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Solvability of a Class of Differential Equations in the Sheaf of Microfunctions with Holomorphic Parameters

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Abstract. The aim of this paper is to give results of solvability of a class of differential equations in the framework of microfunctions with holomorphic parameters. In particular we study transversally elliptic operators and other related operators.

1. Introduction

We study solvability of some class of differential equations in the sheaf of 2-analytic functions, that is, microfunctions with holomorphic parameters. In particular we treat *transversally elliptic operators* and other related operators, which are difficult to study in the former theory of second microlocal analysis.

The theory of the second microlocalization is a very useful method in studying solutions of linear partial differential equations in various situations. M. Kashiwara has constructed the sheaf $\mathcal{C}_V^2$ of 2-microfunctions by applying the microlocalization functor to the sheaf of rings $\mathcal{O}_X$ of holomorphic functions twice. Refer to Kashiwara-Laurent [8] for details.

Since this sheaf is larger than the decomposition of second microlocal singularities of microfunctions, Kataoka-Tose [12] and Kataoka-Okada-Tose [11] gave each definition of a new subsheaf of $\mathcal{C}_V^2$ what is called the sheaf of *small* 2*-microfunctions*. By introducing a bimicrolocalization functor, Schapira-Takeuchi [16] constructed later the same sheaf. In [2] the author also gave elementary reconstruction of the sheaf $\tilde{\mathcal{C}}_V^2$ of small 2-microfunctions based on the idea of K. Kataoka. Using our construction of $\tilde{\mathcal{C}}_V^2$, we reached a result of the theorem of supports, that is, we gave a simple sufficient condition under which a solution complex with coefficients in $\tilde{\mathcal{C}}_V^2$ vanishes locally in the derived category. Our construction of $\tilde{\mathcal{C}}_V^2$ enabled us to estimate the support of solution complexes.

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Let M be an open subset of $\mathbb{R}^n$ with coordinates $x = (x_1, \dots, x_n)$, X a complex neighborhood of M in $\mathbb{C}^n$ with coordinates $z = (z_1, \dots, z_n)$, and $T^*_M X$ the conormal bundle of M in the cotangent vector bundle T^*X of X. We take coordinates of $T^*_M X (\simeq \sqrt{-1}T^*M)$ as $(x, \sqrt{-1}\xi \cdot dx)$ with $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Let V and Σ be the following regular involutive and Lagrangian submanifolds of $T^*_M X$ respectively:

$$V = \left\{ (x, \sqrt{-1}\xi \cdot dx) \in \dot{T}^*_M X; \xi_1 = \cdots = \xi_{n-1} = 0 \right\},$$
$$\Sigma = \left\{ (x, \sqrt{-1}\xi \cdot dx) \in \dot{T}^*_M X; \xi_1 = \cdots = \xi_{n-1} = x_n = 0 \right\},$$

where $\dot{T}^*_M X = T^*_M X \setminus M$. Then we write $x = (x', x_n)$, $\xi = (\xi', \xi_n)$, etc. Let P be a differential operator with analytic coefficients defined on M. Let $p_\circ = (x_\circ, \sqrt{-1}\xi_\circ \cdot dx)$ be a point of Σ with $\sigma(P)(p_\circ) = 0$, where $\sigma(P)$ denotes the principal symbol of P. Assume P is transversally elliptic in a neighborhood of $p_\circ$, that is, P satisfies:

$$|\sigma(P)(x, \sqrt{-1}\xi/|\xi|)| \sim (|x_n| + |\xi'|/|\xi|)^l$$

for some positive integer l. Then Grigis-Schapira-Sjöstrand [4] has given a theorem on the propagation of analytic singularities for this operator P along the bicharacteristic leaf of V passing through $p_\circ$.

On the other hand, we assume:

$$|\sigma(P)(x, \sqrt{-1}\xi/|\xi|)| \sim (|x_n|^k + |\xi'|/|\xi|)^l$$

for some positive integers k and l in a neighborhood of $p_\circ$. Then the author has proved in [2] unique solvability in $\tilde{\mathcal{C}}^2_V$ for this operator P. This result was obtained by using our elementary construction of $\tilde{\mathcal{C}}^2_V$ and the estimate of the support of solution complexes with coefficients in $\tilde{\mathcal{C}}^2_V$. In this case, the structure of solutions of $Pu = f$ in the sheaf $\mathcal{C}_M$ of Sato microfunctions is reduced to that in the sheaf $\mathcal{A}^2_V$ of 2-analytic functions. Therefore our result implies the above theorem due to Grigis-Schapira-Sjöstrand [4] because any section of $\mathcal{A}^2_V$ has the property of the uniqueness of analytic continuation along the bicharacteristic leaves of V. The principal symbol of P studied in [2] is written as:

$$\sigma(P)(x, \xi) = \sum_{|\alpha|=l} a_\alpha(x, \xi)(\xi')^{\alpha'}(x_n)^{k\alpha_n}$$

in a neighborhood of $x_\circ \in M$. Here $a_\alpha(x,\xi)$ are real analytic functions and homogeneous in ξ of degree $m-|\alpha'|$, $\alpha = (\alpha', \alpha_n) = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

As for the property of solvability for those operators, our previous result of [2] is not sufficient. Funakoshi-Kataoka [3] proved solvability for similar operators by using the theory of the Szegö kernel. Wakabayashi [20] also proved local solvability of micro-hyperbolic operators and some second order operators in a different way.

Let $p_\circ = (x_\circ, \sqrt{-1}\xi_\circ \cdot dx)$ be any point of Σ. In connection with those operators we consider the following differential operator of order m with analytic coefficients defined on M:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha| \le m} a_\alpha(x) D_{x'}^{\alpha'} (x_n D_{x_n})^{\alpha_n},$$

where $D_x^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = D_{x_j} = \partial/\partial x_j$. One makes the hypothesis:

$$a_{(0,\dots,0,m)}(x_\circ) \neq 0.$$

Then we have:

$$\mathrm{Ker}(\mathcal{A}_V^2 \underset{P}{\longrightarrow} \mathcal{A}_V^2)_{p_\circ} \subset \mathcal{C}_{Y|X}^{\mathbb{R}}{}_{,p_\circ},$$

where $Y = \{z \in X; z_n = 0\}$, and $\mathcal{C}_{Y|X}^{\mathbb{R}}$ is the sheaf defined by Sato-Kawai-Kashiwara [15]. And furthermore, one makes the hypothesis:

$$a_{(m,0,\dots,0)}(x_\circ) \neq 0.$$

Then we get results of solvability of $Pu = f$ in $\mathcal{A}_V^2$ at $p_\circ$ on some suitable condition of $f \in \mathcal{A}_V^2$.

In this paper, we show these theorems in the following way. For the theorem of the kernel of P, we continue analytically a defining function of a 2-analytic function by means of the Cauchy-Kowalewski theorem. For the theorem of solvability, we turn a defining function of a 2-analytic function into the form of integral representation by means of the Fourier transformation. This is easier to deal with than the original form. For that purpose one extends a domain of the defining function by using the method due to Hörmander [5] so that some growth condition holds. Next, regarding the variable of integration as a parameter, we consider the differential equation with the parameter. Then we get a real solution by superposing a solution with respect to the parameter in the end. At this time, the solution with the parameter needs to be infra-exponential, that is, slowly increasing. For this purpose we find out an approximate solution and estimate the remainder by means of majorant series in the Cauchy-Kowalewski theorem with the parameter.

We give the plan of this paper.

Section 2 is preliminaries of subsequent sections. We review the theory of 2-microlocal analysis, some results about transversally elliptic operators, L^2 estimates and existence theorems for the $\bar{\partial}$ operator due to Hörmander [5], etc.

In Section 3, we give the theorem of solvability of $Pu = f$ in $\mathcal{A}^2_V$. We also give the theorem of the kernel of $P\colon \mathcal{A}^2_V \to \mathcal{A}^2_V$.

In Section 4, we give the proof of the main theorem of solvability, which is decomposed into several steps.

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2. Preliminaries

2.1. 2-microlocal analysis

Let M be an open subset of $\mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_n)$ and X a complex neighborhood of M in $\mathbb{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$. One denotes by $\mathcal{O}_X$ the sheaf of rings of holomorphic functions on X. Let (z, ζ) be the associated coordinates on T^*X with $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$. Then $(x, \sqrt{-1}\xi \cdot dx)$ denotes a point of a conormal bundle $T^*_M X (\simeq \sqrt{-1}T^*M)$ with $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Let V be the following regular involutive submanifold of $T^*_M X$:

$$V = \left\{ (x, \sqrt{-1}\xi \cdot dx) \in \dot{T}^*_M X; \xi_1 = \cdots = \xi_d = 0 \right\} \qquad (1 \le d < n),$$

where $\dot{T}^*_M X = T^*_M X \setminus M$. We put $x = (x', x'')$ with $x' = (x_1, \ldots, x_d)$, $x'' = (x_{d+1}, \ldots, x_n)$, $z = (z', z'')$ with $z' = (z_1, \ldots, z_d)$, $z'' = (z_{d+1}, \ldots, z_n)$, and $\xi = (\xi', \xi'')$ with $\xi' = (\xi_1, \ldots, \xi_d)$, $\xi'' = (\xi_{d+1}, \ldots, \xi_n)$. We set, moreover,

$$N = \left\{ z \in X; \operatorname{Im} z'' = 0 \right\},$$
$$\tilde{V} = T^*_N X.$$

This space $\tilde{V}$ is called a partial complexification of V. It is equipped with the sheaf

$$\mathcal{C}_{\tilde{V}} = \mu_N(\mathcal{O}_X)[n-d]$$

of microfunctions with holomorphic parameters z', where μ_N denotes the functor of Sato's microlocalization along N. Refer to Kashiwara-Schapira [9, 10]. And furthermore, we set

$$Y = \left\{ z \in X; z'' = 0 \right\},$$

and the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of microfunctions on a complex submanifold Y:

$$\mathcal{C}^{\mathbb{R}}_{Y|X} = \mu_Y(\mathcal{O}_X)[n-d].$$

M. Kashiwara constructed the sheaf $\mathcal{C}^2_V$ of 2-microfunctions along V on $T^*_V\tilde{V}$ by

$$\mathcal{C}^2_V = \mu_V(\mathcal{C}_{\tilde{V}})[d].$$

We also define

$$\begin{aligned} \mathcal{A}^2_V &= \mathcal{C}_{\tilde{V}}|_V, \\ \mathcal{B}^2_V &= R\varGamma_V(\mathcal{C}_{\tilde{V}})[d] = \mathcal{C}^2_V|_V. \end{aligned}$$

We call $\mathcal{A}^2_V$ the sheaf of 2-analytic functions along V and $\mathcal{B}^2_V$ the sheaf of 2-hyperfunctions along V. Note that these complexes $\mathcal{C}_{\tilde{V}}$, $\mathcal{C}^{\mathbb{R}}_{Y|X}$, $\mathcal{C}^2_V$ and $\mathcal{B}^2_V$ are concentrated in degree 0.

Concerning $\mathcal{C}^2_V$, there are fundamental exact sequences on V:

$$0 \longrightarrow \mathcal{A}^2_V \longrightarrow \mathcal{B}^2_V \longrightarrow \dot{\pi}_{V*}\left(\mathcal{C}^2_V|_{\dot{T}^*_V\tilde{V}}\right) \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow \mathcal{C}_M|_V \longrightarrow \mathcal{B}^2_V. \tag{2.2}$$

Here $\dot{\pi}_V$ is the restriction of the projection $\pi_V\colon T^*_V\tilde{V} \to V$ to $\dot{T}^*_V\tilde{V}$, and $\mathcal{C}_M(=\mu_M(\mathcal{O}_X)[n])$ is the sheaf of Sato microfunctions on M constructed in Sato-Kawai-Kashiwara [15]. We note that the inclusion $\mathcal{C}_M|_V \hookrightarrow \mathcal{B}^2_V$ is not surjective. Refer to Kataoka-Okada-Tose [11] for a counter-example.

Moreover there exists the canonical spectrum map

$$\mathrm{Sp}^2_V\colon \pi_V^{-1}\mathcal{B}^2_V \longrightarrow \mathcal{C}^2_V$$

on $T^*_V\tilde{V}$. By using Sp^2_V we define

$$\mathrm{SS}^2_V(u) = \mathrm{supp}\left(\mathrm{Sp}^2_V(u)\right)$$

for $u \in \mathcal{B}^2_V$. This subset $\mathrm{SS}^2_V(u)$ is called the second singular spectrum of u along V.

Refer to Kashiwara-Laurent [8] for more details.

The exact sequence (2.1) shows that $\mathcal{C}^2_V$ gives second microlocal analytic singularities for sections of $\mathcal{B}^2_V$. This sheaf $\mathcal{C}^2_V$ is, however, too large to study microfunction solutions of a class of differential equations because the inclusion in (2.2) is not surjective.

For this reason Kataoka-Tose [12] constructed the subsheaf $\overset{\circ}{\mathcal{C}}{}^2_V$ of $\mathcal{C}^2_V|_{\dot{T}^*_V\tilde{V}}$ satisfying the exact sequence

$$0 \longrightarrow \mathcal{A}^2_V \longrightarrow \mathcal{C}_M|_V \longrightarrow \dot{\pi}_{V*}\overset{\circ}{\mathcal{C}}{}^2_V \longrightarrow 0 \tag{2.3}$$

by using the comonoidal transformation. Kataoka-Okada-Tose [11] also constructed $\tilde{\mathcal{C}}^2_V$ as the image sheaf of the morphism

$$\dot{\pi}_V^{-1}(\mathcal{C}_M|_V) \longrightarrow \mathcal{C}^2_V|_{\dot{T}^*_V\tilde{V}}$$

to have the same exact sequence as (2.3). Schapira-Takeuchi [16] constructed later the same sheaf

$$\mathcal{C}_{MN} = \mu_{MN}(\mathcal{O}_X)[n]$$

by using the functor μ_{MN} of Schapira-Takeuchi's bimicrolocalization. Refer also to Takeuchi [19] for details.

Also, the author has given in [2] elementary reconstruction of $\tilde{\mathcal{C}}^2_V$ based on the idea of K. Kataoka in order to show the theorem of supports. One sets

$$\begin{aligned}
\tilde{X} &= X \times (\mathbb{R}^d \setminus \{0\}),\\
H_c &= \left\{(z,\xi') \in \tilde{X}; \langle \mathrm{Im}\, z', \xi'\rangle \le c|\mathrm{Im}\, z''|\right\},\\
G &= \left\{(z,\xi') \in \tilde{X}; \mathrm{Im}\, z'' = 0\right\}
\end{aligned}$$

for $c > 0$, where $\langle \operatorname{Im} z', \xi' \rangle = \operatorname{Im} z_1 \cdot \xi_1 + \cdots + \operatorname{Im} z_d \cdot \xi_d$. We identify

$$\left\{(z', x'', \xi', \sqrt{-1}\xi'' \cdot dx'') \in \dot{T}^*_G \tilde{X}; \operatorname{Im} z' = 0\right\}$$

with $\dot{T}^*_V \tilde{V}$ through the correspondence

$$(x, \xi', \sqrt{-1}\xi'' \cdot dx'') \longleftrightarrow (x, \sqrt{-1}\xi'' \cdot dx'', \sqrt{-1}\xi' \cdot dx').$$

We denote by p the projection $\tilde{X} \to X$ and by i the inclusion $\dot{T}^*_V \tilde{V} \hookrightarrow T^*_G \tilde{X}$.

Definition 2.1. One sets

$$\tilde{\mathcal{C}}^2_V = \varinjlim_c H^n \left(i^{-1} \mu_G R\Gamma_{H_c}(p^{-1}\mathcal{O}_X)\right),$$

where c tends to $+\infty$. One calls $\tilde{\mathcal{C}}^2_V$ the sheaf of small 2-microfunctions along V.

Then we have the essential exact sequence:

$$0 \longrightarrow \mathcal{A}^2_V \longrightarrow \mathcal{C}_M|_V \longrightarrow \dot{\pi}_{V*}\tilde{\mathcal{C}}^2_V \longrightarrow 0. \tag{2.4}$$

We take the following regular involutive submanifold of T^*X:

$$V^{\mathbb{C}} = \left\{(z, \zeta \cdot dz) \in \dot{T}^*X; \zeta_1 = \cdots = \zeta_d = 0\right\} \qquad (1 \le d < n),$$

where $\dot{T}^*X = T^*X \setminus X$. We put $\zeta = (\zeta', \zeta'')$ with $\zeta' = (\zeta_1, \ldots, \zeta_d)$, $\zeta'' = (\zeta_{d+1}, \ldots, \zeta_n)$. For this space, one sets $\tilde{V}^{\mathbb{C}} \subset T^*(X \times X)$ as in Laurent [13]. We note that $V^{\mathbb{C}}$ and $T^*_{V^{\mathbb{C}}}\tilde{V}^{\mathbb{C}}$ are complexifications of V and $T^*_V \tilde{V}$ respectively.

We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators on X. We also denote by $\mathcal{E}_X$ the sheaf of rings of microdifferential operators and by $\mathcal{E}^2_{V^{\mathbb{C}}}$ the sheaf of rings of 2-microdifferential operators. We denote, moreover, by $\sigma(P)$ (resp. $\sigma_{V^{\mathbb{C}}}(P)$) the principal symbol of a microdifferential (resp. 2-microdifferential) operator P. We can regard a microdifferential operator P as a 2-microdifferential operator in a neighborhood of a point of $V^{\mathbb{C}}$:

$$\mathcal{E}_X|_{V^{\mathbb{C}}} \xrightarrow{\sim} \mathcal{D}^2_{V^{\mathbb{C}}} := \mathcal{E}^2_{V^{\mathbb{C}}}|_{V^{\mathbb{C}}}.$$

For this operator P, $\sigma_{V^{\mathbb{C}}}(P)$ is the lowest degree term of the Taylor expansion of $\sigma(P)$ along $V^{\mathbb{C}}$:

$$\begin{cases} \sigma(P)(z,\zeta) = \displaystyle\sum_{|\alpha|\geq m} a_\alpha(z,\zeta'')\zeta'^{\alpha}, \\ \sigma_{V^{\mathbb{C}}}(P)(z,\zeta'',z'^{*}) = \displaystyle\sum_{|\alpha|=m} a_\alpha(z,\zeta'')z'^{*\alpha}. \end{cases}$$

Let U be an open subset of $T^*_{V^{\mathbb{C}}}\tilde{V}^{\mathbb{C}}$. Then, for a 2-microdifferential operator $P \in \mathcal{E}^2_{V^{\mathbb{C}}}(U)$, P is invertible on U if and only if $\sigma_{V^{\mathbb{C}}}(P) \neq 0$ on U.

Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined in a neighborhood of a point of V. One says that $\mathcal{M}$ is *partially elliptic* along V if:

$$\mathrm{Ch}^2_{V^{\mathbb{C}}}(\mathcal{M}) \cap \dot{T}^*_V\tilde{V} = \emptyset.$$

Here the subset $\mathrm{Ch}^2_{V^{\mathbb{C}}}(\mathcal{M})$ of $T^*_{V^{\mathbb{C}}}\tilde{V}^{\mathbb{C}}$ is the microcharacteristic variety of $\mathcal{M}$ along $V^{\mathbb{C}}$. Next let P be a microdifferential operator defined in a neighborhood of a point of V, which is partially elliptic along V. Since this operator P induces an isomorphism $P\colon \mathcal{C}^2_V \xrightarrow{\sim} \mathcal{C}^2_V$, any microfunction (or any 2-hyperfunction) solution of the equation $Pu = 0$ always belongs to $\mathcal{A}^2_V$.

Refer to Laurent [13] and Bony-Schapira [1] for more details.

Let $\mathcal{M}$ be an arbitrary coherent $\mathcal{D}_X$-module, and we denote by $\mathrm{char}(\mathcal{M})$ the characteristic variety of $\mathcal{M}$. Assume $d+1=n$, that is to say, a regular involutive submanifold V is defined by $\xi_1 = \cdots = \xi_{n-1} = 0$. In this case, the author has obtained an estimate of the support of solution complexes with coefficients in $\tilde{\mathcal{C}}^2_V$.

THEOREM 2.2. *Let* $q_\circ = (x_\circ, \pm\sqrt{-1}\,dx_n, \sqrt{-1}\eta'_\circ \cdot dx')$ *be a point of* $\dot{T}^*_V\tilde{V}$. *Then we have*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}^2_V)_{q_\circ} = 0$$

if there exists a positive constant δ *such that*

$$\Big\{(z, (\xi' + \sqrt{-1}\varepsilon\eta')\cdot dz' \pm (\xi_n + \sqrt{-1})\,dz_n) \in T^*X; \\ |z - x_\circ| + |\eta' - \eta'_\circ| < \delta, \quad |\operatorname{Im} z_n| + |\xi| < \varepsilon\delta\Big\} \cap \mathrm{char}(\mathcal{M}) = \emptyset$$

for any ε *with* $0 < \varepsilon < \delta$.

In the proof of Theorem 2.2, we have made use of our construction of $\tilde{\mathcal{C}}_V^2$ and the method of micro-support due to Kashiwara-Schapira [9, 10]. Refer to [2] for the proof of Theorem 2.2.

2.2. Transversally elliptic operators

We review in this subsection some results on transversally elliptic operators. Let $\Sigma \subset V$ be the following Lagrangian submanifold of T_M^*X:

$$\Sigma = \left\{(x, \sqrt{-1}\xi \cdot dx) \in \dot{T}_M^*X; \xi_1 = \cdots = \xi_d = x_{d+1} = \cdots = x_n = 0\right\}.$$

Let P be a differential operator with analytic coefficients defined on M. Let $p_\circ = (x_\circ, \sqrt{-1}\xi_\circ \cdot dx)$ be a point of Σ with $\sigma(P)(p_\circ) = 0$, where $\sigma(P)$ denotes the principal symbol of P.

Theorem 2.3 (Grigis-Schapira-Sjöstrand [4]). *Let Γ_0 be the bicharacteristic leaf of V passing through $p_\circ$. We suppose that for some positive integer l*

$$|\sigma(P)(x, \sqrt{-1}\xi/|\xi|)| \sim (|x''| + |\xi'|/|\xi|)^l \tag{2.5}$$

in a neighborhood W of $p_\circ$ such that $\Gamma_0 \cap W$ is connected. If u is a distribution defined on M, such that $\Gamma_0 \cap \mathrm{WF}_a(Pu) = \emptyset$, then either $\Gamma_0 \cap W \subset \mathrm{WF}_a(u)$ or

$$\Gamma_0 \cap W \cap \mathrm{WF}_a(u) = \emptyset.$$

The operator which satisfies (2.5) is called transversally elliptic in a neighborhood of $p_\circ$. Refer also to Sjöstrand [17, 18].

On the other hand, we consider a case where $d + 1 = n$, that is, V and Σ are defined respectively by $\xi_1 = \cdots = \xi_{n-1} = 0$ and $\xi_1 = \cdots = \xi_{n-1} = x_n = 0$. In this case the author has proved in [2] unique solvability in $\tilde{\mathcal{C}}_V^2$ for some class of partial differential operators by using Theorem 2.2.

Theorem 2.4. *One sets $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. We suppose that for some positive integers k and l*

$$|\sigma(P)(x, \sqrt{-1}\xi/|\xi|)| \sim (|x_n|^k + |\xi'|/|\xi|)^l$$

in a neighborhood of $p_\circ$*. Then for any* $q_\circ \in \dot{\pi}_V^{-1}(p_\circ)$*, one has*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_V^2)_{q_\circ} = 0.$$

In the situation of Theorem 2.4, we can get the following isomorphism in a neighborhood of $p_\circ \in \Sigma$ in V by the fundamental exact sequence (2.4):

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_V^2) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M|_V).$$

This shows that the structure of solutions of $Pu = f$ in $\mathcal{C}_M|_V$ has been reduced to that in $\mathcal{A}_V^2$. Hence, any microfunction solution of $Pu = 0$ always belongs to $\mathcal{A}_V^2$. Since any section of $\mathcal{A}_V^2$ has the property of the uniqueness of analytic continuation along the bicharacteristic leaves of V, we find that Theorem 2.4 implies Theorem 2.3.

Note that the principal symbol of P in Theorem 2.4 is written:

$$\sigma(P)(x,\xi) = \sum_{|\alpha|=l} a_\alpha(x,\xi)(\xi')^{\alpha'}(x_n)^{k\alpha_n}$$

in a neighborhood of $x_\circ \in M$. Here $a_\alpha(x,\xi)$ are real analytic functions and homogeneous in ξ of degree $m - |\alpha'|$, $\alpha = (\alpha', \alpha_n) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

2.3. L^2 estimates and existence theorems for the $\bar{\partial}$ operator

Let Ω be an open set in $\mathbb{C}^n$ and φ a real-valued continuous function in Ω. Recall the $L^2(\Omega, \varphi)$ space of Hörmander, that is, $f \in L^2(\Omega, \varphi)$ if and only if

$$\|f\|_\varphi^2 := \int |f|^2 e^{-\varphi}\, dV < \infty.$$

Here the symbol dV is the standard Euclidean volume form on $\mathbb{C}^n$:

$$dV = dV(z) = (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n).$$

This is a subspace of the space $L^2(\Omega, \mathrm{loc})$ of functions in Ω which are locally square integrable with respect to the Lebesgue measure, and it is clear that every function in $L^2(\Omega, \mathrm{loc})$ belongs to $L^2(\Omega, \varphi)$ for some φ. By $L^2_{(p,q)}(\Omega, \varphi)$ we denote the space of forms of type (p, q) with coefficients in $L^2(\Omega, \varphi)$,

$$f = \sum_{|I|=p}{}' \sum_{|J|=q}{}' f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. We set

$$|f|^2 = \sum_{I,J}{}' |f_{I,J}|^2,$$

and

$$\|f\|^2_\varphi = \int |f|^2 e^{-\varphi}\, dV = \sum_{I,J}{}' \|f_{I,J}\|^2_\varphi.$$

Note that $L^2(\Omega, \varphi)$ is a Hilbert space with this norm. Similarly one defines the space $L^2_{(p,q)}(\Omega, \mathrm{loc})$.

The following theorem is of fundamental importance.

Theorem 2.5 (Hörmander [5]). *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ any plurisubharmonic function in Ω. For every $g \in L^2_{(p,q+1)}(\Omega, \varphi)$ with $\bar\partial g = 0$ there is a solution $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ of the equation $\bar\partial u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi}(1 + |z|^2)^{-2}\, dV \leq \int_\Omega |g|^2 e^{-\varphi}\, dV.$$

3. Solvability in the Sheaf of Microfunctions with Holomorphic Parameters

In this section, we give the theorem of solvability of some class of differential equations in the sheaf of 2-analytic functions, that is, microfunctions with holomorphic parameters. In Theorems 2.3 and 2.4, we have seen the propagation of analytic singularities for each operator along the bicharacteristic leaves of the regular involutive submanifold. However, they are not sufficient to get results of solvability for these operators.

Assume $d+1=n$ in the notation of the previous subsections. We shall use the following notation. Let M be an open subset of $\mathbb{R}^n$ with coordinates $x=(x_1,\dots,x_n)$, X a complex neighborhood of M in $\mathbb{C}^n$ with coordinates $z=(z_1,\dots,z_n)$, and $Y=\{z\in X; z_n=0\}$. The letter $\mathbb{N}$ denotes the set of non-negative integers. We write $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha!=\alpha_1!\dots\alpha_n!$, $D_x^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$, and $D_j=D_{x_j}=\partial/\partial x_j$ for $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$. Let V and Σ be the following regular involutive and Lagrangian submanifolds of T_M^*X respectively:

$$V=\left\{(x,\sqrt{-1}\xi\cdot dx)\in \dot{T}_M^*X; \xi_1=\cdots=\xi_{n-1}=0\right\},$$
$$\Sigma=\left\{(x,\sqrt{-1}\xi\cdot dx)\in \dot{T}_M^*X; \xi_1=\cdots=\xi_{n-1}=x_n=0\right\},$$

where $\dot{T}_M^*X=T_M^*X\setminus M$. So, we put $x=(x',x_n)$ with $x'=(x_1,\dots,x_{n-1})$, $z=(z',z_n)$ with $z'=(z_1,\dots,z_{n-1})$, $\xi=(\xi',\xi_n)$ with $\xi'=(\xi_1,\dots,\xi_{n-1})$, and $\alpha=(\alpha',\alpha_n)$ with $\alpha'=(\alpha_1,\dots,\alpha_{n-1})$.

Let $p_\circ$ be any point of Σ, and let $p_\circ=(x_\circ,\sqrt{-1}\xi_\circ\cdot dx)$, $x_\circ=(x'_\circ,0)=(x_{\circ,1},\dots,x_{\circ,n-1},0)$, $\xi_\circ=(0,\xi_{\circ,n})$. In connection with the transversally elliptic operator in Subsection 2.2, we consider the following differential operator of order m with analytic coefficients defined on M:

$$P(x,D_{x'},x_nD_{x_n})=\sum_{|\alpha|\le m}a_\alpha(x)D_{x'}^{\alpha'}(x_nD_{x_n})^{\alpha_n}. \tag{3.1}$$

Assume P is the restriction to M of a holomorphic differential operator of order m defined on X:

$$P(z,D_{z'},z_nD_{z_n})=\sum_{|\alpha|\le m}a_\alpha(z)D_{z'}^{\alpha'}(z_nD_{z_n})^{\alpha_n}.$$

Note that

$$\begin{cases}\sigma(P)(z,\zeta)=\displaystyle\sum_{|\alpha|=m}a_\alpha(z){\zeta'}^{\alpha'}(z_n\zeta_n)^{\alpha_n},\\ \sigma_{V^{\mathbb{C}}}(P)(z,\zeta_n,z'^*)=a_{(0,\dots,0,m)}(z)(z_n\zeta_n)^m\end{cases}$$

provided $a_{(0,\dots,0,m)}(x_\circ)\neq 0$. Hence, one cannot apply Bony-Schapira's theory to this operator, since P is not partially elliptic along V. See Subsection 2.1 and Bony-Schapira [1].

Now recall that $\mathcal{A}^2_V = \mathcal{C}_{\tilde{V}}|_V = H^1(\mu_N(\mathcal{O}_X))|_V$ and that

$$\begin{aligned}\mathcal{A}^2_{V,p_\circ} &\simeq H^1_Z(\mathcal{O}_X)_{x_\circ} \\ &\simeq \varinjlim_{r>0} \mathcal{O}(D^{n-1}_r \times U_r)/\mathcal{O}(D^n_r).\end{aligned} \tag{3.2}$$

Here we have set the closed subset $Z \subset X$, the open polydisc $D^k_r \subset \mathbb{C}^k$ and the open subset $U_r \subset \mathbb{C}$ respectively by:

$$\begin{aligned} Z &= \{z \in X; \operatorname{Im}(\xi_{\circ,n} z_n) \le 0\}, \\ D^k_r &= \{z \in \mathbb{C}^k; |z_j - x_{\circ,j}| < r, j = 1, \dots, k\}, \\ U_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im}(\xi_{\circ,n} z_n) > 0\} \end{aligned}$$

for $k \le n$ and $r > 0$. Then any germ $f(x) \in \mathcal{A}^2_{V,p_\circ}$ is obtained as boundary value of a holomorphic function:

$$f(x) = b_{D^{n-1}_r \times U_r}(F(z)), \tag{3.3}$$

where $F(z) \in \mathcal{O}(D^{n-1}_r \times U_r)$ for some $r > 0$.

Recall, moreover, the sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of microfunctions on Y defined by: $\mathcal{C}^{\mathbb{R}}_{Y|X} = H^1(\mu_Y(\mathcal{O}_X))$. The stalk of $\mathcal{C}^{\mathbb{R}}_{Y|X}$ at $p_\circ \in \Sigma$ is also written:

$$\begin{aligned}\mathcal{C}^{\mathbb{R}}_{Y|X,p_\circ} &\simeq \varinjlim_{r>0} H^1_{Z_r}(\mathcal{O}_X)_{x_\circ} \\ &\simeq \varinjlim_{r>0} \mathcal{O}(D^{n-1}_r \times V_r)/\mathcal{O}(D^n_r).\end{aligned} \tag{3.4}$$

Here we have set the closed subset $Z_r \subset X$ and the open subset $V_r \subset \mathbb{C}$ respectively by:

$$\begin{aligned} Z_r &= \{z \in X; \operatorname{Im}(\xi_{\circ,n} z_n) \le -r|\operatorname{Re}(\xi_{\circ,n} z_n)|\}, \\ V_r &= \{z_n \in \mathbb{C}; |z_n| < r, \operatorname{Im}(\xi_{\circ,n} z_n) > -r|\operatorname{Re}(\xi_{\circ,n} z_n)|\}. \end{aligned}$$

Also any germ $f(x) \in \mathcal{C}^{\mathbb{R}}_{Y|X,p_\circ}$ is obtained as boundary value of a holomorphic function:

$$f(x) = b_{D^{n-1}_r \times V_r}(F(z)),$$

where $F(z) \in \mathcal{O}(D^{n-1}_r \times V_r)$ for some $r > 0$.

It is clear by the definitions that there exists the exact sequence on Σ concerning these sheaves:

$$0 \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X}|_{\Sigma} \longrightarrow \mathcal{A}^2_V|_{\Sigma}.$$

Now one makes the hypothesis:

$$a_{(0,\dots,0,m)}(x_\circ) \neq 0. \tag{3.5}$$

We can prove the following theorem of the kernel of $P\colon \mathcal{A}^2_V \to \mathcal{A}^2_V$.

THEOREM 3.1. *Assume* (3.5) *for the differential operator* (3.1). *Then:*

$$\mathrm{Ker}(\mathcal{A}^2_V \underset{P}{\longrightarrow} \mathcal{A}^2_V)_{p_\circ} \subset \mathcal{C}^{\mathbb{R}}_{Y|X,p_\circ}.$$

PROOF. We can suppose from the beginning that $x_\circ = 0$, $\xi_\circ = (0, \dots, 0, 1)$. The 2-analytic function $u \in \mathrm{Ker}(\mathcal{A}^2_V \underset{P}{\longrightarrow} \mathcal{A}^2_V)_{p_\circ}$ is written as boundary value of a holomorphic function:

$$u(x) = b_{D_r^{n-1} \times U_r}(U(z)),$$

where $U(z) \in \mathcal{O}(D_r^{n-1} \times U_r)$ for some small $r > 0$. Then one has by the assumption

$$P(x, D_{x'}, x_n D_{x_n})u(x) = b_{D_r^{n-1} \times U_r}(P(z, D_{z'}, z_n D_{z_n})U(z)) = 0$$

in the space $\mathcal{A}^2_{V,p_\circ}$. Therefore the holomorphic function

$$F(z) := P(z, D_{z'}, z_n D_{z_n})U(z) \tag{3.6}$$

can be also an element of $\mathcal{O}(D^n_{r_1})$ for a sufficiently small $r_1 > 0$ by the isomorphisms in (3.2). We can assume that $a_{(0,\dots,0,m)}(z) \equiv 1$ from the hypothesis (3.5) in advance.

One introduces the new local coordinates $(z', w) = (z_1, \dots, z_{n-1}, w)$ with

$$w = \log z_n, \quad -\frac{1}{2}\pi < \arg z_n < \frac{3}{2}\pi.$$

By this local coordinate system, our differential equation (3.6) is turned into:

$$P(z', e^w, D_{z'}, D_w)U(z', e^w) = F(z', e^w)$$

on the unbounded domain $\{(z', w); z' \in D^{n-1}_{r_1}, \operatorname{Re} w < \log r_1, 0 < \operatorname{Im} w < \pi\}$, where the operator P is written as:

$$P(z', e^w, D_{z'}, D_w) = \sum_{|\alpha| \le m} a_\alpha(z', e^w) D^{\alpha'}_{z'} D^{\alpha_n}_w.$$

Consider the Cauchy problem:

$$\tag{3.7} \begin{cases} P(z', e^w, D_{z'}, D_w)V(z', w) = F(z', e^w), \\ D^j_w V = D^j_w U \quad \text{when } w = w_\circ,\ 0 \le j < m \end{cases}$$

for any fixed $w_\circ \in \mathbb{C}$ with $\operatorname{Re} w_\circ < \log r_1 - 1$, $0 < \operatorname{Im} w_\circ < \pi$. Set:

$$\begin{aligned} \Omega_{w_\circ, r} &= \{(z', w) \in \mathbb{C}^n; |z_1| + \cdots + |z_{n-1}| + |w - w_\circ| < r\}, \\ \Omega_{w_\circ, L, r} &= \{(z', w) \in \mathbb{C}^n; |z_1| + \cdots + |z_{n-1}| + L|w - w_\circ| < r\}, \\ \Omega'_r &= \{z' \in \mathbb{C}^{n-1}; |z_1| + \cdots + |z_{n-1}| < r\}. \end{aligned}$$

Note that $a_\alpha(z', e^w) \in \mathcal{O}(\Omega_{w_\circ, r})$, $F(z', e^w) \in \mathcal{O}(\Omega_{w_\circ, r_1})$, and that $D^j_w U|_{w=w_\circ} \in \mathcal{O}(\Omega'_{r_1})$. By the Cauchy-Kowalewski theorem, there exists a unique solution $V(z', w)$ of the Cauchy problem (3.7) that is holomorphic at $(z', w) = (0, w_\circ)$. Moreover, this solution $V(z', w)$ is holomorphic on $\Omega_{w_\circ, L, r_1}$, where L is a constant with $L \ge 1$. Note that we can choose the constant L independent of $w_\circ$, because there exists a constant M such that

$$|a_\alpha(z', e^w)| \le M$$

for any $w_\circ$, $(z', w) \in \bar{\Omega}_{w_\circ, r/2}$, and any α. See Ōshima-Komatsu [14] for the detailed way to choose the constant L.

Then $U(z', e^w)$ extends holomorphically to the domain

$$\left\{(z', w) \in \mathbb{C}^n; z' \in D^{n-1}_{r_2}, \operatorname{Re} w < \log r_1 - 1, -\delta < \operatorname{Im} w < \pi + \delta\right\}$$

through the unique solution $V(z', w)$ of the Cauchy problem (3.7), where r_2 and δ are sufficiently small positive constants. By using the original system of local coordinates, the function $U(z)$ is holomorphic on $D_{r_3}^{n-1} \times V_{r_3}$ for a small $r_3 > 0$. Hence one obtains

$$u(x) = b_{D_{r_3}^{n-1} \times U_{r_3}}(U(z)) = b_{D_{r_3}^{n-1} \times V_{r_3}}(U(z)) \in \mathcal{C}^{\mathbb{R}}_{Y|X,p_\circ}$$

by the isomorphisms in (3.4). $\square$

Now one makes the hypothesis:

$$\begin{cases} a_{(m,0,\dots,0)}(x_\circ) \neq 0, \\ a_{(0,\dots,0,m)}(x_\circ) \neq 0. \end{cases} \tag{3.8}$$

One can obtain the following theorem on the solvability for the operator $P\colon \mathcal{A}^2_V \to \mathcal{A}^2_V$.

THEOREM 3.2. *Assume* (3.8) *for the differential operator* (3.1). *We assume, furthermore, a germ $f \in \mathcal{A}^2_{V,p_\circ}$ represented by* (3.3) *satisfies the following growth condition. There exist positive constants $p < 1$, C such that*

$$|F(z)| \leq C|\operatorname{Im} z_n|^{-p}, \quad z \in D_r^{n-1} \times U_r. \tag{3.9}$$

Then we can find a solution $u \in \mathcal{A}^2_{V,p_\circ}$ of $Pu = f$.

REMARK 3.3. Assume the real analytic functions a_α in (3.1) are constants on M for any α, and the complex constants a_α are not all equal to 0. In this case, for any given $f \in \mathcal{A}^2_{V,p_\circ}$ there exists a solution $u \in \mathcal{A}^2_{V,p_\circ}$ of $Pu = f$. Indeed, we can suppose $p_\circ = (0, \sqrt{-1}\,dx_n)$ and set $u = b_{D_r^{n-1} \times U_r}(U)$, $f = b_{D_r^{n-1} \times U_r}(F)$ for some $r > 0$. It suffices to consider the differential equation:

$$P(D_{z'}, z_n D_{z_n})U(z) = F(z) \tag{3.10}$$

on the complex domain $D_r^{n-1} \times U_r$. By using the system of local coordinates (z', w) in the proof of Theorem 3.1, (3.10) is reduced to the differential equation:

$$P(D_{z'}, D_w)U(z', e^w) = F(z', e^w) \tag{3.11}$$

with constant coefficients on $\{(z', w); z' \in D_r^{n-1}, \operatorname{Re} w < \log r, 0 < \operatorname{Im} w < \pi\}$. Note that this unbounded domain is $\mathbb{C}$ convex (i.e. the intersection of this open set and L is a connected and simply connected open subset of L for every affine complex line L). Therefore the equation (3.11) can be solved globally. Refer to Hörmander [6] for the notion of $\mathbb{C}$ convexity and global existence theorems for analytic differential equations with constant coefficients.

REMARK 3.4. For similar microdifferential equations, we cannot prove solvability by using the method of Section 4.

4. Proof of the Main Theorem

4.1. Integral representation of holomorphic functions

In order to show the existence of solutions in Theorem 3.2, we will make the following steps. First, we turn holomorphic functions into the form of integral representation which is easy to deal with by means of Fourier transformation. Secondly, regarding the variable of integration as a parameter, we solve the differential equation with the parameter. Then we get a real solution by superposing a solution with respect to the parameter. At this time, we have to find a infra-exponential solution with the parameter. For that purpose, we construct an approximate solution with infra-exponential growth order. Thirdly, we estimate the remainder by means of majorant series in the Cauchy-Kowalewski theorem with the parameter.

Now we can suppose $x_\circ = 0$, $\xi_\circ = (0, \dots, 0, 1)$, and $p_\circ = (0, \sqrt{-1}\, dx_n)$. Set:

$$U_r^\infty = \mathbb{P}^1 \setminus \{z_n \in \mathbb{C}; |z_n| \le r, \operatorname{Im} z_n \le 0\}\,.$$

Note that the open set $(D_r^{n-1} \times U_r^\infty) \cup D_r^n \subset \mathbb{C}^{n-1} \times \mathbb{P}^1$ is a Stein manifold. Therefore one can find functions $F_\infty \in \mathcal{O}(D_r^{n-1} \times U_r^\infty)$ and $F_0 \in \mathcal{O}(D_r^n)$ such that $F = F_\infty - F_0$ in $D_r^{n-1} \times U_r$ and $F_\infty(z', \infty) \equiv 0$ by the solvability of the first Cousin problem. Then one obtains:

$$f(x) = b_{D_r^{n-1} \times U_r}(F(z)) = b_{D_r^{n-1} \times U_r}(F_\infty(z))$$

by the isomorphisms in (3.2). In this way it is enough to consider $F_\infty \in \mathcal{O}(D_r^{n-1} \times U_r^\infty)$ which satisfies $F_\infty(z', \infty) \equiv 0$ instead of F.

Next, choose the system of local coordinates $(z', w) = (z_1, \dots, z_{n-1}, w)$ with

$$w = \log z_n, \quad 0 < \arg z_n < \pi,$$

and set $w = u + \sqrt{-1}v$. We choose a C^∞-function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \psi(v) \le 1$ for $v \in \mathbb{R}$, $\psi(v) = 0$ for $v \le \delta_\circ$ and $\psi(v) = 1$ for $v \ge \pi - \delta_\circ$, where $\delta_\circ > 0$ is a small constant. Using this function, we define:

$$G(z', w) = \frac{\partial}{\partial \bar{w}}(F_\infty(z', e^w)\psi(v)) = \frac{i}{2} F_\infty(z', e^w)\psi'(v)$$

for $0 < v < \pi$. We can consider $G(z', w)$ as a C^∞-function on $D_r^{n-1} \times \mathbb{C}$ by setting $G(z', w) \equiv 0$ for $\operatorname{Im} w \in \mathbb{R} \setminus (0, \pi)$.

LEMMA 4.1. *Let q be a positive constant with $0 < p < q < 1$, and set $\varphi(u) = -2qu$. Then $G(z', w) \in L^2(D_r^{n-1} \times \mathbb{C}, \varphi)$ by shrinking D_r^{n-1}, that is, one has*

$$\int_{D_r^{n-1} \times \mathbb{C}} |G(z', w)|^2 e^{-\varphi(u)}\, dV(z', w) < \infty.$$

PROOF. Because of the fact that $F_\infty \in \mathcal{O}(D_r^{n-1} \times U_r^\infty)$ and that $F_\infty(z', \infty) \equiv 0$, there exists a constant C_1 such that one has the inequality

$$|F_\infty(z)| \le C_1 |z_n|^{-1} \quad \text{for} \quad z' \in D_r^{n-1}, \quad |z_n| \gg 1.$$

Then we have the inequality:

$$|F_\infty(z', e^w)| \le C_1 e^{-\operatorname{Re} w} \quad \text{for} \quad z' \in D_r^{n-1}, \quad \operatorname{Re} w \gg 1. \tag{4.1}$$

On the other hand, by the condition (3.9) and the fact that $F = F_\infty - F_0$, there exists $C_2 > 0$ such that

$$|F_\infty(z', e^w)| \le C_2 |\operatorname{Im} e^w|^{-p} \le C_2 (e^{\operatorname{Re} w} \sin \delta_\circ)^{-p} = C_3 e^{-p \operatorname{Re} w} \tag{4.2}$$

for $z' \in D_r^{n-1}$, $\operatorname{Re} w \ll -1$, $\delta_\circ \le \operatorname{Im} w \le \pi - \delta_\circ$, where we set $C_3 = C_2 (\sin \delta_\circ)^{-p}$. Therefore we obtain

$$\begin{aligned} &\int_{D_r^{n-1} \times \mathbb{C}} |G(z', w)|^2 e^{-\varphi(u)}\, dV(z', w) \\ &\qquad \le C' \int_{D_r^{n-1} \times \{w \in \mathbb{C}; \delta_\circ \le v \le \pi - \delta_\circ\}} |F_\infty(z', e^w)|^2 e^{-\varphi(u)}\, dV(z', w). \end{aligned} \tag{4.3}$$

Here we set $C' = 1/4 \max_{\delta_\circ \leq v \leq \pi - \delta_\circ} |\psi'(v)|^2$. We get by the inequalities in (4.1) and (4.2),

$$|F_\infty(z', e^w)|^2 e^{-\varphi(u)} \leq \begin{cases} C_1^2 e^{-2(1-q)u} & \text{if } u \gg 1 \\ C_3^2 e^{2(q-p)u} & \text{if } u \ll -1. \end{cases}$$

Therefore one can find that the integral in the right-hand side of (4.3) is convergent. □

By Lemma 4.1, we can apply Theorem 2.5 due to Hörmander [5] to $G(z', w)\, d\bar{w} \in L^2_{(0,1)}(D_r^{n-1} \times \mathbb{C}, \varphi)$, that is to say, there is a solution $H(z', w) \in L^2(D_r^{n-1} \times \mathbb{C}, \mathrm{loc})$ of the equation $\bar{\partial} H = G\, d\bar{w}$ such that

$$\int_{D_r^{n-1} \times \mathbb{C}} |H|^2 e^{-\varphi} \left(1 + |(z', w)|^2\right)^{-2} dV \leq \int_{D_r^{n-1} \times \mathbb{C}} |G|^2 e^{-\varphi}\, dV.$$

In fact, $H \in L^2(D_r^{n-1} \times \mathbb{C}, \Phi)$, where $\Phi(z', w) := \varphi(u) + 2\log(1 + |(z', w)|^2)$.

Set:

$$\begin{aligned} V &= \{w \in \mathbb{C}; 0 < \operatorname{Im} w < \pi\}, \\ V_+ &= \{w \in \mathbb{C}; \operatorname{Im} w > 0\}, \\ V_- &= \{w \in \mathbb{C}; \operatorname{Im} w < \pi\}, \end{aligned}$$

and

$$\begin{aligned} F_+(z', w) &= F_\infty(z', e^w)(1 - \psi(v)) + H(z', w), \\ F_-(z', w) &= F_\infty(z', e^w)\psi(v) - H(z', w). \end{aligned}$$

We find immediately that $F_\pm \in \mathcal{O}(D_r^{n-1} \times V_\pm)$ and that

$$F_\infty(z', e^w) = F_+(z', w) + F_-(z', w) \quad \text{for} \quad (z', w) \in D_r^{n-1} \times V.$$

Indeed, one obtains on $D_r^{n-1} \times V_\pm$:

$$\begin{aligned} \bar{\partial}(F_+) &= \frac{\partial}{\partial \bar{w}}(F_\infty(1 - \psi))\, d\bar{w} + \bar{\partial} H = -G\, d\bar{w} + \bar{\partial} H = 0, \\ \bar{\partial}(F_-) &= \frac{\partial}{\partial \bar{w}}(F_\infty \psi)\, d\bar{w} - \bar{\partial} H = G\, d\bar{w} - \bar{\partial} H = 0. \end{aligned}$$

We study values of the holomorphic functions $F_\pm$ as $|w| \to \infty$. Let

$$V_{+\delta} = \{w \in \mathbb{C}; \operatorname{Im} w > \delta\},$$
$$V_{-\delta} = \{w \in \mathbb{C}; \operatorname{Im} w < \pi - \delta\}$$

for $\delta > 0$, and set $r_1 = r/2 > 0$.

PROPOSITION 4.2. *For any small positive δ there exists a positive constant C_δ such that one has*

$$|F_\pm(z', w)| \leq C_\delta \left| w^2 e^{-qw} \right| \tag{4.4}$$

for $(z', w) \in D_{r_1}^{n-1} \times V_{\pm\delta}$ with $|w| \gg 1$.

PROOF. First, one has

$$F_+|_{D_r^{n-1} \times V_{+\delta/2}} \in L^2(D_r^{n-1} \times V_{+\delta/2}, \Phi),$$
$$F_-|_{D_r^{n-1} \times V_{-\delta/2}} \in L^2(D_r^{n-1} \times V_{-\delta/2}, \Phi).$$

Indeed, we have the fact that $H \in L^2(D_r^{n-1} \times V_{\pm\delta/2}, \Phi)$ and the inequalities

$$\int_{D_r^{n-1} \times V_{+\delta/2}} |F_\infty(1-\psi)|^2 e^{-\Phi} \, dV$$
$$\leq \int_{D_r^{n-1} \times \{w \in \mathbb{C}; \delta/2 \leq v \leq \pi - \delta_\circ\}} |F_\infty|^2 e^{-\varphi} \, dV < \infty,$$
$$\int_{D_r^{n-1} \times V_{-\delta/2}} |F_\infty \psi|^2 e^{-\Phi} \, dV$$
$$\leq \int_{D_r^{n-1} \times \{w \in \mathbb{C}; \delta_\circ \leq v \leq \pi - \delta/2\}} |F_\infty|^2 e^{-\varphi} \, dV < \infty$$

in the same way as in the proof of Lemma 4.1.

Choose any point $(z'_\circ, w_\circ) \in D_{r_1}^{n-1} \times V_{\pm\delta}$. Since $F_\pm$ is holomorphic on $D_r^{n-1} \times V_\pm$, we have

$$F_\pm(z'_\circ, w_\circ) = \frac{1}{\operatorname{vol}(B((z'_\circ, w_\circ), \delta/2))} \int_{B((z'_\circ, w_\circ), \delta/2)} F_\pm(z', w) \, dV(z', w),$$

where we set the open ball:

$$B((z'_\circ, w_\circ), \delta/2) = \{(z', w) \in \mathbb{C}^n; |(z', w) - (z'_\circ, w_\circ)| < \delta/2\}.$$

Note that $\bar{B}((z'_\circ, w_\circ), \delta/2) \subset D_r^{n-1} \times V_{\pm\delta/2}$. Then we have the inequalities:

$$\begin{aligned}
&|F_\pm(z'_\circ, w_\circ)| \\
&\quad\le \frac{1}{\mathrm{vol}(B((z'_\circ, w_\circ), \delta/2))} \int_{B((z'_\circ, w_\circ),\delta/2)} |F_\pm| e^{-\Phi/2} e^{\Phi/2}\, dV \\
&\quad\le \frac{1}{\mathrm{vol}(B((z'_\circ, w_\circ), \delta/2))} \left(\int_{B((z'_\circ, w_\circ),\delta/2)} |F_\pm|^2 e^{-\Phi}\, dV \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{B((z'_\circ, w_\circ),\delta/2)} e^{\Phi}\, dV \right)^{1/2} \\
&\quad\le \frac{1}{\mathrm{vol}(B((z'_\circ, w_\circ), \delta/2))^{1/2}} \left\| F_\pm|_{D_r^{n-1} \times V_{\pm\delta/2}} \right\|_\Phi \sup_{B((z'_\circ, w_\circ),\delta/2)} e^{\Phi/2}.
\end{aligned}$$

From these inequalities and the fact that $e^{\Phi/2} = e^{-qu}(1 + |(z', w)|^2)$, we can get the required inequality (4.4). □

Now, we define the following holomorphic functions on $D_r^{n-1} \times V_+$:

$$\begin{aligned}
\tilde{F}_+(z', w) &= e^{qw} F_+(z', w), \\
\tilde{F}_-(z', w) &= e^{-qw} F_-(z', \pi i - w).
\end{aligned}$$

Corollary 4.3. *For any small positive δ there exists a positive constant C_δ such that one has*

$$\left| \tilde{F}_\pm(z', w) \right| \le C_\delta \left| w^2 \right| \tag{4.5}$$

for $(z', w) \in D_{r_1}^{n-1} \times V_{+\delta}$ with $|w| \gg 1$.

This corollary shows that the holomorphic function $\tilde{F}_\pm$ is slowly increasing with respect to w. Therefore, the boundary value

$$b_{D_r^{n-1} \times V_+} \left(\tilde{F}_\pm(z', w) \right)$$

represents a slowly increasing Fourier hyperfunction with respect to the variable w. Refer to Kaneko [7] for the notion of Fourier hyperfunctions. We introduce the Fourier transformation of $b_{D_r^{n-1}\times V_+}(\tilde{F}_\pm)$ with respect to w in the following way. First we decompose $\tilde{F}_\pm$ by using

$$\chi_1(w) = \frac{e^w}{1+e^w},$$
$$\chi_2(w) = \frac{1}{1+e^w}$$

into the form of:

$$\tilde{F}_\pm(z', w) = \chi_1(w)\tilde{F}_\pm(z', w) + \chi_2(w)\tilde{F}_\pm(z', w).$$

Then we define

$$G_{\pm j}(z', \zeta) = \int_{\operatorname{Im} w = v} e^{-iw\zeta}\chi_j(w)\tilde{F}_\pm(z', w)\, du \tag{4.6}$$

for an arbitrary fixed v with $0 < v < \pi$ and $j = 1, 2$. Set $\zeta = \xi + \sqrt{-1}\eta$ and define the open subsets:

$$W = \{\zeta \in \mathbb{C}; -1 < \operatorname{Im}\zeta < 1\} \setminus [0, +\infty),$$
$$W_+ = \{\zeta \in \mathbb{C}; 0 < \operatorname{Im}\zeta < 1\},$$
$$W_- = \{\zeta \in \mathbb{C}; -1 < \operatorname{Im}\zeta < 0\}.$$

LEMMA 4.4. *In the preceding situation, the integral transform* (4.6) *is independent of the choice of* $\operatorname{Im} w = v$ *in the path of integration as long as* $0 < v < \pi$. *We have, moreover,*

$$\begin{cases} G_{\pm 1} \in \mathcal{O}(D_{r_1}^{n-1} \times W_-), \\ G_{\pm 2} \in \mathcal{O}(D_{r_1}^{n-1} \times W_+). \end{cases}$$

PROOF. One considers the case of $G_{\pm 1}$. It is all the same to another case. Choose any positive constants c_1, c_2 and c_3 with $-1 < -c_2 < -c_3 < 0$.

Let (z', ζ) be any point of $D^{n-1}_{r_1} \times W_-$ with $|\xi| < c_1$, $-1 < -c_2 < \eta < -c_3 < 0$. Then one has

$$\begin{aligned}
&\left| e^{-iw\zeta} \chi_1(w) \tilde{F}_\pm(z', w) \right| \\
&\quad \le e^{u\eta + v\xi} e^u |1 + e^u(\cos v + i \sin v)|^{-1} \left| \tilde{F}_\pm(z', u + iv) \right| \\
&\quad \le \begin{cases} e^{-c_3 u + c_1 v} |\sin v|^{-1} \left| \tilde{F}_\pm(z', u + iv) \right| & \text{if } u \ge 0 \\ e^{(1-c_2)u + c_1 v} (1 - |\cos v|)^{-1} \left| \tilde{F}_\pm(z', u + iv) \right| & \text{if } u \le 0. \end{cases}
\end{aligned}$$

On the other hand, there exists a positive constant C_v depending on the path of integration $\operatorname{Im} w = v$ such that $|\tilde{F}_\pm(z', u + iv)| \le C_v |u|^2$ for $|u| \gg 1$ by Corollary 4.3. Therefore the integral in (4.6) converges uniformly on every compact subset of $D^{n-1}_{r_1} \times W_-$. Thus one has $G_{\pm 1} \in \mathcal{O}(D^{n-1}_{r_1} \times W_-)$.

It is easy to verify the independence of the integral transform from the choice of v in the path of integration by virtue of Cauchy's integral theorem and Corollary 4.3. □

From this lemma, we can introduce the Fourier transformation:

$$\begin{aligned}
\mathcal{F} b_{D^{n-1}_r \times V_+} \left(\tilde{F}_+ \right) &= b_{D^{n-1}_{r_1} \times W_-}(G_{+1}) + b_{D^{n-1}_{r_1} \times W_+}(G_{+2}), \\
\mathcal{F} b_{D^{n-1}_r \times V_+} \left(\tilde{F}_- \right) &= b_{D^{n-1}_{r_1} \times W_-}(G_{-1}) + b_{D^{n-1}_{r_1} \times W_+}(G_{-2}).
\end{aligned}$$

Using the holomorphic functions $G_{\pm j}$, one defines

$$G_\pm(z', \zeta) = \begin{cases} G_{\pm 2}(z', \zeta) & \text{on } D^{n-1}_{r_1} \times W_+ \\ -G_{\pm 1}(z', \zeta) & \text{on } D^{n-1}_{r_1} \times W_-. \end{cases}$$

Note that the function $\tilde{F}_\pm$ is holomorphic not only on $D^{n-1}_r \times V$ but also on $D^{n-1}_r \times V_+$, and that $\tilde{F}_\pm$ satisfies the growth condition (4.5) in Corollary 4.3. In this special situation we can claim the following proposition on the holomorphic function $G_\pm$.

Proposition 4.5. *The holomorphic function $G_\pm(z', \zeta)$ can be extended to a function in $\mathcal{O}(D^{n-1}_{r_1} \times W)$.*

Proof. We make use of the following function:

$$H_\pm(z', \zeta) = \begin{cases} H_{\pm 2}(z', \zeta) & \text{on } D^{n-1}_{r_1} \times W_+ \\ -H_{\pm 1}(z', \zeta) & \text{on } D^{n-1}_{r_1} \times W_-, \end{cases}$$

where one sets

$$H_{\pm j}(z',\zeta) = \int_{\operatorname{Im} w = v} e^{-iw\zeta}\chi_j(w)\tilde{F}_\pm(z',w)(w+i)^{-4}\,du$$

for an arbitrary fixed v with $0 < v < \pi$. We find immediately that

$$H_{\pm 1} \in \mathcal{O}(D_{r_1}^{n-1} \times W_-),$$
$$H_{\pm 2} \in \mathcal{O}(D_{r_1}^{n-1} \times W_+)$$

and that $H_{\pm j}$ are independent of the choice of v as long as $0 < v < \pi$, similarly as in Lemma 4.4. Note that there is the relation between $G_\pm$ and $H_\pm$ on $D_{r_1}^{n-1} \times (W_+ \cup W_-)$:

$$(D_\zeta + 1)^4 H_\pm(z',\zeta) = G_\pm(z',\zeta). \tag{4.7}$$

The holomorphic functions $H_{\pm j}$ have finite boundary values at any point of $D_{r_1}^{n-1} \times \mathbb{R}$:

$$\begin{aligned} H_{\pm 1}(z',\xi) &= \lim_{W_- \ni \zeta \to \xi} H_{\pm 1}(z',\zeta) \\ &= \int_{\operatorname{Im} w = v} e^{-iw\xi}\chi_1(w)\tilde{F}_\pm(z',w)(w+i)^{-4}\,du, \\ H_{\pm 2}(z',\xi) &= \lim_{W_+ \ni \zeta \to \xi} H_{\pm 2}(z',\zeta) \\ &= \int_{\operatorname{Im} w = v} e^{-iw\xi}\chi_2(w)\tilde{F}_\pm(z',w)(w+i)^{-4}\,du. \end{aligned}$$

For each fixed $\xi \in \mathbb{R}$, $H_{\pm j}(z',\xi)$ is holomorphic on $D_{r_1}^{n-1}$. Furthermore, the functions $-H_{\pm 1}$ and $H_{\pm 2}$ have the same boundary values at $(z',\xi) \in D_{r_1}^{n-1} \times \mathbb{R}$ with $\xi < 0$. We have indeed on $D_{r_1}^{n-1} \times \mathbb{R}$

$$\begin{aligned} H_{\pm 1}(z',\xi) + H_{\pm 2}(z',\xi) &= \int_{\operatorname{Im} w = v} e^{-iw\xi}\tilde{F}_\pm(z',w)(w+i)^{-4}\,du \\ &= e^{v\xi}\int_{\operatorname{Im} w = v} e^{-iu\xi}\tilde{F}_\pm(z',w)(w+i)^{-4}\,du. \end{aligned} \tag{4.8}$$

Since the function $\tilde{F}_\pm(z',w)(w+i)^{-4}$ in (4.8) is holomorphic not only on $D_r^{n-1} \times V$ but also on $D_r^{n-1} \times V_+$, we can choose an arbitrary v in the path of integration as long as $v > 0$. By using the estimation in Corollary 4.3, there exists a positive constant C such that we have the inequalities for $v \gg 1$:

$$\begin{aligned}
&\left| e^{v\xi} \int_{\operatorname{Im} w = v} e^{-iu\xi} \tilde{F}_\pm(z', w)(w+i)^{-4}\, du \right| \\
&\qquad \le e^{v\xi} \int_{-\infty}^{\infty} \left| \tilde{F}_\pm(z', u+iv)(u+iv+i)^{-4} \right| du \\
&\qquad \le e^{v\xi} \int_{-\infty}^{\infty} C|u+iv|^2 |u+iv+i|^{-4}\, du \\
&\qquad \le Ce^{v\xi} \int_{-\infty}^{\infty} \frac{u^2+v^2}{(u^2+v^2)^2}\, du \\
&\qquad \le Ce^{v\xi} \int_{-\infty}^{\infty} \frac{1}{1+u^2}\, du.
\end{aligned}$$

Therefore one obtains

$$H_{\pm 1}(z', \xi) + H_{\pm 2}(z', \xi) = 0 \quad \text{for} \quad \xi < 0,$$

since v is arbitrary as long as $v > 0$ in the preceding inequalities. Thus the function $H_\pm$ is holomorphic on $D_{r_1}^{n-1} \times (W_+ \cup W_-)$ and extended to a continuous function on $D_{r_1}^{n-1} \times W$. Then we find that $H_\pm$ is holomorphic on $D_{r_1}^{n-1} \times W$. We find, furthermore, that $G_\pm$ can be extended to a holomorphic function on the domain $D_{r_1}^{n-1} \times W$ through the relation (4.7). $\square$

Now we consider values of the holomorphic function $G_\pm$ as $\operatorname{Re}\zeta \to \pm\infty$. We can show that $G_\pm$ possesses infra-exponential growth order as $\operatorname{Re}\zeta \to \infty$ and decreases exponentially as $\operatorname{Re}\zeta \to -\infty$.

PROPOSITION 4.6. *For any compact subset K of $(-1, 0) \cup (0, 1)$ and any positive ε there exists a positive constant $C_{K,\varepsilon}$ such that one has*

$$|G_\pm(z', \zeta)| \le C_{K,\varepsilon} e^{\varepsilon \operatorname{Re}\zeta} \tag{4.9}$$

for $z' \in D_{r_1}^{n-1}$, $\operatorname{Re}\zeta > 0$, $\operatorname{Im}\zeta \in K$.

PROOF. It is enough to prove this proposition for $0 < \varepsilon < \pi$. Choose the path of integration $\operatorname{Im} w = \varepsilon$ in (4.6), and let η_1, η_2 be positive constants satisfying $0 < \eta_1 < |\eta| < \eta_2 < 1$ for any $\eta \in K$. Then one has:

$$|G_{\pm 1}(z', \zeta)| \leq \int_{\operatorname{Im} w = \varepsilon} \left| e^{-iw\zeta} \chi_1(w) \tilde{F}_{\pm}(z', w) \right| du$$
$$= \int_{-\infty}^{\infty} e^{u\eta + \varepsilon\xi} \left| \chi_1(u + i\varepsilon) \tilde{F}_{\pm}(z', u + i\varepsilon) \right| du.$$

Let $u_\circ$ be a sufficiently large constant. We decompose this integral into integrals over intervals $(-\infty, -u_\circ]$, $[-u_\circ, u_\circ]$ and $[u_\circ, \infty)$. By Corollary 4.3 the first part is as follows:

$$\int_{-\infty}^{-u_\circ} e^{u\eta + \varepsilon\xi} e^u |1 + e^u(\cos\varepsilon + i \sin\varepsilon)|^{-1} \left| \tilde{F}_{\pm}(z', u + i\varepsilon) \right| du$$
$$\leq e^{\varepsilon\xi} \int_{-\infty}^{-u_\circ} e^{(1-\eta_2)u} (1 - |\cos\varepsilon|)^{-1} C_\varepsilon |u|^2 \, du, \tag{4.10}$$

where C_ε is a positive constant depending on ε. The second estimation is as follows:

$$\int_{-u_\circ}^{u_\circ} e^{u\eta + \varepsilon\xi} \left| \chi_1(u + i\varepsilon) \tilde{F}_{\pm}(z', u + i\varepsilon) \right| du$$
$$\leq 2u_\circ e^{u_\circ + \varepsilon\xi} \max_{|u| \leq u_\circ,\, z' \in \bar{D}_{r_1}^{n-1}} \left| \chi_1(u + i\varepsilon) \tilde{F}_{\pm}(z', u + i\varepsilon) \right|. \tag{4.11}$$

Also the third estimation is as follows:

$$\int_{u_\circ}^{\infty} e^{u\eta + \varepsilon\xi} e^u |1 + e^u(\cos\varepsilon + i \sin\varepsilon)|^{-1} \left| \tilde{F}_{\pm}(z', u + i\varepsilon) \right| du$$
$$\leq e^{\varepsilon\xi} \int_{u_\circ}^{\infty} e^{-\eta_1 u} |\sin\varepsilon|^{-1} C_\varepsilon |u|^2 \, du. \tag{4.12}$$

Then by summing up the right-hand sides of (4.10), (4.11) and (4.12), we can get the required inequality (4.9) for $\operatorname{Im}\zeta < 0$. It is all the same for $\operatorname{Im}\zeta > 0$. □

PROPOSITION 4.7. *For any compact subset K of $(-1, 1)$ and any positive ε there exists a positive constant $C'_{K,\varepsilon}$ such that one has*

$$|G_{\pm}(z', \zeta)| \leq C'_{K,\varepsilon} e^{(\pi - \varepsilon) \operatorname{Re}\zeta}$$

for $z' \in D_{r_1}^{n-1}$, $\operatorname{Re}\zeta < -1$, $\operatorname{Im}\zeta \in K$.

PROOF. We use the function $H_\pm$ in the proof of Proposition 4.5. It is enough to show this proposition for $0 < \varepsilon < \pi$. Choose the path of integration $\operatorname{Im} w = \pi - \varepsilon$ in the definition of $H_{\pm j}$. Then one has:

$$\begin{aligned}
|H_{\pm 1}(z', \zeta)| &\leq \int_{\operatorname{Im} w = \pi - \varepsilon} \left| e^{-iw\zeta} \chi_1(w) \tilde{F}_\pm(z', w)(w+i)^{-4} \right| du \\
&= \int_{-\infty}^{\infty} e^{u\eta + (\pi-\varepsilon)\xi} e^u |1 + e^u(-\cos\varepsilon + i \sin\varepsilon)|^{-1} \\
&\qquad \times \left| \tilde{F}_\pm(z', u + i(\pi - \varepsilon))(u + i(\pi - \varepsilon + 1))^{-4} \right| du.
\end{aligned}$$

In exactly the same way as in the proof of Proposition 4.6, we decompose this integral into integrals over intervals $(-\infty, -u_\circ]$, $[-u_\circ, u_\circ]$ and $[u_\circ, \infty)$ for a sufficiently large constant $u_\circ$. The integral over $(-\infty, -u_\circ]$ is less than or equal to:

$$e^{(\pi-\varepsilon)\xi} \int_{-\infty}^{-u_\circ} (1 - |\cos\varepsilon|)^{-1} C_\varepsilon |u|^{-2}\, du, \tag{4.13}$$

where C_ε is a positive constant depending on ε. The integrals over $[-u_\circ, u_\circ]$ and $[u_\circ, \infty)$ are not greater than

$$2u_\circ e^{u_\circ + (\pi-\varepsilon)\xi} \max_{|u| \leq u_\circ,\, v = \pi - \varepsilon,\, z' \in \bar{D}_{r_1}^{n-1}} \left| \chi_1(w) \tilde{F}_\pm(z', w)(w+i)^{-4} \right| \tag{4.14}$$

and

$$e^{(\pi-\varepsilon)\xi} \int_{u_\circ}^{\infty} |\sin\varepsilon|^{-1} C_\varepsilon |u|^{-2}\, du \tag{4.15}$$

respectively. Then by summing up (4.13), (4.14), (4.15), and estimating $H_{\pm 2}$ in like manner, one has the following estimate. There exists a positive constant C'_ε such that we have

$$|H_\pm(z', \zeta)| \leq C'_\varepsilon e^{(\pi-\varepsilon)\operatorname{Re}\zeta} \tag{4.16}$$

for $z' \in D_{r_1}^{n-1}$, $\operatorname{Re}\zeta < 0$, $-1 < \operatorname{Im}\zeta < 1$.

Let ρ be a positive constant which satisfies $|\eta| + 4\rho < 1$ for all $\eta \in K$. Choose any point $(z'_\circ, \zeta_\circ)$ with $z'_\circ \in D^{n-1}_{r_1}$, $\operatorname{Re}\zeta_\circ < -1$, $\operatorname{Im}\zeta_\circ \in K$, and let $\zeta_\circ = \xi_\circ + \sqrt{-1}\eta_\circ$. Then we have by Cauchy's inequalities:

$$|(D_\zeta + 1)H_\pm(z'_\circ, \zeta_\circ)| \leq \left(\rho^{-1} + 1\right) \sup_{|\zeta-\zeta_\circ|\leq\rho} |H_\pm(z'_\circ, \zeta)|.$$

Hence we have by the relation (4.7) and the inequality (4.16):

$$\begin{aligned}|G_\pm(z'_\circ, \zeta_\circ)| &= \left|(D_\zeta + 1)^4 H_\pm(z'_\circ, \zeta_\circ)\right| \\ &\leq \left(\rho^{-1} + 1\right)^4 \sup_{|\zeta-\zeta_\circ|\leq 4\rho} |H_\pm(z'_\circ, \zeta)| \\ &\leq \left(\rho^{-1} + 1\right)^4 \sup_{z'\in D^{n-1}_{r_1},\, |\xi-\xi_\circ|\leq 4\rho,\, |\eta|<1} |H_\pm(z', \zeta)| \\ &\leq \left(\rho^{-1} + 1\right)^4 C'_\varepsilon e^{(\pi-\varepsilon)(\xi_\circ+4\rho)}.\end{aligned}$$

Thus we set $C'_{K,\varepsilon} = (\rho^{-1} + 1)^4 C'_\varepsilon e^{4\rho(\pi-\varepsilon)} > 0$. $\square$

We have constructed the holomorphic function $G_\pm$ from $\tilde{F}_\pm$ by using the Fourier transformation with the estimation of Propositions 4.6 and 4.7. Now we shall restore $G_\pm$ to the original holomorphic function $\tilde{F}_\pm$ by using Fourier's inversion formula.

Set the infinite path

$$\gamma\colon \zeta = \zeta(t) = \begin{cases}(t-1) + i/2, & t \geq 1\\ 1/2e^{(1-t/2)\pi i}, & -1 \leq t \leq 1\\ -(t+1) - i/2, & t \leq -1.\end{cases}$$

Figure 1 shows the infinite path γ.

PROPOSITION 4.8. *For $(z', w) \in D^{n-1}_{r_1} \times V$, one has*

$$\tilde{F}_\pm(z', w) = \frac{1}{2\pi}\int_\gamma e^{iw\zeta} G_\pm(z', \zeta)\, d\zeta.$$

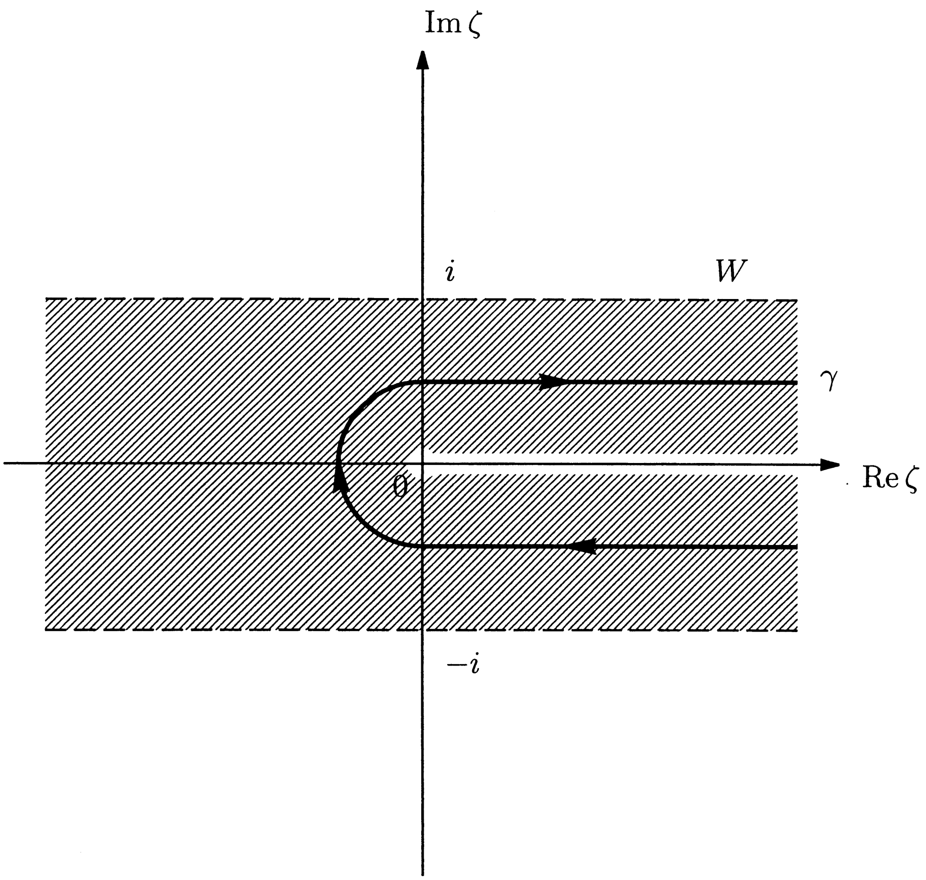

Fig. 1. The path γ.

PROOF. Let η_1 and η_2 be arbitrary fixed constants which satisfy $-1 < \eta_1 < 0 < \eta_2 < 1$. Note that

$$\begin{aligned}
G_{\pm j}(z', \zeta) &= \int_{\operatorname{Im} w = v} e^{-iw\zeta} \chi_j(w) \tilde{F}_{\pm}(z', w)\, du \\
&= e^{v\zeta} \int_{-\infty}^{\infty} e^{-iu\xi} e^{u\eta} \chi_j(u + iv) \tilde{F}_{\pm}(z', u + iv)\, du \\
&= e^{v\zeta} \mathcal{F}_u \left(e^{u\eta} \chi_j(u + iv) \tilde{F}_{\pm}(z', u + iv) \right) (\xi).
\end{aligned}$$

Then:

$$\begin{aligned}
&e^{-w\eta_j}\mathcal{F}_\xi^{-1}\left(e^{-v\xi}G_{\pm j}(z',\xi+i\eta_j)\right)(u)\\
&\quad= e^{-u\eta_j}\mathcal{F}_\xi^{-1}\left(\mathcal{F}_u\left(e^{u\eta_j}\chi_j(u+iv)\tilde{F}_\pm(z',u+iv)\right)(\xi)\right)(u)\\
&\quad= \chi_j(w)\tilde{F}_\pm(z',w).
\end{aligned}$$

Here $\mathcal{F}_u$ denotes the Fourier transformation, and $\mathcal{F}_\xi^{-1}$ the inverse transformation. Therefore for any point $(z',w)\in D_{r_1}^{n-1}\times V$ it follows that:

$$\chi_j(w)\tilde{F}_\pm(z',w)=\frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_j} e^{iw\zeta}G_{\pm j}(z',\zeta)\,d\xi. \tag{4.17}$$

Note that the integral in (4.17) converges. Indeed, when we choose positive constants ε_1, ε_2 with $0<\varepsilon_1<\operatorname{Im}w$, $0<\varepsilon_2<\pi-\operatorname{Im}w$, there exist positive constants C_{ε_1}, C'_{ε_2} such that

$$\begin{aligned}
&\left|e^{iw(\xi+i\eta_j)}G_{\pm j}(z',\xi+i\eta_j)\right|\\
&\qquad\le\begin{cases} e^{-(u\eta_j+v\xi)}C_{\varepsilon_1}e^{\varepsilon_1\xi}=C_{\varepsilon_1}e^{-u\eta_j-(v-\varepsilon_1)\xi} & \text{if } \operatorname{Re}\zeta>0\\ e^{-(u\eta_j+v\xi)}C'_{\varepsilon_2}e^{(\pi-\varepsilon_2)\xi}=C'_{\varepsilon_2}e^{-u\eta_j+(\pi-v-\varepsilon_2)\xi} & \text{if } \operatorname{Re}\zeta<-1\end{cases}
\end{aligned}$$

by the estimates of Propositions 4.6 and 4.7.

Therefore by (4.17):

$$\begin{aligned}
\tilde{F}_\pm(z',w)&=\chi_1(w)\tilde{F}_\pm(z',w)+\chi_2(w)\tilde{F}_\pm(z',w)\\
&=-\frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_1}e^{iw\zeta}G_\pm(z',\zeta)\,d\xi+\frac{1}{2\pi}\int_{\operatorname{Im}\zeta=\eta_2}e^{iw\zeta}G_\pm(z',\zeta)\,d\xi.
\end{aligned}$$

We can deform the path of integration into γ, since the integration as $\operatorname{Re}\zeta\to-\infty$ can be neglected by the exponential decay of the integrand that we have proved in Proposition 4.7. Then we can get the required integral representation of the holomorphic function $\tilde{F}_\pm$. $\square$

By Proposition 4.8 and the definition of $\tilde{F}_\pm$, it follows that

$$\begin{aligned}
e^{qw}F_+(z',w)&=\frac{1}{2\pi}\int_\gamma e^{iw\zeta}G_+(z',\zeta)\,d\zeta,\\
e^{-qw}F_-(z',\pi i-w)&=\frac{1}{2\pi}\int_\gamma e^{iw\zeta}G_-(z',\zeta)\,d\zeta,
\end{aligned}$$

that is,

$$F_+(z', w) = \frac{1}{2\pi} \int_\gamma e^{(i\zeta - q)w} G_+(z', \zeta)\, d\zeta,$$
$$F_-(z', w) = \frac{1}{2\pi} \int_\gamma e^{(i\zeta + q)(\pi i - w)} G_-(z', \zeta)\, d\zeta$$

for $(z', w) \in D_{r_1}^{n-1} \times V$. Then we reach a conclusion of the following representation through the variable z_n in the first situation.

COROLLARY 4.9. *One has $F_\infty(z) = F_+(z', \log z_n) + F_-(z', \log z_n)$ with*

$$\begin{cases} F_+(z', \log z_n) = \dfrac{1}{2\pi} \displaystyle\int_\gamma (z_n)^{i\zeta - q} G_+(z', \zeta)\, d\zeta, \\ F_-(z', \log z_n) = \dfrac{1}{2\pi} \displaystyle\int_\gamma e^{(-\zeta + iq)\pi} (z_n)^{-i\zeta - q} G_-(z', \zeta)\, d\zeta \end{cases} \tag{4.18}$$

for $z' \in D_{r_1}^{n-1}$, $0 < \arg z_n < \pi$.

4.2. Solutions of infra-exponential type

In this subsection, regarding the variable of integration as a parameter, we construct an approximate solution with infra-exponential growth order.

We may assume from the beginning that the holomorphic differential operator $P(z, D_{z'}, z_n D_{z_n})$ satisfies

$$\begin{cases} a_{(m,0,\dots,0)}(z) \neq 0, \\ a_{(0,\dots,0,m)}(z) \equiv (-i)^m \end{cases} \tag{4.19}$$

on the open polydisc D_r^n by the hypothesis (3.8).

By Corollary 4.9, our differential equation is turned into:

$$\begin{cases} Pu_+(x) = b_{D_{r_1}^{n-1} \times U_{r_1}}(F_+(z', \log z_n)), \\ Pu_-(x) = b_{D_{r_1}^{n-1} \times U_{r_1}}(F_-(z', \log z_n)) \end{cases}$$

at $p_\circ = (0, \sqrt{-1}\, dx_n) \in \Sigma$ with the integral representation (4.18). It suffices to consider the differential equation on the complex domain $D_{r_1}^{n-1} \times U_{r_1}$

$$P(z, D_{z'}, z_n D_{z_n}) U_\pm(z) = F_\pm(z', \log z_n).$$

Of course it is enough to show the existence of a holomorphic solution $U_\pm(z)$ on a smaller domain $D^{n-1}_{r'} \times U_{r'}$ for $0 < r' < r_1$.

Now in order to study the existence of $U_\pm(z)$, we consider the differential equations with a parameter ζ on $D^{n-1}_{r_1} \times U_{r_1}$:

$$P(z, D_{z'}, z_n D_{z_n}) U_+(z, \zeta) = (z_n)^{i\zeta - q} G_+(z', \zeta), \tag{4.20}$$

$$P(z, D_{z'}, z_n D_{z_n}) U_-(z, \zeta) = e^{(-\zeta + iq)\pi} (z_n)^{-i\zeta - q} G_-(z', \zeta). \tag{4.21}$$

Here the parameter ζ ranges through the path γ. Note that these equations are equivalent to:

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) \tilde{U}_\pm(z, \zeta) = G_\pm(z', \zeta), \tag{4.22}$$

where we set

$$\begin{aligned} \tilde{U}_+(z, \zeta) &= (z_n)^{-i\zeta + q} U_+(z, \zeta), \\ \tilde{U}_-(z, \zeta) &= e^{(\zeta - iq)\pi} (z_n)^{i\zeta + q} U_-(z, \zeta). \end{aligned}$$

LEMMA 4.10. *For any $\zeta \in \mathbb{C}$ the differential operator $P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)$ is written as:*

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) = \sum_{j=0}^{m} (\pm\zeta)^j P_j(z, D_z), \tag{4.23}$$

where the P_j's are holomorphic differential operators of order $m - j$ defined on D^n_r

$$P_j(z, D_z) = \sum_{|\alpha| \le m-j} a^j_\alpha(z) D^\alpha_z.$$

PROOF. It is easy to verify that

$$\begin{aligned} P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) &= \sum_{|\alpha| \le m} a_\alpha(z) D^{\alpha'}_{z'} (z_n D_{z_n} \pm i\zeta - q)^{\alpha_n} \\ &= \sum_{|\alpha| \le m} \sum_{0 \le j \le m - |\alpha|} (\pm\zeta)^j a^j_\alpha(z) D^\alpha_z, \end{aligned}$$

where $a_\alpha^j(z) \in \mathcal{O}(D_r^n)$. Thus we get the required equation (4.23). $\square$

From the condition (4.19) it follows that

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) = (\pm\zeta)^m + \sum_{j=0}^{m-1} (\pm\zeta)^j P_j(z, D_z).$$

One has a solution

$$U_\pm(z) = \frac{1}{2\pi} \int_\gamma U_\pm(z, \zeta)\, d\zeta$$

if we can find a solution $U_\pm(z, \zeta)$ and this integral is convergent. Then it suffices to show the existence of a solution $\tilde{U}_\pm(z, \zeta)$ of infra-exponential type with respect to ζ on the path γ, so we study the differential equation (4.22) with the parameter ζ.

By the differential equation (4.22) one has formally:

$$\tilde{U}_\pm(z, \zeta) = (\pm\zeta)^{-m} \sum_{k=0}^{\infty} \left(- \sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_\pm(z', \zeta),$$

but in general this infinite sum is divergent. So we introduce another modified function instead of $\tilde{U}_\pm$. Let A be a sufficiently large constant. We set $W_0 = W$ and $W_k = \{\zeta \in W; \operatorname{Re}\zeta > Ak\}$ for $k \in \mathbb{N} \setminus \{0\}$. Then we define:

(4.24) $U_{\pm 1}(z, \zeta)$

$$= (\pm\zeta)^{-m} \sum_{k=0}^{\infty} \chi_{W_k}(\zeta) \left(- \sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_\pm(z', \zeta)$$

for $(z, \zeta) \in D_{r_1}^n \times W$, where χ_{W_k} is the characteristic function of W_k:

$$\chi_{W_k}(\zeta) = \begin{cases} 1 & \text{if } \zeta \in W_k \\ 0 & \text{if } \zeta \notin W_k. \end{cases}$$

We find immediately that the function $U_{\pm 1}$ is well-defined, since the sum in (4.24) is locally finite on $D_{r_1}^n \times W$. The function $U_{\pm 1}(z, \zeta)$ is holomorphic in $D^n_{r_1}$ for each fixed $\zeta \in W$. We can claim that $U_{\pm 1}$ is the function of infra-exponential type as $\operatorname{Re}\zeta \to \infty$. Set $r_2 = r_1/2$.

PROPOSITION 4.11. *For any positive ε there exists a positive constant M_ε such that one has*

$$|U_{\pm 1}(z,\zeta)| \le M_\varepsilon e^{\varepsilon \operatorname{Re}\zeta} \tag{4.25}$$

for $z \in D^n_{r_2}$ and ζ on the path γ.

PROOF. First choose a positive constant M such that

$$\left|a^j_\alpha(z)\right| \le M \quad \text{for} \quad j, \quad \alpha, \quad z \in D^n_{r_1}.$$

Let ζ be any point on the path γ with $\operatorname{Re}\zeta > 1$, and $\rho = r_1/(8k) > 0$ for $k \in \mathbb{N} \setminus \{0\}$. Then we have by Cauchy's inequalities:

$$\begin{aligned}
\max_{z \in \bar{D}^n_{r_2}} &\left| -\sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) G_\pm(z',\zeta) \right| \\
&\le \sum_{0 \le j \le m-1,\, |\alpha| \le m-j} |\zeta|^{j-m} M \max_{z \in \bar{D}^n_{r_2}} |D^\alpha_z G_\pm(z',\zeta)| \\
&\le mM \sum_{|\alpha| \le m} |\zeta|^{-\max\{|\alpha|,1\}} \alpha! \rho^{-|\alpha|} \max_{z \in \bar{D}^n_{r_2+\rho}} |G_\pm(z',\zeta)|.
\end{aligned}$$

Similarly, we have for $k \in \mathbb{N} \setminus \{0\}$:

$$\begin{aligned}
\max_{z \in \bar{D}^n_{r_2}} &\left| \chi_{W_k}(\zeta) \left(-\sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_\pm(z',\zeta) \right| \\
&\le \chi_{W_k}(\zeta) \left(mM \sum_{|\alpha| \le m} |\zeta|^{-\max\{|\alpha|,1\}} \alpha! {r_1}^{-|\alpha|} (8k)^{|\alpha|} \right)^k \\
&\qquad \times \max_{z \in \bar{D}^n_{r_2+k\rho}} |G_\pm(z',\zeta)| \\
&\le \left(mM \sum_{|\alpha| \le m} A^{-1} k^{-\max\{|\alpha|,1\}} \alpha! {r_1}^{-|\alpha|} (8k)^{|\alpha|} \right)^k \max_{z \in \bar{D}^n_{r_2+k\rho}} |G_\pm(z',\zeta)|
\end{aligned}$$

$$\leq A^{-k}\left(mM\sum_{|\alpha|\leq m}\alpha!{r_1}^{-|\alpha|}8^{|\alpha|}\right)^k \sup_{z'\in D^{n-1}_{r_1}}|G_\pm(z',\zeta)|.$$

At the definition of $U_{\pm 1}$ we take a sufficiently large constant A so that we have:

$$B_1 := \sum_{k=0}^{\infty} A^{-k}\left(mM\sum_{|\alpha|\leq m}\alpha!{r_1}^{-|\alpha|}8^{|\alpha|}\right)^k < \infty.$$

Then we get the following inequalities:

$$\begin{aligned}|U_{\pm 1}(z,\zeta)| &\leq |\zeta|^{-m}\sum_{k=0}^{\infty}\left|\chi_{W_k}(\zeta)\left(-\sum_{j=0}^{m-1}(\pm\zeta)^{j-m}P_j(z,D_z)\right)^k G_\pm(z',\zeta)\right| \\ &\leq B_1 \sup_{z'\in D^{n-1}_{r_1}}|G_\pm(z',\zeta)|\end{aligned}$$

for $z\in D^n_{r_2}$ and ζ on the path γ with $\mathrm{Re}\,\zeta > 1$. From these inequalities and Proposition 4.6, we obtain the required inequality (4.25). $\square$

Needless to say, the function $U_{\pm 1}$ is not a solution of (4.22), but it gives sufficient approximation of solutions of (4.22) in the following sense. By its construction, we have:

$$\begin{aligned}&P(z, D_{z'}, z_nD_{z_n}\pm i\zeta - q)U_{\pm 1}(z,\zeta)\\ &\quad= \left((\pm\zeta)^m + \sum_{j=0}^{m-1}(\pm\zeta)^jP_j(z,D_z)\right)U_{\pm 1}(z,\zeta)\\ &\quad= \sum_{k=0}^{\infty}\chi_{W_k}(\zeta)\left(-\sum_{j=0}^{m-1}(\pm\zeta)^{j-m}P_j(z,D_z)\right)^k G_\pm(z',\zeta)\\ &\qquad\quad - \sum_{k=0}^{\infty}\chi_{W_k}(\zeta)\left(-\sum_{j=0}^{m-1}(\pm\zeta)^{j-m}P_j(z,D_z)\right)^{k+1} G_\pm(z',\zeta)\\ &\quad= G_\pm(z',\zeta) - \sum_{k=1}^{\infty}\chi_{W_{k-1}\setminus W_k}(\zeta)\left(-\sum_{j=0}^{m-1}(\pm\zeta)^{j-m}P_j(z,D_z)\right)^k G_\pm(z',\zeta).\end{aligned}$$

Setting:

$$G_{\pm 0}(z,\zeta) = \sum_{k=1}^{\infty} \chi_{W_{k-1}\setminus W_k}(\zeta) \left(-\sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_{\pm}(z',\zeta), \tag{4.26}$$

one has the equation on $D^n_{r_1} \times W$:

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) U_{\pm 1}(z,\zeta) = G_{\pm}(z',\zeta) - G_{\pm 0}(z,\zeta). \tag{4.27}$$

Here the error $G_{\pm 0}(z,\zeta)$ is also a holomorphic function in $D^n_{r_1}$ for each fixed $\zeta \in W$, since the sum in (4.26) is locally finite on $D^n_{r_1} \times W$. Then one can claim that $G_{\pm 0}$ is exponentially decreasing as $\operatorname{Re}\zeta \to \infty$.

PROPOSITION 4.12. *There exist positive constants δ_1 and M_1 such that one has*

$$|G_{\pm 0}(z,\zeta)| \le M_1 e^{-\delta_1 \operatorname{Re}\zeta} \tag{4.28}$$

for $z \in D^n_{r_2}$ and ζ on the path γ.

PROOF. In exactly the same way as in the proof of Proposition 4.11, we estimate $G_{\pm 0}(z,\zeta)$. Similarly, we define a positive constant M. Let ζ be any point on the path γ with $\operatorname{Re}\zeta > A$, and $\rho = r_1/(8(k-1)) > 0$ for $k \in \mathbb{N} \setminus \{0,1\}$. Then we have for $k \in \mathbb{N} \setminus \{0,1\}$:

$$\begin{aligned}
&\max_{z\in \bar{D}^n_{r_2}} \left| \chi_{W_{k-1}\setminus W_k}(\zeta) \left(-\sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_{\pm}(z',\zeta) \right| \\
&\quad \le \chi_{W_{k-1}\setminus W_k}(\zeta) \left(mM \sum_{|\alpha|\le m} |\zeta|^{-\max\{|\alpha|,1\}} \alpha! {r_1}^{-|\alpha|} (8(k-1))^{|\alpha|} \right)^k \\
&\qquad\qquad \times \max_{z\in \bar{D}^n_{r_2+k\rho}} |G_{\pm}(z',\zeta)| \\
&\quad \le \chi_{W_{k-1}\setminus W_k}(\zeta) A^{-k} \left(mM \sum_{|\alpha|\le m} \alpha! {r_1}^{-|\alpha|} 8^{|\alpha|} \right)^k \sup_{z'\in D^{n-1}_{r_1}} |G_{\pm}(z',\zeta)|.
\end{aligned}$$

Hence we also obtain the following estimates:

$$
\begin{aligned}
(4.29)\quad & |G_{\pm 0}(z,\zeta)| \\
&\le \sum_{k=2}^{\infty} \left| \chi_{W_{k-1}\setminus W_k}(\zeta) \left(-\sum_{j=0}^{m-1} (\pm\zeta)^{j-m} P_j(z, D_z) \right)^k G_{\pm}(z',\zeta) \right| \\
&\le e^{-2\delta_1 \operatorname{Re}\zeta} e^{2\delta_1 \operatorname{Re}\zeta} \sup_{z' \in D_{r_1}^{n-1}} |G_{\pm}(z',\zeta)| \\
&\qquad \times \sum_{k=2}^{\infty} \chi_{W_{k-1}\setminus W_k}(\zeta) A^{-k} \left(mM \sum_{|\alpha|\le m} \alpha! {r_1}^{-|\alpha|} 8^{|\alpha|} \right)^k \\
&\le e^{-2\delta_1 \operatorname{Re}\zeta} \sup_{z' \in D_{r_1}^{n-1}} |G_{\pm}(z',\zeta)| \\
&\qquad \times \sum_{k=2}^{\infty} e^{2\delta_1 A k} A^{-k} \left(mM \sum_{|\alpha|\le m} \alpha! {r_1}^{-|\alpha|} 8^{|\alpha|} \right)^k
\end{aligned}
$$

for any positive δ_1.

We have set the constant A so that

$$A^{-1} mM \sum_{|\alpha|\le m} \alpha! {r_1}^{-|\alpha|} 8^{|\alpha|} < 1$$

in the proof of Proposition 4.11. So we choose a sufficiently small $\delta_1 > 0$ so that the following geometric series converges:

$$B_2 := \sum_{k=2}^{\infty} e^{2\delta_1 A k} A^{-k} \left(mM \sum_{|\alpha|\le m} \alpha! {r_1}^{-|\alpha|} 8^{|\alpha|} \right)^k < \infty. \tag{4.30}$$

On the other hand, by Proposition 4.6 there exists a positive constant C_{δ_1} such that one has

$$|G_{\pm}(z',\zeta)| \le C_{\delta_1} e^{\delta_1 \operatorname{Re}\zeta} \quad \text{for} \quad z' \in D_{r_1}^{n-1}. \tag{4.31}$$

By (4.29), (4.30), and (4.31), we obtain the inequality

$$|G_{\pm 0}(z,\zeta)| \le B_2 C_{\delta_1} e^{-\delta_1 \operatorname{Re}\zeta}$$

for $z \in D_{r_2}^{n}$ and ζ on the path γ with $\operatorname{Re}\zeta > A$. Then setting a constant M_1 such that $M_1 > B_2 C_{\delta_1}$, we obtain the required inequality (4.28). □

4.3. The Cauchy-Kowalewski theorem with a parameter

In this subsection, we estimate the remainder by means of majorant series in the Cauchy-Kowalewski theorem with a parameter.

In Subsection 4.2, we have considered the differential equation (4.22) with the parameter ζ, and constructed the approximate solution $U_{\pm 1}$ of infra-exponential type with respect to ζ. Now we show the existence of exponentially decreasing solutions of the differential equation

$$P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)U_{\pm 0}(z,\zeta) = G_{\pm 0}(z,\zeta)$$

by using the classical Cauchy-Kowalewski theorem with a parameter. One sets:

$$\tilde{U}_{\pm} = U_{\pm 0} + U_{\pm 1}.$$

At that time we find immediately that $\tilde{U}_{\pm}$ is a solution of infra-exponential type of (4.22) by the results in the preceding subsection.

First of all set:

$$\Omega_r = \{z \in \mathbb{C}^n; |z_1| + |z_2| + \cdots + |z_n| < r\},$$
$$\Omega_{L,r} = \{z \in \mathbb{C}^n; L|z_1| + |z_2| + \cdots + |z_n| < r\}$$

for $r > 0$, $L \geq 1$. Note that the differential operator

$$\begin{aligned} P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q) &= \sum_{j=0}^{m} (\pm\zeta)^j P_j(z, D_z) \\ &= \sum_{|\alpha| \leq m} \sum_{0 \leq j \leq m-|\alpha|} (\pm 2\zeta)^j 2^{-j} a^j_\alpha(z) D^\alpha_z \end{aligned}$$

is defined on Ω_r and $G_{\pm 0}(z,\zeta) \in \mathcal{O}(\Omega_{r_1})$ for each fixed $\zeta \in W$. Here we have set $r_1 = r/2$, $r_2 = r_1/2$ in the previous subsections. Note, moreover, that the principal symbol of $P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)$ does not depend on the parameter ζ.

By the condition (4.19), we may assume from the beginning that

$$a^0_{(m,0,\ldots,0)}(z) = a_{(m,0,\ldots,0)}(z) \equiv 1.$$

Consider the Cauchy problem:

$$
\begin{cases} P(z, D_{z'}, z_n D_{z_n} \pm i\zeta - q)U_{\pm 0} = G_{\pm 0} \\ D_{z_1}^j U_{\pm 0} = 0 \quad \text{when } z_1 = 0,\, j < m. \end{cases} \tag{4.32}
$$

PROPOSITION 4.13. *There exists $L \geq 1$ such that the Cauchy problem (4.32) has a unique solution $U_{\pm 0} \in \mathcal{O}(\Omega_{L,r_2})$ for any ζ on the path γ. Moreover, there exist $r_3 > 0$, $M_0 > 0$, $\delta_2 > 0$ such that for any $z \in \Omega_{L,r_3}$, any ζ, we have*

$$
|U_{\pm 0}(z, \zeta)| \leq M_0 e^{-\delta_2 \operatorname{Re} \zeta}.
$$

PROOF. We make use of majorant series in the same way as in Ōshima-Komatsu [14]. We consider power series of z for each fixed ζ on the path γ, where $|\zeta| \geq 1/2$.

To begin with, the Cauchy problem (4.32) has a unique power series solution at $z = 0$. It is easy to verify this fact: if

$$
U_{\pm 0} = \sum_{\alpha \in \mathbb{N}^n} u_{\pm\alpha}(\zeta) z^\alpha = \sum_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} u_{\pm\alpha}(\zeta) z_1^{\alpha_1} \dots z_n^{\alpha_n}
$$

is a solution, we find that $u_{\pm\alpha}(\zeta) = 0$ for $\alpha_1 \leq m-1$ by the Cauchy boundary conditions in (4.32). And furthermore, we compare the both sides of the differential equation in (4.32) for $\alpha_1 \geq m$. We find easily that the coefficient of $z_1^{\alpha_1 - m} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ in the left-hand side is $\alpha_1(\alpha_1 - 1) \dots (\alpha_1 - m + 1) u_{\pm\alpha}(\zeta)$ and the remainder which depends only on $u_\beta(\zeta)$ with $\beta = (\beta_1, \dots, \beta_n)$, $\beta_1 \leq \alpha_1 - 1$, $|\beta| \leq |\alpha|$. Thus we can fix the coefficients $u_{\pm\alpha}(\zeta)$ uniquely in turn.

Next, we show that the power series solution $U_{\pm 0}$ is holomorphic in Ω_{L,r_2} for some $L \geq 1$ by the method of majorant series. We set

$$
s = L z_1 + z_2 + \dots + z_n
$$

for $L \geq 1$, and construct a majorant series $V(s, \zeta)$ of $U_{\pm 0}(z, \zeta)$ as a power series of z for each ζ.

There exist positive constants M_1, M_2, δ_1 such that for any j and any α:

$$2^{-j}a^j_\alpha(z) \ll \frac{M_2}{1-{r_1}^{-1}s},$$
$$G_{\pm 0}(z,\zeta) \ll \frac{M_1 e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s}.$$

Indeed, the functions $2^{-j}a^j_\alpha(z) = \sum_\beta a^j_{\alpha,\beta} z^\beta$ are holomorphic in D^n_r, so we can choose a positive constant M_2 such that $|2^{-j}a^j_\alpha(z)| \le M_2$ on $D^n_{r_1}$ for any j and any α. Then by Cauchy's inequalities, we have $|a^j_{\alpha,\beta}| \le M_2{r_1}^{-|\beta|}$ and hence

$$2^{-j}a^j_\alpha(z) \ll \frac{M_2}{(1-r_1^{-1}z_1)\ldots(1-r_1^{-1}z_n)}.$$

It is all the same to this case for $G_{\pm 0}(z,\zeta)$ by using the estimate in Proposition 4.12.

Note the way to set the coefficients $u_{\pm\alpha}(\zeta)$ and the equation

$$D^\alpha_z V(s,\zeta) = L^{\alpha_1} D^{|\alpha|}_s V(s,\zeta).$$

Then, if a power series $V(s,\zeta)$ satisfies:

$$\tag{4.33} \begin{cases} L^m D^m_s V \gg \dfrac{M_2}{1-{r_1}^{-1}s}\left(\dfrac{(m+n-1)!}{m!(n-1)!}-1\right)L^{m-1}D^m_s V \\ \qquad\qquad + \dfrac{M_2}{1-{r_1}^{-1}s}\displaystyle\sum_{k=0}^{m-1}\dfrac{(m-k+1)(k+n-1)!}{k!(n-1)!} \\ \qquad\qquad\qquad\qquad\qquad \times |2\zeta|^{m-k}L^k D^k_s V \\ \qquad\qquad + \dfrac{M_1 e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s}, \\ V \gg 0, \end{cases}$$

it follows that $U_{\pm 0}(z,\zeta) \ll V(s,\zeta)$ for each fixed ζ.

And furthermore, if power series $W_k(s,\zeta)$ $(k=0,\dots,m-1)$ satisfy:

$$
\begin{cases}
LD_sW_{m-1} \gg \dfrac{M_2M_3}{1-{r_1}^{-1}s}D_sW_{m-1} \\
\qquad\qquad\qquad + \dfrac{M_2M_3}{1-{r_1}^{-1}s}\displaystyle\sum_{k=0}^{m-1}|2\zeta|W_k + \dfrac{M_1e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s}, \\
D_sW_k \gg |2\zeta|W_{k+1} \qquad (k=0,\dots,m-2), \\
W_k \gg 0 \qquad (k=0,\dots,m-1),
\end{cases}
\tag{4.34}
$$

then there exists a power series $V(s,\zeta)$ which satisfies (4.33) and

$$
|2\zeta|^{m-k-1}D_s^kV(s,\zeta) \ll L^{1-m}W_k(s,\zeta) \ll W_k(s,\zeta)
\tag{4.35}
$$

for any k. Here we define:

$$
M_3 = \max_{0\le k\le m}\frac{(m-k+1)(k+n-1)!}{k!(n-1)!}.
$$

Indeed, we find that for any non-negative integer $j \ge m$ and power series $W_k(s,\zeta)$ which satisfy (4.34),

$$
\begin{aligned}
&\Big(\text{the coefficient of } s^{j-k} \text{ in } |2\zeta|^{m-k-1}D_s^kV(s,\zeta)\Big) \\
&\qquad = \frac{|2\zeta|^{m-k-1}}{(j-k)\dots(j-m+1)} \times \big(\text{the coefficient of } s^{j-m} \text{ in } D_s^mV(s,\zeta)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\text{the coefficient of } s^{j-k} \text{ in } L^{1-m}W_k(s,\zeta)\Big) \\
&\qquad \ge \frac{|2\zeta|^{m-k-1}}{(j-k)\dots(j-m+1)} \\
&\qquad\qquad \times \big(\text{the coefficient of } s^{j-m} \text{ in } L^{1-m}D_sW_{m-1}(s,\zeta)\big).
\end{aligned}
$$

Set:

$$
L = \max\{2M_2M_3, 1\},
$$

and consider the following ordinary differential equation:

$$
(4.36)\qquad \begin{cases} D_s W = \left(L - \dfrac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1} \left(\dfrac{M_2M_3}{1-{r_1}^{-1}s} m|2\zeta| W \right. \\ \qquad\qquad \left. + \dfrac{M_1 e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s} + (L - M_2M_3)|2\zeta| W \right), \\ W(0,\zeta) = 0. \end{cases}
$$

The functions a, b defined respectively by:

$$
\begin{aligned}
a(s,\zeta) &= \left(L - \frac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1} \left(\frac{M_2M_3}{1-{r_1}^{-1}s} m + (L - M_2M_3)\right) |2\zeta|, \\
b(s,\zeta) &= \left(L - \frac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1} \frac{M_1 e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s}
\end{aligned}
$$

are holomorphic in $D_{r_2} := \{s \in \mathbb{C}; |s| < r_2\}$ for each fixed ζ, since one has:

$$
\left| \frac{M_2M_3}{1-{r_1}^{-1}s} \right| < \frac{M_2M_3}{1-{r_1}^{-1}r_2} = 2M_2M_3 \le L, \quad s \in D_{r_2}.
$$

Moreover, it is clear that $a(s,\zeta) \gg 0$, $b(s,\zeta) \gg 0$ as the Taylor series at $s = 0$. Therefore there exists a unique power series solution of (4.36) so that $W(s,\zeta) \gg 0$.

On the other hand, the solution of (4.36) is written concretely as:

$$
\begin{aligned}
W(s,\zeta) &= \exp\left(\int_0^s a(s_2,\zeta)\, ds_2\right) \int_0^s b(s_1,\zeta) \exp\left(-\int_0^{s_1} a(s_2,\zeta)\, ds_2\right) ds_1 \\
&= \int_0^s b(s_1,\zeta) \exp\left(\int_{s_1}^s a(s_2,\zeta)\, ds_2\right) ds_1, \quad s \in D_{r_2}.
\end{aligned}
$$

Then it follows that $W(s,\zeta) \in \mathcal{O}(D_{r_2})$ for each fixed ζ.

For the solution $W(s,\zeta)$, we set:

$$
W_k(s,\zeta) = W(s,\zeta), \qquad k = 0, \dots, m-1.
$$

Then the power series $W_k(s,\zeta)$ satisfy (4.34). Indeed, one has:

$$\begin{aligned}
LD_sW &= \frac{M_2M_3}{1-{r_1}^{-1}s}D_sW + \frac{M_2M_3}{1-{r_1}^{-1}s}m|2\zeta|W \\
&\qquad + \frac{M_1e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s} + (L-M_2M_3)|2\zeta|W \\
&\gg \frac{M_2M_3}{1-{r_1}^{-1}s}D_sW + \frac{M_2M_3}{1-{r_1}^{-1}s}m|2\zeta|W + \frac{M_1e^{-\delta_1 \operatorname{Re}\zeta}}{1-{r_2}^{-1}s}, \\
D_sW &\gg \left(L - \frac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1}(L-M_2M_3)|2\zeta|W \\
&\gg |2\zeta|W.
\end{aligned}$$

From (4.35), we can obtain for each fixed ζ:

$$U_{\pm 0}(z,\zeta) \ll V(s,\zeta) \ll W(s,\zeta) \tag{4.37}$$

and

$$\begin{gathered}
W(Lz_1+z_2+\cdots+z_n,\zeta) \in \mathcal{O}(\Omega_{L,r_2}), \\
U_{\pm 0}(z,\zeta) \in \mathcal{O}(\Omega_{L,r_2}).
\end{gathered}$$

Finally we estimate the solution $U_{\pm 0}(z,\zeta)$ by using the majorant series $W(s,\zeta)$, that is to say, we show the exponential decay of $U_{\pm 0}(z,\zeta)$ as $\operatorname{Re}\zeta \to \infty$. Set:

$$\begin{aligned}
M_4 &= \sup_{s\in D_{r_2/2}} \left| \left(L - \frac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1} \left(\frac{M_2M_3}{1-{r_1}^{-1}s}m + (L-M_2M_3)\right) \right|, \\
M_5 &= \sup_{s\in D_{r_2/2}} \left| \left(L - \frac{M_2M_3}{1-{r_1}^{-1}s}\right)^{-1} \frac{M_1}{1-{r_2}^{-1}s} \right|.
\end{aligned}$$

Then by the definitions of the functions a, b, one has the following inequalities for $s \in D_{r_2/2}$:

$$\begin{aligned}
|a(s,\zeta)| &\le M_4|2\zeta|, \\
|b(s,\zeta)| &\le M_5 e^{-\delta_1 \operatorname{Re}\zeta}.
\end{aligned}$$

Choose a constant r_3 with $0 < r_3 < \min\{r_2/2, \delta_1/(2M_4)\}$. Then we have for $s \in D_{r_3}$:

$$\begin{aligned}
|W(s,\zeta)| &\le \int_0^s |b(s_1,\zeta)| \exp\left(\int_{s_1}^s |a(s_2,\zeta)|\cdot|ds_2|\right)|ds_1| \\
&\le \int_0^s M_5 e^{-\delta_1 \operatorname{Re}\zeta} \exp\left(\int_{s_1}^s M_4|2\zeta|\cdot|ds_2|\right)|ds_1| \\
&\le r_3 M_5 e^{-\delta_1 \operatorname{Re}\zeta} e^{r_3 M_4 |2\zeta|} \\
&\le r_3 M_5 e^{3r_3M_4} e^{-(\delta_1 - 2r_3M_4)\operatorname{Re}\zeta}.
\end{aligned}$$

We set $M_0 = r_3M_5e^{3r_3M_4} > 0$, $\delta_2 = \delta_1 - 2r_3M_4 > 0$. Thus we obtain from (4.37) for $z \in \Omega_{L,r_3}$:

$$\begin{aligned}
|U_{\pm 0}(z,\zeta)| &\le \sum_\alpha |u_{\pm\alpha}(\zeta)z^\alpha| \\
&\le W(L|z_1| + |z_2| + \cdots + |z_n|, \zeta) \\
&\le M_0 e^{-\delta_2 \operatorname{Re}\zeta}.
\end{aligned}$$

This completes the proof of Proposition 4.13. □

REMARK 4.14. In the situation of Proposition 4.13, we can choose the constant L independent of the parameter ζ because the principal symbol of $P(z, D_{z'}, z_nD_{z_n} \pm i\zeta - q)$ does not depend on ζ. So the domain Ω_{L,r_2} in which the solution is holomorphic does not shrink as $\operatorname{Re}\zeta \to \infty$.

As the corollary of Propositions 4.11 and 4.13, we can get the following result of the existence of solutions of (4.22) with infra-exponential growth order.

COROLLARY 4.15. *There exists $r_4 > 0$ such that for any ζ on the path γ, the differential equation (4.22) has a solution $\tilde{U}_\pm \in \mathcal{O}(D^n_{r_4})$ with the following estimate. For any $\varepsilon > 0$ there exists a constant $M_\varepsilon > 0$ such that one has*

$$\left|\tilde{U}_\pm(z,\zeta)\right| \le M_\varepsilon e^{\varepsilon \operatorname{Re}\zeta}$$

for any $z \in D^n_{r_4}$, any ζ on the path γ.

Proof. We set $r_4 = r_3/(nL) > 0$ and $\tilde{U}_\pm(z,\zeta) = U_{\pm 0}(z,\zeta) + U_{\pm 1}(z,\zeta)$ for $z \in D^n_{r_4}$, ζ on the path γ. From the equations (4.27) and (4.32) it follows that $\tilde{U}_\pm$ is a solution of (4.22). We also get the estimate of infra-exponential type by Propositions 4.11 and 4.13. $\square$

Remark 4.16. Our construction of the solution $\tilde{U}_\pm$ has made it possible to get the results in Corollary 4.15. If we consider (4.22) and the Cauchy boundary conditions directly, we cannot find a solution of infra-exponential type. For this reason one has reduced $G_\pm$ to $G_{\pm 0}$ which is exponentially decreasing, and considered the Cauchy problem (4.32).

Finally we define the following holomorphic functions on $D^{n-1}_{r_4} \times U_{r_4}$ for each fixed ζ on the path γ:

$$\begin{aligned} U_+(z,\zeta) &= (z_n)^{i\zeta - q}\tilde{U}_+(z,\zeta), \\ U_-(z,\zeta) &= e^{(-\zeta+iq)\pi}(z_n)^{-i\zeta-q}\tilde{U}_-(z,\zeta), \end{aligned}$$

where $0 < \arg z_n < \pi$. Then we have the differential equations (4.20) and (4.21) on $D^{n-1}_{r_4} \times U_{r_4}$, as we considered in the preceding subsection.

One defines for $z \in D^{n-1}_{r_4} \times U_{r_4}$:

$$\begin{aligned} U_\pm(z) &= \frac{1}{2\pi}\int_\gamma U_\pm(z,\zeta)\,d\zeta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U_\pm(z,\zeta(t))\zeta'(t)\,dt. \end{aligned}$$

Lemma 4.17. *The function $U_\pm(z,\zeta(t))\zeta'(t)$ is integrable over $\mathbb{R}$ for each $z \in D^{n-1}_{r_4} \times U_{r_4}$. Moreover, it follows that $U_\pm(z) \in \mathcal{O}(D^{n-1}_{r_4} \times U_{r_4})$.*

Proof. First, note that the functions $G_{\pm 0}(z,\zeta(t))$ and $U_{\pm 1}(z,\zeta(t))$ are measurable with respect to $t \in \mathbb{R}$ for each $z \in D^{n-1}_{r_4} \times U_{r_4}$. All coefficients of the Taylor series of $G_{\pm 0}$:

$$G_{\pm 0} = \sum_\alpha g_{\pm\alpha}(\zeta(t))z^\alpha$$

are measurable functions, too. Therefore we also find that $U_{\pm 0}(z,\zeta(t))$ is measurable with respect to $t \in \mathbb{R}$ from its construction in the Cauchy problem (4.32). Thus $\tilde{U}_{\pm}(z,\zeta(t))$ and $U_{\pm}(z,\zeta(t))$ are measurable functions for each fixed $z \in D_{r_4}^{n-1} \times U_{r_4}$.

Secondly, by Corollary 4.15, for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that one has:

$$\begin{aligned}
|U_+(z,\zeta)| &\le e^{\operatorname{Re}((i\zeta - q)\log z_n)} \left|\tilde{U}_+(z,\zeta)\right| \\
&\le M_\varepsilon e^{-2\log|z_n| - (\arg z_n - \varepsilon)\operatorname{Re}\zeta}, \\
|U_-(z,\zeta)| &\le e^{\operatorname{Re}((-\zeta + iq)\pi + (-i\zeta - q)\log z_n)} \left|\tilde{U}_-(z,\zeta)\right| \\
&\le M_\varepsilon e^{-2\log|z_n| - (\pi - \arg z_n - \varepsilon)\operatorname{Re}\zeta}.
\end{aligned}$$

Let z be any point of $D_{r_4}^{n-1} \times U_{r_4}$ such that $2\varepsilon < \arg z_n < \pi - 2\varepsilon$, $\varepsilon < |z_n|$. Then we find that $U_{\pm}(z,\zeta(t))\zeta'(t)$ is integrable over $\mathbb{R}$ for each z from the estimate:

$$|U_{\pm}(z,\zeta(t))\zeta'(t)| \le \begin{cases} M_\varepsilon e^{-2\log\varepsilon - \varepsilon(t-1)}, & t > 1 \\ \pi/4 M_\varepsilon e^{-2\log\varepsilon + \pi/2}, & -1 < t < 1 \\ M_\varepsilon e^{-2\log\varepsilon + \varepsilon(t+1)}, & t < -1. \end{cases}$$

Furthermore, the function $U_{\pm}(z)$ is holomorphic in $D_{r_4}^{n-1} \times U_{r_4}$, because the integral

$$\int_{-\infty}^{\infty} |U_{\pm}(z,\zeta(t))\zeta'(t)|\, dt$$

is locally finite on $D_{r_4}^{n-1} \times U_{r_4}$. $\square$

By using the holomorphic function $U_{\pm}(z)$, we can now obtain a solution in Theorem 3.2.

Integrate the both sides of (4.20) and (4.21) over the path γ. By Corollary 4.9 and Lemma 4.17, we have on $D_{r_4}^{n-1} \times U_{r_4}$:

$$P(z, D_{z'}, z_n D_{z_n}) U_{\pm}(z) = F_{\pm}(z', \log z_n).$$

Then we define the elements of $\mathcal{A}^2_{V,p_\circ}$:

$$u_{\pm}(x) = b_{D_{r_4}^{n-1} \times U_{r_4}}(U_{\pm}(z))$$

and

$$u(x) = u_+(x) + u_-(x).$$

Thus one obtains at $p_\circ$:

$$\begin{aligned} &P(x, D_{x'}, x_n D_{x_n})u(x) \\ &\quad = P(x, D_{x'}, x_n D_{x_n})u_+(x) + P(x, D_{x'}, x_n D_{x_n})u_-(x) \\ &\quad = b_{D_{r_4}^{n-1}\times U_{r_4}}(F_+(z', \log z_n)) + b_{D_{r_4}^{n-1}\times U_{r_4}}(F_-(z', \log z_n)) \\ &\quad = b_{D_{r_4}^{n-1}\times U_{r_4}}(F_\infty(z)) \\ &\quad = f(x). \end{aligned}$$

This completes the proof of Theorem 3.2.

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