Generalized Okubo systems and the middle convolution

（一般大久保型方程式とミドルコンボルーション）

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Generalized Okubo systems and the middle convolution

Hiroshi KAWAKAMI

The present thesis consists of the two part:

I Generalized Okubo systems and the middle convolution

II Confluence of singular points and the Okubo systems

In the first part, we give a generalization, called a generalized Okubo system, of a system of linear differential equations of the Okubo normal form and define a mapping π from a set of generalized Okubo systems to a set of linear differential systems. We consider the operation, the middle convolution introduced by Katz, using π, and show that any system of linear differential equations, not necessarily of the Fuchsian type, with a regular singularity at infinity, can be transformed into generalized Okubo system.

For any non-Fuchsian system, we can construct a Fuchsian system with a parameter ε which tends to the given equation as $\varepsilon \to 0$.

In the second part, we consider a confluence of the convolution in the sense of Katz-Dettweiler-Reiter and we show that convolutions of each equation is compatible with the confluence.

Part I

Generalized Okubo systems and the middle convolution

1 Introduction

We call as an *Okubo system*, a system of linear differential equations of the form

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A\Psi, \tag{1.1}$$

where T is a constant diagonal matrix and A is an arbitrary constant matrix.

We note that an Okubo system appears from the system:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \left(T + \frac{B}{z}\right)\Phi, \tag{1.2}$$

which is called the Birkhoff canonical form of Poincaré rank 1. By means of the Laplace transform

$$\Psi(x) = \int e^{-xz}\Phi(z)\mathrm{d}z, \tag{1.3}$$

the equation (1.2) transforms into the Okubo system (1.1), with $A = -B-I$. In this way, studies on behavior of solutions of (1.2) reduces to those on connection problem of equation (1.1) ([1], [2]).

Moreover the Gauss' hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\beta y = 0 \tag{1.4}$$

is equivalent to an Okubo system of the following form:

$$\left(xI - \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = \begin{pmatrix} \gamma-\alpha-\beta & \beta-\gamma \\ \gamma-\alpha & -\gamma \end{pmatrix}\Psi. \tag{1.5}$$

By regarding system (1.1) as a good generalization of the hypergeometric equation, Okubo studied the case when these equations have no accessory parameter, and representation of monodromy of such a system ([11], [12], [8]); we say, following Katz ([7]) that (1.1) is *rigid*, if the system has no accessory parameter. Since the hypergeometric equation is rigid, it is important for the theory of special functions to obtain all rigid Fuchsian systems and to investigate them as extensions of the Gauss hypergeometric equation. In this paper, we study the Okubo systems from a different point of view.

Katz introduced the operations, called *addition* and *middle convolution*, taking a Fuchsian system to another Fuchsian system, which do not change the number of accessory parameters. He showed the following theorem [7]:

Theorem (Katz) . *Every irreducible rigid Fuchsian system is obtained from rank 1 Fuchsian system by a finite iteration of the addition and the middle convolution.*

This is an algorithm that construct all rigid Fuchsian systems.

Katz described the operations in terms of local systems. Here we follow the terminology of Dettweiler and Reiter ([3]) where the operations of Katz are reformulated in terms of linear algebra.

On the other hand, Yokoyama ([13]) introduced the operations *extension* and *restriction* for Okubo systems, and he proved the following theorem.

Theorem (Yokoyama) . *Every irreducible rigid semisimple Okubo system is obtained from rank 1 Okubo system by a finite iteration of the extension and the restriction.*

In Section 4 we define a mapping π from the set of generalized Okubo systems to the set of linear differential equations which are not necessarily of the Fuchsian type. We give an interpretation of middle convolution of Dettweiler and Reiter version through the mapping π.

We will see in Section 5 that the middle convolution is closely related to transforming given equation into Okubo system (see Proposition 5.2).

In the present article, we will show that any non-Fuchsian system with a regular singular point at infinity into generalized Okubo system.

In Section 2, we recall Katz-Dettweiler-Reiter operations and in Section 3, we introduce a concept of the generalized Okubo system. Section 6 is devoted to the proof of surjectivity of the mapping π.

In what follows we denote by I_k the $k \times k$ identity matrix, O_k the $k \times k$ null matrix, $O_{k,l}$ the $k \times l$ null matrix and N_k the $k \times k$ nilpotent matrix of the following form:

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

2 Katz's operations

In this section, we recall the Katz's operations, in terms of the linear algebraic version, reformulated by Dettweiler and Reiter [3].

For the sake of simplicity, we represent the Fuchsian system of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \left(\frac{A_1}{x - t_1} + \cdots + \frac{A_p}{x - t_p} \right) Y \quad (m \times m)$$

as $A = (A_1, \ldots, A_p)$ in this section.

Definition (addition) . *For $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p$, an operation*

$$A \mapsto (A_1 + \alpha_1 I_m, \ldots, A_p + \alpha_p I_m)$$

is called addition.

Let λ be a complex parameter. We put a $pm \times pm$ matrix G_ν as follows:

$$G_\nu = \begin{pmatrix} O_m & & \cdots & & O_m \\ \vdots & \ddots & & & \vdots \\ A_1 & \cdots & A_\nu + \lambda I_m & \cdots & A_p \\ \vdots & & & \ddots & \vdots \\ O_m & & \cdots & & O_m \end{pmatrix} \quad (\nu = 1, \ldots, p).$$

Definition (convolution) . *The system $(G_1, \ldots, G_p)$ is called convolution with λ of A. We denote this system by $c_\lambda(A)$.*

Let $\mathcal{K}, \mathcal{L}$ be the linear subspaces of $\mathbb{C}^{pm}$:

$$\begin{aligned} \mathcal{K} &:= \begin{pmatrix} \mathrm{Ker}(A_1) \\ \vdots \\ \mathrm{Ker}(A_p) \end{pmatrix}, \\ \mathcal{L} &:= \mathrm{Ker}(G_1 + \cdots + G_p). \end{aligned} \tag{2.1}$$

We remark that $\mathcal{K}, \mathcal{L}$ are $G_1, \ldots, G_p$-invariant subspaces.

Let $\bar{G}_\nu$ be the linear transformation on quotient space $\mathbb{C}^{pm}/(\mathcal{K}+\mathcal{L})$ induced by G_ν.

Definition (middle convolution) . *We call the operation* $A \mapsto (\bar{G}_1, \ldots, \bar{G}_p)$ *middle convolution with* λ *and denote by* mc_λ.

3 Generalized Okubo system

The system of linear differential equations of the form:

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A\Psi \tag{3.1}$$

is called a system of Okubo normal form, in this paper, we call it *Okubo system* in short. Here T is an $n \times n$ constant diagonal matrix and A is an $n \times n$ arbitrary constant matrix. When we suppose

$$T = \begin{pmatrix} t_1 I_{l_1} & & \\ & \ddots & \\ & & t_p I_{l_p} \end{pmatrix},$$

then the system (3.1) has regular singularities at $x = t_1, t_2, \ldots, t_p$, and at $x = \infty$.

When a matrix T is not semisimple, a system of the form (3.1) may have irregular singularities. In the case when T is a Jordan matrix, non-semisimple, call (3.1) *generalized Okubo system.*

In the following of this paper, we assume that the matrix A is semisimple and denote its non-zero eigenvalues by $-\rho_1, \ldots, -\rho_m$, namely, we put

$$A = -GRG^{-1}, \; R = \mathrm{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0). \tag{3.2}$$

Then systems of the form (3.1) can be written in the following form:

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi.$$

We represent such a system as (T, R, G).

Let $\mathrm{Stab}(M)$ be the stabilizer of $M \in M(n, \mathbb{C})$:

$$\mathrm{Stab}(M) = \{g \in GL(n, \mathbb{C}) \mid gM = Mg\}.$$

For a Jordan matrix T and a diagonal matrix $R = \mathrm{diag}(\rho_1, \dots, \rho_m, 0, \dots, 0)$, let $\mathcal{O}(T, R)$ be the following set of systems:

$$\mathcal{O}(T,R) := \{(T,R,G)\}/\underset{\mathcal{O}}{\sim}. \tag{3.3}$$

Here the equivalent relation $\underset{\mathcal{O}}{\sim}$ in (3.3) is defined by

$$G \underset{\mathcal{O}}{\sim} hGg$$

for $h \in \mathrm{Stab}(T)$, $g \in \mathrm{Stab}(R)$. We write the set of Okubo and generalized Okubo systems as follows:

$$\mathcal{GO} := \coprod_{T,R} \mathcal{O}(T,R)$$

where T runs over all Jordan matrices, including diagonal matrices, and R runs over all diagonal matrices of the form (3.2). Similarly, we denote the set of Okubo systems, a subset of $\mathcal{GO}$, by

$$\mathcal{O} := \coprod_{T,R} \mathcal{O}(T,R)$$

where T runs over all diagonal matrices.

To define the set of linear differential systems we put $\Gamma_{(m,p)}$ and $\Gamma^*_{(m,p)}$ as

$$\begin{aligned} \Gamma_{(m,p)} &= \mathbb{C}^p \times (\mathbb{Z}_{\geq 0})^p \times (\mathbb{C}^\times)^m, \\ \Gamma^*_{(m,p)} &= \mathbb{C}^p \times (\mathbb{C}^\times)^m. \end{aligned} \tag{3.4}$$

We regard the set $\Gamma^*_{(m,p)}$ as a subset of $\Gamma_{(m,p)}$ through the inclusion mapping

$$\begin{aligned} &\Gamma^*_{(m,p)} \hookrightarrow \Gamma_{(m,p)} \\ (t_1, \dots, t_p, \rho_1, \dots, \rho_m) &\mapsto (t_1, \dots, t_p, \overbrace{0, \dots, 0}^{p}, \rho_1, \dots, \rho_m). \end{aligned} \tag{3.5}$$

For every element $\gamma = (t_1, \dots, t_p, r_1, \dots, r_p, \rho_1, \dots, \rho_m)$ of $\Gamma_{(m,p)}$, we denote by $\tilde{R}_\gamma$ the $m \times m$ diagonal matrix $\mathrm{diag}(\rho_1, \dots, \rho_m)$. Then we define $\mathcal{E}_\gamma$ by

$$\mathcal{E}_\gamma = \left\{ A(x) = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}} \right. \left| A_\nu^{(-k)} \in M(m, \mathbb{C}),\ A_\nu^{(-r_\nu)} \neq O,\ -\sum_{\nu=1}^{p} A_\nu^{(0)} = \tilde{R}_\gamma \right\} \Big/ \underset{\mathcal{E}_\gamma}{\sim}. \tag{3.6}$$

Here equivalent relation $\underset{\mathcal{E}_\gamma}{\sim}$ in (3.6) is defined by

$$A(x) \underset{\mathcal{E}_\gamma}{\sim} gA(x)g^{-1} \tag{3.7}$$

for some $g \in \mathrm{Stab}(\tilde{R}_\gamma)$. We identify an element $A(x)$ of $\mathcal{E}_\gamma$ with the system of differential equations $\dfrac{\mathrm{d}Y}{\mathrm{d}x} = A(x)Y$.

We put

$$\begin{aligned} \mathcal{E} &:= \coprod_{m,p\in\mathbb{Z}_{\geq 1}} \coprod_{\gamma\in\Gamma_{(m,p)}} \mathcal{E}_\gamma, \\ \mathcal{F} &:= \coprod_{m,p\in\mathbb{Z}_{\geq 1}} \coprod_{\gamma\in\Gamma^*_{(m,p)}} \mathcal{E}_\gamma; \end{aligned} \tag{3.8}$$

namely $\mathcal{E}$ is the set of systems of linear differential equations on $\mathbb{P}^1$ which have regular singularity at infinity, and $\mathcal{F}$ is the set of Fuchsian systems on $\mathbb{P}^1$.

4 Definition of $\pi : \mathcal{GO} \to \mathcal{E}$

In this section we define the mapping $\pi : \mathcal{GO} \to \mathcal{E}$, which is a generalization of correspondence in [9] (cf. Remark 2).

We begin with notations; let $J_k(a)$ be a $k \times k$ Jordan block

$$J_k(a) := aI_k + N_k$$

where $a \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 1}$. For a partition of a positive integer $\lambda = (m_1, \ldots, m_l)$, we put

$$J_\lambda(a) := J_{m_1}(a) \oplus \cdots \oplus J_{m_l}(a).$$

Let $[T, R, G]$ be an arbitrary element of $\mathcal{GO}$, that is, a system of the form

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi. \tag{4.1}$$

Here T is a Jordan matrix of the following form:

$$T = J_{\lambda_1}(t_1) \oplus \cdots \oplus J_{\lambda_p}(t_p) \tag{4.2}$$

$\lambda_1, \dots, \lambda_p$ being *partitions*. We write $n = |\lambda_1| + \cdots + |\lambda_p|$. The matrix R stands for $\mathrm{diag}(\rho_1, \dots, \rho_m, 0, \dots, 0)$. Set $\tilde{R} = \mathrm{diag}(\rho_1, \dots, \rho_m)$. By changing the unknown function of (4.1) into $\Psi = G\tilde{\Psi}$, we have

$$\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = -G^{-1}(xI - T)^{-1}GR\tilde{\Psi}. \tag{4.3}$$

The coefficient matrix of the right-hand side is rewritten into the following form:

$$-G^{-1}(xI - T)^{-1}GR = \sum_{\nu=1}^{p} \sum_{k=0}^{\lambda_{\nu,1}-1} \frac{B_\nu^{(-k)}}{(x - t_\nu)^{k+1}} \tag{4.4}$$

with

$$B_\nu^{(-k)} := -G^{-1} J_\nu^{(-k)} GR, \tag{4.5}$$

where $J_\nu^{(-k)}$ denotes the coefficient matrix of $1/(x - t_\nu)^{k+1}$ in $(xI - T)^{-1}$. We have:

$$J_\nu^{(-k)} = O_{|\lambda_1|+\cdots+|\lambda_{\nu-1}|} \oplus N_{m_{\nu,1}}^k \oplus \cdots \oplus N_{m_{\nu,l_\nu}}^k \oplus O_{|\lambda_{\nu+1}|+\cdots+|\lambda_p|}, \tag{4.6}$$

where we write $\lambda_\nu = (m_{\nu,1}, \dots, m_{\nu,l_\nu})$. Since the last $n - m$ columns of R are zero, the matrix $B_\nu^{(-k)}$ is of the form

$$B_\nu^{(-k)} = \begin{pmatrix} A_\nu^{(-k)} & O_{m,n-m} \\ X_\nu^{(-k)} & O_{n-m,n-m} \end{pmatrix}, \tag{4.7}$$

$A_\nu^{(-k)}$ being some $m \times m$ matrix and $X_\nu^{(-k)}$ some $(n-m) \times m$ matrix. Starting from $[T, R, G]$ we obtain

$$\sum_{\nu=1}^{p} \sum_{k=0}^{\lambda_{\nu 1}-1} \frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}} \in \mathcal{E}. \tag{4.8}$$

From (4.4), an arbitraliness by $\mathrm{Stab}(T)$ does not change (4.8) and that by $\mathrm{Stab}(R)$ induces the modification of $B_\nu^{(-k)}$ like

$$\begin{pmatrix} A_\nu^{(-k)} & O \\ X_\nu^{(-k)} & O \end{pmatrix} \mapsto g^{-1} \begin{pmatrix} A_\nu^{(-k)} & O \\ X_\nu^{(-k)} & O \end{pmatrix} g \quad (g \in \mathrm{Stab}(R)).$$

Here matrix g is parted into blocks such that $\begin{pmatrix} g' & \\ & Z \end{pmatrix}$ where $g' \in \mathrm{Stab}(\tilde{R})$ and $Z \in GL(n-m, \mathbb{C})$. Thus $A_\nu^{(-k)}$ changes as follows:

$$A_\nu^{(-k)} \mapsto (g')^{-1} A_\nu^{(-k)} g'.$$

However, from (3.7), they are precisely equivalent. Therefore this is well-defined and thus we complete the definition of π.

Taken together, the definition of mapping π is the following:

Definition 1. *For $[T, R, G] \in \mathcal{GO}$, we define the mapping $\pi : \mathcal{GO} \to \mathcal{E}$ as follows:*

$$\pi(T, R, G) := \text{the principal } m \times m \text{ part of } (-G^{-1}(xI - T)^{-1} GR) \tag{4.9}$$

Conversely, suppose we are given an system

$$A = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}} \in \mathcal{E}, \quad -\sum_{\nu=1}^{p} A_\nu^{(0)} = \tilde{R}$$

of size m. Fix a Jordan matrix $T = J_{\lambda_1}(t_1) \oplus \cdots \oplus J_{\lambda_p}(t_p)$ of size $n \geq m$ and a diagonal $R = \tilde{R} \oplus O_{n-m,n-m}$. Let us write down an equation which G of $(T, R, G) \in \pi^{-1}(A)$ should satisfy. From (4.7) the matrix G must satisfy

$$-G^{-1} J_\nu^{(-k)} GR = \begin{pmatrix} A_\nu^{(-k)} & O \\ X_\nu^{(-k)} & O \end{pmatrix} \quad (\nu = 1, \ldots, p, \ k = 0, \ldots, r_\nu). \tag{4.10}$$

Since $\pi(T, R, G)$ does not depend on $X_\nu^{(-k)}$'s, we give $(n-m) \times m$ matrices $X_\nu^{(-k)}$'s suitably. For instance, $X_\nu^{(-k)}$'s must satisfy

$$\mathrm{rank}(-G^{-1} J_\nu^{(-k)} GR) = \mathrm{rank} \begin{pmatrix} A_\nu^{(-k)} \\ X_\nu^{(-k)} \end{pmatrix}. \tag{4.11}$$

Multiplying G from the left of both sides of (4.10), we have

$$J_\nu^{(-k)} GR + G \begin{pmatrix} A_\nu^{(-k)} & O \\ X_\nu^{(-k)} & O \end{pmatrix} = 0. \tag{4.12}$$

This is equivalent to

$$\left\{ J_\nu^{(-k)} \otimes R + I \otimes \begin{pmatrix} {}^tA_\nu^{(-k)} & {}^tX_\nu^{(-k)} \\ O & O \end{pmatrix} \right\} \boldsymbol{g} = 0 \tag{4.13}$$

where $\boldsymbol{g}$ is the column vector

$$\boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_n \end{pmatrix}$$

consist of the *row* vectors of G, i.e. $G = {}^t(\boldsymbol{g}_1, \dots, \boldsymbol{g}_n)$. Exchanging the rows of the above equation (4.13) properly, we find that the vector $\boldsymbol{g}$ satisfies the equation $\Omega \boldsymbol{g} = 0$. Here the matrix Ω is defined by

$$\Omega := \Omega_1 \oplus \cdots \oplus \Omega_p$$

where

$$\Omega_\nu := J_\nu^{(0)} \otimes \Omega_\nu^{(0)} + J_\nu^{(-1)} \otimes R_\nu^{(-1)} + \cdots + J_\nu^{(-\lambda_{\nu,1}+1)} \otimes R_\nu^{(-\lambda_{\nu,1}+1)}$$

and

$$\Omega_\nu^{(0)} = \begin{pmatrix} {}^tA_1^{(0)} & {}^tX_1^{(0)} \\ \vdots & \vdots \\ \widehat{{}^tA_\nu^{(0)}} & \widehat{{}^tX_\nu^{(0)}} \\ \vdots & \vdots \\ {}^tA_p^{(-r_p)} & {}^tX_p^{(-r_p)} \end{pmatrix}, \quad R_\nu^{(-k)} = \begin{pmatrix} O_m & O_{m,n-m} \\ \vdots & \vdots \\ \tilde{R} & O_{m,n-m} \\ \vdots & \vdots \\ O_m & O_{m,n-m} \end{pmatrix}.$$

The matrix $\Omega_\nu^{(0)}$ is made by blocks $({}^tA_\nu^{(-k)} {}^tX_\nu^{(-k)})$ $(\nu = 1, \dots, p,\ k = 0, \dots, r_\nu)$ except for $({}^tA_\nu^{(0)} {}^tX_\nu^{(0)})$ and the matrix $R_\nu^{(-k)}$ has only non-null block $\tilde{R}$ at the position correspond to ${}^tA_\nu^{(-k)}$ of $\Omega_\nu^{(0)}$. For example, in the case of $\lambda_1 = (3,3), \lambda_2 = (2)$:

$$T = \begin{pmatrix} t_1 & 1 & 0 \\ & t_1 & 1 \\ & & t_1 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} t_2 & 1 \\ & t_2 \end{pmatrix},$$

Ω is expressed as

$$\Omega = \begin{pmatrix} \Omega_1^{(0)} & R_1^{(-1)} & R_1^{(-2)} \\ & \Omega_1^{(0)} & R_1^{(-1)} \\ & & \Omega_1^{(0)} \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} \Omega_2^{(0)} & R_2^{(-1)} \\ & \Omega_2^{(0)} \end{pmatrix}$$

where

$$\Omega_1^{(0)} = \begin{pmatrix} {}^tA_1^{(-1)} & {}^tX_1^{(-1)} \\ {}^tA_1^{(-2)} & {}^tX_1^{(-2)} \\ {}^tA_2^{(0)} & {}^tX_2^{(0)} \\ {}^tA_2^{(-1)} & {}^tX_2^{(-1)} \end{pmatrix}, \quad \Omega_2^{(0)} = \begin{pmatrix} {}^tA_1^{(0)} & {}^tX_1^{(0)} \\ {}^tA_1^{(-1)} & {}^tX_1^{(-1)} \\ {}^tA_1^{(-2)} & {}^tX_1^{(-2)} \\ {}^tA_2^{(-1)} & {}^tX_2^{(-1)} \end{pmatrix}$$

and

$$R_1^{(-1)} = \begin{pmatrix} \tilde{R} & O \\ O & O \\ O & O \\ O & O \end{pmatrix}, \quad R_1^{(-2)} = \begin{pmatrix} O & O \\ \tilde{R} & O \\ O & O \\ O & O \end{pmatrix}, \quad R_2^{(-1)} = \begin{pmatrix} O & O \\ O & O \\ O & O \\ \tilde{R} & O \end{pmatrix}.$$

Thus we arrive at the following lemma:

Lemma 4.1. $\pi(T, R, G) = A$ *is equivalent to the following equation:* $\Omega \boldsymbol{g} = 0$ *and* $\det G \neq 0$.

5 Relation to the middle convolution

In this section, we investigate the map $\pi|_{\mathcal{O}} : \mathcal{O} \to \mathcal{F}$, and we discuss the relation to the middle convolution.

Let $A = \left[\sum_{\nu=1}^{p} \frac{A_\nu^{(0)}}{x - t_\nu}\right]$ be an element of $\mathcal{F}$ whose matrix size is m. Put $\mathrm{rank} A_\nu^{(0)} = l_\nu$. We can factorize $A_\nu^{(0)}$ into $A_\nu^{(0)} = B_\nu C_\nu$ where B_ν is an $m \times l_\nu$ matrix and C_ν is an $l_\nu \times m$ matrix and $\mathrm{rank} B_\nu = \mathrm{rank} C_\nu = l_\nu$. Put $n = l_1 + \cdots + l_p$.

We define the matrices $T_{\min}$ and $A_{\min}$ as follows:

$$T_{\min} = \begin{pmatrix} t_1 I_{l_1} & & \\ & \ddots & \\ & & t_p I_{l_p} \end{pmatrix}, \quad A_{\min} = \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} (B_1 \dots B_p), \tag{5.1}$$

both are $n \times n$ matrices. Then following proposition holds.

Proposition 5.1. *The minimal size Okubo system in $\pi^{-1}(A)$ uniquely exists up to conjugate action of* $\mathrm{Stab}(T_{\min})$ *and is given as follows:*

$$(xI - T_{\min})\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A_{\min}\Psi. \tag{5.2}$$

In particular, $\pi|_{\mathcal{O}} : \mathcal{O} \to \mathcal{F}$ is surjective.

Remark 1. Since $-(A_1^{(0)} + \cdots + A_p^{(0)}) = \tilde{R}$ $(= \mathrm{diag}(\rho_1, \ldots, \rho_m))$, by comparing the rank of both sides, we get the inequality $n \geq m$.

Proof. First, we show that the Okubo system (5.2) is the minimal size system in $\pi^{-1}(A)$. By the assumption $-(A_1^{(0)} + \cdots + A_p^{(0)}) = \tilde{R}$, we have the relation:

$$(B_1 \ldots B_p)\begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} = -\tilde{R}.$$

Since the matrix $\tilde{R}$ is invertible, we have

$$\mathrm{rank}(B_1 \ldots B_p) = \mathrm{rank}\begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} = m.$$

In this case, from (4.11), we can write $X_\nu^{(0)}$ of (4.7) as $X_\nu^{(0)} = X_\nu C_\nu$. Since $X_1^{(0)} + \cdots + X_p^{(0)} = O$ (corresponds to $-(B_1^{(0)} + \cdots + B_p^{(0)}) = R$ where $R := \tilde{R} \oplus O_{n-m}$), X_ν's satisfy

$$(X_1 \ldots X_p)\begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} = 0. \tag{5.3}$$

We choose $X_1, \ldots, X_p$ so that

$$\mathrm{rank}(X_1, \ldots, X_p) = n - m. \tag{5.4}$$

Now we show that the matrix $\begin{pmatrix} B_1 & \ldots & B_p \\ X_1 & \ldots & X_p \end{pmatrix}$ is invertible. Let us assume that

$${}^t\boldsymbol{u}(B_1 \ldots B_p) + {}^t\boldsymbol{v}(X_1 \ldots X_p) = 0, \tag{5.5}$$

where $\boldsymbol{u} \in \mathbb{C}^m, \boldsymbol{v} \in \mathbb{C}^{n-m}$. By multiplying the matrix $\begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix}$ from the right of both sides of (5.5), we have ${}^t\boldsymbol{u}\tilde{R} = 0$. This implies $\boldsymbol{u} = 0$. By the condition (5.7), we have $\boldsymbol{v} = 0$. Then we can put $G := \begin{pmatrix} B_1 & \dots & B_p \\ X_1 & \dots & X_p \end{pmatrix}^{-1}$.

We also define $l_\nu \times (n-m)$ matrix Y_ν $(\nu = 1, \dots, p)$ by the following equations:

$$(B_1 \dots B_p) \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix} = 0. \tag{5.6}$$

Here we choose $Y_1, \dots, Y_p$ so that

$$\operatorname{rank} \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix} = n - m. \tag{5.7}$$

We set $G' := \begin{pmatrix} C_1 & Y_1 \\ \vdots & \vdots \\ C_p & Y_p \end{pmatrix}$. We can show that G' is also invertible. Then we have

$$\begin{aligned} G^{-1}G' &= \begin{pmatrix} B_1 & \dots & B_p \\ X_1 & \dots & X_p \end{pmatrix} \begin{pmatrix} C_1 & Y_1 \\ \vdots & \vdots \\ C_p & Y_p \end{pmatrix} \\ &= \begin{pmatrix} -\tilde{R} & O \\ O & Z \end{pmatrix} \in \operatorname{Stab}(R) \end{aligned} \tag{5.8}$$

Moreover we have the relation

$$-GRG^{-1} = A_{\min}, \tag{5.9}$$

because

$$
\begin{aligned}
G^{-1}A_{\min} &= \begin{pmatrix} B_1 & \dots & B_p \\ X_1 & \dots & X_p \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_p \end{pmatrix} (B_1 \dots B_p) \\
&= \begin{pmatrix} -\tilde{R} \\ O \end{pmatrix} (B_1 \dots B_p) \\
&= \begin{pmatrix} -\tilde{R} & O \\ O & O \end{pmatrix} \begin{pmatrix} B_1 & \dots & B_p \\ X_1 & \dots & X_p \end{pmatrix} .
\end{aligned}
\tag{5.10}
$$

Therefore the residue matrices arround each finite singularity of (5.2) read as (cf.(4.4)):

$$
\begin{aligned}
B_\nu^{(0)} &= -G^{-1} \begin{pmatrix} O_{l_1+\cdots l_{\nu-1}} & & \\ & I_{l_\nu} & \\ & & O_{l_{\nu+1}+\cdots l_p} \end{pmatrix} GR \\
&= -\begin{pmatrix} B_1 & \dots & B_p \\ X_1 & \dots & X_p \end{pmatrix} \begin{pmatrix} O_{l_1+\cdots l_{\nu-1}} & & \\ & I_{l_\nu} & \\ & & O_{l_{\nu+1}+\cdots l_p} \end{pmatrix} \\
&\quad \times \begin{pmatrix} C_1 & Y_1 \\ \vdots & \vdots \\ C_p & Y_p \end{pmatrix} \begin{pmatrix} -\tilde{R} & O \\ O & Z \end{pmatrix}^{-1} R \\
&= \begin{pmatrix} B_\nu C_\nu & O \\ X_\nu C_\nu & O \end{pmatrix} .
\end{aligned}
\tag{5.11}
$$

By the definition of π, we find this Okubo system lies in the fiber of Fuchsian system A.

Note the following inequality concerned with the rank of $B_\nu^{(0)}$:

$$
\operatorname{rank} \left[G^{-1} \begin{pmatrix} O_{l_1+\cdots l_{\nu-1}} & & \\ & I_{l_\nu} & \\ & & O_{l_{\nu+1}+\cdots l_p} \end{pmatrix} GR \right] \le l_\nu .
$$

Therefore the Okubo system whose size is less than n or different T of same size n can not realize the given rank of $A_\nu^{(0)}$'s. Thus we can conclude that (5.2) is the minimal size.

Next, we consider the uniqueness of the system (5.2). Owing to the Lemma 4.1 and (5.11), G satisfies $\Omega\boldsymbol{g}=0$, where $\Omega=\Omega_1\oplus\cdots\oplus\Omega_p$ and for $\nu=1,\ldots,p$,

$$\Omega_\nu=\begin{pmatrix} {}^tA_1^{(0)} & {}^t(X_1C_1) \\ \vdots & \vdots \\ \widehat{{}^tA_\nu^{(0)}} & \widehat{{}^t(X_\nu C_\nu)} \\ \vdots & \vdots \\ {}^tA_p^{(0)} & {}^t(X_pC_p) \end{pmatrix}=\begin{pmatrix} {}^tC_1 & & & & \\ & \ddots & & & \\ & & \widehat{{}^tC_\nu} & & \\ & & & \ddots & \\ & & & & {}^tC_p \end{pmatrix}\begin{pmatrix} {}^tB_1 & {}^tX_1 \\ \vdots & \vdots \\ \widehat{{}^tB_\nu} & \widehat{{}^tX_\nu} \\ \vdots & \vdots \\ {}^tB_p & {}^tX_p \end{pmatrix}.$$

Here rank of Ω_ν is $n-l_\nu$. Thus a freedom of representation of the solution space of $\Omega\boldsymbol{g}=0$ corresponds to $\mathrm{Stab}(T_{\min})$, that is, this arbitrariness is given by $G\mapsto hG$ where $h\in\mathrm{Stab}(T_{\min})$. We note that, from (5.9), $-GRG^{-1}$ does not depend on a choice of X_ν. Finally it is easily seen that freedom of factorization $A_\nu^{(0)}=B_\nu C_\nu$ corresponds also $\mathrm{Stab}(T_{\min})$. Therefore this minimal size Okubo system is uniquely determined by A up to conjugate action of $\mathrm{Stab}(T_{\min})$. □

Remark 2. In the case of $m=2$, $p=3$ and eigenvalues of $A_\nu^{(0)}$ are $0,\theta_\nu$ ($\nu=1,2,3$), we can parametrize $A_\nu^{(0)}$ generically as

$$\begin{aligned} A_\nu^{(0)}&=\frac{1}{2}\begin{pmatrix} a_\nu b_\nu+\theta_\nu & -a_\nu^2 \\ b_\nu^2-\frac{\theta_\nu^2}{a_\nu^2} & -a_\nu b_\nu+\theta_\nu \end{pmatrix}\\ &=\begin{pmatrix} a_\nu \\ \frac{a_\nu b_\nu-\theta_\nu}{a_\nu} \end{pmatrix}\begin{pmatrix} \frac{a_\nu b_\nu+\theta_\nu}{2a_\nu} & -\frac{a_\nu}{2} \end{pmatrix}. \end{aligned}$$

Then $A_{\min}$ is given as follows:

$$\begin{aligned} A_{\min}&=\begin{pmatrix} \frac{a_1b_1+\theta_1}{2a_1} & -\frac{a_1}{2} \\ \frac{a_2b_2+\theta_2}{2a_2} & -\frac{a_2}{2} \\ \frac{a_3b_3+\theta_3}{2a_3} & -\frac{a_3}{2} \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ \frac{a_1b_1-\theta_1}{a_1} & \frac{a_2b_2-\theta_2}{a_2} & \frac{a_3b_3-\theta_3}{a_3} \end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix} 2\theta_1 & a_2b_1-a_1b_2+\theta_1\frac{a_2}{a_1}+\theta_2\frac{a_1}{a_2} & a_3b_1-a_1b_3+\theta_1\frac{a_3}{a_1}+\theta_3\frac{a_1}{a_3} \\ a_1b_2-a_2b_1+\theta_2\frac{a_1}{a_2}+\theta_1\frac{a_2}{a_1} & 2\theta_2 & a_3b_2-a_2b_3+\theta_2\frac{a_3}{a_2}+\theta_3\frac{a_2}{a_3} \\ a_1b_3-a_3b_1+\theta_3\frac{a_1}{a_3}+\theta_1\frac{a_3}{a_1} & a_2b_3-a_3b_2+\theta_3\frac{a_2}{a_3}+\theta_2\frac{a_3}{a_2} & 2\theta_3 \end{pmatrix}. \end{aligned}$$

This correspondence between $A_\nu^{(0)}$'s and $A_{\min}$ appeared in [4], [9].

In the above settings, we can show the following proposition.

Proposition 5.2. *For any $\lambda \in \mathbb{C}$, the middle convolution of $A = \left[\sum_{\nu=1}^{p} \frac{A_\nu^{(0)}}{x - t_\nu}\right] \in$ $\mathcal{F}$ with λ coincides with the image of following system under π:*

$$(xI - T_{\min})\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (A_{\min} + \lambda)\Psi. \tag{5.12}$$

Proof. We consider the vector space $\mathbb{C}^{pm}/(\mathcal{K} + \mathcal{L})$ as the quotient of $\mathbb{C}^{pm}/\mathcal{K}$ by $\mathcal{L}$ because $\mathcal{K}$ does not depend on the parameter λ (cf. Section 2). First, by a direct computation, we can show that the middle convolution with generic λ (i.e. $\lambda \neq 0, \rho_1, \ldots, \rho_m$) of A coincides with (5.12). This corresponds to dividing $\mathbb{C}^{pm}$ by $\mathcal{K}$.

Next we consider the cases of $\lambda = \rho_\nu$ ($\nu = 1, \ldots, m$). Now we take a quotient by $\mathcal{L}$. We explain the case $\lambda = \rho_1$ and the other cases are quite similar. Then "dividing by $\mathcal{L}$" corresponds to "dividing by $\mathrm{Ker}(A_{\min} + \rho_1)$". Notice that $A_{\min} = -GRG^{-1}$ (see (5.9)), we have

$$\mathrm{Ker}(A_{\min} + \rho_1) = \mathrm{Ker}(R - \rho_1)G^{-1} = \mathbb{C}G \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Namely, the first column of G is a basis of $\mathrm{Ker}(A_{\min} + \rho_1)$. We set $\bar{A}_\nu$ as the $(n-1) \times (n-1)$ submatrix of

$$\begin{aligned} &G^{-1}(O_{l_1+\cdots+l_{\nu-1}} \oplus I_{l_\nu} \oplus O_{l_{\nu+1}+\cdots+l_p})(A_{\min} + \rho_1 I)G \\ &= -G^{-1}(O_{l_1+\cdots+l_{\nu-1}} \oplus I_{l_\nu} \oplus O_{l_{\nu+1}+\cdots+l_p})G(R - \rho_1 I) \end{aligned} \tag{5.13}$$

obtained by eliminating the first column and row, then the system

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \sum_{\nu=1}^{p} \frac{\bar{A}_\nu}{x - t_\nu} Y$$

is $mc_{\rho_1}(A)$, and is also the image of A by π. If there are ρ_ν's which are equal to ρ_1, the proof goes in a same manner.

Finally, when $\lambda = 0$, both $mc_0(A)$ and the image of (5.12) under π coincide with the original system A. Thus we complete the proof. □

Hence, the middle convolution is obtained by the following procedure:

1. Lift a system in $\mathcal{F}$ to $\mathcal{O}$ in the minimal size.

2. Shift its right-hand side with scalar matrix (representing this operation by T_λ).

3. Carry it to $\mathcal{F}$ through π.

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{T_\lambda} & \mathcal{O} \\ {\scriptstyle \pi|_{\mathcal{O}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi|_{\mathcal{O}}} \\ \mathcal{F} & \xrightarrow[mc_\lambda]{} & \mathcal{F} \end{array}$$

The shift of right-hand side of Okubo systems by a scalar matrix is realized by the Euler transformation:

$$\Psi(x) \mapsto \int \Psi(t)(x-t)^\lambda \mathrm{d}t.$$

Therefore, we can state that the middle convolution is "Transformation into Okubo system + Euler transform". By taking the above consideration into account, we can define an analogue of the middle convolution for non-Fuchsian systems by the same procedure.

$$\begin{array}{ccc} \mathcal{GO} & \xrightarrow{T_\lambda} & \mathcal{GO} \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{E} & \xrightarrow[\text{“}mc_\lambda\text{”}]{} & \mathcal{E} \end{array}$$

It is necessary to show the surjectivity of π so that this procedure may work. In the next section we consider the surjectivity of π.

At the end of this section, we see a correspondence of solutions under π. Let Ψ be a solution of generalized Okubo system

$$(xI-T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi.$$

Then from (4.3), $\tilde{\Psi} = G^{-1}\Psi$ satisfies the equation of the following form:

$$\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = \sum\sum \frac{B_\nu^{(-k)}}{(x-t_\nu)^{k+1}}\tilde{\Psi}.$$

From the form of $B_\nu^{(-k)}$ (cf.(4.7)),

$$\psi = \begin{pmatrix} (\tilde{\Psi})_1 \\ \vdots \\ (\tilde{\Psi})_m \end{pmatrix} = \begin{pmatrix} (G^{-1}\Psi)_1 \\ \vdots \\ (G^{-1}\Psi)_m \end{pmatrix}$$

solves the equation $\pi(T, R, G)$. Thus at the level of solutions, the map π induces the following:

$$\Psi \mapsto \begin{pmatrix} (G^{-1}\Psi)_1 \\ \vdots \\ (G^{-1}\Psi)_m \end{pmatrix}. \tag{5.14}$$

6 Surjectivity of π

In this section, we prove the surjectivity of π, that is, any linear differential system which has an regular singularity at infinity can be converted into generalized Okubo system. This is done by considering an extension of convolution of Fuchsian system (see section 2) to non-Fuchsian case.

Let $\displaystyle\sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu}\frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}}$ be an element of $\mathcal{E}$ whose matrix size is m. Put $n = \sum_{\nu=1}^{p} \tilde{r}_\nu$ where $\tilde{r}_\nu := m(r_\nu + 1)$. Let $\tilde{A}_\nu$ be the following $\tilde{r}_\nu \times n$ matrix

$$\tilde{A}_\nu := \begin{pmatrix} & & & O_{mr_\nu, n} & & & \\ A_1^{(-r_1)} & \dots & A_1^{(0)} & \dots & A_p^{(-r_p)} & \dots & A_p^{(0)} \end{pmatrix}.$$

We set the matrices $\tilde{A}, T, \tilde{T}$, and P as

$$\begin{aligned} \tilde{A} &:= \begin{pmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_p \end{pmatrix}, \\ T &:= J_{r_1+1}(t_1)^{\oplus m} \oplus \dots \oplus J_{r_p+1}(t_p)^{\oplus m}, \\ \tilde{T} &:= J_{r_1+1}(t_1) \oplus \dots \oplus J_{r_p+1}(t_p), \\ P &:= P_{(m,r_1+1)} \oplus \dots \oplus P_{(m,r_p+1)}. \end{aligned} \tag{6.1}$$

Here $P_{(i,j)}$ is a permutation matrix of the form

$$P_{(i,j)} = (I_i \otimes \boldsymbol{e}_1, I_i \otimes \boldsymbol{e}_2, \dots, I_i \otimes \boldsymbol{e}_j)$$

where $e_1, \dots, e_j$ are the j-dimensional unit vectors. It is easily seen that $P_{(i,j)}^{-1} = {}^tP_{(i,j)} = P_{(j,i)}$ and thus we can show the relation

$$P_{(i,j)}^{-1}(J_j(t)^k)^{\oplus i}P_{(i,j)} = (J_j(t)^k) \otimes I_i. \tag{6.2}$$

By using this formula, we have $P^{-1}TP = \tilde{T} \otimes I_m$.

Definition 2. *We call the following (generalized) Okubo system:*

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (P\tilde{A}P^{-1} + \lambda I_n)\Psi \tag{6.3}$$

convolution of A with λ, and we denote it by $c_\lambda(A)$.

Remark 3. When $r_1 = \cdots = r_p = 0$, this coincides with the convolution of Fuchsian systems; see section 2.

By changing the unknown Ψ of (6.3) to $\tilde{\Psi} = P^{-1}\Psi$, we can express the equation $c_\lambda(A)$ as follows:

$$(xI_n - \tilde{T} \otimes I_m)\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = (\tilde{A} + \lambda I_n)\tilde{\Psi}. \tag{6.4}$$

We prove the following

Theorem 6.1. *For any element A of $\mathcal{E}$, $c_0(A)$ lies in the fiber of A. In particular, the mapping $\pi : \mathcal{GO} \to \mathcal{E}$ is surjection.*

Proof. We set the $\tilde{r}_\nu \times \tilde{r}_\mu$ matrix $\Gamma_{\nu\mu}$ such that

$$\Gamma_{\nu 1} := \begin{cases} \begin{pmatrix} & O_{mr_1,\tilde{r}_1} & & \\ I_m - \tilde{R} & A_1^{(-1)} & \dots & A_1^{(-r_1)} \end{pmatrix} + K_{r_1+1} \otimes \tilde{R} & (\nu = 1) \\ \begin{pmatrix} & O_{mr_\nu,\tilde{r}_1} & & \\ I_m & A_1^{(-1)} & \dots & A_1^{(-r_1)} \end{pmatrix} & (\nu = 2, \dots, p) \end{cases}, \tag{6.5}$$

and for $\mu = 2, \dots, p$,

$$\Gamma_{\nu\mu} := \begin{cases} \begin{pmatrix} & O_{mr_\nu,\tilde{r}_\nu} & \\ A_\nu^{(0)} & \dots & A_\nu^{(-r_\nu)} \end{pmatrix} + K_{r_\nu+1} \otimes \tilde{R} & (\nu = \mu) \\ \begin{pmatrix} & O_{mr_\nu,\tilde{r}_\mu} & \\ A_\mu^{(0)} & \dots & A_\mu^{(-r_\mu)} \end{pmatrix} & (\nu \neq \mu) \end{cases}, \tag{6.6}$$

where K_r denotes the $r \times r$ anti-diagonal matrix whose non-zero entries are all one and $\tilde{R} = -(A_1^{(0)} + \cdots + A_p^{(0)})$. We set $\tilde{G}$ as follows:

$$\tilde{G} := \begin{pmatrix} \Gamma_{11} & \dots & \Gamma_{1p} \\ \vdots & \ddots & \vdots \\ \Gamma_{p1} & \dots & \Gamma_{pp} \end{pmatrix}. \tag{6.7}$$

Then $\tilde{G}^{-1}$ is given as follows:

$$\tilde{G}^{-1} = \begin{pmatrix} \Gamma^{11} & \dots & \Gamma^{1p} \\ \vdots & \ddots & \vdots \\ \Gamma^{p1} & \dots & \Gamma^{pp} \end{pmatrix}$$

where $\Gamma^{\nu\mu}$ is the $\tilde{r}_\nu \times \tilde{r}_\mu$ matrix defined by the following:

$$\Gamma^{1\mu} := \begin{cases} \begin{pmatrix} -A_1^{(-r_1)}\tilde{R}^{-1} & \dots & -(A_1^{(0)} + I_m)\tilde{R}^{-1} \\ & O_{mr_1,\tilde{r}_1} & \end{pmatrix} + K_{r_1+1} \otimes \tilde{R}^{-1} & (\mu = 1) \\ \begin{pmatrix} -A_\mu^{(-r_\mu)}\tilde{R}^{-1} & \dots & -A_\mu^{(0)}\tilde{R}^{-1} \\ & O_{mr_1,\tilde{r}_\mu} & \end{pmatrix} & (\mu = 2, \dots, p) \end{cases}, \tag{6.8}$$

$$\Gamma^{\nu 1} := \begin{pmatrix} O_{m,mr_\mu} & -\tilde{R}^{-1} \\ O_{mr_\nu,\tilde{r}_\mu} & \end{pmatrix} \quad (\nu = 2, \dots, p), \tag{6.9}$$

and for $\nu, \mu = 2, \dots, p$,

$$\Gamma^{\nu\mu} := \begin{cases} K_{r_\nu+1} \otimes \tilde{R}^{-1} & (\nu = \mu) \\ O_{\tilde{r}_\nu,\tilde{r}_\mu} & (\nu \neq \mu) \end{cases}. \tag{6.10}$$

We set $G := P(I_{r_1+\cdots+r_p+p} \otimes \tilde{R})^{-1}\tilde{G}$. Then we can show the relation:

$$-GRG^{-1} = P\tilde{A}P^{-1},$$

where $R := \tilde{R} \oplus O_{n-m}$. Now let us compute the image of the generalized Okubo system $[T, R, G]$ under the mapping π. Notice that $I \otimes \tilde{R}$ commutes with $P^{-1}TP = \tilde{T} \otimes I_m$, then $P(I \otimes \tilde{R})P^{-1}$ commutes with T. Then from the definition of π,

$$
\begin{aligned}
&G^{-1}(xI - T)^{-1}GR \\
&\quad = \tilde{G}^{-1}(I \otimes \tilde{R})P^{-1}(xI - T)^{-1}P(I \otimes \tilde{R})^{-1}\tilde{G}R \\
&\quad = \tilde{G}^{-1}P^{-1}\{P(I \otimes \tilde{R})P^{-1}\}(xI - T)^{-1}\{P(I \otimes \tilde{R})^{-1}P^{-1}\}P\tilde{G}R \\
&\quad = \tilde{G}^{-1}P^{-1}(xI - T)^{-1}P\tilde{G}R.
\end{aligned}
$$

Here

$$
P^{-1}(xI - T)^{-1}P = \sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu} O_{\sum_{\mu=1}^{\nu-1}\tilde{r}_\mu} \oplus \frac{N^k_{r_\nu+1} \otimes I_m}{(x - t_\nu)^{k+1}} \oplus O_{\sum_{\mu=\nu+1}^{p}\tilde{r}_\mu}. \tag{6.11}
$$

Then the coefficient matrix of $1/(x - t_\nu)^{k+1}$ in $-G^{-1}(xI - T)^{-1}GR$ is

$$
\begin{aligned}
&-\tilde{G}^{-1}O_{\sum_{\mu=1}^{\nu-1}\tilde{r}_\mu} \oplus (N^k_{r_\nu+1} \otimes I_m) \oplus O_{\sum_{\mu=\nu+1}^{p}\tilde{r}_\mu}\tilde{G}R \\
&= -\begin{pmatrix} \Gamma^{1\nu} \\ \vdots \\ \Gamma^{p\nu} \end{pmatrix} (N^k_{r_\nu+1} \otimes I_m)(\Gamma_{\nu 1} \ldots \Gamma_{\nu p})R.
\end{aligned} \tag{6.12}
$$

Therefore the coefficient matrix of $1/(x - t_\nu)^{k+1}$ in the $\pi(T, R, G)$ is the following:

$$
\{\text{the principal } m \times m \text{ part of } -\Gamma^{1\nu}(N^k_{r_\nu+1} \otimes I_m)\Gamma_{\nu 1}\} \times \tilde{R}. \tag{6.13}
$$

Since the inside of $\{\ \}$ of (6.13) is expressed as follows:

$$
\begin{pmatrix} O_{m,mk} & A_\nu^{(-r_\nu)}\tilde{R}^{-1} & \ldots & A_\nu^{(-k)}\tilde{R}^{-1} \\ & * & & \end{pmatrix}
\begin{pmatrix} O_m & \ldots & * \\ \vdots & & \vdots \\ I_m & \ldots & A_1^{(-r_1)} \end{pmatrix}
= \begin{pmatrix} A_\nu^{(-k)}\tilde{R}^{-1} & * \\ * & * \end{pmatrix}, \tag{6.14}
$$

we obtain $\pi(c_0(A)) = A$. $\square$

Remark 4. The assumption that A has at least one regular singularity is not essintial since, by a gauge transformation $Y \to (x - a)^\alpha Y$, we can add $\frac{\alpha}{x-a}$ to A.

Remark 5. For any A the generalized Okubo system of size $n = \sum_{\nu=1}^{p} \tilde{r}_\nu$ lying in $\pi^{-1}(A)$ is unique up to conjugate action of $\mathrm{Stab}(T)$. This can be proved in a similar way to the case of Proposition 5.1.

Remark 6. When the corank of leading terms $A_\nu^{(-r_\nu)}$ ($\nu = 1, \ldots, p$) are all zero, this gives the minimal size generalized Okubo system in the fiber.

Remark 7. There is the following relation between a solution of $A \in \mathcal{E}$ and a solution of its convolution $c_\lambda(A)$. Let Y be a solution of the following equation:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}} Y. \tag{6.15}$$

If we put

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_p(x) \end{pmatrix}$$

where

$$F_\nu(x) := \begin{pmatrix} \frac{Y(x)}{(x-t_\nu)^{r_\nu+1}} \\ \vdots \\ \frac{Y(x)}{x-t_\nu} \end{pmatrix} \qquad (\nu = 1, \ldots, p). \tag{6.16}$$

Then

$$Z(x) = \int_C F(t)(x-t)^\lambda dt$$

with suitable C is a solution of (6.4).

7 Examples

In this section, we compute two examples of non-Fuchsian analog of the middle convolution and discuss the relations to Bäcklund transformations of P_{IV} and P_{V}.

7.1 The fifth Painlevé equation (P_{V})

We consider a system of linear differential equation L_{V} with the following properties:

1. L_{V} has singularities at $x = 0, 1, \infty$,
2. $x = 0$ is a regular singular point,
3. $x = \infty$ is a regular singular point,
4. $x = 1$ is an irregular singular point of Poincaré rank 1.

By means of suitable changes of the dependent variables, the system L_{V} is written as follows:

$$\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{A_1^{(-1)}}{(x-1)^2} + \frac{A_1^{(0)}}{x-1} + \frac{A_0^{(0)}}{x} \right) Y, \\
A_0^{(0)} &= \begin{pmatrix} z_0 + \alpha_3 & -uz_0 \\ (z_0+\alpha_3)/u & -z_0 \end{pmatrix}, \\
A_1^{(-1)} &= \begin{pmatrix} z_1 + t & -vz_1 \\ (z_1+t)/v & -z_1 \end{pmatrix}, \\
A_1^{(0)} &= -A_0^{(0)} - \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 + \alpha_1 - 1 \end{pmatrix},
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
(1-\alpha_1)z_0 &= \lambda^2(\lambda-1)^2\mu^2 \\
&\quad + \{\alpha_0(\lambda-1) - \alpha_2 - t\}\{(\lambda-1)\mu + \alpha_0\}\lambda \\
&\quad + \{\alpha_0\lambda(\lambda-1) - t\}\lambda\mu + \alpha_3(\alpha_1 - 1), \\
(1-\alpha_1)z_1 &= \lambda(\lambda-1)^3\mu^2 \\
&\quad + \{2\alpha_0\lambda^2 - (2\alpha_0 - \alpha_3 + t)\lambda - \alpha_3\}\{(\lambda-1)\mu + \alpha_0\} \\
&\quad - \alpha_0^2\lambda(\lambda-1) + (\alpha_0 + \alpha_1 - 1)t, \\
v &= \frac{\lambda - 1}{\lambda}\frac{z_0}{z_1}u.
\end{aligned} \tag{7.2}$$

The parameter λ represents a position of apparent singular point.

The holonomic deformation of (7.1) is governed by the following system:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial H_{\mathrm{V}}}{\partial \mu}, \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{V}}}{\partial \lambda}, \tag{7.3}$$

here the Hamiltonian $H_{\rm V}$ is given as follows:

$$\begin{aligned} tH_{\rm V} &= \lambda(\lambda-1)^2\mu^2 \\ &\quad + \{(2\alpha_0+\alpha_1)(\lambda-1)^2 + (\alpha_0-\alpha_2+1-t)(\lambda-1)-t\}\mu \\ &\qquad + \alpha_0(\alpha_0+\alpha_1)\lambda. \end{aligned} \tag{7.4}$$

By eliminating μ from (7.3), we obtain the fifth Painlevé equation $P_{\rm V}$:

$$\begin{aligned} \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} \\ &\quad + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda+\frac{\beta}{\lambda}\right) + \gamma\frac{\lambda}{t} + \delta\frac{\lambda(\lambda+1)}{\lambda-1}, \end{aligned} \tag{7.5}$$

where

$$\alpha = \frac{{\alpha_1}^2}{2}, \quad \beta = -\frac{{\alpha_3}^2}{2}, \quad \gamma = \alpha_0-\alpha_2, \quad \delta = -\frac{1}{2},$$

and $\alpha_0+\alpha_1+\alpha_2+\alpha_3=1$.

The Bäcklund transformations of $P_{\rm V}$ are given as follows:

x	α_0	α_1	α_2	α_3	t
$s_0(x)$	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	$\alpha_3+\alpha_0$	t
$s_1(x)$	$\alpha_0+\alpha_1$	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	t
$s_2(x)$	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	t
$s_3(x)$	$\alpha_0+\alpha_3$	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	t
$\varpi(x)$	α_1	α_2	α_3	α_0	t
$\sigma(x)$	α_0	α_3	α_2	α_1	$-t$

Here we omit an expression of λ, μ.

Proposition 7.1. *The minimal size generalized Okubo system in $\pi^{-1}(L_{\rm V})$ is uniquely given as follows:*

$$(xI_3 - T_{\rm V})\frac{d\Psi}{dx} = C_{\rm V}\Psi \tag{7.6}$$

where

$$T_{\rm V} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_{\rm V} = \begin{pmatrix} \alpha_2 - \alpha_0 & -\frac{1}{t}\det A_1^{(0)} & (C_{\rm V})_{13} \\ t & 0 & ((\lambda-1)\mu+\alpha_0)\lambda+\alpha_3 \\ t - \{(\lambda-1)\mu+\alpha_0\}(\lambda-1) & (C_{\rm V})_{32} & \alpha_3 \end{pmatrix}, \tag{7.7}$$

and

$$\begin{aligned} t(C_{\rm V})_{32} &= (\alpha_1-1)z_1 + (\alpha_0+\alpha_1-1)(t-((\lambda-1)\mu+\alpha_0)(\lambda-1)), \\ (C_{\rm V})_{13} &= \frac{1}{t-\{(\lambda-1)\mu+\alpha_0\}(\lambda-1)} \\ &\quad \{(\alpha_1-1)z_0 - (((\lambda-1)\mu+\alpha_0)\lambda+\alpha_3)(C_{\rm V})_{32} - \alpha_3(\alpha_0+\alpha_3)\}. \end{aligned} \tag{7.8}$$

Proof. Let R be $\mathrm{diag}(\alpha_0, \alpha_0+\alpha_1-1, 0)$. Suppose $[T_{\rm V}, R, G] \in \pi^{-1}(L_{\rm V})$. By the Lemma, the matrix G satisfies an equation $\Omega \boldsymbol{g} = 0$. Here the coefficient matrix Ω is given by

$$\Omega = \begin{pmatrix} \Omega_1 & R_1^{(-1)} & \\ & \Omega_1 & \\ & & \Omega_2 \end{pmatrix}$$

where

$$\Omega_1 = \begin{pmatrix} {}^tA_1^{(-1)} & {}^tX_1^{(-1)} \\ {}^tA_0^{(0)} & {}^tX_0^{(0)} \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} {}^tA_1^{(0)} & {}^tX_1^{(0)} \\ {}^tA_1^{(-1)} & {}^tX_1^{(-1)} \end{pmatrix},$$

and

$$R_1^{(-1)} = \begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_0+\alpha_1-1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$X_0^{(0)}$, $X_1^{(0)}$, and $X_1^{(-1)}$ are 1×2 matrices. By a direct computation, we can show that G is uniquely determined by the equation, and the matrix $-GRG^{-1}$ coincides with $C_{\rm V}$. □

By translating the right-hand side of (7.6) with a scalar matrix and send it by π, we obtain a middle convolution of the system (7.1). The middle convolution of (7.1) with α_0 is given as follows:

$$\begin{aligned} \frac{dY}{dx} &= \left(\frac{\bar{A}_1^{(-1)}}{(x-1)^2} + \frac{\bar{A}_1^{(0)}}{x-1} + \frac{\bar{A}_0^{(0)}}{x} \right) Y, \\ \bar{A}_0^{(0)} &= \begin{pmatrix} \bar{z}_0 + \alpha_3 + \alpha_0 & -\bar{u}\bar{z}_0 \\ (\bar{z}_0 + \alpha_3 + \alpha_0)/\bar{u} & -\bar{z}_0 \end{pmatrix}, \\ \bar{A}_1^{(-1)} &= \begin{pmatrix} \bar{z}_1 + t & -\bar{v}\bar{z}_1 \\ (\bar{z}_1 + t)/\bar{v} & -\bar{z}_1 \end{pmatrix}, \\ \bar{A}_1^{(0)} &= -\bar{A}_0^{(0)} - \begin{pmatrix} -\alpha_0 & 0 \\ 0 & \alpha_1 - 1 \end{pmatrix}, \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} \bar{z}_0 &= \frac{\alpha_1 - 1}{\alpha_0 + \alpha_1 - 1} z_0, \\ \bar{z}_1 &= \frac{\alpha_1 - 1}{\alpha_0 + \alpha_1 - 1} z_1, \\ \bar{v} &= \frac{\lambda + \alpha_0/\mu - 1}{\lambda + \alpha_0/\mu} \frac{z_0}{z_1} \bar{u}. \end{aligned} \tag{7.10}$$

By comparing (7.1) and (7.9), we have the transformation

$$\alpha_0 \mapsto -\alpha_0, \quad \alpha_1 \mapsto \alpha_1 + \alpha_0, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_3 + \alpha_0,$$
$$t \mapsto t, \quad \lambda \mapsto \lambda + \frac{\alpha_0}{\mu}, \quad \mu \mapsto \mu.$$

We gain the transformation s_0.

Next we consider an analogue of *addition*. The addition at $x = 0$ is given by

$$(A_1^{(-1)}, A_1^{(0)}, A_0^{(0)}) \mapsto (A_1^{(-1)}, A_1^{(0)}, A_0^{(0)} - \alpha_3 I). \tag{7.11}$$

This induces a transformation

$$\alpha_0 \mapsto \alpha_0 + \alpha_3, \quad \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_2 + \alpha_3, \quad \alpha_3 \mapsto -\alpha_3,$$
$$t \mapsto t, \quad \lambda \mapsto \lambda, \quad \mu \mapsto \mu - \frac{\alpha_3}{\lambda}.$$

This is the simple reflection s_3.

Similarly, the addition at $x = 1$ is given by

$$(A_1^{(-1)}, A_1^{(0)}, A_0^{(0)}) \mapsto (A_1^{(-1)} - tI, A_1^{(0)} - (\alpha_2 - \alpha_0)I, A_0^{(0)}). \tag{7.12}$$

This induces a transformation

$$\alpha_0 \mapsto \alpha_2, \quad \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_0, \quad \alpha_3 \mapsto \alpha_3,$$
$$t \mapsto -t, \quad \lambda \mapsto \lambda, \quad \mu \mapsto \mu - \frac{\alpha_2 - \alpha_0}{\lambda - 1} - \frac{t}{(\lambda - 1)^2}.$$

We write this transformation add_1. It is easy to see that transformation $(add_1)s_0(add_1)$ coincides with the simple reflection s_2.

In order to obtain the simple reflection s_1, we exchange the exponent at infinity. It is realized by a gauge transformation by

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This induces a transformation

$$\alpha_0 \mapsto \alpha_0 + \alpha_1 - 1, \quad \alpha_1 \mapsto -\alpha_1 + 2, \quad \alpha_2 \mapsto \alpha_2 + \alpha_1 - 1, \quad \alpha_3 \mapsto \alpha_3,$$
$$t \mapsto t.$$

Since the expression of transformed λ and μ is complicated, we omitted it. By performing the Schlesinger transformation

$$\alpha_0 \mapsto \alpha_0 - 1, \quad \alpha_1 \mapsto \alpha_1 + 2, \quad \alpha_2 \mapsto \alpha_2 - 1, \quad \alpha_3 \mapsto \alpha_3$$

successively, we obtain s_1.

Let T_1 be the Schlesinger transformation

$$\alpha_0 \mapsto \alpha_0 + 1, \quad \alpha_1 \mapsto \alpha_1 - 1, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_3.$$

It is easy to see the relation $\varpi = T_1 s_1 s_2 s_3$ and $\sigma = (add_1)\varpi^2$.

Remark 8. For a system of linear differential equation, the Schlesinger transformation is a discrete deformation of parameters that does not change the monodromy; see [6].

Then we obtain all the Bäcklund transformations of P_{V} by means of associate linear differential equation L_{V}.

7.2 The fourth Painlevé equation ($P_{\rm IV}$)

Next we consider a system of linear differential equation $L_{\rm IV}$ which has the following properties:

1. $L_{\rm IV}$ has singularities at $x = 0, \infty$,
2. $x = 0$ is an irregular singular point of Poincaré rank 2,
3. $x = \infty$ is a regular singular point.

By means of suitable changes of the dependent variables, the system $L_{\rm IV}$ is written as follows:

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{A_0^{(-2)}}{x^3} + \frac{A_0^{(-1)}}{x^2} + \frac{A_0^{(0)}}{x} \right) Y, \\ A_0^{(-2)} &= \begin{pmatrix} z + 1/2 & -uz \\ (z+1/2)/u & -z \end{pmatrix}, \\ A_0^{(-1)} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\ A_0^{(0)} &= -\begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 + \alpha_1 - 1 \end{pmatrix}, \end{aligned} \tag{7.13}$$

where

$$\begin{aligned} 2(1-\alpha_1)z &= (2\lambda^3\mu + 2\alpha_0\lambda^2 - 2t\lambda - 1)(\lambda\mu + \alpha_0) + \alpha_0 + \alpha_1 - 1, \\ \lambda a_{11} &= -(z + 1/2) + \lambda^3\mu + \alpha_0\lambda^2, \\ a_{12} &= uz/\lambda, \\ a_{21} &= \frac{\lambda}{uz}\left\{ a_{11}(t - a_{11}) - (\alpha_1 - 1)z - \frac{\alpha_0 + \alpha_1 - 1}{2} \right\}, \\ a_{22} &= t - a_{11}. \end{aligned} \tag{7.14}$$

The holonomic deformation of (7.13) is governed by the following system:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial H_{\rm IV}}{\partial \mu}, \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = -\frac{\partial H_{\rm IV}}{\partial \lambda}, \tag{7.15}$$

here the Hamiltonian $H_{\rm IV}$ is

$$H_{\rm IV} = 2\lambda^3\mu^2 + \{2(1 - \alpha_2 + \alpha_0)\lambda^2 - 2t\lambda - 1\}\mu + 2\alpha_0(\alpha_0 + \alpha_1)\lambda.$$

By putting $q = 1/\lambda$, we have the fourth Painlevé equation $P_{\rm IV}$:

$$\frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2 - \alpha)q + \frac{\beta}{q},$$

where

$$\alpha = \alpha_0 - \alpha_2, \quad \beta = -2{\alpha_1}^2,$$

and $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

The Bäcklund transformations of P_{IV} are given as follows:

x	α_0	α_1	α_2	t
$s_0(x)$	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	t
$s_1(x)$	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	t
$s_2(x)$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	t
$\varpi(x)$	α_1	α_2	α_0	t
$\sigma_1(x)$	α_0	α_2	α_1	$\sqrt{-1}t$
$\sigma_2(x)$	α_1	α_0	α_2	$\sqrt{-1}t$

we have the

Proposition 7.2. *The minimal size generalized Okubo system in* $\pi^{-1}(L_{\rm IV})$ *is uniquely given as follows:*

$$(xI_3 - T_{\rm IV})\frac{d\Psi}{dx} = C_{\rm IV}\Psi \tag{7.16}$$

where

$$T_{\rm IV} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C_{\rm IV} = \begin{pmatrix} 2(\alpha_1 - 1)z - \alpha_0 & (C_{\rm IV})_{12} & (C_{\rm IV})_{13} \\ 0 & -2(\alpha_1 - 1)z + \alpha_2 & (C_{\rm IV})_{23} \\ \frac{1}{2} & t & 0 \end{pmatrix}, \tag{7.17}$$
$$(C_{\rm IV})_{13} = 4(\alpha_1 - 1)\lambda\{2\lambda(\lambda\mu + \alpha_0)^2 - \mu\}z,$$
$$(C_{\rm IV})_{23} = -4(\alpha_1 - 1)\lambda(\lambda\mu + \alpha_0)z,$$
$$(C_{\rm IV})_{12} = \frac{((-2(\alpha_1 - 1)z + \alpha_2)(C_{\rm IV})_{13}}{(C_{\rm IV})_{23}} + 2t(2(\alpha_1 - 1)z - \alpha_0).$$

Remark 9. When we perform the Laplace transform

$$\Psi(x) = \int e^{-xz}\Phi(z)\mathrm{d}z$$

to (7.16), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \left(T_{\mathrm{IV}} - \frac{C_{\mathrm{IV}} + I}{z}\right)\Phi.$$

This system is essentially one of the Lax pair of Noumi-Yamada system of type $A_2^{(1)}$, that is, symmetric form of P_{IV} (cf. [10]).

The middle convolution of (7.13) with α_0 is given as follows:

$$\begin{aligned}
\frac{\mathrm{d}Y}{\mathrm{d}x} &= \left(\frac{\bar{A}_0^{(-2)}}{x^3} + \frac{\bar{A}_0^{(-1)}}{x^2} + \frac{\bar{A}_0^{(0)}}{x}\right)Y,\\
\bar{A}_0^{(-2)} &= \begin{pmatrix} \bar{z} + 1/2 & -\bar{u}\bar{z} \\ (\bar{z} + 1/2)/\bar{u} & -\bar{z} \end{pmatrix},\\
\bar{A}_0^{(-1)} &= \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix},\\
\bar{A}_0^{(0)} &= -\begin{pmatrix} -\alpha_0 & 0 \\ 0 & \alpha_1 - 1 \end{pmatrix},
\end{aligned} \tag{7.18}$$

where

$$\begin{aligned}
\bar{z} &= \frac{\alpha_0 + \alpha_2}{\alpha_2} z,\\
\bar{a}_{11} &= \frac{1}{\alpha_2}\left\{(\alpha_0 + \alpha_2)a_{11} - \alpha_0 t\right\},\\
\bar{a}_{12} &= \frac{\bar{u}\bar{z}}{\lambda + \alpha_0/\mu},\\
\bar{a}_{21} &= \frac{\lambda + \alpha_0/\mu}{\bar{u}\bar{z}}\left\{\bar{a}_{11}(t - \bar{a}_{11}) + \alpha_2\bar{z} + \frac{\alpha_0 + \alpha_2}{2}\right\},\\
\bar{a}_{22} &= t - \bar{a}_{11}.
\end{aligned} \tag{7.19}$$

By comparing (7.13) and (7.18), we have the transformation

$$\alpha_0 \mapsto -\alpha_0, \quad \alpha_1 \mapsto \alpha_1 + \alpha_0, \quad \alpha_2 \mapsto \alpha_2, \quad \alpha_3 \mapsto \alpha_3 + \alpha_0,$$
$$t \mapsto t, \quad \lambda \mapsto \lambda + \frac{\alpha_0}{\mu}, \quad \mu \mapsto \mu$$

We obtain the simple reflection s_0. Next we consider an addition. The addition at $x = 0$ is given by

$$(A_0^{(-2)}, A_0^{(-1)}, A_0^{(0)}) \mapsto (A_0^{(-2)} - \frac{1}{2}I, A_0^{(-1)} - tI, A_0^{(0)} - (\alpha_2 - \alpha_0)I). \quad (7.20)$$

Let us change the independent variable $x \to \sqrt{-1}x$, this induces a transformation

$$\alpha_0 \mapsto \alpha_2, \quad \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_0,$$
$$t \mapsto \sqrt{-1}t, \quad \lambda \mapsto -\sqrt{-1}\lambda, \quad \mu \mapsto \frac{\sqrt{-1}}{2\lambda^3}(2\lambda^3\mu - 2(\alpha_2 - \alpha_0)\lambda^2 - 2t\lambda - 1).$$

We denote this transformation by add_0. We can see easily that $s_2 = (add_0)s_0(add_0)$.

We can obtain the simple reflection s_1 by exchanging the exponent at infinity and the Schlesinger transformation (similar to the case of P_{V}). Let T_1 be the Schlesinger transformation

$$\alpha_0 \mapsto \alpha_0 + 1, \quad \alpha_1 \mapsto \alpha_1 - 1, \quad \alpha_2 \mapsto \alpha_2.$$

It is easy to see the relation $\varpi = T_1 s_1 s_2$, $\sigma_1 = \varpi(add_0)$, and $\sigma_2 = (add_0)\varpi$.

Then we obtain all the Bäcklund transformations of P_{IV} by means of associate linear differential equation L_{IV}.

Part II
Confluence of singular points and the Okubo systems

8 Introduction

For a Fuchsian differential equation, the characteristic exponents are defined at each regular singularities. If a Fuchsian equation is completely determined by giving only these characteristic exponents, then we call such a equation *rigid*. Here "rigid" can be paraphrased as "accessory parameter free". Thus, for rigid Fuchsian systems, global behavior of solutions is determined by local behavior.

Since the Gauss' hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{\mathrm{d}y}{\mathrm{d}x}-\alpha\beta y=0$$

is rigid, it is important for the theory of special functions to obtain all rigid Fuchsian systems. Concerning this problem, Katz introduced the operations, called *addition* and *middle convolution*, which takes a Fuchsian system to another Fuchsian system without changing the number of accessory parameters. Here we follow the terminology of Dettweiler and Reiter [3].

Katz [7] showed the following theorem:

Theorem (Katz) . *Every irreducible rigid Fuchsian system is obtained from rank 1 Fuchsian system by a finite iteration of the addition and the middle convolution.*

Here we recall the Katz-Dettweiler-Reiter operations for systems of the Fuchsian type. For the sake of simplicity, we represent the Fuchsian system of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}x}=\left(\frac{A_1}{x-t_1}+\cdots+\frac{A_p}{x-t_p}\right)Y \quad (m\times m)$$

as $A=(A_1,\ldots,A_p)$.

Definition (addition) . *For* $\alpha=(\alpha_1,\ldots,\alpha_p)\in\mathbb{C}^p$, *an operation*

$$A\mapsto(A_1+\alpha_1 I_m,\ldots,A_p+\alpha_p I_m)$$

is called addition.

Let λ be a complex parameter. We put a $pm\times pm$ matrix G_ν as follows:

$$G_\nu=\begin{pmatrix} O_m & & \cdots & & O_m\\ \vdots & \ddots & & & \vdots\\ A_1 & \cdots & A_\nu+\lambda I_m & \cdots & A_p\\ \vdots & & & \ddots & \vdots\\ O_m & & \cdots & & O_m\end{pmatrix} \quad (\nu=1,\ldots,p).$$

Definition (convolution) . *The system* $(G_1,\ldots,G_p)$ *is called convolution with* λ *of* A. *We denote this system by* $c_\lambda(A)$.

Let $\mathcal{K}, \mathcal{L}$ be the following linear subspaces of $\mathbb{C}^{pm}$:

$$\mathcal{K} := \begin{pmatrix} \mathrm{Ker}(A_1) \\ \vdots \\ \mathrm{Ker}(A_p) \end{pmatrix}, \qquad (8.1)$$
$$\mathcal{L} := \mathrm{Ker}(G_1 + \cdots + G_p).$$

We can see easily that $\mathcal{K}, \mathcal{L}$ are $G_1, \ldots, G_p$-invariant subspaces.

Let $\overline{G}_\nu$ be the linear transformation on quotient space $\mathbb{C}^{pm}/(\mathcal{K} + \mathcal{L})$ induced by G_ν.

Definition (middle convolution) . *We call the operation $A \mapsto (\overline{G}_1, \ldots, \overline{G}_p)$ middle convolution with λ and denote by mc_λ.*

In Part I, we extend the notion of convolution for systems, not necessarily of the Fuchsian type. If a system of differential equations has an irregular singular point, $x = a$, then we can construct a system with regular singular points arround $x = a$ with a parameter ε such that these regular singularities merge to the irregular singularity when $\varepsilon \to 0$. That is, a system, not necessarily of the Fuchsian type, defined on $\mathbb{P}^1(\mathbb{C})$ is obtained by confluence of singularities from a system of the Fuchsian type. In Part II, we show that convolutions of each equation is compatible with confluence.

9 Generalized Okubo system

The system of linear differential equations of the form

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = A\Psi \qquad (9.1)$$

is called a system of Okubo normal form, in this paper, we call it *Okubo system* in short. Here T is an $n \times n$ constant diagonal matrix and A is an $n \times n$ arbitrary constant matrix.

When a matrix T is not semisimple, a system of the form (9.1) may have irregular singularities. In the case when T is a Jordan matrix, non-semisimple, call (9.1) *generalized Okubo system.*

In this paper, we assume that the matrix A is semisimple and denote its non-zero eigenvalues by $-\rho_1, \ldots, -\rho_m$, namely, we put

$$A = -GRG^{-1}, \; R = \mathrm{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0). \qquad (9.2)$$

Then systems of the form (9.1) can be written in the following form:

$$(xI - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = -GRG^{-1}\Psi.$$

We represent such a system as (T, R, G).

Let $\mathrm{Stab}(M)$ be the stabilizer of $M \in M(n, \mathbb{C})$:

$$\mathrm{Stab}(M) = \{g \in GL(n, \mathbb{C}) \mid gM = Mg\}.$$

For a Jordan matrix T and a diagonal matrix $R = \mathrm{diag}(\rho_1, \ldots, \rho_m, 0, \ldots, 0)$, let $\mathcal{O}(T, R)$ be the following set of systems:

$$\mathcal{O}(T, R) := \{(T, R, G)\} / \underset{\mathcal{O}}{\sim} . \tag{9.3}$$

Here the equivalent relation $\underset{\mathcal{O}}{\sim}$ in (9.3) is defined by

$$G \underset{\mathcal{O}}{\sim} hGg$$

for $h \in \mathrm{Stab}(T)$, $g \in \mathrm{Stab}(R)$. We write the set of Okubo and generalized Okubo systems as follows:

$$\mathcal{GO} := \coprod_{T,R} \mathcal{O}(T, R)$$

where T runs over all Jordan matrices, including diagonal matrices, and R runs over all diagonal matrices of the form (9.2). Similarly, we denote the set of Okubo systems, a subset of $\mathcal{GO}$, by

$$\mathcal{O} := \coprod_{T,R} \mathcal{O}(T, R)$$

where T runs over all diagonal matrices.

Furthermore, we define another sets of linear differential systems. We put $\Gamma_{(m,p)}$ and $\Gamma^*_{(m,p)}$ as

$$\Gamma_{(m,p)} = \mathbb{C}^p \times (\mathbb{Z}_{\geq 0})^p \times (\mathbb{C}^\times)^m,$$
$$\Gamma^*_{(m,p)} = \mathbb{C}^p \times (\mathbb{C}^\times)^m.$$

We regard the set $\Gamma^*_{(m,p)}$ as a subset of $\Gamma_{(m,p)}$ through the inclusion mapping

$$\Gamma^*_{(m,p)} \hookrightarrow \Gamma_{(m,p)}$$

$$(t_1, \dots, t_p, \rho_1, \dots, \rho_m) \mapsto (t_1, \dots, t_p, \overbrace{0, \dots, 0}^{p}, \rho_1, \dots, \rho_m).$$

For every element $\gamma = (t_1, \dots, t_p, r_1, \dots, r_p, \rho_1, \dots, \rho_m)$ of $\Gamma_{(m,p)}$, we denote an $m \times m$ diagonal matrix $\mathrm{diag}(\rho_1, \dots, \rho_m)$ by $\tilde{R}_\gamma$. Then we define $\mathcal{E}_\gamma$ by

$$\mathcal{E}_\gamma = \left\{ A(x) = \sum_{\nu=1}^{p} \sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}} \right. \left| A_\nu^{(-k)} \in M(m, \mathbb{C}),\ A_\nu^{(-r_\nu)} \neq O,\ -\sum_{\nu=1}^{p} A_\nu^{(0)} = \tilde{R}_\gamma \right\} \Big/ \underset{\mathcal{E}_\gamma}{\sim}. \quad (9.4)$$

Here equivalent relation $\underset{\mathcal{E}_\gamma}{\sim}$ in (9.4) is defined by

$$A(x) \underset{\mathcal{E}_\gamma}{\sim} gA(x)g^{-1} \quad (9.5)$$

where $g \in \mathrm{Stab}(\tilde{R}_\gamma)$. We identify an element $A(x)$ of $\mathcal{E}_\gamma$ with the system of differential equation $\dfrac{\mathrm{d}Y}{\mathrm{d}x} = A(x)Y$.

We set

$$\mathcal{E} = \coprod_{m,p \in \mathbb{Z}_{\geq 1}} \coprod_{\gamma \in \Gamma_{(m,p)}} \mathcal{E}_\gamma,$$

$$\mathcal{F} = \coprod_{m,p \in \mathbb{Z}_{\geq 1}} \coprod_{\gamma \in \Gamma^*_{(m,p)}} \mathcal{E}_\gamma,$$

namely $\mathcal{E}$ is a set of linear differential equations on $\mathbb{P}^1$ which have regular singularity at infinity, and $\mathcal{F}$ is a set of Fuchsian equations on $\mathbb{P}^1$.

Now we give the definition of the mapping $\pi : \mathcal{GO} \to \mathcal{E}$.

Definition 3. *For $[T, R, G] \in \mathcal{GO}$, we define the mapping $\pi : \mathcal{GO} \to \mathcal{E}$ as follows:*

$$\pi(T, R, G) := \textit{the principal } m \times m \textit{ part of } (-G^{-1}(xI - T)^{-1}GR). \quad (9.6)$$

We can show that the definition is well-defined.

10 Confluence

In this section, we define the operation convolution for non-Fuchsian systems. This is an operation which transforms given equation into Okubo system (in larger system size).

For a given non-Fuchsian system, we can construct a Fuchsian system with parameter ε which tends to the given equation as $\varepsilon \to 0$. Then we can consider the convolutions of both systems. We define a compatible confluence which connects both Okubo systems.

Let $A = \displaystyle\sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}}$ be an element of $\mathcal{E}$ whose matrix size is m. Put $n = \sum_{\nu=1}^{p} m(r_\nu + 1)$. Then we can construct a (generalized) Okubo system of size n lying in $\pi^{-1}(A)$ in the following manner.

Let $\tilde{A}_\nu$ be the following $m(r_\nu + 1) \times n$ matrix

$$\tilde{A}_\nu = \begin{pmatrix} & & O_{mr_\nu,n} & & \\ A_1^{(-r_1)} & \dots & A_1^{(0)} & \dots & A_p^{(-r_p)} & \dots & A_p^{(0)} \end{pmatrix}. \tag{10.1}$$

We set further matrices $\tilde{A}$, T, $\tilde{T}$, and P by

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_p \end{pmatrix}, \\ T &= J_{r_1+1}(t_1)^{\oplus m} \oplus \cdots \oplus J_{r_p+1}(t_p)^{\oplus m}, \\ \tilde{T} &= J_{r_1+1}(t_1) \oplus \cdots \oplus J_{r_p+1}(t_p), \\ P &= P_{(m,r_1+1)} \oplus \cdots \oplus P_{(m,r_p+1)}. \end{aligned} \tag{10.2}$$

Here $P_{(i,j)}$ is a permutation matrix of the form

$$P_{(i,j)} = (I_i \otimes \boldsymbol{e}_1, I_i \otimes \boldsymbol{e}_2, \dots, I_i \otimes \boldsymbol{e}_j)$$

where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_j$ are the j-dimensional unit vectors.

Definition 4. *We call the following (generalized) Okubo system:*

$$(xI_n - T)\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (P\tilde{A}P^{-1} + \lambda I_n)\Psi \tag{10.3}$$

convolution of A with λ, and we denote it by $c_\lambda(A)$.

By changing the unknown Ψ of (10.3) to $\tilde{\Psi} = P^{-1}\Psi$, we can express the equation $c_\lambda(A)$ as follows:

$$(xI_n - \tilde{T} \otimes I_m)\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = (\tilde{A} + \lambda I_n)\tilde{\Psi}. \tag{10.4}$$

We proved in Part I that, for any element A of $\mathcal{E}$, $\pi(c_0(A)) = A$.

Now we define a Fuchsian system related to A by

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A(\varepsilon)Y, \quad A(\varepsilon) = \sum_{\nu=1}^{p}\sum_{j=1}^{r_\nu+1} \frac{A_{\nu,j}}{x - s_{\nu,j}}$$

where

$$\begin{cases} A_{\nu,j} = \dfrac{1}{r_\nu + 1}\displaystyle\sum_{k=0}^{r_\nu}(\zeta_\nu^{j-1}\varepsilon)^{-k}A_\nu^{(-k)} \\ s_{\nu,j} = t_\nu + \zeta_\nu^{j-1}\varepsilon \end{cases}. \tag{10.5}$$

Here ζ_ν is a primitive $(r_\nu + 1)$th root of unity. We can show the proposition (see [5]):

Proposition 10.1.

$$\lim_{\varepsilon\to 0}\left(\sum_{\nu=1}^{p}\sum_{j=1}^{r_\nu+1}\frac{A_{\nu,j}}{x - s_{\nu,j}}\right) = \sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu}\frac{A_\nu^{(-k)}}{(x - t_\nu)^{k+1}}.$$

The convolution of the system $A(\varepsilon)$ is given by

$$(xI_n - S(\varepsilon))\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (\bar{A}(\varepsilon) + \lambda I_n)\Psi,$$

where $S(\varepsilon)$ is

$$\begin{pmatrix} s_{1,1}I_m & & \\ & \ddots & \\ & & s_{1,r_1+1}I_m \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} s_{p,1}I_m & & \\ & \ddots & \\ & & s_{p,r_p+1}I_m \end{pmatrix} \tag{10.6}$$

and

$$\bar{A}(\varepsilon) = \begin{pmatrix} A_{1,1} & \dots & A_{1,r_1+1} & A_{2,1} & \dots & A_{2,r_2+1} & \dots & A_{p,1} & \dots & A_{p,r_p+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ A_{1,1} & \dots & A_{1,r_1+1} & A_{2,1} & \dots & A_{2,r_2+1} & \dots & A_{p,1} & \dots & A_{p,r_p+1} \end{pmatrix}. \tag{10.7}$$

The purpose of present article is to prove the following

Theorem 10.2. *The system*

$$(xI_n - S(\varepsilon))\frac{\mathrm{d}\Psi}{\mathrm{d}x} = (\bar{A}(\varepsilon) + \lambda I_n)\Psi$$

tends to the system

$$(xI_n - \tilde{T} \otimes I_m)\frac{\mathrm{d}\tilde{\Psi}}{\mathrm{d}x} = (\tilde{A} + \lambda I_n)\tilde{\Psi}$$

as $\varepsilon \to 0$. *More precisely, there exists a matrix* $j(\varepsilon)$ *which depends on the parameter* ε *such that the following hold:*

$$\lim_{\varepsilon \to 0} j(\varepsilon)S(\varepsilon)j(\varepsilon)^{-1} = \tilde{T} \otimes I_m,$$
$$\lim_{\varepsilon \to 0} j(\varepsilon)\bar{A}(\varepsilon)j(\varepsilon)^{-1} = \tilde{A}.$$

Now we define the following $(r+1) \times (r+1)$ diagonal matrices:

$$\Delta_1^{r+1}(\zeta, \varepsilon) = \mathrm{diag}(1, \dots, \zeta^{\frac{(i-1)(i-2)}{2}}\varepsilon^{i-1}, \dots, \zeta^{\frac{r(r-1)}{2}}\varepsilon^r),$$
$$\Delta_2^{r+1}(\zeta) = \mathrm{diag}(1, \dots, \zeta^{i-1}, \dots, \zeta^r)$$

and $(j+1) \times (j+1)$ matrix

$$G_{j+1}(\zeta) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ & \psi_1(\zeta) & & \\ & & \ddots & \\ & & & \psi_j(\zeta) \end{pmatrix}$$

where $\psi_j(\zeta) := \zeta^j - 1$. Here ζ is a complex parameter.

We will verify the Theorem 10.2 step-by-step in the next section.

11 Proof of the Theorem

11.1 The case $m = 1$, $p = 1$

First, we consider the case of $m = 1$, $p = 1$, i.e.,

$$A = \frac{a^{(0)}}{x-t} + \frac{a^{(-1)}}{(x-t)^2} + \dots + \frac{a^{(-r)}}{(x-t)^{r+1}}$$

where $a^{(-k)} \in \mathbb{C}$. We put a_j and s_j as to (10.5), that is,

$$\begin{cases} a_j = \dfrac{1}{r+1} \displaystyle\sum_{k=0}^{r} (\zeta^{j-1}\varepsilon)^{-k} a^{(-k)} \\ s_j = t + \zeta^{j-1}\varepsilon \quad (j = 1, \dots, r+1) \end{cases}.$$

Here ζ is a primitive $(r+1)$th root of unity. Then from Proposition 10.1, we have

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{r+1} \frac{a_j}{x - s_j} = \sum_{k=0}^{r} \frac{a^{(-k)}}{(x-t)^{k+1}}.$$

In this case, the matrices concerned are

$$\begin{cases} S(\varepsilon) = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_{r+1} \end{pmatrix} \\ \bar{A}(\varepsilon) = \begin{pmatrix} a_1 & \dots & a_{r+1} \\ \vdots & & \vdots \\ a_1 & \dots & a_{r+1} \end{pmatrix} \end{cases},$$

and

$$\begin{cases} \tilde{T} = J_{r+1}(t) \\ \tilde{A} = \begin{pmatrix} & O_{r,r+1} & \\ a^{(-r)} & \dots & a^{(0)} \end{pmatrix} \end{cases}.$$

We put the matrix $j(\varepsilon)$ as

$$j(\varepsilon) = \Delta_1^{r+1}(\zeta, \varepsilon)(I_{r-1} \oplus G_2)(I_{r-2} \oplus G_3) \cdots (I_1 \oplus G_r) G_{r+1} \Delta_2^{r+1}(\zeta).$$

Then we prove the

Proposition 11.1.

$$\lim_{\varepsilon \to 0} j(\varepsilon) S(\varepsilon) j(\varepsilon)^{-1} = \tilde{T}.$$

Proof. Since

$$
(I_k \oplus G_{r-k+1}) \begin{pmatrix} s_1 & \varepsilon & & & & & & \\ & s_2 & \zeta\varepsilon & & & & & \\ & & \ddots & \zeta^{k-2}\varepsilon & & & & \\ & & & s_k & \zeta^{k-1}\varepsilon & \dots & \zeta^{k-1}\varepsilon \\ & & & & s_{k+1} & & \\ & & & & & \ddots & \\ & & & & & & s_{r+1} \end{pmatrix} (I_k \oplus G_{r-k+1})^{-1}
$$

$$
= \begin{pmatrix} s_1 & \varepsilon & & & & & & \\ & s_2 & \zeta\varepsilon & & & & & \\ & & \ddots & \zeta^{k-2}\varepsilon & & & & \\ & & & s_k & \zeta^{k-1}\varepsilon & & & \\ & & & & s_{k+1} & \zeta^{k}\varepsilon & \dots & \zeta^{k}\varepsilon \\ & & & & & s_{k+2} & & \\ & & & & & & \ddots & \\ & & & & & & & s_{r+1} \end{pmatrix},
$$

we obtain

$$
(I_{r-1} \oplus G_2) \cdots G_{r+1} S(\varepsilon) G_{r+1}^{-1} \cdots (I_{r-1} \oplus G_2)^{-1}
= \begin{pmatrix} s_1 & \varepsilon & & & \\ & s_2 & \zeta\varepsilon & & \\ & & \ddots & \ddots & \\ & & & \ddots & \zeta^{r-1}\varepsilon \\ & & & & s_{r+1} \end{pmatrix}.
$$

Hence

$$
j(\varepsilon) S(\varepsilon) j(\varepsilon)^{-1} = \Delta_1^{r+1}(\zeta, \varepsilon) \begin{pmatrix} s_1 & \varepsilon & & & \\ & s_2 & \zeta\varepsilon & & \\ & & \ddots & \ddots & \\ & & & \ddots & \zeta^{r-1}\varepsilon \\ & & & & s_{r+1} \end{pmatrix} \Delta_1^{r+1}(\zeta, \varepsilon)^{-1}.
$$

Here the $(i, i+1)$-entry is

$$\zeta^{\frac{(i-1)(i-2)}{2}}\varepsilon^{i-1}(\zeta^{i-1}\varepsilon)\zeta^{-\frac{i(i-1)}{2}}\varepsilon^{-i} = 1.$$

Therefore we have

$$j(\varepsilon)S(\varepsilon)j(\varepsilon)^{-1} = \begin{pmatrix} s_1 & 1 & & \\ & s_2 & \ddots & \\ & & \ddots & 1 \\ & & & s_{r+1} \end{pmatrix}.$$

By taking a limit $\varepsilon \to 0$, we complete the proof. □

We can show easily the following lemma by an induction:

Lemma 11.2. *For* $p \in \{1, \dots, r-k\}$ *and a primitive* $(r+1)$*th root of unity* ζ*, the relation*

$$\sum_{l=1}^{r-k+1} \zeta^{pl}\psi_l\psi_{l+1}\cdots\psi_{l+k-1} = 0$$

holds.

For $l = 0, \dots, r-1$, we define $a_j[l]$s recursively as follows:

$$a_j[l+1] = \begin{cases} a_j[l] & (j = 1, \dots, l+1) \\ (a_j[l] - a_{l+1}[l])\psi_{j-l-1}^{-1} & (j = l+2, \dots, r+1) \end{cases} \tag{11.1}$$

and $a_j[0] = \zeta^{-j+1}a_j$ $(j = 1, \dots, r+1)$. Then we show the following

Proposition 11.3.

$$j(\varepsilon)\bar{A}(\varepsilon)j(\varepsilon)^{-1} = \frac{(-1)^r(r+1)}{\zeta}\Delta_1^{r+1}(\zeta,\varepsilon)\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ a_1[r] & \dots & a_{r+1}[r] \end{pmatrix}\Delta_1^{r+1}(\zeta,\varepsilon)^{-1}. \tag{11.2}$$

Proof. In general, by virtue of the Lemma 11.2, the following holds:

$$(I_k \oplus G_{r-k+1}) \begin{pmatrix} \mathbf{0}_k \\ \zeta^k \psi_1 \cdots \psi_k a_j \\ \vdots \\ \zeta^r \psi_{r-k+1} \cdots \psi_r a_j \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{k+1} \\ \zeta^{k+1} \psi_1 \cdots \psi_{k+1} a_j \\ \vdots \\ \zeta^r \psi_{r-k} \cdots \psi_r a_j \end{pmatrix},$$

where $\mathbf{0}_k$ is the zero vector in $\mathbb{C}^k$. Thus we obtain

$$(I_{r-1} \oplus G_2) \cdots (I_1 \oplus G_r) G_{r+1} \Delta_2^{r+1}(\zeta) \bar{A}(\varepsilon) = \frac{(-1)^r (r+1)}{\zeta} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ a_1 & \dots & a_{r+1} \end{pmatrix}.$$

Next, by the definition of $a_j[l]$, we have

$$(a_1[k], \dots, a_{r+1}[k]) \times (I_k \oplus G_{r-k+1})^{-1} = (a_1[k+1], \dots, a_{r+1}[k+1]).$$

Taking above into consideration, we obtain (11.2). □

We define $a_j[l]_k$ by the following:

$$a_j[l] = \frac{1}{r+1} \sum_{k=0}^{r} a_j[l]_k \varepsilon^{-k} a^{(-k)},$$

then from (11.1), these satisfy the difference equation:

$$a_j[l+1]_k = (a_j[l]_k - a_{l+1}[l]_k) \psi_{j-l-1}^{-1} \quad (j = l+2, \dots, r+1). \tag{11.3}$$

Then we can show the following

Proposition 11.4. *$a_j[l]_{r-k}$ is expressed explicitly as follows:*

$$a_j[l]_{r-k} = \begin{cases} \zeta^{(j-1)k} \sum_{d_1=1}^{k} \sum_{d_2=1}^{k-d_1} \cdots \sum_{d_l=1}^{k-d_1-\cdots-d_{l-1}} \zeta^{-\sum_{i=1}^{l}(j-i)d_i} & (l \le k \le r,\ l < j) \\ 0 & (\textit{otherwise}) \end{cases}.$$

for $l = 0, \dots, r$, $j = 1, \dots, r+1$.

Proof. Induction on k. □

It follows that

Corollary 11.5. $a_j[j-1]_{r-j+1} = \zeta^{\frac{(j-1)(j-2)}{2}}$.

Hence the $(r+1, j)$-entry of (11.2) reads as

$$\begin{aligned}
&(r+1)\zeta^{-\frac{(j-1)(j-2)}{2}}\varepsilon^{r-j+1}a_j[r] \\
&= (r+1)\zeta^{-\frac{(j-1)(j-2)}{2}}\varepsilon^{r-j+1}a_j[j-1] \\
&= \zeta^{-\frac{(j-1)(j-2)}{2}}\sum_{k=0}^{r-j+1} a_j[j-1]_k\varepsilon^{r-j+1-k}a^{(-k)} \qquad (11.4) \\
&\xrightarrow{\varepsilon\to 0} \zeta^{-\frac{(j-1)(j-2)}{2}}a_j[j-1]_{r-j+1}a^{(-r+j-1)} \\
&= a^{(-r+j-1)},
\end{aligned}$$

we finish the proof in this case.

11.2 The case $m = 1$ and general p

In this subsection, we consider the case $A = \sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu}\frac{a_\nu^{(-k)}}{(x-t_\nu)^{k+1}}$ where $a_\nu^{(-k)} \in \mathbb{C}$. Put $n = \sum_{\nu=1}^{p}(r_\nu+1)$. For $\nu = 1, \ldots, p$, $j = 1, \ldots, r_\nu+1$, let $a_{\nu,j}$, $s_{\nu,j}$ be

$$\begin{cases} a_{\nu,j} = \dfrac{1}{r_\nu+1}\displaystyle\sum_{k=0}^{r_\nu}(\zeta_\nu^{j-1}\varepsilon)^{-k}a_\nu^{(-k)} \\ s_{\nu,j} = t_\nu + \zeta_\nu^{j-1}\varepsilon \end{cases},$$

where ζ_ν is a primitive $(r_\nu+1)$th root of unity. Then from Proposition 10.1, we have

$$\lim_{\varepsilon\to 0}\left(\sum_{\nu=1}^{p}\sum_{j=1}^{r_\nu+1}\frac{a_{\nu,j}}{x-s_{\nu,j}}\right) = \sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu}\frac{a_\nu^{(-k)}}{(x-t_\nu)^{k+1}}.$$

In this case, the matrices concerned are

$$\begin{cases} S(\varepsilon) = \begin{pmatrix} s_{1,1} & & \\ & \ddots & \\ & & s_{1,r_1+1} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} s_{p,1} & & \\ & \ddots & \\ & & s_{p,r_p+1} \end{pmatrix} \\ \bar{A}(\varepsilon) = \begin{pmatrix} a_{1,1} & \dots & a_{1,r_1+1} & a_{2,1} & \dots & a_{2,r_2+1} & \dots & a_{p,1} & \dots & a_{p,r_p+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{1,1} & \dots & a_{1,r_1+1} & a_{2,1} & \dots & a_{2,r_2+1} & \dots & a_{p,1} & \dots & a_{p,r_p+1} \end{pmatrix}, \end{cases}$$

and $\tilde{T} = J_{r_1+1}(t_1) \oplus \cdots \oplus J_{r_p+1}(t_p)$. $\tilde{A}$ is given by the formula (10.1) and (10.2).

Let $G_j^{(\nu)} = G_j(\zeta_\nu)$ and

$$h_\nu(\varepsilon) := \Delta_1^{r_\nu+1}(\zeta_\nu, \varepsilon)(I_{r_\nu-1} \oplus G_2^{(\nu)})(I_{r_\nu-2} \oplus G_3^{(\nu)}) \cdots (I_1 \oplus G_{r_\nu}^{(\nu)}) G_{r_\nu+1}^{(\nu)} \Delta_2^{r_\nu+1}(\zeta_\nu). \quad (11.5)$$

Now we define the matrices $h(\varepsilon)$, $k(\varepsilon)$ by

$$h(\varepsilon) = h_1(\varepsilon) \oplus \cdots \oplus h_p(\varepsilon) \quad (11.6)$$

and

$$k(\varepsilon) = \begin{pmatrix} \frac{\varepsilon^{-r_1}}{r_1+1} I_{r_1+1} & & \\ & \ddots & \\ & & \frac{\varepsilon^{-r_p}}{r_p+1} I_{r_p+1} \end{pmatrix}. \quad (11.7)$$

Let $j(\varepsilon) = k(\varepsilon)h(\varepsilon)$. Since $S(\varepsilon)$ is a direct sum of ones of the case $p = 1$, the following is clear.

Proposition 11.6.

$$\lim_{\varepsilon \to 0} j(\varepsilon) S(\varepsilon) j(\varepsilon)^{-1} = \tilde{T}.$$

Next let us compute $h(\varepsilon)\bar{A}(\varepsilon)h(\varepsilon)^{-1}$. First we note that

$$h(\varepsilon)\bar{A}(\varepsilon) = \begin{pmatrix} \frac{(-1)^{r_1}(r_1+1)}{\zeta_1}\Delta_1^{r_1+1}(\zeta_1,\varepsilon) & & \\ & \ddots & \\ & & \frac{(-1)^{r_p}(r_p+1)}{\zeta_p}\Delta_1^{r_p+1}(\zeta_p,\varepsilon) \end{pmatrix}$$

$$\times \begin{pmatrix} & & & O_{r_1,n} & & & \\ a_{1,1} & \dots & a_{1,r_1+1} & \dots & a_{p,1} & \dots & a_{p,r_p+1} \\ & & & \vdots & & & \\ & & & O_{r_p,n} & & & \\ a_{1,1} & \dots & a_{1,r_1+1} & \dots & a_{p,1} & \dots & a_{p,r_p+1} \end{pmatrix}. \quad (11.8)$$

This can be shown in a similar manner to the previous subsection. Then, by multiplying $h(\varepsilon)^{-1}$ from the right, we obtain

$$h(\varepsilon)\bar{A}(\varepsilon)h(\varepsilon)^{-1} = \begin{pmatrix} \frac{(-1)^{r_1}(r_1+1)}{\zeta_1}\Delta_1^{r_1+1}(\zeta_1,\varepsilon) & & \\ & \ddots & \\ & & \frac{(-1)^{r_p}(r_p+1)}{\zeta_p}\Delta_1^{r_p+1}(\zeta_p,\varepsilon) \end{pmatrix}$$

$$\times \begin{pmatrix} A_{1,1} & \dots & A_{1,p} \\ \vdots & \ddots & \vdots \\ A_{p,1} & \dots & A_{p,p} \end{pmatrix} \begin{pmatrix} \Delta_1^{r_1+1}(\zeta_1,\varepsilon) & & \\ & \ddots & \\ & & \Delta_1^{r_p+1}(\zeta_p,\varepsilon) \end{pmatrix}^{-1}. \quad (11.9)$$

Here the $(r_\nu+1)\times(r_\mu+1)$ matrices $A_{\nu,\mu}$ are given as follows:

$$A_{\nu,\mu} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ a_{\mu,1}[r_\mu] & \dots & a_{\mu,r_\mu+1}[r_\mu] \end{pmatrix},$$

and $a_{\mu,j}[r_\mu]$s are defined by same procedure (11.1) with replacing ζ by ζ_μ. Hence the (ν,μ)-block of $h(\varepsilon)\bar{A}(\varepsilon)h(\varepsilon)^{-1}$ is given by

$$\frac{(-1)^{r_\nu}(r_\nu+1)}{\zeta_\nu}\Delta_1^{r_\nu+1}(\zeta_\nu,\varepsilon)A_{\nu,\mu}\Delta_1^{r_\mu+1}(\zeta_\mu,\varepsilon)^{-1}$$

and its $(r_\nu+1,j)$-entry is written as

$$(r_\nu+1)\zeta_\mu^{-\frac{(j-1)(j-2)}{2}}\varepsilon^{r_\nu-j+1}a_{\mu,j}[r_\mu].$$

Then by considering a conjugate action of $k(\varepsilon)$, the above expression is multiplied by $\frac{r_\mu+1}{r_\nu+1}\varepsilon^{r_\mu-r_\nu}$, together with (11.4), we complete the verification in this case.

11.3 The case general m, p

In this subsection, we consider the most general case: $A = \sum_{\nu=1}^{p}\sum_{k=0}^{r_\nu} \frac{A_\nu^{(-k)}}{(x-t_\nu)^{k+1}}$ where $A_\nu^{(-k)} \in M(m,\mathbb{C})$. The matrices $S(\varepsilon)$, $\bar{A}(\varepsilon)$, $\tilde{T}$, and $\tilde{A}$ are the same as (10.6), (10.7), (10.1), and (10.2). Putting $j(\varepsilon) = (k(\varepsilon)h(\varepsilon)) \otimes I_m$, where $h(\varepsilon)$, $k(\varepsilon)$ are same as (11.6), (11.7). Then the proof goes in quite a similar way to the previous subsections. Thus we can conclude that the Theorem 10.2 is valid.

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