enlargement T, and one can check that the diagram $\omega^\bullet_{\hat{P}}(\mathcal{E})_T$ is independent of the choice of the chart $\mathcal{C}_T$ of T. So, by varying T, we obtain the diagram

$$\omega^\bullet_{\hat{P}}(\mathcal{E}) := [\omega^0_{\hat{P}}(\mathcal{E}) \xrightarrow{d} \omega^1_{\hat{P}}(\mathcal{E}) \xrightarrow{d} \cdots \xrightarrow{d} \omega^q_{\hat{P}}(\mathcal{E}) \xrightarrow{d} \cdots].$$

We prove that the diagram $\omega^\bullet_{\hat{P}}(\mathcal{E})$ forms a complex and that the adjoint map $\mathcal{E} \longrightarrow j_{\hat{P},*}j^*_{\hat{P}}\mathcal{E} = \omega^0_{\hat{P}}(\mathcal{E})$ induces the quasi-isomorphism $\mathcal{E} \xrightarrow{\sim} \omega^\bullet_{\hat{P}}(\mathcal{E})$. To prove it, it suffices to check it on charted affine enlargements $T := (((T, M_T), (Z, M_Z), i, z), \mathcal{C}_T)$ with respect to $\mathcal{C}_0$ such that $\mathcal{E}_T$ is a free $K \otimes_V \mathcal{O}_T$-module. We have

$$\omega^0_{\hat{P}}(\mathcal{E})_T = \varprojlim_n (K \otimes_V \mathcal{O}_{T_{Z,n}((\underline{P}\times T)^{\mathrm{ex}})}) \otimes_{K\otimes\mathcal{O}_T} \mathcal{E}_T.$$

Note that the diagram (2.3.4) is defined over T. By this fact, one can see that $\mathcal{E}_T$ is annihilated by d. So the homomorphism

$$d : \omega^0_{\hat{P}}(\mathcal{E})_T \longrightarrow \omega^1_{\hat{P}}(\mathcal{E})_T = \omega^0_{\hat{P}}(\mathcal{E})_T \otimes_{\mathcal{O}_P} \omega^1_{P/V}$$

is expressed as $d(a \otimes e) = \underline{d}(a) \otimes e$ $(a \in \varprojlim_n (K \otimes_V \mathcal{O}_{T_{Z,n}((\underline{P}\times T)^{\mathrm{ex}})}), e \in \mathcal{E}_T)$, where

$$\underline{d} : \varprojlim_n (K \otimes_V \mathcal{O}_{T_{Z,n}((\underline{P}\times T)^{\mathrm{ex}})}) \longrightarrow \varprojlim_n (K \otimes_V \mathcal{O}_{T_{Z,n}((\underline{P}\times T)^{\mathrm{ex}})}) \otimes_{\mathcal{O}_P} \omega^1_{P/V}$$

is obtained by applying the functor sp_* to the relative differential

$$\tilde{d} : \mathcal{O}_{]Z[^{\log}_{\underline{P}\times T}} \longrightarrow \omega^1_{]Z[^{\log}_{\underline{P}\times T}/]Z[_T},$$

where $\omega^1_{]Z[^{\log}_{\underline{P}\times T}/]Z[_T}$ is defined by $\omega^1_{]Z[^{\log}_{\underline{P}\times T}/]Z[_T} := \omega^1_{]Z[^{\log}_{\underline{P}\times T}}/g_K^*\omega^1_{]Z[_T}$ (g_K : $]Z[^{\log}_{\underline{P}\times T}\longrightarrow]Z[_T$ is the morphism induced by the projection $g : \underline{P}\times T \longrightarrow T$). Since $\tilde{d}\circ\tilde{d} = 0$ holds, we have $d\circ d = 0$. So the diagram $\omega^\bullet_{\hat{P}}(\mathcal{E})$ forms a complex.

To prove that $\mathcal{E} \longrightarrow \omega^\bullet_{\hat{P}}(\mathcal{E})$ is a quasi-isomorphism, it suffices to show that the relative de Rham complex on $]Z[^{\log}_{\underline{P}\times T}$

$$\mathrm{DR} := [\mathcal{O}_{]Z[^{\log}_{\underline{P}\times T}} \xrightarrow{\tilde{d}} \omega^1_{]Z[^{\log}_{\underline{P}\times T}/]Z[_T} \xrightarrow{\tilde{d}} \omega^2_{]Z[^{\log}_{\underline{P}\times T}/]Z[_T} \xrightarrow{\tilde{d}} \cdots]$$

satisfies the quasi-isomorphism $Rg_{K,*}\mathrm{DR} = \mathcal{O}_{]Z[_T}$. Note that we have the diagram

$$\begin{array}{ccc}
(Z, M_Z) & \longrightarrow & ((\underline{P}\times T)^{\mathrm{ex}}, M_{(\underline{P}\times T)^{\mathrm{ex}}}) \\
\Big\| & & \Big\downarrow g^{\mathrm{ex}} \\
(Z, M_Z) & \xrightarrow{\ i\ } & (T, M_T),
\end{array}$$

where horizontal lines are exact closed immersions and g^{ex}(:= the morphism induced by g) is formally log smooth. (Here we denoted the log structure defined on $(\underline{P}\times T)^{\mathrm{ex}}$ defined by the exactification of the charted affine pre-widening $\underline{P}\times T$ by $M_{(\underline{P}\times T)^{\mathrm{ex}}}$.) Since g^{ex} is formally smooth on a neighborhood of Z, we have the isomorphism (by weak fibration theorem)

$$]Z[^{\log}_{\underline{P}\times T} =]Z[_{(\underline{P}\times T)^{\mathrm{ex}}} \stackrel{\sim}{=} D^r_V \times]Z[_T = D^r_A \tag{2.3.8}$$

Zariski locally on T, where r is the relative dimension of g. So, by Theorem B of Kiehl, we have $Rg_{K,*}\mathrm{DR} = g_{K,*}\mathrm{DR}$. So it suffices to prove that the homomorphism $\mathcal{O}_{]Z[_T} \longrightarrow g_{K,*}\mathrm{DR}$ is a quasi-isomorphism. Note that it suffices to prove the quasi-isomorphism

$$\Gamma(T_{1,K}, \mathcal{O}_{]Z[_T}) \xrightarrow{\sim} \Gamma(T_{1,K}, g_{K,*}\mathrm{DR})$$

for any admissible open affinoid $T_{1,K} \subset T_K =]Z[_T$ satisfying $g_K^{-1}(T_{1,K}) \overset{\sim}{=} T_{1,K} \times D_V^r$. By replacing T by T_1, the assertion is reduced to the proof of the quasi-isomorphism

$$\Gamma(]Z[_T, \mathcal{O}_{]Z[_T}) \xrightarrow{\sim} \Gamma(]Z[_T, g_{K,*}\mathrm{DR}) = \Gamma(]Z[^{\log}_{\underline{P}\times T}, \mathrm{DR}) \tag{2.3.9}$$

under the existence of the isomorphism (2.3.8).

Put $T := \mathrm{Spm}\, A$. For $s \in \mathbb{N}$, put $A(s)_K := \Gamma(D_A^r, \mathcal{O}_{D_A^r})$, $\Omega^1_{A(s)_K} := \oplus_{i=1}^s A(s)_K dt_i$ and $\Omega^q_{A(s)_K} := \bigwedge^q \Omega^1_{A(s)_K}$, where $t_1, \cdots, t_s$ are the coordinate of D_A^s. Then, one can see, by the construction of the isomorphism (2.3.8) due to Berthelot, that $\Gamma(]Z[^{\log}_{\underline{P}\times T}, \mathrm{DR})$ is identified with the relative de Rham complex

$$C(A,r) := [0 \to A(r)_K \to \Omega^1_{A(r)_K} \to \Omega^2_{A(r)_K} \to \cdots]$$

via the isomorphism (2.3.8). So, to prove the quasi-isomorphism (2.3.9), it suffices to prove the equations

$$H^q(C(A,r)) = \begin{cases} K \otimes_V A, & q = 0, \\ 0, & q > 0. \end{cases}$$

This is nothing but Lemma 2.2.16. So the proof of the theorem is completed. □

We have the following corollary:

COROLLARY 2.3.6. *Assume we are given the diagram*

$$\begin{array}{ccc} (X,M) & \xrightarrow{i} & (P,L) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ (\mathrm{Spec}\, k, N) & \xrightarrow{\iota} & (\mathrm{Spf}\, V, N) \end{array}$$

as in (2.3.1) (but we do not assume the conditions (a) and (b)), and let $\mathcal{E}$ *be a locally free isocrystal on* $(X/V)^{\log}_{\mathrm{conv,et}}$. *Then there exists uniquely a complex* $\omega^\bullet_P(\mathcal{E})$ *in* $(X/V)^{\log,\sim}_{\mathrm{conv,et}}$ *and a quasi-isomorphism* $\mathcal{E} \xrightarrow{\sim} \omega^\bullet_P(\mathcal{E})$

satisfying the following condition: Let $P' \subset P$ be an open immersion and put $(X', M') := (X, M) \times_{(P,L)} (P', L)$. Then, if the diagram

$$\begin{array}{ccc} (X', M') & \longrightarrow & (P', L) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ (\operatorname{Spec} k, N) & \xrightarrow{\iota} & (\operatorname{Spf} V, N) \end{array}$$

satisfies the conditions (a) and (b), the complex $\omega^\bullet_P(\mathcal{E})|_{(X'/V)^{\log}_{\mathrm{conv,et}}}$ and the quasi-isomorphism $\mathcal{E}|_{(X'/V)^{\log}_{\mathrm{conv,et}}} \overset{\sim}{\longrightarrow} \omega^\bullet_P(\mathcal{E})|_{(X'/V)^{\log}_{\mathrm{conv,et}}}$ are identical with $\omega^\bullet_{\hat{P}'}(\mathcal{E}|_{(X'/V)^{\log}_{\mathrm{conv,et}}})$ and $\mathcal{E}|_{(X'/V)^{\log}_{\mathrm{conv,et}}} \longrightarrow \omega^\bullet_{\hat{P}'}(\mathcal{E}|_{(X'/V)^{\log}_{\mathrm{conv,et}}})$ defined in Theorem 2.3.5 respectively, and this identification is functorial.

Proof. Since P is of Zariski type, the above diagram satisfies the conditions (a) and (b) Zariski locally on P. So, if we prove that the definition of $\omega^\bullet_{\hat{P}}(\mathcal{E})$ is independent of the chart $\mathcal{C}$ which was chosen before the theorem, we can glue the complex $\omega^\bullet_{\hat{P}}(\mathcal{E})$, and if we denote the resulting complex on $(X/V)^{\log}_{\mathrm{conv,et}}$ by $\omega^\bullet_P(\mathcal{E})$, this complex satisfies the required conditions. Hence it suffices to prove that the definition of $\omega^\bullet_{\hat{P}}(\mathcal{E})$ is independent of the chart $\mathcal{C}$.

Let us take another chart $\mathcal{C}'$ of the diagram $(X, M) \hookrightarrow (P, L) \longrightarrow (\operatorname{Spf} V, N)$. To show the required independence, we may assume that there exists a morphism of charts $\mathcal{C} \longrightarrow \mathcal{C}'$. Let $\hat{P}'$ be the exact widening defined as $\hat{P}$ by using the chart $\mathcal{C}'$ instead of $\mathcal{C}$ and let $\omega^\bullet_{\hat{P}'}(\mathcal{E})$ be the complex defined as $\omega^\bullet_{\hat{P}}(\mathcal{E})$ by using the chart $\mathcal{C}'$ instead of $\mathcal{C}$. Then the morphism of charts $\mathcal{C} \longrightarrow \mathcal{C}'$ induces the homomorphism of complexes

$$\omega^\bullet_{\hat{P}}(\mathcal{E}) \longrightarrow \omega^\bullet_{\hat{P}'}(\mathcal{E}).$$

It suffices to prove that the above homomorphism induces an isomorphism on each degree. By the assertion (1) of Theorem 2.3.5, we have

$$\omega^i_{\hat{P}}(\mathcal{E})_T = \varprojlim_n \mathcal{E}_{T_{Z,n}(\hat{P}\times T)^{\mathrm{ex}}} \otimes_{\mathcal{O}_P} \omega^i_{P/V},$$
$$\omega^i_{\hat{P}'}(\mathcal{E})_T = \varprojlim_n \mathcal{E}_{T_{Z,n}(\hat{P}'\times T)^{\mathrm{ex}}} \otimes_{\mathcal{O}_P} \omega^i_{P/V}.$$

Now note that we have the diagram of functors

$$\begin{array}{ccc}
\begin{pmatrix}\text{compatible family of} \\ \text{isocoherent sheaves on} \\ \{T_{Z,n}((\hat{P}\times T)^{\mathrm{ex}})\}_n\end{pmatrix} & \xrightarrow{\Phi} & \begin{pmatrix}\text{coherent sheaf on} \\ ]Z[^{\log}_{\hat{P}\times T}\end{pmatrix} \\
\alpha\downarrow & & \| \\
\begin{pmatrix}\text{compatible family of} \\ \text{isocoherent sheaves on} \\ \{T_{Z,n}((\hat{P}'\times T)^{\mathrm{ex}})\}_n\end{pmatrix} & \xrightarrow{\Phi'} & \begin{pmatrix}\text{coherent sheaf on} \\ ]Z[^{\log}_{\hat{P}\times T}\end{pmatrix},
\end{array}$$

where Φ and Φ' are equivalences of categories and α is the pull-back by the morphism of inductive systems of formal schemes $\{T_{Z,n}((\hat{P}'\times T)^{\mathrm{ex}})\}_n \longrightarrow \{T_{Z,n}((\hat{P}\times T)^{\mathrm{ex}})\}_n$. So we have the isomorphism

$$\begin{aligned}\Phi(\{\mathcal{E}_{T_{Z,n}((\hat{P}\times T)^{\mathrm{ex}})}\otimes_{\mathcal{O}_P}\omega^i_{P/V}\}_n) &= \Phi'\circ\alpha(\{\mathcal{E}_{T_{Z,n}((\hat{P}\times T)^{\mathrm{ex}})}\otimes_{\mathcal{O}_P}\omega^i_{P/V}\}_n) \\ &\xrightarrow{\sim} \Phi'(\{\mathcal{E}_{T_{Z,n}((\hat{P}'\times T)^{\mathrm{ex}})}\otimes_{\mathcal{O}_P}\omega^i_{P/V}\}_n).\end{aligned}$$

By applying the functor sp_* (where sp is the specialization map $]Z[^{\log}_{P\times T}\longrightarrow Z$) to the above isomorphism, we get the isomorphism

$$\varprojlim_n \mathcal{E}_{T_{Z,n}(\hat{P}\times T)^{\mathrm{ex}}}\otimes_{\mathcal{O}_P}\omega^i_{P/V} \xrightarrow{\sim} \varprojlim_n \mathcal{E}_{T_{Z,n}(\hat{P}'\times T)^{\mathrm{ex}}}\otimes_{\mathcal{O}_P}\omega^i_{P/V},$$

that is, the isomorphism

$$\omega^i_{\hat{P}}(\mathcal{E})_T \xrightarrow{\sim} \omega^i_{\hat{P}'}(\mathcal{E})_T.$$

So the proof is finished. □

REMARK 2.3.7. The similar argument shows that the complex $\omega^\bullet_P(\mathcal{E})$ is independent of the choice of the chart $\varphi : Q_V \to N$ of $(\mathrm{Spf}\, V, N)$ chosen above. Details are left to the reader.

Corollary 2.3.8. *Assume we are given the diagram*

$$\begin{array}{ccc} (X,M) & \xrightarrow{i} & (P,L) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ (\operatorname{Spec} k, N) & \xrightarrow{\iota} & (\operatorname{Spf} V, N) \end{array}$$

as in Corollary 2.3.6 and let $\mathcal{E}$ be a locally free isocrystal on $(X/V)^{\log}_{\mathrm{conv,et}}$. Then we have the canonical quasi-isomorphism

$$R\tilde{u}_*\mathcal{E} = \mathrm{sp}_*\mathrm{DR}(]X[^{\log}_P, \mathcal{E}).$$

Proof. By Theorem 2.3.5 and Corollary 2.3.6, we have the diagram

$$R\tilde{u}_*\mathcal{E} \longrightarrow R\tilde{u}_*\omega^{\bullet}_P(\mathcal{E}) \longleftarrow \tilde{u}_*\omega^{\bullet}_P(\mathcal{E}),$$

and the first arrow is a quasi-isomorphism. We prove first that the second arrow is also a quasi-isomorphism. To prove this, we may work Zariski locally. So we may assume the conditions (a) and (b). Then we have

$$\begin{aligned} R\tilde{u}_*\omega^{\bullet}_P(\mathcal{E}) &= R\tilde{u}_* j_{\hat{P},*}(\phi^*_{\hat{P}}(\omega^{\bullet}_{P/V}|_{\hat{P}}) \otimes j^*_{\hat{P}}\mathcal{E}) \\ &= \tilde{u}_* j_{\hat{P},*}(\phi^*_{\hat{P}}(\omega^{\bullet}_{P/V}|_{\hat{P}}) \otimes j^*_{\hat{P}}\mathcal{E}) \quad \text{(Corollary 2.3.4)} \\ &= \tilde{u}_*\omega^{\bullet}_P(\mathcal{E}). \end{aligned}$$

So we obtain the assertion.

Now it suffices to prove the isomorphism (not only the quasi-isomorphism)

$$\tilde{u}_*\omega^{\bullet}_P(\mathcal{E}) \overset{\sim}{=} \mathrm{sp}_*\mathrm{DR}(]X[^{\log}_P, \mathcal{E}).$$

To prove it, it suffices to construct the functorial isomorphism Zariski locally. So we may assume the conditions (a) and (b). Let $\varphi : \mathcal{F} \longrightarrow h_{\hat{P}}$ be as in the proof of Theorem 2.3.5 and let e be the initial object of $(X/V)^{\log,\sim}_{\mathrm{conv,et}}$. Then we have

$$\Gamma(X, \tilde{u}_*\omega^i_P(\mathcal{E})) = \Gamma(X, \tilde{u}_* j_{\hat{P},*}(\phi^*_{\hat{P}}(\omega^i_{P/V}|_{\hat{P}}) \otimes j^*_{\hat{P}}\mathcal{E}))$$

$$
\begin{aligned}
&= \mathrm{Hom}(e, j_{\hat{P},*}(\phi_{\hat{P}}^*(\omega_{P/V}^i|_{\hat{P}}) \otimes j_{\hat{P}}^*\mathcal{E}) \\
&= \mathrm{Hom}(h_{\hat{P}} \xrightarrow{\mathrm{id}} h_{\hat{P}}, \mathcal{F} \xrightarrow{\varphi} h_{\hat{P}}) \\
&= \varprojlim_n \mathcal{E}(T_{X,n}(\hat{P})) \otimes_{\mathcal{O}_P} \omega_{P/V}^i.
\end{aligned}
$$

Hence we have the isomorphism of the global sections of each degree $\Gamma(X, \tilde{u}_*\omega_P^i(\mathcal{E})) \overset{\simeq}{=} \Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^i)$. Since this isomorphism is functorial with respect to $(X,M) \hookrightarrow (P,L)$, we obtain the isomorphism of sheaves $\tilde{u}_*\omega_P^i(\mathcal{E}) \overset{\simeq}{=} \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^i$.

Now we prove that the above isomorphism induces the isomorphism of complexes. Note that, for any charted affine enlargement T with respect to $\mathcal{C}_0$, the diagram (2.3.4) in the proof of Theorem 2.3.5 is defined over the following diagram of affine enlargements:

$$
\begin{array}{ccc}
\underline{P}^m & \longrightarrow & \underline{P}_i^m \\
\downarrow & & \downarrow \\
\underline{P} & \Longleftrightarrow & \underline{P},
\end{array}
$$

where $\underline{P}^m$ is the underlying enlargement of $\underline{P}^{m,i}$. (Note that it is independent of i.) Then, by the construction of the complex $\omega_P^\bullet(\mathcal{E})(T)$ and that of the complex $\mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})$ given in Remark 2.2.9, one obtains the following: for any charted affine enlargement T, the differential

$$\Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^{q-1}) \longrightarrow \Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^q)$$

is compatible with the differential

$$\omega_P^{q-1}(\mathcal{E})(T) \longrightarrow \omega_P^q(\mathcal{E})(T)$$

via the homomorphisms

$$\Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^r) \overset{\simeq}{=} \Gamma(X, \tilde{u}_*\omega_P^r(\mathcal{E})) \longrightarrow \omega_P^r(\mathcal{E})(T) \quad (r = q-1, q).$$

Note that this compatibility is true for any charted affine enlargement T. So the above compatibility implies that the differential

$$\Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^{q-1}) \longrightarrow \Gamma(X, \mathrm{sp}_*\mathrm{DR}(]X[_P^{\log}, \mathcal{E})^q)$$

is compatible with the differential

$$\Gamma(X, \tilde{u}_* \omega_P^{q-1}(\mathcal{E})) \longrightarrow \Gamma(X, \tilde{u}_* \omega_P^q(\mathcal{E}))$$

via the isomorphism $\Gamma(X, \mathrm{sp}_* \mathrm{DR}(]X[_P^{\log}, \mathcal{E})^r) \overset{\simeq}{=} \Gamma(X, \tilde{u}_* \omega_P^r(\mathcal{E}))$ $(r = q - 1, q)$. Hence the complex $\mathrm{sp}_* \mathrm{DR}(]X[_P^{\log}, \mathcal{E})$ is isomorphic to the complex $\tilde{u}_* \omega_P^\bullet(\mathcal{E})$ as complexes on X_{Zar}. So the proof of the corollary is finished. □

COROLLARY 2.3.9 (Log Convergent Poincaré Lemma). *Let us assume given the diagram*

$$(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N),$$

where f is of finite type, ι is the canonical exact closed immersion and assume that $(\mathrm{Spf}\, V, N)$ *admits a chart. Then, for a locally free isocrystal $\mathcal{E}$ on $(X/V)_{\mathrm{conv,et}}^{\log}$, we have the isomorphism*

$$H^i((X/V)_{\mathrm{conv,et}}^{\log}, \mathcal{E}) \overset{\simeq}{=} H_{\mathrm{an}}^i((X/V)^{\log}, \mathcal{E}).$$

PROOF. Take a good embedding system

$$(X, M) \xleftarrow{g} (X^{(\bullet)}, M^{(\bullet)}) \overset{i}{\hookrightarrow} (P^{(\bullet)}, L^{(\bullet)})$$

and let $\mathrm{sp}^{(\bullet)}$ be the specialization map

$$]X^{(\bullet)}[_{P^{(\bullet)}}^{\log} \longrightarrow X^{(\bullet)}.$$

Then we have

$$\begin{aligned} H^i((X/V)_{\mathrm{conv,et}}^{\log}, \mathcal{E}) &\overset{\simeq}{=} H^i((X^{(\bullet)}/V)_{\mathrm{conv,et}}^{\log}, g^*\mathcal{E}) \quad \text{(Proposition 2.1.20)} \\ &\overset{\simeq}{=} H^i(X_{\mathrm{Zar}}^{(\bullet)}, R\tilde{u}_*(g^*\mathcal{E})) \\ &\overset{\simeq}{=} H^i(X_{\mathrm{Zar}}^{(\bullet)}, \mathrm{sp}_*^{(\bullet)} \mathrm{DR}(]X^{(\bullet)}[_{P^{(\bullet)}}^{\log}, g^*\mathcal{E})) \\ &\qquad \text{(Corollary 2.3.8)} \end{aligned}$$

$$\begin{aligned}&\cong H^i(X, Rg_* R\mathrm{sp}_* \mathrm{DR}(]X[^{\log}_P, g^*\mathcal{E}))\\ &\qquad \text{(Theorem B of Kiehl)}\\ &= H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}).\end{aligned}$$

So we are done. □

REMARK 2.3.10. One can see easily that the above isomorphism is independent of the choice of the good embedding system.

REMARK 2.3.11. Let us assume given the diagram

$$(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N),$$

where f is of finite type, N is a fine log structure on $\mathrm{Spf}\, V$ and ι is the canonical exact closed immersion. In this remark, we prove that the isomorphism

$$H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \cong H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E})$$

holds even if $(\mathrm{Spf}\, V, N)$ does not admit a chart.

Let V' be a finite Galois extension of V with Galois group G such that $(\mathrm{Spf}\, V', N)$ admits a chart. Put $X' := X \times_V V'$ and let $g : X' \longrightarrow X$ be the canonical projection. Then we have the isomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \cong H^i_{\mathrm{an}}((X'/V')^{\log}, \mathcal{E})^G,$$

by definition given in Remark 2.2.18. On the other hand, by Corollary 2.3.9, we have the isomorphism

$$H^i((X'/V')^{\log}_{\mathrm{conv,et}}, g^*\mathcal{E}) \cong H^i_{\mathrm{an}}((X'/V')^{\log}, g^*\mathcal{E}),$$

and one can see that this isomorphism is G-equivariant. So it suffices to prove the isomorphism

$$H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \cong H^i((X'/V')^{\log}_{\mathrm{conv,et}}, g^*\mathcal{E})^G.$$

First, since V' is etale over V, we have the canonical equivalence of site $(X'/V')^{\log}_{\mathrm{conv,et}} \simeq (X'/V)^{\log}_{\mathrm{conv,et}}$. Denote the morphism of topoi

$$(X'/V)^{\log,\sim}_{\mathrm{conv,et}} \longrightarrow (X/V)^{\log,\sim}_{\mathrm{conv,et}}$$

induced by g by $g := (g_*, g^*)$, by abuse of notation.

Let I be an injective sheaf on $(X/V)^{\log}_{\text{conv,et}}$. We prove that g^*I is flabby. To prove this, it suffices to prove the vanishing $\check{H}^q(\mathcal{U}, g^*I) = 0\,(q > 0)$ for any enlargement T in $(X'/V)^{\log}_{\text{conv,et}}$ and any covering $\mathcal{U} := \{T_\lambda \longrightarrow T\}_\lambda$ of T (see [Mi, III.2.12]). Since the objects T_λ, T naturally define the objects in $(X/V)^{\log}_{\text{conv,et}}$ and the covering $\mathcal{U}$ naturally defines a covering in $(X/V)^{\log}_{\text{conv,et}}$, we have $\check{H}^q(\mathcal{U}, g^*I) = \check{H}^q(\mathcal{U}, I) = 0\,(q > 0)$, as desired. Hence g^*I is flabby.

Now let us consider the following diagram of topoi:

$$\begin{array}{ccc} (X/V)^{\log,\sim}_{\text{conv,et}} & \xleftarrow{\ g\ } & (X'/V)^{\log,\sim}_{\text{conv,et}} \\ {\scriptstyle u}\big\downarrow & & {\scriptstyle u'}\big\downarrow \\ X_{\text{et}} & \xleftarrow{\ f\ } & X'_{\text{et}}, \end{array}$$

where u' is the functor u for X'. Note that we have the isomorphism $f^* \circ u_* = u'_* \circ g^*$. Let us take an injective resolution $\mathcal{E} \longrightarrow I^\bullet$. Then $g^*\mathcal{E} \longrightarrow g^*I^\bullet$ is a flabby resolution of $g^*\mathcal{E}$. So we have

$$f^*(Ru_*\mathcal{E}) = f^*u_*I^\bullet = u'_*g^*I^\bullet = Ru'_*g^*\mathcal{E}. \tag{2.3.10}$$

Now let us note that we have the Hochschild-Serre spectral sequence

$$H^p(G, H^q(X'_{\text{et}}, f^*Ru_*\mathcal{E})) \Longrightarrow H^{p+q}(X_{\text{et}}, Ru_*\mathcal{E}).$$

Applying (2.3.10), we get the Hochschild-Serre spectral sequence for log convergent cohomology

$$H^p(G, H^q((X'/V)^{\log}_{\text{conv,et}}, g^*\mathcal{E})) \Longrightarrow H^{p+q}((X/V)^{\log}_{\text{conv,et}}, \mathcal{E}).$$

Since $H^q((X'/V)^{\log}_{\text{conv,et}}, g^*\mathcal{E})$'s are $\mathbb{Q}$-vector spaces, the above spectral sequence always degenerates. Hence we get the desired isomorphism

$$H^i((X/V)^{\log}_{\text{conv,et}}, \mathcal{E}) \cong H^i((X'/V)^{\log}_{\text{conv,et}}, g^*\mathcal{E})^G = H^i((X'/V')^{\log}_{\text{conv,et}}, g^*\mathcal{E})^G.$$

2.4. Log convergent cohomology and rigid cohomology

Let us assume given the diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, \text{triv. log str.}) \hookrightarrow (\operatorname{Spf} V, \text{triv. log str.}),$$

where (X, M) is an fs log scheme of Zariski type and f is a proper log smooth morphism of finite type. Put $U := X_{\mathrm{triv}}$ and denote the open immersion $U \hookrightarrow X$ by j. Let $a \in \mathbb{N}, a > 0$ and assume that there exists a lifting $\sigma : \operatorname{Spf} V \longrightarrow \operatorname{Spf} V$ of the a-times iteration of the absolute Frobenius $F_k : \operatorname{Spec} k \longrightarrow \operatorname{Spec} k$. Let $\mathcal{E}$ be an F^a-isocrystal (whose definition will be given in Definition 2.4.2) on (X, M) over $\operatorname{Spf} V$. Then, as we will see in Proposition 2.4.1, 'the restriction of $\mathcal{E}$ to U' has naturally the structure of an overconvergent isocrystal on U over $\operatorname{Spf} V$, which we denote by $j^\dagger \mathcal{E}$. The purpose of this section is to prove the isomorphism

$$H^i_{\mathrm{rig}}(U/K, j^\dagger \mathcal{E}) \cong H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}). \tag{2.4.1}$$

Since we have shown the isomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \cong H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E})$$

in the previous section (we will see that $\mathcal{E}$ is indeed locally free in Proposition 2.4.3), the isomorphism (2.4.1) implies the isomorphism

$$H^i_{\mathrm{rig}}(U/K, j^\dagger \mathcal{E}) \cong H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E})$$

between rigid cohomology and log convergent cohomology. The important part of the proof of the isomorphism (2.4.1) is due to Baldassarri and Chiarellotto([Ba-Ch], [Ba-Ch2]): We reduce the proof of the isomorphism to certain local assertion and prove it by using the theory of p-adic differential equations with log poles on unit disk over smooth affinoid rigid analytic space, which is developped by them.

First, we define the restriction functor from the category of locally free isocrystals on log convergent site to the category of overconvergent isocrystals. Assume we are given the diagram

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N),$$

where f is a morphism between fine log formal V-schemes of finite type, N is a fine log structure on $\operatorname{Spf} V$ such that $(\operatorname{Spf} V, N)$ admits a chart and ι is the canonical exact closed immersion. Put $U := X_{f\text{-triv}}$ and denote the open immersion $U \hookrightarrow X$ by j. Then, we have the canonical restriction functor

$$j^* : I_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I_{\mathrm{conv,et}}((U/V)^{\log}) \simeq I_{\mathrm{conv,et}}(U/V).$$

On the other hand, we can construct the functor

$$r : I^\dagger(U, X) \longrightarrow I_{\mathrm{conv,et}}(U/V)$$

in the following way: Take a diagram

$$X \xleftarrow{g} X^{(\bullet)} \overset{i}{\hookrightarrow} P^{(\bullet)}, \tag{2.4.2}$$

where g is a Zariski hypercovering and i is a closed immersion into a simplicial formal V-scheme such that each $P^{(i)}$ is formally smooth on a neighborhood of $U^{(i)} := X^{(i)} \times_X U$. Then, an object $\mathcal{E} \in I^\dagger(U, X)$ defines a pair $(\mathcal{E}^{(0)}, \varphi)$, where $\mathcal{E}^{(0)}$ is an object in $I^\dagger(U^{(0)}, X^{(0)}, P^{(0)})$ and φ is an isomorphism

$$p_2\mathcal{E}^{(0)} \xrightarrow{\sim} p_1\mathcal{E}^{(0)}$$

in $I^\dagger(U^{(1)}, X^{(1)}, P^{(1)})$ (where $p_i \, (i = 1, 2)$ is the functor $I^\dagger(U^{(0)}, X^{(0)}, P^{(0)}) \longrightarrow I^\dagger(U^{(1)}, X^{(1)}, P^{(1)})$ induced by the i-th projection) which reduces to the identity in $I^\dagger(U^{(0)}, X^{(0)}, P^{(0)})$ and satisfies the cocycle condition in $I^\dagger(U^{(2)}, X^{(2)}, P^{(2)})$. Denote the open immersion $U^{(i)} \hookrightarrow X^{(i)}$ by $j^{(i)}$. First let us define the functor

$$r^{(i)} : I^\dagger(U^{(i)}, X^{(i)}, P^{(i)}) \longrightarrow I_{\mathrm{conv,et}}(U^{(i)}/V)$$

as follows: For $n \in \mathbb{N}$, denote the $(n+1)$-fold fiber product of $P^{(i)}$ over $\operatorname{Spf} V$ by $P^{(i)}(n)$. For a strict neighborhood O of $]U^{(i)}[_{P^{(i)}(n)}$ in $]X^{(i)}[_{P^{(i)}(n)}$, let us denote the inclusion $]U^{(i)}[_{P^{(i)}(n)} \hookrightarrow O$ by α_O. Then pull-back by α_O's induce the functor

$$\alpha^{(i)}(n) : \begin{pmatrix} \text{coherent} \\ j^{(i),\dagger}\mathcal{O}_{]X^{(i)}[_{P^{(i)}(n)}}\text{-modules} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{coherent} \\ \mathcal{O}_{]U^{(i)}[_{P^{(i)}(n)}}\text{-modules} \end{pmatrix}$$

for $i, n \in \mathbb{N}$. Since the functors $\alpha^{(i)}(n)$ are compatible with projections with respect to n, it induces the functor

$$I^\dagger(U^{(i)}, X^{(i)}, P^{(i)}) \longrightarrow \mathrm{Str}''(U^{(i)} \hookrightarrow P^{(i)}/\mathrm{Spf}\, V).$$

By composing it with the equivalence

$$\mathrm{Str}''(U^{(i)} \hookrightarrow P^{(i)}/\mathrm{Spf}\, V) \simeq I_{\mathrm{conv,et}}(U^{(i)}/V),$$

we obtain the functor $r^{(i)}$. Then, since the functors $r^{(i)}$ $(i \in \mathbb{N})$ are compatible with projections with respect to i, $r^{(i)}$'s for $i = 0, 1, 2$ define the functor

$$r : I^\dagger(U, X) \longrightarrow I_{\mathrm{conv,et}}(U/V).$$

One can check that this definition is independent of the choice of the diagram (2.4.2).

Now we construct a functor

$$j^\dagger : I^{\mathrm{lf}}_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I^\dagger(U, X)$$

which is compatible with j^* and r:

Proposition 2.4.1. *Let $(X, M) \longrightarrow (\mathrm{Spec}\, k, N) \hookrightarrow (\mathrm{Spf}\, V, N), U, j, j^*$ and r be as above and assume that (X, M) is of Zariski type. Then there exists a functor*

$$j^\dagger : I^{\mathrm{lf}}_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I^\dagger(U, X)$$

such that $r \circ j^\dagger = j^*$ holds.

Proof. Take a diagram

$$(X, M) \xleftarrow{g} (X^{(\bullet)}, M^{(\bullet)}) \overset{i}{\hookrightarrow} (P^{(\bullet)}, L^{(\bullet)}),$$

where $g := \{g^{(n)}\}_n$ is a Zariski hypercovering satisfying $(g^{(n)})^* M \overset{\simeq}{=} M^{(n)}$ and i is a locally closed immersion into a simplicial fine log formal V-scheme such that each $(P^{(n)}, L^{(n)})$ is formally log smooth over $(\mathrm{Spf}\, V, N)$ of Zariski type and formally smooth (in the classical sense) over $(\mathrm{Spf}\, V, N)$

on a neighborhood of $U^{(n)} := X^{(n)} \times_X U$. (The existence of such a diagram can be shown in a similar way as Proposition 2.2.11. The detail is left to the reader.) Denote the open immersion $U^{(\bullet)} \hookrightarrow X^{(\bullet)}$ by $j^{(\bullet)}$. Since the categories $I^{\mathrm{lf}}_{\mathrm{conv,et}}((X/V)^{\log})$ and $I^\dagger(U,X)$ satisfy the descent property for Zariski open covering of X, it suffices to construct the functors

$$j^{(n),\dagger} : I^{\mathrm{lf}}_{\mathrm{conv,et}}((X^{(n)}/V)^{\log}) \longrightarrow I^\dagger(U^{(n)}, X^{(n)}) = I^\dagger(U^{(n)}, X^{(n)}, P^{(n)})$$

which are compatible with the transition morphisms of simplicial objects such that $r^{(n)} \circ j^{(n),\dagger} = j^{(n),*}$ holds, where $r^{(n)}$ is the functor r for $U^{(n)} \hookrightarrow X^{(n)}$.

First, let us note the following claim:

CLAIM. Let $((X,M) \overset{i}{\hookrightarrow} (P,L))$ be an object in the category $\mathcal{Z}$ introduced in Proposition 2.2.4, and let U be an open set of X such that $i|_{(U,M)}$ is exact. Then there exists a canonical map

$$\varphi :]X[^{\log}_P \longrightarrow]X[_P$$

and there exists a strict neighborhood V of $]U[_P$ in $]X[_P$ such that the morphism $\varphi^{-1}(V) \longrightarrow V$ induced by φ is an isomorphism.

PROOF OF CLAIM. If there exists a factorization

$$(X,M) \overset{i'}{\hookrightarrow} (P',L') \overset{f'}{\longrightarrow} (P,L)$$

such that i' is an exact closed imersion and f' is formally log etale, then we have $]X[^{\log}_P =]X[_{P'}$ by definition. So we define φ as the morphism $]X[_{P'} \longrightarrow]X[_P$ induced by f'. In general case, we can define φ Zariski locally on P, since there exists a factorization as above Zariski locally. Moreover, we can define the morphism $\varphi :]X[^{\log}_P \longrightarrow]X[_P$ globally by gluing this local definition.

Let us prove the latter statement. To prove it, we may work Zariski locally and so we may assume the existence of the above factorization. Then, f' is formally etale on a neighborhood of U. Then, we can apply Theorem 1.3.5 and so there exists a strict neighborhood V of $]U[_P$ in $]X[_P$ such that $\varphi^{-1}(V) \longrightarrow V$ is an isomorphism. □

Let us denote the essential image of the category $I^{\mathrm{lf}}_{\mathrm{conv,et}}((X^{(n)}/V)^{\log})$ via the equivalence of categories

$$I_{\mathrm{conv,et}}((X^{(n)}/V)^{\log}) \simeq \mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))$$

by $\mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}}$. Then $I^{\mathrm{lf}}_{\mathrm{conv,et}}((X^{(n)}/V)^{\log})$ is equivalent to $\mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}}$. One can see that, for any object (E, ϵ) in $\mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}}$, E is a locally free $\mathcal{O}_{]X[^{\log}_P}$-module.

Now we construct a functor

$$\begin{aligned} \underline{j}^{(n),\dagger} :& \mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}} \\ &\longrightarrow I^{\dagger}(U^{(n)}, X^{(n)}, P^{(n)}). \end{aligned}$$

For $m \in \mathbb{N}$, let $(P^{(n)}(m), L^{(n)}(m))$ be the $(m+1)$-fold fiber product of $(P^{(n)}, L^{(n)})$ over $(\mathrm{Spf}\, V, N)$. By the claim, we have the morphism

$$]X^{(n)}[^{\log}_{P^{(n)}(m)} \longrightarrow]X^{(n)}[_{P^{(n)}(m)},$$

which we denote by $\varphi^{(n)}(m)$. Let us take strict neighborhoods $V^{(n)}(m)$ of $]U^{(n)}[_{P^{(n)}(m)}$ in $]X^{(n)}[_{P^{(n)}(m)}$ satisfying the following conditions:

(1) The morphisms $(\varphi^{(n)}(m))^{-1}(V^{(n)}(m)) \longrightarrow V^{(n)}(m)$ are isomorphisms.

(2) For any projections $\alpha^* : P^{(n)}(m) \longrightarrow P^{(n)}(m')$ corresponding to an injective map $\alpha : [0, m'] \hookrightarrow [0, m]$, the induced map of rigid analytic spaces $\alpha^*_K :]X^{(n)}[_{P^{(n)}(m)} \longrightarrow]X^{(n)}[_{P^{(n)}(m')}$ satisfies $V^{(n)}(m) \subset (\alpha^*_K)^{-1}(V^{(n)}(m'))$.

Let

$$\begin{aligned} p_i :&]X^{(n)}[_{P^{(n)}(1)} \longrightarrow]X^{(n)}[_{P^{(n)}} \quad (i = 1, 2), \\ p_{ij} :&]X^{(n)}[_{P^{(n)}(2)} \longrightarrow]X^{(n)}[_{P^{(n)}(1)} \quad (1 \leq i < j \leq 3) \end{aligned}$$

be the projections and let

$$\Delta :]X^{(n)}[_{P^{(n)}} \longrightarrow]X^{(n)}[_{P^{(n)}(1)}$$

be the diagonal map. Put $\overline{V}^{(n)}(0) := \Delta^{-1}(V^{(n)}(1))$. Then, by the above conditions on $V^{(n)}(m)$'s, any object (E, ϵ) in the category $\mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))$ defines the following data:

(1) A locally free sheaf E on $V^{(n)}(0)$.

(2) An isomorphism $\epsilon : p_2^* E \xrightarrow{\sim} p_1^* E$ on $V^{(n)}(1)$ satisfying $p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon) = p_{13}^*(\epsilon)$ on $V^{(n)}(2)$ and $\Delta^*(\epsilon) = \mathrm{id}$ on $\overline{V}^{(n)}(0)$.

By pushing these data to $]X^{(n)}[_{P^{(n)}}$, we obtain an object in the category $I^\dagger(U^{(n)}, X^{(n)}, P^{(n)})$. So we have constructed the functor

$$\mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}} \longrightarrow I^\dagger(U^{(n)}, X^{(n)}, P^{(n)}),$$

which is the definition of the functor $\underline{j}^{(n),\dagger}$.

Let us define the functor

$$j^{(n),\dagger} : I^{\mathrm{lf}}_{\mathrm{conv,et}}((X^{(n)}/V)^{\log}) \longrightarrow I^\dagger(U^{(n)}, X^{(n)}) = I^\dagger(U^{(n)}, X^{(n)}, P^{(n)})$$

by the composite

$$I^{\mathrm{lf}}_{\mathrm{conv,et}}((X^{(n)}/V)^{\log}) \simeq \mathrm{Str}''((X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})/(\mathrm{Spf}\, V, N))^{\mathrm{lf}} \xrightarrow{\underline{j}^{(n),\dagger}} I^\dagger(U^{(n)}, X^{(n)}, P^{(n)}).$$

Then one can check easily that this functor has the required properties. Hence the assertion is proved. □

Now we prove the comparison theorem between the analytic cohomology and the rigid cohomology. In the following in this section, we consider the following situation: We are given a diagram

$$(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, \mathrm{triv.\, log\, str.}) \hookrightarrow (\mathrm{Spf}\, V, \mathrm{triv.\, log\, str.}),$$

where (X, M) is an fs log scheme and f is a proper log smooth morphism of finite type. In the following, we write $(\mathrm{Spec}\, k, \mathrm{triv.\, log\, str.})$ and $(\mathrm{Spf}\, V, \mathrm{triv.\, log\, str.})$ simply by $\mathrm{Spec}\, k, \mathrm{Spf}\, V$, respectively. Put $U := X_{\mathrm{triv}}$ and denote the open immersion $U \hookrightarrow X$ by j.

First we define the notion of F^a-isocrystals on $(X/V)^{\log}_{\mathrm{conv,et}} := ((X, M)/V)_{\mathrm{conv,et}}$.

Definition 2.4.2. Let the notations be as above and let $F_X : (X,M) \longrightarrow (X,M)$, $F_k : \operatorname{Spec} k \longrightarrow \operatorname{Spec} k$ be the absolute Frobenius endomorphisms. Let $a \in \mathbb{N}, a > 0$ and assume there exists a morphism $\sigma : \operatorname{Spf} V \longrightarrow \operatorname{Spf} V$ which coincides with F_k^a modulo the maximal ideal of V. Then we have the following commutative diagram:

$$\begin{array}{ccc} (X,M) & \xrightarrow{F_X^a} & (X,M) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} k & \xrightarrow{F_k^a} & \operatorname{Spec} k \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spf} V & \xrightarrow{\sigma} & \operatorname{Spf} V. \end{array}$$

For an isocrystal $\mathcal{E}$ on log convergent site $((X,M)/V)_{\text{conv,et}}$, denote the pull-back of E by (F_X^a, F_k^a, σ) in the above diagram by $F^{a,*}E$. An F^a-isocrystal on log convergent site $((X,M)/V)_{\text{conv,et}}$ with respect to σ is a pair $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is an isocrystal on $((X,M)/V)_{\text{conv,et}}$ and Φ is an isomorphism $F^{a,*}E \longrightarrow E$.

Next proposition assures that an F^a-isocrystal is in fact a locally free isocrystal:

Proposition 2.4.3. *Let the notations be as in the above definition and let $(\mathcal{E}, \Phi)$ be an F^a-isocrystal on $((X,M)/V)_{\text{conv,et}}$. Then $\mathcal{E}$ is a locally free isocrystal.*

Proof. It suffices to prove the following: For any $x \in X$, there exists a formally log smooth lifting $i : (X,M) \hookrightarrow (P,L)$ over $\operatorname{Spf} V$ defined on an etale neighborhood of $\bar{x}$(:= a geometric point with image x) such that the value of $\mathcal{E}$ on the enlargement $P := ((P,L),(X,M),i,\mathrm{id})$ is a locally free $K \otimes_V \mathcal{O}_P$-module around $\bar{x}$. So we may assume that (X,M) admits a chart $R_X \to M$ such that $R_{\bar{x}} \overset{\simeq}{=} M_{\bar{x}}/\mathcal{O}_{X,\bar{x}}^{\times}$ holds and that X is smooth over $\operatorname{Spec} k[R]$ ([Kf, (3.1.1)]). By shrinking X if necessary, there exists an integer r such that X is etale over $\operatorname{Spec} k[R \oplus \mathbb{N}^r]$ and x is defined by the locus $\{a = 0 \,|\, a \in R \oplus \mathbb{N}^r\}$. So there exists a formal scheme P which is etale over $\operatorname{Spf} V\{R \oplus \mathbb{N}^r\}$ such that P modulo π coincides with X. Then, if we define

L to be the log structure on P associated to the monoid homomorphism $R \to V\{R \oplus \mathbb{N}^r\} \to \Gamma(P, \mathcal{O}_P)$, we obtain a lifting (P, L) of (X, M) which is formally log smooth over $\mathrm{Spf}\, V$. By shrinking P, we may assume P is affine. Let $\tau : (P, L) \longrightarrow (P, L)$ be the unique morphism which lifts $F_X : (X, M) \longrightarrow (X, M)$ and compatible with $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ and the homomorphism $R \oplus \mathbb{N}^r \longrightarrow R \oplus \mathbb{N}^r$ defined by 'multiplication by p^a'.

Put $P = \mathrm{Spf}\, A$ and let $\mathfrak{m}$ be the maximal ideal of $K \otimes A$ defined by $R \oplus \mathbb{N}^r$. Denote the value of $\mathcal{E}$ at P (regarded as a $K \otimes A$-module) by E. It suffices to show that the localization $E_{\mathfrak{m}}$ of E at $\mathfrak{m}$ is a flat $(K \otimes A)_{\mathfrak{m}}$-module. To show this, it suffices to show that $E/\mathfrak{m}^n E$ is a flat $(K \otimes A)/\mathfrak{m}^n$-module for any $n \geq 1$.

It is well-known that $K' := (K \otimes A)/\mathfrak{m}$ is a finite extension field of K. Choose an isomorphism $(K')^t \cong E/\mathfrak{m}E$ and let $f : ((K \otimes A)/\mathfrak{m}^n)^t \longrightarrow E/\mathfrak{m}^n E$ be a lifting of this isomorphism. Let I be the kernel of f. Then I is contained in $(\mathfrak{m}/\mathfrak{m}^n)^t$ and we have the isomorphism

$$((K \otimes A)/\mathfrak{m}^n)^t / I \xrightarrow{\sim} E/\mathfrak{m}^n E. \tag{2.4.3}$$

Let us tensor the both sides with $\otimes_{K \otimes A, \tau^n} (K \otimes A)/\mathfrak{m}^n$. Since there exists a natural surjection

$$((K \otimes A)/\mathfrak{m}^n)^t / I \longrightarrow ((K \otimes A)/\mathfrak{m})^t,$$

we have the surjection

$$\begin{aligned} ((K \otimes A)/\mathfrak{m}^n)^t / I \otimes_{K \otimes A, \tau^n} & (K \otimes A)/\mathfrak{m}^n \\ & \longrightarrow ((K \otimes A)/\mathfrak{m} \otimes_{K \otimes A, \tau^n} (K \otimes A)/\mathfrak{m}^n)^t. \end{aligned}$$

For any $x \in \mathfrak{m}$, we have $\tau^n(x) \in \mathfrak{m}^{p^{an}} \subset \mathfrak{m}^n$. Hence the right hand side is isomorphic to $((K \otimes A)/\mathfrak{m}^n)^t$ and so the left hand side is also isomorphic to $((K \otimes A)/\mathfrak{m}^n)^t$. Therefore, if we tensor the left hand side of the isomorphism (2.4.3) with $\otimes_{K \otimes A, \tau^n} (K \otimes A)/\mathfrak{m}^n$, it becomes a free $(K \otimes A)/\mathfrak{m}^n$-module.

On the other hand, E comes from an F^a-isocrystal. So, if we tensor the right hand side of the isomorphism (2.4.3) with $\otimes_{K \otimes A, \tau^n} (K \otimes A)/\mathfrak{m}^n$, it is again isomorphic to $E/\mathfrak{m}^n E$. Hence $E/\mathfrak{m}^n E$ is a free (hence flat) $(K \otimes A)/\mathfrak{m}^n$-module, as desired. □

Now we state the main theorem in this section:

Theorem 2.4.4. *Let*

$$(X,M) \xrightarrow{f} \operatorname{Spec} k \xhookrightarrow{\iota} \operatorname{Spf} V$$

be as above and assume that (X,M) *is of Zariski type. Put* $U := X_{\mathrm{triv}}$ *and denote the open immersion* $U \hookrightarrow X$ *by* j. *Let* $\mathcal{E}$ *be one of the following*:

(1) $\mathcal{E} = \mathcal{K}_{X/V}$ *holds.*
(2) *There exists a lifting* $\sigma : \operatorname{Spf} V \longrightarrow \operatorname{Spf} V$ *of the* a*-times iteration of the absolute Frobenius on* $\operatorname{Spec} k$ $(a > 0)$ *and* $\mathcal{E}$ *is an* F^a*-isocrystal on* $(X/V)^{\log}_{\mathrm{conv,et}}$ *with respect to* σ.

Then we have the isomorphism

$$H^i_{\mathrm{rig}}(U/K, j^\dagger \mathcal{E}) \cong H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}).$$

First, let us recall the following result, which is an immediate consequence of results due to Kempf–Knudsen–Mumford–Saint-Donat ([KKMS]) and Kato ([Kk2]):

Proposition 2.4.5. *Let* (X,M) *be an fs log scheme of Zariski type which is log smooth over* $\operatorname{Spec} k$. *Then there exists a morphism* $f : (\overline{X}, \overline{M}) \longrightarrow (X,M)$ *of fs log schemes of Zariski type given by 'subdivision of fan' which satisfies the following conditions*:

(1) f *is proper, birational, log etale,* $\overline{X}$ *is regular, and* $\overline{X}_{\mathrm{triv}}$ *is isomorphic to* X_{triv}.
(2) *For any point* $x \in \overline{X}$, *there exists a natural number* $r(x)$ *such that* $\overline{M}_{\bar{x}}/\mathcal{O}^\times_{\overline{X},\bar{x}} \cong \mathbb{N}^{r(x)}$ *holds.*
(3) *Let* U *be an open set of* X *and suppose we have a formally log smooth lifting* $i : (U,M) \hookrightarrow (P,L)$ *over* $\operatorname{Spf} V$ *such that* (P,L) *is of Zariski type. Then we can construct the following diagram*:

$$\begin{array}{ccc} (f^{-1}(U), \overline{M}) & \xrightarrow{\bar{i}} & (\overline{P}, \overline{L}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (U,M) & \xrightarrow{i} & (P,L) \end{array}$$

Here $\overline{i}$ is a formally log smooth lifting over $\mathrm{Spf}\, V$*, g is formally log etale, $(\overline{P}, \overline{L})$ is of Zariski type and $Rg_*\mathcal{O}_{\overline{P}} = \mathcal{O}_P$ holds.*

Proof. We only sketch the outline of the proof. The details are left to the reader.

First, via the equivalence of categories in Corollary 1.1.11, we may work in the category of fs log schemes with respect to Zariski topology. (Note that all the argument in [Kk2] is done in the category of fs log schemes with respect to Zariski topology.)

Let (F, M_F) be the fan ([Kk2, (9.3)]) associated to (X, M). (Note that (X, M) is log regular by [Kk2, (8.3)].) Then there exists a proper subdivision $\varphi : (\overline{F}, M_{\overline{F}}) \longrightarrow (F, M_F)$ such that, for any $x \in \overline{F}$, there exists a natural number $r(x)$ satisfying $M_{\overline{F},x} \cong \mathbb{N}^{r(x)}$ ([KKMS, I,Theorem 11], [Kk2, (9.8)]). Then, we can define 'the fiber product' $(\overline{X}, \overline{M}) := (X, M) \times_{(F,M_F)} (\overline{F}, M_{\overline{F}})$ ([Kk2, (9.10)]). Then it is known that 'the projection' $f : (\overline{X}, \overline{M}) \longrightarrow (X, M)$ satisfies the conditions (1) and (2).

Let us prove the assertion (3). Note that we have the morphism of monoidal spaces

$$(P, L/\mathcal{O}_P^{\times}) = (U, M/\mathcal{O}_X^{\times}) \longrightarrow (F, M_F).$$

So we can define 'the completed fiber product'

$$(\overline{P}, \overline{L}) := (P, L) \hat{\times}_{(F,M_F)} (\overline{F}, M_{\overline{F}}),$$

and the projection $g : (\overline{P}, \overline{L}) \longrightarrow (P, L)$ fits into the diagram in the statement (3) of the proposition. The assertion $Rg_*\mathcal{O}_{\overline{P}} = \mathcal{O}_P$ follows from the equation $Rf_*\mathcal{O}_{f^{-1}(U)} = \mathcal{O}_U$, which is proved in [Kk2, (11.3)]. □

Now we begin the proof of Theorem 2.4.4:

Proof of Theorem 2.4.4. Since the proof is long, we divide it into three steps.

Step 1. First, we construct the homomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U/K, j^{\dagger}\mathcal{E}).$$

Let us take an open covering $X = \bigcup_{i \in I} X_i$ of X by finite number of open subschemes and a formally log smooth lifting $(X_i, M) \hookrightarrow (P_i, L_i)$ over $\mathrm{Spf}\, V$ such that (P_i, L_i) is of Zariski type and the value of $\mathcal{E}$ at the enlargement $P_i := ((P_i, L_i), (X_i, M), (X_i, M) \hookrightarrow (P_i, L_i), (X_i, M) \hookrightarrow (X, M))$ is a free $K \otimes_V \mathcal{O}_{P_i}$-module. Then we put $(X^{(0)}, M^{(0)}) := (\coprod_{i \in I} X_i, M|_{\coprod_{i \in I} X_i})$, $(P^{(0)}, L^{(0)}) := \coprod_{i \in I} (P_i, L_i)$ and denote the exact closed immersion $(X^{(0)}, M^{(0)}) \hookrightarrow (P^{(0)}, L^{(0)})$ induced by the morphisms $(X_i, M) \hookrightarrow (P_i, L_i)\, (i \in I)$ by $i^{(0)}$. For $n \in \mathbb{N}$, let $(X^{(n)}, M^{(n)})$ (resp. $(P^{(n)}, L^{(n)})$) be the $(n+1)$-fold fiber product of $(X^{(0)}, M^{(0)})$ (resp. $(P^{(0)}, L^{(0)})$) over (X, M) (resp. $\mathrm{Spf}\, V$). Then one can form the good embedding system

$$(X, M) \overset{g}{\leftarrow} (X^{(\bullet)}, M^{(\bullet)}) \overset{i^{(\bullet)}}{\to} (P^{(\bullet)}, L^{(\bullet)})$$

naturally. (Here $i^{(\bullet)}$ is the morphism induced by the fiber products of $i^{(0)}$.) Let $\varphi^{(n)} :]X^{(n)}[^{\log}_{P^{(n)}} \longrightarrow]X^{(n)}[_{P^{(n)}}$ be the morphism defined in the claim in the proof of Proposition 2.4.1.

Suppose for the moment that the morphism $i^{(n)} : (X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})$ has a factorization

$$(X^{(n)}, M^{(n)}) \hookrightarrow (\tilde{P}^{(n)}, \tilde{L}^{(n)}) \longrightarrow (P^{(n)}, L^{(n)}), \tag{2.4.4}$$

where the first arrow is an exact closed immersion and the second arrow is formally log etale. Then we have $]X^{(n)}[^{\log}_{P^{(n)}} =]X^{(n)}[_{\tilde{P}^{(n)}}$. Now note that $\tilde{P}^{(n)}$ is formally smooth over $\mathrm{Spf}\, V$ on a neighborhood of $U^{(n)} := X^{(n)} \times_X U$. So $j^\dagger \mathcal{E}$ defines the de Rham complex $\mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger \mathcal{E})$ on $]X^{(n)}[_{\tilde{P}^{(n)}}$. By definition of $j^\dagger \mathcal{E}$, we have the isomorphism

$$\mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger \mathcal{E}) = \varinjlim_O \alpha_{O,*} \alpha_O^* \mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, \mathcal{E}),$$

where O runs through the strict neighborhoods of $]U^{(n)}[_{\tilde{P}^{(n)}}$ in $]X^{(n)}[_{\tilde{P}^{(n)}}$ and we denoted the inclusion $O \hookrightarrow]X^{(n)}[_{\tilde{P}^{(n)}}$ by α_O. So we have the canonical homomorphism

$$\mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger \mathcal{E}).$$

Now let us consider the case that the morphism $i^{(n)}$ does not necessarily have the factorization (2.4.4). Since $(X^{(n)}, M^{(n)})$ and $(P^{(n)}, L^{(n)})$ are of Zariski type, the morphism $i^{(n)}$ has has the factorization (2.4.4) Zariski locally. Then one can define the de Rham complex $\mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger\mathcal{E})$ locally, and one can see that this complex can be glued and gives the de Rham complex on $]X^{(n)}[^{\log}_{P^{(n)}}$, which we denote by $\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, j^\dagger\mathcal{E})$. Moreover, one can glue the homomorphism

$$\mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger\mathcal{E})$$

and hence it gives the homomorphism

$$\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, j^\dagger\mathcal{E}).$$

Since this construction is compatible with respect to n, we can define the homomorphism

$$\mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, j^\dagger\mathcal{E}). \tag{2.4.5}$$

Now we prove the quasi-isomorphism

$$R\varphi^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, j^\dagger\mathcal{E}) \overset{\sim}{=} \mathrm{DR}(]X^{(\bullet)}[_{P^{(\bullet)}}, j^\dagger\mathcal{E}). \tag{2.4.6}$$

First we may replace $\bullet$ by n. Then, since we can work Zariski locally on $P^{(n)}$, we may assume the existence of the factorization (2.4.4). Then, since the morphism $\tilde{P}^{(n)} \longrightarrow P^{(n)}$ is formally etale on a neighborhood of $U^{(n)}$, the assertion follows from Proposition 1.3.10. Hence, by applying $R\varphi^{(\bullet)}_*$ to the homomorphism (2.4.5), we obtain the homomorphism

$$R\varphi^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(\bullet)}[_{P^{(\bullet)}}, j^\dagger\mathcal{E}). \tag{2.4.7}$$

By applying the functor $H^i(X, Rg_* R\mathrm{sp}^{(\bullet)}_* -)$ (where $\mathrm{sp}^{(\bullet)}$ is the specialization map $]X^{(\bullet)}[_{P^{(\bullet)}} \longrightarrow X^{(\bullet)}$) to the homomorphism (2.4.7), we obtain the homomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E}).$$

Step 2. In this step, we reduce the theorem to the case that X is regular and M is the log structure associated to a simple normal crossing divisor. Let (F, M_F) be the fan associated to (X, M) and take a proper subdivision $(\overline{F}, M_{\overline{F}}) \longrightarrow (F, M_F)$ of (F, M_F) such that, for any $x \in \overline{F}$, there exists a natural number $r(x)$ such that $M_{\overline{F},x} \cong \mathbb{N}^{r(x)}$ holds. Let $(\overline{X}, \overline{M})$ be $(X, M) \times_{(F,M_F)} (\overline{F}, M_{\overline{F}})$ and let h be 'the projection' $(\overline{X}, \overline{M}) \longrightarrow (X, M)$. (Then $\overline{X}$ is regular and $\overline{M}$ is the log structure associated to a simple normal crossing divisor.) Let us denote the open immersion $U = \overline{X}_{\mathrm{triv}} \hookrightarrow \overline{X}$ by $\overline{j}$, and denote the restriction of $\mathcal{E}$ to $(\overline{X}/V)^{\log}_{\mathrm{conv,et}}$ by $\overline{\mathcal{E}}$.

Take a diagram

$$(X, M) \overset{g}{\leftarrow} (X^{(\bullet)}, M^{(\bullet)}) \overset{i^{(\bullet)}}{\to} (P^{(\bullet)}, L^{(\bullet)})$$

as in Step 1. Put $(\overline{X}^{(\bullet)}, \overline{M}^{(\bullet)}) := (\overline{X}, \overline{M}) \times_{(X,M)} (X^{(\bullet)}, M^{(\bullet)})$, and let $(\overline{P}^{(0)}, \overline{L}^{(0)})$ be $(P^{(0)}, L^{(0)}) \times_{(F,M_F)} (\overline{F}, M_{\overline{F}})$. For $n \in \mathbb{N}$, let $(\overline{P}^{(n)}, \overline{L}^{(n)})$ be the $(n+1)$-fold fiber product of $(\overline{P}^{(0)}, \overline{L}^{(0)})$ over $\operatorname{Spf} V$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
(\overline{X}, \overline{M}) & \xleftarrow{\overline{g}} & (\overline{X}^{(\bullet)}, \overline{M}^{(\bullet)}) & \xrightarrow{\overline{i}^{(\bullet)}} & (\overline{P}^{(\bullet)}, \overline{L}^{(\bullet)}) \\
{\scriptstyle h}\downarrow & & {\scriptstyle h^{(\bullet)}}\downarrow & & {\scriptstyle \tilde{h}^{(\bullet)}}\downarrow \\
(X, M) & \xleftarrow{g} & (X^{(\bullet)}, M^{(\bullet)}) & \xrightarrow{i^{(\bullet)}} & (P^{(\bullet)}, L^{(\bullet)}).
\end{array}$$

Now let us note that the log scheme $(\overline{P}^{(n)}, \overline{L}^{(n)})$ is of Zariski type by Proposition 2.4.5. Moreover, $\overline{g}$ is a Zariski hypercovering and $\overline{X}_{\mathrm{triv}} = U$ holds. Hence, by the argument of Step 1, the above diagram induces the following commutative diagram of cohomologies:

$$\begin{array}{ccc}
H^i_{\mathrm{an}}((\overline{X}/V)^{\log}, \overline{\mathcal{E}}) & \longrightarrow & H^i_{\mathrm{rig}}(U/K, \overline{j}^\dagger \overline{\mathcal{E}}) \\
{\scriptstyle h^*}\uparrow & & \uparrow \\
H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) & \longrightarrow & H^i_{\mathrm{rig}}(U/K, j^\dagger \mathcal{E}).
\end{array}$$

One can easily see that $\overline{j}^\dagger \overline{\mathcal{E}}$ is identical with $j^\dagger \mathcal{E}$. Hence the right vertical arrow is an isomorphism. If the homomorphism h^* is an isomorphism, we may replace (X, M) by $(\overline{X}, \overline{M})$ to prove the theorem, that is, the proof of Step 2 is done. So, in the following, we prove that h^* is an isomorphism.

Let $\mathrm{sp}^{(n)}$ be the specialization map $]X^{(n)}[^{\log}_{P^{(n)}} \longrightarrow X^{(n)}$ and let $\psi^{(n)}$ be the morphism $]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}} \longrightarrow]X^{(n)}[_{P^{(n)}}$ induced by $\tilde{h}^{(n)}$. To prove h^* is isomorphic, it suffices to prove that the morphism

$$R\mathrm{sp}^{(n)}_* \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}) \longrightarrow R\mathrm{sp}^{(n)}_* R\psi^{(n)}_* \mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}}, \overline{\mathcal{E}}) \tag{2.4.8}$$

is a quasi-isomorphism. Let us consider the following diagram

$$\begin{array}{ccc} (X^{(n)}, M^{(n)}) & \xrightarrow{i^{(n)}} & (P^{(n)}, L^{(n)}) \\ \Big\| & & \pi \Big\downarrow \\ (X^{(n)}, M^{(n)}) & \xrightarrow{\nu^{(n)}} & (P^{(0)}, L^{(0)}), \end{array} \tag{2.4.9}$$

where π is the first projection and $\nu^{(n)}$ is the composite

$$(X^{(n)}, M^{(n)}) \xrightarrow{\text{1-st proj.}} (X^{(0)}, M^{(0)}) \overset{i^{(0)}}{\hookrightarrow} (P^{(0)}, L^{(0)}).$$

Then $\nu^{(n)}$ is a locally closed immersion and π is formally log smooth. Let $\mathrm{sp}^{(n)}_0$ be the specialization map

$$]X^{(n)}[^{\log}_{P^{(0)}} \longrightarrow X^{(n)}$$

and let π_K be the morphism

$$]X^{(n)}[^{\log}_{P^{(n)}} \longrightarrow]X^{(n)}[^{\log}_{P^{(0)}}$$

induced by π. Then we have

$$\begin{aligned} R\mathrm{sp}^{(n)}_* \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}) &= R\mathrm{sp}^{(n)}_{0,*} R\pi_{K,*} \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}) \\ &= R\mathrm{sp}^{(n)}_{0,*} \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(0)}}, \mathcal{E}), \end{aligned}$$

since π is formally log smooth. On the other hand, let us consider the following diagram:

$$\begin{array}{ccc} (\overline{X}^{(n)}, \overline{M}^{(n)}) & \xrightarrow{\overline{i}^{(n)}} & (\overline{P}^{(n)}, \overline{L}^{(n)}) \\ \Big\| & & \overline{\pi} \Big\downarrow \\ (\overline{X}^{(n)}, \overline{M}^{(n)}) & \xrightarrow{\overline{\nu}^{(n)}} & (\overline{P}^{(0)}, \overline{L}^{(0)}), \end{array} \tag{2.4.10}$$

where $\overline{\pi}$ is the first projection and $\overline{\nu}^{(n)}$ is the composite

$$(\overline{X}^{(n)}, \overline{M}^{(n)}) \overset{\text{1-st proj.}}{\longrightarrow} (\overline{X}^{(0)}, \overline{M}^{(0)}) \overset{\overline{i}^{(0)}}{\hookrightarrow} (\overline{P}^{(0)}, \overline{L}^{(0)}).$$

Then the diagrams (2.4.9) and (2.4.10) are compatible with the projections $h^{(\bullet)}, \tilde{h}^{(\bullet)}$ $(\bullet = 0, n)$. Let $\overline{\pi}_K$ be the morphism

$$]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}} \longrightarrow]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(0)}}$$

induced by $\overline{\pi}$. Then one has the natural morphism of rigid analytic spaces

$$\psi_0^{(n)} :]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(0)}} \longrightarrow]X^{(n)}[^{\log}_{P^{(0)}}$$

induced by $\tilde{h}^{(0)}$. Then we have $\pi_K \circ \psi^{(n)} = \psi_0^{(n)} \circ \overline{\pi}_K$. So we obtain the quasi-isomorphisms

$$\begin{aligned} R\mathrm{sp}^{(n)}_* R\psi^{(n)}_* \mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}}, \overline{\mathcal{E}}) &= R\mathrm{sp}^{(n)}_{0,*} R\psi^{(n)}_{0,*} R\overline{\pi}_{K,*} \mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}}, \overline{\mathcal{E}}) \\ &= R\mathrm{sp}^{(n)}_{0,*} R\psi^{(n)}_{0,*} \mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(0)}}, \overline{\mathcal{E}}). \end{aligned}$$

Hence the assertion of Step 2 is reduced to the following claim: The homomorphism

$$\text{(2.4.11)} \qquad \mathrm{DR}(]X^{(n)}[_{P^{(0)}}, \mathcal{E}) \longrightarrow R\psi^{(n)}_{0,*} \mathrm{DR}(]\overline{X}^{(n)}[_{\overline{P}^{(0)}}, \overline{\mathcal{E}})$$

is a quasi-isomorphism. By shrinking $P^{(0)}$, we may assume that $(P^{(0)}, L^{(0)})$ is a formally log smooth lifting of $(X^{(n)}, M^{(n)})$ over $\mathrm{Spf}\, V$ and that the diagram

$$\begin{array}{ccc} (\overline{X}^{(n)}, \overline{M}^{(n)}) & \xrightarrow{\overline{\nu}^{(n)}} & (\overline{P}^{(0)}, \overline{L}^{(0)}) \\ \downarrow & & \downarrow \tilde{h}^{(0)} \\ (X^{(n)}, M^{(n)}) & \xrightarrow{\nu^{(n)}} & (P^{(0)}, L^{(0)}) \end{array}$$

is Cartesian. Then, since $\mathcal{E}$ is locally free, the above claim follows from the quasi-isomorphism $R\tilde{h}^{(0)}_* \mathcal{O}_{\overline{P}^{(0)}} = \mathcal{O}_{P^{(0)}}$, which is proved in Proposition 2.4.5. Hence the proof of Step 2 is now finished.

Step 3. In this step, we reduce the assertion to a certain local assertion. Let X be a smooth scheme over k, M be the log structure on X associated to a simple normal crossing divisor on X, and take a diagram

$$(X,M) \overset{g}{\leftarrow} (X^{(\bullet)}, M^{(\bullet)}) \overset{i^{(\bullet)}}{\rightarrow} (P^{(\bullet)}, L^{(\bullet)})$$

as in Step 1. Let us denote the specialization map $]X^{(\bullet)}[^{\log}_{P(\bullet)} \longrightarrow X^{(\bullet)}$ by $\mathrm{sp}^{(\bullet)}$. Then the homomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E})$$

is obtained by applying $H^i(X, Rg_*R\mathrm{sp}^{(\bullet)}_* -) = H^i(]X^{(\bullet)}[^{\log}_{P(\bullet)}, -)$ to the homomorphism

$$\mathrm{DR}(]X^{(\bullet)}[^{\log}_{P(\bullet)}, \mathcal{E}) \longrightarrow \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P(\bullet)}, j^\dagger\mathcal{E}).$$

Note that we have the spectral sequence

$$E_1^{p,q} = H^q(]X^{(p)}[^{\log}_{P(p)}, -) \Longrightarrow H^{p+q}(]X^{(\bullet)}[^{\log}_{P(\bullet)}, -).$$

Hence it suffices to prove that the homomorphism

$$H^i(]X^{(n)}[^{\log}_{P(n)}, \mathrm{DR}(]X^{(n)}[^{\log}_{P(n)}, \mathcal{E})) \longrightarrow H^i(]X^{(n)}[^{\log}_{P(n)}, \mathrm{DR}(]X^{(n)}[^{\log}_{P(n)}, j^\dagger\mathcal{E}))$$

is an isomorphism for any $i, n \in \mathbb{N}$. Since the assertion is Zariski local on $P^{(n)}$ (which follows from the spectral sequence induced by Zariski hypercovering of $P^{(n)}$), we may assume that the closed immersion $(X^{(n)}, M^{(n)}) \hookrightarrow (P^{(n)}, L^{(n)})$ admits a factorization

$$(X^{(n)}, M^{(n)}) \hookrightarrow (\tilde{P}^{(n)}, \tilde{L}^{(n)}) \longrightarrow (P^{(n)}, L^{(n)}),$$

where the first arrow is an exact closed immersion and the second arrow is formally log etale. (Then we have $\mathrm{DR}(]X^{(n)}[^{\log}_{P(n)}, j^\dagger\mathcal{E}) = \mathrm{DR}(]X^{(n)}[_{\tilde{P}(n)}, j^\dagger\mathcal{E})$.) Let us consider the following diagram

(2.4.12)
$$\begin{array}{ccc} (X^{(n)}, M^{(n)}) & \xrightarrow{i^{(n)}} & (P^{(n)}, L^{(n)}) \\ \| & & \downarrow{\scriptstyle \pi} \\ (X^{(n)}, M^{(n)}) & \xrightarrow{\nu^{(n)}} & (P^{(0)}, L^{(0)}), \end{array}$$

where the notations are as those in Step 2. Let π_K be as in Step 2. Then, since π is formally log smooth, we have

$$H^i(]X^{(n)}[^{\log}_{P^{(n)}}, \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E})) \cong H^i(]X^{(n)}[_{P^{(0)}}, \mathrm{DR}(]X^{(n)}[_{P^{(0)}}, \mathcal{E})).$$

On the other hand, since the composite

$$\tilde{P}^{(n)} \longrightarrow P^{(n)} \longrightarrow P^{(0)}$$

is formally smooth on a neighborhood of $U^{(n)} := X^{(n)} \times_X U$, we have

$$\begin{aligned} H^i(]X^{(n)}[^{\log}_{P^{(n)}}, \mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, j^\dagger\mathcal{E})) &= H^i(]X^{(n)}[_{\tilde{P}^{(n)}}, \mathrm{DR}(]X^{(n)}[_{\tilde{P}^{(n)}}, j^\dagger\mathcal{E})) \\ &\cong H^i(]X^{(n)}[_{P^{(0)}}, \mathrm{DR}(]X^{(n)}[_{P^{(0)}}, j^\dagger\mathcal{E})). \end{aligned}$$

Hence it suffices to prove that the homomorphism

$$H^i(]X^{(n)}[_{P^{(0)}}, \mathrm{DR}(]X^{(n)}[_{P^{(0)}}, \mathcal{E})) \longrightarrow H^i(]X^{(n)}[_{P^{(0)}}, \mathrm{DR}(]X^{(n)}[_{P^{(0)}}, j^\dagger\mathcal{E}))$$

is an isomorphism. By shrinking $P^{(0)}$, we may assume that $P^{(0)}$ is a formally smooth lifting of $X^{(n)}$ and that the log structure $L^{(0)}$ is defined by relative simple normal crossing divisor over $\mathrm{Spf}\, V$. Hence the proof of the theorem is reduced to the proposition below. So the proof of the theorem is finished modulo the following proposition. □

PROPOSITION 2.4.6. *Let X be an affine scheme which is smooth of finite type over k, and let P be a formally smooth lifting over* $\mathrm{Spf}\, V$. *Let $(t_1, \cdots, t_n)$ be a lifting of a regular parameter of X to P. Let $r \leq n$ be a natural number and put $D_i := \{t_i = 0\} \subset P, D := \bigcup_{i=1}^r D_i$, $Z_i := \{t_i = 0\} \subset X, Z := \bigcup_{i=1}^r Z_i$. Let L be the log structure on P associated to the pair (P, D) and let M be the log structure on X associated to the pair (X, Z). Put $U := X - Z$ and denote the open immersion $U \hookrightarrow X$ by j. Let $\mathcal{E}$ be one of the following*:

(1) $\mathcal{E} = \mathcal{K}_{X/V}$.
(2) *There exists a lifting $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ of the a-times iteration of absolute Frobenius of* $\mathrm{Spec}\, k$ *($a > 0$) and $\mathcal{E}$ is an F^a-isocrystal on $((X, M)/V)_{\mathrm{conv,et}}$ with respect to σ such that the value of $\mathcal{E}$ on the enlargement $((X, M), (P, L), (X, M) \hookrightarrow (P, L), \mathrm{id})$ is a free $K \otimes_V \mathcal{O}_P$-module.*

For $\lambda \in \Gamma, |\pi| < \lambda < 1$*, let* U_λ *be* $P_K - [Z]_{P,\lambda}$ *and denote the open immersion* $U_\lambda \hookrightarrow P_K$ *by* j_λ*. Then the canonical homomorphism*

$$\begin{aligned} H^i(P_K, \mathrm{DR}(P_K, \mathcal{E})) &\longrightarrow H^i(P_K, \mathrm{DR}(P_K, j^\dagger \mathcal{E})) \\ &= H^i(P_K, \varinjlim_{\lambda \to 1} j_{\lambda,*} j_\lambda^* \mathrm{DR}(P_K, \mathcal{E})) \end{aligned}$$

is an isomorphism.

In the following, we give a proof of the above proposition. The argument is essentially due to Baldassarri and Chiarellotto ([Ba-Ch], [Ba-Ch2]). It suffices to prove the homomorphism

$$H^i(P_K, \mathrm{DR}(P_K, \mathcal{E})) \longrightarrow H^i(P_K, j_{\lambda,*} j_\lambda^* \mathrm{DR}(P_K, \mathcal{E}))$$

is an isomorphism for $\lambda \in \Gamma, |\pi| < \lambda < 1$. First, we consider a certain nice admissible covering of P_K ([Ba-Ch, (4.2)]):

Proposition 2.4.7 (Baldassarri-Chiarellotto). *Let* $\eta \in \Gamma, \lambda < \eta < 1$*. For a subset* $\mathcal{S}$ *of* $[1, r]$*, put*

$$\begin{aligned} P_{\mathcal{S},\eta} &:= \{x \in P_K \,|\, |t_i(x)| < 1 \text{ for } i \in \mathcal{S}, |t_i(x)| \geq \eta \text{ for } i \in [1, r] - \mathcal{S}\}, \\ V_{\mathcal{S},\eta} &:= \{x \in P_K \,|\, |t_i(x)| = 0 \text{ for } i \in \mathcal{S}, |t_i(x)| \geq \eta \text{ for } i \in [1, r] - \mathcal{S}\}. \end{aligned}$$

Then, $V_{\mathcal{S},\eta}$ *is a smooth rigid analytic space* (*for definition, see* [Be3]) *and there exists a retraction* $q_S : P_{\mathcal{S},\eta} \longrightarrow V_{\mathcal{S},\eta}$ *of the inclusion* $V_{\mathcal{S},\eta} \longrightarrow P_{\mathcal{S},\eta}$ *such that* $P_{\mathcal{S},\eta}$ *is a trivial bundle whose fiber is an open disk of radius 1 of dimension* $s := |\mathcal{S}|$*.* (*That is,* q_S *induces the isomorphism* $P_{\mathcal{S},\eta} \stackrel{\sim}{=} V_{\mathcal{S},\eta} \times D_K^s$*.*) *Via the identification* $P_{\mathcal{S},\eta} \stackrel{\sim}{=} V_{\mathcal{S},\eta} \times D_K^s$*, we have* $P_{\mathcal{S},\eta} \cap U_\lambda = V_{\mathcal{S},\eta} \times C_{K,\lambda}^s$*, where* $C_{K,\lambda}^s$ *is the open annulus* $\{x \in D_K^s \,|\, \lambda < |t_i(x)| < 1\}$ *of dimension s. Moreover, we have an admissible covering*

$$\bigcup_{\mathcal{S} \subset [1,r]} P_{\mathcal{S},\eta} = P_K.$$

When $\mathcal{E}$ is in the case (2) in Proposition 2.4.6, let $\sigma_K : \mathrm{Spm}\, K \longrightarrow \mathrm{Spm}\, K$ be the morphism induced by σ and let $\sigma_{P_K} : P_K \longrightarrow P_K$ be the morphism compatible with σ_K and satisfying $\sigma_{P_K}^*(t_i) = t_i^{p^a}$. Then we have $\sigma_{P_K}(P_{\mathcal{S},\eta^{1/p^a}}) \subset P_{\mathcal{S},\eta}, \sigma_{P_K}(V_{\mathcal{S},\eta^{1/p^a}}) \subset V_{\mathcal{S},\eta}$. On the other hand, let $\sigma_{D_K^s}$ be the morphism $D_K^s \longrightarrow D_K^s$ compatible with σ_K and satisfying $\sigma_{D_K^s}^*(t_i) = t_i^{p^a}$. Then the argument of [Ba-Ch, (4.2)] shows that, via the identification $P_{\mathcal{S},\eta} \overset{\sim}{=} V_{\mathcal{S},\eta} \times D_K^s$, the morphism σ_{P_K} on the left hand side is identified with the morphism $\sigma_{P_K} \times_{\sigma_K} \sigma_{D_K^s}$ on the right hand side.

Now we reduce Proposition 2.4.6 to a certain assertion on the open disk over smooth affinoid rigid analytic space. To do this, we make some observations.

First, let (E, ∇) be the integrable log connection on $P_{\mathcal{S},\eta} \overset{\sim}{=} V_{\mathcal{S},\eta} \times D_K^s$ associated to $\mathcal{E}$. Then, since $j^\dagger\mathcal{E}$ is overconvergent, the integrable connection is overconvergent in the sense of Baldassarri-Chiarellotto ([Ba-Ch], [Ba-Ch2]), by [Ba-Ch, §6, pp. 41–42]. Moreover, in the case (2) in Proposition 2.4.6, the log connection (E, ∇) has an F^a-structure, that is, we have an isomorphism $\sigma_{P_K}^*(E, \nabla) \overset{\sim}{=} (E, \nabla)$ on $P_{\mathcal{S},\eta^{1/p^a}}$.

Next, for $m \in \mathbb{N}$, let I_m be the set

$$\{(\mathcal{S}_0, \cdots, \mathcal{S}_m) \,|\, \mathcal{S}_j \subset [1, r]\}.$$

For $\mathbb{S} := (\mathcal{S}_0, \cdots, \mathcal{S}_m) \in I_m$, denote the set $\bigcap_{j=0}^m \mathcal{S}_j$ by $\underline{\mathbb{S}}$ and let $P_{\mathbb{S},\eta}$ be the rigid analytic space $\bigcap_{j=0}^m P_{\mathcal{S}_j,\eta}$. Then we have the spectral sequence

$$E_1^{p,q} = \bigoplus_{\mathbb{S} \in I_p} H^q(P_{\mathbb{S},\eta}, -) \Longrightarrow H^{p+q}(P_K, -).$$

So it suffices to prove that the homomorphism

$$H^i(P_{\mathbb{S},\eta}, \mathrm{DR}(P_K, \mathcal{E})) \longrightarrow H^i(P_{\mathbb{S},\eta}, j_{\lambda,*}j_\lambda^*\mathrm{DR}(P_K, \mathcal{E})) \quad (\mathbb{S} \in I_m, m \geq 0)$$

are isomorphisms for some η satisfying $\lambda < \eta < 1, \eta \in \Gamma$. Note that we have

$$P_{\mathbb{S},\eta} := \{x \in P_K \,|\, |t_i(x)| < 1 \text{ for } i \in \bigcup_{j=0}^m \mathcal{S}_j,\, |t_i(x)| \geq \eta \text{ for } i \in [1, r] - \underline{\mathbb{S}}\}.$$

Let us define $V_{\mathbb{S},\eta}$ by

$$V_{\mathbb{S},\eta} := \{x \in P_K \,|\, |t_i(x)| = 0 \text{ for } i \in \underline{\mathbb{S}}, |t_i(x)| \geq \eta \text{ for } i \in [1, r] - \underline{\mathbb{S}}, \\ |t_i(x)| < 1 \text{ for } i \in \bigcup_{j=0}^m \mathcal{S}_j - \underline{\mathbb{S}}\}.$$

Then, one can see that $P_{\mathbb{S},\eta} \subset P_{\underline{\mathbb{S}},\eta}$ and $V_{\mathbb{S},\eta} \subset V_{\underline{\mathbb{S}},\eta}$ are admissible open sets. Moreover, the retraction $q_{\underline{\mathbb{S}}} : P_{\underline{\mathbb{S}},\eta} \longrightarrow V_{\underline{\mathbb{S}},\eta}$ of Proposition 2.4.7 induces the retraction $P_{\mathbb{S},\eta} \longrightarrow V_{\mathbb{S},\eta}$ and it induces the isomorphism $P_{\mathbb{S},\eta} \overset{\sim}{=} V_{\mathbb{S},\eta} \times D^s_K$, where $s = |\underline{\mathbb{S}}|$. Note moreover that $V_{\mathbb{S},\eta^{1/p^b}} = V_{\mathbb{S},\eta} \cap V_{\underline{\mathbb{S}},\eta^{1/p^b}}$ is quasi-Stein for any $b \in \mathbb{N}$.

From the above observations, we can reduce the proof of Proposition 2.4.6 to the following:

Proposition 2.4.8. *Let A be a Tate algebra such that $S := \mathrm{Spm}\, A$ is smooth* (*in the sense of* [Be3]) *and let $s \in \mathbb{N}$. Denote the open immersion $S \times C^s_{K,\lambda} \hookrightarrow S \times D^s_K =: T$ by j and the projection $T \longrightarrow S$ by π. Let $\underline{S} \subset S$ be an admissible open set which is quasi-Stein and put $\underline{T} := \underline{S} \times D^s_V \subset T$. Let ω^1 be the log differential module*

$$\pi^*\Omega^1_S \oplus \bigoplus_{i=1}^{s} \mathcal{O}_T \mathrm{dlog}\, t_i$$

and put $\omega^q := \wedge^q \omega^1$. Then $\omega^\bullet$ forms the log de Rham complex. Then:

(1) *The canonical homomorphism*

$$H^i(\underline{T}, \omega^\bullet) \longrightarrow H^i(\underline{T}, j_*j^*\omega^\bullet)$$

is an isomorphism.

(2) *Assume there exists a sequence of affinoid admissible open sets*

$$S \supset S_1 \supset S_2 \supset \cdots$$

and a finite flat morphism $\sigma_S : S_1 \longrightarrow S$ over σ_K satisfying the following conditions:

(a) *$\sigma_S(S_n) \subset S_{n-1}$ holds.*

(b) *$\underline{S}_n := \underline{S} \cap S_n$ is quasi-Stein for any n.*

Put $T_n := S_n \times D^s_V$, $\underline{T}_n := \underline{S}_n \times D^s_V$ and let $\sigma_T := \sigma_S \times_{\sigma_K} \sigma_{D^s_V}$. Let E be an $\mathcal{O}_T$-module of the form π^*F, where F is a locally free A-module and let $\nabla : E \longrightarrow E \otimes \omega^1$ be an integrable log connection on T which is overconvergent in the sense of Baldassarri-Chiarellotto. Assume moreover that there exists an isomorphism

$$\Phi : \sigma_T^*(E, \nabla) \overset{\sim}{=} (E, \nabla)|_{T_1}.$$

Denote the de Rham complex associated to (E, ∇) by $\mathrm{DR}(E, \nabla)$. Then, for sufficiently large integer n, the canonical homomorphism

$$H^i(\underline{T}_n, \mathrm{DR}(E, \nabla)) \longrightarrow H^i(\underline{T}_n, j_* j^* \mathrm{DR}(E, \nabla))$$

is an isomorphism.

Remark 2.4.9. To deduce Proposition 2.4.6 from Proposition 2.4.8, it suffices to apply the proposition in the case where $S = V_{\mathbb{S},\eta}, S_n = V_{\underline{\mathbb{S}},\eta^{1/p^{an}}}$ and $\underline{S} = V_{\underline{\mathbb{S}},\eta}$ hold and (E, ∇) is the integrable log connection on $P_{\underline{\mathbb{S}},\eta}$ associated to $\mathcal{E}$.

To prove Proposition 2.4.8, we recall the theory of p-adic differential equations with log pole on unit open disk over a smooth affinoid rigid analytic space, which is due to Baldassarri-Chiarellotto ([Ba-Ch2]).

Let A, S, T, π be as in Proposition 2.4.8 and let (E, ∇) be an $\mathcal{O}_T$ module of the form $\pi^* F$ (F is a locally free A-module) with an overconvergent integrable log connection. Let $o : S \longrightarrow T$ be the zero section and put $\omega^1_{T/S} := \omega^1/\pi^*\Omega^1_S$. Then, by taking the residue along $t_i \frac{\partial}{\partial t_i}$ to the composite

$$E \xrightarrow{\nabla} E \otimes \omega^1 \longrightarrow E \otimes \omega^1_{T/S},$$

we obtain the A-linear endomorphism $o^*E \longrightarrow o^*E$, which we denote by φ_i. Then we have the following([Ba-Ch2, (1.5.3)]):

Lemma 2.4.10 (Baldassarri-Chiarellotto). *There exists a polynomial Q_i with coefficient K such that $Q_i(\varphi_i) = 0$ holds.*

For $1 \leq i \leq s$, let P_i be the minimal monic polymonial such that $P_i(\varphi_i) = 0$ holds. Define $\Lambda(E, \nabla) \subset \overline{K}^s$ (where $\overline{K}$ is the algebraic closure of K) by $\Lambda(E, \nabla) := \{(\xi_1, \cdots, \xi_s) \,|\, P_i(\xi_i) = 0\}$.

In the following, we assume $\Lambda(E, \nabla) \subset K^s$ holds. (By replacing K by a finite extension, we may assume it.) For $\xi := (\xi_1, \cdots, \xi_s) \in K^s$, we call $(E = \pi^* F, \nabla)$ a ξ-simple object if $\nabla(f) = \sum_{i=1}^s \xi_i f \mathrm{dlog}\, t_i$ holds for $f \in F$. Then we have the following ([Ba-Ch2, (6.5.2)]):

Proposition 2.4.11 (Baldassarri-Chiarellotto). *Let us fix a subgroup $\Sigma \subset K$ which contains $\mathbb{Z}$ and which does not contain a p-adic Liouville number. Then, if $\Lambda(E,\nabla)$ is contained in Σ^s, (E,∇) can be written as a successive extension of ξ-simple objects for $\xi \in \Lambda(E,\nabla)$.*

Moreover, we have the following ([Ba-Ch, §6]):

Proposition 2.4.12 (Baldassarri-Chiarellotto). *Let (E,∇) be a ξ-simple object, where $\xi := (\xi_1, \cdots, \xi_s)$ and each ξ_i is a p-adic non-Liouville number which is not contained in $\mathbb{Z}_{>0}$. Then the canonical homomorphism*

$$H^i(T, \mathrm{DR}(E,\nabla)) \longrightarrow H^i(T, j_*j^*\mathrm{DR}(E,\nabla))$$

is an isomorphism.

Outline of proof. By taking an admissible covering of S, we can reduce to the case where F is a free A-module of finite rank. Since T and the inclusion $T' := S \times C^s_{A,\lambda} \hookrightarrow T$ are quasi-Stein, we have

$$H^i(T, \mathrm{DR}(E,\nabla)) = H^i(\Gamma(T, \mathrm{DR}(E,\nabla))),$$
$$H^i(T, j_*j^*\mathrm{DR}(E,\nabla)) = H^i(\Gamma(T', j^*\mathrm{DR}(E,\nabla))).$$

Let us prepare some notations: Put $A_T := \Gamma(T, \mathcal{O}_T), A_{T'} := \Gamma(T', \mathcal{O}_{T'})$. Let $\overline{F} := \Gamma(S, F)$ and $\overline{E} := \Gamma(T, \overline{E}) = \overline{F} \otimes_A A_T$. Let $\overline{\omega}^1 := \oplus_{i=1}^s A_T \mathrm{dlog}\, t_i$ and let $\overline{\omega}^q$ be the q-th exterior power of $\overline{\omega}^1$ over A_T. Let $\overline{\mathrm{DR}}$ be the relative log de Rham complex

$$0 \to \overline{E} \overset{\overline{\nabla}}{\to} \overline{E} \otimes \overline{\omega}^1 \overset{\overline{\nabla}}{\to} \cdots,$$

where $\overline{\nabla}$ is defined by $\overline{\nabla}(f) := \sum_{i=1}^s \xi_i f \mathrm{dlog}\, t_i$ $(f \in \overline{F})$. Put $\overline{\mathrm{DR}}' := \overline{\mathrm{DR}} \otimes_{A_T} A_{T'}$. Finally, put $\Omega^q_A := \Gamma(\mathrm{Spm}\, A, \Omega^q_{\mathrm{Spm}\, A})$.

Then, by introducing the filtration of Katz-Oda type on $\Gamma(T, \mathrm{DR}(E,\nabla))$ and $\Gamma(T', j^*\mathrm{DR}(E,\nabla))$, we obtain the following commutative diagram:

$$\begin{array}{ccc} E_1^{p,q} = H^q(\overline{\mathrm{DR}}) \otimes \Omega^p_A & \Longrightarrow & H^{p+q}(\Gamma(T, \mathrm{DR}(E,\nabla))) \\ \downarrow & & \downarrow \\ E_1^{p,q} = H^q(\overline{\mathrm{DR}}') \otimes \Omega^p_A & \Longrightarrow & H^{p+q}(\Gamma(T', j^*\mathrm{DR}(E,\nabla))). \end{array}$$

So it suffices to prove that the canonical homomorphism $\overline{\mathrm{DR}} \longrightarrow \overline{\mathrm{DR}}'$ is a quasi-isomorphism. One can prove it by constructing a homotopy explicitly ([Ba-Ch, (6.6)]). □

Now we give a proof of Proposition 2.4.8.

PROOF OF PROPOSITION 2.4.8. First we prove the assertion (1). Let $\underline{S} = \bigcup_{j=1}^{\infty} \underline{S}_j$ be an admissible covering of $\underline{S}$ by increasing admissible open affinoid rigid analytic spaces. Put $\underline{T}_j := \underline{S}_j \times D_V^s$. Then, we have

$$\begin{aligned} H^i(\underline{T}, \omega^\bullet) &= H^i(\Gamma(\underline{T}, \omega^\bullet)) \\ &= H^i(\varprojlim_j \Gamma(\underline{T}_j, \omega^\bullet)), \end{aligned}$$

$$\begin{aligned} H^i(\underline{T}, j_*j^*\omega^\bullet) &= H^i(\Gamma(\underline{S} \times C_{V,\lambda}^s, \omega^\bullet)) \\ &= H^i(\varprojlim_j \Gamma(\underline{S}_j \times C_{V,\lambda}^s, \omega^\bullet)). \end{aligned}$$

One can see, by using the quasi-Steinness of $\underline{T}_j$ and $\underline{T}$, that the projective system $\{\Gamma(\underline{T}_j, \omega^q)\}_j$ satisfies $\varprojlim^1 \{\Gamma(\underline{T}_j, \omega^q)\}_j = 0$. By the same reason, we have $\varprojlim^1 \{\Gamma(\underline{S}_j \times C_{V,\lambda}^s, \omega^q)\}_j = 0$. So we get the following diagram, where the horizontal lines are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim^1 H^{i-1}(\Gamma(\underline{T}_j, \omega^\bullet)) & \longrightarrow & H^i(\underline{T}, \omega^\bullet) & \longrightarrow & \varprojlim H^i(\Gamma(\underline{T}_j, \omega^\bullet)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim^1 H^{i-1}(\Gamma(\underline{S}_j \times C_{V,\lambda}^s, \omega^\bullet)) & \longrightarrow & H^i(\underline{T}, j_*j^*\omega^\bullet) & \longrightarrow & \varprojlim H^i(\Gamma(\underline{S}_j \times C_{V,\lambda}^s, \omega^\bullet)) & \longrightarrow & 0. \end{array}$$

Hence it suffices to prove that the canonical homomorphism

$$H^i(\Gamma(\underline{T}_j, \omega^\bullet)) \longrightarrow H^i(\Gamma(\underline{S}_j \times C_{V,\lambda}^s, \omega^\bullet))$$

is isomorphism for any i and j. By the quasi-Steinness of $\underline{T}_j$ and $\underline{S}_j$, this homomorphism is nothing but the homomorphism

$$H^i(\underline{T}_j, \omega^\bullet) \longrightarrow H^i(\underline{T}_j, j_*j^*\omega^\bullet).$$

Since $\underline{S}_j$ is a smooth admissible rigid analytic space and the trivial log connection $(\mathcal{O}_{\underline{T}_j}, d)$ is a $(0, \cdots, 0)$-simple object, the above homomorphism is isomorphic by Proposition 2.4.12. So the proof of the assertion (1) is finished.

Next let us prove the assertion (2). First, let us prove the following claim:

Claim. we have $\Lambda((E,\nabla)|_{T_n}) = \{(0,\cdots,0)\}$ for sufficiently large n.

Proof of Claim. Let $\varphi_{i,n}$ be the endomorphism φ_i for (E_n, ∇_n) and let $P_{i,n}$ be the minimal monic polynomial with coefficient K such that $P_{i,n}(\varphi_{i,n}) = 0$ holds. Since $\varphi_{i,n}$ is the restriction of φ_i to $\mathrm{End}(i^*E|_{S_n})$, we have $P_{i,n'}(\varphi_{i,n}) = 0$ if $n \geq n'$ holds. Hence $P_{i,n'}$ is divisible by $P_{i,n}$. So $P_{i,n}$'s are the same for sufficiently large n. We denote it by $P_{i,\infty}$.

We prove that $P_{i,\infty}$ has no non-zero root. Let us assume the contrary and let a be the non-zero root such that $|a|$ is minimal. Let $\tau : K \longrightarrow K$ be the endomorphism induced by $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$. Since $\sigma_T^*(E_n, \nabla_n) = (E_{n+1}, \nabla_{n+1})$ holds, we have $\tau(P_{i,\infty})(p\varphi_{i,n+1}) = 0$, where τ acts on $K[x]$ by the action on coefficients. Put $Q(x) := \tau(P_{i,\infty})(px)$. Then $Q(x)$ is divisible by $P_{i,\infty}$. Since the degrees are the same, we have $Q(a) = 0$. Then we have

$$P_{i,\infty}(p\tau^{-1}(a)) = \tau^{-1}(\tau(P_{i,\infty})(pa)) = \tau(Q(a)) = 0.$$

Since $0 < |p\tau^{-1}(a)| = p^{-1}|a| < |a|$ holds, the above equation contradicts to the definition of a. Hence the assertion is proved. □

Now take an integer n satisfying the conclusion of the claim and let $\underline{S}_n = \bigcup_{j=1}^{\infty} \underline{S}_{n,j}$ be an admissible covering of $\underline{S}_n$ by increasing admissible open affinoid rigid analytic spaces. Put $\underline{T}_{n,j} := \underline{S}_{n,j} \times D_V^s$. Then, by the argument for the proof of the assertion (1), it suffices to prove that the homomorphism

$$H^i(\underline{T}_{n,j}, \mathrm{DR}(E,\nabla)) \longrightarrow H^i(\underline{T}_{n,j}, j_*j^*\mathrm{DR}(E,\nabla))$$

is an isomorphism for any j. Since $\underline{S}_{n,j}$ is a smooth admissible rigid analytic space and we have $\Lambda((E,\nabla)|_{\underline{T}_{n,j}}) \subset \Lambda((E,\nabla)|_{T_n}) = \{(0,\cdots,0)\}$, the above homomorphism is isomorphic by Propositions 2.4.11 and 2.4.12. So the proof of the assertion (2) is finished and the proof of proposition is now completed. □

Since the proof of Proposition 2.4.8 is finished, the proof of Proposition 2.4.6 is also finished and the proof of Theorem 2.4.4 is now completed.

Corollary 2.4.13. *Under the assumption of Theorem 2.4.4, we have the isomorphism*

$$H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \overset{\sim}{=} H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E}).$$

Proof. It is immediate from Theorem 2.4.4 and Corollary 2.3.9. □

Remark 2.4.14. Theorem 2.4.4 is true for more general $\mathcal{E}$'s: In fact, in the notations of Propositions 2.4.8 and the paragraphs following it, the conditions $\Lambda(E, \nabla) \subset \Sigma^s$ and $\xi_i \notin \mathbb{Z}_{>0}\,(1 \leq i \leq s)$ for any $\xi := (\xi_1, \cdots, \xi_s) \in \Lambda(E, \nabla)$ are the only conditions which we need to apply the results of Baldassarri and Chiarellotto.

Chapter 3. Applications

Throughout this chapter, let k be a perfect field of characteristic $p > 0$, let W be the Witt ring of k and let V be a totally ramified finite extension of W. Denote the fraction field of W by K_0 and the fraction field of V by K. For $n \in \mathbb{N}$, put $W_n := W \otimes \mathbb{Z}/p^n\mathbb{Z}$. In this chapter, we give some applications of the results in the previous chapter to rigid cohomologies and crystalline fundamental groups.

In Section 3.1, we prove results on finiteness of rigid cohomologies with coefficients. In Section 3.2, we give an alternative proof of Berthelot-Ogus theorem for fundamental groups, which was proved in the previous paper [Shi]. We remark that the condition is slightly weakened. In Section 3.3, we give the affirmative answer to the following problem, which we asked in [Shi]: Let X be a proper smooth scheme over k, D a normal crossing divisor and M the log structure associated to the pair (X, D). Put $U := X - D$ and let x be a k-valued point in U. Then, does the crystalline fundamental group $\pi_1^{\mathrm{crys}}((X, M)/\mathrm{Spf}\, W, x)$ depend only on U and x? We use the comparison between rigid cohomology and log convergent cohomology to solve this problem.

3.1. Notes on finiteness of rigid cohomology

In this section, we prove some results on finiteness of rigid cohomologies with coefficients. In the case of trivial coefficient, the finiteness of rigid cohomologies is proved by Berthelot ([Be4]). In the case of curves, the finiteness of rigid cohomologies with coefficients (under certain condition) is proved by Crew ([Cr2]). In the case that the coefficient is a unit-root overconvergent F^a-isocrystal, the finiteness is proved by Tsuzuki ([Ts2]). Here we prove the finiteness in the case that the coefficient is an overconvergent F^a-isocrystal which can be extended to an F^a-isocrystal on log compactification. This result, together with a version of quasi-unipotent conjecture, allows us to prove (the conjectual) finiteness result in the case that the coefficient is an overconvergent F^a-isocrystal.

Before proving the finiteness result of rigid cohomologies, we prove the comparison theorem between log convergent cohomology and log crystalline cohomology, which is the key to the proof of the finiteness results. Let us consider the following situation

$$(X, M) \xrightarrow{f} (\operatorname{Spec} k, N) \overset{\iota}{\hookrightarrow} (\operatorname{Spf} V, N), \tag{3.1.1}$$

where f is a log smooth morphism of finite type between fine log schemes, N is a fine log structure on $\operatorname{Spf} V$ and ι is the canonical exact closed immersion. In the previous paper ([Shi, §5.3]), we defined a functor

$$\Phi : I_{\mathrm{conv,et}}((X/W)^{\log}) \longrightarrow I_{\mathrm{crys}}((X/W)^{\log}).$$

(In the previous paper, we assumed that f is integral, but to define the functor Φ, this condition is not necessary.) For an object $\mathcal{E} = K \otimes \mathcal{F}$ in $I_{\mathrm{crys}}((X/W)^{\log})$ ($\mathcal{F} \in C_{\mathrm{crys}}((X/W)^{\log})$), we put

$$H^i((X/W)^{\log}_{\mathrm{crys}}, \mathcal{E}) := \mathbb{Q} \otimes_{\mathbb{Z}} H^i((X/W)^{\log}_{\mathrm{crys}}, \mathcal{F}).$$

Then we have the following theorem. (It is a log version of a result of Berthelot [Og2, (0.7.7)].)

Theorem 3.1.1. *Let the notations be as above. Then, for* $\mathcal{E} \in I^{\mathrm{lf}}_{\mathrm{conv,et}}((X/W)^{\log})$*, we have the isomorphism*

$$H^i((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \overset{\simeq}{=} H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E})).$$

REMARK 3.1.2. Here we remark that we may assume that $(\mathrm{Spf}\, W, N)$ has a chart to prove the above theorem. Indeed, let W' be a finite Galois extension of W with Galois group G such that $(\mathrm{Spf}\, W', N)$ has a chart, and let

$$(X', M) \longrightarrow (\mathrm{Spec}\, k', N) \hookrightarrow (\mathrm{Spf}\, W', N)$$

be the base change of the diagram (3.1.1) by $(\mathrm{Spf}\, W', N) \longrightarrow (\mathrm{Spf}\, W, N)$. Let $\mathcal{E}'$ be the restriction of $\mathcal{E}$ to $I^{\mathrm{lf}}_{\mathrm{conv,et}}((X'/W')^{\log})$. Then, by Remark 2.3.11, we have the isomorphism

$$H^i((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \overset{\simeq}{=} H^i((X'/W')^{\log}_{\mathrm{conv,et}}, \mathcal{E}')^G,$$

and one can prove in the same way the isomorphism

$$H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E})) \overset{\simeq}{=} H^i((X'/W')^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}'))^G.$$

One can see (by construction which we will give below) that the homomorphism

$$H^i((X'/W')^{\log}_{\mathrm{conv,et}}, \mathcal{E}') \longrightarrow H^i((X'/W')^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}'))$$

is G-equivariant. So, if it is isomorphic, we obtain the desired isomorphism

$$H^i((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \overset{\simeq}{=} H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}))$$

by taking the G-fixed part. So we may replace W by W', that is, we may assume that $(\mathrm{Spf}\, W, N)$ admits a chart. So, in the following, we assume this condition.

Before giving a proof of the theorem, we recall some properties of Φ and we make some observations in local situation.

First, let us consider the following situation: Let (X, M) be as above and assume that it is of Zariski type, and assume we are given a closed immersion $i : (X, M) \hookrightarrow (P, L)$ into a fine log formal V-scheme (P, L) of Zariski type which is formally log smooth over $(\mathrm{Spf}\, W, N)$. Assume moreover that the diagram

$$(X, M) \overset{i}{\hookrightarrow} (P, L) \longrightarrow (\mathrm{Spf}\, W, N)$$

admits a chart

$$(Q_V \to N, R_P \to L, S_X \to M, Q \to R \overset{\alpha}{\to} S), \tag{3.1.2}$$

such that α^{gp} is surjective. Let $(P(n), L(n))$ be the $(n+1)$-fold fiber product of (P, L) over $(\mathrm{Spf}\, W, N)$ and denote the closed immersion $(X, M) \hookrightarrow (P(n), L(n))$ induced by i by $i(n)$. Let $R(n)$ be the $(n+1)$-fold push out (in the category of fine monoids) of R over Q and let $\alpha(n) : R(n) \longrightarrow S$ be the homomorphism defined by $(r_1, \cdots, r_{n+1}) \mapsto \alpha(r_1 \cdots r_{n+1})$. (Then $(P(n), L(n))$ has a chart $R(n)_{P(n)} \to L(n)$.) Put $\tilde{R}(n) := (\alpha(n)^{\mathrm{gp}})^{-1}(S)$, $\tilde{P}(n) := P(n) \hat{\times}_{\mathrm{Spf}\, \mathbb{Z}_p\{R(n)\}} \mathrm{Spf}\, \mathbb{Z}_p\{\tilde{R}(n)\}$ and let $\tilde{L}(n)$ be the pull-back of the canonical log structure on $\mathrm{Spf}\, \mathbb{Z}_p\{\tilde{R}(n)\}$ to $\tilde{P}(n)$. Then the morphism $i(n)$ has the factorization

$$(X, M) \hookrightarrow (\tilde{P}(n), \tilde{L}(n)) \longrightarrow (P(n), L(n)),$$

where the first arrow is an exact closed immersion and the second arrow is formally log etale. Let $\{(T_{X,m}(\tilde{P}(n)), L_{X,m}(\tilde{P}(n)))\}_m$ be the system of universal enlargements of the exact pre-widening $((X, M), (\tilde{P}(n), \tilde{L}(n)), i(n), \mathrm{id})$, and let $(D(n), M_{D(n)})$ be the p-adically completed log PD-envelope of (X, M) in $(P(n), L(n))$. Then, in the previous paper, we have shown that there exists a morphism $(D(n), M_{D(n)}) \longrightarrow (T_{X,N}(\tilde{P}(n)), L_{X,N}(\tilde{P}(n)))$ for sufficiently large N induced by the universality of blow-ups. One can see that the induced morphism $\beta(n) : (D(n), M_{D(n)}) \longrightarrow \{(T_{X,m}(\tilde{P}(n)), L_{X,m}(\tilde{P}(n)))\}_m$ to the inductive system is independent of the choice of N. The morphism $\beta(n)\,(n = 0, 1, 2)$ is compatible with projections and diagonals, and so they induce the functor

$$\begin{aligned} \Psi : I_{\mathrm{conv,et}}((X/W)^{\log}) &\simeq \mathrm{Str}'((X, M) \hookrightarrow (P, L)) \\ &\longrightarrow \mathrm{HPDI}((X, M) \hookrightarrow (P, L)). \end{aligned}$$

(For the definition of $\mathrm{HPDI}((X, M) \hookrightarrow (P, L))$, see [Shi, (4.3.1)].) On the other hand, we defined the fully-faithful functor

$$\Lambda : I_{\mathrm{crys}}((X/W)^{\log}) \longrightarrow \mathrm{HPDI}((X, M) \hookrightarrow (P, L))$$

in [Shi, §4.3]. Then the functor

$$\Phi : I_{\mathrm{conv,et}}((X/W)^{\log}) \longrightarrow I_{\mathrm{crys}}((X/W)^{\log})$$

is characterized by the equality $\Psi = \Lambda \circ \Phi$.

Let us define the functor

$$\gamma(n) : \begin{pmatrix} \text{coherent sheaf on} \\ ]X[^{\log}_{P(n)} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{isocoherent sheaf} \\ \text{on } D(n) \end{pmatrix}$$

by the composite

$$\begin{pmatrix} \text{coherent sheaf on} \\ ]X[^{\log}_{P(n)} \end{pmatrix} \simeq \begin{pmatrix} \text{compatible family of} \\ \text{isocoherent sheaves on} \\ \{T_{X,m}(\tilde{P}(n))\}_m \end{pmatrix} \overset{\beta(n)^*}{\longrightarrow} \begin{pmatrix} \text{isocoherent sheaf} \\ \text{on } D(n) \end{pmatrix}.$$

Then the functor

$$I_{\mathrm{conv,et}}((X/W)^{\log}) \simeq \mathrm{Str}''((X,M) \hookrightarrow (P,L)) \longrightarrow \mathrm{HPDI}((X,M) \hookrightarrow (P,L))$$

induced by $\gamma(n)\,(n = 0,1,2)$ is identical with the functor Ψ.

Now let us consider the following situation: Let (X, M) be as above and assume that it is of Zariski type, and assume we are given a closed immersion $i : (X, M) \hookrightarrow (P, L)$ into a fine log formal V-scheme (P, L) of Zariksi type which is formally log smooth over $(\mathrm{Spf}\, V, N)$. (But we do not assume the existence of the chart as in the previous paragraph.) Since the diagram $(X, M) \hookrightarrow (P, L) \longrightarrow (\mathrm{Spf}\, W, N)$ admits a chart as in the previous paragraph Zariski locally, we can define the functor $\gamma(n)\,(n \in \mathbb{N})$ Zariski locally. Moreover, one can check that the definition of $\gamma(n)$ is independent of the chart. So we can glue the functor $\gamma(n)$, and so the functor $\gamma(n)$ is defined globally. Hence we have the functor

$$\begin{aligned}\Psi : I_{\mathrm{conv,et}}((X/W)^{\log}) &\simeq \mathrm{Str}''((X,M) \hookrightarrow (P,L)) \\ &\longrightarrow \mathrm{HPDI}((X,M) \hookrightarrow (P,L))\end{aligned}$$

also in this case, and we have the equality $\Phi \circ \Lambda = \Psi$.

Keep the notations of the previous paragraph and let $\mathcal{E}$ be an object in the category $I_{\mathrm{conv,et}}((X/W)^{\log})$. Let (E, ϵ) be the corresponding object in $\mathrm{Str}''((X, M) \hookrightarrow (P, L))$. Then (E, ϵ) defines the integrable log connection

$$\nabla : E \longrightarrow E \otimes \omega^1_{]X[^{\log}_P}$$

on $]X[^{\log}_P$ and the associated log de Rham complex $\mathrm{DR}(]X[^{\log}_P, \mathcal{E})$. On the other hand, let (E', ϵ') be an object in $\mathrm{HPDI}((X, M) \hookrightarrow (P, L))$. Let D^1 be the first infinitesimal neighborhood of D in $D(1)$. Then we have $\mathrm{Ker}(\mathcal{O}_{D^1} \longrightarrow \mathcal{O}_D) \overset{\sim}{=} \omega^1_D := \omega^1_{P/W}|_D$, by [Kk, (6.5)]. Let us denote the pull-back of the isomorphism ϵ' to D^1 by $\epsilon'_1 : \mathcal{O}_{D^1} \otimes E' \overset{\sim}{\longrightarrow} E' \otimes \mathcal{O}_{D^1}$. Then, one can define the log connection

$$\nabla' : E' \longrightarrow E' \otimes \omega^1_D$$

by $\nabla'(e) := \epsilon'_1(1 \otimes e) - e \otimes 1$. Assume that (E', ϵ') comes from an object $\mathcal{E}$ in $I_{\mathrm{crys}}((X/W)^{\log})$. Then the log connection (E', ∇') is integrable: Indeed, put $\mathcal{E} := K_0 \otimes \mathcal{F}$ ($\mathcal{F} \in C_{\mathrm{crys}}((X/W)^{\log})$ and let $\mathcal{F}_n$ be the restriction of $\mathcal{F}$ to $C_{\mathrm{crys}}((X/W_n)^{\log})$. Then, by [Kk, (6.2)], we have the log connections $\nabla'_n : \mathcal{F}_{n,D} \longrightarrow \mathcal{F}_{n,D} \otimes \omega^1_D$ and they are integrable. Since we have $(E', \nabla') := K_0 \otimes_W \varprojlim_n (\mathcal{F}_{n,D}, \nabla'_{n,D})$, (E', ∇') is also integrable. In this case, we denote the log de Rham complex associated to (E', ∇') by $\mathrm{DR}(D, \mathcal{E})$.

Now let $\mathcal{E}$ be an object in $I_{\mathrm{conv,et}}((X/W)^{\log})$ and let (E, ϵ) be the associated object in $\mathrm{Str}''((X, M) \hookrightarrow (P, L))$. On the other hand, put $(E', \epsilon') := \Psi(\mathcal{E}) = \Lambda(\Phi(\mathcal{E}))$. Then we have the integrable log connection (E, ∇) on $]X[^{\log}_P$ associated to (E, ϵ) and the integrable log connection (E', ∇') on D associated to (E', ϵ'). Now let us note that, for any coherent sheaf F on $]X[^{\log}_{P(n)}$, we have the functorial homomorphism $\mathrm{sp}_* F \longrightarrow \gamma(n)(F)$ on $D_{\mathrm{Zar}} = X_{\mathrm{Zar}}$. In particular, we have the homomorphism

$$\mathrm{sp}_*(E \otimes \omega^q_{]X[^{\log}_P}) \longrightarrow E' \otimes \omega^q_D$$

for $q \in \mathbb{N}$. Let us prove the following lemma:

LEMMA 3.1.3. *With the above notations, the following diagram is commutative*:

$$\begin{array}{ccc} \mathrm{sp}_*E & \xrightarrow{\mathrm{sp}_*\nabla} & \mathrm{sp}_*(E \otimes \omega^1_{]X[^{\log}_P}) \\ \big\downarrow & & \big\downarrow \\ E' & \xrightarrow{\nabla'} & E' \otimes \omega^1_D. \end{array} \tag{3.1.3}$$

PROOF. We may work Zariski locally. So we may assume the existence of the chart (3.1.2). So we can define the fine log formal schemes $(\tilde{P}(n), \tilde{L}(n))$ as above. Let $\tilde{P}^1$ be the first infinitesimal neighborhood of $\tilde{P}(0)$ in $\tilde{P}(1)$. One can define the functor

$$\gamma^1 : \begin{pmatrix} \text{coherent sheaf on} \\ ]X[^{\log}_{\tilde{P}^1} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{isocoherent sheaf} \\ \text{on } D^1 \end{pmatrix}$$

in the same way as the functor $\gamma(n)$. Let $p_i :]X[_{\tilde{P}^1} \longrightarrow]X[_{\tilde{P}(0)} =]X[^{\log}_P$ $(i = 1, 2)$ be the projections and denote the pull-back of ϵ to $]X[_{\tilde{P}^1}$ by $\epsilon_1 : p_2^*E \xrightarrow{\sim} p_1^*E$. By the definitions of ∇ and ∇', it suffices to show the equality $\gamma^1(\epsilon_1) = \epsilon'_1$. Let us consider the following commutative diagram:

$$\begin{array}{ccc} D(1) & \longrightarrow & \{T_{X,m}(\tilde{P}(1))\}_m \\ \big\uparrow & & \big\uparrow \\ D^1 & \longrightarrow & \{T_{X,m}(\tilde{P}^1)\}_m. \end{array}$$

One can see that $\gamma^1(\epsilon_1)$ is obtained by pulling back ϵ by the composite

$$D^1 \longrightarrow \{T_{X,m}(\tilde{P}^1)\}_m \longrightarrow \{T_{X,m}(\tilde{P}(1))\}_m,$$

and that ϵ'_1 is obtained by pulling back ϵ by the composite

$$D^1 \longrightarrow D(1) \longrightarrow \{T_{X,m}(\tilde{P}(1))\}_m.$$

So the assertion follows from the commutativity of the above diagram. □

By the diagram (3.1.3), we obtain the morphism of complexes

$$\mathrm{sp}_*\mathrm{DR}(]X[^{\log}_P, \mathcal{E}) \longrightarrow \mathrm{DR}(D, \Phi(\mathcal{E})). \tag{3.1.4}$$

Now let us note the following lemma:

Lemma 3.1.4. *Let the notations be as above and let $\mathcal{E} := K_0 \otimes \mathcal{F}$ be an object in $I_{\mathrm{crys}}((X/W)^{\log})$. Denote the projection $(X/W)^{\log,\sim}_{\mathrm{crys}} \longrightarrow X^{\sim}_{\mathrm{et}} \longrightarrow X^{\sim}_{\mathrm{Zar}}$ by $\tilde{u}$. Then we have*

$$\mathbb{Q} \otimes_{\mathbb{Z}} R\tilde{u}_*\mathcal{F} = \mathrm{DR}(D, \mathcal{E}).$$

Proof. Denote the canonical morphism of topoi $X^{\sim}_{\mathrm{et}} \longrightarrow X^{\sim}_{\mathrm{Zar}}$ by ϵ. Then we have $R\epsilon_*\epsilon^*\mathrm{DR}(D, \mathcal{E}) = \mathrm{DR}(D, \mathcal{E})$, since each term of $\mathrm{DR}(D, \mathcal{E})$ is isocoherent. Let u be the projection $(X/W)^{\log,\sim}_{\mathrm{crys}} \longrightarrow X^{\sim}_{\mathrm{et}}$. Then it suffices to prove the quasi-isomorphism

$$\mathbb{Q} \otimes Ru_*\mathcal{F} = \epsilon^*\mathrm{DR}(D, \mathcal{E}).$$

Let $\mathcal{F}_n$ be the restriction of $\mathcal{F}$ to $C_{\mathrm{crys}}((X/W_n)^{\log})$. Then, by [Kk, (6.4)] and the limit argument of [Be-Og, (7.23)], we have

$$Ru_*\mathcal{F} = \varprojlim_n \epsilon^*\mathrm{DR}(D, \mathcal{F}_n) = \epsilon^* \varprojlim_n \mathrm{DR}(D, \mathcal{F}_n).$$

By tensoring with $\mathbb{Q}$ over $\mathbb{Z}$, we obtain the assertion. □

Now we give a proof of Theorem 3.1.1:

Proof of Theorem 3.1.1. Take a good embedding system

$$(X, M) \overset{g}{\leftarrow} (X^{(\bullet)}, M^{(\bullet)}) \overset{i^{(\bullet)}}{\rightarrow} (P^{(\bullet)}, L^{(\bullet)}).$$

(Note that there exists a good embedding system, since we have assumed that $(\mathrm{Spf}\, W, N)$ admits a chart in Remark 3.1.2.) Let us denote the specialization map $]X^{(\bullet)}[^{\log}_{P^{(\bullet)}} \longrightarrow X^{(\bullet)}$ by $\mathrm{sp}^{(\bullet)}$. Let $(D^{(n)}, M_{D^{(n)}})$ be the p-adically completed log PD-envelope of $(X^{(n)}, M^{(n)})$ in $(P^{(n)}, L^{(n)})$.

Denote the restriction of $\mathcal{E}$ (resp. $\Phi(\mathcal{E})$) to $I_{\mathrm{conv,et}}((X^{(\bullet)}/W)^{\log})$ (resp. $I_{\mathrm{crys}}((X^{(\bullet)}/W)^{\log})$) by $\mathcal{E}^{(\bullet)}$ (resp. $\Phi(\mathcal{E})^{(\bullet)}$). Then, since one has $\Phi(\mathcal{E}^{(\bullet)}) = \Phi(\mathcal{E})^{(\bullet)}$, we have the homomorphism

$$\text{(3.1.5)} \qquad \mathrm{sp}_*^{(\bullet)}\mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}^{(\bullet)}) \longrightarrow \mathrm{DR}(D^{(\bullet)}, \Phi(\mathcal{E})^{(\bullet)})$$

induced by the homomorphism (3.1.4). By applying $H^i(X, Rg_*-)$, we obtain the homomorphism

$$\text{(3.1.6)} \qquad H^i_{\mathrm{an}}((X/W)^{\log}, \mathcal{E}) \longrightarrow H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}))$$

(The expression of the right hand side follows from Lemma 3.1.4 and the cohomological descent.) By Corollary 2.3.9, it suffices to prove that the homomorphism (3.1.6) is an isomorphism. To prove this, it suffices to show the homomorphism

$$\text{(3.1.7)} \qquad \mathrm{sp}_*^{(n)}\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}^{(n)}) \longrightarrow \mathrm{DR}(D^{(n)}, \Phi(\mathcal{E})^{(n)})$$

induced by (3.1.5) is a quasi-isomorphism. To prove this, we may assume that $P^{(n)}$ and $X^{(n)}$ are affine. Let us take a formally log smooth lifting $(X^{(n)}, M^{(n)}) \hookrightarrow (\underline{P}^{(n)}, \underline{L}^{(n)})$ of $(X^{(n)}, M^{(n)})$ over $(\mathrm{Spf}\, W, N)$. Then, both sides of (3.1.7) is unchanged if we replace $(P^{(n)}, L^{(n)})$ by $(\underline{P}^{(n)}, \underline{L}^{(n)})\hat{\times}_{(\mathrm{Spf}\, W,N)}(P^{(n)}, L^{(n)})$ and then by $(\underline{P}^{(n)}, \underline{L}^{(n)})$. So we may assume that $(P^{(n)}, L^{(n)})$ is a formally log smooth lifting of $(X^{(n)}, M^{(n)})$ over $(\mathrm{Spf}\, W, N)$. Then, it is easy to see that $\mathrm{sp}_*^{(n)}\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}^{(n)})$ is identical with $\mathrm{DR}(D^{(n)}, \Phi(\mathcal{E})^{(n)})$ in this case. So the assertion is proved and hence the proof of the theorem is now finished. □

We have the following corollary (cf. [Og2, (0.7.9)]):

Corollary 3.1.5. *Let N be a fine log structure on* $\mathrm{Spf}\, W$ *and denote the pull-back of it to* $\mathrm{Spf}\, V$ *by the same letter. Assume given the following diagram*

$$(X, M) \xrightarrow{f} (\mathrm{Spec}\, k, N) \overset{\iota}{\hookrightarrow} (\mathrm{Spf}\, V, N),$$

where f is a proper log smooth morphism between fine log formal schemes and ι is the canonical exact closed immersion. Then, for $\mathcal{E} \in$

$I_{\mathrm{conv,et}}((X/V)^{\log})$, *the log convergent cohomology* $H^i((X/V)^{\log}_{\mathrm{conv}}, \mathcal{E})$ *is a finite-dimensional* K*-vector space.*

PROOF. Let V' be the Galois closure of $W \subset V$ with Galois group G. Then, by the base-change property of log convergent cohomology with respect to $\mathrm{Spf}\, V' \longrightarrow \mathrm{Spf}\, V$, we may assume that $V = V'$ holds. For $g \in G$, let $\sigma_g : I_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I_{\mathrm{conv,et}}((X/V)^{\log})$ be the functor induced by $g : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ and put $\mathcal{E}' := \bigoplus_{g \in G} \sigma_g(\mathcal{E})$. Then $\mathcal{E}$ is a direct summand of $\mathcal{E}'$ and there exists an object $\mathcal{E}_0$ in $I_{\mathrm{conv,et}}((X/W)^{\log})$ such that $\mathcal{E}'$ is the pull-back of $\mathcal{E}_0$ to $I_{\mathrm{conv,et}}((X/V)^{\log})$. Again by base-change property, we are reduced to show the finite dimensionality of the K_0-vector space $H^i((X/W)^{\log}_{\mathrm{conv}}, \mathcal{E}_0)$. By Theorem 3.1.1, we have the isomorphism

$$H^i((X/W)^{\log}_{\mathrm{conv}}, \mathcal{E}_0) \stackrel{\sim}{=} H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}_0)),$$

and the finite dimensionality of $H^i((X/W)^{\log}_{\mathrm{crys}}, \Phi(\mathcal{E}_0))$ follows from the argument in [Be-Og, §7] (which is valid also in the case of log crystalline cohomology). □

REMARK 3.1.6. The assumption 'N is defined on $\mathrm{Spf}\, W$' can be weakened to the assumption 'N is defined on $\mathrm{Spf}\, V_1$, where V_1 is a sub complete discrete valuation ring of V with absolute ramification index $< p-1$', since the theory of log crystalline cohomology works well over such a base. We expect that the above corollary is valid without these assumptions. If one has the reasonable theory of log crystalline site with level m ($m \in \mathbb{N}$), one will be able to remove these assumptions.

COROLLARY 3.1.7. *Let* (X, M) *be an fs log scheme of Zariski type which is proper and log smooth over* k. *Let* $\mathcal{E}$ *be one of the following*:

(1) $\mathcal{E} = \mathcal{K}_{X/V}$.
(2) *There exists a lifting* $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ *of the* a*-times iteration of the absolute Frobenius of* $\mathrm{Spec}\, k$ $(a > 0)$ *and* $\mathcal{E}$ *is a locally free* F^a*-isocrystal on* (X, M) *over* $\mathrm{Spf}\, V$ *with respect to* σ.

Put $U := X_{\mathrm{triv}}$ *and denote the open immersion* $U \hookrightarrow X$ *by* j. *Then the rigid cohomology group* $H^i_{\mathrm{rig}}(U/K, j^\dagger\mathcal{E})$ *is a finite-dimensional* K*-vector space.*

PROOF. Immediate from Theorem 2.4.4 and Corollary 3.1.5. □

Now we state a version of quasi-unipotent conjecture on overconvergent F^a-isocrystals and prove that this conjecture implies the finiteness of rigid cohomology in the case that the coefficient is an overconvergent F^a-isocrystal.

CONJECTURE 3.1.8 (A version of quasi-unipotent conjecture). *Let us assume that there exists a lifting $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ of the a-times iteration of the absolute Frobenius of $\mathrm{Spec}\, k$. Let X be a smooth scheme of finite type over k and let $\mathcal{E}$ be an overconvergent F^a-isocrystal on X over $\mathrm{Spf}\, V$ with respect to σ. Then, there exist*:

(1) *A proper surjective generically etale morphism $f : X_1 \longrightarrow X$,*
(2) *An open immersion $j : X_1 \hookrightarrow \overline{X}_1$ into a projective smooth variety such that $D := \overline{X}_1 - X_1$ is a simple normal crossing divisor,*
(3) *An F^a-isocrystal $\mathcal{F}$ on $(\overline{X}_1, M)$ (where M is the log structure associated to the pair $(\overline{X}_1, D)$) over $\mathrm{Spf}\, V$,*

such that $j^\dagger \mathcal{F} \overset{\sim}{=} f^* \mathcal{E}$ holds in $I^\dagger(X_1)$.

This conjecture is true in the case that $\mathcal{E}$ is trivial by the alteration theorem of de Jong ([dJ]). Moreover, this conjecture is true if $\mathcal{E}$ is a unit-root overconvergent F^a-isocrystal ([Ts1]). This conjecture can be regarded as a p-adic analogue of the quasi-unipotentness of l-adic sheaves. Then we have the following theorem:

THEOREM 3.1.9. *Let us assume that Conjecture 3.1.8 is true. Let us assume that there exists a lifting $\sigma : \mathrm{Spf}\, V \longrightarrow \mathrm{Spf}\, V$ of the a-times iteration of the absolute Frobenius of $\mathrm{Spec}\, k$. Let X be a smooth scheme of finite type over k and let $\mathcal{E}$ be an overconvergent F^a-isocrystal on X with respect to σ. Then the rigid cohomology $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ is a finite dimensional K-vector space.*

PROOF. The method of proof is the same as that of [Be4, Théorème 3.1] and [Ts2, Theorem 6.1.1]. So we only sketch the outline.

By induction, we prove the following two assertions:

$(a)_n$ $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$ is finite dimensional for any X which is smooth over k with $\dim X \leq n$ and for any overconvergent F^a-isocrystal $\mathcal{E}$ on X.

$(b)_n$ $H^i_{Z,\mathrm{rig}}(X/K,\mathcal{E})$ is finite dimensional for any closed immersion $Z \hookrightarrow X$ of a scheme Z of finite type over k with $\dim Z \leq n$ into a smooth scheme X over k and for any overconvergent F^a-isocrystal $\mathcal{E}$ on X.

First we prove the implication $(b)_{n-1} + (a)_n \implies (b)_n$. To prove $(b)_n$, we may assume that Z is reduced. By long exact sequence of excision and the hypothesis $(b)_{n-1}$, we may assume that the closed immersion $Z \hookrightarrow X$ is a smooth pair, and that there exists a lifting $\mathcal{Z} \hookrightarrow \mathcal{X}$ of $Z \hookrightarrow X$ over $\mathrm{Spec}\, V$ such that $\mathcal{Z}, \mathcal{X}$ are smooth over V and that there exists an etale morphism $\mathcal{X} \longrightarrow \mathrm{Spec}\, V[\mathbb{N}^d]$ for some d. Then, by [Ts2, Theorem 4.1.1], we have the isomorphism $H^i_{Z,\mathrm{rig}}(X/K,\mathcal{E}) \cong H^{i-2c}_{\mathrm{rig}}(Z/K,\mathcal{E}|_Z)$, where $c := \dim X - \dim Z$. So the hypothesis $(a)_n$ implies the finite-dimensionality of $H^i_{Z,\mathrm{rig}}(X/K,\mathcal{E})$.

Next, we prove the implication $(b)_n \implies (a)_{n+1}$. Let $X, \mathcal{E}$ be as in the assertion $(a)_{n+1}$, and let $f : X_1 \longrightarrow X, j : X_1 \hookrightarrow \overline{X}_1, M$ and $\mathcal{F}$ as in Conjecture 3.1.8. Let $U := \mathrm{Spec}\, A_0 \subset X$ be an affine open subscheme such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is etale. Then it is finite etale. Put $f^{-1}(U) := \mathrm{Spec}\, B_0$. First, we have the isomorphism

$$H^i_{\mathrm{rig}}(X_1/K, f^*\mathcal{E}) \cong H^i_{\mathrm{rig}}(X_1/K, j^\dagger\mathcal{F})$$

and it is finite-dimensional by Corollary 3.1.7. By long exact sequence of excision and the hypothesis $(b)_n$, the group $H^i_{\mathrm{rig}}(U_1/K, f^*\mathcal{E})$ is finite-dimensional. Now note the following claim:

CLAIM. The homomorphism

$$f^* : H^i_{\mathrm{rig}}(U/K,\mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U_1/K, f^*\mathcal{E})$$

is an injection into a direct summand.

PROOF OF CLAIM. The proof is almost the same as that in [Be4, Proposition 3.6]. Let $\varphi_0 : A_0 \longrightarrow B_0$ be the ring homomorphism induced by f. Then we have the weakly completed V-algebras $A^\dagger, B^\dagger$ and a ring homomorphism $\varphi : A^\dagger \longrightarrow B^\dagger$ lifting φ_0. By [Be4, Proposition 3.6], φ is finite flat. On the other hand, it is known that one can associate to $\mathcal{E}$ the de Rham complex $E \otimes \Omega^\bullet_{A^\dagger}$ (where E is a finitely generated $A^\dagger$-module) such that $H^i_{\mathrm{rig}}(U/K, \mathcal{E}) = H^i(E \otimes \Omega^\bullet_{A^\dagger})$ holds (for example, see [Ts2, (2.2)]). Via this correspondence, the de Rham complex associated to $f^*\mathcal{E}$ is identical with $(E \otimes \Omega^\bullet_{A^\dagger}) \otimes_{A^\dagger} B^\dagger$. Since $B^\dagger$ is finite flat over $A^\dagger$, one can define the trace morphism

$$\mathrm{tr} : (E \otimes \Omega^\bullet_{A^\dagger}) \otimes_{A^\dagger} B^\dagger \longrightarrow E \otimes \Omega^\bullet_{A^\dagger}$$

such that the composite with the natural inclusion

$$E \otimes \Omega^\bullet_{A^\dagger} \longrightarrow (E \otimes \Omega^\bullet_{A^\dagger}) \otimes_{A^\dagger} B^\dagger$$

(note that it corresponds to f^*) is equal to the multiplication by $[B^\dagger : A^\dagger]$. Hence $H^i_{\mathrm{rig}}(U/K, \mathcal{E}) = H^i(E \otimes \Omega^\bullet_{A^\dagger})$ is a direct summand of $H^i((E \otimes \Omega^\bullet_{A^\dagger}) \otimes_{A^\dagger} B^\dagger) = H^i_{\mathrm{rig}}(U_1/K, f^*\mathcal{E})$ via the homomorphism f^*. □

By the claim, $H^i_{\mathrm{rig}}(U/K, \mathcal{E})$ is finite-dimensional. Again by the long exact sequence of excision and the hypothesis $(b)_n$, we obtain the finite-dimensionality of $H^i_{\mathrm{rig}}(X/K, \mathcal{E})$, as desired. □

3.2. A remark on Berthelot-Ogus theorem for fundamental groups

In this section, we give an alternative proof of Berthelot-Ogus theorem for fundamental groups, which was proved in the previous paper [Shi]. We remark that here we prove the theorem under a slightly weaker assumption.

The main result in this section is as follows:

THEOREM 3.2.1 (Berthelot-Ogus theorem for π_1). *Assume we are given the following commutative diagram of fine log schemes*

$$\begin{array}{ccccc}
(X_k, M) & \hookrightarrow & (X, M) & \hookleftarrow & (X_K, M) \\
\downarrow & & f \downarrow & & \downarrow \\
(\mathrm{Spec}\, k, N) & \hookrightarrow & (\mathrm{Spec}\, V, N) & \hookleftarrow & (\mathrm{Spec}\, K, N) \\
& \searrow & \downarrow & & \\
& & (\mathrm{Spec}\, W, N), & &
\end{array}$$

where the two squares are Cartesian, f is proper and log smooth. Assume moreover that $H^0_{\mathrm{dR}}((X, M)/(\mathrm{Spf}\, V, N)) = V$ holds, and that we are given a V-valued point x of $X_{f\text{-triv}}$. Denote the special fiber (*resp. generic fiber*) *of x by x_k (resp. x_K). Then there exists a canonical isomorphism of pro-algebraic groups*

$$\pi_1^{\mathrm{crys}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k) \times_{K_0} K \overset{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\mathrm{Spec}\, K, N), x_K).$$

Remark 3.2.2. We remark here the difference of the assumption in the above theorem and that in the Berthelot-Ogus theorem for fundamental groups proved in the previous paper ([Shi, (5.3.2)]). First, we do not need the assumption 'f is integral' to prove the theorem, which we needed in [Shi, (5.3.2)]. Second, we do not need the assumption 'X_k is reduced' to prove the theorem, which we also needed in [Shi, (5.3.2)].

Proof. Let $\hat{X}$ be the p-adic completion of X. First we prove the equivalence of categories

$$\mathcal{N}C((X_K, M)/(\mathrm{Spec}\, K, N)) \simeq \mathcal{N}I_{\mathrm{conv,et}}((X_k, M)/(\mathrm{Spf}\, V, N)).$$

Let α be the composite

$$\begin{aligned}\mathcal{N}I_{\mathrm{conv,et}}((X_k, M)/(\mathrm{Spf}\, V, N)) &\simeq \mathcal{N}I_{\mathrm{conv}}((\hat{X}, M)/(\mathrm{Spf}\, V, N)) \\ &\xrightarrow{\bar{\iota}} \mathcal{N}I_{\mathrm{inf}}((\hat{X}, M)/(\mathrm{Spf}\, V, N)),\end{aligned}$$

where $\bar{\iota}$ is the functor defined in [Shi, §5.2]. Since we have the equivalence of categories

$$\mathcal{N}I_{\mathrm{inf}}((\hat{X}, M)/(\mathrm{Spf}\, V, N)) \simeq \mathcal{N}C((X_K, M)/(\mathrm{Spec}\, K, N)) \tag{3.2.1}$$

by [Shi, (3.2.16)], it suffices to prove that the functor α gives an equivalence of categories. By [Shi, (5.2.2)], we may work formally etale locally on $\hat{X}$. So we may assume that $(\hat{X}, M)$ is affine of Zariski type. (Then so is (X_k, M).) To prove that α gives an equivalence of categories, it suffices to show that there exists canonical isomorphisms

$$H^i((X_k/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \xrightarrow{\sim} H^i((\hat{X}/V)^{\log}_{\mathrm{inf}}, \alpha(\mathcal{E})) \quad (i = 0, 1) \tag{3.2.2}$$

for any $\mathcal{E} \in \mathcal{N}I_{\mathrm{conv,et}}((X_k/V)^{\log})$. By log convergent Poincaré lemma and Theorem B of Kiehl, we have the isomorphism

$$H^i((X_k/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \stackrel{\simeq}{=} H^i(X, \mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})).$$

(Strictly speaking, we should enlarge V and take Galois-invariant part if $(\operatorname{Spf} V, N)$ does not admit a chart. But the above isomorphism remains true also in this case.) On the other hand, the object corresponding to $\alpha(\mathcal{E})$ via the equivalence (3.2.1) is nothing but the formal log connection $\mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})^0 \longrightarrow \mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})^1$. By the interpretation of extension class in the category $\mathcal{N}\hat{C}((\hat{X}, M)/(\operatorname{Spf} V, N))$, we have

$$\begin{aligned} H^i((X/V)^{\log}_{\inf}, \mathcal{E}) &\stackrel{\simeq}{=} H^i(X, \mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})^0 \longrightarrow \mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})^1) \\ &\stackrel{\simeq}{=} H^i(X, \mathrm{sp}_*\mathrm{DR}(\hat{X}_K, \mathcal{E})) \end{aligned}$$

for $i = 0, 1$. Hence we have the canonical isomorphism (3.2.2). So we obtained the equivalence of categories

$$\mathcal{N}C((X_K, M)/(\operatorname{Spec} K, N)) \simeq \mathcal{N}I_{\mathrm{conv,et}}((X_k, M)/(\operatorname{Spf} V, N)).$$

Since we have $H^0_{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N)) = K$, we obtain the isomorphism

$$H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/V}) \stackrel{\simeq}{=} K$$

and so the above two categories are neutral Tannakian. (The neutrality follows from the existence of the base points x_K and x_k.) So we obtain the isomorphism of fundamental groups

$$\text{(3.2.3)} \quad \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k) \stackrel{\simeq}{=} \pi_1^{\mathrm{dR}}((X_K, M)/(\operatorname{Spec} K, N), x_K).$$

Next, since we have the base-change property

$$H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/V}) \stackrel{\simeq}{=} H^0((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/W}) \otimes_{K_0} K,$$

we have $H^0((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/W}) = K_0$. So we have the base-change property of fundamental groups

$$\text{(3.2.4)} \quad \begin{aligned} &\pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} V, N), x_k) \\ &\stackrel{\simeq}{=} \pi_1^{\mathrm{conv}}((X_k, M)/(\operatorname{Spf} W, N), x_k) \times_{K_0} K, \end{aligned}$$

by [Shi, (5.1.13)]. By the isomorphisms (3.2.3) and (3.2.4), the proof is reduced to the proof of the isomorphism

$$\pi_1^{\mathrm{conv}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k) \overset{\sim}{=} \pi_1^{\mathrm{crys}}((X_k, M)/(\mathrm{Spf}\, W, N), x_k),$$

that is, to the proof of the equivalence of the functor

$$\overline{\Phi} : \mathcal{N}I_{\mathrm{conv,et}}((X_k, M)/(\mathrm{Spf}\, W, N)) \longrightarrow \mathcal{N}I_{\mathrm{crys}}((X_k, M)/(\mathrm{Spf}\, W, N))$$

defined in [Shi, (5.3.1)]. This follows from the canonical isomorphisms

$$H^i((X/W)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \overset{\sim}{=} H^i((X/W)^{\log}_{\mathrm{crys}}, \overline{\Phi}(\mathcal{E})) \quad (i = 0, 1)$$

for $\mathcal{E} \in \mathcal{N}I_{\mathrm{conv,et}}((X/W)^{\log})$, which is proved in Theorem 3.1.1. □

3.3. Independence of compactification for crystalline fundamental groups

In the previous paper (Shi, [§4.2]), we considered the following problem:

Problem 3.3.1. Let X be a connected proper smooth scheme over k and let $D \subset X$ be a normal crossing divisor. Denote the log structure on X associated to D by M. Put $U := X - D$, and let x be a k-valued point of U. Then, is the crystalline fundamental group $\pi_1^{\mathrm{crys}}((X, M)/\mathrm{Spf}\, W, x)$ of (X, M) over $\mathrm{Spf}\, W$ with base point x independent of the choice of the compactification (X, D) of U as above?

In the previous paper, we gave the affirmative answer under the condition $\dim U \leq 2$. We needed this condition because we used the resolution of singularities due to Abhyankar ([A]) and the theorem on the structure of proper birational morphism between surfaces due to Shafarevich ([Sha]). In this section, we give the affirmative answer to the above question in general case. We use the category of nilpotent overconvergent isocrystals to prove this problem.

First, we recall the rigid fundamental group for a k-scheme, which is due to Chiarellotto and Le Stum ([Ch], [Ch-LS], [Ch-LS2]). Let X be a scheme which is separated and of finite type over k. Then, by [Be3, 2.3.3(iii)], the category $I^\dagger(X)$ of overconvergent isocrystals is an abelian tensor category, and it is easy to check that it is rigid. Let us denote the unit object of the category $I^\dagger(X)$ by $\mathcal{O}$. (In local situation, it is identical with $(j^\dagger \mathcal{O}_{]X[_P}, \mathrm{id})$.) By [Be3, (2.2.7)] and [Ch-LS2, (1.3.1)], we have the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{I^\dagger(X)}(\mathcal{E}, \mathcal{E}') &= H^0_{\mathrm{rig}}(X/K, \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E}')), \\ \mathrm{Ext}_{I^\dagger(X)}(\mathcal{E}, \mathcal{E}') &= H^1_{\mathrm{rig}}(X/K, \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E}')), \end{aligned}$$

for $\mathcal{E}, \mathcal{E}' \in I^\dagger(X)$. Let us note the following lemma:

LEMMA 3.3.2. *Let X be a scheme which is separated and of finite type over k and assume that $H^0_{\mathrm{rig}}(X/K, \mathcal{O})$ is a field. Then the category $\mathcal{N}I^\dagger(X)$(:= the nilpotent part* ([Shi, (1.1.9)]) *of the abelian tensor category $I^\dagger(X)$) is a Tannakian category.*

PROOF. Since $I^\dagger(X)$ is an abelican tensor category and $\mathrm{End}(\mathcal{O}) = H^0_{\mathrm{rig}}(X/K, \mathcal{O})$ is a field, the category $\mathcal{N}I^\dagger(X)$ is an abelian category by [Shi, (1.2.1)]. Moreover, one can check easily that $\mathcal{N}I^\dagger(X)$ has a tensor structure which makes it a rigid abelian tensor category. So it suffices to prove the existence of the fiber functor $\mathcal{N}I^\dagger(X) \longrightarrow \mathrm{Vec}_L$ for a field L.

Let $X \subset \overline{X}$ be a compactification of X, and let $\overline{U} \subset \overline{X}$ be an affine open subset. Put $U := \overline{U} \cap X$. (Note that it is not empty.) Take a closed immersion $\overline{U} \hookrightarrow P$ of $\overline{U}$ into a formal V-scheme P which is formally log smooth over $\mathrm{Spf}\, V$. Since the specialization map $P_K \longrightarrow P$ is surjective by [Be3, (1.1.5)], the admissible open set $]U[_P$ in $]\overline{U}[_P$ is non-empty. Let $\mathrm{Spm}\, A \subset]U[_P$ be a non-empty admissible open set of $]U[_P$ and take a maximal ideal I of A. Put $L := A/I$. Then we can define the functor $\xi : \mathcal{N}I^\dagger(X) \longrightarrow \mathrm{Vec}_L$ as the composite

$$\begin{aligned} \mathcal{N}I^\dagger(X) &\longrightarrow (A\text{-modules}) \\ &\longrightarrow \mathrm{Vec}_L, \end{aligned}$$

where the first arrow is defined by the evaluation at the rigid analytic space $\mathrm{Spm}\, A$ and the second arrow is induced by the ring homomorphism $A \longrightarrow A/I = L$. Then one can see that this functor is an exact tensor functor, noting the fact that the essential image of the first arrow consists of free modules. Then, by [D2, (2.10)], ξ is faithful. Hence ξ is a fiber functor and so we are done. □

So we can define the rigid fundamental group as follows. This definition is due to Chiarellotto and Le Stum ([Ch], [Ch-LS], [Ch-LS2]).

Definition 3.3.3 (Definition of π_1^{rig}). Let X be a scheme which is separated and of finite type over k and let x be a k-valued point of X. Then we define the rigid fundamental group of X over K with base point x by

$$\pi_1^{\mathrm{rig}}(X/K, x) := G(\mathcal{N}I^\dagger(X), \omega_x),$$

where ω_x is the fiber functor

$$\mathcal{N}I^\dagger(X) \longrightarrow \mathcal{N}I^\dagger(x) \simeq \mathrm{Vec}_K$$

induced by the closed immersion $x \hookrightarrow X$ and the notation $G(\cdots)$ is as in Theorem 1.1.8 in [Shi].

The main result in this section is as follows:

Theorem 3.3.4. *Let (X, M) be one of the following*:

1. *(X, M) is an fs log scheme of Zariski type which is proper and log smooth over k.*
2. *X is a proper smooth scheme of finite type over k and M is the log structure associated to a normal crossing divisor on X.*

Put $U := X_{\mathrm{triv}}$ and denote the open immersion $U \hookrightarrow X$ by j. Then there exists the canonical equivalence of categories

$$\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log}) \simeq \mathcal{N}I^\dagger(U).$$

In particular, for a k-valued point x of U, we have the isomorphism

$$\pi_1^{\mathrm{conv}}((X, M)/\mathrm{Spf}\, V, x) \overset{\simeq}{=} \pi_1^{\mathrm{rig}}(U/K, x).$$

Proof. First, let us consider the case (1). In this case, we prove that the functor

$$j^\dagger : I_{\mathrm{conv,et}}((X/V)^{\log}) \longrightarrow I^\dagger(U, X) = I^\dagger(U)$$

induces the equivalence of categories. To prove it, it suffices to prove that the homomorphisms induced by $j^\dagger$

$$H^i((X/V)^{\log}_{\mathrm{conv}}, \mathcal{E}) \longrightarrow H^i_{\mathrm{rig}}(U/K, j^\dagger \mathcal{E}) \quad (i = 0, 1)$$

are isomorphisms for any $\mathcal{E} \in \mathcal{N}I_{\mathrm{conv,et}}((X, M)/(\mathrm{Spf}\, V, N))$. By five lemma, it suffices to prove that the homomorphisms

$$H^i((X/V)^{\log}_{\mathrm{conv}}, \mathcal{K}_{X/V}) \longrightarrow H^i_{\mathrm{rig}}(U/K, \mathcal{O})$$

are isomorphisms for any $i \in \mathbb{N}$. It is already proved in Theorems 2.4.4. So we are done.

Next, let us consider the case (2). Note that there exists a log etale proper birational morphism

$$f : (\overline{X}, \overline{M}) \longrightarrow (X, M)$$

which is defined by a composition of blow-ups whose centers are smooth self-intersections of irreducible components of D such that $\overline{X}$ is smooth and $\overline{M}$ is the log structure associated to a simple normal crossing divisor $f^{-1}(D)_{\mathrm{red}}$ on $\overline{X}$. Then we have $\overline{X}_{\mathrm{triv}} = X_{\mathrm{triv}} = U$ and $Rf_* \mathcal{O}_{\overline{X}} = \mathcal{O}_X$. Now let us note the following proposition:

Proposition 3.3.5. *Let* $f : (\overline{X}, \overline{M}) \longrightarrow (X, M)$ *be as above and let* $\mathcal{E}$ *be an object in* $I^{\mathrm{lf}}_{\mathrm{conv,et}}((X/V)^{\log})$. *Denote the restriction of* $\mathcal{E}$ *to* $I^{\mathrm{lf}}_{\mathrm{conv,et}}((\overline{X}/V)^{\log})$ *by* $\overline{\mathcal{E}}$. *Then the homomorphism*

$$H^i((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \longrightarrow H^i((\overline{X}/V)^{\log}_{\mathrm{conv,et}}, \overline{\mathcal{E}})$$

induced by f *is an isomorphism.*

Proof. By log convergent Poincaré lemma, it suffices to show that the homomorphism

$$H^i_{\mathrm{an}}((X/V)^{\log}, \mathcal{E}) \longrightarrow H^i_{\mathrm{an}}((\overline{X}/V)^{\log}, \overline{\mathcal{E}})$$

induced by f is an isomorphism.

Let us take the diagram

$$(X, M) \xleftarrow{g} (X^{(0)}, M^{(0)}) \overset{i^{(0)}}{\hookrightarrow} (P^{(0)}, L^{(0)}),$$

where g is an etale covering such that $M^{(0)}$ is associated to a simple normal crossing divisor on $X^{(0)}$, and $(P^{(0)}, L^{(0)})$ is a log smooth lifting of $(X^{(0)}, M^{(0)})$ over $\operatorname{Spf} V$ such that $P^{(0)}$ is formally log smooth over $\operatorname{Spf} V$ and that $L^{(0)}$ is associated to a relative simple normal crossing divisor on $P^{(0)}$ over $\operatorname{Spf} V$. For $n \in \mathbb{N}$, let $(X^{(n)}, M^{(n)})$ (resp. $(P^{(n)}, L^{(n)})$) be the $(n+1)$-fold fiber product of $(X^{(0)}, M^{(0)})$ (resp. $(P^{(0)}, L^{(0)})$) over (X, M) (resp. $\operatorname{Spf} V$), and put $(\overline{X}^{(n)}, \overline{M}^{(n)}) := (\overline{X}, \overline{M}) \times_{(X,M)} (X^{(n)}, M^{(n)})$. Then the morphism

$$(\overline{X}^{(0)}, \overline{M}^{(0)}) \longrightarrow (X^{(0)}, M^{(0)})$$

is induced by a subdivision of fan. So this morphism fits into a Cartesian diagram

$$\begin{array}{ccc}
(\overline{X}^{(0)}, \overline{M}^{(0)}) & \xrightarrow{\bar{i}^{(0)}} & (\overline{P}^{(0)}, \overline{L}^{(0)}) \\
\big\downarrow & & {\scriptstyle g^{(0)}}\big\downarrow \\
(X^{(0)}, M^{(0)}) & \xrightarrow{i^{(0)}} & (P^{(0)}, L^{(0)}),
\end{array}$$

where $g^{(0)}$ is a formally log etale morphism. Then we have $Rg^{(0)}_* \mathcal{O}_{\overline{P}^{(0)}} = \mathcal{O}_{P^{(0)}}$. Let $(\overline{P}^{(n)}, \overline{L}^{(n)})$ be the $(n+1)$-fold fiber product of $(\overline{P}^{(0)}, \overline{L}^{(0)})$ over $\operatorname{Spf} V$. Let $g^{(n)}$ be the morphism $(\overline{P}^{(n)}, \overline{M}^{(n)}) \longrightarrow (P^{(n)}, L^{(n)})$ induced by $g^{(0)}$ and denote the associated morphism of rigid analytic spaces $]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}} \longrightarrow]X^{(n)}[^{\log}_{P^{(n)}}$ by $g^{(n)}_K$. Let $\mathrm{sp}^{(n)} :]X^{(n)}[^{\log}_{P^{(n)}} \longrightarrow X^{(n)}$ be the specialization map. Then, it suffices to prove the quasi-isomorphism

$$R\mathrm{sp}^{(\bullet)}_* Rg^{(\bullet)}_{K,*} \mathrm{DR}(]\overline{X}^{(\bullet)}[^{\log}_{\overline{P}^{(\bullet)}}, \overline{\mathcal{E}}) \cong R\mathrm{sp}^{(\bullet)}_* \mathrm{DR}(]X^{(\bullet)}[^{\log}_{P^{(\bullet)}}, \mathcal{E}).$$

To prove this, we may replace $\bullet$ by n. Since $(X^{(n)}, M^{(n)})$ is etale (in classical sense) over $(X^{(0)}, M^{(0)})$, there exists a formally etale morphism $(P^{(n)}_1, L^{(n)}_1) \longrightarrow (P^{(0)}, L^{(0)})$ such that $(P^{(n)}_1, L^{(n)}_1) \times_{(P^{(0)}, L^{(0)})} (X^{(0)}, M^{(0)}) =$

$(X^{(n)}, M^{(n)})$ holds. Let $(P_2^{(n)}, L_2^{(n)})$ be the $(n+1)$-fold fiber product of $(P_1^{(n)}, L_1^{(n)})$ over $\mathrm{Spf}\, V$. Then we have the commutative diagram

$$\begin{array}{ccccc}
(X^{(n)}, M^{(n)}) & = & (X^{(n)}, M^{(n)}) & = & (X^{(n)}, M^{(n)}) \\
\downarrow & & \downarrow & & \downarrow \\
(P_1^{(n)}, L_1^{(n)}) & \xleftarrow{\pi_1} & (P_2^{(n)}, L_2^{(n)}) & \xrightarrow{\pi_2} & (P^{(n)}, L^{(n)}),
\end{array}$$

where the vertical arrows are closed immersions and $\pi_i\,(i=1,2)$ are formally log smooth. Let $\mathrm{sp}_i^{(n)} :]X^{(n)}[^{\log}_{P_i^{(n)}} \longrightarrow X^{(n)}$ be the specialization map $(i=1,2)$. Then we have the quasi-isomorphisms

$$\begin{aligned}
R\mathrm{sp}_*^{(n)}\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}}, \mathcal{E}) &= R\mathrm{sp}_{2,*}^{(n)}\mathrm{DR}(]X^{(n)}[^{\log}_{P_2^{(n)}}, \mathcal{E}) \\
&= R\mathrm{sp}_{1,*}^{(n)}\mathrm{DR}(]X^{(n)}[^{\log}_{P_1^{(n)}}, \mathcal{E}).
\end{aligned}$$

On the other hand, let $(\overline{P}_1^{(n)}, \overline{L}_1^{(n)}) := (P_1^{(n)}, L_1^{(n)}) \times_{(P^{(0)}, L^{(0)})} (\overline{P}^{(0)}, \overline{L}^{(0)})$ and let $(\overline{P}_2^{(n)}, \overline{L}_2^{(n)})$ be the $(n+1)$-fold fiber product of $(\overline{P}_1^{(n)}, \overline{L}_1^{(n)})$ over $\mathrm{Spf}\, V$. Denote the morphism

$$(\overline{P}_i^{(n)}, \overline{L}_i^{(n)}) \longrightarrow (P_i^{(n)}, L_i^{(n)})$$

induced by $g^{(n)}$ by $g_i^{(n)}$ and denote the associated morphism of rigid analytic spaces $]\overline{X}^{(n)}[^{\log}_{\overline{P}_i^{(n)}} \longrightarrow]X^{(n)}[^{\log}_{P_i^{(n)}}$ by $g_{i,K}^{(n)}$. Then, since the morphisms

$$\begin{aligned}
(\overline{P}_2^{(n)}, \overline{L}_2^{(n)}) &\longrightarrow (\overline{P}^{(n)}, \overline{L}^{(n)}), \\
(\overline{P}_2^{(n)}, \overline{L}_2^{(n)}) &\longrightarrow (\overline{P}_1^{(n)}, \overline{L}_1^{(n)}),
\end{aligned}$$

induced by π_1, π_2 are formally log smooth, we have the quasi-isomorphisms

$$\begin{aligned}
R\mathrm{sp}_*^{(n)} Rg_{K,*}^{(n)}\mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}}, \overline{\mathcal{E}}) &= R\mathrm{sp}_{2,*}^{(n)} Rg_{2,K,*}^{(n)}\mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}_2^{(n)}}, \overline{\mathcal{E}}) \\
&= R\mathrm{sp}_{1,*}^{(n)} Rg_{1,K,*}^{(n)}\mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}_1^{(n)}}, \overline{\mathcal{E}}).
\end{aligned}$$

So it suffices to prove the quasi-isomorphism

$$R\mathrm{sp}^{(n)}_{1,*}Rg^{(n)}_{1,K,*}\mathrm{DR}(]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}_1},\overline{\mathcal{E}}) \cong R\mathrm{sp}^{(n)}_{1,*}\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}_1},\mathcal{E}).$$

Now let us note the equalities $]\overline{X}^{(n)}[^{\log}_{\overline{P}^{(n)}_1}=\overline{P}^{(n)}_{1,K}$, $]X^{(n)}[^{\log}_{P^{(n)}_1}=P^{(n)}_{1,K}$. Since the each term of $\mathrm{DR}(]X^{(n)}[^{\log}_{P^{(n)}_1},\mathcal{E})$ is a locally free $\mathcal{O}_{P^{(n)}_{1,K}}$-module, it suffices to prove that $Rg^{(n)}_{1,K,*}\mathcal{O}_{\overline{P}^{(n)}_{1,K}}=\mathcal{O}_{P^{(n)}_{1,K}}$ holds. This follows from the flat base change and the quasi-isomorphism $Rg_*\mathcal{O}_{\overline{P}^{(0)}}=\mathcal{O}_{P^{(0)}}$, which we already remarked. Hence the assertion is proved and so the proof of the proposition is finished. □

Proof of Theorem 3.3.4 (continued). By Proposition 3.3.5, the functor

$$f^*:\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log})\longrightarrow\mathcal{N}I_{\mathrm{conv,et}}((\overline{X}/V)^{\log})$$

induced by f^* is an equivalence of categories. Let us denote the open immersion $U\hookrightarrow\overline{X}$ by $\overline{j}$. We define the functor

$$j^\dagger_f:\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log})\longrightarrow\mathcal{N}I^\dagger(U)$$

by the composite $\overline{j}^\dagger\circ f^*$. Then, since f^* and $\overline{j}^\dagger$ are equivalences, $j^\dagger_f$ is also an equivalence of categories.

Note that the functor $j^\dagger_f$ *a propri* depends on the choice of f. We prove that the functor $j^\dagger_f$ is in fact independent of the choice of f. Note that we have the following commutative diagram for any f as above:

$$\begin{array}{ccc}\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log}) & \xrightarrow{j^\dagger_f} & \mathcal{N}I^\dagger(U)\\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle r}\\ \mathcal{N}I_{\mathrm{conv,et}}(U/V) & = & \mathcal{N}I_{\mathrm{conv,et}}(U/V),\end{array}\tag{3.3.1}$$

where the functors j^* and r are the functors defined in Section 2.4. Suppose for the moment the injectivity of the homomorphism

$$H^1((X/V)^{\log}_{\mathrm{conv,et}},\mathcal{K}_{X/V})\longrightarrow H^1((U/V)_{\mathrm{conv,et}},\mathcal{K}_{U/V}),\tag{3.3.2}$$

which we prove in the proposition below. Note that we have the isomorphism

$$H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/V}) \overset{\simeq}{=} H^0((U/V)_{\mathrm{conv,et}}, \mathcal{K}_{U/V}).$$

Indeed, since we may replace k by a finite extension to prove it, we may assume the existence of a k-valued point of U, and in this case, both sides are equal to K by the argument in the proof of [Shi, (5.1.11)]. Then, by five lemma, one can see that the homomorphism

$$H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \longrightarrow H^0((U/V)_{\mathrm{conv,et}}, \mathcal{E})$$

is an isomorphism for any $\mathcal{E}$ in $\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log})$. Hence j^* is fully-faithful. Then the diagram (3.3.1) implies the fact that the functor $j^{\dagger}_f$ is independent of the choice of f. So the proof is reduced to the injectivity of the homomorphism (3.3.2). Hence the proof of the theorem is finished modulo the proposition below. □

Proposition 3.3.6. *Let X be a smooth scheme of finite type over k and and let M be the log structure associated to a normal crossing divisor D on X. Put $U := X - D$ and denote the open immersion $U \hookrightarrow X$ by j. Then the homomorphism*

$$j^* : H^1((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X/V}) \longrightarrow H^1((U/V)_{\mathrm{conv,et}}, \mathcal{K}_{U/V})$$

is injective.

Proof. By considering the interpretation by the extension class, it suffices to show the following claim: Let

$$0 \to \mathcal{K}_{X/V} \to \mathcal{E} \to \mathcal{K}_{X/V} \to 0 \tag{3.3.3}$$

be an exact sequence in $I_{\mathrm{conv,et}}((X/V)^{\log})$ and assume it splits in the category $I_{\mathrm{conv,et}}(U/V)$. Let $s : \mathcal{K}_{U/V} \longrightarrow \mathcal{E}|_U$ be a splitting in $I_{\mathrm{conv,et}}(U/V)$. Then there exists a splitting $\tilde{s} : \mathcal{K}_{X/V} \longrightarrow \mathcal{E}$ in $I_{\mathrm{conv,et}}((X/V)^{\log})$ which extends s.

First we prove that it suffices to prove this assertion etale locally. Let $X^{(0)} := \coprod_{i \in I} X_i \longrightarrow X$ be an etale covering by schemes of finite type over k with $|I| < \infty$ and let $M^{(0)}$ be the pull-back of M to $X^{(0)}$. Let

$(X^{(1)}, M^{(1)}) := (X^{(0)}, M^{(0)}) \times_{(X,M)} (X^{(0)}, M^{(0)})$ and put $U^{(i)} := X^{(i)} \times_X U \, (i = 0, 1)$. Then claim 2 in [Shi, (5.1.11)] implies the injectivity of the homomorphism

$$H^0((X^{(1)}/V)^{\log}_{\mathrm{conv,et}}, \mathcal{K}_{X^{(1)}/V}) \longrightarrow H^0((U^{(1)}/V)_{\mathrm{conv,et}}, \mathcal{K}_{U^{(1)}/V}).$$

So the homomorphism

$$j^{(1),*} : H^0((X^{(1)}/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) \longrightarrow H^0((U^{(1)}/V)_{\mathrm{conv,et}}, \mathcal{E})$$

is injective for $\mathcal{E} \in \mathcal{N}I_{\mathrm{conv,et}}((X^{(1)}/V)^{\log})$. Assume we are given an exact sequence as in (3.3.3) and denote the pull-back of $\mathcal{E}$ to $I_{\mathrm{conv,et}}(X^{(i)}/V) \, (i = 0, 1)$ by $\mathcal{E}^{(i)}$. Let us consider the following diagram, where the vertical lines are exact:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}) & \xrightarrow{j^*} & H^0((U/V)_{\mathrm{conv,et}}, \mathcal{E}|_U) \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha'}\downarrow \\
H^0((X^{(0)}/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E}^{(0)}) & \xrightarrow{j^{(0),*}} & H^0((U^{(0)}/V)_{\mathrm{conv,et}}, \mathcal{E}|_{U^{(0)}}) \\
{\scriptstyle\beta}\downarrow & & {\scriptstyle\beta'}\downarrow \\
H^0((X^{(1)}/V)^{\log}_{\mathrm{conv,et}}, E) & \xrightarrow{j^{(1),*}} & H^0_{\mathrm{conv}}((U^{(1)}/V)_{\mathrm{conv,et}}, E).
\end{array}$$

Now assume we are given a splitting $s : \mathcal{K}_{U/V} \longrightarrow \mathcal{E}|_U$ in $I_{\mathrm{conv,et}}(U/V)$. Then, if we know the proposition for each $X_i \, (i \in I)$, there exists $\tilde{s}^{(0)} : \mathcal{K}_{X^{(0)}/V} \longrightarrow \mathcal{E}^{(0)}$ in $I_{\mathrm{conv,et}}((X^{(0)}/V)^{\log})$ satisfying $j^{(0),*}(\tilde{s}^{(0)}) = \alpha'(s)$. Then we have

$$j^{(1),*}(\beta(\tilde{s}^{(0)})) = \beta'(j^{(0),*}(\tilde{s}^{(0)})) = \beta'(\alpha'(s)) = 0.$$

Since $j^{(1),*}$ is injective, we have $\beta(\tilde{s}^{(0)}) = 0$ and so there exists an element $\tilde{s} \in H^0((X/V)^{\log}_{\mathrm{conv,et}}, \mathcal{E})$ such that $\alpha(\tilde{s}) = \tilde{s}^{(0)}$ holds. For such an element $\tilde{s}$, we have

$$\alpha'(j^*(\tilde{s})) = j^{(0),*}(\alpha(\tilde{s})) = j^{(0),*}(\tilde{s}^{(0)}) = \alpha'(s)$$

and so $j^*(\tilde{s}) = s$ holds as desired, since α' is injective.

So we may assume that there exists a formally log smooth lifting of (X, M) to an affine fine log formal V-scheme (P, L), a lifting $\{t_1, \cdots, t_n\}$ of a regular parameter of X to P and an integer $r \leq n$ such that L is associated to the relative simple normal crossing divisor $\{t_1 \cdots t_r = 0\}$ on P over $\mathrm{Spf}\, V$. Put $Q := P_{\mathrm{triv}}$ and let K' be the fraction field of the integral domain $\Gamma(P, \mathcal{O}_P)/(t_1, \cdots, t_n)$. Then, by the argument of [Cr1, §4], there exists the following diagram:

$$\begin{array}{ccc} \Gamma(P_K, \mathcal{O}_{P_K}) & \longrightarrow & K'[[t]]^{\mathrm{b}} \\ \big\downarrow & & \big\downarrow \\ \Gamma(Q_K, \mathcal{O}_{Q_K}) & \xrightarrow{\ i\ } & K'((t)), \end{array}$$

where the rings $K'[[t]]^{\mathrm{b}}$ and $K'((t))$ are defined as follows: (Here we use multi-index notation for t_i's.)

$$K'[[t]]^{\mathrm{b}} := \{\sum_{l \in \mathbb{N}^n} a_l t^l \mid a_l \in K', |a_l| \text{ is bounded.}\}.$$

$$K'((t)) := \{\sum_{l \in \mathbb{Z}^n} a_l t^l \mid a_l \in K', |a_l| \text{ is bounded}, |a_l| \to 0 \ (m(l) \to -\infty)\}.$$

Here, for $l = (l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n$, $m(l)$ is defined by

$$m(l) := \min\{l_1, l_2, \cdots, l_n\}.$$

By the same method as in [Cr1, (4.7.1)], one can show the equation

$$\mathrm{Im}(i) \cap K'[[t]]^{\mathrm{b}} = \Gamma(P_K, \mathcal{O}_{P_K}). \tag{3.3.4}$$

Now we prove the proposition in the above situation. Assume we are given an exact sequence (3.3.3) which splits in $I_{\mathrm{conv,et}}(U/V)$. By results of [Shi, Chap. 5], $\mathcal{N}I_{\mathrm{conv,et}}((X/V)^{\log})$ is equivalent to the category of isocoherent sheaves on (P, L) with nilpotent integrable formal log connections, and $\mathcal{N}I_{\mathrm{conv,et}}(U/V)$ is equivalent to the category of isocoherent sheaves on Q with nilpotent integrable formal connections. Let

$$\nabla : E \longrightarrow E \otimes \omega^1_{(P,L)/V}$$

be the integrable formal log connection corresponding to $\mathcal{E}$. Since P is affine, E is isomorphic to $(K \otimes \mathcal{O}_P)^{\oplus 2}$ as sheaves, and by this identification, the connection ∇ can be expressed by a matrix of the form

$$\begin{pmatrix} d & \eta \\ 0 & d \end{pmatrix},$$

where $\eta \in K \otimes \omega^1_{(P,L)/V}$ and d is the usual differential on $K \otimes \mathcal{O}_P$. Since the exact sequence (3.3.3) splits in the category $I_{\mathrm{conv,et}}(U/V)$, there exists an element $f \in \Gamma(Q_K, \mathcal{O}_{Q_K})$ such that $\eta = df$ holds. If we can show that f belongs to $\Gamma(P_K, \mathcal{O}_{P_K})$, we are done. By the equation (3.3.4), it suffices to show that $i(f) \in K'[[t]]^{\mathrm{b}}$ holds.

The element $i(\eta)$ can be expressed as follows:

$$i(\eta) = \sum_{i=1}^{r} \eta_i t_i^{-1} dt_i + \sum_{i=r+1}^{n} \eta_i dt_i. \tag{3.3.5}$$

Here $\eta_i \in K'[[t]]^{\mathrm{b}}$ $(1 \leq i \leq n)$. Let e_i be the multi-index $(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$ of length n. For any multi-index $m = (m_1, m_2, \cdots, m_n)$ of length n, the equation

$$dt^m = \sum_{\substack{1 \leq i \leq n \\ m_i \neq 0}} m_i t^{m-e_i} dt_i \tag{3.3.6}$$

holds. In particular, the term $t^m dt_i$ with $m_i = -1$ does not appear in the right hand side. Since $d(i(f)) = i(\eta)$ holds, one can see from the expression (3.3.5) that $i(\eta)$ is in fact in $\bigoplus_{i=1}^n K'[[t]]^{\mathrm{b}} dt_i$. Let m be a multi-index $(m_1, m_2, \cdots, m_n)$ such that $m_{i_0} < 0$ holds for some $1 \leq i_0 \leq n$, and let a_m be the coefficient of t^m in $i(f)$. Note that for any multi-indices l, m of length n, the equation

$$t^{l-e_i} dt_i = t^{m-e_j} dt_j$$

implies $i = j$ and $l = m$. So, by the equation (3.3.6), the coefficient of $t^{m-e_{i_0}} dt_{i_0}$ in $i(\eta) = d(i(f))$ is equal to $a_m m_{i_0}$. Since $i(\eta) \in \bigoplus_{i=1}^n K'[[t]]^{\mathrm{b}} dt_i$ holds, we obtain $a_m m_{i_0} = 0$. So $a_m = 0$ holds for such m. Therefore $i(f) \in K'[[t]]^{\mathrm{b}}$ holds and the proof of the proposition is now finished. $\square$

Since the proof of Proposition 3.3.6 is finished, the proof of Theorem 3.3.4 is completed.

As a corollary, we can give the affirmative answer to Problem 3.3.1:

COROLLARY 3.3.7. *In the situation of Problem 3.3.1, the crystalline fundamental group* $\pi_1^{\mathrm{crys}}((X,M)/\mathrm{Spf}\,W,x)$ *depends only on* U *and* x*, that is, Problem 3.3.1 is true.*

PROOF. It is immediate from Theorem 3.3.4 and the isomorphism

$$\pi_1^{\mathrm{conv}}((X/W)^{\log},x) \overset{\sim}{=} \pi_1^{\mathrm{crys}}((X/W)^{\log},x), \tag{3.3.7}$$

which follows from [Shi, (5.3.1)]. (One can also deduce the isomorphism (3.3.7) from the isomorphisms of cohomologies

$$H^i((X/W)^{\log}_{\mathrm{conv,et}},\mathcal{K}_{X/W}) \overset{\sim}{=} H^i((X/W)^{\log}_{\mathrm{crys}},K_0\otimes\mathcal{O}_{X/W}) \quad (i\in\mathbb{N})$$

of Theorem 3.1.1.) □

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(Received January 14, 1998)
(Revised December 25, 2000)

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