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The Regularity of Continuous Weak Solutions of Certain Nonlinear Elliptic Systems

By Xu Bin

Abstract. In this paper, we show the locally Hölder continuity of the continuous weak solutions for certain nonlinear elliptic systems. Furthermore we give an application to another kind of elliptic systems.

1. Introduction

In this paper, we shall study the regularity of the continuous weak solutions to nonlinear elliptic systems in divergence form:

$$\sum_{i,j=1}^{n}\sum_{\beta=1}^{N}\partial_i(A_{\alpha\beta}^{ij}(x,u)\partial_j u^\beta) = f_\alpha(x,u(x),\mathcal{D}u),\ \ \alpha = 1,2,\cdots,N\ , \tag{1}$$

the characteristic property of which is that the right hand side grows at most quadratically in the derivative $\mathcal{D}u$. A weak solution u of (1) is a function of class $H^1 \cap L^\infty$ such that

$$\sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{N}\int_\Omega A_{\alpha\beta}^{ij}\partial_j u^\beta \partial_i \phi^\alpha dx = -\sum_{\alpha=1}^{N}\int_\Omega f_\alpha \phi^\alpha dx \tag{2}$$

for every $\phi \in H_0^1 \cap L^\infty(\Omega, R^N)$, in which $\Omega \subset R^n$ is a bounded domain. The weak harmonic mappings between Riemannian manifolds form a special example (cf [5]).

In [4] Giaquinta, M. and Giusti, E. discussed the partial regularity of the weak solutions of (1). In [4] the coefficients $A_{\alpha\beta}^{ij}$ and the right-hand side f of (1) are assumed to satisfy the following three conditions:

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$(\mathcal{Q}1)$ For all $|u| \le M,\ x \in \overline{\Omega}$ and $\xi \in R^{nN}$, there are constants $\lambda > 0$ and $L > 0$ depending on M such that

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}^{ij}(x,u)\xi_i^\alpha \xi_j^\beta \ge \lambda |\xi|^2, \quad |A_{\alpha\beta}^{ij}| \le L;$$

$(\mathcal{Q}2)$ the coefficients $A_{\alpha\beta}^{ij}$ are continuous functions of their arguments and the inequality

$$|A_{\alpha\beta}^{ij}(x,u) - A_{\alpha\beta}^{ij}(y,v)| \le \omega_M(|x-y|^2 + |u-v|^2)$$

holds for all $x, y \in \overline{\Omega}$ and $|u|, |v| \le M$, where ω_M is an increasing, concave, bounded and continuous function and $\omega_M(0) = 0$;
$(\mathcal{Q}3)$ the function f is of quadratic growth, i.e. there are constants $a > 0$ and $b > 0$ depending on M such that

$$|f(x,u,p)| \le a|p|^2 + b$$

for all $x \in \overline{\Omega}$, $|u| \le M$ and $p \in R^{nN}$. Then Giaquinta, M. and Giusti, E. showed that for any $\alpha \in (0,\ 1)$ the C_{loc}^α-regular set Ω_0 of the weak solutions u of (1) has the form

$$\Omega_0 = \{x_1 \in \Omega : \liminf_{\rho \to 0} \rho^{2-n} \int_{B(x_1,\rho)} |\mathcal{D}u|^2 dx = 0\}, \tag{3}$$

provided that for any $x \in \Omega$, there exist $r > 0$ and $q > 2$ such that the inequality

$$(⨍_{B(x,r)} |\mathcal{D}u|^q dx)^{\frac{1}{q}} \le C\{⨍_{B(x,r)} (1 + |\mathcal{D}u|^2) dx\}^{\frac{1}{2}} \tag{4}$$

holds, where

$$⨍_B f dx = \frac{1}{|B|} \int_B f dx\ .$$

In [4] the inequality (4) can be deduced from the assumptions $(\mathcal{Q}1)$, $(\mathcal{Q}2)$, $(\mathcal{Q}3)$ and

$$\|u\|_{L^\infty(\Omega)} < \frac{\lambda}{2a}.$$

In [6] Hildebrandt, S. and Widman, K. studied the regularity of bounded weak solutions to a class of elliptic systems with the principal part given by a diagonal matrix:

$$\sum_{i,j=1}^{n} \partial_i (A^{ij}(x,u,\mathcal{D}u)\partial_j u^\alpha) = f^\alpha(x,u(x),\mathcal{D}u), \quad 1 \le \alpha \le N, \tag{5}$$

whose right hand side is also of quadratic growth as in ($\mathcal{Q}3$). There exists a constant $\lambda > 0$ depending on M such that

$$\sum_{i,j=1}^{n} A^{ij}(x,u,\mathcal{D}u)\xi_j\xi_j \ge \lambda|\xi|^2,$$

for all $x \in \Omega,\ |u| \le M,\ \mathcal{D}u \in R^{nN}$ and $\xi \in R^n$. In [6] Hildebrandt-Widman simplified Wiegner's proof (cf [9], [10], [11]) to the conjecture that the Hölder continuity of the weak solution $u \in H^1 \cap L^\infty(\Omega, R^N)$ of (5) follows from an inequality

$$\|u\|_{L^\infty(\Omega)} < \frac{\lambda}{a}\ .$$

In [8] Sperner, E. also proved the Hölder regularity of small weak solution to certain nonlinear elliptic systems whose principal part can be decomposed into non-vanishing diagonal part and a general part.

In section 3 we shall prove the continuity of a weak solution u of (1) implies its Hölder continuity if the elliptic systems (1) satisfy ($\mathcal{Q}1$), ($\mathcal{Q}2$), ($\mathcal{Q}3$). More detailedly, we have the follwing

THEOREM 1.1. *Assume conditions* $(\mathcal{Q}1), (\mathcal{Q}2), (\mathcal{Q}3)$ *hold, then a continuous weak solution u of nonlinear elliptic system (1) is locally Hölder continuous in Ω with any exponent $\alpha \in (0,\ 1)$. Furthermore, assume that the coefficients $A^{ij}_{\alpha\beta}$ are Hölder continuous with exponent σ in Ω, then $\mathcal{D}u$ is locally Hölder continuous in Ω with exponent σ.*

From Theorem 1.1 we immediately obtain a well-known result that the continous weak harmonic mappings between smooth Riemannian manifolds are harmonic. It was used by Hildebrandt, S., Kaul, H. and Widman, K.

in [5]. As another application of this theorem, we consider the following system

$$\sum_{i=1}^{n} \partial_i(A_\alpha^i(\mathcal{D}u)) = 0, \ \alpha = 1, 2, \cdots, N \ , \tag{6}$$

where the coefficients A_α^i are given differentiable functions. Furthermore, set

$$A_{\alpha\beta}^{ij}(\cdot) = \frac{\partial A_\alpha^i}{\partial p_j^\beta}(\cdot)$$

and assume that there are constants $\lambda > 0$ and $L > 0$ such that

$$(\mathcal{H}1) \qquad |A_{\alpha\beta}^{ij}(p)| \le L, \ \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}^{ij}(p)\xi_i^\alpha \xi_j^\beta \ge \lambda |\xi|^2,$$

$$(\mathcal{H}2) \qquad |A_{\alpha\beta}^{ij}(p) - A_{\alpha\beta}^{ij}(q)| \le \omega(|p-q|^2)$$

for all $p, q \in R^{nN}$ and $\xi \in R^{nN}$, where the function ω has the same properties as the one in the condition $(\mathcal{Q}2)$. A weak solution of (6) is a function $u \in H^1(\Omega, R^N)$ such that

$$\sum_{i=1}^{n} \sum_{\alpha=1}^{N} \int_\Omega A_\alpha^i(\mathcal{D}u)\partial_i \phi^\alpha dx = 0 \tag{7}$$

for every $\phi \in H_0^1(\Omega, R^N)$. In [3] it is proved that a weak solution u of (6) has partial regularity as $\mathcal{D}u \in C_{loc}^\alpha(\Omega_0, R^{nN})$ for each $0 < \alpha < 1$ where

$$\Omega_0 = \{x_1 \in \Omega : \liminf_{\rho \to 0} \rho^{2-n} \int_{B(x_1,\rho)} |\mathcal{D}^2 u|^2 dx = 0\}.$$

With the help of Theorem 1.1 we shall in section 2 prove the following

Corollary 1.1. *Let $u \in H^1(\Omega, R^N)$ be a C^1 weak solution of (6) and the assumptions $(\mathcal{H}1), (\mathcal{H}2)$ hold, then $\mathcal{D}u \in C_{loc}^\alpha(\Omega, R^{nN})$ for each $0 < \alpha < 1$.*

Example 2.1 shows that C^1-regularity of u in Corollary 1.1 cannot be weakened to Lipschitz continuity. Also in Section 2 we shall give an energy estimate of continuous weak solutions of (1), which forms a preliminary lemma for Theorem 1.1.

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2. The Preliminary Lemma, the Proof of Corollary 1.1 and an Example

LEMMA 2.1. *Let $u \in H^1 \cap L^\infty(\Omega, R^N)$ be a continuous weak solution of (1) and the assumptions $(\mathcal{Q}1), (\mathcal{Q}3)$ hold, then for any $\epsilon > 0$, there exists $\rho > 0$ depending on ϵ, n, N, the structural constants in $(\mathcal{Q}1), (\mathcal{Q}3)$ and on the modulus of continuity of u, such that*

$$\int_{B(x_1,\rho)} |\mathcal{D}u|^2 \eta^2 dx \leq \epsilon(\int_{B(x_1,\rho)} \eta^2 dx + \int_{B(x_1,\rho)} |\mathcal{D}\eta|^2 dx) \tag{8}$$

whenever $B(x_1, \rho) \subset \Omega,\ \eta \in C_0^\infty(B(x_1, \rho))$. In particular we have

$$\lim_{\rho \to 0} \rho^{2-n} \int_{B(x_1,\rho)} |\mathcal{D}u|^2 dx = 0 \quad \textit{for any } x_1 \in \Omega. \tag{9}$$

PROOF. We choose the test function $\phi(x) = (u(x) - u(x_1))\eta^2(x)$ in (2), and obtain

$$\begin{aligned}\int_{B(x_1,\rho)} A^{ij}_{\alpha\beta} \partial_j u^\beta \partial_i u^\alpha \eta^2 dx = & - \int_{B(x_1,\rho)} 2A^{ij}_{\alpha\beta} \partial_j u^\beta (u^\alpha - u^\alpha(x_1)) \eta \partial_i \eta dx \\ & - \int_{B(x_1,\rho)} f_\alpha (u^\alpha - u^\alpha(x_1)) \eta^2 dx\end{aligned}$$

Using $(\mathcal{Q}1), (\mathcal{Q}3)$, we have

$$\begin{aligned}\lambda \int_{B(x_1,\rho)} |\mathcal{D}u|^2 \eta^2 dx \leq & \sup_{B(x_1,\rho)} |u(x) - u(x_1)| \times \\ (\int_{B(x_1,\rho)} 2L|\mathcal{D}u||\mathcal{D}\eta|\eta dx + & \int_{B(x_1,\rho)} (a|\mathcal{D}u|^2 + b)\eta^2 dx\ ).\end{aligned} \tag{10}$$

By the Cauchy inequality,

$$2 \int |\mathcal{D}u||\mathcal{D}\eta|\eta dx \leq \int |\mathcal{D}u|^2 \eta^2 dx + \int |\mathcal{D}\eta|^2 dx$$

holds. Substituting the above inequality to (10), since u is continuous, we can choose ρ sufficiently small so that the inequality (8) holds.

In inequality (8), letting η be a cutoff function such that

$$\eta \equiv 1 \text{ in } B(x_1, \frac{\rho}{2}); \quad 0 \le \eta \le 1 \text{ in } B(x_1, \rho); \ |\mathcal{D}\eta| \le \frac{C}{\rho}, \tag{11}$$

where C is a constant only dependening on n, we complete the proof. $\square$

Now we prove Corollary 1.1 using Theorem 1.1. From [1], we see that $u \in H^2_{loc}(\Omega, R^N)$ and

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \int_\Omega A^{ij}_{\alpha\beta}(\mathcal{D}u)\partial_j(\partial_l u^\beta)\partial_i \phi^\alpha dx = 0, \ l = 1, 2, \cdots, n \tag{12}$$

for all $\phi \in H^1_0(\Omega, R^N)$, spt $\phi \subset \Omega$. Set $v = v^{k,\beta} = (\partial_k u^\beta)$ and $A^{ij}_{(\alpha,k)(\beta,l)} = \delta_{kl} A^{ij}_{\alpha\beta}$, where $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$ if $k \ne l$. By (12), v satisfies the elliptic system

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} \sum_{k,l=1}^{n} \int_\Omega A^{ij}_{(\alpha,k)(\beta,l)}(v)\partial_j v^{\beta,l} \partial_i \phi^{\alpha,k} dx = 0 \tag{13}$$

for all $\phi \in H^1_0(\Omega, R^{nN})$, spt $\phi \subset \Omega$. It is clear that (13) has the form of the elliptic system (2). From the assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ and the differentiability of A^i_α, if $v = \mathcal{D}u$ is continuous on Ω , we have $v = \mathcal{D}u \in C^\alpha_{loc}(\Omega, R^N)$ for each $0 < \alpha < 1$ by virtue of Theorem 1.1.

Example 2.1. In [7], Necăs, J., John, O. and Stara, J. give the systems of the form (6) with analytic coefficients satisfying the strong elliptic condition $(\mathcal{H}1)$ for $n \ge 3$ and for which $u(x) = (u_{ij})$ where

$$u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{ij}|x|$$

is a weak solution. Remark that u is a Lipschitz function but not of class C^1. In this sense the assumption of C^1 regularity of u in Corollary 1.1 is sharp for validity of the higher regularity $\mathcal{D}u \in C^\alpha_{loc}(\Omega, R^{nN})$.

Now we write this example in detail. Let $\Omega = \{x \in R^n : |x| < 1\}, n \geq 3$ and set

$$\begin{aligned}\partial_k u_{ij} &= \frac{\partial u_{ij}}{\partial x_k},\ \nabla_j u = \sum_{i=1}^n \partial_i u_{ij} \\ \|\nabla u\|^2 &= \sum_{j=1}^n (\nabla_j u)^2,\ (\nabla u, \nabla v) = \sum_{j=1}^n \nabla_j u \nabla_j v.\end{aligned}$$

For a fixed real γ, let

$$\begin{aligned}\nabla_{ijk} u &= \partial_k u_{ij} + \gamma(\delta_{ij}\nabla_k u + \delta_{ik}\nabla_j u + \delta_{jk}\nabla_i u) \\ \|\delta u\|^2 &= \sum_{i,j,k=1}^n (\nabla_{ijk} u)^2,\ (\delta u, \delta v) = \sum_{i,j,k=1}^n \nabla_{ijk} u \nabla_{ijk} v.\end{aligned}$$

If the numbers λ, γ, ν satisfy the following conditions

$$\begin{aligned}\lambda &= \{1 + (n - \frac{1}{n})^2\}(n - \frac{1}{n})^{-2}(\frac{1}{n-1} - \gamma), \\ \nu &= -(n - \frac{1}{n})^{-5}\{3\gamma^2(n+1)(n - \frac{1}{n}) \\ &+ \gamma(n^2 + 3n + 2) + 1 + \frac{1}{n}\} \times \{1 + (n - \frac{1}{n})^2\}^2\ ,\end{aligned}$$

then the systems of which $u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{ij}|x|$ is a weak solution can be written as

$$\begin{aligned}\sum_{k=1}^n \partial_k\{\partial_k u_{ij} &+ \gamma(\delta_{ij}\nabla_k u + \delta_{ik}\partial_j u_{kk}) + \lambda \nabla_i u \nabla_j u \nabla_k u (1 + \|\nabla u\|^2)^{-1} \\ &+ \delta_{ik}\nabla_j u\{\gamma(4 + 3\gamma(n+2)) + 3\gamma\lambda\|\nabla u\|^2(1 + \|\nabla u\|^2)^{-1} \\ &+ \nu\|\nabla u\|^4(1 + \|\nabla u\|^2)^{-2}\}\} = 0\end{aligned}$$

3. Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into three steps.

Step 1. For any $x \in \Omega$ there exist $r > 0$ and $q > 2$, such that the inequality

$$(14)\qquad (\fint_{B(x,r)} |\mathcal{D}u|^q dx)^{\frac{1}{q}} \leq C\{\fint_{B(x,r)} (1 + |\mathcal{D}u|^2) dx\}^{\frac{1}{2}}$$

holds. In order to show (14), we need the following

FACT 3.1 ([2]). *Suppose that $q, C \in (1,\infty)$, that Q is an n-cube in R^n, that $g : R^n \to [0,\infty]$ is locally L^q in R^n, and that*

$$M(g^q) \le CM(g)^q$$

holds a.e. in Q, where

$$M(g)(x) = \sup_B \fint_B g dx$$

for each $x \in Q$ and n-balls B with center at x. Then g is L^{q_1}-integrable in Q with

$$\fint_Q g^{q_1} dx \le \frac{c}{q+c-q_1}(\fint_Q g^q dx)^{\frac{q_1}{q}}$$

for any $q_1 \in [q,\ q+c)$, where c is a positive constant depending only on q, C and n.

Taking $r > 0$ such that $B(x,r) \subset\subset \Omega$, we shall prove (14) as follows. For any $x_1 \in B(x,r)$ and $B(x_1,\rho) \subset \Omega$, choosing the test function $\phi = \eta^2(u-u_\rho)$ in (2), where η is the cutoff function satisfying (11) and

$$u_\rho = \fint_{B(x_1,\rho)} u dx,$$

we can obtain by $(\mathcal{Q}1), (\mathcal{Q}3)$

$$\lambda \int_{B(x_1,\rho)} \eta^2 |\mathcal{D}u|^2 dx \le 2L \int_{B(x_1,\rho)} \eta |\mathcal{D}\eta||\mathcal{D}u||u-u_\rho| dx + \int_{B(x_1,\rho)} (a|\mathcal{D}u|^2 + b)\eta^2 |u-u_\rho| dx. \tag{15}$$

Since u is continuous in Ω, when $0 < \rho \le 1$ is small enough, we have

$$|u(x) - u_\rho| \le \frac{\lambda}{4a}, \quad \text{for all } x \in B(x_1,\rho)$$

so that

$$a|\mathcal{D}u|^2\eta^2|u-u_\rho| \le \frac{\lambda}{4}|\mathcal{D}u|^2\eta^2 \tag{16}$$

holds. Then by the Cauchy inequality,

$$\begin{aligned}\eta|\mathcal{D}\eta||\mathcal{D}u||u-u_\rho| &\le \frac{C}{\rho^2}|u-u_\rho|^2+\frac{\lambda}{4}\eta^2||\mathcal{D}u|^2\ ,\\ b\eta^2|u-u_\rho| &\le C(|u-u_\rho|^2+1)\end{aligned}$$

holds. Substituting (16) and the above two inequalities to (15), then using the Sobolev-Poincaré inequality, we have

$$\begin{aligned}\int_{B(x_1,\rho)}\eta^2|\mathcal{D}u|^2dx &\le \frac{C}{\rho^2}\Big(\int_{B(x_1,\rho)}|u-u_\rho|^2dx+\rho^{n+2}\Big)\\ &\le \frac{C}{\rho^2}\{\Big(\int_{B(x_1,\rho)}|\mathcal{D}u|^p dx\Big)^{\frac{2}{p}}+\rho^{n+2}\},\end{aligned}$$

where $p=\frac{2n}{n+2}$ and C is dependent on n, N, λ, L, a, b. Dividing by ρ^n on both sides of the above inequality, we obtain

$$\fint_{B(x_1,\frac{\rho}{2})}|\mathcal{D}u|^2dx \le C\{\Big(\fint_{B(x_1,\rho)}|\mathcal{D}u|^p dx\Big)^{\frac{2}{p}}+\fint_{B(x_1,\rho)}dx\}. \tag{17}$$

Then by the inequality

$$a^\alpha+1\le(a+1)^\alpha,\ \text{for all } \alpha\ge 1 \text{ and } a\ge 0,$$

we rewrite (17) to

$$\begin{aligned}\fint_{B(x_1,\frac{\rho}{2})}(|\mathcal{D}u|+1)^2dx &\le C\{\Big(\fint_{B(x_1,\rho)}|\mathcal{D}u|^p dx\Big)^{\frac{2}{p}}+\fint_{B(x_1,\rho)}dx\}\\ &\le C\{\fint_{B(x_1,\rho)}(|\mathcal{D}u|+1)^p dx\}^{\frac{2}{p}}.\end{aligned}$$

Setting $q=\frac{2}{p}$ and

$$g(y)=\begin{cases}(1+|\mathcal{D}u(y)|)^p, & \text{if } y\in\Omega,\\ 0, & \text{if } y\notin\Omega\ ,\end{cases}$$

we can see that the inequality

$$\fint_{B(x_1,\frac{\rho}{2})}g^q dx\le C\{\fint_{B(x_1,\rho)}g dx\}^q$$

holds. Taking the supremum to all $\rho > 0$, we get

$$M(g^q) \le CM(g)^q, \text{ for any } x_1 \in B(x,r), \tag{18}$$

where C may depend on the positive number $dist(x, \partial\Omega) - r$. From (18) and Fact 3.1, there exists $q_1 > q = \frac{2}{p}$ with

$$\fint_{B(x,r)} g^{q_1} dx \le C\{\fint_{B(x,r)} g^q dx\}^{\frac{q_1}{q}}. \tag{19}$$

By the definition of g we obtain (14) from (19).

Step 2. We firstly quote a fact as follows.

FACT 3.2 (cf [4]). *Let u be a weak solution of the elliptic system (1) in Ω, with coefficients $A^{ij}_{\alpha\beta}$ and right-hand side f satisfying the assumptions $(\mathcal{Q}1), (\mathcal{Q}2)$ and $(\mathcal{Q}3)$. For any $x \in \Omega$, assume that there exist $r > 0$ and $q > 2$, such that the inequality*

$$(\fint_{B(x,r)} |\mathcal{D}u|^q dx)^{\frac{1}{q}} \le C\{\fint_{B(x,r)} (1 + |\mathcal{D}u|^2) dx\}^{\frac{1}{2}} \tag{20}$$

holds, then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^\alpha_{loc}(\Omega_0)$ for every $\alpha \in (0,\ 1)$, and

$$\Omega_0 = \{x_1 \in \Omega : \liminf_{\rho \to 0} \rho^{2-n} \int_{B(x_1,\rho)} |\mathcal{D}u|^2 dx = 0\}.$$

On the other hand, by Lemma 2.1 the open subset Ω_0 of Ω becomes Ω if u is continuous. Therefore u has local Hölder continuity in Ω.

Step 3. If in addition to the hypothesis of Fact 3.2, we assume that the coefficients $A^{ij}_{\alpha\beta}$ are Hölder continuous in Ω with exponent σ, it was also showed in [4] that the derivatives of u are locally Hölder continuous with the same exponent in Ω_0. The proof is completed by the result $\Omega_0 = \Omega$ in Step 2.

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Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Menguro-ku
Tokyo 153-8914, Japan
E-mail: xubin@ms.u-tokyo.ac.jp