J. Math. Sci. Univ. Tokyo
8 (2001), 463–499.

Optimal Partial Regularity for Nonlinear Elliptic Systems of Higher Order

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Abstract. We consider the question of partial regularity for weak solutions to homogeneous nonlinear elliptic systems of higher order in divergence form. As well as characterizing the singular set, we show optimal regularity for the solution on its regular set, optimality here being determined by the regularity of the coefficients.

0. Introduction

In this paper we are concerned with the regularity of weak solutions to homogeneous nonlinear elliptic systems of higher order in divergence form, i.e. we consider systems of the type

$$(0\text{-}1)\quad \sum_{i=1}^{N}\sum_{|\alpha|=m_i}\int_\Omega A_i^\alpha(\,\cdot\,,du,D^{\mathbf{m}}u)D^\alpha\varphi^i\,dx=0\quad\text{for all }\varphi\in C_0^\infty(\Omega,\mathbb{R}^N).$$

Here Ω is a bounded domain in $\mathbb{R}^n$, each $m_i\geq 1$ is an integer, $\mathbf{m}=(m_1,\dots m_N)$, u takes values in $\mathbb{R}^N$, $D^{\mathbf{m}}u$ stands for $\{D^\alpha u_i\}$ with $i=1,\dots,N$ and $|\alpha|=m_i$, and similarly du stands for $\{D^\alpha u_i\}$ with $i=1,\dots,N$ and $|\alpha|\leq m_i-1$. We assume that the coefficients $A_i^\alpha(x,\xi,\nu)$ are differentiable with respect to ν with bounded continuous derivatives, and further that they satisfy a uniform strong ellipticity condition. Moreover, we assume that $(x,\xi)\mapsto(1+|\nu|)^{-1}A(x,\xi,\nu)$ is Hölder continuous (uniformly in ν) for some exponent $s\in(0,1)$. Under these assumptions our **main result** can be stated as follows:
Let $u\in H^{\mathbf{m},2}(\Omega,\mathbb{R}^N)$ be a weak solution to (0-1) in Ω. Then there exists an open set $\Omega_0\subset\Omega$ such that $u\in C^{\mathbf{m},s}(\Omega_0,\mathbb{R}^N)$ and $\mathcal{L}^n(\Omega\setminus\Omega_0)=0$.
Here $H^{\mathbf{m},2}(\Omega,\mathbb{R}^N)$ stands for $H^{m_1,2}(\Omega,\mathbb{R})\times\ldots\times H^{m_N,2}(\Omega,\mathbb{R})$, with $C^{\mathbf{m},s}$ being defined analogously, and $\mathcal{L}^n$ denotes n-dimensional Lebesgue measure

2000 *Mathematics Subject Classification.* 35J45, 35G20, 35D10, 35J50.

on $\mathbb{R}^n$. A simple example shows that the demonstrated Hölder continuity of $D^{\mathbf{m}}u$ on the regular set Ω_0 is optimal (see Example 1.2).

The simplest case of systems of the form (0-1) are second-order systems (i.e., the case $m_i = 1$ for $i = 1, \dots n$). We note here that we make no new contribution to the theory in this case, and refer the reader to the monographs of Giaquinta [G1, G2] for results and discussions. In further discussion, we assume that at least one of the equations is of fourth or higher order.

In 1978 and 1979 Giaquinta–Modica obtained regularity results for weak solutions of (0-1). The first result [GM1] concerns the case of a quasilinear leading part, i.e. $A(x,\xi,\nu) = \tilde{A}(x,\xi)\nu$. In that case it is sufficient to assume that the coefficients $\tilde{A}(x,\xi)$ are bounded and continuous, and that a strong ellipticity condition holds. Then, in the homogeneous case it is shown that a weak solution u is of class $C^{\mathbf{m-1},\alpha}(\Omega_0,\mathbb{R}^N)$ for all $\alpha \in (0,1)$, for some open set $\Omega_0 \subset \Omega$ whose complement has vanishing $(n-q)$-dimensional Hausdorff measure for some q strictly greater than 2. The second paper [GM2] treats the case of general nonlinear systems, as in (0-1), under the assumptions given above on the coefficients $A(x,\xi,\nu)$. Giaquinta–Modica showed that a weak solution u of (0-1) admits Hölder continuous $\mathbf{m}$-th order derivatives $D^{\mathbf{m}}u$ for some positive Hölder exponent on an open set Ω_0 with $\mathcal{L}^n(\Omega \setminus \Omega_0) = 0$. In principle one would expect to be able to improve this result to obtain $C^{\mathbf{m},s}$-regularity on Ω_0. In the case $m_i = 1$ for $i = 1, \dots, N$ this can be done by interpreting the system (0-1) as a nonlinear system of the form

$$\sum_{i=1}^{N} \sum_{|a|=m_i} \int_\Omega B_i^\alpha(x, D^{\mathbf{m}}u(x)) D^\alpha \varphi^i(x)\, dx = 0, \tag{0-2}$$

with $B(x,\nu)$ given by $A(x,du(x),\nu)$, and applying the results of Hamburger concerning systems of that particular form, i.e., [Ha, Theorem 1.1, Theorem 1.2]. If one could generalize Hamburger's results to systems of the form (0-2) of higher order one would the be able to conclude the optimal regularity on the regular set Ω_0, i.e., $C^{\mathbf{m},s}$-regularity. Apart from other things one advantage of the method used in our paper is the fact that the optimal regularity is obtained in one step.

The existing proof combines three essential elements. The first element is an inequality of Caccioppoli type. For a given polynominal $P(x) = (P_1(x), \dots, P_N(x))$ of degree $\mathbf{m}$, this allows one to control the mean square deviation of $D^{\mathbf{m}}u$ from $D^{\mathbf{m}}P$ on a ball in terms of the mean square deviation of u from P on a concentric larger ball. If P is chosen such that $D^{\mathbf{m}}P$ is chosen to be the mean value of $D^{\mathbf{m}}u$ on that ball, this quantity is often called the excess of u. The second step of proof then consists in a way of improving the integrability of $D^{\mathbf{m}}(u-P)$. This is achieved by using the Sobolev-Poincaré inequality on the left-hand side of Caccioppoli's inequality to obtain a reverse Hölder-type inequality for $D^{\mathbf{m}}(u-P)$, i.e. an L^2-L^q estimate with some $q < 2$. Then, Gehring's Lemma in a form established by Giaquinta & Modica [GM1] implies higher L^p-integrability of $D^{\mathbf{m}}(u-P)$ for some $p > 2$. The third step in the regularity proof is excess improvement, i.e. assuming that the excess Φ is small on some given ball one has to show that the excess on a concentric smaller ball is substantially smaller than Φ. This discrete excess decay estimate leads to a growth condition for the excess on concentric balls as a function of the radius of the ball. From this excess growth condition a so called ε-regularity theorem follows along well-known lines, yielding also the partial regularity result.

The essential new feature in which our proof differs from the previous one is the method of improving the excess. In the work of Giaquinta–Modica, this is done by linearizing the system and freezing the coefficients on some ball with small excess, and then solving the associated Dirichlet problem for the elliptic constant coefficient operator (with coefficients Γ, say) on that ball with boundary values given by u. Our method here is to replace the solution u of our system (0-1) which is known to be approximatively Γ-harmonic (see Lemma 3.1), by a closest Γ-harmonic function h (i.e. harmonic with respect to the elliptic constant coefficient operator of order $2\mathbf{m}$ defined by Γ). This is made possible by a simple Lemma (see Lemma 3.2) analogous to [Si2, Lemma 21.1]. In order to show that the $\mathbf{m}$-jet of h at the center of the ball gives the desired excess improvement we use Caccioppoli's inequality which is established in Lemma 2.1. This technique, which has its origin in Simon's proof of the Allard regularity theorem [Si2], avoids the use of reverse Hölder inequalities (which, in principle, are the obstacle for proving the optimal regularity result with respect to the Hölder exponent in one step; see the remark above). This technique was applied by the first and third authors in [DG] to obtain optimal partial regularity in a single step in the case of second order systems. A similar technique was used by the first author and Steffen in [DS], where optimal boundary regularity is established for almost minimizing rectifiable currents of general elliptic integrands. Finally, we mention that Kronz [Kr] has obtained a similar partial regularity result for minimizers of quasi-convex variational integrals $\int_\Omega f(D^m u)\,dx$ of higher order $m \geq 2$ with the technique of Γ-harmonic approximation described above. It is worth noting that his paper also deals with the case of integrands of nonquadratic growth, i.e. $f(\nu) \leq a|\nu|^p + b$ for some exponent $p > 2$, using the same technique.

We close this section by briefly summarizing the notation we use in this paper. As noted above, we consider a domain $\Omega \subset \mathbb{R}^n$, and maps from Ω to $\mathbb{R}^N$, where we take $n \geq 2$, $N \geq 1$. For a given set X we denote by $\mathcal{L}^n(X)$ its n–dimensional Lebesgue measure. We write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$, and further set $B_\rho = B_\rho(0)$, $B = B_1$. For bounded $X \subset \mathbb{R}^n$ we denote the average of a given $g \in L^1(X)$ by $⨍_X g\,dx$, i.e. $⨍_X g\,dx = \frac{1}{\mathcal{L}^n(X)} \int_X g\,dx$. In particular, we write $g_{x_0,\rho} = ⨍_{B_\rho(x_0)} g\,dx$. We let $\boldsymbol{\alpha}_n$ denote the volume of the unit–ball in $\mathbb{R}^n$, i.e. $\boldsymbol{\alpha}_n = \mathcal{L}^n(B)$. We write $\mathcal{L}(k,n)$ for the space of k-linear maps on $\mathbb{R}^n$.

Throughout the paper we consider a fixed domain $\Omega \subset \mathbb{R}^n$. We further consider a fixed multiindex $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{N}^N$, and define the space

$$H^{\mathbf{m},2}(\Omega, \mathbb{R}^N) = H^{m_1,2}(\Omega, \mathbb{R}) \times \ldots \times H^{m_N,2}(\Omega, \mathbb{R}).$$

We set $m = \max_{1 \leq k \leq N} m_k$. Given polynomials $R^1(x), \ldots, R^N(x)$ defined on $\mathbb{R}^n$, where the degree of R^k is j_k, for $k = 1, \ldots, N$ we let $R(x)$ denote the (vector-valued) polynomial given by $(R^1(x), \ldots, R^N(x))$, which we term a *polynomial of degree* $\mathbf{j}$ (on $\mathbb{R}^n$), where $\mathbf{j}$ is the multiindex $(j_1, \ldots, j_N)$.

For later use we recall also some elementary calculations useful in connection with multiindices. For given $m \in \mathbb{N} \cup \{0\}$ we have

$$\operatorname{card}\{|\alpha| = m\} = \binom{n+m-1}{m} \quad \text{and} \quad \operatorname{card}\{|\alpha| \leq m\} = \binom{n+m}{m}.$$

Further, setting

$$\begin{aligned} \ell &= \sum_{i=1}^N \operatorname{card}\{\alpha : |\alpha| \leq m_i - 1\} = \sum_{i=1}^N \binom{n+m_i-1}{m_i-1}, \quad \text{and} \\ \tau &= \sum_{i=1}^N \operatorname{card}\{\alpha : |\alpha| = m_i\} = \sum_{i=1}^N \binom{n+m_i-1}{m_i}, \end{aligned}$$

we note the estimates $\ell \leq N(n+1)^{m-1}$ and $\tau \leq N(n+1)^m$.

1. Assumptions, Preliminary Lemmas and the Partial Regularity Theorem

We wish to consider a weak solution $u \in H^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ of the equation

$$\text{(1-1)} \qquad \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_\Omega A_i^\alpha(x, du(x), D^{\mathbf{m}}u(x)) D^\alpha \varphi^i(x)\, dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N),$$

where

$$du(x) = \Big\{ D^\gamma u^k(x) \Big\}_{k=1,\ldots,N}^{|\gamma| \le m_k - 1} \quad \text{and} \quad D^{\mathbf{m}}u(x) = \Big\{ D^\alpha u^i(x) \Big\}_{i=1,\ldots,N}^{|\alpha|=m_i}.$$

For later use we also abbreviate $\Big\{ D^\alpha u^i(x) \Big\}_{i=1,\ldots,N}^{|\alpha|=m_i-1}$ by $D^{\mathbf{m-1}}u(x)$. Note that du and $D^{\mathbf{m}}u$ take values in $\mathbb{R}^\ell$ and $\mathbb{R}^\tau$, respectively. The coefficients $A_i^\alpha(x,\xi,\nu)$, $|\alpha| = m_i$, $i = 1, \ldots, N$, are defined on $\Omega \times \mathbb{R}^\ell \times \mathbb{R}^\tau$. They define linear maps $A(x,\xi,\nu) : \mathbb{R}^\tau \to \mathbb{R}$ and $\frac{\partial A}{\partial \nu}(x,\xi,\nu) : (\mathbb{R}^\tau)^2 \to L(\mathbb{R}^\tau, \mathbb{R})$; we denote their operator norms by $|\cdot|$. We impose the following structure conditions:

(H1) $A(x,\xi,\nu)$ is differentiable with respect to ν with bounded, continuous derivatives, i.e. for some $L > 0$ there holds

$$\Big| \frac{\partial A}{\partial \nu}(x,\xi,\nu) \Big| \le L \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^\ell,\ \nu \in \mathbb{R}^\tau;$$

(H2) A is **uniformly strongly elliptic**, i.e. for some $\lambda > 0$ we have

$$\Big(\frac{\partial A}{\partial \nu}(x,\xi,\nu)\eta \Big) \eta \ge \lambda |\eta|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^\ell,\ \nu,\ \eta \in \mathbb{R}^\tau;$$

(H3) There exist $s \in (0,1)$ and a nondecreasing function $\kappa\colon [0,\infty) \to [1,\infty)$ such that

$$|A(x,\xi,\nu) - A(\tilde{x},\tilde{\xi},\nu)| \le \kappa(|\xi|) \Big(|x - \tilde{x}|^2 + |\xi - \tilde{\xi}|^2 \Big)^{s/2} (1 + |\nu|).$$

for all x, $\tilde{x} \in \Omega$, ξ, $\tilde{\xi} \in \mathbb{R}^\ell$, and $\nu \in \mathbb{R}^\tau$.

From (H2) we obtain

$$(A_i^\alpha(x,\xi,\nu) - A_i^\alpha(x,\xi,\tilde\nu))\,(\nu_i^\alpha - \tilde\nu_i^\alpha) \ge \lambda|\nu-\tilde\nu|^2\,. \tag{1-2}$$

Further, (H1) guarantees the existence of a modulus of continuity ω: $[0,\infty)\times[0,\infty)\to[0,\infty)$ with $\omega(t,0)=0$ for all t such that $t\mapsto\omega(t,s)$ is nondecreasing for fixed s, $s\mapsto\omega^2(t,s)$ is concave and nondecreasing for fixed t, and such that for all x, $\tilde x\in\Omega$, ξ, $\tilde\xi\in\mathbb{R}^\ell$, ν, $\tilde\nu\in\mathbb{R}^\tau$ with $|\xi|+|\nu|\le M$ we have

$$\Big|\frac{\partial A}{\partial\nu}(x,\xi,\nu) - \frac{\partial A}{\partial\nu}(\tilde x,\tilde\xi,\tilde\nu)\Big| \le \omega(M,|x-\tilde x|^2+|\xi-\tilde\xi|^2+|\nu-\tilde\nu|^2)\,. \tag{1-3}$$

We will prove the following **partial regularity theorem** concerning weak solutions of (1-1).

THEOREM 1.1. *On $\Omega\times\mathbb{R}^\ell\times\mathbb{R}^\tau$ consider coefficients $A(x,\xi,\nu)$ satisfying (H1), (H2), and (H3) and let $u\in H^{\mathbf{m},2}(\Omega,\mathbb{R}^N)$ be a weak solution of (1-1). Then there exists a relatively closed set* $\mathrm{Sing}(u)$ *such that $u\in C^{\mathbf{m},s}(\Omega\setminus\mathrm{Sing}(u),\mathbb{R}^N)$. Moreover* $\mathrm{Sing}(u)\subset\Sigma_1\cup\Sigma_2$ *where*

$$\Sigma_1 = \Big\{x_0\in\Omega : \liminf_{\rho\to0} ⨍_{B_\rho(x_0)} |D^{\mathbf{m}}u-(D^{\mathbf{m}}u)_{x_0,\rho}|^2\,dx>0\Big\} \quad \textit{and}$$

$$\Sigma_2 = \Big\{x_0\in\Omega : \limsup_{\rho\to0}\sum_{k=1}^N\sum_{|\alpha|\le m_k}|(D^\alpha u^k)_{x_0,\rho}|=\infty\Big\}.$$

The following (essentially one-dimensional) example shows that the asserted Hölder continuity of $D^{\mathbf{m}}u$ on the regular set is optimal:

Example 1.2. Let $m\ge1$, $n\ge2$, $N=1$, $\Omega=B$, $s\in(0,1)$, and

$$A(x,\xi,\nu) := \frac{\nu}{1+|x_1|^s}\,.$$

Then (H1)–(H3) are fulfilled, and

$$u(x) = \mathrm{sign}(x_1)^m|x_1|^{m+s} + \binom{m+s}{m}x_1^m$$

solves the equation (1-1). We have $\Sigma_1 = \Sigma_2 = \emptyset$, and u is of class $C^{m,s}$, but no more regular, on B.

We note here that any $\varphi \in H_0^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ is admissible as test function in the weak formulation of our system. This is a straightforward consequence of (H1) and (H3).

We also record here for later use a standard result. Consider $u \in H^k(\Omega, \mathbb{R})$ with $k \geq 1$. Then there exists a unique polynomial Q of degree less than or equal to k such that

$$\int_{B_\rho(x_0)} D^\alpha(u - Q)\, dx = 0$$

for all α with $|\alpha| \leq k$. This polynomial takes the form

$$Q(x) = \sum_{|\alpha|\leq k} \sum_{|\alpha+\beta|\leq k} \frac{b_\beta}{\alpha!} \rho^{|\beta|} (D^{\alpha+\beta} u)_{x_0,\rho} (x - x_0)^\alpha \tag{1-4}$$

with coefficients b_β depending on k and n only. Indeed, the constants b_β can be defined recursively by

$$\begin{cases} b_0 = 1\,, & \\ b_\sigma = 0\,, & \text{if } |\sigma| = 1, \\ b_\sigma = -\displaystyle\sum_{0<\beta\leq\sigma} \frac{b_{\sigma-\beta}}{\beta!} \fint_B x^\beta\, dx\,, & \text{if } |\sigma| \geq 2. \end{cases} \tag{1-5}$$

2. A Caccioppoli-type Inequality

THEOREM 2.1. *Let $B_\rho(x_0) \subset\subset \Omega$, with $\rho \leq 1$. Consider an arbitrary solution $u \in H^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ of (1-1), where the structure conditions (H1), (H2) and (H3) are valid, and an arbitrary polynomial P of degree at most* $\mathbf{m}$. *Then there holds:*

$$\int_{B_{\rho/2}(x_0)} |D^{\mathbf{m}}(u - P)|^2\, dx \leq c_1 \Bigg[\sum_{i=1}^{N} \sum_{|\gamma|\leq m_i - 1} \rho^{2(|\gamma| - m_i)} \int_{B_\rho(x_0)} |D^\gamma(u - P)|^2\, dx$$

$$+\boldsymbol{\alpha}_n\Big(\kappa(\ell p)(1+\ell p)\Big)^{\frac{2}{1-s}}\rho^{n+2s}\Bigg],$$

for a constant $c_1 \geq 1$ depending only on n, N, m, λ, and L. Here p is given by

$$p = \sum_{k=1}^{N}\sum_{|\gamma|\leq m_k}|D^\gamma P^k(x_0)|.$$

PROOF. We consider a fixed cut-off function $\eta \in C_c^\infty(B_\rho(x_0),[0,1])$ with $\eta \equiv 1$ on $B_{\rho/2}(x_0)$, and c_2 depending only on n and m such that $|D^\gamma\eta| \leq (c_2\rho)^{-|\gamma|}$ for $|\gamma| \leq m$ (recalling that $m = \max_{i=1,\dots,N} m_i$). Note that we have the elementary estimate

$$|D^\beta(\eta^{2m})| \leq c_3\rho^{-|\beta|}\eta^{2m-|\beta|} \quad \text{for } |\beta| \leq m\,, \tag{2-1}$$

for c_3 depending only on n and m. Taking the function φ with components $\varphi^i = \eta^{2m}(u^i - P^i)$, $i = 1,\dots,N$, as a test-function in (1-1), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\int_{B_\rho(x_0)} A_i^\alpha(\,\cdot\,,du,D^{\mathbf{m}}u)D^\alpha(u^i-P^i)\eta^{2m}\,dx\\
&\quad= -\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\\
&\qquad\cdot\int_{B_\rho(x_0)} A_i^\alpha(\,\cdot\,,du,D^{\mathbf{m}}u)D^\gamma(u^i-P^i)D^{\alpha-\gamma}(\eta^{2m})\,dx\,.
\end{aligned}$$

By the definition of φ we further have

$$\begin{aligned}
&-\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\int_{B_\rho(x_0)} A_i^\alpha(x,du,D^{\mathbf{m}}P)D^\alpha(u^i-P^i)\eta^{2m}\,dx\\
&\quad= \sum_{i=1}^{N}\sum_{|\alpha|=m_i}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\\
&\qquad\cdot\int_{B_\rho(x_0)} A_i^\alpha(\,\cdot\,,du,D^{\mathbf{m}}P)D^\gamma(u^i-P^i)D^{\alpha-\gamma}(\eta^{2m})\,dx\\
&\qquad-\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\int_{B_\rho(x_0)} A_i^\alpha(\,\cdot\,,du,D^{\mathbf{m}}P)D^\alpha\varphi^i\,dx\,.
\end{aligned}$$

Finally we note

$$0 = \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B_\rho(x_0)} A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}} P) D^\alpha \varphi^i \, dx.$$

Combining these three equations, we arrive at the inequality

$$\sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} u) - A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} P)] \cdot D^\alpha(u^i - P^i)\eta^{2m} \, dx \leq I + II + III + IV + V\,, \tag{2-2}$$

where I–V are defined as follows:

$$I = \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} u) - A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} P)] \cdot D^\gamma(u^i - P^i) D^{\alpha-\gamma}(\eta^{2m}) \, dx \Big|,$$

$$II = \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} P) - A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}} P)] \cdot D^\alpha(u^i - P^i)\eta^{2m} \, dx \Big|,$$

$$III = \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}} P) - A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}} P)] \cdot D^\gamma(u^i - P^i) D^{\alpha-\gamma}(\eta^{2m}) \, dx \Big|,$$

$$IV = \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}} P) - A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}} P)] \cdot D^\alpha(u^i - P^i)\eta^{2m} \, dx \Big|,$$

and

$$V = \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma}$$

$$\cdot \int_{B_\rho(x_0)} [A_i^\alpha(\cdot\,, dP, D^{\mathbf{m}}P) - A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}}P)] \cdot D^\gamma(u^i - P^i) D^{\alpha-\gamma}(\eta^{2m})\, dx \Big| .$$

Using (H1) we have

$$I \le L \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)| \cdot \Big[\sum_{i=1}^N \sum_{|\alpha|=m_i} \Big(\sum_{\gamma<\alpha} \binom{\alpha}{\gamma} D^\gamma(u^i - P^i) D^{\alpha-\gamma}\eta^{2m} \Big)^2 \Big]^{1/2} dx .$$

Applying the elementary inequality $a_1^2 + \ldots + a_\ell^2 \le (a_1 + \ldots + a_\ell)^2 \le \ell(a_1^2 + \ldots + a_\ell^2)$ (where ℓ denotes the number of components of du, as defined in Section 1), Young's inequality, and (2-1) we see, for $\varepsilon > 0$ to be specified later:

$$\begin{aligned} I &\le L \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)| \sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |D^\gamma(u^i - P^i)| |D^{\alpha-\gamma}\eta^{2m}|\, dx \\ &\le Lc_3 \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)| \\ &\quad \cdot \sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |D^\gamma(u^i - P^i)| \rho^{-(m_i-|\gamma|)} \eta^{2m-(m_i-|\gamma|)}\, dx \\ &\le \varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m}\, dx \\ &\quad + \frac{c_3^2 L^2}{\varepsilon} \Big[\sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} \rho^{|\gamma|-m_i} \int_{B_\rho(x_0)} |D^\gamma(u^i - P^i)|\, dx \Big]^2 \\ &= \varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m}\, dx \\ &\quad + \frac{c_3^2 (m!)^2 L^2 \ell}{\varepsilon} \sum_{i=1}^N \sum_{j=0}^{m_i-1} \sum_{|\gamma|=j} \rho^{2(j-m_i)} \int_{B_\rho(x_0)} |D^\gamma(u^i - P^i)|^2\, dx . \end{aligned}$$

To estimate II we first note that $|dP| \le \ell p$ and recall $D^\beta P^i(x) = D^\beta P^i(x_0)$ for $|\beta| = m_i$; hence $|D^{\mathbf{m}}P| \le p$. We introduce the function $\widetilde{\kappa}$ from $[0, \infty)$ to

$[0,\infty)$, defined by $\widetilde{\kappa}(s)=\kappa(\ell s)$. Now using (H3), the fact that $\rho\le 1$, and repeatedly using Young's inequality, we see

$$\begin{aligned}
II &\le \int_{B_\rho(x_0)} \kappa(|dP|)|d(u-P)|^s\,(1+|D^{\mathbf{m}}P|)\,|D^{\mathbf{m}}(u-P)|\,\eta^{2m}\,dx \\
&\le \widetilde{\kappa}(p)(1+p)\int_{B_\rho(x_0)} |d(u-P)|^s\,|D^{\mathbf{m}}(u-P)|\,\eta^{2m}\,dx \\
&\le \varepsilon\int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2\eta^{2m}\,dx \\
&\quad + \frac{1}{\varepsilon}\widetilde{\kappa}^2(p)(1+p)^2\int_{B_\rho(x_0)} \rho^{2s}\Big(\frac{|d(u-P)|}{\rho}\Big)^{2s}\,dx \\
&\le \varepsilon\int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2\eta^{2m}\,dx + \frac{1}{\varepsilon}\Big(\widetilde{\kappa}(p)(1+p)\Big)^{\frac{2}{1-s}}\boldsymbol{\alpha}_n\rho^{n+\frac{2s}{1-s}} \\
&\quad + \frac{1}{\varepsilon\rho^2}\int_{B_\rho(x_0)} |d(u-P)|^2\,dx \\
&\le \varepsilon\int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2\eta^{2m}\,dx + \frac{1}{\varepsilon}\Big(\widetilde{\kappa}(p)(1+p)\Big)^{\frac{2}{1-s}}\boldsymbol{\alpha}_n\rho^{n+\frac{2s}{1-s}} \\
&\quad + \frac{1}{\varepsilon}\sum_{i=1}^{N}\sum_{j=0}^{m_i-1}\sum_{|\gamma|=j}\rho^{2(j-m_i)}\int_{B_\rho(x_0)} |D^\gamma(u^i-P^i)|^2\,dx
\end{aligned}$$

We argue similarly to estimate III:

$$\begin{aligned}
III &\le \widetilde{\kappa}(p)(1+p)\int_{B_\rho(x_0)} |d(u-P)|^s \\
&\qquad \cdot\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}|D^\gamma(u^i-P^i)||D^{\alpha-\gamma}\eta^{2m}|\,dx \\
&\le \Big(\widetilde{\kappa}(p)(1+p)\Big)^2\int_{B_\rho(x_0)} \rho^{2s}\Big(\frac{|d(u-P)|}{\rho}\Big)^{2s}\,dx \\
&\quad + \Big(\sum_{i=1}^{N}\sum_{|\alpha|=m_i}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma}\int_{B_\rho(x_0)} |D^\gamma(u^i-P^i)||D^{\alpha-\gamma}\eta^{2m}|\,dx\Big)^2 \\
&\le \Big(\widetilde{\kappa}(p)(1+p)\Big)^{\frac{2}{1-s}}\boldsymbol{\alpha}_n\rho^{n+\frac{2s}{1-s}} + \frac{1}{\rho^2}\int_{B_\rho(x_0)} |d(u-P)|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&\quad + \ell c_3^2 \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma}^2 \rho^{2(|\gamma|-m_i)} \int_{B_\rho(x_0)} |D^\gamma(u^i - P^i)|^2 \, dx \\
&\leq \Big(1 + (m!)^2 \ell c_3^2\Big) \sum_{i=1}^{N} \sum_{j=0}^{m_i-1} \sum_{|\gamma|=j} \rho^{2(j-m_i)} \int_{B_\rho(x_0)} |D^\gamma(u^i - P^i)|^2 \, dx \\
&\quad + \Big(\widetilde{\kappa}(p)(1+p)\Big)^{\frac{2}{1-s}} \boldsymbol{\alpha}_n \rho^{n+\frac{2s}{1-s}} .
\end{aligned}
$$

To estimate IV we use again (H3) and Young's inequality and obtain

$$
\begin{aligned}
IV &\leq \varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m} \, dx \\
&\quad + \frac{1}{\varepsilon} \widetilde{\kappa}^2(p)(1+p)^2 \int_{B_\rho(x_0)} \Big[|x-x_0|^2 + |dP - dP(x_0)|^2\Big]^s \, dx .
\end{aligned}
$$

Noting

$$
\begin{aligned}
\text{(2-3)} \qquad & |dP - dP(x_0)| \\
&\leq \sum_{i=1}^{N} \sum_{|\gamma|\leq m_i-1} |D^\gamma P^i - D^\gamma P^i(x_0)| \\
&= \sum_{i=1}^{N} \sum_{|\gamma|\leq m_i-1} \Big| \sum_{\substack{|\alpha|\leq m_i \\ \alpha>\gamma}} \frac{1}{(\alpha-\gamma)!} D^\alpha P^i(x_0)(x-x_0)^{\alpha-\gamma} \Big| \\
&\leq \rho \sum_{i=1}^{N} \sum_{|\gamma|\leq m_i-1} \sum_{\substack{|\alpha|\leq m_i \\ \alpha>\gamma}} |D^\alpha P^i(x_0)| \leq \ell \rho p ,
\end{aligned}
$$

we can further estimate IV (recalling also $\rho \leq 1$)

$$
\begin{aligned}
IV &\leq \varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m} \, dx + \frac{1}{\varepsilon} \widetilde{\kappa}^2(p)(1+p)^2(1+\ell^2 p^2)^s \boldsymbol{\alpha}_n \rho^{n+2s} \\
&\leq \varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m} \, dx + \frac{1}{\varepsilon} \widetilde{\kappa}^2(p)(1+\ell p)^{2(1+s)} \boldsymbol{\alpha}_n \rho^{n+2s} .
\end{aligned}
$$

Finally, we estimate V, again using (H3) and (2-1), to obtain the estimate

$$
V \leq \Big| \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B_\rho(x_0)} [A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}}P) - A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}}P)]
$$

$$
\begin{aligned}
&\qquad\qquad \cdot \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} D^\gamma(u^i - P^i) D^{\alpha-\gamma}\eta^{2m}\,dx \Big| \\
\le\;& \kappa(|dP(x_0)|)(1+p) \int_{B_\rho(x_0)} \Big[|x-x_0|^2 + |dP - dP(x_0)|^2 \Big]^{s/2} \\
&\qquad \cdot \sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |D^\gamma(u^i-P^i)||D^{\alpha-\gamma}\eta^{2m}|\,dx \\
\le\;& \kappa(p)(1+\ell p)^{1+s}\rho^s \\
&\cdot \int_{B_\rho(x_0)} \sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma} |D^\gamma(u^i-P^i)||D^{\alpha-\gamma}\eta^{2m}|\,dx \\
\le\;& \kappa(p)^2(1+\ell p)^{2(1+s)} \boldsymbol{\alpha}_n \rho^{n+2s} \\
&+ \ell \sum_{i=1}^N \sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} \binom{\alpha}{\gamma}^2 \int_{B_\rho(x_0)} |D^\gamma(u^i-P^i)|^2 |D^{\alpha-\gamma}\eta^{2m}|^2\,dx \\
\le\;& \kappa(p)^2(1+\ell p)^{2(1+s)} \boldsymbol{\alpha}_n \rho^{n+2s} \\
&+ \ell (m!)^2 \sum_{i=1}^N \sum_{j=0}^{m_i-1} \sum_{|\gamma|=j} \rho^{2(j-m_i)} \int_{B_\rho(x_0)} |D^\gamma(u^i-P^i)|^2\,dx\,.
\end{aligned}
$$

Combining these estimates and applying (1-2) to the left hand side of (2-2), we arrive at

$$
\begin{aligned}
&\lambda \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2 \eta^{2m}\,dx \le I + II + III + IV + V \\
\le\;& 3\varepsilon \int_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2\eta^{2m}\,dx \\
&+ 2\Big(1+\frac{1}{\varepsilon}\Big)\Big(\widetilde{\kappa}(p)(1+\ell p)\Big)^{\frac{2}{2-s}} \boldsymbol{\alpha}_n \rho^{n+2s} \\
&+ \Big(\ell c_3^2 (m!)^2 + 1\Big)\Big(\frac{L^2}{\varepsilon} + 2\Big) \sum_{i=1}^N \sum_{j=0}^{m_i-1} \sum_{|\gamma|=j} \rho^{2(j-m_i)} \\
&\cdot \int_{B_\rho(x_0)} |D^\gamma(u^i-P^i)|^2\,dx\,.
\end{aligned}
$$

Choosing $\varepsilon := \lambda/6$ we obtain the desired conclusion with c_1 appropriately chosen. □

3. Approximately m-harmonic Functions

The next lemma is required in order to be able to apply the harmonic approximation technique. For a function $f \in H^{\mathbf{m},2}(B_\rho(x_0), \mathbb{R}^N)$ we set $\Phi(x_0, \rho, f) := \fint_{B_\rho(x_0)} |D^{\mathbf{m}}(u-f)|^2\,dx$.

Lemma 3.1. *Let $u \in H^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ be a weak solution of (1-1). Then for every polynomial P of degree $\mathbf{m}$ and every ball $B_\rho(x_0) \subset\subset \Omega$ with $\rho \le 1$, there holds:*

$$\Big|\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_{B_\rho(x_0)} \frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0, dP(x_0), D^{\mathbf{m}}P) D^\beta(u^j - P^j) D^\alpha \varphi^i\,dx\Big|$$
$$\le \Big[2\Phi(x_0,\rho,P) + \omega(p, \Phi(x_0,\rho,P))\sqrt{\Phi(x_0,\rho,P)} + 4(\kappa(\ell p)(1+\ell p))^{\frac{2}{1-s}}\rho^s$$
$$+ 2\sum_{i=1}^N \sum_{|\gamma|\le m_i-1} \rho^{2(|\gamma|-m_i)} \fint_{B_\rho(x_0)} |D^\gamma(u^i-P^i)|^2\,dx\Big] \sup_{B_\rho(x_0)} |D^{\mathbf{m}}\varphi|,$$

for all $\varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N)$, where $p = \sum_{i=1}^N \sum_{|\alpha|=m_i} |D^\alpha P^i(x_0)|$.

Proof. We begin by assuming $|D^{\mathbf{m}}\varphi| \le 1$. From (1-1) we have, using (1-3):

$$\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_{B_\rho(x_0)} \frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0, dP(x_0), D^{\mathbf{m}}P) D^\beta(u^j-P^j) D^\alpha\varphi^i\,dx \tag{3-1}$$
$$= \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_{B_\rho(x_0)} \Big[\int_0^1 \Big(\frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0, dP(x_0), D^{\mathbf{m}}P) -$$
$$\frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0, dP(x_0), D^{\mathbf{m}}(P + t(u-P)))\Big)\,dt\Big]$$
$$\cdot D^\beta(u^j-P^j) D^\alpha\varphi^i\,dx$$
$$+ \sum_{i=1}^N \sum_{|\alpha|=m_i} \fint_{B_\rho(x_0)} \Big(A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}}u)$$
$$- A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}}u)\Big) D^\alpha\varphi^i\,dx$$
$$= I + II + III,$$

where I denotes the first term of the right-hand side of (3-1) and II and III are defined as follows

$$\begin{aligned} II &= \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \fint_{B_\rho(x_0)} (A_i^\alpha(x_0, dP(x_0), D^{\mathbf{m}}u) \\ &\qquad\qquad - A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}}u))D^\alpha\varphi^i\,dx\,, \\ III &= \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \fint_{B_\rho(x_0)} (A_i^\alpha(\,\cdot\,, dP, D^{\mathbf{m}}u) - A_i^\alpha(\,\cdot\,, du, D^{\mathbf{m}}u))D^\alpha\varphi^i\,dx\,. \end{aligned}$$

From (1-3), $\sup_{B_\rho(x_0)} |D^{\mathbf{m}}\varphi| \le 1$, Cauchy-Schwarz's inequality the concavity of the function $s \mapsto \omega(t,s)^2$, and Jensen's inequality we infer

$$\begin{aligned} |I| &\le \fint_{B_\rho(x_0)} \omega(|dP(x_0)| + |D^{\mathbf{m}}P(x_0)|, |D^{\mathbf{m}}(u-P)|^2)|D^{\mathbf{m}}(u-P)|\,dx \\ &\le \Big(\fint_{B_\rho(x_0)} \omega^2(p, |D^{\mathbf{m}}(u-P)|^2)\,dx\Big)^{1/2}\Big(\fint_{B_\rho(x_0)} |D^{\mathbf{m}}(u-P)|^2\,dx\Big)^{1/2} \\ &\le \omega(p, \Phi(x_0,\rho,P))\Phi^{1/2}(x_0,\rho,P)\,. \end{aligned}$$

The second term can be estimated using (H3), Young's inequality and (2-3) via:

$$\begin{aligned} |II| &\le \kappa(|dP(x_0)|) \\ &\qquad \cdot \fint_{B_\rho(x_0)} \Big(|x-x_0|^2 + |dP - dP(x_0)|^2\Big)^{s/2}(1 + |D^{\mathbf{m}}u|)\,dx \\ &\le \kappa(p)(1+\ell p)^s \rho^s \fint_{B_\rho(x_0)} (1 + p + |D^{\mathbf{m}}(u-P)|)\,dx \\ &\le \kappa(p)(1+\ell p)^{1+s}\rho^s + \kappa^2(p)(1+\ell p)^{2s}\rho^{2s} + \Phi(x_0,\rho,P) \\ &\le 2\kappa^2(p)(1+\ell p)^{1+s}\rho^s + \Phi(x_0,\rho,P)\,; \end{aligned}$$

in the last line we also have used $\rho \le 1$ and $\kappa \ge 1$. Similarly we obtain, since $|dP| \le \ell p$ on $B_\rho(x_0)$:

$$\begin{aligned} |III| &\le \fint_{B_\rho(x_0)} \kappa(|dP|)|d(u-P)|^s(1 + |D^{\mathbf{m}}u|)\,dx \\ &\le \kappa(\ell p)(1+p) \fint_{B_\rho(x_0)} |d(u-P)|^s\,dx \end{aligned}$$

$$
\begin{aligned}
&\quad + \kappa(\ell p) \fint_{B_\rho(x_0)} |d(u-P)|^s |D^{\mathbf{m}}(u-P)|\, dx \\
&\le \kappa(\ell p)(1+p) \fint_{B_\rho(x_0)} |d(u-P)|^s\, dx \\
&\quad + \kappa^2(\ell p) \fint_{B_\rho(x_0)} |d(u-P)|^{2s}\, dx + \Phi(x_0,\rho,P) \\
&= (\kappa(\ell p)(1+p))^{\frac{2}{2-s}} \rho^{\frac{2s}{2-s}} + \frac{2}{\rho^2} \fint_{B_\rho(x_0)} |d(u-P)|^2\, dx \\
&\quad + (\kappa^2(\ell p)\rho^{2s})^{\frac{1}{1-s}} + \Phi(x_0,\rho,P) \\
&\le \frac{2}{\rho^2} \fint_{B_\rho(x_0)} |d(u-P)|^2\, dx + \Phi(x_0,\rho,P) + 2(\kappa(\ell p)(1+p))^{\frac{2}{1-s}} \rho^s .
\end{aligned}
$$

Combining these estimates in (3-1) yields the desired conclusion for φ with $|D^{\mathbf{m}}\varphi| \le 1$; a simple rescaling argument then yields the result for arbitrary $\varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N)$. $\square$

The next result, the $\mathbf{m}$-harmonic approximation lemma, is central to our technique. We refer the reader to the introduction for more comments on this technique, and confine ourselves here to noting that, in the case of a second-order system the result was given in a more general form in [DS, Lemma 3.3] (cf. [Si1, Section 1.6] for the case of Laplace's equation; see also [DG, Lemma 2.1] for the $\mathbf{m} = 1$ analogue).

LEMMA 3.2 ($\mathbf{m}$-harmonic approximation lemma). *For any given $\varepsilon > 0$, there exists $\delta = \delta(n, N, \lambda, L, \mathbf{m}, \varepsilon) \in (0,1]$ with the following property: for any given coefficients $\{a_{ij}^{\alpha\beta}\}$, ($1 \le i,j \le N$, $|\alpha| = m_i$, $|\beta| = m_j$) satisfying:*

$$
\text{(3-2)} \qquad \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} a_{ij}^{\alpha\beta} \nu_i^\alpha \nu_j^\beta \ge \lambda |\nu|^2 \quad \textit{for all } \nu \in \mathbb{R}^\tau, \textit{ and}
$$

$$
\text{(3-3)} \qquad \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} a_{ij}^{\alpha\beta} \nu_i^\alpha \tilde\nu_j^\beta \le L\, |\nu|\, |\tilde\nu| \quad \textit{for all } \nu,\, \tilde\nu \in \mathbb{R}^\tau,
$$

for any $g \in H^{\mathbf{m},2}(B_\rho(x_0), \mathbb{R}^N)$ satisfying

$$
\text{(3-4)} \qquad \fint_{B_\rho(x_0)} |D^{\mathbf{m}} g|^2\, dx \le 1, \quad \textit{and}
$$

$$\Big|\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_{B_\rho(x_0)} a_{ij}^{\alpha\beta} D^\beta g^j D^\alpha \varphi^i \,dx\Big| \le \delta \sup_{B_\rho(x_0)} |D^{\mathbf{m}}\varphi| \tag{3-5}$$
$$\text{for all } \varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N)\,,$$

there exists a function $v \in H^{\mathbf{m},2}(B_\rho(x_0), \mathbb{R}^N)$ with the following properties:

$$\fint_{B_\rho(x_0)} |D^{\mathbf{m}} v|^2 \,dx \le 1; \tag{3-6}$$

$$\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_{B_\rho(x_0)} a_{ij}^{\alpha\beta} D^\beta v^j D^\alpha \varphi^i dx = 0 \tag{3-7}$$
$$\text{for all } \varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N);$$

$$\text{and } \rho^{-2}\fint_{B_\rho(x_0)} |d(v-g)|^2 \,dx \le \varepsilon\,. \tag{3-8}$$

PROOF. We assume first that $x_0 = 0$ and $\rho = 1$. Were the conclusion false, we could find $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exist coefficients ${}^k a_{ij}^{\alpha\beta}$ and functions $g_k \in H^{\mathbf{m},2}(B, \mathbb{R}^N)$, so that for all k we have

$$\fint_B |D^{\mathbf{m}} g_k|^2 \,dx \le 1, \tag{3-9}$$

$$\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\Big|\fint_B {}^k a_{ij}^{\alpha\beta} D^\beta g_k^j D^\alpha \varphi^i \,dx\Big| \le \frac{1}{k}\sup_B |D^{\mathbf{m}}\varphi| \tag{3-10}$$
$$\text{for all } \varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N); \quad \text{and}$$

$$\fint_B |d(v_k - g_k)|^2 \,dx \ge \varepsilon \quad \text{for all } v_k \in \mathcal{H}_k. \tag{3-11}$$

Here $\mathcal{H}_k$ denotes the (nonempty) set of all $h \in H^{m,2}(B, \mathbb{R}^N)$ for which

$$\fint_B |D^{\mathbf{m}} h|^2 \,dx \le 1 \quad \text{and}$$
$$\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_B {}^k a_{ij}^{\alpha\beta} D^\beta h^j D^\alpha \varphi^i \,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B, \mathbb{R}^N);$$

the second of these conditions expresses the fact that h is $({}^k a_{ij}^{\alpha\beta}, \mathbf{m})$-*harmonic* (in B). Clearly, we may assume $\fint_B dg_k\,dx = 0$ for all k (by simply replacing g_k with $g_k - Q_k$ where Q_k is the unique polynomial of degree at most $\mathbf{m}-\mathbf{1}$ satisfying $\int_B d(g_k - Q_k)dx = 0$) and apply Poincaré's inequality (see [G1, Chapter III]) and Rellich's theorem to obtain, after passing to a subsequence, $g_k \to g$ strongly in $H^{\mathbf{m}-\mathbf{1},2}(B,\mathbb{R}^N)$ and weakly in $H^{\mathbf{m},2}(B,\mathbb{R}^N)$ and ${}^k a_{ij}^{\alpha\beta} \to a_{ij}^{\alpha\beta}$. Then $\fint_B |D^{\mathbf{m}} g|^2\,dx \le 1$, and from

$$\begin{aligned}
&\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B a_{ij}^{\alpha\beta} D^\beta g^j D^\alpha \varphi^i\,dx \\
&\quad = \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B a_{ij}^{\alpha\beta} \Big(D^\beta g^j - D^\beta g_k^j\Big) D^\alpha \varphi^i\,dx \\
&\qquad + \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B (a_{ij}^{\alpha\beta} - {}^k a_{ij}^{\alpha\beta}) D^\beta g_k^j D^\alpha \varphi^i\,dx \\
&\qquad + \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B {}^k a_{ij}^{\alpha\beta} D^\beta g_k^j D^\alpha \varphi^i\,dx
\end{aligned}$$

and (3-10) we infer that g is $(a_{ij}^{\alpha\beta}, \mathbf{m})$-harmonic in B.

We denote by V_k the unique solution of the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B {}^k a_{ij}^{\alpha\beta} D^\beta V^j D^\alpha \varphi^i\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B,\mathbb{R}^N), \\ dV = dg \qquad \text{on } \partial B. \end{cases}$$

Then using the ellipticity condition (3-2), the $(a_{ij}^{\alpha\beta}, \mathbf{m})$-harmonicity of g, and $\fint_B |D^{\mathbf{m}} g|^2\,dx \le 1$ we have

$$\begin{aligned}
&\lambda \fint_B |D^{\mathbf{m}}(V_k - g)|^2\,dx \\
&\quad \le \sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B {}^k a_{ij}^{\alpha\beta} D^\beta (V_k^j - g^j) D^\alpha (V_k^i - g^i)\,dx \\
&\quad = -\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B {}^k a_{ij}^{\alpha\beta} D^\beta g^j D^\alpha (V_k^i - g^i)\,dx
\end{aligned}$$

$$= \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \fint_B (a_{ij}^{\alpha\beta} - {}^k a_{ij}^{\alpha\beta}) D^\beta g^j D^\alpha (V_k^i - g^i)\, dx$$
$$\leq ||a - {}^k a|| \fint_B |D^{\mathbf{m}} g|\, |D^{\mathbf{m}}(V_k - g)|\, dx$$
$$\leq ||a - {}^k a|| \Big(\fint_B |D^{\mathbf{m}}(V_k - g)|^2\, dx \Big)^{1/2},$$

which in view of ${}^k a_{ij}^{\alpha\beta} \to a_{ij}^{\alpha\beta}$ implies the convergence of V_k to g strongly in $H^{\mathbf{m},2}(B, \mathbb{R}^N)$. This in turn implies $\|V_k - g_k\|_{H^{\mathbf{m-1},2}} \leq \|V_k - g\|_{H^{\mathbf{m-1},2}} + \|g - g_k\|_{H^{\mathbf{m-1},2}} \to 0$ as $k \to \infty$. The same assertion is true for $v_k = \min(1, \|D^{\mathbf{m}} V_k\|_{L^2}^{-1}) V_k$, contradicting (3-11).

The general case follows immediately from a simple scaling argument. □

The last result of this section is a standard estimate for solutions of elliptic systems with constant coefficients (see [Ca, Teorema 9.2]).

THEOREM 3.3. *Consider coefficients* $\{a_{ij}^{\alpha\beta}\}$ *with ellipticity constant* $\lambda > 0$ *and upper bound* L *as in Lemma 3.2, and* $h \in H^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ *which is* $(a_{ij}^{\alpha\beta}, \mathbf{m})$*-harmonic. Then for a constant* $c_4 \geq 1$ *depending only on* n, N, λ, L, *and* $\mathbf{m}$, *there holds:*

$$\rho^{-2} \sup_{B_{\rho/2}(x_0)} |D^{\mathbf{m}} h|^2 + \sup_{B_{\rho/2}(x_0)} |D^{\mathbf{m+1}} h|^2 \leq c_4 \rho^{-2} \fint_{B_\rho(x_0)} |D^{\mathbf{m}} h|^2\, dx.$$

4. Proof of the Main Theorem

For this section we consider a fixed, but arbitrary solution $u \in H^{\mathbf{m},2}(\Omega, \mathbb{R}^N)$ to (1-1), where we assume that the structure-conditions (H1)–(H3) are valid. As in Section 3., for $B_\rho(x_0) \subset\subset \Omega$ and a function $f \in H^{\mathbf{m},2}(B_\rho(x_0), \mathbb{R}^N)$ write $\Phi(x_0, \rho, f)$ for $\fint_{B_\rho(x_0)} |D^{\mathbf{m}}(u - f)|^2\, dx$. For a given $B_\rho(x_0) \subset\subset \Omega$ we have from Section 1. the existence of a unique polynomial P of degree at most $\mathbf{m}$ satisfying:

(4-1) $$\fint_{B_\rho(x_0)} D^\alpha (u^i - P^i)\, dx = 0 \quad \text{for all } i = 1, \ldots, N,\ |\alpha| \leq m_i.$$

From (1-4) we recall that P has the form

$$P^i(x) = \sum_{|\alpha|\le m_i} \sum_{|\alpha+\beta|\le m_i} \frac{b^i_\beta}{\alpha!}(D^{\alpha+\beta}u)_{x_0,\rho}\rho^{|\beta|}(x-x_0)^\alpha\,, \tag{4-2}$$

with the coefficients depending only on m_i and n. For this polynomial P, we define $p_{x_0,\rho}$ by

$$p_{x_0,\rho} = \sum_{j=1}^{N} \sum_{|\gamma|\le m_j} |D^\gamma P^j(x_0)| = \sum_{j=1}^{N} p^j_{x_0,\rho}\,.$$

We further write $\Phi(x_0,\rho)$ for $\Phi(x_0,\rho,P)$, where P is determined by (4-1).

We begin by proving two propositions which establish growth controls on $p_{x_0,\rho}$ and $\Phi(x_0,\rho)$ under suitable smallness-conditions.

PROPOSITION 4.1. *For $\rho,\theta \in (0,1]$ and $B_\rho(x_0) \subset\subset \Omega$ there exists a constant c_5 depending only on n, N, and m such that there holds:*

$$p_{x_0,\theta\rho} \le (1 + c_5\rho\theta^{-n/2})p_{x_0,\rho} + c_5\theta^{-n/2}\Phi^{1/2}(x_0,\rho)\,.$$

PROOF. We begin by noting, for $|\gamma| \le m_k$ and $x \in B_\rho(x_0)$, that there holds $|D^\gamma P^k(x)| \le p_{x_0,\rho}$, and hence:

$$|(D^\gamma P^k)_{x_0,\rho}| \le p_{x_0,\rho}. \tag{4-3}$$

For such γ, we estimate, using the fact that $(DD^\gamma u^k)_{x_0,\rho}(x-x_0)$ has mean value 0 on balls centered at x_0, and (repeatedly) using Poincaré's inequality:

$$\begin{aligned}
&\Big|(D^\gamma u^k)_{x_0,\rho} - (D^\gamma u^k)_{x_0,\theta\rho}\Big| \\
&= \Big|\fint_{B_{\theta\rho}(x_0)} \Big(D^\gamma u^k - (D^\gamma u^k)_{x_0,\rho} - (DD^\gamma u^k)_{x_0,\rho}(x-x_0)\Big)\,dx\Big| \\
&\le \theta^{-n/2}\Big(\fint_{B_\rho(x_0)} \Big|D^\gamma u^k - (D^\gamma u^k)_{x_0,\rho} - (DD^\gamma u^k)_{x_0,\rho}(x-x_0)\Big|^2\,dx\Big)^{1/2} \\
&\le \theta^{-n/2}c_P\rho\Big(\fint_{B_\rho(x_0)} \Big|DD^\gamma u^k - (DD^\gamma u^k)_{x_0,\rho}\Big|^2\,dx\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \theta^{-n/2} c_P \rho \Big[\Big(\fint_{B_\rho(x_0)} \Big| DD^\gamma u^k - (DD^\gamma u^k)_{x_0,\rho} \\
& \qquad - (D^2 D^\gamma u^k)_{x_0,\rho}(x-x_0) \Big|^2 \, dx \Big)^{1/2} \\
& \qquad + \rho \Big| (D^2 D^\gamma u^k)_{x_0,\rho} \Big| \Big] \\
\leq\ & \theta^{-n/2} c_P \rho \Big[c_P \rho \Big(\fint_{B_\rho(x_0)} \Big| D^2 D^\gamma u^k - (D^2 D^\gamma u^k)_{x_0,\rho} \Big|^2 \, dx \Big)^{1/2} \\
& \qquad + \rho \Big| (D^2 D^\gamma u^k)_{x_0,\rho} \Big| \Big] \, .
\end{aligned}
$$

Here c_P denotes the constant from the Poincaré inequality; note that $c_P \leq 2^{2n}$. Iterating this and recalling the choice of P (i.e. (4-1)) yields

$$
\begin{aligned}
& \Big| (D^\gamma u^k)_{x_0,\rho} - (D^\gamma u^k)_{x_0,\theta\rho} \Big| \\
& \quad \leq\ \theta^{-n/2} (c_P \rho)^{m_k - |\gamma|} \\
& \qquad \cdot \Big[\fint_{B_\rho(x_0)} \Big| D^{m_k-|\gamma|} D^\gamma u^k - (D^{m_k-|\gamma|} D^\gamma u^k)_{x_0,\rho} \Big|^2 \, dx \Big]^{1/2} \\
& \qquad\quad + \theta^{-n/2} \sum_{j=2}^{m_k-|\gamma|} c_P^{j-1} \rho^j |(D^j D^\gamma P^k)_{x_0,\rho}| \, ,
\end{aligned}
$$

with the summation obviously being set to 0 in the case that $|\gamma| = m_k - 1$. In the case $|\gamma| \leq m_k - 1$ we can further estimate (recalling also that $\rho \leq 1$)

$$
\begin{aligned}
& \Big| (D^\gamma u^k)_{x_0,\rho} - (D^\gamma u^k)_{x_0,\theta\rho} \Big| \\
& \leq\ \theta^{-n/2} c_P^m \rho \Big[\Big(\fint_{B_\rho(x_0)} \Big| D^{m_k-|\gamma|} D^\gamma u^k - (D^{m_k-|\gamma|} D^\gamma u^k)_{x_0,\rho} \Big|^2 \, dx \Big)^{1/2} \\
& \qquad\qquad + \sum_{j=2}^{m_k-|\gamma|} |(D^j D^\gamma P^k)_{x_0,\rho}| \Big] \\
& \leq\ \theta^{-n/2} c_P^m \rho \Big[\Big(\fint_{B_\rho(x_0)} \Big| D^{m_k} u^k - (D^{m_k} u^k)_{x_0,\rho} \Big|^2 \, dx \Big)^{1/2} \\
& \qquad\qquad + \sum_{|\sigma| \leq m_k} |(D^\sigma P^k)_{x_0,\rho}| \Big] \\
& \leq\ \theta^{-n/2} (n+1)^m c_P^m \rho \Big[\Big(\fint_{B_\rho(x_0)} \Big| D^{m_k} u^k - (D^{m_k} u^k)_{x_0,\rho} \Big|^2 \, dx \Big)^{1/2} + p^k_{x_0,\rho} \Big] \, .
\end{aligned}
$$

Summing over $0 \le |\gamma| \le m_k - 1$, $k = 1, \dots, N$ yields

$$
\begin{aligned}
(4\text{-}4)\qquad & \sum_{k=1}^{N} \sum_{|\gamma| \le m_k - 1} \left| (D^\gamma u^k)_{x_0,\rho} - (D^\gamma u^k)_{x_0,\theta\rho} \right| \\
& \le \theta^{-n/2} (n+1)^{2m-1} c_P^m \rho \\
& \quad \cdot \Big[\sum_{k=1}^{N} \Big(\fint_{B_\rho(x_0)} \left| D^{m_k} u^k - (D^{m_k} u^k)_{x_0,\rho} \right|^2 dx \Big)^{1/2} + p_{x_0,\rho} \Big] \\
& \le \theta^{-n/2} c_6 \rho \Big[\Phi^{1/2}(x_0,\rho) + p_{x_0,\rho} \Big] ,
\end{aligned}
$$

where $c_6 = (n+1)^{2m-1} c_P^m \sqrt{N}$. We further have:

$$
(4\text{-}5)\qquad \left| (D^{\mathbf{m}} u)_{x_0,\theta\rho} - (D^{\mathbf{m}} u)_{x_0,\rho} \right| \le \theta^{-n/2} \Phi^{1/2}(x_0,\rho) .
$$

We denote by $\widetilde{P}$ the polynomial of degree at most $\mathbf{m}$ on $B_{\theta\rho}(x_0)$ defined by (4-1). Then

$$
\begin{aligned}
p_{x_0,\theta\rho} & \le p_{x_0,\rho} + \sum_{k=1}^{N} \sum_{|\gamma| \le m_k} |D^\gamma P^k(x_0) - D^\gamma \widetilde{P}^k(x_0)| \\
& \le p_{x_0,\rho} + \sqrt{N(n+1)^m} |(D^{\mathbf{m}} u)_{x_0,\theta\rho} - (D^{\mathbf{m}} u)_{x_0,\rho}| + I \\
& \le p_{x_0,\rho} + \sqrt{N(n+1)^m} \theta^{-n/2} \Phi^{1/2}(x_0,\rho) + I ,
\end{aligned}
$$

where

$$
I = \sum_{k=1}^{N} \sum_{|\gamma| \le m_k - 1} |D^\gamma P^k(x_0) - D^\gamma \widetilde{P}^k(x_0)| ;
$$

Here we have used (4-5) in obtaining the last inequality.

Now, we recall that

$$
\begin{aligned}
D^\gamma P^k(x_0) & = \sum_{|\beta+\gamma| \le m_k} b^k_\beta \rho^{|\beta|} (D^{\beta+\gamma} u^k)_{x_0,\rho} , \\
D^\gamma \tilde{P}^k(x_0) & = \sum_{|\beta+\gamma| \le m_k} b^k_\beta (\theta\rho)^{|\beta|} (D^{\beta+\gamma} u^k)_{x_0,\theta\rho} .
\end{aligned}
$$

Noting from (1-5) that $b_0^k = 1$ and using (4-4), this yields:

$$\begin{aligned} I &= \sum_{k=1}^{N} \sum_{|\gamma|\le m_k-1} \sum_{|\beta+\gamma|\le m_k} |b_\beta^k| \rho^{|\beta|} \Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} - \theta^{|\beta|} (D^{\beta+\gamma}u^k)_{x_0,\theta\rho} \Big| \\ &\le \sum_{k=1}^{N} \sum_{|\gamma|\le m_k-1} \sum_{\substack{|\beta+\gamma|\le m_k \\ |\beta|>0}} |b_\beta^k| \rho^{|\beta|} \Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} - \theta^{|\beta|} (D^{\beta+\gamma}u^k)_{x_0,\theta\rho} \Big| \\ &\quad + \theta^{-n/2} c_6 \rho \Big[\Phi^{1/2}(x_0,\rho) + p_{x_0,\rho} \Big] , \end{aligned}$$

Denoting $\max |b_\beta^k|$ by $\mathfrak{B}$ (note that $\mathfrak{B}$ depends on n and m only) we deduce using (4-4), (4-5), (4-3), and the choice of P (see (4-1))

$$\begin{aligned} &\sum_{k=1}^{N} \sum_{|\gamma|\le m_k-1} \sum_{\substack{|\beta+\gamma|\le m_k \\ |\beta|>0}} |b_\beta^k| \rho^{|\beta|} \Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} - \theta^{|\beta|} (D^{\beta+\gamma}u^k)_{x_0,\theta\rho} \Big| \\ &\le \sum_{k=1}^{N} \sum_{|\gamma|\le m_k-1} \sum_{\substack{|\beta+\gamma|\le m_k \\ |\beta|>0}} |b_\beta^k| \Big[(\theta\rho)^{|\beta|} \Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} - (D^{\beta+\gamma}u^k)_{x_0,\theta\rho} \Big| \\ &\qquad\qquad + \rho^{|\beta|} (1-\theta^{|\beta|}) \Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} \Big| \Big] \\ &= \mathfrak{B}\rho \sum_{k=1}^{N} \sum_{|\gamma|\le m_k-1} \sum_{\substack{|\beta+\gamma|\le m_k \\ |\beta|>0}} \Big[\Big| (D^{\beta+\gamma}u^k)_{x_0,\rho} - (D^{\beta+\gamma}u^k)_{x_0,\theta\rho} \Big| \\ &\qquad\qquad + \Big| (D^{\beta+\gamma}P^k)_{x_0,\rho} \Big| \Big] \\ &\le \mathfrak{B}\rho (n+1)^{m-1} \sum_{k=1}^{N} \sum_{|\gamma|\le m_k} \Big[\Big| (D^{\gamma}u^k)_{x_0,\rho} - (D^{\gamma}u^k)_{x_0,\theta\rho} \Big| + \Big| (D^{\gamma}P^k)_{x_0,\rho} \Big| \Big] \\ &\le \mathfrak{B}\theta^{-n/2} (n+1)^{2m} N \Big(1 + (c_P(n+1))^m \Big) \Big[\Phi^{1/2}(x_0,\rho) + p_{x_0,\rho} \Big] \rho \end{aligned}$$

This implies

$$I \le 2\mathfrak{B}\theta^{-n/2} (n+1)^{2m} N \Big(1 + (c_P(n+1))^m \Big) \rho \Big[\Phi^{1/2}(x_0,\rho) + p_{x_0,\rho} \Big]$$

from which the desired estimate easily follows with $c_5 = 3\mathfrak{B}(n+1)^{2m} N (1 + (c_P(n+1))^m)$. $\square$

The second growth-estimate requires additional smallness assumptions. We let

$$K(p) = [\kappa(\ell p)(1+\ell p)]^{\frac{2}{1-s}}.$$

PROPOSITION 4.2. *To a given $t \in (s,1)$ there exist positive constants $\delta \in (0,1]$, $\theta \in (0,1/4]$, c_7, $c_8 \geq 1$ such that, on every ball $B_\rho(x_0) \subset\subset \Omega$ with $\rho \leq 1$ for which the smallness-conditions*

$$\Phi(x_0,\rho) + \omega^2(p_{x_0,\rho}, \Phi(x_0,\rho)) \leq \delta^2/2, \tag{4-6}$$

$$c_7 K^2(p_{x_0,\rho})\rho^{2s} \leq \delta^2 \tag{4-7}$$

are satisfied, there holds:

$$\Phi(x_0,\theta\rho) \leq \theta^{2t}\Phi(x_0,\rho) + \Big[\delta^{-2}K^2(p_{x_0,\rho}) + 2^n c_1 K(1+c_8 p_{x_0,\rho})\Big]\rho^{2s}. \tag{4-8}$$

PROOF. Consider $\varepsilon > 0$ to be determined later, and let $\delta = \delta(n,N,\lambda, L,\mathbf{m},\varepsilon)$ be the corresponding constant from Lemma 3.2. We define, for $c_9 = 4c_P^{m+1}$

$$\chi = 2c_9\Big(\Phi(x_0,\rho,P) + \delta^{-2}K^2(p_{x_0,\rho})\rho^{2s}\Big)^{1/2}, \quad \text{and} \tag{4-9}$$

$$w = \chi^{-1}(u-P).$$

Here the polynomial P of degree less than or equal to $\mathbf{m}$ is chosen to satisfy

$$\fint_{B_\rho(x_0)} D^\alpha(u^k - P^k)\,dx = 0 \quad \text{for all } k = 1,\dots,N,\ |\alpha| \leq m_k. \tag{4-10}$$

With this choice we can use Poincaré's inequality to deduce

$$\begin{aligned}
&\sum_{i=1}^N \sum_{|\gamma|\leq m_i} \rho^{2(|\gamma|-m_i)} \fint_{B_\rho(x_0)} |D^\gamma(u^i - P^i)|^2\,dx \\
&\leq \sum_{i=1}^N \Big(c_P + c_P^2 + \cdots + c_P^{m_i}\Big) \fint_{B_\rho(x_0)} |D^{m_i}(u^i-P^i)|^2\,dx \\
&\leq c_P \frac{c_P^m - 1}{c_P - 1}\Phi(x_0,\rho) \leq c_P^{m+1}\Phi(x_0,\rho)\,.
\end{aligned}$$

In view of Lemma 3.1 we obtain, writing p for $p_{x_0,\rho}$,

$$\begin{aligned}
&\Big|\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_{B_\rho(x_0)}\frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0,dP(x_0),D^{\mathbf{m}}P)D^\beta(u^j-P^j)D^\alpha\varphi^i\,dx\Big| \\
&\quad\le \Big[2\Phi(x_0,\rho)+\omega(p,\Phi(x_0,\rho))\Phi^{1/2}(x_0,\rho) \\
&\qquad\quad +4K(p)\rho^s+2c_P^{m+1}\Phi(x_0,\rho)\Big]\sup_{B_\rho(x_0)}|D^{\mathbf{m}}\varphi| \\
&\quad\le c_9\Big[\Phi(x_0,\rho)+\omega(p,\Phi(x_0,\rho))\Phi^{1/2}(x_0,\rho)+K(p)\rho^s\Big]\sup_{B_\rho(x_0)}|D^{\mathbf{m}}\varphi|\,.
\end{aligned}$$

Dividing the above equation through by χ yields

$$\begin{aligned}
&\Big|\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_{B_\rho(x_0)}\frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0,dP(x_0),D^{\mathbf{m}}P)D^\beta w^j D^\alpha\varphi^i\,dx\Big| \\
&\quad\le \Big[\Phi(x_0,\rho)+\omega^2(p,\Phi(x_0,\rho))+\frac{1}{2}\delta^2\Big]^{1/2}\sup_{B_\rho(x_0)}|D^{\mathbf{m}}\varphi|
\end{aligned}\tag{4-11}$$

for any $\varphi\in C_c^\infty(B_\rho(x_0),\mathbb{R}^N)$, as well as

$$\fint_{B_\rho(x_0)}|D^{\mathbf{m}}w|^2\,dx\le 1\,.\tag{4-12}$$

We now set

$$a_{ij}^{\alpha\beta}=\frac{\partial A_i^\alpha}{\partial \nu_j^\beta}(x_0,dP(x_0),D^{\mathbf{m}}P).\tag{4-13}$$

In view of (H1) and (H2), the coefficients $a_{ij}^{\alpha\beta}$ fulfill the conditions (3-2) and (3-3). Now, if there holds

$$\Phi(x_0,\rho)+\omega^2(p_{x_0,\rho},\Phi(x_0,\rho))\le\frac{1}{2}\delta^2\tag{4-14}$$

we can apply Lemma 3.2 to conclude the existence of $h\in H^{\mathbf{m},2}(B_\rho(x_0),\mathbb{R}^N)$ satisfying:

$$\fint_{B_\rho(x_0)}|D^{\mathbf{m}}h|^2\,dx\le 1\,;\tag{4-15}$$

$$\sum_{i,j=1}^{N}\sum_{|\alpha|=m_i}\sum_{|\beta|=m_j}\fint_{B_\rho(x_0)} a_{ij}^{\alpha\beta} D^\beta h^j D^\alpha \varphi^i \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(B_\rho(x_0),\mathbb{R}^N); \tag{4-16}$$

$$\rho^{-2}\fint_{B_\rho(x_0)} |D^{\mathbf{m-1}}(w-h)|^2 \, dx \le \varepsilon\,. \tag{4-17}$$

From Theorem 3.3 and (4-15) we have the estimate

$$\begin{aligned}\rho^{-2}\sup_{B_{\rho/2}(x_0)}|D^{\mathbf{m}}h|^2 + \sup_{B_{\rho/2}(x_0)}|D^{\mathbf{m+1}}h|^2 \\ \le c_4\rho^{-2}\fint_{B_\rho(x_0)}|D^{\mathbf{m}}h|^2\,dx \le c_4\rho^{-2}.\end{aligned} \tag{4-18}$$

Now consider $\theta \in (1, 1/4]$. From Taylor's theorem applied to h on $B_{2\theta\rho}(x_0)$ we have

$$\sup_{x\in B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}h(x) - D^{\mathbf{m-1}}h(x_0) - D^{\mathbf{m}}h(x_0)(x-x_0)|^2 \le 16c_4\theta^4\rho^2.$$

In view of (4-17) and (4-18) we thus see

$$\begin{aligned}
&(2\theta\rho)^{-2}\fint_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}w(x) - D^{\mathbf{m-1}}h(x_0) - D^{\mathbf{m}}h(x_0)(x-x_0)|^2\,dx\\
&\le 2(2\theta\rho)^{-2}\Big[\fint_{B_{2\theta\rho}(x_0)}|D^{\mathbf{m-1}}(w-h)|^2\,dx\\
&\qquad + \fint_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}h(x) - D^{\mathbf{m-1}}h(x_0)) - D^{\mathbf{m}}h(x_0)(x-x_0)|^2\,dx\Big]\\
&\le 2(2\theta)^{-n-2}\varepsilon + 2(2\theta\rho)^{-2}16c_4\theta^4\rho^2\\
&= 2^{-n-1}\theta^{-n-2}\varepsilon + 8c_4\theta^2\,.
\end{aligned}$$

Multiplying this through by χ^2 yields

$$\begin{aligned}
(2\theta\rho)^{-2}\fint_{B_{2\theta\rho}(x_0)}\Big|&D^{\mathbf{m-1}}(u(x)-P(x))\\
&-\chi\Big(D^{\mathbf{m-1}}h(x_0) + D^{\mathbf{m}}h(x_0)(x-x_0)\Big)\Big|^2\,dx\\
\le\ & 32c_9^2c_4(\theta^{-n-2}\varepsilon+\theta^2)[\Phi(x_0,\rho) + \delta^{-2}K^2(p_{x_0,\rho})\rho^{2s}]\,.
\end{aligned} \tag{4-19}$$

We now define a second polynomial Q of degree at most $\mathbf{m}$ by requiring

$$D^{\mathbf{m}}Q = D^{\mathbf{m}}P + \chi D^{\mathbf{m}}h(x_0)\,, \qquad \text{and} \tag{4-20}$$

$$⨍_{B_{2\theta\rho}(x_0)} d(u-Q)\,dx = 0\,. \tag{4-21}$$

Note that (4-20) and (4-21) uniquely determine Q. In particular (4-21) implies that $D^{\mathbf{m-1}}(u-Q)$ has mean-value 0 on $B_{2\theta\rho}(x_0)$. In view of (4-19), this allows us to deduce

$$\begin{aligned}(2\theta\rho)^{-2} &⨍_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}(u-Q)|^2\,dx \\ \le\ & (2\theta\rho)^{-2} ⨍_{B_{2\theta\rho}(x_0)} \Big| D^{\mathbf{m-1}}u - D^{\mathbf{m-1}}P(x_0) - \chi D^{\mathbf{m-1}}h(x_0) \\ & \qquad - \Big(D^{\mathbf{m}}P + \chi D^{\mathbf{m}}h(x_0)\Big)(x-x_0)\Big|^2\,dx \\ \le\ & 32c_9^2c_4\Big(\theta^{-n-2}\varepsilon + \theta^2\Big)\Big(\Phi(x_0,\rho) + \delta^{-2}K^2(p_{x_0,\rho})\rho^{2s}\Big)\,.\end{aligned} \tag{4-22}$$

We now apply Theorem 2.1 on $B_{2\theta\rho}(x_0)$, with P replaced by Q. For $q_{x_0,2\theta\rho}$ defined by

$$q_{x_0,2\theta\rho} = \sum_{j=1}^{N} \sum_{|\gamma|\le m_j} |D^\gamma Q^j(x_0)|$$

we have

$$\begin{aligned}&\int_{B_{\theta\rho}(x_0)} |D^{\mathbf{m}}(u-Q)|^2\,dx \\ &\quad\le\ c_1\Big[\sum_{i=1}^{N}\sum_{j=0}^{m_i-1}(2\theta\rho)^{2(j-m_i)} \\ &\qquad\quad \cdot \int_{B_{2\theta\rho}(x_0)} |D^j(u^i-Q^i)|^2\,dx + \boldsymbol{\alpha}_n(2\theta\rho)^{n+2s}K(q_{x_0,2\theta\rho})\Big]\,.\end{aligned}$$

As before the choice of Q, i.e. the fact that $\int_{B_{2\theta\rho}(x_0)} D^\alpha(u^i-Q^i)\,dx = 0$ for $i=1,\dots,N$, $|\alpha|\le m_i-1$, allows us to apply Poincaré's inequality iteratively to deduce

$$\sum_{i=1}^{N}\sum_{j=0}^{m_i-1}(2\theta\rho)^{2(j-m_i)}\int_{B_{2\theta\rho}(x_0)} |D^j(u^i-Q^i)|^2\,dx$$

$$\begin{aligned} &\le (1 + c_P + \cdots + c_P^{m-1})(2\theta\rho)^{-2} \int_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}(u-Q)|^2\,dx \\ &\le c_P^m (2\theta\rho)^{-2} \int_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}(u-Q)|^2\,dx\,. \end{aligned}$$

Inserting this into Caccioppoli-type inequality we obtain

$$\begin{aligned} &\int_{B_{\theta\rho}(x_0)} |D^{\mathbf{m}}(u-Q)|^2\,dx \\ &\quad \le c_1\Big[c_P^m (2\theta\rho)^{-2} \int_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}(u-Q)|^2\,dx + \boldsymbol{\alpha}_n (2\theta\rho)^{n+2s} K(q_{x_0,2\theta\rho})\Big] \end{aligned}$$

Multiplying through by $\boldsymbol{\alpha}_n(\theta\rho)^n$ and using (4-22) we arrive at

$$\begin{aligned} \Phi(x_0,\theta\rho) &\le \fint_{B_{\theta\rho}(x_0)} |D^{\mathbf{m}}(u-Q)|^2\,dx \\ &\le 2^n c_1 \Big[\frac{c_P^m}{(2\theta\rho)^2} \fint_{B_{2\theta\rho}(x_0)} |D^{\mathbf{m-1}}(u-Q)|^2\,dx \\ &\quad + (2\theta\rho)^{2s} K(q_{x_0,2\theta\rho})\Big] \\ &= 2^{n+6} c_1 c_P^m c_9^2 c_4 \Big(\theta^{-n-2}\varepsilon + \theta^2\Big)\Big(\Phi(x_0,\rho) + \delta^{-2} K^2(p_{x_0,\rho})\rho^{2s}\Big) \\ &\quad + 2^n c_1 \rho^{2s} K(q_{x_0,2\theta\rho})\,. \end{aligned}$$

We now choose $t \in (s,1)$ and then $\theta \in (0,1/4]$ sufficiently small such that

$$2^{n+7} c_1 c_P^m c_9^2 c_4 \theta^2 \le \theta^{2t}\,. \tag{4-23}$$

With this choice of θ, we set $\varepsilon = \theta^{n+4}$. Note that this also fixes δ. With these choices, we have

$$\begin{aligned} \Phi(x_0,\theta\rho) &\le \theta^{2t}\Big(\Phi(x_0,\rho) + \delta^{-2} K^2(p_{x_0,\rho})\rho^{2s}\Big) \\ &\quad + 2^n c_1 K(q_{x_0,2\theta\rho})\rho^{2s}\,. \end{aligned} \tag{4-24}$$

The next step is to control $q_{x_0,2\theta\rho}$ in terms of $p_{x_0,\rho}$, in order to refine the estimate (4-24). For $|\gamma| \le m_k - 1$ we have, using (4-21) and (4-11):

$$|D^\gamma Q^k(x_0)| \le \Big|\fint_{B_{2\theta\rho}(x_0)} D^\gamma P^k(x)\,dx\Big|$$

$$
\begin{aligned}
&\quad + \Big| \fint_{B_{2\theta\rho}(x_0)} (D^\gamma Q^k(x_0) - D^\gamma P^k(x))\, dx \Big| \\
\leq&\quad p_{x_0,\rho} + \Big| \fint_{B_{2\theta\rho}(x_0)} (D^\gamma u^k - D^\gamma P^k)\, dx \Big| \\
&\quad + \Big| \fint_{B_{2\theta\rho}(x_0)} (D^\gamma Q^k - D^\gamma Q^k(x_0))\, dx \Big| \\
\leq&\quad p_{x_0,\rho} + (2\theta)^{-n/2} \Big(\fint_{B_\rho(x_0)} |D^\gamma (u^k - P^k)|^2 dx \Big)^{1/2} \\
&\quad + \fint_{B_{2\theta\rho}(x_0)} |D^\gamma Q^k - D^\gamma Q^k(x_0)|\, dx
\end{aligned}
$$

Summing over $k = 1, \ldots, N$, $|\gamma| \leq m_k - 1$ we obtain, using Poincaré's inequality iteratively:

$$
\begin{aligned}
&\sum_{k=1}^N \sum_{|\gamma| \leq m_k - 1} |D^\gamma Q^k(x_0)| \\
&\quad \leq N(n+1)^{m-1} p_{x_0,\rho} + \sqrt{N(n+1)^{m-1}} (2\theta)^{-n/2} \\
&\qquad \cdot \Big[\sum_{k=1}^N \sum_{j=1}^{m_k-1} \fint_{B_\rho(x_0)} |D^j(u^k - P^k)|^2\, dx \Big]^{1/2} + R \\
&\quad \leq N(n+1)^{m-1} p_{x_0,\rho} \\
&\qquad + \sqrt{N(n+1)^{m-1}} (2\theta)^{-n/2} \\
&\qquad \cdot \Big[\sum_{k=1}^N \big(c_P \rho^2 + \cdots + (c_P \rho^2)^{m_k} \big) \fint_{B_\rho(x_0)} |D^{m_k}(u^k - P^k)|^2\, dx \Big]^{1/2} + R \\
&\quad \leq N(n+1)^{m-1} p_{x_0,\rho} \\
&\qquad + \sqrt{N(n+1)^{m-1}} (2\theta)^{-n/2} c_P^{(m+1)/2} \Phi(x_0,\rho)^{1/2} + R\,,
\end{aligned}
$$

where

$$
R = \sum_{k=1}^N \sum_{|\gamma| \leq m_k - 1} \fint_{B_{2\theta\rho}(x_0)} |D^\gamma Q^k - D^\gamma Q^k(x_0)|\, dx\,.
$$

To estimate R we observe that for $x \in B_{2\theta\rho}(x_0)$

$$
|D^\gamma Q^k(x) - D^\gamma Q^k(x_0)| \;=\; \Big| \sum_{\substack{|\alpha| \leq m_k \\ \alpha > \gamma}} \frac{1}{(\alpha - \gamma)!} D^\alpha Q^k(x_0)(x - x_0)^{\alpha - \gamma} \Big|
$$

$$\leq \quad 2\theta\rho \sum_{\substack{|\alpha|\leq m_k\\ \alpha>\gamma}} |D^\alpha Q^k(x_0)| \leq 2\theta\rho q_{x_0,2\theta\rho}\,,$$

which yields

$$R \leq 2\theta\rho N(n+1)^{m-1} q_{x_0,2\theta\rho}\,.$$

This implies

$$\text{(4-25)} \quad \sum_{k=1}^{N} \sum_{|\gamma|\leq m_k-1} |D^\gamma Q^k(x_0)| \leq N(n+1)^{m-1}\Big[p_{x_0,\rho} + (2\theta)^{-n/2} c_P^{(m+1)/2} \Phi(x_0,\rho)^{1/2} + 2\theta\rho q_{x_0,2\theta\rho}\Big]\,.$$

Finally, we estimate $q_{x_0,2\theta\rho}$ using (4-20), (4-25), (4-18), and Cauchy-Schwarz's inequality to obtain

$$\begin{aligned} q_{x_0,2\theta\rho} &\leq \sum_{k=1}^{N} \sum_{|\gamma|=m_k} \Big(|D^\gamma P^k(x_0)| + \chi|D^\gamma h(x_0)|\Big) \\ &\quad + \sum_{k=1}^{N} \sum_{|\gamma|\leq m_k-1} |D^\gamma Q^k(x_0)| \\ &\leq \sqrt{N(n+1)^m c_4}\,\chi + (1+N(n+1)^{m-1}) p_{x_0,\rho} \\ &\quad + N(n+1)^{m-1}\Big[(2\theta)^{-n/2} c_P^{(m+1)/2} \Phi(x_0,\rho)^{1/2} + 2\theta q_{x_0,2\theta\rho}\Big]\,. \end{aligned}$$

Choosing θ such that

$$\text{(4-26)} \qquad 4N(n+1)^{m-1}\theta \leq 1$$

we see in view of (4-9) (i.e. the definition of χ) that

$$q_{x_0,2\theta\rho} \leq 2c_{10}\chi + 2(1+N(n+1)^{m-1}) p_{x_0,\rho}\,,$$

where $c_{10} = \sqrt{N(n+1)^m c_4} + N(n+1)^{m-1}(2\theta)^{-n/2} c_P^{(m+1)/2}$. Now if

$$\text{(4-27)} \qquad \Phi(x_0,\rho) + \delta^{-2} K^2(p_{x_0,\rho})\rho^{2s} \leq \frac{1}{16 c_9^2 c_{10}^2}$$

we find

$$\text{(4-28)} \qquad q_{x_0,2\theta\rho} \leq 1 + c_8 p_{x_0,\rho}\,,$$

where $c_8 = 2(1 + N(n+1)^{m-1})$. Inserting this into (4-24) we finally arrive at

$$\Phi(x_0, \theta\rho) \le \theta^{2t}\Phi(x_0, \rho) + \Big[\delta^{-2}K^2(p_{x_0,\rho}) + 2^n c_1 K(1 + c_8 p_{x_0,\rho})\Big]\rho^{2s}\,.$$

Now, if we assume the further smallness condition $\delta^2 \le \frac{1}{16c_9^2c_{10}^2}$, which entails no loss of generality, and choose ρ such that there holds

$$\delta^{-2}K^2(p_{x_0,\rho})\rho^{2s} \le \frac{1}{32c_9^2c_{10}^2}\,,$$

we see that the smallness conditions (4-14) and (4-27) are fulfilled in view of (4-6) and (4-7) by letting $c_7 = 32c_9^2c_{10}^2$, completing the proof. □

The next step is to be able to find conditions sufficient to enable us to iterate Proposition 4.2. We define a function $H : [0, \infty) \to [0, \infty)$ by

$$H(s) = \delta^{-2}K^2(s) + 2^n c_1 K(1 + c_8 s)\,.$$

To a given $s > 0$ we can find $\Phi_0(s) > 0$ sufficiently small such that

$$\omega^2(2s, 2\Phi_0(s)) + 2\Phi_0(s) \le \frac{1}{2}\delta^2\,, \quad \text{and} \tag{4-29}$$

$$8c_7\Phi_0^{1/2}(s) \le (1 - \theta^s)\theta^{n/2}s\,. \tag{4-30}$$

Further we can find $\rho_0(s) \in (0, 1]$ sufficiently small that there holds:

$$c_7K^2(2s)\rho_0^{2s}(s) \le \delta^2; \tag{4-31}$$

$$H(2s)\rho_0^{2s}(s) \le (\theta^{2s} - \theta^{2t})\Phi_0(s) \quad \text{and} \tag{4-32}$$

$$\exp\Big[\frac{c_7\rho_0(s)}{\theta^{n/2}(1-\theta)}\Big] \le 3/2\,. \tag{4-33}$$

Lemma 4.3. *For a given $p_0 > 0$ and $B_\rho(x_0) \subset\subset \Omega$, suppose that there holds:*

(i) $p_{x_0,\rho} \le p_0$;

(ii) $\rho \le \rho_0(p_0)$; *and*

(iii) $\Phi(x_0, \rho) \le \Phi_0(p_0)$.

Then for each $k \in \mathbb{N} \cup \{0\}$ and θ as in the proof of Proposition 4.2 the estimates (4-6) and (4-7) are satisfied on $B_{\theta^k\rho}(x_0)$. Moreover, the limit

$$\Upsilon_{x_0} = \lim_{j\to\infty} (D^{\mathbf{m}}u)_{x_0,\theta^j\rho}$$

exists, and the inequality

$$⨍_{B_\rho(x_0)} |D^{\mathbf{m}}u - \Upsilon_{x_0}|^2\, dx \le \mathrm{const}\Big[\Big(\frac{r}{\rho}\Big)^{2s}\Phi(x_0,\rho) + r^{2s}\Big]$$

is valid for all $0 < r \le \rho$, where $\mathrm{const} = \mathrm{const}(p_0)$.

Proof. We write $(4\text{-}6)_k$ (respectively $(4\text{-}7)_k$ for (4-6) (respectively (4-7)) with ρ replaced by $\theta^k\rho$. We will prove the following for $k \in \mathbb{N} \cup \{0\}$:

(I_k) $\Phi(x_0, \theta^k\rho) \le 2\Phi_0(p_0)$;

$(I\!I_k)$ $p_{x_0,\theta^k\rho} \le 2p_0$.

We see that (I_k), $(I\!I_k)$ combined with (4-29) imply $(4\text{-}6)_k$, and $(I\!I_k)$, (ii) and (4-31) show $(4\text{-}7)_k$. We further note that (I_k) follows from the weaker condition

(I'_k) $\Phi(x_0,\theta^k\rho) \le \theta^{2ks}\Big(\Phi(x_0,\rho) + \dfrac{H(2\rho_0)}{\theta^{2s}-\theta^{2t}}\,\rho^{2s}\Big)$

in view of (ii), (iii) and (4-32). Thus we need to show (I'_k) and $(I\!I_k)$ for all $k \in \mathbb{N} \cup \{0\}$. We do so by induction.

For $k = 0$ we have that (I'_0) is trivial, and $(I\!I_0)$ follows immediately from (i).

In order to show (I'_k) and $(I\!I_k)$ for $k \in \mathbb{N}$ we assume (I'_μ) and $(I\!I_\mu)$ for $\mu = 0,\dots,k-1$. Then the hypotheses of Proposition 4.2 are satisfied on each $B_{\theta^\mu\rho}(x_0)$, $\mu = 0,\dots,k-1$. Thus we can apply Proposition 4.2 repeatedly to conclude with the help of the $(I\!I_\mu)$'s

$$\begin{aligned}\Phi(x_0,\theta^k\rho) &\le \theta^{2kt}\Phi(x_0,\rho) + \sum_{\mu=0}^{k-1} H(p_{x_0,\theta^{k-\mu-1}\rho})\theta^{2\mu t+2(k-\mu-1)s}\rho^{2s}\\ &\le \theta^{2kt}\Phi(x_0,\rho) + H(2p_0)\sum_{\mu=0}^{k-1}\theta^{2\mu t+2(k-\mu-1)s}\rho^{2s}\end{aligned}$$

$$
\begin{aligned}
&= \theta^{2kt}\Phi(x_0,\rho) + H(2p_0)\theta^{2(k-1)s}\rho^{2s}\sum_{\mu=0}^{k-1}\theta^{2\mu(t-s)} \\
&\le \theta^{2ks}\Big(\Phi(x_0,\rho) + \frac{H(2p_0)}{\theta^{2s}-\theta^{2t}}\,\rho^{2s}\Big)
\end{aligned}
$$

establishing (I'_k).

To show $(\mathit{I\!I}_k)$ we begin by noting the elementary inequality (for $c \ge 0$)

$$
\prod_{j=0}^{\infty}(1+c\theta^j\rho) = \exp\sum_{j=0}^{\infty}\log(1+c\theta^j\rho) \le \exp\sum_{j=0}^{\infty}c\rho\theta^j = \exp\Big[\frac{c\rho}{(1-\theta)}\Big].
$$

Given (ii) (recall that $\rho_0 \le 1$), we can iterate Proposition 4.1. Using in turn the (I'_μ)'s, (i), (ii), (iii) and (4-33) this yields:

$$
\begin{aligned}
p_{x_0,\theta^k\rho} &\le \prod_{\mu=0}^{k-1}(1+\theta^{-n/2}c_5\theta^\mu\rho)\Big[p_{x_0,\rho} + c_5\theta^{-n/2}\sum_{\mu=0}^{k-1}\Phi^{1/2}(x_0,\theta^\mu\rho)\Big] \\
&\le \exp\Big[\frac{c_5\rho}{\theta^{n/2}(1-\theta)}\Big] \\
&\quad\cdot\Big[p_{x_0,\rho} + c_5\theta^{-n/2}\sum_{\mu=0}^{k-1}\theta^{\mu s}\Big(\Phi^{1/2}(x_0,\rho) + \rho^s\sqrt{\frac{H(2p_0)}{\theta^{2s}-\theta^{2t}}}\Big)\Big] \\
&\le \exp\Big[\frac{c_5\rho_0(p_0)}{\theta^{n/2}(1-\theta)}\Big] \\
&\quad\cdot\Big[p_{x_0,\rho} + \frac{c_5}{\theta^{n/2}(1-\theta^s)}\Big(\Phi^{1/2}(p_0) + \sqrt{\rho_0(p_0)^{2s}\frac{H(2p_0)}{\theta^{2s}-\theta^{2t}}}\Big)\Big] \\
&\le \exp\Big[\frac{c_5\rho_0(p_0)}{\theta^{n/2}(1-\theta)}\Big]\Big[p_0 + \frac{2c_5\Phi^{1/2}(p_0)}{\theta^{n/2}(1-\theta^s)}\Big] \\
&\le \exp\Big[\frac{c_5\rho_0(p_0)}{\theta^{n/2}(1-\theta)}\Big]\frac{5p_0}{4} < 2p_0\,,
\end{aligned}
$$

which proves $(\mathit{I\!I}_k)$.

We next want to show that $(D^{\mathbf{m}}u)_{x_0,\theta^j\rho}$ converges to some limit Υ_{x_0}. Denoting by $P_{x_0,\theta^j\rho}$ the polynomial associated to u via (4-10) on the ball $B_{x_0,\theta^j\rho}(x_0)$ we have $(D^{\mathbf{m}}u)_{x_0,\theta^j\rho} = (D^{\mathbf{m}}P)_{x_0,\theta^j\rho} = D^{\mathbf{m}}P_{x_0,\theta^j\rho}(x_0)$. Hence, arguing as in the proof of (II_k) above we deduce for $k > j$

$$\begin{aligned} |(D^{\mathbf{m}}u)_{x_0,\theta^j\rho} - (D^{\mathbf{m}}u)_{x_0,\theta^k\rho}| &\le \sum_{\mu=j+1}^{k} |(D^{\mathbf{m}}u)_{x_0,\theta^\mu\rho} - (D^{\mathbf{m}}u)_{x_0,\theta^{\mu-1}\rho}| \\ &\le \frac{\theta^{js}}{\theta^{n/2}(1-\theta^s)}\Big[\Phi(x_0,\rho) + \frac{H(2p_0)\rho^{2s}}{\theta^{2s}-\theta^{2t}}\Big]^{1/2}, \end{aligned}$$

showing that $((D^{\mathbf{m}}u)_{x_0,\theta^j\rho})_{j\in\mathbb{N}}$ is a Cauchy-sequence. Therefore the limit $\Upsilon_{x_0} = \lim_{j\to\infty}(D^{\mathbf{m}}u)_{x_0,\theta^j\rho}$ exists and from the above estimate we infer for $j \in \mathbb{N}\cup\{0\}$

$$|(D^{\mathbf{m}}u)_{x_0,\theta^j\rho} - \Upsilon_{x_0}| \le \frac{\theta^{js}}{\theta^{n/2}(1-\theta^s)}\Big[\Phi(x_0,\rho) + \frac{H(2p_0)\rho^{2s}}{\theta^{2s}-\theta^{2t}}\Big]^{1/2}. \tag{4-34}$$

Now, for $0 < r \le \rho$ we find j such that $\theta^{j+1}\rho < r \le \theta^j\rho$. Using (I'_j) and (4-34) we finally arrive at

$$\begin{aligned} &⨍_{B_r(x_0)} |D^{\mathbf{m}}u - \Upsilon_{x_0}|^2\,dx \\ &\quad\le 2\theta^{-n}\Big[⨍_{B_{\theta^j\rho}(x_0)} |D^{\mathbf{m}}u - (D^{\mathbf{m}}u)_{x_0,\theta^j\rho}|^2\,dx + |(D^{\mathbf{m}}u)_{x_0,\theta^j\rho} - \Upsilon_{x_0}|^2\Big] \\ &\quad\le \frac{4\theta^{2js}}{\theta^{2n}(1-\theta^s)^2}\Big[\Phi(x_0,\rho) + \frac{H(2p_0)\rho^{2s}}{\theta^{2s}-\theta^{2t}}\Big] \\ &\quad\le \frac{4}{\theta^{2n+2s}(1-\theta^s)^2}\Big(\frac{r}{\rho}\Big)^{2s}\Big[\Phi(x_0,\rho) + \frac{H(2p_0)\rho^{2s}}{\theta^{2s}-\theta^{2t}}\Big] \\ &\quad\le \mathrm{const}\Big[\Big(\frac{r}{\rho}\Big)^{2s}\Phi(x_0,\rho) + r^{2s}\Big], \end{aligned}$$

which proves the desired estimate. □

PROOF OF THEOREM 1.1. Suppose $x_0 \in \Omega$ satisfies

$$\liminf_{\rho\to 0} ⨍_{B_\rho(x_0)} |D^{\mathbf{m}}u - (D^{\mathbf{m}}u)_{x_0,\rho}|^2\,dx = 0 \quad \text{and} \tag{4-35}$$

$$\limsup_{\rho\to 0} \sum_{k=1}^{N} \sum_{|\alpha|\le m_k} |(D^\alpha u^k)_{x_0,\rho}| < \infty\,. \tag{4-36}$$

We first show that $p_{x_0,\rho}$ can be estimated in terms of $\sum_{k=1}^{N}\sum_{|\alpha|\le m_k} |(D^\alpha u^k)_{x_0,\rho}|$. For $k = 1, \ldots, N$, $|\alpha| \le m_k$, we have

$$\begin{aligned} |D^\alpha P^k(x_0)| &\le \Big| ⨍_{B_\rho(x_0)} D^\alpha P^k\, dx\Big| + ⨍_{B_\rho(x_0)} |D^\alpha P^k - D^\alpha P^k(x_0)|\, dx \\ &= |(D^\alpha u^k)_{x_0,\rho}| + ⨍_{B_\rho(x_0)} |D^\alpha P^k - D^\alpha P^k(x_0)|\, dx\,. \end{aligned}$$

Arguing as in the proof of Lemma 4.3 we deduce for $|\alpha| \le m_k - 1$ and $x \in B_\rho(x_0)$

$$\begin{aligned} |D^\alpha P^k(x) - D^\alpha P^k(x_0)| &\le \rho \sum_{\substack{|\beta|\le m_k\\ \beta>\alpha}} |D^\beta P^k(x_0)| \\ &\le \rho \sum_{|\beta|\le m_k} |D^\beta P^k(x_0)| = \rho\, p^k_{x_0,\rho}\,, \end{aligned}$$

so that

$$|D^\alpha P^k(x_0)| \le |(D^\alpha u^k)_{x_0,\rho}| + \rho\, p^k_{x_0,\rho}\,.$$

Summing over $|\alpha| \le m_k - 1$, $k = 1, \ldots, N$, yields

$$\sum_{k=1}^{N}\sum_{|\alpha|\le m_k-1} |D^\alpha P^k(x_0)| \le \sum_{k=1}^{N}\sum_{|\alpha|\le m_k-1} |(D^\alpha u^k)_{x_0,\rho}| + N(n+1)^{m-1}\rho\, p_{x_0,\rho}\,.$$

Recalling that $D^\alpha P^k(x_0) = (D^\alpha u)_{x_0,\rho}$ for $k = 1, \ldots, N$, $|\alpha| = m_k$, we obtain

$$\Big(1 - N(n+1)^{m-1}\rho\Big) p_{x_0,\rho} \le \sum_{k=1}^{N}\sum_{|\alpha|\le m_k} |(D^\alpha u)_{x_0,\rho}|\,, \tag{4-37}$$

which is the desired bound on $p_{x_0,\rho}$ for ρ sufficiently small. It is now standard to deduce Theorem 1.1 from (4-35), (4-36), (4-37) and Lemma 4.3; see for example the start of Chapter IV in [G1], specifically Theorem 1.1 and Main Lemma 1.1 in that chapter. □

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(Received January 15, 2001)

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