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The Stickelberger Elements and the Cyclotomic Units in the Cyclotomic $\mathbb{Z}_p$-Extensions

By Takae Tsuji

Abstract. For an odd prime number p and a cyclotomic field K, we will describe a relation between the Stickelberger element and the cyclotomic unit which are defined with respect to the cyclotomic $\mathbb{Z}_p$-extension over K. This is a generalization of a theorem of Iwasawa and Coleman

1. Introduction

Let p be an odd prime number and N an integer prime to p such that $N \not\equiv 2 \bmod 4$. We put $K_n := \mathbb{Q}(\zeta_{Np^{n+1}})$ for all $n \geq 0$ and $K_\infty := \bigcup K_n$. Here $\zeta_{Np^{n+1}}$ denotes a primitive Np^{n+1}-th root of unity. The Stickelberger element $\boldsymbol{\theta}_N$ and the cyclotomic unit $\boldsymbol{\eta}_N$ are defined with respect to the cyclotomic $\mathbb{Z}_p$-extension K_∞/K_0 as below. Our purpose of this note is to describe a relation between the Stickelberger element $\boldsymbol{\theta}_N$ and the cyclotomic unit $\boldsymbol{\eta}_N$.

First we recall the definition of $\boldsymbol{\theta}_N$ and $\boldsymbol{\eta}_N$. The Stickelberger element $\theta_{Np^{n+1}} \in \mathbb{Q}_p[\mathrm{Gal}(K_n/\mathbb{Q})]$ is defined by

$$\theta_{Np^{n+1}} := \sum_{\substack{1 \leq a \leq Np^{n+1} \\ (a,pN)=1}} \Big(\frac{a}{Np^{n+1}} - \frac{1}{2} \Big) \sigma_a^{-1}|_{K_n},$$

where σ_a denotes the element of $\mathrm{Gal}(K_\infty/\mathbb{Q})$ satisfying $\sigma_a(\zeta_{Np^n}) = (\zeta_{Np^n})^a$ for all $n \geq 0$. We put

$$\boldsymbol{\theta}_N := (\theta_{Np^{n+1}})_{n \geq 0}.$$

For every integer c prime to Np, it is known (cf. [W, Lemma 6.9]) that

$$(1 - c\sigma_c^{-1})\, \boldsymbol{\theta}_N \in \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/\mathbb{Q})]] := \varprojlim \mathbb{Z}_p[\mathrm{Gal}(K_n/\mathbb{Q})],$$

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where the projective limit is taken with respect to the restriction maps. For every $s \in \mathbb{N}$, we fix a primitive s-th root ζ_s of unity with the property that $\zeta_{st}^t = \zeta_s$. We define the cyclotomic unit

$$\eta_{Np^{n+1}} := 1 - \zeta_N \zeta_{p^{n+1}} \in K_n^\times.$$

We shall regard $\eta_{Np^{n+1}}$ as an element of Φ_n, the p-adic completion of $(K_n \otimes \mathbb{Q}_p)^\times$. We put

$$\boldsymbol{\eta}_N := ((\eta_{Np^{n+1}})^{\mathrm{Fr}_p^{-n}})_{n \geq 0} \in \varprojlim \Phi_n,$$

where the projective limit is taken with respect to the relative norms and Fr_p denotes the Frobenius element of p in $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. Let $\mathcal{U}_{K_n}$ denote the p-adic completion of $(\mathcal{O}_{K_n} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. Put $\mathcal{U}_{K_\infty} := \varprojlim \mathcal{U}_{K_n}$, where the projective limit is taken with respect to the relative norms. Then $\boldsymbol{\eta}_N \in \mathcal{U}_{K_\infty}$ if $N \neq 1$, and $\boldsymbol{\eta}_1^{1-\sigma} \in \mathcal{U}_{K_\infty}$ for any $\sigma \in \mathrm{Gal}(K_\infty/\mathbb{Q})$.

In [Iw], Iwasawa proved a beautiful theorem which describes a relation between the Stickelberger element $\boldsymbol{\theta}_1$ and the cyclotomic unit $\boldsymbol{\eta}_1$. After that, in [C1] and [C2], Coleman gave a simpler proof of the above theorem by defining a $\mathbb{Z}_p$-homomorphism

$$\Psi : \mathcal{U}_{\mathbb{Q}(\zeta_{p^\infty})} \longrightarrow \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]],$$

which is almost isomorphism. Here $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup \mathbb{Q}(\zeta_{p^n})$. The above mentioned theorem is stated as follows.

Theorem 1.1 (Iwasawa, Coleman [C2, Proposition 6]). *For every integer c prime to p, we have*

$$\Psi(\boldsymbol{\eta}_1^{1-\sigma_c}) = (1 - c\sigma_c)\, \boldsymbol{\theta}_1^*,$$

where $\theta \mapsto \theta^*$ is the involution of $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]]$ induced by $\sigma \mapsto \sigma^{-1}$ for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$.

We shall extend this result for $\boldsymbol{\theta}_N$ and $\boldsymbol{\eta}_N$ $(N > 1)$. Similarly to Theorem 1.1, our main theorem (Theorem 2.1) is also stated using the homomorphism Ψ, which is also defined for $\mathcal{U}_{\mathbb{Q}(\zeta_{Np^\infty})}$. We give two different proofs. The first one (in §2) is a modification of Coleman's method, and is direct one in the sense that we use only some properties of Ψ (and the definitions of $\boldsymbol{\theta}_N$ and $\boldsymbol{\eta}_N$). On the other hand, both $\boldsymbol{\theta}_N$ and $\boldsymbol{\eta}_N$ are related to the Kubota-Leopoldt p-adic L-functions. We see in §4 that Theorem 2.1 is also induced from these classical relations.

2. The Results

We use the same notation as in the Introduction. Put $G_\infty := \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ and $\Delta := \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Then $\mathrm{Gal}(K_\infty/\mathbb{Q}) \cong G_\infty \times \Delta$ since N is prime to p. We write $\widehat{\mathcal{O}} := \mathbb{Z}[\zeta_N] \otimes_{\mathbb{Z}} \mathbb{Z}_p$, the p-adic completion of the integer ring of $\mathbb{Q}(\zeta_N)$. Coleman proved that there exists a $\mathbb{Z}_p$-homomorphism

$$\Psi : \mathcal{U}_{K_\infty} \longrightarrow \widehat{\mathcal{O}}[[G_\infty]],$$

with the property that $\Psi(u^\sigma) = \kappa(\sigma)\sigma\Psi(u)$ for all $u \in \mathcal{U}_{K_\infty}$ and all $\sigma \in \mathrm{Gal}(K_\infty/\mathbb{Q})$, where $\kappa : \mathrm{Gal}(K_\infty/\mathbb{Q}) \longrightarrow \mathbb{Z}_p^\times$ is the p-cyclotomic character (cf. [C1], [C2] and [G, §2]). To compare $(1 - c\sigma_c^{-1})\,\boldsymbol{\theta}_N \in \mathbb{Z}_p[\Delta][[G_\infty]]$ and $\boldsymbol{\eta}_N \in \mathcal{U}_{K_\infty}$ by using Ψ, we need to determine a $\mathbb{Z}_p[\Delta]$-isomorphism $\widehat{\mathcal{O}} \xrightarrow{\sim} \mathbb{Z}_p[\Delta]$, that is, to fix a generator of $\widehat{\mathcal{O}}$ as $\mathbb{Z}_p[\Delta]$-module. For such a generator, we can take Leopoldt's "Basiszahl" defined as follows: For a positive integer m, we put

$$\mathcal{D}_m := \{d \in \mathbb{N} \mid d \mid m,\ s(m) \mid d\},$$

where $s(m)$ denotes the product of all distinct prime divisors of m. We define an element z_N of $\widehat{\mathcal{O}}$ by

$$z_N := \sum_{d \in \mathcal{D}_N} \zeta_d.$$

By using z_N, Leopoldt [Leo] (see also [Let]) described the structure of $\mathbb{Z}[\zeta_N]$ as $\mathbb{Z}[\Delta]$-module. (Actually, in [Leo] and [Let], the Galois module structure of the ring of integers of an abelian number field is described by using "Basiszahl".) Using this result, we see that z_N generates the $\mathbb{Z}_p[\Delta]$-module $\widehat{\mathcal{O}}$ (see also Lemma 3.1). Our main theorem is to describe the relation between $(1 - c\sigma_c^{-1})\,\boldsymbol{\theta}_N$ and $\boldsymbol{\eta}_N$ using Ψ and z_N.

To state our result, we need some notation. We define the Stickelberger element $\boldsymbol{\theta}_d$ and the cyclotomic unit $\boldsymbol{\eta}_d$, for $d \mid N$, as in the case where $d = N$. For an integer c prime to p, let γ_c denote the element of G_∞ satisfying $\gamma_c(\zeta_{p^n}) = \zeta_{p^n}^c$ for all $n \geq 0$. For a prime number l, let δ_l denote a Frobenius element of l in Δ. If d is a divisor of N, we denote by $\mathrm{Cor}_{N,d}$ the corestriction map from $\mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})][[G_\infty]]$ to $\mathbb{Z}_p[\Delta][[G_\infty]]$ induced by, for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$,

$$\tau \mapsto \sum_{\delta|_{\mathbb{Q}(\zeta_d)}=\tau} \delta,$$

where δ runs over automorphisms in Δ whose restriction to $\mathbb{Q}(\zeta_d)$ is τ. Let μ denote the Möbius μ-function.

Definition. We define $\mathfrak{S}_N$ by

$$\mathfrak{S}_N := \gamma_N^{-1} \sum_{d|N} \gamma_d \Big(\prod_{l|N, l\nmid d} (1-l)\delta_l^{-1} \Big) \sum_{d' \in \mathcal{D}_d} \frac{\mu(d/d')}{[\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_{d'})]} \mathrm{Cor}_{N,d'}(\boldsymbol{\theta}_{d'}),$$

where d runs over all divisors of N and l over all prime divisors of N which do not divide d, and $\mathfrak{H}_N$ by

$$\mathfrak{H}_N := \prod_{d \in \mathcal{D}_N} \prod_{d'|d} (\boldsymbol{\eta}_{d/d'})^{\frac{\mu(d')\gamma_{d'}}{d'}},$$

where d' runs over all divisors of d and we regard $1/d'$ as an element of $\mathbb{Z}_p$.

Since $\delta_l^{-1}\mathrm{Cor}_{N,d'}(\boldsymbol{\theta}_{d'}) = \mathrm{Cor}_{N,d'}(\delta_l^{-1}|_{\mathbb{Q}(\zeta_{d'})}\boldsymbol{\theta}_{d'})$, $\mathfrak{S}_N$ does not depend on the choice of δ_l. We note that $\mathfrak{H}_N^{1-\gamma_c} \in \mathcal{U}_{K_\infty}$ and $(1-c\gamma_c^{-1})\boldsymbol{\theta}_N \in \mathbb{Z}_p[\Delta][[G_\infty]]$ for all c prime to p.

If N is square-free, the above definitions are just as

$$\mathfrak{S}_N = \sum_{d|N} \mu(d)\sigma_d^{-1}\mathrm{Cor}_{N,N/d}(\boldsymbol{\theta}_{N/d}), \quad \mathfrak{H}_N = \prod_{d|N} (\boldsymbol{\eta}_{N/d})^{\frac{\mu(d)\gamma_d}{d}},$$

where σ_d denotes an element of $\mathrm{Gal}(K_\infty/\mathbb{Q})$ satisfying $\sigma_d(\zeta_{(N/d)p^{n+1}}) = (\zeta_{(N/d)p^{n+1}})^d$.

The main result of this note is the following.

Theorem 2.1. *For every integer c prime to p, we have the following equations in $\widehat{\mathcal{O}}[[G_\infty]]$:*

$$\Psi(\mathfrak{H}_N^{1-\gamma_c}) = (1-c\gamma_c)\,\boldsymbol{\theta}_N^* z_N,$$

and

$$\Psi(\boldsymbol{\eta}_N) = \mathfrak{S}_N^* z_N.$$

Here $\theta \mapsto \theta^*$ is the involution of $\mathbb{Z}_p[\Delta][[G_\infty]]$ induced by $\sigma \mapsto \sigma^{-1}$ for any $\sigma \in \Delta \times G_\infty$.

Remark. The description of $\boldsymbol{\theta}_N^*$ (resp. $\Psi(\boldsymbol{\eta}_N)$) by $\Psi(\boldsymbol{\eta}_d)$ (resp. $\boldsymbol{\theta}_d^*$) with $d \mid N$ and z_N is not unique. Indeed, there are relations between $\boldsymbol{\theta}_{dl}$ and $\boldsymbol{\theta}_d$ (resp. $\boldsymbol{\eta}_{dl}$ and $\boldsymbol{\eta}_d$), for a prime number l, as follows:

$$\boldsymbol{\theta}_{dl}|_{\mathbb{Q}(\zeta_{dp^\infty})} = \begin{cases} \boldsymbol{\theta}_d & l \mid d \\ (1-\gamma_l^{-1}\delta_l^{-1})\,\boldsymbol{\theta}_d & l \nmid d \end{cases} \tag{2.1}$$

and

$$N_{dl,d}(\boldsymbol{\eta}_{dl}) = \begin{cases} \boldsymbol{\eta}_d^{\gamma_l} & l \mid d \\ \boldsymbol{\eta}_d^{(\gamma_l-\delta_l^{-1})} & l \nmid d, \end{cases} \tag{2.2}$$

where $N_{dl,d}$ denotes the norm map from $\mathbb{Q}(\zeta_{dlp^\infty})$ to $\mathbb{Q}(\zeta_{dp^\infty})$.

3. Proof of Theorem 2.1

In this section we prove Theorem 2.1 by using the properties of $\Psi : \mathcal{U}_{K_\infty} \longrightarrow \widehat{\mathcal{O}}[[G_\infty]]$. We need the following lemma. Although this follows from Leopoldt's Theorem and the facts which are used in his proof ([Leo], see also [Let]), we give a proof for the convenience of the readers.

Lemma 3.1. *The element $z_N = \sum_{d\in\mathcal{D}_N} \zeta_d$ generates the additive $\mathbb{Z}_p[\Delta]$-module $\widehat{\mathcal{O}}$ which is free of rank one. Furthermore, for a positive divisor m of N, we have*

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_m)}(z_N) = [\mathbb{Q}(\zeta_N):\mathbb{Q}(\zeta_m)]\Big(\prod_{l\mid N, l\nmid m} \frac{\delta_l^{-1}}{1-l}\Big)(z_m) \tag{3.1}$$

and

$$\zeta_m = \Big(\prod_{l\mid N, l\nmid m}(1-l)\delta_l\Big)\sum_{d\in\mathcal{D}_m} \frac{\mu(m/d)}{[\mathbb{Q}(\zeta_N):\mathbb{Q}(\zeta_d)]}\mathrm{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_d)}(z_N). \tag{3.2}$$

Proof. The additive $\mathbb{Z}_p[\Delta]$-module $\widehat{\mathcal{O}}$ is generated by ζ_m with $m \mid N$. Therefore the first assertion follows from (3.2).

For the equation (3.1), it suffices to verify the case where $m = N/q$ with a prime divisor q of N. Let d be a divisor of N. When $d \nmid N/q$, the minimal polynomial of ζ_d over $\mathbb{Q}(\zeta_{d/q})$ is $(X^q - \zeta_{d/q})/(X - \delta_q{}^{-1}(\zeta_{d/q})) = \sum_{j=0}^{q-1} X^{q-1-i}(\delta_q{}^{-1}(\zeta_{d/q}))^i$ (resp. $X^q - \zeta_{d/q}$) if $q^2 \nmid d$ (resp. $q^2 \mid d$). Then we have the following:

$$\mathrm{Tr}_{N,N/q}(\zeta_d) = \begin{cases} -\delta_q{}^{-1}(\zeta_{d/q}) & d \nmid N/q,\ q^2 \nmid d, \\ 0 & d \nmid N/q,\ q^2 \mid d, \\ [\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_{N/q})]\zeta_d & d \mid N/q, \end{cases}$$

where $\mathrm{Tr}_{N,N/q} = \mathrm{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{N/q})}$. If we assume $q^2 \nmid N$, we have $\mathcal{D}_{N/q} = \{d/q \mid d \in \mathcal{D}_N\}$ and, for any $d \in \mathcal{D}_N$, $d \nmid N/q$ and $q^2 \nmid d$. On the other hand, if we assume $q^2 \mid N$, we have $\mathcal{D}_{N/q} = \{d \in \mathcal{D}_N \mid d \mid (N/q)\}$ and $q^2 \mid d$ for any $d \in \mathcal{D}_N$ with $d \nmid N/q$. Hence we obtain

$$\mathrm{Tr}_{N,N/q}(z_N) = \begin{cases} -\delta_q{}^{-1}(z_{N/q}) & q^2 \nmid N, \\ [\mathbb{Q}(\zeta_N) : \mathbb{Q}(\zeta_{N/q})]z_{N/q} & q^2 \mid N. \end{cases}$$

This proves the equation (3.1) for $m = N/q$.

The equation (3.2) follows from the equation (3.1) and the following equality:

$$\begin{aligned} \zeta_m &= \sum_{d' \in \mathcal{D}_m} \Big(\sum_{d'' \mid (m/d')} \mu(d'') \Big) \zeta_{d'} \\ &= \sum_{d' \in \mathcal{D}_m} \Big(\sum_{d \mid m,\ d' \mid d} \mu\Big(\frac{m}{d}\Big) \Big) \zeta_{d'} \\ &= \sum_{d \in \mathcal{D}_m} \mu\Big(\frac{m}{d}\Big) \sum_{d' \in \mathcal{D}_d} \zeta_{d'} \\ &= \sum_{d \in \mathcal{D}_m} \mu\Big(\frac{m}{d}\Big) z_d. \end{aligned}$$

We complete the proof. □

Proof of Theorem 2.1. First, we briefly recall the definition of the map $\Psi : \mathcal{U}_{K_\infty} \longrightarrow \widehat{\mathcal{O}}[[G_\infty]]$. For details, see [C1], [C2] and [G, §2]. Let $u = (u_n)$ be an element of $\mathcal{U}_{K_\infty}$. There exists a unique power series $f_u(X)$ in $\widehat{\mathcal{O}}[[X]]$ satisfying

$$f_u(\zeta_{p^{n+1}} - 1) = u_n^{\mathrm{Fr}_p^n}. \tag{3.3}$$

Let φ be an endomorphism of $\widehat{\mathcal{O}}[[X]]$ defined by

$$(\varphi f)(X) = f^{\mathrm{Fr}_p}((1+X)^p - 1),$$

where Fr_p acts on $f(X)$ via the coefficients. Let D be the derivation $(1+X)\frac{d}{dX}$ of $\widehat{\mathcal{O}}[[X]]$. Then there is a unique element $\Psi(u)$ of $\widehat{\mathcal{O}}[[G_\infty]]$ satisfying

$$(1-\varphi)D\log f_u(X)|_{X=\zeta_{p^{n+1}}-1} = \Psi(u)\zeta_{p^{n+1}}$$

for all $n \geq 0$, which is the definition of $\Psi : \mathcal{U}_{K_\infty} \longrightarrow \widehat{\mathcal{O}}[[G_\infty]]$.

Put

$$f_N(X) := 1 - \zeta_N(1+X)$$

and

$$\tilde{f}_N(X) := \prod_{d \in \mathcal{D}_N} \prod_{d' | d} (1 - (\zeta_d(1+X))^{d'})^{\frac{\mu(d')}{d'}}.$$

One can easily verify that $f_N(X)$ (resp. $\tilde{f}_N(X)$) satisfies (3.3) with respect to $\boldsymbol{\eta}_N$ (resp. to $\mathfrak{H}_N$). It suffices to show the following two equations

$$(1-\varphi)D\log f_N(X)|_{X=\zeta_{p^{n+1}}-1} = \mathfrak{S}_N^* z_N \zeta_{p^{n+1}}, \tag{3.4}$$

$$(1-\varphi)D\log \tilde{f}_N(X)|_{X=\zeta_{p^{n+1}}-1} = \boldsymbol{\theta}_N^* z_N \zeta_{p^{n+1}}. \tag{3.5}$$

As in the proof of Theorem 1.1 given in [C2], we use the following.

Lemma 3.2 (cf. [C2, Proposition 5], [G, Lemma 2.15]). *For $m \geq 1$ and $n \geq 1$, we have*

$$\frac{\zeta_m\zeta_{p^{n+1}}}{\zeta_m\zeta_{p^{n+1}} - 1} - \frac{\zeta_m^p\zeta_{p^n}}{\zeta_m^p\zeta_{p^n} - 1} = \sum_{\substack{1 \leq a \leq mp^{n+1} \\ (a,p)=1}} \frac{a}{mp^{n+1}} (\zeta_m\zeta_{p^{n+1}})^a.$$

We first prove the equation (3.4). By Lemma 3.2, we have

$$
\begin{aligned}
(1-\varphi)D\log f_N(X)|_{X=\zeta_{p^{n+1}}-1} &= \frac{\zeta_N\zeta_{p^{n+1}}}{\zeta_N\zeta_{p^{n+1}}-1} - \frac{\zeta_N^p\zeta_{p^n}}{\zeta_N^p\zeta_{p^n}-1} \\
&= \sum_{\substack{1\le a\le Np^{n+1} \\ (a,p)=1}} \frac{a}{Np^{n+1}}(\zeta_N\zeta_{p^{n+1}})^a \\
&= \sum_{d|N}\sum_{\substack{1\le b\le dp^{n+1} \\ (b,pd)=1}} \frac{b}{dp^{n+1}}(\zeta_d\zeta_{p^{n+1}}^{\frac{N}{d}})^b \\
&= \gamma_N\sum_{d|N}\gamma_d^{-1}\theta^*_{dp^{n+1}}\zeta_d\zeta_{p^{n+1}}.
\end{aligned}
$$

By the equation (3.2), we have

$$
\theta^*_{dp^{n+1}}\zeta_d\zeta_{p^{n+1}} = \Big(\prod_{l|N, l\nmid d}(1-l)\delta_l\Big)\sum_{d'\in\mathcal{D}_d}\frac{\mu(d/d')}{[\mathbb{Q}(\zeta_N):\mathbb{Q}(\zeta_{d'})]}\theta^*_{dp^{n+1}}\mathrm{Tr}_{N,d'}(z_N)\zeta_{p^{n+1}},
$$

where $\mathrm{Tr}_{N,d'} = \mathrm{Tr}_{\mathbb{Q}(\zeta_N)/\mathbb{Q}(\zeta_{d'})}$. Since every prime divisor of d is a divisor of d' in $\mathcal{D}_d$, by the equation (2.1), we obtain

$$
\begin{aligned}
\theta^*_{dp^{n+1}}\mathrm{Tr}_{N,d'}(z_N)\zeta_{p^{n+1}} &= \theta^*_{d'p^{n+1}}\mathrm{Tr}_{N,d'}(z_N)\zeta_{p^{n+1}} \\
&= \mathrm{Cor}_{N,d'}(\theta^*_{d'p^{n+1}})z_N\zeta_{p^{n+1}}.
\end{aligned}
$$

Combining the above equalities, we obtain the equation (3.4).

For the equation (3.5), by Lemma 3.2, we have

$$
\begin{aligned}
&(1-\varphi)D\log\Big(\prod_{d'|d}(1-(\zeta_d(1+X))^{d'})^{\frac{\mu(d')}{d'}}\Big)\Big|_{X=\zeta_{p^{n+1}}-1} \\
&\qquad = \sum_{d'|d}\mu(d')\Big(\frac{(\zeta_d\zeta_{p^{n+1}})^{d'}}{(\zeta_d\zeta_{p^{n+1}})^{d'}-1} - \frac{(\zeta_d^p\zeta_{p^n})^{d'}}{(\zeta_d^p\zeta_{p^n})^{d'}-1}\Big) \\
&\qquad = \sum_{d'|d}\mu(d')\sum_{\substack{1\le a\le (d/d')p^{n+1} \\ (a,p)=1}}\frac{ad'}{dp^{n+1}}(\zeta_d\zeta_{p^{n+1}})^{ad'}
\end{aligned}
$$

$$= \sum_{\substack{1 \le b \le dp^{n+1} \\ (b,p)=1}} \Big(\sum_{d' | (b,d)} \mu(d') \Big) \frac{b}{dp^{n+1}} (\zeta_d \zeta_{p^{n+1}})^b$$
$$= \sum_{\substack{1 \le b \le dp^{n+1} \\ (b,dp)=1}} \frac{b}{dp^{n+1}} (\zeta_d \zeta_{p^{n+1}})^b$$
$$= \theta^*_{dp^{n+1}} \zeta_d \zeta_{p^{n+1}}.$$

Therefore, by using the equation (2.1), we obtain

$$\begin{aligned} D(1-\varphi)\log(\tilde{f}_N(X))|_{X=\zeta_{p^{n+1}}-1} &= \sum_{d \in \mathcal{D}_N} \theta^*_{dp^{n+1}} \zeta_d \zeta_{p^{n+1}} \\ &= \sum_{d \in \mathcal{D}_N} \theta^*_{Np^{n+1}} \zeta_d \zeta_{p^{n+1}} \\ &= \theta^*_{Np^{n+1}} z_N \zeta_{p^{n+1}}. \end{aligned}$$

This completes the proof. □

4. p-Adic L-Function

In this section, we review how to connect the Stickelberger element $\boldsymbol{\theta}_N$ and the cyclotomic unit $\boldsymbol{\eta}_N$ with the values of the Dirichlet L-function at negative integer respectively. Then we see that Theorem 2.1 is also induced by using the above connection.

Let χ be a primitive Dirichlet character with values in $\overline{\mathbb{Q}}_p^\times$, whose conductor divides N. We shall regard χ as a character of $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Let $\mathbb{Z}_p[\chi]$ be the ring generated by the values of χ over $\mathbb{Z}_p$. We also denote by χ a natural map

$$\mathbb{Z}_p[\Delta][[G_\infty]] \longrightarrow \mathbb{Z}_p[\chi][[G_\infty]] \tag{4.1}$$

induced by χ. Let $\kappa : G_\infty \longrightarrow \mathbb{Z}_p^\times$ denote the p-cyclotomic character. For every integer $r \ge 0$, one can extend the character κ^r of G_∞ to a homomorphism

$$\mathbb{Z}_p[\chi][[G_\infty]] \longrightarrow \mathbb{Z}_p[\chi].$$

Let $c > 1$ be an integer prime to p. We put

$$\kappa^r\chi(\boldsymbol{\theta}_N^*) = (1 - c^{1+r})^{-1}\kappa^r\chi((1 - c\gamma_c)\,\boldsymbol{\theta}_N^*),$$

which is independent of c. The following theorem is well known.

Theorem 4.1 (Iwasawa cf. [W, Theorem 7.10]). *For every $r \geq 0$, we have*

$$\kappa^r\chi(\boldsymbol{\theta}_N^*) = \prod_{l|Np}(1 - \chi(l)l^r)L(-r, \chi),$$

where l runs over all prime divisors of Np and $L(*, \chi)$ denotes the Dirichlet L-function.

By Lemma 3.1, the correspondence $z_N \mapsto 1$ induces an isomorphism $\widehat{\mathcal{O}} \xrightarrow{\sim} \mathbb{Z}_p[\Delta]$. Let

$$\chi_{z_N} : \widehat{\mathcal{O}}[[G_\infty]] \longrightarrow \mathbb{Z}_p[\chi][[G_\infty]]$$

be the map by composing the above isomorphism $\widehat{\mathcal{O}} \xrightarrow{\sim} \mathbb{Z}_p[\Delta]$ with the map (4.1). Since $\boldsymbol{\eta}_1$ is not in $\mathcal{U}_{K_\infty}$, we write $\kappa^r\chi_{z_N}(\Psi(\boldsymbol{\eta}_1))$ for $(1 - c^{r+1})^{-1}\kappa^r\chi_{z_N}(\Psi(\boldsymbol{\eta}_1^{1-\gamma_c}))$, which is independent of c. Let f_χ denote the conductor of χ and put $f_{\chi,N} = f_\chi \prod_l' l$, where l runs over all prime divisors of N such that $l \nmid f_\chi$. The following theorem is known (cf. e.g. [G, Theorem 2.13], [P, Proposition 3.1.4] and [T, Theorem 4.3 and §7]).

Theorem 4.2. *For every $r \geq 0$, we have*

$$\kappa^r\chi_{z_N}(\Psi(\boldsymbol{\eta}_N)) = \Big(\frac{N}{f_{\chi,N}}\Big)^r \prod_{l|N}(1 - \chi(l)l^{r+1})(1 - \chi(p)p^r)L(-r, \chi),$$

where l runs over all prime divisors of N.

Let θ and θ' be two elements of $\mathbb{Z}_p[\chi][[G_\infty]]$. If $\kappa^r(\theta) = \kappa^r(\theta')$ for all $r \geq 0$, we have $\theta = \theta'$. Thus, combining the above two theorems, we obtain the following.

Proposition 4.3. *For any character χ of Δ and any integer c prime to p, we have*

$$\chi((1 - c\gamma_c)\,\boldsymbol{\theta}_N^*) = \prod_{l|N}(1 - \chi(l)\gamma_l)\chi_{z_{f_\chi}}(\Psi(\boldsymbol{\eta}_{f_\chi}^{1-\gamma_c})),$$

and

$$\chi_{z_N}(\Psi(\boldsymbol{\eta}_N^{1-\gamma_c})) = \frac{\gamma_N}{\gamma_{f_\chi,N}} \prod_{l|N}(1-\chi(l)l\gamma_l)\chi((1-c\gamma_c)\boldsymbol{\theta}^*_{f_\chi}).$$

We will see that the above proposition gives the same relation as Theorem 2.1. We have

$$\chi(\mathfrak{S}^*_N) = \chi(\mathfrak{S}^*_N|_{\mathbb{Q}(\zeta_{f_\chi p^\infty})})$$

and

$$\chi_{z_N}(\Psi(\mathfrak{H}_N^{1-\gamma_c})) = \frac{\prod_{l|N, l\nmid f_\chi}\chi(l)(1-l)}{[\mathbb{Q}(\zeta_N):\mathbb{Q}(\zeta_{f_\chi})]}\chi_{z_{f_\chi}}(\Psi(N_{N,f_\chi}(\mathfrak{H}_N^{1-\gamma_c}))).$$

Here the second equation follows from Lemma 3.1 and the property that $\Psi(N_{N,f_\chi}(u)) = \mathrm{Tr}_{N,f_\chi}(\Psi(u))$ for any $u \in \mathcal{U}_{K_\infty}$. By using the relations (2.1) and (2.2), we have

$$\mathfrak{S}_{dl}|_{\mathbb{Q}(\zeta_{dp^\infty})} = \begin{cases} \gamma_l^{-1}\mathfrak{S}_d & l \mid d \\ (1-l\gamma_l^{-1}\delta_l^{-1})\mathfrak{S}_d & l \nmid d \end{cases}$$

and

$$N_{dl,d}(\mathfrak{H}_{dl}) = \begin{cases} \mathfrak{H}_d^l & l \mid d \\ \mathfrak{H}_d^{(l^{-1}\gamma_l - \delta_l^{-1})} & l \nmid d \end{cases}$$

for a prime number l. One can see that $\chi((1-c\gamma_c)\mathrm{Cor}_{f_\chi,d}(\boldsymbol{\theta}^*_d)) = 0$ and $\chi_{z_{f_\chi}}(\Psi(\boldsymbol{\eta}_d^{1-\gamma_c}) = 0$ for a proper divisor d of f_χ. Then, we obtain $\chi((1-c\gamma_c)\mathfrak{S}^*_{f_\chi}) = \chi((1-c\gamma_c)\boldsymbol{\theta}^*_{f_\chi})$ and $\chi_{z_{f_\chi}}(\Psi(\mathfrak{H}_{f_\chi}^{1-\gamma_c})) = \chi_{z_{f_\chi}}(\Psi(\boldsymbol{\eta}_{f_\chi}^{1-\gamma_c}))$, by the definition of $\mathfrak{S}_N$ and $\mathfrak{H}_N$. Therefore, we obtain

$$\chi((1-c\gamma_c)\mathfrak{S}^*_N) = \frac{\gamma_N}{\gamma_{f_\chi,N}} \prod_{l|N}(1-\chi(l)l\gamma_l)\chi((1-c\gamma_c)\boldsymbol{\theta}^*_{f_\chi}),$$

and

$$\chi_{z_N}(\Psi(\mathfrak{H}_N^{1-\gamma_c})) = \prod_{l|N}(1-\chi(l)\gamma_l)\chi_{z_{f_\chi}}(\Psi(\boldsymbol{\eta}_{f_\chi}^{1-\gamma_c})).$$

Hence the above proposition states that $\chi((1-c\gamma_c)\boldsymbol{\theta}^*_N) = \chi_{z_N}(\Psi(\mathfrak{H}_N^{1-\gamma_c}))$ and $\chi_{z_N}(\Psi(\boldsymbol{\eta}_N^{1-\gamma_c})) = \chi((1-c\gamma_c)\mathfrak{S}^*_N)$ hold, for all character χ of Δ.

These relations show Theorem 2.1, that is $(1 - c\gamma_c)\,\boldsymbol{\theta}_N^* z_N = \Psi(\mathfrak{H}_N^{1-\gamma_c})$ and $\Psi(\boldsymbol{\eta}_N) = \mathfrak{S}_N^* z_N$.

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Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, JAPAN
E-mail: ttsuji@ms.u-tokyo.ac.jp