

**Brauer groups, Mackey and Tambara
functors on profinite groups,
and
2-dimensional homological algebra**

(Brauer 群、プロ有限群上の Mackey 及び丹原関手
と
2次元ホモロジー代数)

BRAUER GROUPS, MACKEY AND TAMBARA FUNCTORS ON PROFINITE GROUPS, AND 2-DIMENSIONAL HOMOLOGICAL ALGEBRA

HIROYUKI NAKAOKA

ABSTRACT. In this thesis, we investigated categories and functors related to Brauer groups.

In 1986, E.T.Jacobson defined the Brauer ring $B(E, D)$ for a finite Galois field extension E/D , whose unit group canonically contains the Brauer group of D . In Part 1, we investigate the structure of $B(E, D)$. More generally, we determine the structure of the F -Burnside ring for any additive functor F . This result enables us to calculate Brauer rings for some extensions. We illustrate how this isomorphism provides Green-functor theoretic meanings for the properties of the Brauer ring shown by Jacobson, and compute the Brauer ring of the extension \mathbb{C}/\mathbb{R} .

For any finite étale covering of schemes, we can associate two homomorphisms of Brauer groups, namely the pull-back and the norm map. For any connected scheme X , if we take the Galois category \mathcal{C} of finite étale coverings over X , we see these homomorphisms make Brauer groups into a bivariant functor (= Mackey functor) on \mathcal{C} . As a corollary, restricting to a finite Galois covering of schemes, we obtain a cohomological Mackey functor on its Galois group. This is a generalization of the result for rings by Ford [12].

The Tambara functor was defined by Tambara in the name of TNR-functor, to treat certain ring-valued Mackey functors on a finite group. Recently Brun revealed the importance of Tambara functors in the Witt-Burnside construction. In Part 3, we define the Tambara functor on the Mackey system of Bley and Boltje. Yoshida's generalized Burnside ring functor is the first example. Consequently, we can consider a Tambara functor on any profinite group. In relation with the Witt-Burnside construction, we can give a Tambara-functor structure on Elliott's functor \mathbf{V}_M , which generalizes the completed Burnside ring functor of Dress and Siebeneicher.

Recently, symmetric categorical groups are used for the study of the Brauer groups of symmetric monoidal categories. As a part of these efforts, some algebraic structures of the 2-category of symmetric categorical groups SCG are investigated. In Part 4, we consider a 2-categorical analogue of an abelian category, in such a way that it contains SCG as an example. As the main theorem in this part, we construct a long cohomology 2-exact sequence from any extension of complexes in such a 2-category. Our axiomatic and self-dual definition will enable us to simplify various kind of arguments related to the 2-dimensional homological algebra, by analogy with abelian categories.

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Part 1. Structure of the Brauer ring of a field extension

1. INTRODUCTION FOR PART 1

For the general theory of Mackey and Green functors, see [4]. Throughout this part, we fix a finite group G , and use the following notation:

- $H \leq G$ means that H is a subgroup of G .
- For any $K \leq H \leq G$ and any $g \in G$, ${}^gH := gHg^{-1}$, $H^g := g^{-1}Hg$, and $\ell_{g,H} : G/{}^gH \rightarrow G/H$ is the G -map defined by $\ell_{g,H}(g' \cdot {}^gH) = g'g \cdot H$, $p_K^H : G/K \rightarrow G/H$ is the canonical projection.
- $\text{Mack}(G)$ and $\text{Green}(G)$ denote the category of Mackey functors and Green functors respectively.
- For any group M , $\mathbb{Z}[M]$ denotes its group ring over \mathbb{Z} , and similarly for \mathbb{Q} .

Most of the following arguments will work well even if the codomain of Green (and several other) functors is the category of R -modules $R\text{-Mod}$ instead of $\text{Ab} = \mathbb{Z}\text{-Mod}$, for any commutative ring R with 1. But we restrict ourselves to the case of $R = \mathbb{Z}$, for the sake of simplicity. Monoids, rings, and Green functors are equipped with 1, but not assumed to be commutative, unless otherwise specified.

For a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \circlearrowleft & \downarrow \\ Y & \longrightarrow & W \end{array},$$

we use a small square \square to indicate that it is a pull-back diagram:

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \square & \downarrow \\ Y & \longrightarrow & W \end{array}$$

The Brauer ring $B(E, D)$ of a finite Galois field extension E/D was defined by Jacobson in [16]. $B(E, D)$ can be regarded as an example of the F -Burnside ring, where F is an additive functor $F : \mathcal{G} \rightarrow \text{Ab}$. By using Chen's result (Corollary 3.4) in [7], for any trivial field extension E/E , we can see that the Brauer ring $B(E, E)$ is naturally isomorphic to the group ring of the Brauer group $\text{Br}(E)$:

$$B(E, E) \cong \mathbb{Z}[\text{Br}(E)]$$

(see also Proposition 2.8 and Remark 2.9 in this paper).

In the following, we will define several types of additive functors, and by the adjoint properties concerning these functors, we will see the structure of the F -Burnside ring as follows:

Theorem 3.13 . For any $F \in \text{Ob}(\text{Madd}(G))$, there is a natural isomorphism of Green functors

$$(\mathbb{Z}[\mathcal{R}_F])_+ \xrightarrow{\cong} \mathcal{A}_F.$$

As a corollary, the structure of the Brauer ring $B(E, D)$ can be seen as follows:

Corollary 4.1 . For any finite Galois extension E/D of fields with Galois group G , we have a ring isomorphism

$$B(E, D) \cong \left(\bigoplus_{H \leq G} \mathbb{Z}[\text{Br}(E^H)] \right) / (I(\mathbb{Z}[G]) \cdot \bigoplus_{H \leq G} \mathbb{Z}[\text{Br}(E^H)]).$$

This is a generalization of the above isomorphism $B(E, E) \cong \mathbb{Z}[\text{Br}(E)]$.

2. DEFINITION OF THE ADDITIVE FUNCTOR AND THE BRAUER RING

In this section, we recall the construction of the F -Burnside ring, defined by Jacobson [16], and introduce the Brauer ring.

We fix a finite group G , and let \mathcal{G} be the category of finite G -sets and G -maps. Set denotes the category of small sets. A contravariant functor $E : \mathcal{G} \rightarrow \text{Set}$ is said to be additive if the canonical map $(E(i_X), E(i_Y)) : E(X \amalg Y) \rightarrow E(X) \times E(Y)$ induced by the inclusions i_X, i_Y is bijective for any $X, Y \in \text{Ob}(\mathcal{G})$. Let $j_{X,Y}$ denote the inverse bijection. $E(\emptyset)$ consists of one element. $\text{Sadd}(G)$ denotes the category of additive functors from \mathcal{G} to Set , whose morphisms are natural transformations.

Definition 2.1. Let E be in $\text{Ob}(\text{Sadd}(G))$. For any $S \in \text{Ob}(\mathcal{G})$, category (G, S, E) is defined as follows:

$$\text{Ob}(G, S, E) = \{(Y, \phi, u) \mid Y \in \text{Ob}(\mathcal{G}), \phi \in \mathcal{G}(Y, S), u \in E(Y)\},$$

$$\text{Morph}_{(G, S, E)}((Y, \phi, u), (Z, \psi, v)) = \{\alpha \in \mathcal{G}(Y, Z) \mid \phi = \psi \circ \alpha, E(\alpha)(v) = u\}.$$

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & Z \\ & \searrow \phi & \swarrow \psi \\ & & S \end{array} \quad \begin{array}{ccc} E(\alpha) : E(Z) & \longrightarrow & E(Y) \\ \cup & & \cup \\ v & \longmapsto & u \end{array}$$

For any $(Y, \phi, u), (Z, \psi, v) \in \text{Ob}(G, S, E)$, we define their sum as follows:

Sum: $(Y, \phi, u) + (Z, \psi, v) := (Y \amalg Z, \phi \cup \psi : Y \amalg Z \rightarrow S, u \amalg v := j_{Y,Z}((u, v)))$. With this sum, we define a group $\mathcal{M}_E(S)$ as the Grothendieck group of the category (G, S, E) . For any object (Y, ϕ, u) , we write its image in $\mathcal{M}_E(S)$ as $[Y, \phi, u]$.

Remark 2.2. \mathcal{M}_E becomes a Mackey functor by the following definition:

Covariant part: For any $f \in \mathcal{G}(S, T)$, $\mathcal{M}_{E*}(f) : \mathcal{M}_E(S) \rightarrow \mathcal{M}_E(T)$, $[Y, \phi, u] \mapsto [Y, f \circ \phi, u]$.

Contravariant part: For any $f \in \mathcal{G}(S, T)$, $\mathcal{M}_E^*(f) : \mathcal{M}_E(T) \rightarrow \mathcal{M}_E(S)$, $[Z, \psi, v] \mapsto [S \times_T Z, \pi_S, E(\pi_Z)(v)]$.

$$\begin{array}{ccc} S \times_T Z & \xrightarrow{\pi_Z} & Z \\ \pi_S \downarrow & \square & \downarrow \psi \\ S & \xrightarrow{f} & T \end{array}$$

We abbreviate $\mathcal{M}_E(G/H)$ to $\mathcal{M}_E(H)$ for any $H \leq G$. The correspondence $E \mapsto \mathcal{M}_E$ is a functor from $\text{Sadd}(G)$ to $\text{Mack}(G)$. Indeed, for any morphism $\eta : E_1 \rightarrow E_2$ in $\text{Sadd}(G)$, we obtain a sum-preserving functor $(G, S, E_1) \rightarrow (G, S, E_2)$ for any $S \in \text{Ob}(\mathcal{G})$, and thus obtain a set of homomorphisms $\mathcal{M}_\eta(H) : \mathcal{M}_{E_1}(H) \rightarrow \mathcal{M}_{E_2}(H)$ ($H \leq G$), which form a morphism of Mackey functors $\mathcal{M}_\eta : \mathcal{M}_{E_1} \rightarrow \mathcal{M}_{E_2}$.

Let \mathcal{E} denote the forgetful functor from $\text{Mack}(G)$ to $\text{Sadd}(G)$; so if M is a Mackey functor for G and if X is a finite G -set, then $\mathcal{E}(M)(X)$ is the set $M(X)$, and if $f : X \rightarrow Y$ is a map in \mathcal{G} , then $\mathcal{E}(M)(f) : M(Y) \rightarrow M(X)$ is the map $M^*(f)$.

Proposition 2.3. *The functor $E \mapsto \mathcal{M}_E$ is left adjoint to \mathcal{E} .*

Proof. Let $E \in \text{Ob}(\text{Sadd}(G))$ and $M \in \text{Ob}(\text{Mack}(G))$. A morphism of Mackey functors $\Phi : \mathcal{M}_E \rightarrow M$ is a collection of group homomorphisms $\Phi_S : \mathcal{M}_E(S) \rightarrow M(S)$ for all finite G -sets S , which are compatible with the Mackey structure. This implies

$$\Phi_S([Y, \phi, u]) = M_*(\phi) \circ \Phi_Y([Y, \text{id}, u])$$

for any $Y \in \text{Ob}(G)$, $u \in E(Y)$ and $\phi \in \mathcal{G}(Y, S)$.

$$\begin{array}{ccc} \mathcal{M}_E(Y) & \xrightarrow{\Phi_Y} & M(Y) \\ \mathcal{M}_{E_*}(\phi) \downarrow & \circlearrowleft & \downarrow M_*(\phi) \\ \mathcal{M}_E(S) & \xrightarrow{\Phi_S} & M(S) \end{array}$$

It follows that if we define $\theta_Y : E(Y) \rightarrow M(Y)$ by $\theta_Y(u) = \Phi_Y([Y, \text{id}_Y, u])$, then Φ is determined by θ as

$$(2.1) \quad \Phi_S([Y, \phi, u]) = M_*(\phi)(\theta_Y(u)).$$

Conversely, for a given set of maps $\theta = (\theta_Y)_{Y \in \text{Ob}(G)}$, define $\Phi = (\Phi_S)_{S \in \text{Ob}(G)}$ by (2.1). Then, Φ is a morphism of Mackey functors if and only if θ is a morphism in $\text{Sadd}(G)$. To see this, since Φ defined by (2.1) is always natural with respect to the covariant part of the Mackey functors, it suffices to show that the following (A) and (B) are equivalent.

- (A) $\Phi_S \mathcal{M}_E^*(f)([Z, \psi, v]) = M^*(f) \Phi_T([Z, \psi, v])$
 $(\forall f \in \mathcal{G}(S, T), \forall [Z, \psi, v] \in \mathcal{M}_E(T))$
- (B) $\theta_S(E(f)(v)) = M^*(f)(\theta_T(v)) \quad (\forall f \in \mathcal{G}(S, T), \forall v \in E(T))$

Since

$$\begin{aligned} \Phi_S \mathcal{M}_E^*(f)([Z, \psi, v]) &= \Phi_S([S \times_T Z, \pi_S, E(\pi_Z)(v)]) \\ &= M_*(\pi_S)(\theta_{S \times_T Z}(E(\pi_Z)(v))) \end{aligned}$$

and

$$\begin{aligned} M^*(f) \Phi_T([Z, \psi, v]) &= M^*(f) M_*(\psi) \theta_Z(v) \\ &= M_*(\pi_S) M^*(\pi_Z)(\theta_Z(v)), \end{aligned}$$

$$\begin{array}{ccc} S \times_T Z & \xrightarrow{\pi_Z} & Z \\ \pi_S \downarrow & \square & \downarrow \psi \\ S & \xrightarrow{f} & T \end{array}$$

we have

$$(A) \Leftrightarrow M_*(\pi_S)(\theta_{S \times_T Z}(E(\pi_Z)(v))) = M_*(\pi_S)(M^*(\pi_Z)\theta_Z(v)).$$

Obviously this follows from (B), and conversely (B) follows from this equality if we put $Z = T$ and $\psi = \text{id}_T$. \square

Let $\mathcal{Madd}(G)$ denote the category of additive contravariant functors from \mathcal{G} to the category Mon of monoids.

Remark 2.4. Let $F \in \text{Ob}(\mathcal{S}\text{add}(G))$. The following are equivalent.

(1) $F \in \text{Ob}(\mathcal{M}\text{add}(G))$.

(2) F is equipped with cross product maps

$$F(X) \times F(Y) \ni (u, v) \mapsto u \times v \in F(X \times Y)$$

which are functorial in an obvious way in both X and Y , and associative. Moreover there exists a unit element $\varepsilon_F \in F(\bullet)$ (\bullet denotes the one-element set).

Proof. (1) \Rightarrow (2)

For any $X, Y \in \text{Ob}(\mathcal{G})$, by using the product in the monoid $F(X \times Y)$, we define

$$u \times v := F(p_X)(u) \cdot F(p_Y)(v),$$

where $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ are the projections.

(2) \Rightarrow (1)

For any $X \in \text{Ob}(\mathcal{G})$, we define the monoid structure on $F(X)$ by

$$u \cdot v := F(\Delta_X)(u \times v),$$

where $\Delta_X : X \rightarrow X \times X$ is the diagonal map.

Zero element is given by $F(X \rightarrow \bullet)(\varepsilon_F)$. \square

Now if F is in $\text{Ob}(\mathcal{M}\text{add}(G))$, then \mathcal{M}_F has an additional Green functor structure: In the category (G, S, F) , we can define the product of two objects (Y, ϕ, u) and (Z, ψ, v) by $(Y, \phi, u) \cdot (Z, \psi, v) := (Y \times_S Z, \phi \circ \pi_Y = \psi \circ \pi_Z, F(\pi_Y)(u) \cdot F(\pi_Z)(v))$, where π_Y and π_Z are the projections of the fiber product $Y \times_S Z$ of Y, Z over S , and $F(\pi_Y)(u) \cdot F(\pi_Z)(v)$ is the product of $F(\pi_Y)(u)$ and $F(\pi_Z)(v)$ in the monoid $F(Y \times_S Z)$.

$$\begin{array}{ccc} Y \times_S Z & \xrightarrow{\pi_Z} & Z \\ \pi_Y \downarrow & \square & \downarrow \psi \\ Y & \xrightarrow{\phi} & S \end{array}$$

Thus $\mathcal{M}_F(S)$ has a natural ring structure, defined by

$$[Y, \phi, u] \cdot [Z, \psi, v] := [(Y, \phi, u) \cdot (Z, \psi, v)].$$

Equivalently, in the view of Remark 2.4, we can describe the Green functor structure on \mathcal{M}_F by the maps

$$\begin{array}{ccc} \mathcal{M}_F(S) \times \mathcal{M}_F(T) & \longrightarrow & \mathcal{M}_F(S \times T) \\ \cup & & \cup \\ ([Y, \phi, u], [Z, \psi, v]) & \longmapsto & [Y \times Z, \phi \times \psi, u \times v] \end{array}$$

($\forall S, T \in \text{Ob}(\mathcal{G})$) (cf. section 2.2 in [4]), where $u \times v$ is the cross product of $u \in F(Y)$, $v \in F(Z)$. From now on, if F is an object of $\mathcal{M}\text{add}(G)$, the Green functor \mathcal{M}_F will be denoted by \mathcal{A}_F . \mathcal{A}_F is called the F -Burnside ring functor [16]. If F is commutative, i.e. $F(X)$ is a commutative monoid for each $X \in \text{Ob}(\mathcal{G})$, then \mathcal{A}_F becomes a commutative Green functor.

Let $\mathcal{F} : \text{Green}(G) \rightarrow \mathcal{M}\text{add}(G)$ be the forgetful functor, i.e. for any $A \in \text{Ob}(\text{Green}(G))$, $\mathcal{F}(A)(X) = A(X)$ ($\forall X \in \text{Ob}(\mathcal{G})$), $\mathcal{F}(A)(f) = A^*(f)$ ($\forall f : X \rightarrow Y$ in \mathcal{G}), and the cross product on $\mathcal{F}(A)$ is the cross product on A .

Proposition 2.5. (cf. Theorem 5.11 in [16]) The functor $F \mapsto \mathcal{A}_F$ from $\mathcal{M}\text{add}(G)$ to $\text{Green}(G)$ is left adjoint to \mathcal{F} .

Proof. Let $F \in \text{Ob}(\mathcal{M}\text{add}(G))$ and $A \in \text{Ob}(\text{Green}(G))$. By Proposition 2.3, there is a one-to-one correspondence between $\Phi \in \text{Mack}(G)(\mathcal{A}_F, A)$ and $\theta \in \text{Sadd}(G)(F, \mathcal{F}A)$. So it suffices to show that under this correspondence, Φ is a morphism of Green functors if and only if θ is a morphism in $\mathcal{M}\text{add}(G)$.

Since

$$\begin{aligned}\Phi_S([Y, \phi, u]) \times \Phi_T([Z, \psi, v]) &= A_*(\phi)(\theta_Y(u)) \times A_*(\psi)(\theta_Z(v)) \\ \Phi_{S \times T}([Y \times Z, \phi \times \psi, u \times v]) &= A_*(\phi \times \psi)(\theta_{Y \times Z}(u \times v)),\end{aligned}$$

for any $S, T \in \text{Ob}(\mathcal{G})$, $[Y, \phi, u] \in \mathcal{A}_F(S)$, $[Z, \psi, v] \in \mathcal{A}_F(T)$, Φ is a morphism of Green functors if and only if

$$A_*(\phi)(\theta_Y(u)) \times A_*(\psi)(\theta_Z(v)) = A_*(\phi \times \psi)(\theta_{Y \times Z}(u \times v))$$

for any $[Y, \phi, u] \in \mathcal{A}_F(S)$, $[Z, \psi, v] \in \mathcal{A}_F(T)$. This is equivalent to

$$\begin{aligned}\theta_{Y \times Z}(u \times v) &= \theta_Y(u) \times \theta_Z(v) \\ (\forall Y, Z \in \text{Ob}(\mathcal{G}), \forall u \in F(Y), \forall v \in F(Z)),\end{aligned}$$

which is equal to the fact that θ is a morphism in $\mathcal{M}\text{add}(G)$. \square

Let $\mathcal{G}\text{add}(G)$ be the category of additive contravariant functors from \mathcal{G} to the category Grp of groups. If F is an object of $\mathcal{M}\text{add}(G)$, then F belongs to $\text{Ob}(\mathcal{G}\text{add}(G))$ if and only if

$$F(X) \in \text{Ob}(\text{Grp}) \quad (\forall X \in \text{Ob}(\mathcal{G})).$$

For any $F \in \text{Ob}(\mathcal{M}\text{add}(G))$, if we define

$$\mathcal{U}F(X) := \{u \in F(X) \mid u \text{ is invertible}\}$$

for any X , then $\mathcal{U}F = (\mathcal{U}F(X))_{X \in \text{Ob}(\mathcal{G})}$ naturally forms an element $\mathcal{U}F$ in $\mathcal{G}\text{add}(G)$. Moreover for any $F_1 \in \text{Ob}(\mathcal{G}\text{add}(G))$ and $F_2 \in \text{Ob}(\mathcal{M}\text{add}(G))$, we have a natural isomorphism

$$\mathcal{G}\text{add}(G)(F_1, \mathcal{U}F_2) \cong \mathcal{M}\text{add}(G)(F_1, F_2).$$

Thus, if we abbreviate $R^\times := \mathcal{U} \circ \mathcal{F}(R)$ for any $R \in \text{Ob}(\text{Green}(G))$, we obtain the next corollary of Proposition 2.5.

Corollary 2.6. *For any $F \in \text{Ob}(\mathcal{G}\text{add}(G))$ and any $R \in \text{Ob}(\text{Green}(G))$, there is a natural isomorphism*

$$\mathcal{G}\text{add}(G)(F, R^\times) \cong \text{Green}(G)(\mathcal{A}_F, R)$$

Let $\text{Add}(G)$ denote the category of additive contravariant functor from \mathcal{G} to the category Ab of abelian groups. Morphisms are natural transformations.

For $F = \widetilde{\text{Br}}_{E/D}$ constructed below, its F -Burnside ring is called the Brauer ring.

Example 2.7. Let E/D be a finite Galois extension of fields with Galois group G . For any $S \in \text{Ob}(\mathcal{G})$, put $\widetilde{\text{Br}}_{E/D}(S) := \text{Br}(\mathcal{G}(S, E))$ where $\mathcal{G}(S, E)$ is regarded as a commutative ring by the pointwise operations, and $\text{Br}(\mathcal{G}(S, E))$ is its Brauer group. Recall that by taking the Brauer group of commutative rings, we obtain a covariant functor $\text{Br} : (\text{CommRng}) \rightarrow \text{Ab}$ from the category of commutative rings (CommRng) to Ab . For any $f \in \mathcal{G}(S, T)$, we have a ring homomorphism $f^* : \mathcal{G}(T, E) \rightarrow \mathcal{G}(S, E)$ defined by the pullback, and if we put $\widetilde{\text{Br}}_{E/D}(f : S \rightarrow T) := (\text{Br}(f^*) : \widetilde{\text{Br}}_{E/D}(T) \rightarrow \widetilde{\text{Br}}_{E/D}(S))$, we obtain an additive

functor $\widetilde{\text{Br}}_{E/D} \in \text{Madd}(G)$ (in fact, $\widetilde{\text{Br}}_{E/D} \in \text{Add}(G)$). As in [16], we abbreviate the $\widetilde{\text{Br}}_{E/D}$ -Burnside ring functor $\mathcal{A}_{\widetilde{\text{Br}}_{E/D}}$ to \mathcal{A}_{Br} , and we call this functor the Brauer ring functor. In particular, we write its value at G as $B(E, D) := \mathcal{A}_{\text{Br}}(G)$.

When the extension is trivial (i.e. G is trivial, $E = D$), we have the following structure theorem by Chen [7].

Proposition 2.8. (Corollary 3.4 in [7]) *There is a natural isomorphism*

$$\mathbb{Z}[\text{Br}(E)] \cong B(E, E)$$

($B(E, E)$ is denoted by $B(E)$ in [7]), compatible with the inclusions of $\text{Br}(E)$ into the multiplicative unit groups.

$$\begin{array}{ccc} & \text{Br}(E) & \\ & \swarrow \quad \searrow & \\ \mathbb{Z}[\text{Br}(E)] & \xrightarrow{\cong} & B(E, E) \end{array}$$

Remark 2.9. *Indeed, Chen defined the Brauer ring $B(R)$ for any commutative ring R , and showed $\mathbb{Z}[\text{Br}(R)] \cong B(R)$ for any connected ring R (the word ‘connected’ means that $\text{Spec}(R)$ is connected).*

Remark 2.10. *For any $H \leq G$, $\mathcal{G}(G/H, E)$ is naturally isomorphic to the fixed field E^H . With this identification, we can easily show that $(\ell_{g,H})^* : E^H \rightarrow E^{(gH)} = g \cdot (E^H)$ is equal to the multiplication by g (we write this as $(\ell_{g,H})^* = g : E^H \rightarrow g \cdot (E^H)$) for any $g \in G$. So, we have $\widetilde{\text{Br}}_{E/D}(\ell_{g,H}) = \text{Br}(g) : \text{Br}(E^H) \rightarrow \text{Br}(g \cdot (E^H))$.*

3. STRUCTURE OF THE F -BURNSIDE RING

We recall the definition of a restriction functor from [2].

Definition 3.1. A restriction functor is a triple $(\mathcal{R}, c, \text{res})$ where $\mathcal{R}, c, \text{res}$ are
 \mathcal{R} : a family of abelian groups $(\mathcal{R}(H))_{H \leq G}$,
 c : a family of conjugation homomorphisms $c_{g,H} : \mathcal{R}(H) \rightarrow \mathcal{R}(gH)$ ($g \in G$, $H \leq G$),
 res : a family of restriction homomorphisms $\text{res}_K^H : \mathcal{R}(H) \rightarrow \mathcal{R}(K)$ ($K \leq H \leq G$),
which satisfy the following conditions:

- (R1) $c_{h,H} = \text{res}_H^H = \text{id}_{\mathcal{R}(H)}$ ($\forall H \leq G, \forall h \in H$)
- (R2) $c_{g'g,H} = c_{g',gH} \circ c_{g,H}$ ($\forall g, g' \in G, \forall H \leq G$)
- (R3) $c_{g,K} \circ \text{res}_K^H = \text{res}_{gK}^{gH} \circ c_{g,H}$ ($\forall g \in G, \forall K \leq H \leq G$)

We sometimes abbreviate $(\mathcal{R}, c, \text{res})$ to \mathcal{R} . A morphism $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ of restriction functors is a family $(\Phi_H : \mathcal{R}(H) \rightarrow \mathcal{S}(H))_{H \leq G}$ of abelian group homomorphisms, compatible with conjugations and restrictions. We write the category of restriction functors $\text{Res}(G)$.

Definition 3.2. Let \mathcal{R} be a restriction functor. A stable basis of \mathcal{R} is a family of subsets $\mathcal{B} = (\mathcal{B}(H))_{H \leq G}$ such that $\mathcal{B}(H) \subset \mathcal{R}(H)$ is a basis for each $H \leq G$, and $c_{g,H}(\mathcal{B}(H)) = \mathcal{B}(gH)$ for any $g \in G$ and any $H \leq G$.

There is a correspondence between additive functors and restriction functors.

Proposition 3.3. *Let F be an object in $\text{Add}(G)$. If we put $\mathcal{R}_F(H) := F(G/H)$, $c_{g,H} = F(\ell_{g,H})$, $\text{res}_K^H := F(p_K^H)$ for each $g \in G$ and $K \leq H \leq G$, then $(\mathcal{R}_F, c, \text{res})$ is a restriction functor.*

Proof. (R1) is trivial. (R2) and (R3) follows from the compatibility of corresponding $\ell_{g,H}$'s and p_K^H 's. \square

For any $F_1, F_2 \in \text{Ob}(\text{Add}(G))$ and any $\varphi \in \text{Add}(G)(F_1, F_2)$, define $\mathcal{R}_\varphi \in \text{Res}(G)(\mathcal{R}_{F_1}, \mathcal{R}_{F_2})$ by $(\mathcal{R}_\varphi)_H = \varphi_{G/H}$. Thus we obtain a functor $\text{Add}(G) \rightarrow \text{Res}(G)$. We claim this functor gives an equivalence of the categories. A similar argument seems to be well-known in the case of Mackey functors, but we include this proof for the reader's convenience.

Proposition 3.4. *The above functor $F \mapsto \mathcal{R}_F$, $\varphi \mapsto \mathcal{R}_\varphi$ gives an equivalence of categories $\text{Add}(G) \xrightarrow{\sim} \text{Res}(G)$.*

Proof. We construct a quasi-inverse functor from $\text{Res}(G)$ to $\text{Add}(G)$ as follows. Suppose that \mathcal{R} is a restriction functor. If X is a finite G -set, then G acts on the abelian group $V = V_{\mathcal{R}}(X) := \bigoplus_{x \in X} \mathcal{R}(G_x)$, where G_x denotes the stabilizer group of x in X : If $x \in X$ and $u \in \mathcal{R}(G_x)$, denote by u_x the image of u in V , and set $g \cdot u_x = (c_{g, G_x}(u))_{gx}$. This makes sense since $G_{gx} = {}^g G_x$. Then define

$$F_{\mathcal{R}}(X) := (V_{\mathcal{R}}(X))_G,$$

as the group of co-invariants, i.e. the quotient of V by the subgroup generated by the elements $(c_{g, G_x}(u))_{gx} - u_x$, for $g \in G$ and $x \in X$. Denote by $[u_x]$ the image of u_x in this quotient.

If $f : X \rightarrow Y$ is a morphism in \mathcal{G} , then define $F_{\mathcal{R}}(f) : F_{\mathcal{R}}(Y) \rightarrow F_{\mathcal{R}}(X)$ by

$$F_{\mathcal{R}}(f)([u_y]) = \sum_{x \in [G_y \backslash f^{-1}(y)]} [(\text{res}_{G_x}^{G_y}(u))_x],$$

where $[G_y \backslash f^{-1}(y)]$ is a set of representatives of G_y -orbits of $f^{-1}(y)$. This makes sense since $G_x \leq G_y$ if $f(x) = y$. The right hand side does not depend on the choice of a set of representatives $[G_y \backslash f^{-1}(y)]$, since for any $x \in f^{-1}(y)$ and any $g \in G_y$, we have

$$\begin{aligned} [(\text{res}_{G_{gx}}^{G_y}(u))_{gx}] &= [(\text{res}_{G_x}^{G_y} \circ c_{g, G_y}(u))_{gx}] \\ &= [(c_{g, G_x} \circ \text{res}_{G_x}^{G_y}(u))_{gx}] = [(\text{res}_{G_x}^{G_y}(u))_x]. \end{aligned}$$

$F_{\mathcal{R}}(f)([u_y])$ is well-defined, i.e. $F_{\mathcal{R}}(f)([u_y])$ does not depend on the choice of a representative of $[u_y]$. Indeed, for any $g \in G$ we have

$$\begin{aligned} F_{\mathcal{R}}(f)([(c_{g, G_y}(u))_{gy}]) &= \sum_{x \in [G_y \backslash f^{-1}(y)]} [(\text{res}_{G_{gx}}^{G_{gy}} \circ c_{g, G_y}(u))_{gx}] \\ &= \sum_{x \in [G_y \backslash f^{-1}(y)]} [(c_{g, G_x} \circ \text{res}_{G_x}^{G_y}(u))_{gx}] \\ &= \sum_{x \in [G_y \backslash f^{-1}(y)]} [(\text{res}_{G_x}^{G_y}(u))_x] = F_{\mathcal{R}}(f)([u_y]). \end{aligned}$$

Here we used the fact that $\{gx \mid x \in [G_y \backslash f^{-1}(y)]\}$ is a set of representatives of $G_{gy} \backslash f^{-1}(gy)$ for any fixed $g \in G$.

$F_{\mathcal{R}}$ is a contravariant functor, since for any $X \xrightarrow{f} Y \xrightarrow{f'} Z$ in \mathcal{G} , we have

$$\begin{aligned} (F_{\mathcal{R}}(f) \circ F_{\mathcal{R}}(f'))([u_z]) &= \sum_{y \in [G_z \setminus f'^{-1}(z)]} \sum_{x \in [G_y \setminus f^{-1}(y)]} [(\text{res}_{G_x}^{G_z}(u))_x] \\ &= \sum_{x \in [G_z \setminus (f' \circ f)^{-1}(z)]} [(\text{res}_{G_x}^{G_z}(u))_x] \\ &= (F_{\mathcal{R}}(f' \circ f))([u_z]) \quad (\forall z \in Z, \forall u \in \mathcal{R}(G_z)). \end{aligned}$$

$F_{\mathcal{R}}$ is additive, since for any sum diagram $X \xrightarrow{i_X} X \amalg Y \xrightarrow{i_Y} Y$ in \mathcal{G} , we have

$$F_{\mathcal{R}}(X \amalg Y) = (V_{\mathcal{R}}(X \amalg Y))_G = (V_{\mathcal{R}}(X) \oplus V_{\mathcal{R}}(Y))_G = V_{\mathcal{R}}(X)_G \oplus V_{\mathcal{R}}(Y)_G$$

and

$$\begin{aligned} F_{\mathcal{R}}(i_X) : F_{\mathcal{R}}(X \amalg Y) &\longrightarrow F_{\mathcal{R}}(X) \\ [u_x] &\longmapsto [u_x] \quad (\forall x \in X, \forall u \in \mathcal{R}(G_x)) \\ [v_y] &\longmapsto 0 \quad (\forall y \in Y, \forall v \in \mathcal{R}(G_y)). \end{aligned}$$

This assignment $\mathcal{R} \mapsto F_{\mathcal{R}}$ gives in fact a functor $\text{Res}(G) \rightarrow \text{Add}(G)$. Indeed, for any morphism $\Phi = (\Phi_H : \mathcal{R}(H) \rightarrow \mathcal{S}(H))_{H \leq G} \in \text{Res}(G)(\mathcal{R}, \mathcal{S})$, we have a natural set of morphisms

$$V_{\Phi, X} : V_{\mathcal{R}}(X) \rightarrow V_{\mathcal{S}}(X)$$

defined simply by the direct sum, and since $V_{\Phi, X}$ is compatible with G -action on $V_{\mathcal{R}}(X)$ and $V_{\mathcal{S}}(X)$, we obtain a natural transformation

$$F_{\Phi} = (F_{\Phi, X} : F_{\mathcal{R}}(X) \rightarrow F_{\mathcal{S}}(X))_{X \in \text{Ob}(\mathcal{G})}$$

induced by $V_{\Phi, X}$.

This functor $\mathcal{R} \mapsto F_{\mathcal{R}}$ is a quasi-inverse of the functor $F \mapsto \mathcal{R}_F$. Indeed, since

$$\mathcal{R}_{F_{\mathcal{R}}}(H) = F_{\mathcal{R}}(G/H) = \left(\bigoplus_{x \in G/H} \mathcal{R}(G_x) \right)_G$$

for any $H \leq G$, the natural morphism

$$\mathcal{R}(H) \xrightarrow{\cong} \mathcal{R}(G_{1_G \cdot H}) \hookrightarrow \bigoplus_{x \in G/H} \mathcal{R}(G_x) \xrightarrow{\text{quotient}} \left(\bigoplus_{x \in G/H} \mathcal{R}(G_x) \right)_G$$

gives a natural isomorphism $\mathcal{R} \xrightarrow{\cong} \mathcal{R}_{F_{\mathcal{R}}}$. Here, the first isomorphism is the identification of $\mathcal{R}(H)$ with the component of $\bigoplus_{x \in G/H} \mathcal{R}(G_x)$ at $x = 1_G \cdot H \in G/H$. And conversely, since

$$F_{\mathcal{R}_F}(X) = \left(\bigoplus_{x \in X} \mathcal{R}_F(G_x) \right)_G = \left(\bigoplus_{x \in X} F(G/G_x) \right)_G,$$

any set of representatives $\{x_1, \dots, x_\ell\}$ of G -orbits of X defines a morphism

$$F(X) \xrightarrow{\cong} \bigoplus_{1 \leq i \leq \ell} F(G/G_{x_i}) \hookrightarrow \bigoplus_{x \in X} F(G/G_x) \rightarrow \left(\bigoplus_{x \in X} F(G/G_x) \right)_G,$$

which gives a natural isomorphism $F \xrightarrow{\cong} F_{\mathcal{R}_F}$. Note that this morphism does not depend on the choice of $\{x_1, \dots, x_\ell\}$, since

$$[u_{gx_i}] = [(c_{g, G_{x_i}}(u))_{gx_i}] = [u_{x_i}]$$

for any $g \in G_{x_i}$ and $u \in F(G/G_{x_i}) = F(G/G_{gx_i})$. \square

Finally let $\mathcal{R}\text{add}(\mathcal{G})$ be the category of additive contravariant functors from \mathcal{G} to the category of rings: The word additive for such a functor R means that for any object X and Y of \mathcal{G} , the map

$$(R(i_X), R(i_Y)) : R(X \amalg Y) \rightarrow R(X) \times R(Y)$$

is a ring isomorphism. Equivalently, R is an object in $\text{Add}(\mathcal{G})$, together with cross product maps

$$R(X) \times R(Y) \rightarrow R(X \times Y)$$

for any $X, Y \in \text{Ob}(\mathcal{G})$, which are natural in X and Y , bilinear, and associative. There is a unit element $\varepsilon \in R(\bullet)$.

Definition 3.5. A restriction functor $(\mathcal{R}, c, \text{res})$ is an algebra restriction functor if $\mathcal{R}(H)$ is a ring for each $H \leq G$, and conjugation and restriction homomorphisms are ring homomorphisms.

In the definition of a morphism $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ of restriction functors, if moreover \mathcal{R}, \mathcal{S} are algebra restriction functors and Φ_H are ring homomorphisms for all $H \leq G$, Φ is said to be a morphism of algebra restriction functors. Thus we have the category of algebra restriction functors $\text{Res}_{\text{alg}}(\mathcal{G})$. From each restriction functor $(\mathcal{R}, c, \text{res})$, we can construct an algebra restriction functor $(\mathbb{Z}[\mathcal{R}], c, \text{res})$ by putting $(\mathbb{Z}[\mathcal{R}])(H) := \mathbb{Z}[\mathcal{R}(H)]$ for each $H \leq G$. Conjugation and restriction homomorphisms of $\mathbb{Z}[\mathcal{R}]$ are canonically induced by those of \mathcal{R} . In the same way as $\text{Res}(\mathcal{G}) \xrightarrow{\sim} \text{Add}(\mathcal{G})$, $\mathcal{R}\text{add}(\mathcal{G})$ is shown to be equivalent to $\text{Res}_{\text{alg}}(\mathcal{G})$.

Here we recall the definition of the functor $-_+ : \text{Res}(\mathcal{G}) \rightarrow \text{Mack}(\mathcal{G})$. For a restriction functor $(\mathcal{R}, c, \text{res})$, put $S_{\mathcal{R}}(H) := \bigoplus_{K \leq H} \mathcal{R}(K)$. Then, H acts on $S_{\mathcal{R}}(H)$ by ${}^h x := c_{h,K}(x)$ ($\forall x \in \mathcal{R}(K) \subset S_{\mathcal{R}}(H), \forall h \in H$) and we put $\mathcal{R}_+(H) := S_{\mathcal{R}}(H)_H := S_{\mathcal{R}}(H) / (I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H))$ for any $H \leq G$, where $I(\mathbb{Z}[H]) \subset \mathbb{Z}[H]$ is the augmentation ideal defined by $I(\mathbb{Z}[H]) = \{\sum_{h \in H} m_h h \mid \sum_{h \in H} m_h = 0, m_h \in \mathbb{Z}\}$. We write $[K, x]_H := x + I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H)$ for any $x \in \mathcal{R}(K) \subset S_{\mathcal{R}}(H)$.

Remark 3.6. The submodule $I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H) \subset S_{\mathcal{R}}(H)$ is generated by $\{x - {}^h x \mid x \in \mathcal{R}(K), h \in H\}$.

Definition 3.7. For any restriction functor $(\mathcal{R}, c, \text{res})$, $\mathcal{R}_+ \in \text{Mack}(\mathcal{G})$ is defined as follows:

$$\mathcal{R}_+(H) = S_{\mathcal{R}}(H) / (I(\mathbb{Z}[H]) \cdot S_{\mathcal{R}}(H)) \text{ as above.}$$

$$c_{+g,H} : \mathcal{R}_+(H) \rightarrow \mathcal{R}_+({}^g H), [K, x]_H \mapsto [{}^g K, {}^g x]_{{}^g H}.$$

$$\text{res}_{+K}^H : \mathcal{R}_+(H) \rightarrow \mathcal{R}_+(K), [L, x]_H \mapsto \sum_{h \in K \setminus H/L} [K \cap {}^h L, \text{res}_{K \cap {}^h L}^{{}^h L}({}^h x)]_K.$$

$$\text{ind}_{+K}^H : \mathcal{R}_+(K) \rightarrow \mathcal{R}_+(H), [L, x]_K \mapsto [L, x]_H.$$

With an appropriate definition for morphisms (see [2]), we obtain a functor $-_+ : \text{Res}(\mathcal{G}) \rightarrow \text{Mack}(\mathcal{G})$, which restricts to a functor $-_+ : \text{Res}_{\text{alg}}(\mathcal{G}) \rightarrow \text{Green}(\mathcal{G})$, and makes the following diagram commutative :

$$\begin{array}{ccc} \text{Res}_{\text{alg}}(\mathcal{G}) & \xrightarrow{-_+} & \text{Green}(\mathcal{G}) \\ \text{forgetful} \downarrow & \circlearrowleft & \downarrow \text{forgetful} \\ \text{Res}(\mathcal{G}) & \xrightarrow{-_+} & \text{Mack}(\mathcal{G}) \end{array}$$

Here, for $\mathcal{R} \in \text{Ob}(\text{Res}_{\text{alg}}(G))$, the ring structure on $\mathcal{R}_+(H)$ is defined by

$$(3.1) \quad [K, x]_H \cdot [L, y]_H := \sum_{h \in K \backslash H/L} [K \cap {}^h L, \text{res}_{K \cap {}^h L}^K(x) \cdot \text{res}_{K \cap {}^h L}^{{}^h L}({}^h y)]_H$$

for each $H \leq G$.

Remark 3.8. Let \mathcal{S} be a restriction functor. Assume \mathcal{S} has a stable basis $\mathcal{B} = (\mathcal{B}(H))_{H \leq G}$. If we choose a set of representatives R_H for the H -orbits of the H -sets $\{(K, x) \mid K \leq H, x \in \mathcal{B}(K)\}$, then, for each $H \leq G$, $\mathcal{S}_+(H)$ is a free \mathbb{Z} -module with a basis $\{[K, x]_H \mid (K, x) \in R_H\}$.

Now, when $\mathcal{S} = \mathbb{Z}[\mathcal{R}]$, if we take $\mathcal{B}(H) := \mathcal{R}(H)$, then \mathcal{B} is a stable basis for \mathcal{S} . As a corollary we obtain a \mathbb{Z} -basis of $\mathbb{Z}[\mathcal{R}]_+(H)$ as follows:

Corollary 3.9. For each $H \leq G$, $\mathbb{Z}[\mathcal{R}]_+(H)$ is a free \mathbb{Z} -module over the basis $\{[K, x]_H \mid (K, x) \in R_H\}$, where R_H is a set of representatives for the H -orbits of $\{(K, x) \mid K \leq H, x \in \mathcal{R}(K)\}$.

For the functor $-_+$, the following adjoint property is known.

Remark 3.10. (Proposition I.4.1 in [2]) The functor

$$\begin{aligned} -_+ : \text{Res}(G) &\longrightarrow \text{Mack}(G) \\ (\text{resp. } : \text{Res}_{\text{alg}}(G) &\longrightarrow \text{Green}(G)) \end{aligned}$$

is left adjoint to the forgetful functor

$$\begin{aligned} \mathcal{O} : \text{Mack}(G) &\longrightarrow \text{Res}(G) \\ (\text{resp. } : \text{Green}(G) &\longrightarrow \text{Res}_{\text{alg}}(G)). \end{aligned}$$

There is a forgetful functor $gr : \text{Green}(G) \rightarrow \mathcal{R}\text{add}(G)$, obtained by forgetting the covariant part of the structure of Green functors. In the same way, we obtain a commutative diagram of categories and forgetful functors

$$(3.2) \quad \begin{array}{ccccc} \text{Green}(G) & \xrightarrow{gr} & \mathcal{R}\text{add}(G) & \xrightarrow{rm} & \mathcal{M}\text{add}(G) \\ \downarrow gm & \circ & \downarrow ra & \circ & \downarrow ms \\ \text{Mack}(G) & \xrightarrow{ma} & \text{Add}(G) & \xrightarrow{as} & \mathcal{S}\text{add}(G). \end{array}$$

Remark 3.11. Let \mathcal{R} be a restriction functor for G , and set $F := F_{\mathcal{R}}$. Then for any $X \in \text{Ob}(G)$, the module $\mathcal{R}_+(X)$ is isomorphic to the quotient of $\mathcal{A}_F(X)$ by the elements of the form

$$(Z, \phi, u + v) - (Z, \phi, u) - (Z, \phi, v)$$

where $\phi : Z \rightarrow X$ is a morphism in \mathcal{G} , and where $u, v \in \mathcal{R}(Z)$. Moreover, the family of projection maps

$$\pi_X : \mathcal{A}_F(X) \rightarrow \mathcal{R}_+(X)$$

is a morphism of Mackey functors $\mathcal{A}_F \rightarrow \mathcal{R}_+$.

Proof. By letting $\mathcal{R}_\#(X)$ be the quotient of $\mathcal{A}_F(X)$ as above, we obtain a quotient Mackey functor $\mathcal{R}_\#$ of \mathcal{A}_F . Remark that there is a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{E} & & \\
 & & \circ & & \\
 \text{Mack}(G) & \xrightarrow{ma} & \text{Add}(G) & \xrightarrow{as} & \text{Sadd}(G). \\
 & \searrow \mathcal{O} & \circ & \nearrow \mathcal{R} \mapsto F_{\mathcal{R}} & \\
 & & \text{Res}(G) & &
 \end{array}$$

By Proposition 2.3, there is a functorial isomorphism

$$\begin{array}{ccc}
 \text{Mack}(G)(\mathcal{A}_{F_{\mathcal{R}}}, M) & \xrightarrow{\cong} & \text{Sadd}(G)(F_{\mathcal{R}}, \mathcal{E}M) \\
 \downarrow \Psi & & \downarrow \Psi \\
 \Phi & \xrightarrow{\quad} & \theta
 \end{array}$$

in the notation in the proof of Proposition 2.3. Since

$$\Phi([Z, \phi, u]) = M_*(\phi)\theta_Z(u),$$

we have

Φ factors $\mathcal{R}_\#$

$$\begin{aligned}
 &\Leftrightarrow \Phi_S([Z, \phi, u+v] - [Z, \phi, u] - [Z, \phi, v]) = 0 \\
 &\quad (\forall \phi \in \mathcal{G}(Z, S), \forall u, v \in F_{\mathcal{R}}(Z)) \\
 &\Leftrightarrow M_*(\phi)\theta_Z(u+v) - M_*(\phi)\theta_Z(u) - M_*(\phi)\theta_Z(v) = 0 \\
 &\quad (\forall \phi \in \mathcal{G}(Z, S), \forall u, v \in F_{\mathcal{R}}(Z)) \\
 &\Leftrightarrow \theta_Z(u+v) - \theta_Z(u) - \theta_Z(v) = 0 \quad (\forall Z \in \text{Ob}(\mathcal{G}), \forall u, v \in F_{\mathcal{R}}(Z)) \\
 &\Leftrightarrow \theta \in \text{Add}(G)(F_{\mathcal{R}}, ma(M)).
 \end{aligned}$$

Thus, we obtain a functorial isomorphism

$$\begin{aligned}
 \text{Mack}(G)(\mathcal{R}_\#, M) &\xrightarrow{\cong} \text{Add}(G)(F_{\mathcal{R}}, ma(M)) \\
 &= \text{Add}(G)(F_{\mathcal{R}}, F_{\mathcal{O}(M)}) \\
 &\cong \text{Res}(G)(\mathcal{R}, \mathcal{O}(M)).
 \end{aligned}$$

So, the functor $-_\# : \text{Res}(G) \rightarrow \text{Mack}(G)$ is left adjoint to \mathcal{O} , and must agree with $-_+$. \square

In diagram (3.2), the composition $as \circ ma$ is the forgetful functor \mathcal{E} . So the left adjoint of \mathcal{E} is the composition of the left adjoint of as , followed by the left adjoint of ma . The left adjoint of as is the “free abelian group functor”, sending an object E of $\text{Sadd}(G)$ to the additive functor $\mathbb{Z}[E]$, defined in the obvious way by $(\mathbb{Z}[E])(X) = \mathbb{Z}[E(X)]$, for any G -set X . The left adjoint of ma is the composition

$$\text{Add}(G) \xrightarrow{\cong} \text{Res}(G) \xrightarrow{-_+} \text{Mack}(G).$$

By the uniqueness of the left adjoint of \mathcal{E} , it follows that for any additive contravariant functor $E \in \text{Ob}(\text{Sadd}(G))$, there is a natural isomorphism of Mackey functors

$$(\mathcal{R}_{\mathbb{Z}[E]})_+ \xrightarrow{\cong} \mathcal{M}_E.$$

Similarly, the composition $rm \circ gr$ is equal to the forgetful functor \mathcal{F} . A similar argument shows that for any $F \in \text{Ob}(\mathcal{M}\text{add}(G))$, there is a natural isomorphism of Green functors

$$(\mathcal{R}_{\mathbb{Z}[F]})_+ \xrightarrow{\cong} \mathcal{A}_F.$$

Thus we obtained the following adjoint isomorphisms.

Proposition 3.12. (1) *For any $E \in \text{Ob}(\mathcal{S}\text{add}(G))$, there is a natural isomorphism of Mackey functors*

$$(\mathcal{R}_{\mathbb{Z}[E]})_+ \xrightarrow{\cong} \mathcal{M}_E.$$

(2) *For any $F \in \text{Ob}(\mathcal{M}\text{add}(G))$, there is a natural isomorphism of Green functors*

$$(\mathcal{R}_{\mathbb{Z}[F]})_+ \xrightarrow{\cong} \mathcal{A}_F.$$

Since obviously $\mathcal{R}_{\mathbb{Z}[F]} \cong \mathbb{Z}[\mathcal{R}_F]$ for any $F \in \text{Ob}(\text{Add}(G))$, we have the following structure theorem for F -Burnside rings.

Theorem 3.13. *For any $F \in \text{Ob}(\text{Add}(G))$, there is a natural isomorphism of Green functors*

$$(\mathbb{Z}[\mathcal{R}_F])_+ \xrightarrow{\cong} \mathcal{A}_F.$$

4. APPLICATIONS

We state some results obtained from Theorem 3.13.

First, we see the structure of the Brauer ring. By Theorem 3.13, especially we have $\mathbb{Z}[\mathcal{R}_F]_+(G) \xrightarrow{\cong} \mathcal{A}_F(G)$. By putting $F = \widetilde{\text{Br}}_{E/D}$, we obtain the following:

Corollary 4.1. *For any finite Galois extension E/D of fields with Galois group G , we have a ring isomorphism*

$$B(E, D) \cong \left(\bigoplus_{H \leq G} \mathbb{Z}[\text{Br}(E^H)] \right) / \left(I(\mathbb{Z}[G]) \cdot \bigoplus_{H \leq G} \mathbb{Z}[\text{Br}(E^H)] \right),$$

where the ring structure of the right hand side is defined by (3.1) in Definition 3.7. When G is trivial, this is nothing other than Proposition 2.8.

As mentioned in [16], if F is the trivial functor, then \mathcal{A}_F is canonically isomorphic to the (ordinary) Burnside ring functor Ω . We can also induce this isomorphism from Theorem 3.13, since there is a canonical isomorphism $\Omega \cong \underline{\mathbb{Z}}_+$, where $\underline{\mathbb{Z}}$ is the constant algebra restriction functor with value \mathbb{Z} (see Example I.2.3 in [2]).

Theorem 3.13 gives us the structure of the F -Burnside ring functors, and allows us to deduce some properties of them. We also remark here that, conversely, this isomorphism gives an explicit categorical meaning (Definition 2.1) to the functor \mathcal{S}_+ , in the case where $\mathcal{S} = \mathbb{Z}[\mathcal{R}]$ for a certain \mathcal{R} .

For any algebra restriction functor \mathcal{A} (in fact, being an algebra conjugation functor is enough (cf. [2])), we have a Green functor \mathcal{A}^+ defined by $\mathcal{A}^+(H) = \left(\prod_{K \leq H} \mathcal{A}(K) \right)^H$. Here H acts on $\prod_{K \leq H} \mathcal{A}(K)$ by conjugation, similarly as in the definition of \mathcal{A}_+ . And $\mathcal{A}^+(H)$ has a canonical ring structure, induced by the componentwise multiplication of $\prod_{K \leq H} \mathcal{A}(K)$. This construction gives us a functor $-^+ : \text{Res}_{\text{alg}}(G) \rightarrow \text{Green}(G)$. There is a natural Green functor morphism $\rho^{\mathcal{A}} : \mathcal{A}_+ \rightarrow \mathcal{A}^+$, called the mark morphism. As in Proposition I.3.2 in [2], for any $H \leq G$, there exists a map $\sigma_H^{\mathcal{A}} : \mathcal{A}^+(H) \rightarrow \mathcal{A}_+(H)$ such that $\sigma_H^{\mathcal{A}} \circ \rho_H^{\mathcal{A}} = |H| \cdot \text{id}$, $\rho_H^{\mathcal{A}} \circ \sigma_H^{\mathcal{A}} = |H| \cdot \text{id}$. Since $\mathbb{Z}[\mathcal{R}_F]_+(H)$ is free for any $H \leq G$ (and so, it has no $|H|$ -torsion), we obtain the following proposition.

Proposition 4.2. *For each $H \leq G$, the component of the mark morphism at H*

$$\rho_H : \mathbb{Z}[\mathcal{R}_F]_+(H) \rightarrow \mathbb{Z}[\mathcal{R}_F]^+(H)$$

is injective.

As the (componentwise) scalar extension of $\mathbb{Z}[\mathcal{R}_F]$ by \mathbb{Q} , the functor $\mathbb{Q}[\mathcal{R}_F]$ has a simpler structure as in [16]. We can realize this with the mark morphism. Since $|H|$ is invertible in \mathbb{Q} for any $H \leq G$, we obtain the following isomorphism of Green functors.

Proposition 4.3. *The mark morphism $\rho : \mathbb{Q}[\mathcal{R}_F]_+ \rightarrow \mathbb{Q}[\mathcal{R}_F]^+$ is an isomorphism.*

From this, we obtain the following:

Corollary 4.4. *(Theorem 3.13 in [16]) For any additive functor F , there is an isomorphism*

$$\mathcal{A}_F(G) \otimes \mathbb{Q} \cong \prod_{a \in P(G)} \mathbb{Q}[F(G/G_a)]^{N_H(G_a)},$$

where $N_G(K)$ denotes the normalizer of K in G for each $K \leq G$.

Proof. In Proposition 4.3, the domain $\mathbb{Q}[\mathcal{R}_F]_+$ is naturally isomorphic to $\mathbb{Z}[\mathcal{R}_F]_+ \otimes \mathbb{Q}$ (see Lemma I.5.1 in [2]). As for the codomain $\mathbb{Q}[\mathcal{R}_F]^+$, we have an isomorphism

$$\begin{aligned} \mathbb{Q}[\mathcal{R}_F]^+(H) &= \left(\prod_{K \leq H} \mathbb{Q}[\mathcal{R}_F](K) \right)^H \\ &= \{(x_K)_{K \leq H} \in \prod_{K \leq H} \mathbb{Q}[\mathcal{R}_F](K) \mid h(x_K) = x_{hK} \quad (\forall K \leq H, \forall h \in H)\} \\ &\cong \{(x_a)_{a \in P(H)} \in \prod_{a \in P(H)} \mathbb{Q}[\mathcal{R}_F](H_a) \mid h(x_a) = x_a \quad (\forall h \in N_H(H_a))\} \\ &= \prod_{a \in P(H)} (\mathbb{Q}[\mathcal{R}_F](H_a))^{N_H(H_a)} \end{aligned}$$

for each $H \leq G$. Thus, we obtain

$$\begin{aligned} \mathcal{A}_F(G) \otimes \mathbb{Q} &\cong (\mathbb{Z}[\mathcal{R}_F]_+(G)) \otimes \mathbb{Q} \\ &\cong \prod_{a \in P(G)} (\mathbb{Q}[\mathcal{R}_F](G_a))^{N_H(G_a)} = \prod_{a \in P(G)} \mathbb{Q}[F(G/G_a)]^{N_H(G_a)}. \end{aligned}$$

□

Theorem 3.13 also enables us to calculate the Brauer ring for some (non-trivial) finite Galois extensions. Here we consider the case of \mathbb{C}/\mathbb{R} .

Corollary 4.5. *We have*

$$B(\mathbb{C}, \mathbb{R}) \cong \mathbb{Z}[X, Y]/(X^2 - 1, Y^2 - 2Y, XY - Y).$$

Proof. We abbreviate $\widetilde{\text{Br}}_{\mathbb{C}/\mathbb{R}}$ to $\widetilde{\text{Br}}$. Since $G = \text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\text{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, we can write them $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$ and $\text{Br}(\mathbb{R}) = \{1, h\}$. By Corollary 4.1, we have

$$B(\mathbb{C}, \mathbb{R}) \cong (\mathbb{Z}[\text{Br}(\mathbb{R})] \oplus \mathbb{Z}[\text{Br}(\mathbb{C})]) / I(\mathbb{Z}[G]) \cdot (\mathbb{Z}[\text{Br}(\mathbb{R})] \oplus \mathbb{Z}[\text{Br}(\mathbb{C})]).$$

We have

$$I(\mathbb{Z}[G]) = \{k \cdot 1 + \ell \cdot \sigma \mid k, \ell \in \mathbb{Z}, k + \ell = 0\} = \{k \cdot (1 - \sigma) \mid k \in \mathbb{Z}\}.$$

By the definition of the conjugation of $\mathcal{R}_{\widetilde{\text{Br}}}$, for any $H \leq G$ we have $c_{\sigma, H} = \widetilde{\text{Br}}(\ell_{\sigma, H}) = \text{Br}(\sigma) : \text{Br}(\mathbb{C}^H) \xrightarrow{\cong} \text{Br}(\sigma \cdot (\mathbb{C}^H))$. So both the maps

$$\widetilde{\text{Br}}(\ell_{\sigma, G}) : \text{Br}(\mathbb{R}) \rightarrow \text{Br}(\mathbb{R}) \quad (\cong \mathbb{Z}/2\mathbb{Z})$$

$$\widetilde{\text{Br}}(\ell_{\sigma, \{1\}}) : \text{Br}(\mathbb{C}) \rightarrow \text{Br}(\mathbb{C}) \quad (= 0)$$

are identities, and we obtain $I(\mathbb{Z}[G]) \cdot (\mathbb{Z}[\text{Br}(\mathbb{R})] \oplus \mathbb{Z}[\text{Br}(\mathbb{C})]) = 0$. Thus $B(\mathbb{C}, \mathbb{R})$ is equal to $\mathbb{Z}[\text{Br}(\mathbb{R})] \oplus \mathbb{Z}[\text{Br}(\mathbb{C})]$ as a module.

Finally we compute its ring structure. To distinguish, let e and f denote the unit element of $\text{Br}(\mathbb{R})$ and $\text{Br}(\mathbb{C})$ respectively. Then in the notation after Definition 3.2, we have $B(\mathbb{C}, \mathbb{R}) = \mathbb{Z} \cdot [G, e]_G \oplus \mathbb{Z} \cdot [G, h]_G \oplus \mathbb{Z} \cdot [\{1\}, f]_G$. And for this basis $\{[G, e], [G, h], [\{1\}, f]\}$ of $B(\mathbb{C}, \mathbb{R})$ (we omit the subscript G), their multiplications are calculated by the formula (3.1) in Definition 3.7 as follows:

$$[G, e]^2 = [G, e], [G, h]^2 = [G, e], [G, e] \cdot [G, h] = [G, h],$$

$$[G, h] \cdot [\{1\}, f] = [\{1\}, f], [G, e] \cdot [\{1\}, f] = [\{1\}, f], [\{1\}, f]^2 = 2[\{1\}, f].$$

So, if we put $X = [G, h]$ and $Y = [\{1\}, f]$, then $B(\mathbb{C}, \mathbb{R})$ becomes isomorphic to $\mathbb{Z}[X, Y]/(X^2 - 1, Y^2 - 2Y, XY - Y)$. \square

Part 2. Mackey-functor structure on the Brauer groups of a finite Galois covering of schemes

5. INTRODUCTION FOR PART 2

In this part, any scheme X is assumed to be Noetherian. $\pi(X)$ denotes its étale fundamental group. Any morphism is locally of finite type, unless otherwise specified. As in [21], X_{et} denotes the small étale site, consisting of étale morphisms of finite type over X . If $\mathcal{U} = (U_i \xrightarrow{f_i} X)_{i \in I}$ is a covering in this site, we write $\mathcal{U} \in \text{Cov}_{\text{et}}(X)$. Besides, $\mathcal{U} \prec \mathcal{V}$ means \mathcal{U} is a refinement of \mathcal{V} .

As for a finite étale covering, an étale fundamental group and a Galois category, we follow the terminology in [22]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to show the following:

Theorem (Theorem 10.6). Let S be a connected scheme. Let (FEt/S) denote the category of finite étale coverings over S . Then, the Brauer group functor Br forms a cohomological Mackey functor on (FEt/S) .

Once we construct the norm map of the Brauer groups compatibly with that of cohomology groups (Theorem 9.1)

$$\begin{array}{ccc} \text{Br}(Y) & \xrightarrow{\exists N_\pi} & \text{Br}(X) \\ \downarrow & \circlearrowleft & \downarrow \\ H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) & \xrightarrow{\text{norm}} & H_{\text{et}}^2(X, \mathbb{G}_{m,X}), \end{array}$$

we can obtain Theorem 10.6 as an easy corollary of the fact that $H_{\text{et}}^2(-, \mathbb{G}_m)$ is a cohomological Mackey functor. In this part, we construct the norm map by an elementary way using fpqc descent. Note that in general the Brauer group does not agree with the cohomological Brauer group, and one must show the existence of N_π anyway.

As a corollary, we obtain the following generalization of the result by Ford [12], which was shown for rings.

Corollary (Corollary 11.2). Let $\pi : Y \rightarrow X$ be a finite Galois covering of schemes with Galois group G . Then the correspondence

$$H \leq G \mapsto \text{Br}(Y/H)$$

forms a cohomological Mackey functor on G . Here, $H \leq G$ means H is a subgroup of G .

6. PRELIMINARIES

To fix the notation, we recall several facts in this section. If \mathcal{C} is a category and X is an object in \mathcal{C} , we abbreviately write $X \in \mathcal{C}$. If $f : X \rightarrow Y$ is a morphism in \mathcal{C} , we write $f \in \mathcal{C}(X, Y)$ or $f \in \text{Mor}_{\mathcal{C}}(X, Y)$.

Monoidal categories, monoidal functors and monoidal transformations are always assumed to be symmetric.

For a scheme X , $\text{q-Coh}(X)$ denotes the category of quasi-coherent modules over \mathcal{O}_X .

Fppqc descent

Definition 6.1. Let $X' \rightarrow X$ be an fpqc morphism of schemes. Put $X^{(2)} := X' \times_X X'$, $X^{(3)} := X' \times_X X' \times_X X'$ and let

$$\begin{aligned} p_i &: X^{(2)} \rightarrow X' & (i = 1, 2) \\ p_{ij} &: X^{(3)} \rightarrow X^{(2)} & (i, j \in \{1, 2, 3\}) \end{aligned}$$

be the projections. Define a category $\text{q-Coh}(X' \rightarrow X)$ as follows :

- an object in $\text{q-Coh}(X' \rightarrow X)$ is a pair (\mathcal{F}, φ) of a sheaf $\mathcal{F} \in \text{q-Coh}(X')$ and an isomorphism $\varphi : p_1^* \mathcal{F} \xrightarrow{\cong} p_2^* \mathcal{F}$ in $\text{q-Coh}(X^{(2)})$ satisfying the cocycle condition.
- a morphism from (\mathcal{F}, φ) to (\mathcal{G}, ψ) is a morphism $\alpha \in \text{q-Coh}(X')(\mathcal{F}, \mathcal{G})$, such that

$$p_2^* \alpha \circ \varphi = \psi \circ p_1^* \alpha.$$

For any (\mathcal{F}, φ) and $(\mathcal{G}, \psi) \in \text{q-Coh}(X' \rightarrow X)$, let $\varphi \otimes \psi$ be the abbreviation of

$$p_1^*(\mathcal{F} \otimes_{\mathcal{O}_{X'}} \mathcal{G}) \xrightarrow{\cong} p_1^* \mathcal{F} \otimes_{\mathcal{O}_{X^{(2)}}} p_1^* \mathcal{G} \xrightarrow{\varphi \otimes \psi} p_2^* \mathcal{F} \otimes_{\mathcal{O}_{X^{(2)}}} p_2^* \mathcal{G} \xrightarrow{\cong} p_2^*(\mathcal{F} \otimes_{\mathcal{O}_{X'}} \mathcal{G}).$$

Then, $\text{q-Coh}(X' \rightarrow X)$ has a canonical symmetric monoidal structure defined by

$$(\mathcal{F}, \varphi) \otimes (\mathcal{G}, \psi) := (\mathcal{F} \otimes_{\mathcal{O}_{X'}} \mathcal{G}, \varphi \otimes \psi).$$

Remark 6.2. Let $f : X' \rightarrow X$ be an fpqc morphism of schemes. The pull-back functor by f

$$f^* : \text{q-Coh}(X) \rightarrow \text{q-Coh}(X')$$

factors through $\text{q-Coh}(X' \rightarrow X)$:

$$\begin{array}{ccc} & \text{q-Coh}(X' \rightarrow X) & \\ \exists \underline{f^*} \nearrow & \circlearrowleft & \searrow U \\ \text{q-Coh}(X) & & \text{q-Coh}(X') \\ & \xleftarrow{f^*} & \end{array}$$

where U is the forgetful functor. By the fpqc descent, $\underline{f^*}$ is an equivalence.

In fact, U is a monoidal functor, and $\underline{f^*}$ is a monoidal equivalence.

Contravariant nature of the Brauer group

Remark 6.3. Let $\pi : Y \rightarrow X$ be a finite étale covering. For any abelian sheaf \mathcal{G} on $Y_{\text{ét}}$ and any positive integer q , the following composition of the canonical morphisms is an isomorphism :

$$c : H_{\text{ét}}^q(X, \pi_* \mathcal{G}) \rightarrow H_{\text{ét}}^q(Y, \pi^* \pi_* \mathcal{G}) \rightarrow H_{\text{ét}}^q(Y, \mathcal{G})$$

Remark 6.4. For any scheme X , there exists a natural monomorphism

$$\chi_X : \text{Br}(X) \hookrightarrow \text{Br}'(X) := H_{\text{ét}}^2(X, \mathbb{G}_{m,X})_{\text{tor}},$$

such that for any morphism $\pi : Y \rightarrow X$,

$$\begin{array}{ccc} \text{Br}(X) & \xrightarrow{\pi^*} & \text{Br}(Y) \\ \downarrow \chi_X & \circlearrowleft & \downarrow \chi_Y \\ H_{\text{ét}}^2(X, \mathbb{G}_{m,X}) & \xrightarrow{\pi^*} & H_{\text{ét}}^2(Y, \mathbb{G}_{m,Y}) \end{array}$$

is a commutative diagram.

Here $\pi^* : \text{Br}(X) \rightarrow \text{Br}(Y)$ is the pull-back of Azumaya algebras, while $\pi^* : H_{\text{et}}^2(X, \mathbb{G}_{m,X}) \rightarrow H_{\text{et}}^2(Y, \mathbb{G}_{m,Y})$ is defined as the composition of the canonical morphism $c : H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y}) \rightarrow H_{\text{et}}^2(Y, \mathbb{G}_{m,Y})$ and $H_{\text{et}}^2(\pi_{\sharp}) : H_{\text{et}}^2(X, \mathbb{G}_{m,X}) \rightarrow H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y})$, where $\pi_{\sharp} : \mathbb{G}_{m,X} \rightarrow \pi_* \mathbb{G}_{m,Y}$ is the canonical (structural) homomorphism of étale sheaves on X .

7. NORM FUNCTOR

In this section, we construct a monoidal functor

$$\mathcal{N}_{\pi} : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X)$$

which we call the *norm functor*, for any finite étale covering $\pi : Y \rightarrow X$.

Trivial case

Definition 7.1. Let X be a scheme, and let

$$\nabla = \nabla_{X,d} : \coprod_{1 \leq k \leq d} X_k \rightarrow X \quad (X_k = X \ (1 \leq \forall k \leq d))$$

be the folding map. We define the norm functor

$$\mathcal{N}_{\nabla} : \text{q-Coh}(\coprod_{1 \leq k \leq d} X_k) \rightarrow \text{q-Coh}(X)$$

by

$$\mathcal{N}_{\nabla}(\mathcal{G}) := \mathcal{G}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{G}|_{X_d}$$

for any $\mathcal{G} \in \text{q-Coh}(\coprod_{1 \leq k \leq d} X_k)$, and similarly for morphisms.

Remark 7.2. \mathcal{N}_{∇} is a monoidal functor.

Remark 7.3. For any automorphism $\tau : \coprod_{1 \leq k \leq d} X_k \xrightarrow{\cong} \coprod_{1 \leq k \leq d} X_k$ compatible with ∇ , there is a natural monoidal isomorphism

$$\mathcal{N}_{\nabla} \circ \tau^* \cong \mathcal{N}_{\nabla}.$$

Proof. Left to the reader. □

Definition 7.4. Let $\pi : Y \rightarrow X$ be a finite étale covering. Assume there exists an isomorphism

$$\eta : \coprod_{1 \leq k \leq d} X_k \xrightarrow{\cong} Y$$

compatible with π and $\nabla_{X,d}$. We define the norm functor \mathcal{N}_{π} by

$$\mathcal{N}_{\pi} := \mathcal{N}_{\nabla} \circ \eta^*.$$

Remark 7.5. By Remark 7.3, \mathcal{N}_{π} does not depend on the choice of trivialization η .

Remark 7.6. Let $\pi : Y \rightarrow X$ be a finite étale covering with a trivialization $\eta : \coprod_{1 \leq k \leq d} X_k \xrightarrow{\cong} Y$, as in Definition 7.4. Let $f : X' \rightarrow X$ be any morphism and take the pull-back :

$$(7.1) \quad \begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \pi' \downarrow & \square & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

Then by pulling η back by f , we obtain an isomorphism

$$\eta' : \coprod_{1 \leq k \leq d} X'_k \xrightarrow{\cong} Y', \quad (X'_k = X' \ (1 \leq k \leq d))$$

compatible with π' and $\nabla_{X',d}$:

$$\begin{array}{ccccc} & & Y' & \xrightarrow{g} & Y \\ & \nearrow \eta' & \downarrow & & \downarrow \eta \\ \coprod_d X' & \xrightarrow{\coprod_d f} & \coprod_d X & & \downarrow \pi \\ \nabla \searrow & & \downarrow \pi' & & \nabla \searrow \\ & & X' & \xrightarrow{f} & X \end{array}$$

(all faces are commutative)

Proposition 7.7. Let $\pi : Y \rightarrow X$ be a finite étale covering with a trivialization.

(i) For any morphism $f : X' \rightarrow X$, if we take the pull-back as in (7.1), then there exists a natural monoidal isomorphism

$$\theta_\pi^f : f^* \circ \mathcal{N}_\pi \xrightarrow{\cong} \mathcal{N}_{\pi'} \circ g^*.$$

Moreover, θ is natural in f :

(ii) For any other morphism $f' : X'' \rightarrow X'$, if we take the pull-back

$$\begin{array}{ccc} Y'' & \xrightarrow{g'} & Y' \\ \pi'' \downarrow & \square & \downarrow \pi' \\ X'' & \xrightarrow{f'} & X' \end{array}$$

then we have

$$(7.2) \quad \theta_\pi^{f \circ f'} = (\theta_{\pi'}^{f'} \circ g^*) \cdot (f'^* \circ \theta_\pi^f).$$

$$\begin{array}{ccc} (f \circ f')^* \mathcal{N}_\pi & \xrightarrow{\theta_\pi^{f \circ f'}} & \mathcal{N}_{\pi''} (g \circ g')^* \\ \cong \searrow & \circ & \cong \searrow \\ f'^* f^* \mathcal{N}_\pi & & \mathcal{N}_{\pi'} g'^* g^* \\ f'^* \circ \theta_\pi^f \searrow & & \theta_{\pi'}^{f'} \circ g^* \searrow \\ & f'^* \mathcal{N}_{\pi'} g^* & \end{array}$$

Proof. (i) This follows from Remark 7.6, since we have

$$\begin{aligned}
f^* \mathcal{N}_\pi(\mathcal{G}) &= f^*((\eta^* \mathcal{G})|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} (\eta^* \mathcal{G})|_{X_d}) \\
&\cong f^*((\eta^* \mathcal{G})|_{X_1}) \otimes_{\mathcal{O}_{X'}} \cdots \otimes_{\mathcal{O}_{X'}} f^*((\eta^* \mathcal{G})|_{X_d}) \\
&\cong ((\coprod_d f)^* \eta^* \mathcal{G})|_{X'_1} \otimes_{\mathcal{O}_{X'}} \cdots \otimes_{\mathcal{O}_{X'}} ((\coprod_d f)^* \eta^* \mathcal{G})|_{X'_d} \\
&\cong (\eta'^*(g^* \mathcal{G}))|_{X'_1} \otimes_{\mathcal{O}_{X'}} \cdots \otimes_{\mathcal{O}_{X'}} (\eta'^*(g^* \mathcal{G}))|_{X'_d} \\
&= \mathcal{N}_{\pi'}(g^* \mathcal{G})
\end{aligned}$$

for any $\mathcal{G} \in \text{q-Coh}(Y)$, in the notation of Remark 7.6.

(ii) This follows from the trivial case

$$\begin{array}{ccccc}
\coprod_{1 \leq k \leq d} X''_k & \xrightarrow{g'} & \coprod_{1 \leq k \leq d} X'_k & \xrightarrow{g} & \coprod_{1 \leq k \leq d} X_k \\
\downarrow \nabla_{X'',d} & \square & \downarrow \nabla_{X',d} & \square & \downarrow \nabla_{X,d} \\
X'' & \xrightarrow{f'} & X' & \xrightarrow{f} & X \\
& & (g = \coprod_d f, g' = \coprod_d f') & &
\end{array}$$

where, (7.2) is obviously satisfied. \square

Constant degree case

Remark 7.8. *Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d . There exists an fpqc morphism $f : X' \rightarrow X$ such that the pull-back of π by f becomes trivial :*

$$\begin{array}{ccccc}
& & Y' & \xrightarrow{g} & Y \\
& \cong & \downarrow \pi' & \square & \downarrow \pi \\
\coprod_d X' & \xrightarrow{\quad} & X' & \xrightarrow{f} & X \\
& \searrow \nabla_{X',d} & & &
\end{array}$$

f can be also taken as a surjective étale morphism.

Proposition 7.9. *In the notation of Remark 7.8,*

$$\mathcal{N}_{\pi'} \circ g^* : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X')$$

factors through $\text{q-Coh}(X' \xrightarrow{f} X) \xrightarrow{U} \text{q-Coh}(X)$ in Remark 6.2.

Proof. For the convenience, we abbreviate two functors

$$\begin{aligned}
g^* &: \text{q-Coh}(Y) \rightarrow \text{q-Coh}(Y') \\
\mathcal{N}_{\pi'} \circ g^* &: \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X')
\end{aligned}$$

respectively to

$$\begin{aligned}
\tilde{\quad} &: \text{q-Coh}(Y) \rightarrow \text{q-Coh}(Y') \\
\overline{\quad} &: \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X').
\end{aligned}$$

These are monoidal. Put

$$\begin{aligned}
X^{(2)} &:= X' \times_X X', \quad Y^{(2)} := Y' \times_Y Y', \\
X^{(3)} &:= X' \times_X X' \times_X X', \quad Y^{(3)} := Y' \times_Y Y' \times_Y Y',
\end{aligned}$$

and denote the projections by

$$\begin{aligned} p_i &: X^{(2)} \rightarrow X', \quad q_i : Y^{(2)} \rightarrow Y' \quad (i = 1, 2), \\ p_{ij} &: X^{(3)} \rightarrow X^{(2)}, \quad q_{ij} : Y^{(3)} \rightarrow Y^{(2)} \quad (1 \leq i < j \leq 3), \\ p'_i &: X^{(3)} \rightarrow X', \quad q'_i : Y^{(3)} \rightarrow Y' \quad (i = 1, 2, 3). \end{aligned}$$

Pulling π back by these projections, we obtain finite étale coverings $\pi^{(2)} : Y^{(2)} \rightarrow X^{(2)}$ and $\pi^{(3)} : Y^{(3)} \rightarrow X^{(3)}$.

$$\begin{array}{ccccccc} Y^{(3)} & \xrightarrow{q_{ij}} & Y^{(2)} & \xrightarrow{q_\ell} & Y' & \xrightarrow{g} & Y \\ \pi^{(3)} \downarrow & \square & \downarrow \pi^{(2)} & \square & \pi' \downarrow & \square & \downarrow \pi \\ X^{(3)} & \xrightarrow{p_{ij}} & X^{(2)} & \xrightarrow{p_\ell} & X' & \xrightarrow{f} & X \end{array} \quad (1 \leq i < j \leq 3, \ell = 1, 2)$$

Remark that each of $\pi^{(2)}$ and $\pi^{(3)}$ has a trivialization. It suffices to show the following:

Claim 7.10. *For each $\mathcal{F} \in \text{q-Coh}(Y)$, there is a canonical isomorphism*

$$\phi_{\mathcal{F}} : p_1^* \overline{\mathcal{F}} \xrightarrow{\cong} p_2^* \overline{\mathcal{F}}$$

satisfying the cocycle condition, such that for any morphism $\alpha \in \text{q-Coh}(Y)(\mathcal{F}, \mathcal{G})$

$$(7.3) \quad p_2^* \overline{\alpha} \circ \phi_{\mathcal{F}} = \phi_{\mathcal{G}} \circ p_1^* \overline{\alpha}$$

is satisfied.

Proof. (Proof of Claim 7.10) Since $\tilde{\mathcal{F}} = g^* \mathcal{F}$, there is a canonical isomorphism $\psi_{\mathcal{F}} : q_1^* \tilde{\mathcal{F}} \xrightarrow{\cong} q_2^* \tilde{\mathcal{F}}$ such that

$$(7.4) \quad q_{13}^* \psi_{\mathcal{F}} = q_{23}^* \psi_{\mathcal{F}} \circ q_{12}^* \psi_{\mathcal{F}}.$$

Define $\phi_{\mathcal{F}} : p_1^* \overline{\mathcal{F}} \xrightarrow{\cong} p_2^* \overline{\mathcal{F}}$ by

$$\begin{aligned} \phi_{\mathcal{F}} &:= (\theta_{\pi'}^{p_2})^{-1} \circ \mathcal{N}_{\pi^{(2)}}(\psi_{\mathcal{F}}) \circ \theta_{\pi'}^{p_1} \\ &= (p_1^* \mathcal{N}_{\pi'} \tilde{\mathcal{F}} \xrightarrow{\theta_{\pi'}^{p_1}} \mathcal{N}_{\pi^{(2)}} q_1^* \tilde{\mathcal{F}} \xrightarrow{\mathcal{N}_{\pi^{(2)}}(\psi_{\mathcal{F}})} \mathcal{N}_{\pi^{(2)}} q_2^* \tilde{\mathcal{F}} \xrightarrow{(\theta_{\pi'}^{p_2})^{-1}} p_2^* \mathcal{N}_{\pi'} \tilde{\mathcal{F}}) \end{aligned}$$

By (7.2) and the naturality of θ , we have a commutative diagram

$$\begin{array}{ccc} p_i^* \overline{\mathcal{F}} & \xrightarrow{p_{ij}^* \phi_{\mathcal{F}}} & p_j^* \overline{\mathcal{F}} \\ \theta_{\pi'}^{p_i} \downarrow & \circ & \downarrow \theta_{\pi'}^{p_j} \\ \mathcal{N}_{\pi^{(3)}} q_i^* \tilde{\mathcal{F}} & \xrightarrow{\mathcal{N}_{\pi^{(3)}}(q_{ij}^* \psi_{\mathcal{F}})} & \mathcal{N}_{\pi^{(3)}} q_j^* \tilde{\mathcal{F}} \end{array}$$

for each $1 \leq i < j \leq 3$. Thus $p_{13}^* \phi_{\mathcal{F}} = p_{23}^* \phi_{\mathcal{F}} \circ p_{12}^* \phi_{\mathcal{F}}$ follows from (7.4). (7.3) can be shown easily. \square

\square

Remark 7.11. For any $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$, we have a commutative diagram

$$\begin{array}{ccc}
 p_1^*(\overline{\mathcal{F} \otimes \mathcal{G}}) & \xrightarrow{\phi_{\mathcal{F} \otimes \mathcal{G}}} & p_2^*(\overline{\mathcal{F} \otimes \mathcal{G}}) \\
 \searrow \cong & & \searrow \cong \\
 p_1^*(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}) & \circ & p_2^*(\overline{\mathcal{F}} \otimes \overline{\mathcal{G}}) \\
 \searrow \cong & & \searrow \cong \\
 p_1^*\overline{\mathcal{F}} \otimes p_1^*\overline{\mathcal{G}} & \xrightarrow{\phi_{\mathcal{F}} \otimes \phi_{\mathcal{G}}} & p_2^*\overline{\mathcal{F}} \otimes p_2^*\overline{\mathcal{G}} .
 \end{array}$$

From this, we can see easily that the factorization of $\mathcal{N}_{\pi'} \circ g^*$

$$\underline{\mathcal{N}_{\pi'} \circ g^*} : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X' \rightarrow X)$$

becomes a monoidal functor.

Corollary 7.12. Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d and $f : X' \rightarrow X$ be an fpqc morphism trivializing π . Then we have a monoidal functor

$$\mathcal{N}_{\pi}^f : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X)$$

uniquely up to a natural monoidal isomorphism, such that (in the notation of Remark 7.8), there is a natural monoidal isomorphism

$$(7.5) \quad \underline{f^*} \circ \mathcal{N}_{\pi}^f \cong \underline{\mathcal{N}_{\pi'} \circ g^*},$$

and thus $f^* \circ \mathcal{N}_{\pi}^f \cong \mathcal{N}_{\pi'} \circ g^*$.

Proof. This follows from Remark 6.2 and Remark 7.11. \square

Proposition 7.13. Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d . If $f_1 : X_1 \rightarrow X$ and $f_2 : X_2 \rightarrow X$ are fpqc morphisms trivializing π , then there exists a natural monoidal isomorphism

$$\mathcal{N}_{\pi}^{f_1} \cong \mathcal{N}_{\pi}^{f_2}.$$

Proof. We may assume f_2 factors through f_1 :

$$\begin{array}{ccc}
 & \exists f_3 \rightarrow & X_1 \\
 X_2 & \nearrow & \searrow f_1 \\
 & \circ & X \\
 & \searrow f_2 & \nearrow
 \end{array}$$

Pulling π back by f_i for each $i = 1, 2$, we obtain diagrams

$$\begin{array}{ccc}
 \coprod_{1 \leq k \leq d} X_{i,k} & \xrightarrow{\eta_i} & Y_i \xrightarrow{g_i} Y \\
 \cong \searrow & \circ & \downarrow \pi_i \quad \square \quad \downarrow \pi \\
 \nabla_{X_i, d} \searrow & & X_i \xrightarrow{f_i} X
 \end{array}, \quad
 \begin{array}{ccc}
 Y_2 & \xrightarrow{g_3} & Y_1 \\
 \pi_2 \downarrow & \square & \downarrow \pi_1 \\
 X_2 & \xrightarrow{f_3} & X_1
 \end{array}$$

where $X_{i,k} = X_i$ ($1 \leq \forall k \leq d$). Let

$$\begin{array}{ccc}
 X_1^{(2)} & \xrightarrow{p_2} & X_1 \\
 p_1 \downarrow & \square & \downarrow f_1 \\
 X_1 & \xrightarrow{f_1} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 X_2^{(2)} & \xrightarrow{p'_2} & X_2 \\
 p'_1 \downarrow & \square & \downarrow f_2 \\
 X_2 & \xrightarrow{f_2} & X
 \end{array}$$

be pull-backs, and let $f_3^{(2)} : X_2^{(2)} \rightarrow X_1^{(2)}$ be the induced morphism:

$$\begin{array}{ccc} X_2^{(2)} & \xrightarrow{f_3^{(2)}} & X_1^{(2)} \\ p_i' \downarrow & \circlearrowleft & \downarrow p_i \\ X_2 & \xrightarrow{f_3} & X_1 \end{array} \quad (i = 1, 2)$$

Using (7.2), we can show that the natural monoidal isomorphism

$$\Xi_{\mathcal{F}} := (f_3^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} \xrightarrow{\theta_{\pi_1}^{f_3} \circ g_1^*} \mathcal{N}_{\pi_2} g_3^* g_1^* \mathcal{F} \xrightarrow{\cong} \mathcal{N}_{\pi_2} g_2^* \mathcal{F}) \quad (\mathcal{F} \in \text{q-Coh}(Y))$$

is compatible with descent data

$$\begin{aligned} \phi_{\mathcal{F}} : p_1^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} &\xrightarrow{\cong} p_2^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} \\ \phi'_{\mathcal{F}} : p_1'^* \mathcal{N}_{\pi_2} g_2^* \mathcal{F} &\xrightarrow{\cong} p_2'^* \mathcal{N}_{\pi_2} g_2^* \mathcal{F} \end{aligned}$$

defined in Claim 7.10:

$$\begin{array}{ccc} f_3^{(2)*} p_1^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} & \xleftarrow{\cong} p_1'^* f_3^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} \xrightarrow{p_1'^* \Xi_{\mathcal{F}}} & p_1'^* \mathcal{N}_{\pi_2} g_2^* \mathcal{F} \\ f_3^{(2)*} \phi_{\mathcal{F}} \downarrow & \circlearrowleft & \downarrow \phi'_{\mathcal{F}} \\ f_3^{(2)*} p_2^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} & \xleftarrow{\cong} p_2'^* f_3^* \mathcal{N}_{\pi_1} g_1^* \mathcal{F} \xrightarrow{p_2'^* \Xi_{\mathcal{F}}} & p_2'^* \mathcal{N}_{\pi_2} g_2^* \mathcal{F} \end{array} \quad (i = 1, 2)$$

Thus Proposition 7.13 follows from Remark 6.2. \square

Definition 7.14. Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d . By Proposition 7.13, \mathcal{N}_{π}^f is uniquely determined as a monoidal functor up to a natural monoidal isomorphism, independently of the fpqc morphism f trivializing π . We denote this functor simply by \mathcal{N}_{π} , and call it the *norm functor* for π .

Proposition 7.15. Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d . Let $f : X' \rightarrow X$ be a morphism, and take the pull-back as in diagram (7.1). Then there exists a natural monoidal isomorphism

$$\theta_{\pi}^f : f^* \mathcal{N}_{\pi} \xrightarrow{\cong} \mathcal{N}_{\pi'} g^*.$$

Proof. Let $u : U \rightarrow X$ be an fpqc morphism trivializing π , and take the pull-backs

$$\begin{array}{cccc} \begin{array}{ccc} V & \xrightarrow{v} & Y \\ \varpi \downarrow & \square & \downarrow \pi \\ U & \xrightarrow{u} & X \end{array}, & \begin{array}{ccc} U' & \xrightarrow{f_U} & U \\ u' \downarrow & \square & \downarrow u \\ X' & \xrightarrow{f} & X \end{array}, & \begin{array}{ccc} V' & \xrightarrow{g_V} & V \\ v' \downarrow & \square & \downarrow v \\ Y' & \xrightarrow{g} & Y \end{array}, & \begin{array}{ccc} V' & \xrightarrow{v'} & Y' \\ \varpi' \downarrow & \square & \downarrow \pi' \\ U' & \xrightarrow{u'} & X' \end{array}. \end{array}$$

Remark

$$\begin{array}{ccc} V' & \xrightarrow{g_V} & V \\ \varpi' \downarrow & \square & \downarrow \varpi \\ U' & \xrightarrow{f_U} & U \end{array}$$

is also a pull-back diagram.

By Proposition 7.7, there is a natural monoidal isomorphism

$$\theta_{\varpi}^{f_U} : f_U^* \mathcal{N}_{\varpi} \xrightarrow{\cong} \mathcal{N}_{\varpi'} g_V^*.$$

Let $X = \coprod_{1 \leq i \leq n} X_i$ be the decomposition into the connected open components. For any $\mathcal{F} \in \text{q-Coh}(X)$, we have a canonical decomposition

$$\mathcal{F} = \bigoplus_{1 \leq i \leq n} \mathcal{F}|_{X_i} = \mathcal{F}|_{X_1} \oplus \cdots \oplus \mathcal{F}|_{X_n}.$$

Regarding this decomposition, for any $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(X)$, we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}|_{X_1} \otimes_{\mathcal{O}_X} \mathcal{G}|_{X_1}) \oplus \cdots \oplus (\mathcal{F}|_{X_n} \otimes_{\mathcal{O}_X} \mathcal{G}|_{X_n}).$$

Definition 7.18. Let $\pi : Y \rightarrow X$ be a finite étale covering, and let $X = \coprod_{1 \leq i \leq n} X_i$ be the decomposition into the connected open components. Put $Y_i := \pi^{-1}(X_i)$, and let $\pi_i : Y_i \rightarrow X_i$ be the restriction of π onto Y_i .

We define the *norm functor*

$$\mathcal{N}_\pi : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X)$$

by

$$\mathcal{N}_\pi(\mathcal{G}) := \mathcal{N}_{\pi_1}(\mathcal{G}|_{Y_1}) \oplus \cdots \oplus \mathcal{N}_{\pi_n}(\mathcal{G}|_{Y_n})$$

for each $\mathcal{G} \in \text{q-Coh}(Y)$.

By the arguments so far, we obtain the following:

Proposition 7.19. Let $\pi : Y \rightarrow X$ be a finite étale covering.

(i) For any morphism $f : X' \rightarrow X$, if we take the pull-back

$$(7.6) \quad \begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \pi' \downarrow & \square & \downarrow \pi \\ X' & \xrightarrow{f} & X, \end{array}$$

then there exists a natural monoidal isomorphism $\theta_\pi^f : f^* \circ \mathcal{N}_\pi \xrightarrow{\cong} \mathcal{N}_{\pi'} \circ g^*$.

(ii) For any other morphism $f' : X'' \rightarrow X'$, if we take the pull-back

$$\begin{array}{ccc} Y'' & \xrightarrow{g'} & Y' \\ \pi'' \downarrow & \square & \downarrow \pi' \\ X'' & \xrightarrow{f'} & X', \end{array}$$

then we have $\theta_\pi^{f \circ f'} = (\theta_{\pi'}^{f'} \circ g^*) \cdot (f'^* \circ \theta_\pi^f)$.

Proof. This immediately follows from Proposition 7.15 and Corollary 7.16. \square

Remark 7.20. \mathcal{N}_π is uniquely determined up to a natural monoidal isomorphism, by Definition 7.1 and Proposition 7.19.

Proposition 7.21. Let $\pi : Y \rightarrow X$ be a finite étale covering of constant degree d . For any positive integer m , we have :

$$\mathcal{N}_\pi(\mathcal{O}_Y^{\oplus m}) \cong \mathcal{O}_X^{\oplus m^d}$$

Proof. Take an fpqc morphism $f : X' \rightarrow X$ trivializing π :

$$\begin{array}{ccc} \coprod_d X' = Y' & \xrightarrow{g} & Y \\ \nabla = \pi' \downarrow & \square & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

Then we have an isomorphism

$$\begin{aligned} \beta : \mathcal{N}_{\pi'} g^*(\mathcal{O}_Y^{\oplus m}) &\cong \mathcal{N}_{\pi'}(\mathcal{O}_{Y'}^{\oplus m}) = (\mathcal{O}_{Y'}^{\oplus m})|_{X'_1} \otimes_{X'_1} \cdots \otimes_{X'_1} (\mathcal{O}_{Y'}^{\oplus m})|_{X'_d} \\ &\cong \mathcal{O}_{X'}^{\oplus m} \otimes_{X'} \cdots \otimes_{X'} \mathcal{O}_{X'}^{\oplus m} \cong \mathcal{O}_{X'}^{\oplus m^d} \cong f^*(\mathcal{O}_{X'}^{\oplus m^d}). \end{aligned}$$

This β satisfies the commutativity of

$$\begin{array}{ccc} p_1^* \mathcal{N}_{\pi'} g^*(\mathcal{O}_Y^{\oplus m}) & \xrightarrow{p_1^* \beta} & p_1^* f^*(\mathcal{O}_{X'}^{\oplus m^d}) \\ \phi \downarrow \cong & \circlearrowleft & \cong \downarrow \text{can.} \\ p_2^* \mathcal{N}_{\pi'} g^*(\mathcal{O}_Y^{\oplus m}) & \xrightarrow{p_2^* \beta} & p_2^* f^*(\mathcal{O}_{X'}^{\oplus m^d}) \end{array}$$

where $\phi := \phi_{\mathcal{O}_Y^{\oplus m}}$ is the isomorphism defined in Claim 7.10. \square

Corollary 7.22. *Let $\pi : Y \rightarrow X$ be a finite étale covering. If $\mathcal{E} \in \text{q-Coh}(Y)$ is locally free of finite rank, then so is $\mathcal{N}_\pi(\mathcal{E}) \in \text{q-Coh}(X)$.*

Proof. By Proposition 7.19, we may assume X is affine and connected. Then Y is also affine, and π is of constant degree. Remark \mathcal{E} is locally free of finite rank if and only if there is an integer m and an epimorphism $s : \mathcal{O}_Y^{\oplus m} \rightarrow \mathcal{E}$. Take an fpqc morphism $f : X' \rightarrow X$ trivializing π :

$$\begin{array}{ccc} \coprod_d X' = Y' & \xrightarrow{g} & Y \\ \nabla = \pi' \downarrow & \square & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

By the definition of \mathcal{N}_∇ , it can be easily seen that $\mathcal{N}_\nabla g^*(s)$ becomes epimorphic. Thus $f^* \mathcal{N}_\pi(s)$ is epimorphic. Since f is fully faithful, $\mathcal{N}_\pi(s) : \mathcal{N}_\pi(\mathcal{O}_Y^{\oplus m}) \rightarrow \mathcal{N}_\pi(\mathcal{E})$ also becomes epimorphic. \square

8. NORM MAPS

Norm map of the Brauer group

Definition 8.1. Let $\pi : Y \rightarrow X$ be a finite étale covering. For any $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$, we define a morphism

$$\delta_\pi = \delta_{\pi, (\mathcal{F}, \mathcal{G})} : \mathcal{N}_\pi \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G})$$

as follows :

Let $e = \text{ev}_{\mathcal{F}, \mathcal{G}} : \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{F} \rightarrow \mathcal{G}$ be the evaluation morphism, i.e., the morphism corresponding to $\text{id}_{\text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})}$ under the adjoint isomorphism. Define

ξ_π as the composition

$$\xi_\pi := (\mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{N}_\pi \mathcal{F} \xrightarrow{\cong} \mathcal{N}_\pi(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{F}) \xrightarrow{\mathcal{N}_\pi(e)} \mathcal{N}_\pi(\mathcal{G})).$$

By the adjoint isomorphism

$$\begin{aligned} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G}) \\ & \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}), \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G})), \end{aligned}$$

we obtain δ_π corresponding to ξ_π .

Remark 8.2. To define δ_π , we only used the monoidality of \mathcal{N}_π . In fact, for any monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between closed symmetric monoidal categories, we can define a natural transformation

$$\delta_F : F[-, -]_{\mathcal{C}} \rightarrow [F(-), F(-)]_{\mathcal{D}},$$

where $[-, -]_{\mathcal{C}}$ and $[-, -]_{\mathcal{D}}$ are the right adjoint of $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$, respectively.

The following proposition also follows from general arguments on monoidal functors. We omit its proof.

Proposition 8.3. In Definition 8.1, if $\mathcal{F} = \mathcal{G}$, then

$$\delta_\pi : \mathcal{N}_\pi \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F}) \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F})$$

is a monoid morphism.

Remark 8.4. Let $\pi : Y \rightarrow X$ be a finite étale covering, and take the pull-back by a morphism $f : X' \rightarrow X$ as in diagram (7.6). Let $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$. From θ_π^f , we obtain an isomorphism

$$I_\theta : \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G}) \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G})$$

such that for any $\mathcal{E} \in \text{q-Coh}(X')$, the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{H}om(\mathcal{E}, \mathcal{H}om(\mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G})) & \xrightarrow{I_\theta \circ -} & \mathcal{H}om(\mathcal{E}, \mathcal{H}om(f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G})) \\ \text{adj.} \swarrow \cong & \circ & \searrow \cong \text{adj.} \\ \mathcal{H}om(\mathcal{E} \otimes \mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G}) & & \mathcal{H}om(\mathcal{E} \otimes f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G}) \\ & \searrow - \circ (\text{id} \otimes \theta_\pi^f) & \swarrow \theta_\pi^f \circ - \\ & \mathcal{H}om(\mathcal{E} \otimes f^* \mathcal{N}_\pi \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G}) & \end{array}$$

Proposition 8.5. In the notation of Remark 8.4, assume f is flat and \mathcal{F} is locally free of finite rank. Remark there exist canonical natural isomorphisms

$$\begin{aligned} c_1 : g^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) & \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_{Y'}}(g^* \mathcal{F}, g^* \mathcal{G}), \\ c_2 : f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G}) & \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G}). \end{aligned}$$

Then, the following diagram is commutative :

$$\begin{array}{ccc}
f^* \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) & \xrightarrow{f^* \delta_\pi} & f^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G}) \\
\theta_\pi^f \downarrow \cong & & \cong \downarrow c_2 \\
\mathcal{N}_{\pi'} g^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) & \circlearrowleft & \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G}) \\
\mathcal{N}_{\pi'}(c_1) \downarrow \cong & & \cong \uparrow I_\theta \\
\mathcal{N}_{\pi'} \mathcal{H}om_{\mathcal{O}_{Y'}}(g^* \mathcal{F}, g^* \mathcal{G}) & \xrightarrow{\delta_{\pi'}} & \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G})
\end{array}$$

Proof. Put

$$e := \text{ev}_{\mathcal{F}, \mathcal{G}} : \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{F} \rightarrow \mathcal{G},$$

$$e' := \text{ev}_{g^* \mathcal{F}, g^* \mathcal{G}} : \mathcal{H}om_{\mathcal{O}_{Y'}}(g^* \mathcal{F}, g^* \mathcal{G}) \otimes_{\mathcal{O}_{Y'}} g^* \mathcal{F} \rightarrow g^* \mathcal{G}.$$

Remark that

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_{Y'}}(g^* \mathcal{F}, g^* \mathcal{G}) \otimes_{\mathcal{O}_{Y'}} g^* \mathcal{F} & \xrightarrow{\cong} & g^*(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{F}) \\
& \searrow e' & \swarrow g^* e \\
& & g^* \mathcal{G}
\end{array}$$

is commutative. Put

$$u := I_\theta \circ \delta_{\pi'} \circ \mathcal{N}_{\pi'}(c_1) \circ \theta_\pi^f,$$

$$v := c_2 \circ f^* \delta_\pi,$$

and let μ and ν be their images under the adjoint isomorphism

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}), \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G})) \\
\cong \rightarrow \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} f^* \mathcal{N}_\pi \mathcal{F}, f^* \mathcal{N}_\pi \mathcal{G}),
\end{array}$$

respectively. It suffices to show $\mu = \nu$.

Put $u_0 := \delta_{\pi'} \circ \mathcal{N}_{\pi'}(c_1) \circ \theta_\pi^f$, and let μ_0 be its image under

$$\begin{array}{ccc}
\mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_{\pi'} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}), \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G})) \\
\cong \rightarrow \mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{N}_{\pi'} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} \mathcal{N}_{\pi'} g^* \mathcal{F}, \mathcal{N}_{\pi'} g^* \mathcal{G}).
\end{array}$$

Since $\mu = I_\theta \circ \mu_0$, by Remark 8.4 we have

$$(8.1) \quad \mu_0 \circ (\text{id} \otimes \theta_\pi^f) = \theta_\pi^f \circ \mu.$$

$$\begin{array}{ccc}
f^* \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} f^* \mathcal{N}_\pi \mathcal{F} & \xrightarrow{\mu} & f^* \mathcal{N}_\pi \mathcal{G} \\
\text{id} \otimes \theta_\pi^f \downarrow & \circlearrowleft & \downarrow \theta_\pi^f \\
f^* \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} \mathcal{N}_{\pi'} g^* \mathcal{F} & \xrightarrow{\mu_0} & \mathcal{N}_{\pi'} g^* \mathcal{G}
\end{array}$$

By the definition of $\xi_{\pi'}$ and the naturality of the adjoint isomorphism, we can show easily

$$(8.2) \quad \mu_0 = \xi_{\pi'} \circ (\mathcal{N}_{\pi'}(c_1) \otimes \text{id}) \circ (\theta_\pi^f \otimes \text{id}).$$

$$\begin{array}{ccc}
f^* \mathcal{N}_\pi \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} \mathcal{N}_{\pi'} g^* \mathcal{F} & \xrightarrow{\mu_0} & \mathcal{N}_{\pi'} g^* \mathcal{G} \\
\theta_\pi^f \otimes \text{id} \downarrow & \circlearrowleft & \uparrow \xi_{\pi'} \\
\mathcal{N}_{\pi'} g^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_{X'}} \mathcal{N}_{\pi'} g^* \mathcal{F} & \xrightarrow{\mathcal{N}_{\pi'}(c_1) \otimes \text{id}} & \mathcal{N}_{\pi'} \mathcal{H}om(g^* \mathcal{F}, g^* \mathcal{G}) \otimes_{\mathcal{O}_{X'}} \mathcal{N}_{\pi'} g^* \mathcal{F}
\end{array}$$

On the other hand, we have a commutative diagram

$$(8.3) \quad \begin{array}{ccc} f^* \mathcal{N}_\pi \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes f^* \mathcal{N}_\pi \mathcal{F} & \xrightarrow{\nu} & f^* \mathcal{N}_\pi \mathcal{G} \\ \cong \searrow & \circlearrowleft & \nearrow f^* \xi_\pi \\ & & f^*(\mathcal{N}_\pi \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{N}_\pi \mathcal{F}) . \end{array}$$

Moreover, since θ_π^f is a natural monoidal transformation, the following diagram is commutative:

$$(8.4) \quad \begin{array}{ccc} f^* \mathcal{N}_\pi \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes f^* \mathcal{N}_\pi \mathcal{F} & \xrightarrow{\theta_\pi^f \otimes \theta_\pi^f} & \mathcal{N}_{\pi'} g^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{N}_{\pi'} g^* \mathcal{F} \\ \cong \downarrow & \circlearrowleft & \downarrow \cong \\ f^* \mathcal{N}_\pi (\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{G}) & \xrightarrow{\theta_\pi^f} & \mathcal{N}_{\pi'} g^* (\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{G}) \\ f^* \mathcal{N}_\pi (e) \downarrow & \circlearrowleft \theta_\pi^f & \downarrow \mathcal{N}_{\pi'} g^* (e) \\ f^* \mathcal{N}_\pi \mathcal{G} & \xrightarrow{\theta_\pi^f} & \mathcal{N}_{\pi'} g^* \mathcal{G} \end{array}$$

From (8.1), (8.2), (8.3), (8.4), we obtain $\mu = \nu$. \square

Corollary 8.6. *Let $\pi : Y \rightarrow X$ be a finite étale covering, and let $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$. If \mathcal{F} is locally free of finite rank, then*

$$\delta_\pi : \mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G})$$

is an isomorphism.

Proof. Let $U \subset X$ be any open subscheme. Put $V := \pi^{-1}(U)$ and let $\varpi : V \rightarrow U$ be the restriction of π onto V .

By Proposition 8.5, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{N}_\pi \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}))|_U & \xrightarrow{(\delta_\pi)|_U} & (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{F}, \mathcal{N}_\pi \mathcal{G}))|_U \\ \cong \downarrow & \circlearrowleft & \downarrow \cong \\ \mathcal{N}_\varpi \mathcal{H}om_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V) & \xrightarrow{\delta_\varpi} & \mathcal{H}om_{\mathcal{O}_U}(\mathcal{N}_\varpi(\mathcal{F}|_V), \mathcal{N}_\varpi(\mathcal{G}|_V)) . \end{array}$$

Thus by taking an affine open cover of X , we may assume X is affine and connected. Moreover, again by Proposition 8.5, replacing X by its finite étale covering $X' \rightarrow X$, we may assume Y is trivial over X , i.e., $\pi = \nabla_{X,d} : Y = \coprod_{1 \leq k \leq d} X_k \rightarrow X$.

Since Y is affine, any $\mathcal{F} \in \text{q-Coh}(Y)$ can be identified with $\Gamma(Y, \mathcal{F})$, which is a $\Gamma(Y, \mathcal{O}_Y)$ -module. Similarly for the sheaves on X . For any $\mathcal{F}, \mathcal{G} \in \text{q-Coh}(Y)$, under this identification $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})$ is regarded as $\text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})$, and $e = \text{ev}_{\mathcal{F}, \mathcal{G}} : \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_Y} \mathcal{F} \rightarrow \mathcal{G}$ is given by

$$e(\varphi \otimes x) = \varphi(x) \quad (\forall \varphi \in \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}), \forall x \in \Gamma(Y, \mathcal{F})) .$$

Similarly, it can be easily seen that ξ_π

$$\begin{aligned}
& (\mathrm{Hom}(\mathcal{F}, \mathcal{G})|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Hom}(\mathcal{F}, \mathcal{G})|_{X_d}) \otimes_{\mathcal{O}_X} (\mathcal{F}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}|_{X_d}) \\
& \quad \cong \\
& (\mathrm{Hom}(\mathcal{F}|_{X_1}, \mathcal{G}|_{X_1}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Hom}(\mathcal{F}|_{X_d}, \mathcal{G}|_{X_d})) \otimes_{\mathcal{O}_X} (\mathcal{F}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}|_{X_d}) \\
& \quad \downarrow \\
& \mathcal{G}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{G}|_{X_d}
\end{aligned}$$

is given by

$$\xi_\pi((\varphi_1 \otimes \cdots \otimes \varphi_d) \otimes (x_1 \otimes \cdots \otimes x_d)) = \varphi_1(x_1) \otimes \cdots \otimes \varphi_d(x_d)$$

for any $\varphi_k \in \mathrm{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}|_{X_k}, \mathcal{G}|_{X_k})$ and $x_k \in \Gamma(X_k, \mathcal{F}|_{X_k})$ ($1 \leq \forall k \leq d$).

Correspondingly, δ_π is given by

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_{X_1}, \mathcal{G}|_{X_1}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_{X_d}, \mathcal{G}|_{X_d}) & \ni & \varphi_1 \otimes \cdots \otimes \varphi_d \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}|_{X_d}, \mathcal{G}|_{X_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{G}|_{X_d}) & \ni & \varphi_1 \otimes \cdots \otimes \varphi_d,
\end{array}$$

which is isomorphic. \square

Corollary 8.7. *In particular, for any locally free $\mathcal{E} \in \mathrm{q}\text{-Coh}(Y)$ of finite rank,*

$$\delta_\pi : \mathcal{N}_\pi(\mathrm{End}_{\mathcal{O}_Y}(\mathcal{E})) \rightarrow \mathrm{End}_{\mathcal{O}_X}(\mathcal{N}_\pi(\mathcal{E}))$$

is an isomorphism of \mathcal{O}_X -algebras.

Remark 8.8. *Let $\pi : Y \rightarrow X$ be a finite étale covering. For any surjective étale morphism $g : V \rightarrow Y$, there exists a surjective étale morphism $f : U \rightarrow X$ such that*

$$\mathrm{pr}_Y : U \times_X Y \rightarrow Y$$

factors through g .

Proposition 8.9. *Let $\pi : Y \rightarrow X$ be a finite étale covering. If $\mathcal{A} \in \mathrm{q}\text{-Coh}(Y)$ is an Azumaya algebra on Y , then $\mathcal{N}_\pi(\mathcal{A})$ becomes an Azumaya algebra on X .*

Proof. Since \mathcal{N}_π is monoidal, $\mathcal{N}_\pi(\mathcal{A})$ becomes an \mathcal{O}_X -algebra. By Proposition 7.22, $\mathcal{N}_\pi(\mathcal{A})$ is locally free of finite rank. Let $g : V \rightarrow Y$ be a surjective étale morphism such that there exists a locally free sheaf $\mathcal{E} \in \mathrm{q}\text{-Coh}(Y)$ of finite rank, with an isomorphism of \mathcal{O}_V -algebras

$$(8.5) \quad g^* \mathcal{A} \cong g^* \mathrm{End}_{\mathcal{O}_Y}(\mathcal{E}).$$

By Remark 8.8, replacing g if necessary, we may assume there exists a surjective étale morphism $f : U \rightarrow X$ such that g is the pull-back of f by π :

$$\begin{array}{ccc}
V & \xrightarrow{g} & Y \\
\varpi \downarrow & \square & \downarrow \pi \\
U & \xrightarrow{f} & X
\end{array}$$

By (8.5) and Corollary 8.7, we obtain an isomorphism of \mathcal{O}_U -algebras

$$\begin{aligned} f^* \mathcal{N}_\pi(\mathcal{A}) &\xrightarrow{\cong} \mathcal{N}_{\varpi} g^* \mathcal{A} \xrightarrow{\cong} \mathcal{N}_{\varpi} g^*(\text{End}_{\mathcal{O}_Y}(\mathcal{E})) \\ &\xrightarrow{\cong} f^* \mathcal{N}_\pi(\text{End}_{\mathcal{O}_Y}(\mathcal{E})) \xrightarrow{\cong} f^*(\text{End}_{\mathcal{O}_X}(\mathcal{N}_\pi(\mathcal{E}))), \end{aligned}$$

which shows $\mathcal{N}_\pi(\mathcal{A})$ is an Azumaya algebra on X . \square

Corollary 8.10. *Let $\pi : Y \rightarrow X$ be a finite étale covering. Norm functor $\mathcal{N}_\pi : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X)$ induces a group homomorphism*

$$N_\pi : \text{Br}(Y) \rightarrow \text{Br}(X),$$

which we call the norm map.

Proof. This follows from Corollary 8.7 and Proposition 8.9. \square

Norm map of the cohomology group

Remark 8.11. *Remark there is a natural isomorphism*

$$\gamma_X : \Gamma(X, \mathbb{G}_{m,X}) \xrightarrow{\cong} \text{Aut}_{\mathcal{O}_X}(\mathcal{O}_X)$$

for each scheme X . If $\pi : Y \rightarrow X$ is a finite étale covering, from the norm functor $\mathcal{N}_\pi : \text{q-Coh}(Y) \rightarrow \text{q-Coh}(X)$, we obtain a group homomorphism

$$\overline{N}_\pi : \text{Aut}_{\mathcal{O}_Y}(\mathcal{O}_Y) \xrightarrow{\mathcal{N}_\pi} \text{Aut}_{\mathcal{O}_X}(\mathcal{N}_\pi(\mathcal{O}_Y)) \xrightarrow{\cong} \text{Aut}_{\mathcal{O}_X}(\mathcal{O}_X).$$

Thus we can define a group homomorphism

$$N_\pi(X) := \gamma_X^{-1} \circ \overline{N}_\pi \circ \gamma_Y : \Gamma(Y, \mathbb{G}_{m,Y}) \rightarrow \Gamma(X, \mathbb{G}_{m,X}).$$

Proposition 8.12. *Let $\pi : Y \rightarrow X$ be a finite étale covering, and $f : U \rightarrow X$ be any étale morphism of finite type. Take the pull-back diagram*

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \\ \varpi \downarrow & \square & \downarrow \pi \\ U & \xrightarrow{f} & X. \end{array}$$

We define $N_\pi(U) : \Gamma_{\text{et}}(U, \pi_* \mathbb{G}_{m,Y}) \rightarrow \Gamma_{\text{et}}(U, \mathbb{G}_{m,X})$ by

$$N_\pi(U) := N_{\varpi}(U) : \Gamma(V, \mathcal{O}_V^\times) \rightarrow \Gamma(U, \mathcal{O}_U^\times).$$

Then the set of group homomorphisms $\{N_\pi(U) \mid (U \xrightarrow{f} X) \in X_{\text{et}}\}$ gives a homomorphism of abelian sheaves $N_\pi : \pi_* \mathbb{G}_{m,Y} \rightarrow \mathbb{G}_{m,X}$ on X_{et} .

Proof. Let $f' : U' \rightarrow X$ be another étale morphism of finite type, and $u : U' \rightarrow U$ be an étale morphism over X . It suffices to show the commutativity of

$$\begin{array}{ccc} \Gamma_{\text{et}}(V, \mathbb{G}_{m,Y}) & \xrightarrow{N_{\varpi}(U)} & \Gamma_{\text{et}}(U, \mathbb{G}_{m,X}) \\ v^* \downarrow & \circlearrowleft & \downarrow u^* \\ \Gamma_{\text{et}}(V', \mathbb{G}_{m,Y}) & \xrightarrow{N_{\varpi'}(U')} & \Gamma_{\text{et}}(U', \mathbb{G}_{m,X}), \end{array}$$

for the pull-back diagram

$$\begin{array}{ccccc}
 & & g' & & \\
 & & \circlearrowleft & & \\
 V' & \xrightarrow{v} & V & \xrightarrow{g} & Y \\
 \varpi' \downarrow & \square & \downarrow \varpi & \square & \downarrow \pi \\
 U' & \xrightarrow{u} & U & \xrightarrow{f} & X \\
 & & \circlearrowright & & \\
 & & f' & &
 \end{array}$$

This immediately follows from the fact that θ is a natural monoidal isomorphism. \square

Definition 8.13. Let $\pi : Y \rightarrow X$ be a finite étale covering. By Proposition 8.12, we obtain a homomorphism $H_{\text{et}}^2(N_\pi) : H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y}) \rightarrow H_{\text{et}}^2(X, \mathbb{G}_{m,X})$. We define the norm map of cohomology groups as the composition of this map with the canonical isomorphism $c^{-1} : H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) \xrightarrow{\cong} H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y})$, and abbreviate it by $N_\pi : H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) \rightarrow H_{\text{et}}^2(X, \mathbb{G}_{m,X})$.

9. COMPATIBILITY OF THE NORM MAPS

In this section, we show the following:

Theorem 9.1. *For any finite étale covering $\pi : Y \rightarrow X$, we have a commutative diagram*

$$\begin{array}{ccc}
 \text{Br}(Y) & \xrightarrow{N_\pi} & \text{Br}(X) \\
 \downarrow \chi_Y & \circlearrowleft & \downarrow \chi_X \\
 H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) & \xrightarrow{N_\pi} & H_{\text{et}}^2(X, \mathbb{G}_{m,X}).
 \end{array}$$

Remark 9.2. *By definition of N_π , this is nothing other than the commutativity of the following diagram:*

$$\begin{array}{ccc}
 \text{Br}(Y) & \xrightarrow{N_\pi} & \text{Br}(X) \\
 \chi_Y \swarrow & & \searrow \chi_X \\
 H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) & \circlearrowleft & H_{\text{et}}^2(X, \mathbb{G}_{m,X}) \\
 \cong \searrow & & \nearrow H_{\text{et}}^2(N_\pi) \\
 & H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y}) &
 \end{array}$$

Remark also that we may assume X is connected.

Remark 9.3. *For any finite étale covering $\pi : Y \rightarrow X$,*

$$\pi_* : \mathbf{S}(Y_{\text{et}}) \rightarrow \mathbf{S}(X_{\text{et}})$$

is exact. Here, $\mathbf{S}(X_{\text{et}})$ denotes the category of abelian sheaves on X_{et} . Thus we have natural homomorphisms

$$\pi_* : H_{\text{et}}^q(Y, \mathbb{G}_{m,Y}) \rightarrow H_{\text{et}}^q(X, \pi_* \mathbb{G}_{m,Y}) \quad (\forall q \geq 0).$$

It can be easily seen that this gives the inverse of

$$c : H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y}) \rightarrow H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}).$$

Proposition 9.4. *Let $\pi : Y \rightarrow X$ be a finite étale covering of a connected scheme X . For any $\mathbb{G}_{m,Y}$ -gerbe F on $Y_{\text{ét}}$, if we define a fibered category π_*F over $X_{\text{ét}}$ by*

$$(\pi_*F)(U) = F(U \times_X Y) \quad (\forall U \in X_{\text{ét}}),$$

*in a natural way, then π_*F becomes a $\pi_*\mathbb{G}_{m,Y}$ -gerbe on $X_{\text{ét}}$. This defines a group homomorphism $\pi_* : H_g^2(Y, \mathbb{G}_{m,Y}) \rightarrow H_g^2(X, \pi_*\mathbb{G}_{m,Y})$, where H_g^2 denotes the non-abelian cohomology of Giraud.*

Proof. Since F is a stack fibered in groupoid, it can be easily seen that so is π_*F . Thus, to show π_*F is a gerbe, it suffices to show the following:

(a) π_*F is locally connected

(b) π_*F is locally non-empty

(a) For any $U \in X_{\text{ét}}$ and any $a_1, a_2 \in \pi_*F(U) = F(V)$ ($V := U \times_X Y$), there exists a surjective étale morphism $V' \xrightarrow{v} V$ of finite type such that $v^*a_1 \cong v^*a_2$ in $F(V')$. By Remark 8.8, there exists a surjective étale morphism $U' \xrightarrow{u} U$ of finite type such that $U' \times_X Y = U' \times_U V \xrightarrow{\text{pr}_V} V$ factors through v :

$$(9.1) \quad \begin{array}{ccc} & & V' \\ & \nearrow \exists w & \searrow v \\ U' \times_U V & & V \\ & \searrow \text{pr}_V & \nearrow \end{array}$$

Thus we have $w^*v^*a_1 \cong w^*v^*a_2$ in $F(U' \times_U V)$, namely, $u^*a_1 \cong u^*a_2$ in $\pi_*F(U')$.

(b) For any $U \in X_{\text{ét}}$, let $V' \xrightarrow{v} V = U \times_X Y$ be a surjective étale morphism of finite type, such that $F(V') \neq \emptyset$. Take $U' \xrightarrow{u} U$ satisfying (9.1) as above. If we put $W_1 := w(U' \times_U V)$ and $W_2 := V' \setminus W_1$, then each W_i is an open subscheme of V' ($i = 1, 2$), satisfying

$$V' = W_1 \amalg W_2.$$

Thus we have $F(V') \simeq F(W_1) \times F(W_2)$. In particular, $F(W_1) \neq \emptyset$. Since $w : U' \times_U V \rightarrow W_1$ is surjective étale, $\pi_*F(U') = F(U' \times_U V) \neq \emptyset$ follows from $F(W_1) \neq \emptyset$.

Thus π_*F is a gerbe, which is obviously bound by $\pi_*\mathbb{G}_{m,Y}$. \square

Remark 9.5. ([21]) *Let X be a scheme. For any Azumaya algebra A on X , let F_A denote a fibered category over $X_{\text{ét}}$, whose fiber $F_A(U)$ over $U \in X_{\text{ét}}$ is defined as follows :*

- An object is a pair (\mathcal{E}, α) , where $\mathcal{E} \in \text{q-Coh}(U)$ is locally free of finite rank, $\alpha : \text{End}_{\mathcal{O}_U}(\mathcal{E}) \xrightarrow{\cong} A|_U$ is an isomorphism of \mathcal{O}_U -algebras.

- A morphism $(\mathcal{E}, \alpha) \rightarrow (\mathcal{E}', \alpha')$ is an isomorphism $\mathcal{E} \xrightarrow{\cong} \mathcal{E}'$ compatible with α and α' .

Then F_A becomes a gerbe, bound by $\mathbb{G}_{m,X}$. (Indeed, multiplication by elements of $\Gamma(U, \mathcal{O}_U^\times)$ gives an isomorphism $\Gamma(U, \mathcal{O}_U^\times) \xrightarrow{\cong} \text{Aut}_{F_A}(\mathcal{E}, \alpha)$.) This gives the natural monomorphism $\chi_X : \text{Br}(X) \hookrightarrow H_g^2(X, \mathbb{G}_{m,X})$.

Lemma 9.6. *Let $\pi : Y \rightarrow X$ be a finite étale covering, and let A be an Azumaya algebra on Y .*

(i) For any $U \in X_{\text{et}}$, let

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \\ \varpi \downarrow & \square & \downarrow \pi \\ U & \xrightarrow{f} & X \end{array}$$

be a pull-back diagram. We define a functor

$$\mathcal{N}_{\varpi/\pi} : F_{\mathcal{A}}(V) \rightarrow F_{\mathcal{N}_{\pi}\mathcal{A}}(U)$$

by $\mathcal{N}_{\varpi/\pi}(\mathcal{E}, \alpha) = (\mathcal{N}_{\varpi}(\mathcal{E}), \beta)$, where β is the composition

$$\text{End}_{\mathcal{O}_U}(\mathcal{N}_{\varpi}(\mathcal{E})) \xrightarrow{\cong} \mathcal{N}_{\varpi}(\text{End}_{\mathcal{O}_V}(\mathcal{E})) \xrightarrow{\mathcal{N}_{\varpi}(\alpha)} \mathcal{N}_{\varpi}(g^*\mathcal{A}) \xrightarrow{\cong} f^*\mathcal{N}_{\pi}\mathcal{A}.$$

For any morphism $u : U' \rightarrow U$ in X_{et} , if we take the pull-back

$$\begin{array}{ccc} V' & \xrightarrow{v} & V \\ \varpi' \downarrow & \square & \downarrow \varpi \\ U' & \xrightarrow{u} & U, \end{array}$$

then we have a natural isomorphism $u^*\mathcal{N}_{\varpi/\pi} \cong \mathcal{N}_{\varpi'/\pi}v^* : F_{\mathcal{A}}(V) \rightarrow F_{\mathcal{N}_{\pi}\mathcal{A}}(U')$.

(ii) $\mathcal{N}_{\varpi/\pi}$ makes the following diagram commutative :

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_V^{\times}) & \xrightarrow{N_{\varpi}(U)} & \Gamma(U, \mathcal{O}_U^{\times}) \\ \downarrow \cong & \circ & \downarrow \cong \\ \text{Aut}_{F_{\mathcal{A}}(V)}(\mathcal{E}, \alpha) & \xrightarrow{N_{\varpi/\pi}} & \text{Aut}_{F_{\mathcal{N}_{\pi}\mathcal{A}}(U)}(\mathcal{N}_{\varpi/\pi}(\mathcal{E}, \alpha)) \end{array}$$

Proof. (i) This is induced from the natural monoidal isomorphism

$$\theta_{\varpi}^u : u^*\mathcal{N}_{\varpi} \xrightarrow{\cong} \mathcal{N}_{\varpi'}v^* : \text{q-Coh}(V) \rightarrow \text{q-Coh}(U').$$

(ii) This follows from the commutativity of

$$\begin{array}{ccc} \Gamma(V, \mathcal{O}_V^{\times}) & \xrightarrow{N_{\varpi}(U)} & \Gamma(U, \mathcal{O}_U^{\times}) & \text{Aut}_{F_{\mathcal{A}}(V)}(\mathcal{E}, \alpha) & \xrightarrow{N_{\varpi/\pi}} & \text{Aut}_{F_{\mathcal{N}_{\pi}\mathcal{A}}(U)}(\mathcal{N}_{\varpi/\pi}(\mathcal{E}, \alpha)) \\ \downarrow & \circ & \downarrow & \downarrow & \circ & \downarrow \\ \text{Aut}_{\mathcal{O}_V}(\mathcal{E}) & \xrightarrow{N_{\varpi}} & \text{Aut}_{\mathcal{O}_U}(\mathcal{N}_{\varpi}(\mathcal{E})) & \text{Aut}_{\mathcal{O}_V}(\mathcal{E}) & \xrightarrow{N_{\varpi}} & \text{Aut}_{\mathcal{O}_U}(\mathcal{N}_{\varpi}(\mathcal{E})) \end{array}$$

□

Remark 9.7. ([14] Proposition 3.1.5) Let X be a scheme. For any morphism $u : \mathcal{F} \rightarrow \mathcal{G}$ in $\mathbf{S}(X_{\text{et}})$, we have a group homomorphism $H_g^2(u) : H_g^2(X, \mathcal{F}) \rightarrow H_g^2(X, \mathcal{G})$, compatible with $H_{\text{et}}^2(u)$. If F is an \mathcal{F} -gerbe and G is a \mathcal{G} -gerbe, then $H_g^2(u)(F) = G$ in $H_g^2(X, \mathcal{G})$ if and only if there exists a morphism of gerbes $F \rightarrow G$ bound by u .

By the above arguments, Theorem 9.1 is reduced to the following:

Proposition 9.8. *Let $\pi : Y \rightarrow X$ be a finite étale covering of a connected scheme X . The following diagram is commutative :*

$$\begin{array}{ccc}
 \text{Br}(Y) & \xrightarrow{N_\pi} & \text{Br}(X) \\
 \chi_Y \swarrow & & \searrow \chi_X \\
 H_g^2(Y, \mathbb{G}_{m,Y}) & \circlearrowleft & H_g^2(X, \mathbb{G}_{m,X}) \\
 \pi_* \searrow & & \nearrow H_g^2(N_\pi) \\
 & H_g^2(X, \pi_* \mathbb{G}_{m,Y}) &
 \end{array}$$

Proof. By (i) in Lemma 9.6, if we associate a functor $\mathcal{N}_\pi(U) := \mathcal{N}_\pi : \pi_* F_{\mathcal{A}}(U) \rightarrow F_{\mathcal{N}_\pi \mathcal{A}}(U)$ to each $U \in X_{\text{ét}}$, then these functors form a morphism of fibered categories $\mathcal{N}_\pi : \pi_* F_{\mathcal{A}} \rightarrow F_{\mathcal{N}_\pi \mathcal{A}}$. By (ii) in Lemma 9.6, this is bound by $N_\pi : \pi_* \mathbb{G}_{m,Y} \rightarrow \mathbb{G}_{m,X}$. Thus we have $H_g^2(N_\pi)(\pi_* F_{\mathcal{A}}) = F_{\mathcal{N}_\pi \mathcal{A}}$. \square

10. BRAUER-MACKEY FUNCTOR ON THE GALOIS CATEGORY

Let Ab be the category of abelian groups. For any profinite group G , let $G\text{-Sp}$ denote the category of finite discrete G -spaces and continuous equivariant G -maps.

Definition 10.1. Let \mathcal{C} be a Galois category, with fundamental functor F . In other words, there exists a profinite group $\pi(\mathcal{C})$ such that F gives an equivalence from \mathcal{C} to $\pi(\mathcal{C})\text{-Sp}$. (For the precise definition of Galois category, see [22]).

A cohomological Mackey functor on \mathcal{C} is a pair of functors $M = (M^*, M_*)$ from \mathcal{C} to Ab , where M^* is contravariant and M_* is covariant, satisfying the following conditions :

- (0) $M^*(X) = M_*(X) (= M(X)) \quad (\forall X \in \text{Ob}(\mathcal{C}))$.
(1) (Additivity) For each coproduct $X \xrightarrow{i_X} X \amalg Y \xleftarrow{i_Y} Y$ in \mathcal{C} , canonical morphism

$$(M^*(i_X), M^*(i_Y)) : M(X \amalg Y) \rightarrow M(X) \oplus M(Y)$$

is an isomorphism.

- (2) (Mackey condition) For any pull-back diagram

$$\begin{array}{ccc}
 Y' & \xrightarrow{\varpi'} & Y \\
 \pi \downarrow & \square & \downarrow \pi' \\
 X' & \xrightarrow{\varpi} & X
 \end{array}
 ,$$

the following diagram is commutative :

$$\begin{array}{ccc}
 M(Y) & \xrightarrow{M^*(\varpi')} & M(Y') \\
 M_*(\pi) \downarrow & \circlearrowleft & \downarrow M_*(\pi') \\
 M(X) & \xrightarrow{M^*(\varpi)} & M(X')
 \end{array}$$

- (3) (Cohomological condition) For any morphism $\pi : X \rightarrow Y$ in \mathcal{C} with X and Y connected (i.e. not decomposable into non-trivial coproducts), we have

$$M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg \pi$$

where $\deg \pi := \#F(Y)/\#F(X)$.

$$\begin{array}{ccc}
 & M(Y) & \\
 M^*(\pi) \nearrow & \circ & \searrow M_*(\pi) \\
 M(X) & & M(X) \\
 & \xrightarrow{\deg \pi} &
 \end{array}$$

Definition 10.2. Let M and N be Mackey functors on \mathcal{C} . A morphism $f : M \rightarrow N$ is a collection $\{f(X) \mid X \in \mathcal{C}\}$ of homomorphisms in Ab , which is natural with respect to each of the covariant and the contravariant part of M and N . With the objectwise composition, we define the category of cohomological Mackey functors $\text{Mack}_c(\mathcal{C})$.

A standard example is the cohomological Mackey functor on a profinite group (see [1]):

Definition 10.3. Let G be a profinite group, and let $\mathcal{C} = G\text{-Sp}$, $F = \text{id}_{\mathcal{C}}$. A cohomological Mackey functor on \mathcal{C} is simply called a cohomological Mackey functor on G , and their category is denoted by $\text{Mack}_c(G)$.

Remark 10.4. Any object X in $G\text{-Sp}$ is a direct sum of transitive G -sets of the form G/H , where H is a open subgroup of G . So a Mackey functor on G is equal to the following datum :

- an abelian group $M(H)$ for each open $H \leq G$, with structure maps:
 - a homomorphism $\text{res}_K^H : M(H) \rightarrow M(K)$ for each open $K \leq H \leq G$,
 - a homomorphism $\text{cor}_K^H : M(K) \rightarrow M(H)$ for each open $K \leq H \leq G$,
 - a homomorphism $c_{g,H} : M(H) \rightarrow M({}^gH)$ for each open $H \leq G$ and $g \in G$,
- where ${}^gH := gHg^{-1}$, satisfying certain compatibilities (cf. [1]). Here, $M(G/H)$ is abbreviated to $M(H)$ for any open subgroup $H \leq G$.

Definition 10.5. Let G be a finite group, and let G^{op} be its opposite group. For any Mackey functor $M = (M, \text{res}, \text{cor}, c) \in \text{Mack}_c(G)$ (in the notation of Remark 10.4), we define its *opposite Mackey functor* M^{op} by

$$\begin{aligned}
 M^{\text{op}}(H^{\text{op}}) &:= M(H) & (H \leq G) \\
 \text{res}_{K^{\text{op}}}^{H^{\text{op}}} &:= \text{res}_K^H & (K \leq H \leq G) \\
 \text{cor}_{K^{\text{op}}}^{H^{\text{op}}} &:= \text{cor}_K^H & (K \leq H \leq G) \\
 c_{g, H^{\text{op}}} &:= c_{g^{-1}, H} & (g \in G, H \leq G).
 \end{aligned}$$

This gives an isomorphism of categories

$$\text{op} : \text{Mack}_c(G) \rightarrow \text{Mack}_c(G^{\text{op}}).$$

For any finite étale covering $\pi : Y \rightarrow X$, put $\text{Br}^*(\pi) := \pi^*$ and $\text{Br}_*(\pi) := N_\pi$. Then we obtain a cohomological Mackey functor Br (and Br' , $H_{\text{et}}^2(-, \mathbb{G}_m)$) as follows. Remark that for any connected scheme S , the category (FEt/S) of finite étale coverings over S becomes a Galois category [22].

Theorem 10.6. For any connected scheme S , we have a sequence of cohomological Mackey functors $\text{Br} \hookrightarrow \text{Br}' \hookrightarrow H_{\text{et}}^2(-, \mathbb{G}_m)$ on (FEt/S) .

Proof. We only show Mackey and cohomological conditions. Since π^* and N_π are compatible with inclusions $\text{Br}(X) \hookrightarrow \text{Br}'(X) \hookrightarrow H_{\text{et}}^2(X, \mathbb{G}_{m,X})$, it suffices to show for $H_{\text{et}}^2(-, \mathbb{G}_m)$.

Mackey condition

Let

$$(10.1) \quad \begin{array}{ccc} Y & \xleftarrow{\varpi'} & Y' \\ \pi \downarrow & \square & \downarrow \pi' \\ X & \xleftarrow{\varpi} & X' \end{array}$$

be a pull-back diagram in (FEt/S) . For any étale morphism of finite type $f : U \rightarrow X$, take the pull-back of (10.1) by f :

$$\begin{array}{ccc} V & \xleftarrow{\varpi'_U} & V' \\ \pi_U \downarrow & \square & \downarrow \pi'_U \\ U & \xleftarrow{\varpi_U} & U' \end{array}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} \pi_* \mathbb{G}_{m,Y} & \xrightarrow{\pi_*(\varpi'_\sharp)} & \pi_* \varpi'_* \mathbb{G}_{m,Y'} & \cong & \varpi_* \pi'_* \mathbb{G}_{m,Y'} \\ N_\pi \downarrow & & \circ & & \downarrow \varpi_*(N_{\pi'}) \\ \mathbb{G}_{m,X} & \xrightarrow{\varpi_\sharp} & \varpi_* \mathbb{G}_{m,X'} & & \end{array}$$

which implies the Mackey condition.

Cohomological conditionFor any finite étale covering $\varpi : V \rightarrow U$ of constant degree d , the composition

$$\begin{aligned} \text{Aut}_{\mathcal{O}_U}(\mathcal{O}_U) &\xrightarrow{\varpi^*} \text{Aut}_{\mathcal{O}_V}(\varpi^* \mathcal{O}_U) \cong \text{Aut}_{\mathcal{O}_V}(\mathcal{O}_V) \\ &\xrightarrow{N_\varpi} \text{Aut}_{\mathcal{O}_U}(\mathcal{N}_\varpi \mathcal{O}_V) \cong \text{Aut}_{\mathcal{O}_U}(\mathcal{O}_U) \end{aligned}$$

is equal to the multiplication by d . This follows from the trivial case $\nabla : \coprod_d U \rightarrow U$ via fpqc descent. From this, we can see

$$N_\pi \circ \pi_\sharp : \mathbb{G}_{m,X} \rightarrow \mathbb{G}_{m,X}$$

is equal to the multiplication by $d = \deg(\pi)$

$$\begin{array}{ccccc} \mathbb{G}_{m,X} & \xrightarrow{\pi_\sharp} & \pi_* \mathbb{G}_{m,Y} & \xrightarrow{N_\pi} & \mathbb{G}_{m,X} \\ & \searrow & \circ & \nearrow & \\ & & d & & \end{array}$$

Thus we obtain $N_\pi \circ \pi^* = d$.

$$\begin{array}{ccccc} & & H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) & & \\ & \nearrow \pi^* & \uparrow \cong \uparrow c & \searrow N_\pi & \\ H_{\text{et}}^2(X, \mathbb{G}_{m,X}) & \xrightarrow{H_{\text{et}}^2(\pi_\sharp)} & H_{\text{et}}^2(X, \pi_* \mathbb{G}_{m,Y}) & \xrightarrow{H_{\text{et}}^2(N_\pi)} & H_{\text{et}}^2(X, \mathbb{G}_{m,X}) \\ & \searrow & \circ & \nearrow & \\ & & d & & \end{array}$$

□

Remark 10.7. We gave the structure of a cohomological Mackey functor to Br , by using the Mackey-functor structure on $H_{\text{et}}^2(-, \mathbb{G}_m)$. In fact, it does not seem difficult at all to show that $H_{\text{et}}^q(-, \mathbb{G}_m)$ becomes a Mackey functor for any non-negative integer q , which we do not need here.

11. RESTRICTION TO A FINITE GALOIS COVERING

In the previous section, we obtained a cohomological Mackey functor Br on FEt/S . Pulling back by a quasi-inverse \mathcal{S} of the fundamental functor

$$F : \text{FEt}/S \xrightarrow{\cong} \pi(S)\text{-Sp},$$

we obtain a Mackey functor on $\pi(S)$:

Corollary 11.1. *There is a sequence of cohomological Mackey functors*

$$\text{Br} \circ \mathcal{S} \hookrightarrow \text{Br}' \circ \mathcal{S} \hookrightarrow H_{\text{et}}^2(-, \mathbb{G}_m) \circ \mathcal{S}$$

on $\pi(S)$, where $\text{Br} \circ \mathcal{S} := (\text{Br}^* \circ \mathcal{S}, \text{Br}_* \circ \mathcal{S})$ (and similarly for Br' , $H_{\text{et}}^2(-, \mathbb{G}_m)$).

Corollary 11.2. *Let X be a connected scheme. For any finite Galois covering $\pi : Y \rightarrow X$ with $\text{Gal}(Y/X) = G$, there exists a cohomological Mackey functor Br on G which satisfies*

$$\text{Br}(H) \cong \text{Br}(Y/H) \quad (\forall H \leq G),$$

with structure maps ϖ^* and N_{ϖ} for each intermediate covering ϖ . (We abbreviate $\text{Br}(G/H)$ to $\text{Br}(H)$, as in Remark 10.4.)

Proof. By the projection $\text{pr} : \pi(X) \rightarrow G^{\text{op}}$, we can regard any finite G^{op} -set naturally as a finite $\pi(X)$ -space, to obtain a functor

$$G^{\text{op}}\text{-Sp} \rightarrow \pi(X)\text{-Sp}.$$

Pulling back by this functor, and taking the opposite Mackey functor, we obtain

$$\begin{array}{ccc} \text{Mack}_c(\pi(X)) & \longrightarrow & \text{Mack}_c(G^{\text{op}}) \xrightarrow{\text{op}} \text{Mack}_c(G) \\ \downarrow \Psi & & \downarrow \Psi \\ M & \longrightarrow & M_G. \end{array}$$

In terms of subgroups of G , this satisfies $M_G(H) = M(\text{pr}^{-1}(H^{\text{op}}))$ for each subgroup $H \leq G$.

Applying this to $\text{Br} \circ \mathcal{S}$, we obtain $\mathcal{B}\text{r} := (\text{Br} \circ \mathcal{S})_G \in \text{Mack}_c(G)$. Since the equivalence $\mathcal{S} : (\pi(X)\text{-Sp}) \xrightarrow{\cong} (\text{FEt}/X)$ satisfies $\mathcal{S}(\pi(X)/\text{pr}^{-1}(H^{\text{op}})) \cong Y/H$, we have $\mathcal{B}\text{r}(H) \cong \text{Br}(Y/H)$. □

Corollary 11.3. *Let $\pi : Y \rightarrow X$ be a finite Galois covering of a connected scheme X , with Galois group G . By a similar way, we can define Br' (and also $(H_{\text{et}}^2(-, \mathbb{G}_m) \circ \mathcal{S})_G$). Since $\text{Mack}_c(G)$ is an abelian category with objectwise (co-)kernels (see for example [4]), we can take the quotient Mackey functor $\text{Br}' / \text{Br} \in \text{Mack}_c(G)$, which satisfies $(\text{Br}' / \text{Br})(H) \cong (\text{Br}'(Y/H)) / (\text{Br}(Y/H))$.*

12. APPENDIX 1

In this section, we briefly introduce some easy applications of Bley and Boltje's theorem to $\mathcal{B}r$.

Let ℓ be a prime number. For any abelian group A , let $A(\ell) := \{m \in A \mid \exists e \in \mathbb{N}_{\geq 0}, \ell^e m = 0\}$ be the ℓ -primary part. This is a \mathbb{Z}_ℓ -module.

Definition 12.1 ([1]). For any finite group H ,
 H is ℓ -hypoelementary \Leftrightarrow $\underset{\text{def}}{H}$ has a normal ℓ -subgroup with a cyclic quotient.
 H is hypoelementary \Leftrightarrow $\underset{\text{def}}{H}$ is ℓ -hypoelementary for some prime ℓ .

Fact 12.2 ([1]). Let M be a cohomological Mackey functor on a finite group G .
 (i) Let ℓ be a prime number. If $H \leq G$ is not ℓ -hypoelementary, then there is a natural isomorphism of \mathbb{Z}_ℓ -modules

$$\bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{odd}}} M(U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{even}}} M(U)(\ell)^{|U|}.$$

(ii) If $H \leq G$ is not hypoelementary and $M(U)$ is torsion for any subgroup $U \leq H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{odd}}} M(U)^{|U|} \cong \bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{even}}} M(U)^{|U|}.$$

Here, $|U|$ denotes the order of U .

Applying this theorem to $\mathcal{B}r$, we obtain the following relations for the Brauer groups of intermediate coverings:

Corollary 12.3. Let X be a connected scheme and $\pi : Y \rightarrow X$ be a finite Galois covering with $\text{Gal}(Y/X) = G$.

(i) Let ℓ be a prime number. If $H \leq G$ is not ℓ -hypoelementary, then there is a natural isomorphism of \mathbb{Z}_ℓ -modules

$$\bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{odd}}} \text{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{even}}} \text{Br}(Y/U)(\ell)^{|U|}.$$

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{odd}}} \text{Br}(Y/U)^{|U|} \cong \bigoplus_{\substack{U=H_0 < \dots < H_n=H \\ n:\text{even}}} \text{Br}(Y/U)^{|U|}.$$

Definition 12.4. Let G be a finite group. For any subgroups $U \leq H \leq G$, put

$$\mu(U, H) := \sum_{U=H_0 < \dots < H_n=H} (-1)^n, \quad \text{Möbius function.}$$

If m (resp. m_ℓ) is an additive invariant of abelian groups (resp. \mathbb{Z}_ℓ -modules) which is finite on Brauer groups, we obtain the following equations:

Corollary 12.5. Let $\pi : Y \rightarrow X$ as before, $G = \text{Gal}(Y/X)$.

(i) If $H \leq G$ is not ℓ -hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m_\ell(\text{Br}(Y/U)(\ell)) = 0.$$

(ii) If $H \leq G$ is not hypoelementary,

$$\sum_{U \leq H} |U| \cdot \mu(U, H) \cdot m(\text{Br}(Y/U)) = 0.$$

Example 1

For a prime ℓ and an abelian group A , its corank is defined as $\text{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$, where $T_\ell(A) = \varprojlim_n \text{Ker}(\ell^n : A \rightarrow A)$. Here we denote this by $\text{rk}_\ell(A)$:

$$\text{rk}_\ell(A) := \text{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$$

$\text{Br}(X)(\ell)$ is known to be of finite corank, for example in the following cases ([15]):

- (C1) k : a separably closed or finite field, X : of finite type $/k$, and proper or smooth $/k$, or $\text{char}(k) = 0$ or $\dim X \leq 2$.
- (C2) X : of finite type $/\text{Spec}(\mathbb{Z})$, and smooth $/\text{Spec}(\mathbb{Z})$ or proper over $\exists \text{open } \subset \text{Spec}(\mathbb{Z})$.

Remark that if Y/X is a finite étale covering and if X satisfies (C1) or (C2), then so does Y .

Example 12.6. Assume X satisfies (C1) or (C2). If a subgroup $H \leq G$ is not ℓ -hypoelementary, we have an equation

$$\sum_{U \leq H} |U| \mu(U, H) \cdot \text{rk}_\ell(\text{Br}(Y/H)(\ell)) = 0.$$

Example 2

By Gabber's lemma (Lemma 4 in [13]), for any finite étale covering $\pi : Y \rightarrow X$ of a connected scheme X , $\pi^* : \text{Br}'(X)/\text{Br}(X) \rightarrow \text{Br}'(Y)/\text{Br}(Y)$ is monomorphic. In particular, if $\text{Br}(Y) \subset \text{Br}(Y)'$ is of finite index, then so is $\text{Br}(X) \subset \text{Br}(X)'$.

Example 12.7. Assume Y satisfies $[\text{Br}'(Y) : \text{Br}(Y)] < \infty$. Then for any non-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U| \mu(U, H) \cdot [\text{Br}'(Y/U) : \text{Br}(Y/U)] = 0.$$

13. APPENDIX 2

Theorem 9.1 can be shown by using Čech cohomology, if we assume the following:

Assumption 13.1. For any finite subset F of X , there exists an affine open subscheme $U \subset X$ containing F .

Remark that if X satisfies Assumption 13.1, then so does any finite étale covering Y over X .

Proof. (Another proof of Theorem 9.1) First, we briefly recall the construction of

$$\chi_Y : \text{Br}(Y) \hookrightarrow H_{\text{et}}^2(Y, \mathbb{G}_{m,Y})$$

using Čech cohomology (cf. [21]). For any Azumaya algebra \mathcal{A} on Y , there exists a surjective étale morphism $g : V \rightarrow Y$, a locally free $\mathcal{E} \in \text{q-Coh}(Y)$ of finite rank, and an isomorphism of \mathcal{O}_V -algebras

$$\phi : g^* \mathcal{A} \xrightarrow{\cong} g^* \text{End}_{\mathcal{O}_Y}(\mathcal{E}).$$

Take the pull-back

$$V \times_Y V =: \begin{array}{ccc} V^{(2)} & \xrightarrow{q_2} & V \\ q_1 \downarrow & \square & \downarrow g \\ V & \xrightarrow{g} & Y \end{array},$$

and put

$$\begin{aligned} q^{(2)} &:= g \circ q_1 = g \circ q_2, \\ \phi^{(2)} &:= (\text{End}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*} \mathcal{E}) \xrightarrow{\cong} q^{(2)*} \text{End}_{\mathcal{O}_Y}(\mathcal{E}) \xrightarrow{q_1^* \phi^{-1}} q^{(2)*} \mathcal{A} \\ &\xrightarrow{q_2^* \phi} q^{(2)*} \text{End}_{\mathcal{O}_Y}(\mathcal{E}) \xrightarrow{\cong} \text{End}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*} \mathcal{E})). \end{aligned}$$

Then, since $q^{(2)*} \text{End}_{\mathcal{O}_Y}(\mathcal{E})$ is an Azumaya algebra on $V^{(2)}$, there exists a surjective étale morphism $W \rightarrow V^{(2)}$ and an element $c \in \Gamma(W, \text{End}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*} \mathcal{E}) |_{\check{W}}^\times)$ such that $\phi^{(2)}$ is the inner automorphism defined by c :

$$\phi^{(2)} |_W = \text{Inn}(c)$$

By Assumption 13.1, there exists a surjective étale morphism $V' \xrightarrow{g'} Y$ which factors through V

$$\begin{array}{ccc} & V & \\ v \nearrow & \circ & \searrow g \\ V' & & Y \\ & \searrow g' & \end{array}$$

such that the induced morphism

$$V'^{(2)} := V' \times_Y V' \xrightarrow{v^{(2)}} V^{(2)}$$

factors through W .

So, by replacing $V \xrightarrow{g} Y$ by $V' \xrightarrow{g'} Y$, we may assume the existence of a quartet

$$(\mathcal{V}, \mathcal{E}, \phi, c)$$

which satisfies

$$\begin{aligned} \mathcal{V} &= (V \xrightarrow{g} Y), \text{ surjective étale morphism of finite type,} \\ \mathcal{E} &\in \text{q-Coh}(Y), \text{ locally free of finite rank,} \\ \phi &: g^* \mathcal{A} \xrightarrow{\cong} g^* \text{End}_{\mathcal{O}_Y}(\mathcal{E}), \text{ } \mathcal{O}_V\text{-algebra isomorphism,} \\ c &\in \Gamma(V^{(2)}, \text{End}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*} \mathcal{E})^\times), \phi^{(2)} = \text{Inn}(c). \end{aligned}$$

We call $(\mathcal{V}, \mathcal{E}, \phi, c)$ a *compatible trivialization* of \mathcal{A} . Remark that for any refinement of \mathcal{V}

$$\mathcal{V}' = (V' \xrightarrow{g'} V) \quad \begin{array}{ccccc} & & V & \xrightarrow{g} & Y \\ & \nearrow v & \circ & & \\ V' & & & \searrow g' & \end{array} ,$$

we obtain an induced compatible trivialization of \mathcal{A} on \mathcal{V}'

$$(\mathcal{V}', \mathcal{E}, v^* \phi, v^{(2)*} c).$$

Let $q_{ij} : V^{(3)} = V \times_Y V \times_Y V \rightarrow V^{(2)}$ ($1 \leq i < j \leq 3$) be the projections to the (i, j) -th components, and put $q^{(3)} := q^{(2)} \circ q_{ij}$. If we put

$$\chi := q_{12}^* c \cdot q_{13}^* c^{-1} \cdot q_{23}^* c \in \Gamma(V^{(3)}, q^{(3)*} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times),$$

then in $\text{Aut}_{\mathcal{O}_{V^{(3)}}}(q^{(3)*} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E}))$, we have

$$\begin{aligned} \text{Inn}(\chi) &= \text{Inn}(q_{12}^* c) \cdot \text{Inn}(q_{13}^* c^{-1}) \cdot \text{Inn}(q_{23}^* c) \\ &= q_{12}^* \phi^{(2)} \circ q_{13}^* \phi^{(2)-1} \circ q_{23}^* \phi^{(2)} \\ &= \text{id}. \end{aligned}$$

So χ is in the center of $\Gamma(V^{(3)}, q^{(3)*} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times)$, i.e.,

$$\chi \in Z(\Gamma(V^{(3)}, q^{(3)*} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times)) = \Gamma(V^{(3)}, \mathcal{O}_{V^{(3)}}^\times).$$

Thus we obtain a 2-cocycle $\chi_Y^{\mathcal{V}}(\mathcal{A})$ in the Čech complex $\check{C}^\bullet(\mathcal{V}/Y, \mathbb{G}_{m,Y})$, which defines $\chi_Y(\mathcal{A}) \in \check{H}_{\text{et}}^2(Y, \mathbb{G}_{m,Y})$.

For any Azumaya algebra \mathcal{A} on Y , take a compatible trivialization $(\mathcal{V}, \mathcal{E}, \phi, c)$. By Lemma 8.8, there exists $\mathcal{U} = (U \xrightarrow{f} X) \in \text{Cov}_{\text{et}}(X)$ such that $\pi^* \mathcal{U} \prec \mathcal{V}$. Here, $\pi^* \mathcal{U} \in \text{Cov}_{\text{et}}(Y)$ denotes the covering induced by pull-back by π .

So, replacing $(\mathcal{V}, \mathcal{E}, \phi, c)$ by the induced compatible trivialization on $\pi^* \mathcal{U}$, we may assume $\mathcal{V} = \pi^* \mathcal{U}$.

Take the pull-back

$$\begin{array}{ccccccc} V^{(3)} & \xrightarrow{q_{ij}} & V^{(2)} & \xrightarrow{q_i} & V & \xrightarrow{g} & Y \\ \downarrow \varpi^{(3)} & \square & \downarrow \varpi^{(2)} & \square & \downarrow \varpi & \square & \downarrow \pi \\ U^{(3)} & \xrightarrow{p_{ij}} & U^{(2)} & \xrightarrow{p_i} & U & \xrightarrow{f} & X \\ & & & & \circ & & \\ & & & & \circ & \xrightarrow{p^{(2)}} & \\ & & & & \circ & \xrightarrow{p^{(3)}} & \end{array} \quad \begin{array}{l} (\ell = 1, 2) \\ (1 \leq i < j \leq 3) \end{array}$$

Then we obtain a compatible trivialization $(\mathcal{U}, \mathcal{N}_\pi \mathcal{E}, N_\varpi \phi, N_{\varpi^{(2)}}(c))$ of $\mathcal{N}_\pi(\mathcal{A})$, defined as follows:

$$\begin{aligned} N_\varpi(\phi) &:= (f^* \mathcal{N}_\pi \mathcal{A} \xrightarrow{\theta_\pi^f} N_\varpi g^* \mathcal{A} \xrightarrow{N_\varpi(\phi)} N_\varpi g^* \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})) \\ &\xrightarrow{(\theta_\pi^f)^{-1}} f^* \mathcal{N}_\pi \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E}) \xrightarrow{f^* \delta_\pi} f^* \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{N}_\pi \mathcal{E}) \end{aligned}$$

$$\begin{array}{ccc}
\Gamma(V^{(2)}, \mathcal{E}nd_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E})^\times) & \xrightarrow{N_{\varpi^{(2)}}} & \Gamma(U^{(2)}, \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times) \\
\parallel & \circ & \parallel \\
\text{Aut}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E}) & & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E}) \\
\searrow N_{\varpi^{(2)}} & & \swarrow c(\theta_\pi^{p^{(2)}}) \\
& \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E}) &
\end{array}$$

Here $c(\theta_\pi^{p^{(2)}})$ is the conjugation by $\theta_\pi^{p^{(2)}}$.

By Remark 8.4, there is an induced group isomorphism

$$I_\theta^\times : \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E})^\times \xrightarrow{\cong} \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times.$$

Claim 13.2. $(\mathcal{U}, \mathcal{N}_\pi\mathcal{E}, N_\varpi\phi, N_{\varpi^{(2)}}(c))$ is a compatible trivialization of $\mathcal{N}_\pi(\mathcal{A})$.

Proof. (Proof of Claim 13.2) It suffices to show

$$(13.1) \quad (N_\varpi(\phi))^{(2)} = \text{Inn}(N_{\varpi^{(2)}}(c)).$$

Using Proposition 7.19 (ii) and Proposition 8.5, we can show easily

$$(13.2) \quad (N_\varpi(\phi))^{(2)} = I_\theta^\times \circ \delta_{\varpi^{(2)}} \circ \mathcal{N}_{\varpi^{(2)}}(\phi^{(2)}) \circ \delta_{\varpi^{(2)}}^{-1} \circ I_\theta^{\times-1}.$$

By Remark 8.4, we have a commutative diagram

$$\begin{array}{ccc}
\text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E}) & \xrightarrow{c(\theta_\pi^{p^{(2)}})^{-1}} & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E}) \\
\parallel & \circ & \parallel \\
\Gamma(U^{(2)}, \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E})^\times) & \xrightarrow{\Gamma(U^{(2)}, I_\theta^\times)} & \Gamma(U^{(2)}, \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times) \\
\downarrow \text{Inn} & \circ & \downarrow \text{Inn} \\
\text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E})^\times) & \xrightarrow{c(I_\theta^\times)} & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times).
\end{array}$$

Thus we obtain the following commutative diagram:

$$\begin{array}{ccc}
\Gamma(V^{(2)}, \mathcal{E}nd_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E})^\times) & \xrightarrow{\text{Inn}} & \text{Aut}_{\mathcal{O}_{V^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E})^\times) \\
\parallel & & \downarrow N_{\varpi^{(2)}} \\
\text{Aut}_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E}) & & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}\mathcal{E}nd_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E})^\times) \\
\downarrow N_{\varpi^{(2)}} & & \downarrow c(\delta_{\varpi^{(2)}}) \\
\text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E}) & \circ & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(\mathcal{N}_{\varpi^{(2)}}q^{(2)*}\mathcal{E})^\times) \\
\downarrow c(\theta_\pi^{p^{(2)}})^{-1} & & \downarrow c(I_\theta^\times) \\
\text{Aut}_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E}) & & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times) \\
\parallel & & \parallel \\
\Gamma(U^{(2)}, \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times) & \xrightarrow{\text{Inn}} & \text{Aut}_{\mathcal{O}_{U^{(2)}}}(\mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}\mathcal{N}_\pi\mathcal{E})^\times)
\end{array}$$

By (13.2), this means (13.1). \square

We have

$$\check{H}_{\text{et}}^2(\mathcal{V}/Y, \mathbb{G}_{m,Y}) = \check{H}_{\text{et}}^2(\pi^*U/Y, \mathbb{G}_{m,Y}) = \check{H}_{\text{et}}^2(U/X, \pi_*\mathbb{G}_{m,Y}),$$

and the canonical natural isomorphism

$$c^{-1} : H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) \xrightarrow{\cong} H_{\text{et}}^2(X, \pi_*\mathbb{G}_{m,Y})$$

fits into the following commutative diagram:

$$\begin{array}{ccc} \check{H}_{\text{et}}^2(\mathcal{V}/Y, \mathbb{G}_{m,Y}) & \xrightarrow[\cong]{\text{id}} & \check{H}_{\text{et}}^2(U/X, \pi_*\mathbb{G}_{m,Y}) \\ \text{can.} \downarrow & \circlearrowleft & \downarrow \text{can.} \\ H_{\text{et}}^2(Y, \mathbb{G}_{m,Y}) & \xrightarrow[\cong]{c^{-1}} & H_{\text{et}}^2(X, \pi_*\mathbb{G}_{m,Y}) \end{array}$$

So, it suffices to show

$$\check{H}_{\text{et}}^2(U/X, N_\pi)(\chi_Y^\vee(\mathcal{A})) = \chi_X^{\mathcal{U}}(N_\pi(\mathcal{A})).$$

Similarly as $N_{\varpi(2)}$, we can construct a homomorphism

$$N_{\varpi(3)} : \Gamma(V^{(3)}, q^{(3)*}\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times) \rightarrow \Gamma(U^{(3)}, p^{(3)*}\mathcal{E}nd_{\mathcal{O}_X}(N_\pi\mathcal{E})^\times),$$

compatible with $N_{\varpi(2)}$ and $N_{\varpi(3)} : \Gamma(V^{(3)}, \mathcal{O}_{V^{(3)}}^\times) \rightarrow \Gamma(U^{(3)}, \mathcal{O}_{U^{(3)}}^\times)$.

$$\begin{array}{ccc} \Gamma(V^{(2)}, \mathcal{E}nd_{\mathcal{O}_{V^{(2)}}}(q^{(2)*}\mathcal{E})^\times) & \xrightarrow{N_{\varpi(2)}} & \Gamma(U^{(2)}, \mathcal{E}nd_{\mathcal{O}_{U^{(2)}}}(p^{(2)*}N_\pi\mathcal{E})^\times) \\ \downarrow \cong & & \downarrow \cong \\ \Gamma(V^{(2)}, q^{(2)*}\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times) & \circlearrowleft & \Gamma(U^{(2)}, p^{(2)*}\mathcal{E}nd_{\mathcal{O}_X}(N_\pi\mathcal{E})^\times) \\ \downarrow q_{ij}^* & & \downarrow p_{ij}^* \\ \Gamma(V^{(3)}, q^{(3)*}\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times) & \xrightarrow{\exists N_{\varpi(3)}} & \Gamma(U^{(3)}, p^{(3)*}\mathcal{E}nd_{\mathcal{O}_X}(N_\pi\mathcal{E})^\times) \\ \uparrow \cong & & \uparrow \cong \\ Z(\Gamma(V^{(3)}, q^{(3)*}\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})^\times)) & \circlearrowleft & Z(\Gamma(U^{(3)}, p^{(3)*}\mathcal{E}nd_{\mathcal{O}_X}(N_\pi\mathcal{E})^\times)) \\ \cong \uparrow & & \cong \uparrow \\ \Gamma(V^{(3)}, \mathcal{O}_{V^{(3)}}^\times) & \xrightarrow{N_{\varpi(3)}} & \Gamma(U^{(3)}, \mathcal{O}_{U^{(3)}}^\times) \end{array}$$

From this, we have

$$\begin{aligned} \check{H}_{\text{et}}^2(U/X, N_\pi)(\chi_Y^\vee(\mathcal{A})) &= N_{\varpi(3)}(q_{12}^*c \cdot q_{13}^*c^{-1} \cdot q_{23}^*c) \\ &= N_{\varpi(3)}(q_{12}^*c) \cdot N_{\varpi(3)}(q_{13}^*c^{-1}) \cdot N_{\varpi(3)}(q_{23}^*c) \\ &= (p_{12}^*N_{\varpi(2)}(c)) \cdot (p_{13}^*N_{\varpi(2)}(c)^{-1}) \cdot (p_{23}^*N_{\varpi(2)}(c)) \\ &= \chi_X^{\mathcal{U}}(N_\pi(\mathcal{A})). \end{aligned}$$

□

Part 3. Tambara functors on profinite groups and generalized Burnside functors

14. INTRODUCTION FOR PART 3

The Tambara functor was defined by Tambara in [30] for any finite group G , in the name of TNR-functors. Roughly speaking, a Tambara functor on G is a ring-valued Mackey functor with multiplicative transfers, satisfying certain compatibility conditions for exponential diagrams. Recently, Brun revealed that Tambara functors play an important role in the Witt-Burnside construction [6].

As Mackey functors admit a Lindner-type description (see [19]), the category of Tambara functors is equivalent to the category of product-preserving functors $[U, (\text{Set})_0]$ from a certain category U to the category of sets [30]. This enables us a more functorial treatment of fixed point functors, cohomology ring functors, and Burnside ring functors, as examples of Tambara functors.

On the other hand, to consider Mackey functors on a possibly infinite group G , Bley and Boltje defined in [1] general Mackey systems for arbitrary groups, on which Mackey functors are defined. This general class of functors include ordinary Mackey functors on finite groups, Mackey functors on profinite groups (so-called G -modulations), and has several applications in number theory as shown in [1]. A Mackey system $(\mathcal{C}, \mathcal{O})$ for an arbitrary group G is a pair of family of subgroups in G with certain conditions, each of them is closed under conjugation and finite intersections.

Independently, for any finite group G and any conjugation-closed family \mathfrak{X} of subgroups in G , Yoshida has defined in [33] the generalized Burnside ring $\Omega(G, \mathfrak{X})$, which has several properties similar to the ordinary Burnside ring $\Omega(G)$. It is shown in [33] that if \mathfrak{X} is moreover closed under (necessarily finite) intersections, $\Omega(G, \mathfrak{X})$ is equal to the Grothendieck ring of a category associated to (G, \mathfrak{X}) , and becomes a subring of $\Omega(G)$. We only consider the case where $\mathfrak{X} := \mathcal{C}$ is also closed under (finite) intersections, and in this part we generalize $\Omega(G, \mathcal{C})$ to a Mackey functor $\Omega_{(\mathcal{C}, \mathcal{O})}$ on any Mackey system $(\mathcal{C}, \mathcal{O})$ for an arbitrary group G .

In this part, we consider a generalization of Tambara functors, namely, we define a Tambara functor on any Mackey system with certain conditions. As a consequence, we can consider a Tambara functor on a profinite group. Our main theorem (Theorem 18.16) enables us to construct Tambara functors, for example, we make the above Burnside functor $\Omega_{(\mathcal{C}, \mathcal{O})}$ into a Tambara functor $\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})}$. In relation with the Witt-Burnside construction, on any profinite group G , we give a Tambara-functor structure to Elliott's functor \mathbf{V}_M , where M is an arbitrary multiplicative monoid. This functor is closely related to the Witt-Burnside construction as shown in [10], which generalizes the completed Burnside ring functor considered by Dress and Siebeneicher in [8].

In section 15, after fixing our notation, we introduce some known results and preparative properties concerning Mackey functors on Mackey systems. In section 16, we show any Mackey functor on a Mackey system admits a Lindner-type definition. In this context, the above Burnside functor $\Omega_{(\mathcal{C}, \mathcal{O})}$ can be easily regarded as a Mackey functor. In section 17, we define Tambara systems and (semi-)Tambara functors on them, generalizing the case of finite groups. In section 18, we show how a semi-Tambara functor gives rise to a Tambara functor in Theorem 18.16. The theorem is as follows, and proven in a similar way as the finite-group case:

Theorem 18.16 . *Let S be a semi-Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$.*

With $(\gamma S)(X) = K_0 S(X)$ and $\tilde{\xi}_+$, $\tilde{\eta}_{\bullet}$, $\tilde{\zeta}^$ appropriately defined, γS becomes a Tambara functor.*

By virtue of this theorem, we can show that the Burnside functor $\Omega_{(\mathcal{C}, \mathcal{O})}$ becomes a Tambara functor. As a further example concerning the Witt-Burnside construction, we make \mathbf{V}_M into a Tambara functor on a profinite group G .

15. PRELIMINARIES

First we fix a notation. For any group G , $H \leq G$ means that H is a subgroup of G . For any subgroup $H \leq G$ and any $g \in G$, define ${}^g H := gHg^{-1}$ and $H^g := g^{-1}Hg$. ${}_G \text{Set}$ denotes the category of G -sets and equivariant maps, and ${}_G \text{set}$ denotes the category of *finite* G -sets, which is a full subcategory of ${}_G \text{Set}$. If X is a G -set and $x \in X$, let G_x denote the stabilizer group of x in X . In this part, monoids are assumed to be commutative and have an additive unit 0. A homomorphism of monoids preserves 0. Semi-rings are assumed to be commutative both for the addition and the multiplication, and have an additive unit 0 and a multiplicative unit 1. A homomorphism of semi-rings preserves 0 and 1. For any category \mathcal{K} and any objects $X, Y \in \text{Ob}(\mathcal{K})$, the set of morphisms from X to Y in \mathcal{K} is denoted by $\mathcal{K}(X, Y)$.

The following definitions are based on [1]. When we consider a Mackey functor, we will only treat the case of a \mathbb{Z} -Mackey functor and call it simply a Mackey functor.

Definition 15.1. (Definition 2.1 in [1])

Let G be an arbitrary group. A Mackey system for G is a pair $(\mathcal{C}, \mathcal{O})$ with the following property.

- \mathcal{C} is a set of subgroups of G , closed under conjugation and finite intersections,
- $\mathcal{O} = \{\mathcal{O}(H)\}_{H \in \mathcal{C}}$ is a family of subsets $\mathcal{O}(H) \subseteq \mathcal{C}(H) := \{U \in \mathcal{C} \mid U \leq H\}$,

which satisfies

- (i) $[H : U] < \infty$
- (ii) $\mathcal{O}(U) \subseteq \mathcal{O}(H)$
- (iii) $\mathcal{O}(gHg^{-1}) = g\mathcal{O}(H)g^{-1}$
- (iv) $U \cap V \in \mathcal{O}(V)$

for all $H \in \mathcal{C}$, $U \in \mathcal{O}(H)$, $V \in \mathcal{C}(H)$, and $g \in G$.

Example 15.2. Let \mathcal{C} be a set of subgroups of G , closed under conjugation and finite intersections.

- (1) If we define $\mathcal{O}_{\mathcal{C}}$ by

$$\mathcal{O}_{\mathcal{C}}(H) := \{U \in \mathcal{C}(H) \mid [H : U] < \infty\} \quad (\forall H \in \mathcal{C}),$$

then $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ becomes a Mackey system for G .

- (2) If we define \mathcal{O}_d by

$$\mathcal{O}_d(H) = \{H\} \quad (\forall H \in \mathcal{C}),$$

then $(\mathcal{C}, \mathcal{O}_d)$ becomes a Mackey system for G .

Remark 15.3. *Both $\mathcal{O}_{\mathcal{C}}$ and \mathcal{O}_d satisfy $H \in \mathcal{O}_{\mathcal{C}}(H)$ and $H \in \mathcal{O}_d(H)$ for any $H \in \mathcal{C}$. In the following, we often impose the condition*

$$(15.1) \quad H \in \mathcal{O}(H) \quad (\forall H \in \mathcal{C})$$

to a Mackey system $(\mathcal{C}, \mathcal{O})$. If we fix \mathcal{C} , then $(\mathcal{C}, \mathcal{O}_c)$ (resp. $(\mathcal{C}, \mathcal{O}_d)$) is the largest (resp. smallest) Mackey system, among all the Mackey systems $(\mathcal{C}, \mathcal{O})$ satisfying (15.1).

Definition 15.4. In (1) in Example 15.2, if in particular G is a topological group and \mathcal{C} is the set of all closed (resp. open) subgroups of G , we call $(\mathcal{C}, \mathcal{O}_c)$ the natural (resp. open-natural) Mackey system for G . If G is a profinite group, then this definition of the (resp. open-)natural Mackey system agrees with the definition of the (resp. finite) natural Mackey system in [1].

Definition 15.5 (cf. Definition 2.3. in [1]). Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system for an arbitrary group G . A semi-Mackey functor M on $(\mathcal{C}, \mathcal{O})$ is a function which assigns

- a monoid $M(H)$ to each $H \in \mathcal{C}$,
- a homomorphism of monoids $c_H^g : M(H) \rightarrow M({}^gH)$ to each $H \in \mathcal{C}$ and each $g \in G$,
- a homomorphism of monoids $r_I^H : M(H) \rightarrow M(I)$ to each pair $I \leq H$ in \mathcal{C} ,
- a homomorphism of monoids $t_I^H : M(I) \rightarrow M(H)$ to each $H \in \mathcal{C}$ and each $I \in \mathcal{O}(H)$,

in a compatible way as in [1]. If all the $M(H)$ are abelian groups, then M is called a Mackey functor. The maps c_H^g, r_I^H, t_I^H are called conjugations, restrictions, and transfers, respectively.

A morphism of (semi-)Mackey functors $f : M \rightarrow N$ is a set of monoid homomorphisms $f = \{f_H : M(H) \rightarrow N(H)\}_{H \in \mathcal{C}}$, which are compatible with the conjugations, restrictions, and transfers in the obvious sense. We write the category of semi-Mackey functors (resp. Mackey functors) as $\mathcal{SMack}_{(\mathcal{C}, \mathcal{O})}$ (resp. $\mathcal{Mack}_{(\mathcal{C}, \mathcal{O})}$). Note that $\mathcal{Mack}_{(\mathcal{C}, \mathcal{O})}$ is a full subcategory of $\mathcal{SMack}_{(\mathcal{C}, \mathcal{O})}$.

Remark 15.6. For a finite group G , if we regard G as a discrete topological group, both the natural and open-natural Mackey systems are

$$\begin{aligned} \mathcal{C} &= \{\forall \text{ subgroup of } G\} \\ \mathcal{O}(H) &= \mathcal{C}(H) = \{\forall \text{ subgroup of } H\} \quad (\forall H \leq G). \end{aligned}$$

A (resp. semi-)Mackey functor on this Mackey system is nothing other than a (resp. semi-)Mackey functor on G . Thus the Mackey functor theory on finite groups is contained in that on Mackey systems.

Definition 15.7 (Definition 2.6. in [1]). Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system for G .

(1) ${}_G\text{Set}_{\mathcal{C}}$ is defined to be a full subcategory of ${}_G\text{Set}$, whose objects are those $X \in \text{Ob}({}_G\text{Set})$ which satisfy

$$G_x \in \mathcal{C} \text{ for any } x \in X.$$

(2) ${}_G\text{Set}_{\mathcal{C}, \mathcal{O}}$ is defined to be a category with the same objects as ${}_G\text{Set}_{\mathcal{C}}$, whose morphisms from X to Y are those $f \in {}_G\text{Set}_{\mathcal{C}}(X, Y)$ satisfying the following properties:

- (i) f has finite fibers (i.e. $f^{-1}(y)$ is a finite set for any $y \in Y$),
- (ii) $G_x \in \mathcal{O}(G_{f(x)})$ for any $x \in X$.

Remark 15.8. Let \mathcal{C} be a set of subgroups of G closed under conjugation and finite intersections, and consider the Mackey system $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. Then for any $f \in {}_G\text{Set}_{\mathcal{C}}(X, Y)$, we have

$$f \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}(X, Y) \Leftrightarrow f \text{ has finite fibers.}$$

Proof. The necessity is trivial. Conversely, assume f has finite fibers. For any $x \in X$, let X_x denote the orbit through x in X . Since $G_{f(x)}/G_x \cong X_x \subseteq f^{-1}(f(x))$ and $f^{-1}(f(x))$ is finite by the assumption, we have $[G_{f(x)} : G_x] = \#X_x < \infty$, i.e. $G_x \in \mathcal{O}_{\mathcal{C}}(G_{f(x)})$. \square

Remark 15.9. *If the Mackey system $(\mathcal{C}, \mathcal{O})$ satisfies (15.1), then for any two objects X, Y in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$, any injective map ι from X to Y in ${}_{\mathcal{C}}\text{Set}$*

$$\iota : X \rightarrow Y$$

belongs to ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$.

In particular, isomorphisms, inclusions are morphisms in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$.

Moreover, the folding maps

$$\nabla : X \amalg X \rightarrow X$$

are morphisms in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$.

Remark 15.10. *Let $(\mathcal{C}, \mathcal{O})$ be an arbitrary Mackey system. For any pull-back diagram in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ g' \downarrow & \square & \downarrow g \\ Y' & \xrightarrow{f} & Y \end{array},$$

if g belongs to ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$, then g' also belongs to ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$.

Definition 15.11 (cf. Definition 2.6. in [1]). *Category $\mathcal{SBif}_{(\mathcal{C}, \mathcal{O})}$ (resp. $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$) is defined as follows:*

An object M in $\mathcal{SBif}_{(\mathcal{C}, \mathcal{O})}$ (resp. $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$) is a function which assigns

- a monoid (resp. abelian group) $M(X)$ to each $X \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}})$,

- a monoid morphism $f^ : M(Y) \rightarrow M(X)$ to each $f \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}}(X, Y)$,*

- a monoid morphism $g_ : M(X) \rightarrow M(Y)$ to each $g \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$,*

in such a way that the following conditions are satisfied:

(i) *We have*

$$(g' \circ g)_* = g'_* \circ g_*, \quad (f' \circ f)^* = f^* \circ f'^*$$

for all composable pairs in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$ and ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$ respectively, and

$$\text{id}^* = \text{id}_* = \text{id}.$$

(ii) *(Mackey condition) If*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ f' \downarrow & \square & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

is a pull-back diagram in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$ where $g \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$, then

$$f^* \circ g_* = g'_* \circ f'^*.$$

(iii) *For any direct sum decomposition $X = \coprod_{\lambda \in \Lambda} X_{\lambda}$ in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$, the natural map*

$$(i_{\lambda}^*)_{\lambda \in \Lambda} : M(X) \rightarrow \prod_{\lambda \in \Lambda} M(X_{\lambda})$$

is an isomorphism of sets, where $i_\lambda : X_\lambda \hookrightarrow X$ ($\lambda \in \Lambda$) are the inclusions.

$M(\emptyset)$ consists of a single element.

For any $M, N \in \text{Ob}(\text{SBif}_{(\mathcal{C}, \mathcal{O})})$ (or $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$), a morphism φ from M to N is a collection of monoid homomorphisms $\varphi_X : M(X) \rightarrow N(X)$ ($X \in \mathcal{G}\text{Set}_{\mathcal{C}}$), compatible with all g_* and f^* . Remark that $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$ is a full subcategory of $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$.

Remark 15.12. *An object in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$ (resp. $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$) is nothing other than a pair of functors $M = (M^*, M_*)$ which satisfies $M^*(X) = M_*(X) (= M(X))$ ($\forall X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})$), where*

- $M^* : \mathcal{G}\text{Set}_{\mathcal{C}} \rightarrow (\text{Mon})$ (resp. $\mathcal{G}\text{Set}_{\mathcal{C}} \rightarrow (\text{Ab})$) is a contravariant functor with $M^*(f) = f^*$,

- $M_* : \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}} \rightarrow (\text{Mon})$ (resp. $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}} \rightarrow (\text{Ab})$) is a covariant functor with $M_*(g) = g_*$,

which satisfies the above condition (ii) and (iii).

In this view, a collection $(\varphi_X : M(X) \rightarrow N(X))_{X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})}$ of monoid homomorphisms is a morphism in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$ if and only if it is a natural transformation with respect to each of the covariant and the contravariant part.

Remark 15.13. *Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system satisfying (15.1), and M be an object in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$. Let $X = \coprod_{1 \leq i \leq n} X_i$ be a finite direct sum of objects in $\mathcal{G}\text{Set}_{\mathcal{C}}$, and let $\iota_i : X_i \hookrightarrow X$ be the inclusion ($1 \leq i \leq n$). The inverse of the isomorphism*

$$\eta := (\iota_i^*)_{1 \leq i \leq n} : M(X) \xrightarrow{\cong} \prod_{1 \leq i \leq n} M(X_i) = \bigoplus_{1 \leq i \leq n} M(X_i)$$

is

$$\sum_{1 \leq i \leq n} \iota_{i*} : \bigoplus_{1 \leq i \leq n} M(X_i) \rightarrow M(X).$$

Proof. Since

$$\begin{array}{ccc} X_i & \xleftarrow{\text{id}} & X_i \\ \downarrow \iota_i & \square & \downarrow \text{id} \\ X & \xleftarrow{\iota_i} & X_i \end{array} \quad \text{and} \quad \begin{array}{ccc} X_i & \xleftarrow{\quad} & \emptyset \\ \downarrow \iota_i & \square & \downarrow \\ X & \xleftarrow{\iota_j} & X_j \end{array} \quad (i \neq j)$$

are pull-back diagrams, we have

$$\begin{aligned} \iota_i^* \circ \iota_{i*} &= \text{id} \\ \iota_i^* \circ \iota_{j*} &= 0 \quad (i \neq j). \end{aligned}$$

So we have

$$\eta \circ \sum_{1 \leq i \leq n} \iota_{i*} = \text{id}.$$

Since η is an isomorphism, this means $\eta^{-1} = \sum_{1 \leq i \leq n} \iota_{i*}$. □

Corollary 15.14. *Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system satisfying (15.1), and M be an object in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$. If the pull-back of $f \in \mathcal{G}\text{Set}_{\mathcal{C}}(X, Z)$ and $g \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}(Y, Z)$ is*

written as

$$\begin{array}{ccc}
 X & \xleftarrow{\bigcup_{1 \leq i \leq n} k_i} & \prod_{1 \leq i \leq n} W_i \\
 \downarrow f & \square & \downarrow \bigcup_{1 \leq i \leq n} h_i \\
 Z & \xleftarrow{g} & Y
 \end{array}$$

where $k_i \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}(W_i, X)$ and $h_i \in \mathcal{G}\text{Set}_{\mathcal{C}}(W_i, Y)$, then we have

$$f^* \circ g_* = \sum_{1 \leq i \leq n} k_{i*} \circ h_i^*.$$

Proof. Put

$$\begin{aligned}
 W &:= \prod_{1 \leq i \leq n} W_i, \\
 k &:= \bigcup_{1 \leq i \leq n} k_i, \\
 h &:= \bigcup_{1 \leq i \leq n} h_i,
 \end{aligned}$$

and let $\iota_i : W_i \hookrightarrow W$ be the inclusion ($1 \leq i \leq n$). If we put $\eta := (\iota_i^*)_{1 \leq i \leq n} : M(W) \xrightarrow{\cong} \prod_{1 \leq i \leq n} M(W_i)$, Then by Remark 15.13, we have

$$\begin{aligned}
 f^* \circ g_* &= k_* \circ h^* = k_* \circ \eta^{-1} \circ \eta \circ h^* \\
 &= k_* \circ \sum_{1 \leq i \leq n} \iota_{i*} \circ ((\iota_i^*)_{1 \leq i \leq n}) \circ h^* = \sum_{1 \leq i \leq n} k_{i*} \circ h_i^*.
 \end{aligned}$$

□

Remark 15.15. Let $K_0 : (\text{Mon}) \rightarrow (\text{Ab})$ be the group completion functor. For any $M = (M^*, M_*) \in \text{Ob}(\text{SBif}_{(\mathcal{C}, \mathcal{O})})$, we define $\gamma M \in \text{Ob}(\text{Bif}_{(\mathcal{C}, \mathcal{O})})$ by $\gamma M := (K_0 \circ M^*, K_0 \circ M_*)$. For any morphism $\varphi = (\varphi_X : M(X) \rightarrow N(X))_{X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})} \in \text{SBif}_{(\mathcal{C}, \mathcal{O})}(M, N)$, we define $\gamma\varphi \in \text{Bif}_{(\mathcal{C}, \mathcal{O})}(\gamma M, \gamma N)$ by

$$\gamma\varphi := (K_0(\varphi_X) : K_0 M(X) \rightarrow K_0 N(X))_{X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})}.$$

Thus, we obtain a functor $\gamma : \text{SBif}_{(\mathcal{C}, \mathcal{O})} \rightarrow \text{Bif}_{(\mathcal{C}, \mathcal{O})}$.

Proposition 15.16. Let $M = (M^*, M_*)$ be an object in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$. For any isomorphism $v : V \rightarrow V'$ in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}$, we have

$$M_*(v)M^*(v) = \text{id}_{M(V')}.$$

Proof. From the pull-back diagram

$$\begin{array}{ccc}
 V & \xleftarrow{\text{id}} & V \\
 \downarrow v & \square & \downarrow \text{id} \\
 V' & \xleftarrow{v} & V
 \end{array}$$

we obtain

$$M^*(v)M_*(v) = M_*(\text{id})M^*(\text{id}) = \text{id}$$

by the Mackey condition. Since $M_*(v)$ is an isomorphism, this means $M^*(v) = M_*(v)^{-1}$. \square

As Theorem 2.7. in [1], the following can be shown:

Theorem 15.17. *Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system for G . Then*

- (i) $\mathcal{SMack}_{(\mathcal{C}, \mathcal{O})}$ is equivalent to $\mathcal{SBif}_{(\mathcal{C}, \mathcal{O})}$,
- (ii) $\text{Mack}_{(\mathcal{C}, \mathcal{O})}$ is equivalent to $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$.

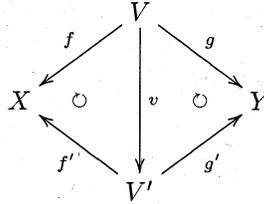
By this equivalence, we often identify (semi-)Mackey functors with objects in $(\mathcal{S})\text{Bif}_{(\mathcal{C}, \mathcal{O})}$.

16. A LINDNER-TYPE DEFINITION OF MACKEY FUNCTORS

First, we introduce a well known fact (Fact 16.1), essentially due to Lindner [19]. For any group G , let ${}_G\text{set}$ denote the category of finite G -sets and G -maps, and let $\text{Sp}({}_G\text{set})$ denote the span category of ${}_G\text{set}$ (i.e. the classifying category of the bicategory of spans in ${}_G\text{set}$ (cf. [20])). By definition, $\text{Ob}({}_G\text{set}) = \text{Ob}(\text{Sp}({}_G\text{set}))$ and

$$\begin{aligned} \text{Sp}({}_G\text{set})(X, Y) &= \{\text{span from } X \text{ to } Y \text{ in } {}_G\text{set}\} / \underset{\text{equivalence}}{\sim} \\ &= \{(X \xleftarrow{f} V \xrightarrow{g} Y) \mid f, g \text{ are morphisms in } {}_G\text{set}\} / \underset{\text{equiv.}}{\sim} \end{aligned}$$

for any $X, Y \in \text{Ob}(\text{Sp}({}_G\text{set}))$, where $(f, V, g) := (X \xleftarrow{f} V \xrightarrow{g} Y)$ and (f', V', g') are equivalent if and only if there exists an isomorphism $v \in {}_G\text{set}(V, V')$ such that $f = f' \circ v$, $g = g' \circ v$.



If we want to indicate v , we write

$$(f, V, g) \xrightarrow[v]{\sim} (f', V', g')$$

instead of $(f, V, g) \sim (f', V', g')$. We write the equivalence class of $(f, V, g) = (X \xleftarrow{f} V \xrightarrow{g} Y)$ as $[f, V, g] = [X \xleftarrow{f} V \xrightarrow{g} Y]$. For any $X, Y \in \text{Ob}(\text{Sp}({}_G\text{set}))$, the set of morphisms $\text{Sp}({}_G\text{set})(X, Y)$ has a natural monoid structure. (This is an example of Proposition 16.5 (ii).)

For any category \mathcal{K} , let $[\mathcal{K}, (\text{Set})]_0$ denote the category of covariant functors from \mathcal{K} to the category (Set) of sets, preserving an arbitrary product (whenever the product exists in \mathcal{K}). Here by the word ‘arbitrary product’, we mean a product $\prod_{\lambda \in \Lambda} X_\lambda$ of any objects $X_\lambda \in \text{Ob}(\mathcal{K})$, indexed by an arbitrary set Λ . If $\mathcal{K} = \text{Sp}({}_G\text{set})$, we can view $[\mathcal{K}, (\text{Set})]_0$ also as the category of certain *contravariant* functors by the self-dual nature of $\text{Sp}({}_G\text{set})$. But we use this covariant way, in view of analogy with Tambara functors in the later sections.

Fact 16.1 (cf. Theorem 4 in [19]). Let G be a finite group.

- (1) $\mathcal{SMack}(G)$ is equivalent to $[\mathrm{Sp}(G\mathrm{set}), (\mathrm{Set})]_0$.
- (2) There exists a unique category $\mathcal{B}(G\mathrm{set})$ with arbitrary finite products and a functor $\kappa : \mathrm{Sp}(G\mathrm{set}) \rightarrow \mathcal{B}(G\mathrm{set})$ with the following properties:
 - (a) $\mathrm{Ob}(\mathcal{B}(G\mathrm{set})) = \mathrm{Ob}(\mathrm{Sp}(G\mathrm{set}))$.
 - (b) κ preserves arbitrary products.
 - (c) $\mathcal{B}(G\mathrm{set})(X, Y) = K_0(\mathrm{Sp}(G\mathrm{set})(X, Y))$ for any X, Y in $\mathrm{Ob}(\mathcal{B}(G\mathrm{set}))$, where K_0 denotes the group completion, and the maps of κ

$$\kappa_{X, Y} : \mathrm{Sp}(G\mathrm{set})(X, Y) \rightarrow \mathcal{B}(G\mathrm{set})(X, Y)$$

are the completion maps.

- (3) $\mathrm{Mack}(G)$ is equivalent to $[\mathcal{B}(G\mathrm{set}), (\mathrm{Set})]_0$.

Remark 16.2. For any set of objects $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathrm{Sp}(G\mathrm{set})$ (or $\mathcal{B}(G\mathrm{set})$), their product is of the form $\prod_{\lambda \in \Lambda} X_\lambda$ (cf. Proposition 16.5 (1)). Since $\mathrm{Ob}(\mathrm{Sp}(G\mathrm{set})) = \mathrm{Ob}(\mathcal{B}(G\mathrm{set})) = \mathrm{Ob}(G\mathrm{set})$ consists of finite G -sets, when we consider the product of $(X_\lambda)_{\lambda \in \Lambda}$, then X_λ must be equal to \emptyset except finite $\lambda \in \Lambda$. So, a functor $F : \mathrm{Sp}(G\mathrm{set}) \rightarrow (\mathrm{Set})$ (or $\mathcal{B}(G\mathrm{set}) \rightarrow (\mathrm{Set})$) preserves arbitrary products if and only if F preserves finite products.

Definition 16.3. Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system for an arbitrary group G which satisfies (15.1). We define a category $\mathcal{S} = \mathrm{Sp}(\mathcal{C}, \mathcal{O})$ as follows:

- $\mathrm{Ob}(\mathcal{S}) = \mathrm{Ob}(G\mathrm{Set}_{\mathcal{C}})$,
 - $\mathcal{S}(X, Y)$
- $$= \{(f, V, g) \mid V \in \mathrm{Ob}(G\mathrm{Set}_{\mathcal{C}}), f \in G\mathrm{Set}_{\mathcal{C}}(V, X), g \in G\mathrm{Set}_{\mathcal{C}, \mathcal{O}}(V, Y)\} / \sim_{\text{equiv.}}$$

for any $X, Y \in \mathrm{Ob}(\mathcal{S})$, where $(f, V, g) = (X \xleftarrow{f} V \xrightarrow{g} Y)$ and (f', V', g') are equivalent if and only if there exists an isomorphism v such that $f = f' \circ v$, $g = g' \circ v$.

Let $[f, V, g] = [X \xleftarrow{f} V \xrightarrow{g} Y]$ denote the equivalence class of (f, V, g) . Composition in \mathcal{S} is defined by

$$\begin{aligned} [h, W, k] \circ [f, V, g] &= [\begin{array}{ccccc} & & V \times_Y W & & \\ & & \swarrow h' & \diamond & \searrow g' \\ X & \xleftarrow{f} & V & & W & \xrightarrow{k} & Z \\ & & \searrow g & & \swarrow h & & \end{array}] \\ &= [f \circ h', V \times_Y W, k \circ g'] \end{aligned}$$

for any $[f, V, g] \in \mathcal{S}(X, Y)$ and $[h, W, k] \in \mathcal{S}(Y, Z)$, where

$$\begin{array}{ccc} V \times_Y W & \xrightarrow{g'} & W \\ h' \downarrow & \square & \downarrow h \\ V & \xrightarrow{g} & Y \end{array}$$

is the pull-back diagram.

To distinguish, we write the morphism in \mathcal{S} by an arrow $X \rightarrow Y$.

For any $f \in \mathcal{G}\text{Set}_{\mathcal{C}}(X, Y)$ and $g \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$, we use the notation

$$\begin{aligned} R_f &:= [f, X, \text{id}_X] : Y \rightarrow X \\ T_g &:= [\text{id}_X, X, g] : X \rightarrow Y. \end{aligned}$$

Any morphism $[f, V, g]$ in $\mathcal{S}(X, Y)$ has a decomposition $[f, V, g] = T_g \circ R_f$.

Remark 16.4. (i) *We have*

$$T_{(g' \circ g)} = T_{g'} \circ T_g, \quad R_{(f' \circ f)} = R_f \circ R_{f'}$$

for all composable pairs in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}$ and $\mathcal{G}\text{Set}_{\mathcal{C}}$ respectively, and

$$T_{\text{id}} = \text{id}, \quad R_{\text{id}} = \text{id}.$$

(ii) *If*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ f' \downarrow & \square & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

is a pull-back diagram in $\mathcal{G}\text{Set}_{\mathcal{C}}$ where $g \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$, then we have

$$R_f \circ T_g = T_{g'} \circ R_{f'}.$$

Proof. These can be easily checked directly from the definition of the composition law. \square

In the rest of this section, let G be an arbitrary group and let $(\mathcal{C}, \mathcal{O})$ denote a Mackey system satisfying (15.1), unless otherwise specified.

Proposition 16.5. (i) *For any $X_\lambda \in \text{Ob}(\mathcal{S})$ ($\lambda \in \Lambda$), if we put $X := \coprod_{\lambda \in \Lambda} X_\lambda$, then*

$$(R_{i_\lambda} : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

is their product in \mathcal{S} , where $i_\lambda : X_\lambda \hookrightarrow X$ is the inclusion ($\forall \lambda \in \Lambda$). \emptyset is the terminal object in \mathcal{S} .

(ii) *For any $A, X \in \text{Ob}(\mathcal{S})$, $\mathcal{S}(A, X)$ has a monoid structure defined by*

$$[f_1, V_1, g_1] + [f_2, V_2, g_2] = [f_1 \cup f_2, V_1 \amalg V_2, g_1 \cup g_2]$$

for any $[f_1, V_1, g_1], [f_2, V_2, g_2] \in \mathcal{S}(A, X)$. $[A \leftarrow \emptyset \rightarrow X]$ is the zero in this monoid, and we abbreviate this morphism to 0. Moreover, with this monoid structure, the functor

$$\mathcal{S}(-, X) : \mathcal{S} \rightarrow (\text{Set})$$

factors through the subcategory (Mon) of (Set) .

Proof. (i) It suffices to show that for any $Y \in \text{Ob}(\mathcal{S})$ and any set of morphisms $[f_\lambda, V_\lambda, g_\lambda] : Y \rightarrow X_\lambda$ ($\lambda \in \Lambda$), there exists a unique morphism $[f, V, g] : Y \rightarrow X$ such that

$$R_{i_\lambda} \circ [f, V, g] = [f_\lambda, V_\lambda, g_\lambda] \quad (\forall \lambda \in \Lambda).$$

$$\begin{array}{ccc}
 Y & & \\
 \downarrow [f, V, g] & \searrow [f_\lambda, V_\lambda, g_\lambda] & \\
 X & \xrightarrow{R_{i_\lambda}} & X_\lambda
 \end{array}$$

As for the existence, we can see easily that the morphism

$$[f, V, g] := \left[\bigcup_{\lambda \in \Lambda} f_\lambda, \prod_{\lambda \in \Lambda} V_\lambda, \prod_{\lambda \in \Lambda} g_\lambda \right]$$

satisfies the above commutativity condition. To prove the uniqueness, let $[f', V', g']$ be a morphism satisfying

$$(16.1) \quad R_{i_\lambda} \circ [f', V', g'] = [f_\lambda, V_\lambda, g_\lambda] \quad (\forall \lambda \in \Lambda).$$

Put

$$\begin{aligned}
 V'_\lambda &:= g'^{-1}(X_\lambda), \\
 g'_\lambda &:= g'|_{V'_\lambda}, \\
 i'_\lambda &:= V'_\lambda \hookrightarrow V' \quad \text{inclusion.}
 \end{aligned}$$

Then since $R_{i_\lambda} \circ [f', V', g'] = [f_\lambda, V_\lambda, g_\lambda]$, condition (16.1) is equal to the fact that there exists an isomorphism $v_\lambda : V_\lambda \xrightarrow{\cong} V'_\lambda$ for each $\lambda \in \Lambda$, such that $g_\lambda = g'_\lambda \circ v_\lambda$, $f_\lambda = f' \circ i'_\lambda \circ v_\lambda$.

$$\begin{array}{ccc}
 & V_\lambda & \\
 f_\lambda \swarrow & \downarrow v_\lambda & \searrow g_\lambda \\
 Y & & X_\lambda \\
 f' \circ i'_\lambda \swarrow & \downarrow & \searrow g'_\lambda \\
 & V'_\lambda &
 \end{array}$$

By taking the direct sum, we obtain an isomorphism

$$\prod_{\lambda \in \Lambda} v_\lambda : \prod_{\lambda \in \Lambda} V_\lambda \xrightarrow{\cong} \prod_{\lambda \in \Lambda} V'_\lambda,$$

which makes the following diagram commutative:

$$\begin{array}{ccc}
 & \prod_{\lambda \in \Lambda} V_\lambda & \\
 \bigcup_{\lambda \in \Lambda} f_\lambda \swarrow & \downarrow \prod_{\lambda \in \Lambda} v_\lambda & \searrow \prod_{\lambda \in \Lambda} g_\lambda \\
 Y & & \prod_{\lambda \in \Lambda} X_\lambda \\
 \bigcup_{\lambda \in \Lambda} (f' \circ i'_\lambda) \swarrow & \downarrow & \searrow \prod_{\lambda \in \Lambda} g'_\lambda \\
 & \prod_{\lambda \in \Lambda} V'_\lambda &
 \end{array}$$

But since we have

$$\prod_{\lambda \in \Lambda} V'_\lambda = V', \quad \bigcup_{\lambda \in \Lambda} (f' \circ i'_\lambda) = f', \quad \prod_{\lambda \in \Lambda} g'_\lambda = g',$$

this means $(f, V, g) \xrightarrow[\prod_{\lambda \in \Lambda} v_\lambda]{\sim} (f', V', g')$.

Proof. This follows from Proposition 16.5 (i), Remark 16.4 (i) and the commutativity of the following diagrams:

$$\begin{array}{ccc}
 X \amalg X \amalg X & \xrightarrow{\nabla_X \amalg \text{id}} & X \amalg X \\
 \text{id} \amalg \nabla_X \downarrow & \circlearrowleft & \downarrow \nabla_X \\
 X \amalg X & \xrightarrow{\nabla_X} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \amalg X & \xrightarrow{\nabla_X} & X \\
 \text{tw} \searrow & \circlearrowleft & \nearrow \nabla_X \\
 & X \amalg X &
 \end{array}$$

$$\begin{array}{ccc}
 X \amalg X & \xrightarrow{\nabla_X} & X \\
 \uparrow \iota_2 & \circlearrowleft & \uparrow \nabla_X \\
 X & \xrightarrow{\iota_1} & X \amalg X
 \end{array}$$

Here, $\text{tw} : X \amalg X \rightarrow X \amalg X$ is the twisting map and satisfies

$$T_{\text{tw}} = \text{tw} : X \times X \rightarrow X \times X,$$

and $\iota_i : X \hookrightarrow X \amalg X$ is the inclusion into the i -th component ($i = 1, 2$). \square

Remark 16.7. *By Yoneda's lemma, an object $X \in \text{Ob}(\mathcal{C})$ is a monoid object if and only if the functor*

$$\mathcal{C}(-, X) : \mathcal{C} \rightarrow (\text{Set})$$

factors through the subcategory (Mon) of (Set) .

By Proposition 16.6, every object $X \in \text{Ob}(\mathcal{S})$ is a monoid object in \mathcal{S} . The corresponding monoid structure on $\mathcal{S}(A, X)$ ($\forall A \in \text{Ob}(\mathcal{S})$) is nothing other than (ii) in Proposition 16.5.

An arrow $f \in \mathcal{C}(X, Y)$ between monoid objects is called a morphism of monoid objects if it makes the following diagrams commutative:

$$\begin{array}{ccc}
 X \times X & \xrightarrow{m_X} & X \\
 f \times f \downarrow & \circlearrowleft & \downarrow f \\
 Y \times Y & \xrightarrow{m_Y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 & e_X \nearrow & X \\
 \emptyset & \circlearrowleft & \downarrow f \\
 & e_Y \searrow & Y
 \end{array}$$

Again by Yoneda's lemma, this is equivalent to that

$$f \circ - : \mathcal{C}(-, X) \rightarrow \mathcal{C}(-, Y)$$

is a natural transformation of functors $\mathcal{C} \rightarrow (\text{Mon})$.

Proposition 16.8. (i) *For any $f \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}}(X, Y)$,*

$$R_f : Y \rightarrow X$$

is a morphism of monoid objects.

(ii) *For any $g \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$,*

$$T_g : X \rightarrow Y$$

is a morphism of monoid objects.

Proof. Since $[f, V, g] = T_g \circ R_f$, to show (i) and (ii) is equivalent to show that any morphism in \mathcal{S} is a morphism of monoid objects. But this can be shown in the same way as in the proof of (ii) in Proposition 16.5. \square

Definition 16.9. Let $F : \mathcal{C} \rightarrow (\text{Set})$ be a functor. We say F preserves finite products if, for any finite product $(p_k : X \rightarrow X_k)_{1 \leq k \leq n}$, the induced map

$$(F(p_k))_{1 \leq k \leq n} : F(X) \rightarrow \prod_{1 \leq k \leq n} F(X_k)$$

is an isomorphism.

We can see easily the following:

Proposition 16.10. Let $F : \mathcal{C} \rightarrow (\text{Set})$ be a functor which preserves finite products.

- (i) If $X \in \text{Ob}(\mathcal{C})$ is a monoid object, then $F(X)$ becomes a monoid.
- (ii) If $f \in \mathcal{C}(X, Y)$ is a morphism of monoid objects, then $F(f) : F(X) \rightarrow F(Y)$ becomes a monoid homomorphism.
- (iii) If Y is a monoid object in \mathcal{C} , then for any $X \in \text{Ob}(\mathcal{C})$,

$$F_{X,Y} : \mathcal{C}(X, Y) \rightarrow (\text{Set})(F(X), F(Y))$$

is a monoid homomorphism, where the monoid structure of the right hand side is the one induced from that of $F(Y)$.

Remark 16.11. For any category \mathcal{C} with finite products, we can define commutative group objects, semi-ring objects, and ring objects in \mathcal{C} , in the same way. They satisfy the analogous statement of Remark 16.7, and Proposition 16.10 for any finite-product-preserving functor F .

Theorem 16.12. $\text{SMack}_{(\mathcal{C}, \mathcal{O})}$ is equivalent to $[\text{Sp}(\mathcal{C}, \mathcal{O}), (\text{Set})]_0$.

Proof. In view of Theorem 15.17, we show that there exists an isomorphism of categories

$$\text{SBif}_{(\mathcal{C}, \mathcal{O})} \xrightarrow{\cong} [\mathcal{S}, (\text{Set})]_0.$$

In one direction, for any $M = (M^*, M_*) \in \text{Ob}(\text{SBif}_{(\mathcal{C}, \mathcal{O})})$, we associate a covariant functor $E : \mathcal{S} \rightarrow (\text{Set})$ by

$$\begin{aligned} E(X) &:= M(X) \quad (\forall X \in \text{Ob}(\mathcal{S})) \\ E([f, V, g]) &:= M_*(g)M^*(f) \quad (\forall [f, V, g] \in \mathcal{S}(X, Y)). \end{aligned}$$

This definition of $E([f, V, g])$ is well-defined. Indeed, for any

$$(f, V, g) \xrightarrow[v]{\sim} (f', V', g')$$

we have

$$\begin{aligned} M_*(g)M^*(f) &= M_*(g' \circ v)M^*(f' \circ v) \\ &= M_*(g')M_*(v)M^*(v)M^*(f') \\ &\stackrel{\text{Prop. 15.16}}{=} M_*(g')M^*(f'). \end{aligned}$$

In other words, E is defined by

$$\begin{aligned} E(R_f) &= M^*(f) \quad (\forall f \in {}_G\text{Set}_{\mathcal{C}}(X, Y)) \\ E(T_g) &= M_*(g) \quad (\forall g \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)) \end{aligned}$$

on morphisms. The associativity and unit law for morphisms can be checked easily.

For any set of objects $(X_\lambda)_{\lambda \in \Lambda}$ in ${}_G\text{Set}_{\mathcal{C}}$, since M preserves arbitrary products, we have an isomorphism

$$(M^*(i_\lambda))_{\lambda \in \Lambda} : M(X) \xrightarrow{\cong} \prod_{\lambda \in \Lambda} M(X_\lambda)$$

where $X := \prod_{\lambda \in \Lambda} X_\lambda$ and $i_\lambda : X_\lambda \hookrightarrow X$ is the inclusion. But this is nothing other than the isomorphism

$$(E(R_{i_\lambda}))_{\lambda \in \Lambda} : E(X) \xrightarrow{\cong} \prod_{\lambda \in \Lambda} E(X_\lambda),$$

which means that E preserves arbitrary products. Thus E becomes an object of $[\mathcal{S}, (\text{Set})]_0$. If $E, F \in \text{Ob}([\mathcal{S}, (\text{Set})]_0)$ are the objects corresponding to $M, N \in \text{Ob}(\text{SBif}_{(\mathcal{C}, \mathcal{O})})$ as above, then for any morphism $\varphi : M \rightarrow N$ in $\text{SBif}_{(\mathcal{C}, \mathcal{O})}$, naturally the same φ

$$(\varphi_X : E(X) \rightarrow F(X))_{X \in \text{Ob}(\mathcal{S})}$$

gives a morphism $\varphi : E \rightarrow F$ in $[\mathcal{S}, (\text{Set})]_0$.

In the other direction, if we are given an object $E \in \text{Ob}([\mathcal{S}, (\text{Set})]_0)$, then we define $M = (M^*, M_*)$ by

$$\begin{aligned} M(X) &:= E(X) \quad (\forall X \in \text{Ob}({}_G\text{Set}_{\mathcal{C}})), \\ M^*(f) &:= E(R_f) \quad (\forall f \in {}_G\text{Set}_{\mathcal{C}}(X, Y)), \\ M_*(g) &:= E(T_g) \quad (\forall g \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)). \end{aligned}$$

By Remark 16.4 (i) and Proposition 16.10, we can see easily that M^* is a contravariant functor from ${}_G\text{Set}_{\mathcal{C}}$ to (Mon) , and M_* is a covariant functor from ${}_G\text{Set}_{\mathcal{C}, \mathcal{O}}$ to (Mon) . Conditions (ii) in Definition 15.11 follows from (ii) in Remark 16.4. By the same argument as before, we can see that M preserves arbitrary products since E does. If $M, N \in \text{Ob}(\text{SBif}_{(\mathcal{C}, \mathcal{O})})$ are the objects corresponding to $E, F \in \text{Ob}([\mathcal{S}, (\text{Set})]_0)$ and $\varphi : E \rightarrow F$ is a morphism in $[\mathcal{S}, (\text{Set})]_0$, then φ_X is a monoid homomorphism for each $X \in \text{Ob}(\mathcal{S})$. In fact, for each $X \in \text{Ob}(\mathcal{S})$, the diagram

$$\begin{array}{ccc} E(X \amalg X) & \xrightarrow[\cong]{(E(R_{i_1}), E(R_{i_2}))} & E(X) \times E(X) \\ \varphi_{X \amalg X} \downarrow & \circlearrowleft & \downarrow \varphi_X \times \varphi_X \\ F(X \amalg X) & \xrightarrow[\cong]{(F(R_{i_1}), F(R_{i_2}))} & F(X) \times F(X) \end{array}$$

is commutative where $\iota_j : X \hookrightarrow X \amalg X$ is the inclusion into j -th component ($j = 1, 2$). From this, we can show that the diagram

$$\begin{array}{ccc} E(X) \times E(X) & \xrightarrow{\varphi_X \times \varphi_X} & F(X) \times F(X) \\ \text{product} \downarrow & \circlearrowleft & \downarrow \text{product} \\ E(X) & \xrightarrow{\varphi_X} & F(X) \end{array}$$

is commutative, which means that φ_X is a monoid homomorphism. Thus, for any morphism $\varphi : E \rightarrow F$ in $[\mathcal{S}, (\text{Set})]_0$, naturally the same φ

$$(\varphi_X : M(X) \rightarrow N(X))_{X \in \text{Ob}(\mathcal{C}\text{Set}_{\mathcal{C}})}$$

gives a morphism in $\mathcal{SBif}_{\mathcal{C}, \mathcal{O}}$. It can be seen easily that these correspondences $M \leftrightarrow E$ gives an isomorphism of categories

$$\mathcal{SBif}_{(\mathcal{C}, \mathcal{O})} \xrightarrow{\cong} [\text{Sp}(\mathcal{C}, \mathcal{O}), (\text{Set})]_0.$$

□

Definition 16.13. We define a category $\mathcal{B} = \mathcal{B}(\mathcal{C}, \mathcal{O})$ as follows:

- $\text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{S})$,
- $\mathcal{B}(X, Y) = K_0(\mathcal{S}(X, Y))$ for any $X, Y \in \text{Ob}(\mathcal{B})$.

So, any morphism in $\mathcal{B}(X, Y)$ can be written as the difference of the images $\langle \alpha_1 \rangle$, $\langle \alpha_2 \rangle$ of two morphisms α_1 and α_2 in $\mathcal{S}(X, Y)$. \mathcal{B} actually becomes a category with the composition law defined by

$$\begin{aligned} & (\langle \beta_1 \rangle - \langle \beta_2 \rangle) \circ (\langle \alpha_1 \rangle - \langle \alpha_2 \rangle) \\ & := \langle \beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 \rangle - \langle \beta_1 \circ \alpha_2 + \beta_2 \circ \alpha_1 \rangle, \end{aligned}$$

where $\alpha_i \in \mathcal{S}(X, Y)$, $\beta_i \in \mathcal{S}(Y, Z)$ ($i = 1, 2$). The well-definedness of this composition can be shown easily, via the fact that the composition in \mathcal{S}

$$\circ : \mathcal{S}(X, Y) \times \mathcal{S}(Y, Z) \rightarrow \mathcal{S}(X, Z)$$

is biadditive. By this definition, the composition in \mathcal{B}

$$\circ : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \rightarrow \mathcal{B}(X, Z)$$

also becomes biadditive.

There is a functor $\kappa : \mathcal{S} \rightarrow \mathcal{B}$ defined by $\kappa(F : X \rightarrow Y) := (\langle F \rangle : X \rightarrow Y)$. For any $X_\lambda \in \text{Ob}(\mathcal{S})$ ($\lambda \in \Lambda$), since $(R_{i_\lambda} : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$ is their product in \mathcal{S} , we have an isomorphism of sets

$$(- \circ R_{i_\lambda})_{\lambda \in \Lambda} : \mathcal{S}(Y, X) \xrightarrow{\cong} \prod_{\lambda \in \Lambda} \mathcal{S}(Y, X_\lambda)$$

for each object $Y \in \text{Ob}(\mathcal{S})$. By taking K_0 of this isomorphism, we obtain an isomorphism for each $Y \in \text{Ob}(\mathcal{B}) = \text{Ob}(\mathcal{S})$

$$\begin{aligned} (- \circ \langle R_{i_\lambda} \rangle)_{\lambda \in \Lambda} : \mathcal{B}(Y, X) & \xrightarrow{\cong} K_0\left(\prod_{\lambda \in \Lambda} \mathcal{S}(Y, X_\lambda)\right) \\ & = \prod_{\lambda \in \Lambda} \mathcal{B}(Y, X_\lambda), \end{aligned}$$

which means that

$$(\langle R_{i_\lambda} \rangle : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

is the product of $(X_\lambda)_{\lambda \in \Lambda}$. Thus κ preserves arbitrary products.

Theorem 16.14. $\text{Mack}_{(\mathcal{C}, \mathcal{O})}$ is equivalent to $[\mathcal{B}, (\text{Set})]_0$.

Proof. As before, we show the equivalence of $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$ with $[\mathcal{B}, (\text{Set})]_0$. Let $\tau : [\mathcal{S}, (\text{Set})]_0 \xrightarrow{\cong} \mathcal{SBif}_{(\mathcal{C}, \mathcal{O})}$ denote the isomorphism constructed in Theorem 16.12. Let $\kappa^\# : [\mathcal{B}, (\text{Set})]_0 \rightarrow [\mathcal{S}, (\text{Set})]_0$ be the composition by κ .

$$\begin{array}{ccc} [\mathcal{B}, (\text{Set})]_0 & & \text{Bif}_{(\mathcal{C}, \mathcal{O})} \\ \kappa^\# \downarrow & & \downarrow \text{fully faithful} \\ [\mathcal{S}, (\text{Set})]_0 & \xrightarrow[\tau]{\cong} & \mathcal{SBif}_{(\mathcal{C}, \mathcal{O})} \end{array}$$

Let E be any object in $\text{Ob}([\mathcal{B}, (\text{Set})]_0)$. Since $X \in \mathcal{B}$ is a commutative group object by Remark 16.11, $(\kappa^\# E)(X) = E \circ \kappa(X)$ becomes an abelian group. So we have $\tau(\kappa^\#(\text{Ob}([\mathcal{B}, (\text{Set})]_0))) \subset \text{Ob}(\text{Bif}_{(\mathcal{C}, \mathcal{O})})$. Thus we obtain a functor

$$\bar{\tau} : [\mathcal{B}, (\text{Set})]_0 \rightarrow \text{Bif}_{(\mathcal{C}, \mathcal{O})}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} [\mathcal{B}, (\text{Set})]_0 & \xrightarrow{\bar{\tau}} & \text{Bif}_{(\mathcal{C}, \mathcal{O})} \\ \kappa^\# \downarrow & \circlearrowleft & \downarrow \text{fully faithful} \\ [\mathcal{S}, (\text{Set})]_0 & \xrightarrow[\tau]{\cong} & \mathcal{SBif}_{(\mathcal{C}, \mathcal{O})} \end{array}$$

Let E be any element in $[\mathcal{S}, (\text{Set})]_0$. Then $\tau(E)$ belongs to $\text{Bif}_{(\mathcal{C}, \mathcal{O})}$ if and only if $E(X)$ is an abelian group for any $X \in \text{Ob}(\mathcal{S})$. In this case, if we define $\bar{E} \in \text{Ob}([\mathcal{B}, (\text{Set})]_0)$ by

$$\begin{aligned} \bar{E}(X) &:= E(X) \quad (\forall X \in \text{Ob}(\mathcal{B})) \\ \bar{E}(\langle \alpha_1 \rangle - \langle \alpha_2 \rangle) &:= E(\langle \alpha_1 \rangle) - E(\langle \alpha_2 \rangle) \quad (\forall \alpha_1, \alpha_2 \in \mathcal{S}(X, Y)), \end{aligned}$$

then we have $E = \kappa^\#(\bar{E})$. Thus $\bar{\tau}$ is essentially surjective.

To show $\bar{\tau}$ is fully faithful, it suffices to show that $\kappa^\#$ is fully faithful. Let $T_1, T_2 \in \text{Ob}([\mathcal{B}, (\text{Set})]_0)$ be any elements, and consider the map

$$\begin{array}{ccc} [\mathcal{B}, (\text{Set})]_0(T_1, T_2) & \longrightarrow & [\mathcal{S}, (\text{Set})]_0(\kappa^\# T_1, \kappa^\# T_2) \\ \cup & & \cup \\ (\varphi_X)_{X \in \text{Ob}(\mathcal{B})} & \longmapsto & (\varphi_{\kappa(X)})_{X \in \text{Ob}(\mathcal{S})} \end{array}$$

We show this map is bijective. Injectivity is trivial, since κ is identity on objects. To show surjectivity, let $\varphi = (\varphi_X : \kappa^\# T_1(X) \rightarrow \kappa^\# T_2(X))_{X \in \text{Ob}(\mathcal{S})}$ be any element in $[\mathcal{S}, (\text{Set})]_0(\kappa^\# T_1, \kappa^\# T_2)$. By the naturality of φ , we have

$$\kappa^\# T_2(\alpha) \circ \varphi_X = \varphi_Y \circ \kappa^\# T_1(\alpha)$$

for any $\alpha \in \mathcal{S}(X, Y)$. Remark that $\kappa^{\sharp}T_j(\alpha) = T_j(\langle \alpha \rangle)$ ($j = 1, 2$).

$$\begin{array}{ccc} T_1(X) & \xrightarrow{\varphi_X} & T_2(X) \\ T_1(\langle \alpha \rangle) \downarrow & \circ & \downarrow T_2(\langle \alpha \rangle) \\ T_1(Y) & \xrightarrow{\varphi_Y} & T_2(Y) \end{array}$$

It suffices to show φ is also natural with respect to any element $\langle \alpha_1 \rangle - \langle \alpha_2 \rangle \in \mathcal{B}(X, Y)$ ($\forall \alpha_1, \alpha_2 \in \mathcal{S}(X, Y)$). By (the proof of) Theorem 16.12, φ_X is a monoid homomorphism ($\forall X \in \text{Ob}(\mathcal{S}) = \text{Ob}(\mathcal{B})$). Moreover,

$$(T_j)_{X, Y} : \mathcal{B}(X, Y) \longrightarrow (\text{Ab})(T_j(X), T_j(Y))$$

is a group homomorphism (cf. Proposition 16.10 (ii)) ($\forall j = 1, 2$). From these, we can show easily

$$T_2(\langle \alpha_1 \rangle - \langle \alpha_2 \rangle) \circ \varphi_X = \varphi_Y \circ T_1(\langle \alpha_1 \rangle - \langle \alpha_2 \rangle).$$

□

Example 16.15. Let $(\mathcal{C}, \mathcal{O})$ be a Mackey system for G which satisfies (15.1) and $G \in \mathcal{C}$. In this case, one-point set $\{*\} = G/G$ is the terminal object in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$. By Theorem 16.14, we obtain a Mackey functor $\Omega_{(\mathcal{C}, \mathcal{O})}$ corresponding to the Hom-functor $\mathcal{B}(\{*\}, -) \in [\mathcal{B}, (\text{Set})]_0$. For each $X \in \text{Ob}(\mathcal{S})$, since the monoid

$$\begin{aligned} & \mathcal{S}(\{*\}, X) \\ &= \{(\{*\} \leftarrow V \xrightarrow{g} X) \mid V \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}}), g \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}(V, X)\} / \sim_{\text{equiv.}} \\ &= \{(V \xrightarrow{g} X) \mid V \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}), g \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}(V, X)\} / \sim_{\text{equiv.}} \end{aligned}$$

is the set of isomorphism classes of the comma category ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}/X$, we have

$$\mathcal{B}(\{*\}, X) = K_0({}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}/X),$$

where K_0 denotes the Grothendieck group of ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}/X$. Thus $\Omega_{(\mathcal{C}, \mathcal{O})}$ satisfies

$$\Omega_{(\mathcal{C}, \mathcal{O})}(X) = K_0({}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}/X)$$

for each $X \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}})$.

Definition 16.16. We call $\Omega_{(\mathcal{C}, \mathcal{O})}$ the Burnside functor on the Mackey system $(\mathcal{C}, \mathcal{O})$.

Remark 16.17. $\Omega_{(\mathcal{C}, \mathcal{O})}(X)$ is in fact a ring, with a multiplication induced from the fiber product over X in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}}$. In particular if G is finite and $\mathcal{O} = \mathcal{O}_{\mathcal{C}}$, then $\Omega_{(\mathcal{C}, \mathcal{O})}(\{*\}) = \Omega_{(\mathcal{C}, \mathcal{O})}(G/G)$ agrees with the generalized Burnside ring $\Omega(G, \mathcal{C})$ defined in [33].

17. TAMBARA FUNCTORS ON MACKEY SYSTEMS

Let G be an arbitrary group. We consider a triplet $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$, where $(\mathcal{C}, \mathcal{O}_+)$ and $(\mathcal{C}, \mathcal{O}_\bullet)$ are Mackey systems for G . For $X \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}})$, we define a category $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+|X}$ as follows:

$$\begin{aligned} - \text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_+|X}) &= \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_+}/X) = \{(A \xrightarrow{\xi} X) \mid A \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_+}) = \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}}), \\ & \xi \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(A, X)\}, \end{aligned}$$

- $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X}((A \xrightarrow{\xi} X), (A' \xrightarrow{\xi'} X)) = \{\zeta \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}}(A, A') = {}_{\mathcal{G}}\text{Set}(A, A') \mid \xi' \circ \zeta = \xi\}$
for any $(A \xrightarrow{\xi} X), (A' \xrightarrow{\xi'} X)$ in $\text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X})$.

Remark that for any $X \in \text{Ob}({}_{\mathcal{G}}\text{Set}_{\mathcal{C}})$, $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X}$ is a full subcategory of ${}_{\mathcal{G}}\text{Set}/X$.

Definition 17.1. We call $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ a Tambara system if it satisfies the following condition:

For any $\eta \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$, the pull-back functor defined by η

$$X \times_Y - : \mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | Y} \rightarrow \mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X}$$

has a right adjoint

$$\varpi_\eta : \mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X} \rightarrow \mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | Y}.$$

If ϖ_η exists, we write the object $\varpi_\eta((A \xrightarrow{\xi} X)) \in \mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | Y}$ abbreviately as $(\varpi_\eta(A) \xrightarrow{v(\xi)} Y)$.

As in the case where G is finite [30], it can be easily seen for any (possibly infinite) group G that the pull-back functor defined by $f \in {}_{\mathcal{G}}\text{Set}(X, Y)$

$$X \times_Y - : {}_{\mathcal{G}}\text{Set}/Y \rightarrow {}_{\mathcal{G}}\text{Set}/X$$

always has a right adjoint. The construction is as follows:

- For any $(A \xrightarrow{p} X) \in \text{Ob}({}_{\mathcal{G}}\text{Set}/X)$, we define

$$\begin{aligned} \Pi_f(A) &:= \{(y, \sigma) \mid y \in Y, \sigma : f^{-1}(y) \rightarrow A \text{ map}, p \circ \sigma = \text{id}_{f^{-1}(y)}\}, \\ q : \Pi_f(A) &\rightarrow Y, \quad q(y, \sigma) := y. \end{aligned}$$

$\Pi_f(A)$ is a G -set by $g \cdot (y, \sigma) = (gy, {}^g\sigma)$, where ${}^g\sigma$ is defined by $({}^g\sigma)(x) = g\sigma(g^{-1}x)$ for any $x \in f^{-1}(gy)$. We sometimes write this q as $\pi(p)$ or $\pi_f(p)$. We define abbreviately

$$\Pi_f((A \xrightarrow{p} X)) := (\Pi_f(A) \xrightarrow{q} Y) \quad (\in \text{Ob}({}_{\mathcal{G}}\text{Set}/Y)).$$

- For any morphism $a : (A \xrightarrow{p} X) \rightarrow (A' \xrightarrow{p'} X')$ in ${}_{\mathcal{G}}\text{Set}/X$, we define a morphism

$$\Pi_f(a) : \Pi_f((A \xrightarrow{p} X)) \rightarrow \Pi_f((A' \xrightarrow{p'} X'))$$

in ${}_{\mathcal{G}}\text{Set}/Y$ by $\Pi_f(a)((y, \sigma)) := (y, a \circ \sigma)$.

Then, this functor

$$\Pi_f : {}_{\mathcal{G}}\text{Set}/X \rightarrow {}_{\mathcal{G}}\text{Set}/Y$$

is right adjoint to $X \times_Y -$.

Lemma 17.2. Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ be a triplet where $(\mathcal{C}, \mathcal{O}_+)$ and $(\mathcal{C}, \mathcal{O}_\bullet)$ are Mackey systems. This triplet $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ becomes a Tambara system if it satisfies the following condition for any $X, Y \in \text{Ob}({}_{\mathcal{G}}\text{Set}_{\mathcal{C}})$ and any $\eta \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$:

For any $(A \xrightarrow{\xi} X)$ in $\text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X})$,

$$(17.1) \quad \Pi_\eta(A) \in \text{Ob}({}_{\mathcal{G}}\text{Set}_{\mathcal{C}}), \quad \pi(\xi) \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(\Pi_\eta(A), Y).$$

Proof. Since $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X}$ (resp. $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | Y}$) is a full subcategory of ${}_{\mathcal{G}}\text{Set}/X$ (resp. ${}_{\mathcal{G}}\text{Set}/Y$), the triplet $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ becomes a Tambara system if for any $\eta \in {}_{\mathcal{G}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$ and any $(A \xrightarrow{\xi} X) \in \text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | X})$, the object $(\Pi_\eta(A) \xrightarrow{\pi(\xi)} Y) \in \text{Ob}({}_{\mathcal{G}}\text{Set}/Y)$ belongs to $\text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_+ | Y})$. This is equivalent to condition (17.1). \square

Thus if condition (17.1) is satisfied, then ϖ_η can be taken as Π_η .

Definition 17.3. Let $\tilde{\mathcal{C}}$ be a set of subsets of G , which is closed under left and right translation, finite intersections and finite unions (hence $\emptyset \in \tilde{\mathcal{C}}$). To $\tilde{\mathcal{C}}$, we associate a set \mathcal{C} of subgroups of G by

$$\mathcal{C} := \{H \in \tilde{\mathcal{C}} \mid H \leq G\}.$$

Then \mathcal{C} is closed under conjugation and finite intersections. In this case, we say ‘ \mathcal{C} arises from $\tilde{\mathcal{C}}$ ’. Whenever we say ‘ \mathcal{C} arises from $\tilde{\mathcal{C}}$ ’, we always assume that $\tilde{\mathcal{C}}$ satisfies the above condition.

For a given $\tilde{\mathcal{C}}$, we often consider a Mackey system $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, where \mathcal{C} arises from $\tilde{\mathcal{C}}$ as above.

Example 17.4. Let G be a topological group. If $\tilde{\mathcal{C}} = \{\forall \text{closed subset of } G\}$ (resp. $\{\forall \text{open subset of } G\}$), then the above $(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is the natural (resp. open-natural) Mackey system for G .

From a Mackey system satisfying (15.1), we can construct a Tambara system.

Proposition 17.5. *Let $(\mathcal{C}, \mathcal{O}_{\bullet})$ be a Mackey system satisfying (15.1), and assume \mathcal{C} arises from some $\tilde{\mathcal{C}}$, then $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$ becomes a Tambara system.*

Proof. By Lemma 17.2, it suffices to show

- (1) $\Pi_{\eta}(A) \in \text{Ob}({}_{\mathcal{C}}\text{Set}_{\mathcal{C}})$
- (2) $\pi(\xi) \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}(\Pi_{\eta}(A), Y)$

for any $\eta \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}(X, Y)$ and any $(A \xrightarrow{\xi} X)$ in $\text{Ob}(\mathcal{S}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}|X})$.

proof of (1)

This is equivalent to

$$G_{(y, \sigma)} \in \tilde{\mathcal{C}} \quad (\forall (y, \sigma) \in \Pi_{\eta}(A)).$$

We have

$$\begin{aligned} & G_{(y, \sigma)} \\ &= \{g \in G \mid g \cdot (y, \sigma) = (y, \sigma)\} \\ &= \bigcap_{x \in \eta^{-1}(y)} \{g \in G_y \mid g\sigma(x) = \sigma(gx)\} \\ &= \bigcap_{x \in \eta^{-1}(y)} \left(\bigcup_{a \in \xi^{-1}(\eta^{-1}(y))} (\{g \in G_y \mid g\sigma(x) = a\} \cap \{g \in G_y \mid \sigma(gx) = a\}) \right) \\ &= \bigcap_{x \in \eta^{-1}(y)} \left(\bigcup_{a \in \xi^{-1}(\eta^{-1}(y))} (L_{x, a} \cap R_{x, a}) \right), \end{aligned}$$

where

$$\begin{aligned} L_{x, a} &:= \{g \in G_y \mid g\sigma(x) = a\}, \\ R_{x, a} &:= \{g \in G_y \mid \sigma(gx) = a\}. \end{aligned}$$

So, it suffices to show

$$L_{x, a}, R_{x, a} \in \tilde{\mathcal{C}} \quad (\forall x \in \eta^{-1}(y), \forall a \in \xi^{-1}(\eta^{-1}(y))).$$

If $L_{x,a} = \emptyset$, then $L_{x,a} \in \tilde{\mathcal{C}}$. Otherwise we can take an element $g_0 \in L_{x,a}$, and

$$\begin{aligned} L_{x,a} &= \{g \in G_y \mid g\sigma(x) = g_0\sigma(x)\} \\ &= G_y \cap g_0 \cdot G_{\sigma(x)} \in \tilde{\mathcal{C}}. \end{aligned}$$

If $\sigma^{-1}(a) = \emptyset$, then $R_{x,a} = \emptyset \in \tilde{\mathcal{C}}$. Otherwise, since σ is injective, there exists a unique element $x_0 \in \eta^{-1}(y)$ such that $\sigma(x_0) = a$. So we have

$$R_{x,a} = \{g \in G_y \mid gx = x_0\}.$$

By a similar argument for $L_{x,a}$, we obtain $R_{x,a} \in \tilde{\mathcal{C}}$.

proof of (2)

By Remark 15.8, it suffices to show that $\pi(\xi)$ has finite fibers. But since

$$\pi(\xi)^{-1}(y) = \{\sigma \mid \sigma : \eta^{-1}(y) \rightarrow A \text{ map, } \xi \circ \sigma = \text{id}_{\eta^{-1}(y)}\}$$

for any $y \in Y$, this follows from the fact that ξ has finite fibers. \square

Tambara systems we mainly consider are of this type.

Example 17.6. If G is a topological group and $(\mathcal{C}, \mathcal{O}_\bullet)$ is the natural (resp. open-natural) Mackey system, then $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$ is a Tambara system. We call this $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$ the natural (resp. open-natural) Tambara system.

Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ be a Tambara system and let $\xi \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(A, X)$ and $\eta \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$ be any elements. We construct a certain commutative diagram from ξ and η , called an exponential diagram. Take ϖ_η of the object $(A \xrightarrow{\xi} X)$ in $\mathcal{S}_{\mathcal{C}, \mathcal{O}_+|X}$, and take the fiber product of η and $v(\xi)$.

$$\begin{array}{ccc} & & A \\ & \swarrow \xi & \\ X & \xleftarrow{v(\xi)'} & X \times_Y \varpi_\eta(A) \\ \eta \downarrow & \square & \downarrow \eta' \\ Y & \xleftarrow{v(\xi)} & \varpi_\eta(A) \end{array}$$

By the adjoint isomorphism

$$\begin{aligned} \mathcal{S}_{\mathcal{C}, \mathcal{O}_+|X}((X \times_Y \varpi_\eta(A) \xrightarrow{v(\xi)'} X), (A \xrightarrow{\xi} X)) \\ \cong \mathcal{S}_{\mathcal{C}, \mathcal{O}_+|Y}((\varpi_\eta(A) \xrightarrow{v(\xi)} Y), (\varpi_\eta(A) \xrightarrow{v(\xi)} Y)), \end{aligned}$$

there exists a morphism $\zeta : X \times_Y \varpi_\eta(A) \rightarrow A$ corresponding to $\text{id}_{\varpi_\eta(A)}$. Thus we obtain a commutative diagram in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$

$$(17.2) \quad \begin{array}{ccc} X & \xleftarrow{\xi} & A \xleftarrow{\zeta} X \times_Y \varpi_\eta(A) \\ \eta \downarrow & \circlearrowleft & \downarrow \eta' \\ Y & \xleftarrow{v(\xi)} & \varpi_\eta(A) \end{array}$$

where η' is the pull-back of η by $v(\xi)$.

Definition 17.7. A commutative diagram

$$(17.3) \quad \begin{array}{ccccc} X & \xleftarrow{\xi} & Z & \xleftarrow{\zeta} & X' \\ \eta \downarrow & & \circ & & \downarrow \eta' \\ Y & \xleftarrow{\pi} & & & Y' \end{array}$$

in ${}_G\text{Set}_{\mathcal{C}}$ where π, ξ are morphisms in ${}_G\text{Set}_{\mathcal{C}, \mathcal{O}_+}$ and η, η' are in ${}_G\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}$ is called an exponential diagram if it is isomorphic to one of the diagrams (17.2) constructed above. We write

$$\begin{array}{ccccc} X & \xleftarrow{\quad} & Z & \xleftarrow{\quad} & X' \\ \downarrow & & \text{exp} & & \downarrow \\ Y & \xleftarrow{\quad} & & & Y' \end{array}$$

to indicate that it is an exponential diagram.

Example 17.8. Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)$ be a Tambara system where $(\mathcal{C}, \mathcal{O}_+)$ and $(\mathcal{C}, \mathcal{O}_\bullet)$ satisfy (15.1). The following commutative diagrams are exponential diagrams.

(i)

$$\begin{array}{ccccc} X & \xleftarrow{\xi} & A & \xleftarrow{\text{id}} & A \\ \text{id} \downarrow & & \text{exp} & & \downarrow \text{id} \\ X & \xleftarrow{\xi} & & & A \end{array}$$

(ii)

$$\begin{array}{ccccc} X \amalg X & \xleftarrow{\text{id} \amalg \nabla} & X \amalg X \amalg X & \xleftarrow{(\nabla \amalg \text{id} \amalg \text{id}) \circ (\text{id} \amalg \text{tw} \amalg \text{id})} & X \amalg X \amalg X \amalg X \\ \nabla \downarrow & & \text{exp} & & \downarrow \nabla \amalg \nabla \\ X & \xleftarrow{\nabla} & & & X \amalg X \end{array}$$

(iii) Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$ be a Tambara system as in Proposition 17.5. The exponential diagram constructed from $\nabla \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}(X \amalg X, X)$ and $\eta \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$ is

$$\begin{array}{ccccc} X & \xleftarrow{\nabla} & X \amalg X & \xleftarrow{\gamma \amalg \gamma'} & U \amalg U' \\ \eta \downarrow & & \text{exp} & & \downarrow \tau \cup \tau' \\ Y & \xleftarrow{\sigma} & & & V \end{array},$$

where

$$\begin{aligned} V &= \{(y, C) \mid y \in Y, C \subset \eta^{-1}(y)\}, \\ U &= \{(x, C) \mid x \in X, C \subset \eta^{-1}(\eta(x)), x \in C\}, \\ U' &= \{(x, C) \mid x \in X, C \subset \eta^{-1}(\eta(x)), x \notin C\}, \end{aligned}$$

$$\begin{aligned}
\gamma &: U \rightarrow X, & (x, C) &\mapsto x, \\
\gamma' &: U' \rightarrow X, & (x, C) &\mapsto x, \\
\tau &: U \rightarrow V, & (x, C) &\mapsto (\eta(x), C), \\
\tau' &: U' \rightarrow V, & (x, C) &\mapsto (\eta(x), C), \\
\sigma &: V \rightarrow Y, & (y, C) &\mapsto y.
\end{aligned}$$

These are in $\mathcal{G}\text{Set}_{\mathcal{C}}$. Moreover, it can be easily checked that τ, τ' are morphisms in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}$, and σ is a morphism in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}$.

Definition 17.9. In the notation of (iii) in Remark 17.8, the commutative diagrams

$$\begin{array}{ccc}
X & \xleftarrow{\gamma} & U \\
\eta \downarrow & \circlearrowleft & \downarrow \tau \\
Y & \xleftarrow{\sigma} & V
\end{array}
, \quad
\begin{array}{ccc}
X & \xleftarrow{\gamma'} & U' \\
\eta \downarrow & \circlearrowleft & \downarrow \tau' \\
Y & \xleftarrow{\sigma} & V
\end{array}$$

are called T-diagram and F-diagram of η , respectively.

Now we define a (semi-)Tambara functor on a Mackey system. For the original definition of a Tambara functor on a finite group, see [30].

Definition 17.10. Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})$ be a Tambara system. A semi-Tambara functor S on $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})$ is a function which assigns

- a semi-ring $S(X)$ to each $X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})$,
- a homomorphism of additive monoids $\xi_+ : S(X) \rightarrow S(Y)$ to each $\xi \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(X, Y)$,
- a homomorphism of multiplicative monoids $\eta_{\bullet} : S(X) \rightarrow S(Y)$ to each $\eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}(X, Y)$,
- a semi-ring homomorphism $\zeta^* : S(Y) \rightarrow S(X)$ to each $\zeta \in \mathcal{G}\text{Set}_{\mathcal{C}}(X, Y)$,

in such a way that the following conditions are satisfied:

- (i) For any direct sum decomposition $X = \coprod_{\lambda \in \Lambda} X_{\lambda}$ in $\mathcal{G}\text{Set}_{\mathcal{C}}$, the natural map

$$(i_{\lambda}^*)_{\lambda \in \Lambda} : S(X) \rightarrow \prod_{\lambda \in \Lambda} S(X_{\lambda})$$

is an isomorphism of sets, where $i_{\lambda} : X_{\lambda} \rightarrow X$ ($\lambda \in \Lambda$) are the inclusions. $S(\emptyset)$ consists of a single element.

- (ii)

$$(\xi' \circ \xi)_+ = \xi'_+ \circ \xi_+, \quad (\eta' \circ \eta)_{\bullet} = \eta'_{\bullet} \circ \eta_{\bullet}, \quad (\zeta' \circ \zeta)^* = \zeta^* \circ \zeta'^*$$

for all composable pairs in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}$, $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}$, $\mathcal{G}\text{Set}_{\mathcal{C}}$ respectively, and

$$\text{id}_+ = \text{id}_{\bullet} = \text{id}^* = \text{id}.$$

- (iii) If

$$\begin{array}{ccc}
X' & \xrightarrow{\xi'} & Y' \\
\zeta' \downarrow & \square & \downarrow \zeta \\
X & \xrightarrow{\xi} & Y
\end{array}$$

is a pull-back diagram in $\mathcal{G}\text{Set}_{\mathcal{C}}$ where $\xi \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(X, Y)$, then $\zeta^* \circ \xi_+ = \xi'_+ \circ \zeta'^*$.

(iv) If

$$\begin{array}{ccc} X' & \xrightarrow{\eta'} & Y' \\ \zeta' \downarrow & \square & \downarrow \zeta \\ X & \xrightarrow{\eta} & Y \end{array}$$

is a pull-back diagram in $\mathcal{G}\text{Set}_{\mathcal{C}}$ where $\eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$, then $\zeta^* \circ \eta_\bullet = \eta'_\bullet \circ \zeta'^*$.

(v) For any exponential diagram (17.3), $\eta_\bullet \circ \xi_+ = \pi_+ \circ \eta'_\bullet \circ \zeta^*$.

If all $S(X)$ are rings, S is called a Tambara functor.

Remark 17.11. The conditions from (i) to (iv) can be written in terms of Mackey functors as follows:

(I) The morphisms ξ_+ , η_\bullet , ζ^* yield functors

- a covariant functor $S_+ : \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+} \rightarrow (\text{Set})$,

- a covariant functor $S_\bullet : \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet} \rightarrow (\text{Set})$,

- a contravariant functor $S^* : \mathcal{G}\text{Set}_{\mathcal{C}} \rightarrow (\text{Set})$.

(II) The pairs (S^*, S_+) and (S^*, S_\bullet) are Mackey functors on $(\mathcal{C}, \mathcal{O}_+)$ and $(\mathcal{C}, \mathcal{O}_\bullet)$, with respect to the additive and multiplicative structures of $S(X)$ ($X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})$), respectively.

Proposition 17.12. Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$ be a Tambara system as in Proposition 17.5, and let S be a semi-Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$. For any $\eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$, consider the T - and F -diagrams as in Definition 17.9:

$$\begin{array}{ccc} X & \xleftarrow{\gamma} & U \\ \eta \downarrow & \circlearrowleft & \downarrow \tau \\ Y & \xleftarrow{\sigma} & V \end{array}, \quad \begin{array}{ccc} X & \xleftarrow{\gamma'} & U' \\ \eta \downarrow & \circlearrowleft & \downarrow \tau' \\ Y & \xleftarrow{\sigma} & V \end{array}$$

Then we have

$$\eta_\bullet(x + y) = \sigma_+((\tau_\bullet \circ \gamma^*(x)) \cdot (\tau'_\bullet \circ \gamma'^*(y))) \quad (\forall x, y \in S(X)).$$

Proof. Let $\iota_i : X \hookrightarrow X \amalg X$ and $\iota'_i : V \hookrightarrow V \amalg V$ be the inclusions into the i -th components ($i = 1, 2$). Since

$$\begin{array}{ccc} X & \xleftarrow{\nabla} & X \amalg X \xleftarrow{\gamma \amalg \gamma'} & U \amalg U' \\ \eta \downarrow & & \text{exp} & \downarrow \tau \cup \tau' \\ Y & \xleftarrow{\sigma} & V \end{array},$$

is an exponential diagram, we have

$$\eta_\bullet \circ \nabla_+ = \sigma_+ \circ (\tau \cup \tau')_\bullet \circ (\gamma \amalg \gamma')^* = \sigma_+ \circ \nabla_\bullet \circ (\tau \amalg \tau')_\bullet \circ (\gamma \amalg \gamma')^*.$$

By Remark 15.13, for any $x, y \in S(X)$ we have

$$(\iota_1^*, \iota_2^*)^{-1}(x, y) = \iota_{1+}(x) + \iota_{2+}(y) = \iota_{1\bullet}(x) \cdot \iota_{2\bullet}(y).$$

So we have

$$\eta_\bullet \circ \nabla_+((\iota_1^*, \iota_2^*)^{-1}(x, y)) = \eta_\bullet \circ \nabla_+(\iota_{1+}(x) + \iota_{2+}(y)) = \eta_\bullet(x + y),$$

and

$$\begin{aligned}
& \nabla_{\bullet} \circ (\tau \amalg \tau')_{\bullet} \circ (\gamma \amalg \gamma')^* ((\iota_1^*, \iota_2^*)^{-1}(x, y)) \\
&= \nabla_{\bullet} \circ (\tau \amalg \tau')_{\bullet} \circ (\gamma \amalg \gamma')^* (\iota_{1\bullet}(x) \cdot \iota_{2\bullet}(y)) \\
&= \nabla_{\bullet} (\iota'_{1\bullet}(\tau_{\bullet} \circ \gamma^*(x)) \cdot \iota'_{2\bullet}(\tau'_{\bullet} \circ \gamma'^*(y))) \\
&= (\tau_{\bullet} \circ \gamma^*(x)) \cdot (\tau'_{\bullet} \circ \gamma'^*(y)).
\end{aligned}$$

Thus we obtain

$$\eta_{\bullet}(x + y) = \sigma_+((\tau_{\bullet} \circ \gamma^*(x)) \cdot (\tau'_{\bullet} \circ \gamma'^*(y))).$$

□

Definition 17.13. Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})$ be a Tambara system, and S, T be two Tambara functors (resp. semi-Tambara functors). A morphism φ from S to T is a collection of semi-ring morphisms $\varphi_X : S(X) \rightarrow T(X)$ ($X \in \text{Ob}({}_G\text{Set}_{\mathcal{C}})$), which commute with all $\xi_+, \eta_{\bullet}, \zeta^*$.

$\text{Tam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})}$ (resp. $\text{STam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})}$) denotes the category of Tambara (resp. semi-Tambara) functors and their morphisms. $\text{Tam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})}$ is a full subcategory of $\text{STam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})}$.

Definition 17.14. Let $(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})$ be a Tambara system, where $(\mathcal{C}, \mathcal{O}_+)$ and $(\mathcal{C}, \mathcal{O}_{\bullet})$ are Mackey systems satisfying (15.1). We define a category $\mathcal{U} = \mathcal{U}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_{\bullet})}$ as follows:

- $\text{Ob}(\mathcal{U}) = \text{Ob}({}_G\text{Set}_{\mathcal{C}})$
- For any $X, Y \in \text{Ob}(\mathcal{U})$, the set of morphisms $\mathcal{U}(X, Y)$ is

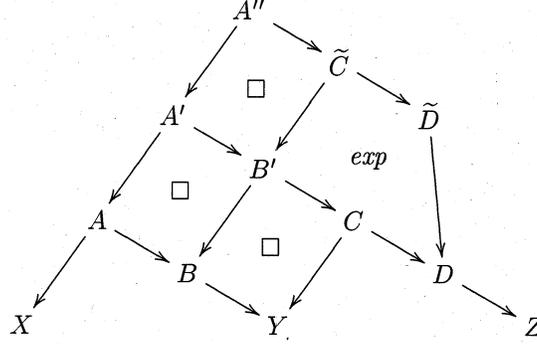
$$\{(X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y) \mid \begin{array}{l} A, B \in \text{Ob}({}_G\text{Set}_{\mathcal{C}}), \xi \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_+}(B, Y) \\ \eta \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}(A, B), \zeta \in {}_G\text{Set}_{\mathcal{C}}(A, X) \end{array}\} / \sim_{\text{equiv.}}$$

where $(X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y)$ and $(X \xleftarrow{\zeta'} A' \xrightarrow{\eta'} B' \xrightarrow{\xi'} Y)$ are equivalent if and only if there exists a pair of isomorphisms $a : A \rightarrow A'$ and $b : B \rightarrow B'$ such that $\xi = \xi' \circ b$, $b \circ \eta = \eta' \circ a$, $\zeta = \zeta' \circ a$.

$$\begin{array}{ccccc}
& & A & \xrightarrow{\eta} & B & & \\
& \zeta & \swarrow & & \searrow & \xi & \\
X & & & & & & Y \\
& \zeta' & \swarrow & & \searrow & \xi' & \\
& & A' & \xrightarrow{\eta'} & B' & &
\end{array}$$

Let $[X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y]$ denote the equivalence class of $(X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y)$.

Composition law is defined by $[Y \leftarrow C \rightarrow D \rightarrow Z] \circ [X \leftarrow A \rightarrow B \rightarrow Y] = [X \leftarrow A'' \rightarrow \tilde{D} \rightarrow Z]$, with the morphisms appearing in the following diagram:



Then \mathcal{U} becomes in fact a category. This can be shown in the same way as in the case of finite groups [30]. We write $X \rightarrow Y$ to indicate the morphism in \mathcal{U} .

For any $X, Y \in \text{Ob}(\mathcal{U})$, we use the notation

$$- T_\xi := [X \xleftarrow{\text{id}} X \xrightarrow{\text{id}} X \xrightarrow{\xi} Y] \text{ for any } \xi \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(X, Y),$$

$$- N_\eta := [X \xleftarrow{\text{id}} X \xrightarrow{\eta} Y \xrightarrow{\text{id}} Y] \text{ for any } \eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y),$$

$$- R_\zeta := [X \xleftarrow{\zeta} Y \xrightarrow{\text{id}} Y \xrightarrow{\text{id}} Y] \text{ for any } \zeta \in \mathcal{G}\text{Set}_{\mathcal{C}}(Y, X).$$

As follows, \mathcal{U} has analogous properties as in the case of a finite group [30].

Remark 17.15. (cf. Proposition 7.2. in [30])

(i) Any morphism $[X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y]$ in $\mathcal{U}(X, Y)$ admits a decomposition $[X \xleftarrow{\zeta} A \xrightarrow{\eta} B \xrightarrow{\xi} Y] = T_\xi \circ N_\eta \circ R_\zeta$.

(ii) $\xi \mapsto T_\xi$ defines a covariant functor $T : \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+} \rightarrow \mathcal{U}$. Similarly, we have a covariant functor $N : \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet} \rightarrow \mathcal{U}$ and a contravariant functor $R : \mathcal{G}\text{Set}_{\mathcal{C}} \rightarrow \mathcal{U}$.

(iii) If

$$\begin{array}{ccc} X' & \xrightarrow{\xi'} & Y' \\ \zeta' \downarrow & \square & \downarrow \zeta \\ X & \xrightarrow{\xi} & Y \end{array}$$

is a pull-back diagram in $\mathcal{G}\text{Set}_{\mathcal{C}}$ where ξ is a morphism in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}$, then $R_\zeta \circ T_\xi = T_{\xi'} \circ R_{\zeta'}$.

If

$$\begin{array}{ccc} X' & \xrightarrow{\eta'} & Y' \\ \zeta' \downarrow & \square & \downarrow \zeta \\ X & \xrightarrow{\eta} & Y \end{array}$$

is a pull-back diagram in $\mathcal{G}\text{Set}_{\mathcal{C}}$ where η is a morphism in $\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}$, then $R_\zeta \circ N_\eta = N_{\eta'} \circ R_{\zeta'}$.

(iv) If

$$\begin{array}{ccccc}
 X & \xleftarrow{\xi} & Z & \xleftarrow{\zeta} & X' \\
 \eta \downarrow & & \circ & & \downarrow \eta' \\
 Y & \xleftarrow{\pi} & & & Y'
 \end{array}$$

is an exponential diagram, then $N_\eta \circ T_\xi = T_\pi \circ N_{\eta'} \circ R_\zeta$.

Proof. These can be easily checked directly from the definition of the composition law. \square

Proposition 17.16. (cf. Proposition 7.5. in [30])

(i) For any set of objects $(X_\lambda)_{\lambda \in \Lambda}$ in \mathcal{U} ,

$$(R_{i_\lambda} : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

is their product in \mathcal{U} , where $X := \prod_{\lambda \in \Lambda} X_\lambda$ and $i_\lambda : X_\lambda \hookrightarrow X$ is the inclusion. \emptyset is the terminal object in \mathcal{U} . Thus, arbitrary products exist in \mathcal{U} .

(ii) Any $X \in \text{Ob}(\mathcal{U})$ has a structure of a semi-ring object of \mathcal{U} with addition T_∇ , additive unit T_{i_\emptyset} , multiplication N_∇ , multiplicative unit N_{i_\emptyset} .

(iii) For any $\xi \in {}_G\text{Set}_{C, \mathcal{O}_+}(X, Y)$, T_ξ preserves the additive structure on X and Y .

For any $\eta \in {}_G\text{Set}_{C, \mathcal{O}_\bullet}(X, Y)$, N_η preserves the multiplicative structure on X and Y .

For any $\zeta \in {}_G\text{Set}_C(X, Y)$, R_ζ preserves the semi-ring structure on X and Y .

Proof. proof of (i)

This is proved essentially in the same way as in the proof of Proposition 16.5 (i). For any $Y \in \text{Ob}(\mathcal{U})$ and any set of morphisms

$$f_\lambda = [Y \xleftarrow{\zeta_\lambda} A_\lambda \xrightarrow{\eta_\lambda} B_\lambda \xrightarrow{\xi_\lambda} X_\lambda] : Y \rightarrow X_\lambda \quad (\lambda \in \Lambda),$$

we define $f : Y \rightarrow X$ by

$$f := [Y \xleftarrow{\zeta} \prod_{\lambda \in \Lambda} A_\lambda \xrightarrow{\eta} \prod_{\lambda \in \Lambda} B_\lambda \xrightarrow{\xi} X],$$

where $\zeta = \bigcup_{\lambda \in \Lambda} \zeta_\lambda$, $\eta = \prod_{\lambda \in \Lambda} \eta_\lambda$, $\xi = \prod_{\lambda \in \Lambda} \xi_\lambda$. Then, by virtue of Example 17.8 (i), we have

$$R_{i_\lambda} \circ f = f_\lambda \quad (\forall \lambda \in \Lambda).$$

For any morphism $f' = [Y \xleftarrow{\zeta'} A' \xrightarrow{\eta'} B' \xrightarrow{\xi'} X]$ which satisfies

$$(17.4) \quad R_{i_\lambda} \circ f' = f_\lambda \quad (\forall \lambda \in \Lambda),$$

put

$$\begin{aligned}
 A'_\lambda &:= (\xi' \circ \eta')^{-1}(X_\lambda) \\
 B'_\lambda &:= \xi'^{-1}(X_\lambda) \\
 \zeta'_\lambda &:= \zeta'|_{A'_\lambda} : A'_\lambda \rightarrow Y \\
 \eta'_\lambda &:= \eta'|_{A'_\lambda} : A'_\lambda \rightarrow B'_\lambda \\
 \xi'_\lambda &:= \xi'|_{B'_\lambda} : B'_\lambda \rightarrow X_\lambda.
 \end{aligned}$$

Then we have $R_{i_\lambda} \circ f' = [Y \xleftarrow{\zeta'_\lambda} A'_\lambda \xrightarrow{\eta'_\lambda} B'_\lambda \xrightarrow{\xi'_\lambda} X]$, and by (17.4), there exists a pair of isomorphisms (a_λ, b_λ) such that

$$\begin{array}{ccccc} & & A_\lambda & \xrightarrow{\eta_\lambda} & B_\lambda & & & & \\ & \zeta_\lambda & \downarrow & & \downarrow & \xi_\lambda & & & \\ Y & \circlearrowleft & & & & \circlearrowright & & & X_\lambda \\ & \zeta'_\lambda & \downarrow a_\lambda & \circlearrowleft & \circlearrowright b_\lambda & \downarrow & & & \\ & & A'_\lambda & \xrightarrow{\eta'_\lambda} & B'_\lambda & & & & \end{array}$$

By taking the direct sum, we obtain isomorphisms

$$\begin{aligned} a &= \prod_{\lambda \in \Lambda} a_\lambda : A \rightarrow A' \\ b &= \prod_{\lambda \in \Lambda} b_\lambda : B \rightarrow B', \end{aligned}$$

which makes the following diagram commutative:

$$\begin{array}{ccccc} & & A & \xrightarrow{\eta} & B_\lambda & & & & \\ & \zeta & \downarrow & & \downarrow & \xi & & & \\ Y & \circlearrowleft & & & & \circlearrowright & & & X \\ & \zeta' & \downarrow a & \circlearrowleft & \circlearrowright b & \downarrow & & & \\ & & A' & \xrightarrow{\eta'} & B' & & & & \end{array}$$

This means $f = f'$.

proof of (ii)

By Proposition 16.6, it suffices to show the distributive law. But this follows from Remark 17.15 (iv), applied to the exponential diagram (ii) in Example 17.8.

proof of (iii)

This immediately follows from Proposition 16.8. \square

Theorem 17.17. (cf. Proposition 7.7. in [30])

There is an isomorphism of categories

$$\mathcal{STam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)} \xrightarrow{\cong} [\mathcal{U}, (\text{Set})]_0.$$

Proof. Objects $S \in \text{Ob}(\mathcal{STam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)})$ and $\Sigma \in \text{Ob}([\mathcal{U}, (\text{Set})]_0)$ correspond to each other by

$$\begin{aligned} S(X) &= \Sigma(X) \quad (\forall X \in \text{Ob}({}_\mathcal{G}\text{Set}_\mathcal{C}) = \text{Ob}(\mathcal{U})) \\ \xi_+ &= \Sigma(T_\xi) \quad (\forall \xi \in {}_\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(X, Y)) \\ \eta_\bullet &= \Sigma(N_\eta) \quad (\forall \eta \in {}_\mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)) \\ \zeta^* &= \Sigma(R_\zeta) \quad (\forall \zeta \in {}_\mathcal{G}\text{Set}_\mathcal{C}(X, Y)). \end{aligned}$$

For any semi-Tambara functors S_1, S_2 and corresponding Σ_1, Σ_2 in $[\mathcal{U}, (\text{Set})]_0$, their morphisms $\varphi = (\varphi_X : S_1(X) \rightarrow S_2(X))_{X \in \text{Ob}({}_\mathcal{G}\text{Set}_\mathcal{C})}$ in $\mathcal{STam}_{(\mathcal{C}, \mathcal{O}_+, \mathcal{O}_\bullet)}$ and $\varphi = (\varphi_X : \Sigma_1(X) \rightarrow \Sigma_2(X))_{X \in \text{Ob}(\mathcal{U})}$ in $[\mathcal{U}, (\text{Set})]_0$ correspond in the obvious way. Details can be checked by using (the proof of) Theorem 16.12. \square

By this theorem, we identify semi-Tambara functors and their morphisms with the objects and morphisms in $[\mathcal{U}, (\text{Set})]_0$.

18. FROM SEMI-TAMBARA FUNCTORS TO TAMBARA FUNCTORS

Throughout this section, let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$ be a Tambara system as in Proposition 17.5. As in the case of a finite group [30], we construct a Tambara functor γS from a semi-Tambara functor S .

We first recall the definition and some properties of algebraic maps from [8].

Definition 18.1. (section 5.6. in [8]) Let A be an additive monoid, M be an abelian group, and $\varphi : A \rightarrow M$ be a map. For any n elements $a_1, a_2, \dots, a_n \in A$, a map $D_{(a_1, a_2, \dots, a_n)}\varphi : A \rightarrow M$ is defined by

$$D_{(a_1, \dots, a_n)}\varphi(x) = \sum_{k=0}^n \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} (-1)^{n-k} \varphi(x + a_{i_1} + \dots + a_{i_k}) \right).$$

When $n = 1$, we abbreviate $D_{(a)}\varphi$ to $D_a\varphi$. Remark that $D_a\varphi(x) = \varphi(x+a) - \varphi(x)$. The map φ is said to be algebraic if either $\varphi \equiv 0$ or there exists a positive integer n such that

$$D_{(a_1, \dots, a_n)}\varphi \equiv 0 \quad (\forall a_1, \dots, a_n \in A).$$

For any algebraic map $\varphi : A \rightarrow M$, its degree is defined by

$$\deg \varphi := \max\{n \in \mathbb{N}_{\geq 0} \mid \exists a_1, \dots, a_n \in A \text{ such that } D_{(a_1, \dots, a_n)}\varphi \not\equiv 0\}$$

if $\varphi \not\equiv 0$, and $\deg \varphi := -1$ if $\varphi \equiv 0$.

By this definition, φ is algebraic of degree $\leq n$ if and only if $D_a\varphi$ is algebraic of degree $\leq n - 1$ for all $a \in A$.

Remark 18.2. (i) $D_{(a_1, \dots, a_n)}\varphi$ does not depend on the order of a_1, \dots, a_n .

(ii) φ is algebraic of degree ≤ 0 if and only if φ is constant.

(iii) φ is algebraic of degree ≤ 1 if and only if φ is a sum of a constant map and an additive homomorphism.

Especially, an additive homomorphism is an algebraic map of degree ≤ 1 .

Proposition 18.3. (Lemma 5.6.15 in [8]) Let $\varphi : A \rightarrow M$ be an algebraic map from an additive monoid A to an abelian group M . Let $\kappa_A : A \rightarrow K_0A$ be the group completion of A . Then there exists a unique extension $\tilde{\varphi} : K_0A \rightarrow M$ of φ as an algebraic map, i.e. a unique algebraic map $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \kappa_A = \varphi$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & M \\ & \searrow \kappa_A & \nearrow \tilde{\varphi} \\ & & K_0A \end{array}$$

Moreover this $\tilde{\varphi}$ satisfies $\deg \tilde{\varphi} = \deg \varphi$.

From this proposition and Lemma 5.6.13 in [8], we have the following:

Remark 18.4. Let A be an additive monoid, and M, N be abelian groups. If $\varphi : A \rightarrow M$ and $\psi : M \rightarrow N$ are algebraic maps of degree m and n respectively, then $\psi \circ \varphi : A \rightarrow N$ becomes algebraic of degree $\leq mn$.

For the later use, we want to use the word ‘algebraic map’ also in the case where the codomain is an additive monoid. So, we make the following definition.

Definition 18.5. Let $\varphi : A \rightarrow B$ be a map between additive monoids A and B . We say φ is algebraic of degree n if $\kappa_B \circ \varphi : A \rightarrow K_0B$ is algebraic of degree n , where $\kappa_B : B \rightarrow K_0B$ is the group completion map.

Remark 18.6. Let A be a semi-ring. For any element $a \in A$, the multiplication map by a

$$\begin{array}{ccc} A & \longrightarrow & A \\ \cup & & \cup \\ x & \longmapsto & ax \end{array}$$

is algebraic of degree ≤ 1 , with respect to the additive structure of A .

Proof. This follows from Remark 18.2 (iii). \square

Proposition 18.7. Let $\varphi : A \rightarrow B$ be an algebraic map between additive monoids A and B . Then there exists a unique extension $\tilde{\varphi} : K_0A \rightarrow K_0B$ of φ as an algebraic map, i.e. a unique algebraic map $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \kappa_A = \kappa_B \circ \varphi$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \kappa_A \downarrow & \circlearrowleft & \downarrow \kappa_B \\ K_0A & \xrightarrow{\tilde{\varphi}} & K_0B \end{array}$$

Moreover this $\tilde{\varphi}$ satisfies $\deg \tilde{\varphi} = \deg \varphi$.

Proof. By Proposition 18.3, there exists a unique algebraic map $\widetilde{(\kappa_B \circ \varphi)} : K_0A \rightarrow K_0B$ such that $\widetilde{(\kappa_B \circ \varphi)} \circ \kappa_A = \kappa_B \circ \varphi$.

$$\begin{array}{ccc} A & \xrightarrow{\kappa_B \circ \varphi} & K_0B \\ \kappa_A \searrow & \circlearrowleft & \nearrow \widetilde{(\kappa_B \circ \varphi)} \\ & K_0A & \end{array}$$

If we abbreviate $\widetilde{(\kappa_B \circ \varphi)}$ to $\tilde{\varphi}$, then this $\tilde{\varphi}$ satisfies the desired conditions. \square

Remark 18.8. Let S be a semi-Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$.

(i) For any $\xi \in \mathcal{O}\text{Set}_{\mathcal{C}, \mathcal{O}_+}(X, Y)$, $\xi_+ : S(X) \rightarrow S(Y)$ has an extension

$$\tilde{\xi}_+ : K_0S(X) \rightarrow K_0S(Y)$$

as an algebraic map, which is also an additive homomorphism.

(ii) For any $\zeta \in \mathcal{O}\text{Set}_{\mathcal{C}}(X, Y)$, $\zeta^* : S(Y) \rightarrow S(X)$ has an extension

$$\tilde{\zeta}^* : K_0S(Y) \rightarrow K_0S(X)$$

as an algebraic map, which is also a semi-ring homomorphism.

Proof. Indeed, the natural extensions $\tilde{\xi}_+ = K_0(\xi_+)$ and $\tilde{\zeta}^* = K_0(\zeta^*)$ give the desired maps, where K_0 denotes the functor from the category of semi-rings to the category of commutative rings, defined by the completion of semi-rings. \square

We want to extend the maps η_{\bullet} ($\eta \in \mathcal{O}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}(X, Y)$).

Remark 18.9. Let A, B, C be additive monoids. If $\varphi : A \rightarrow B$ and $\psi : B \rightarrow C$ are algebraic maps of degree m and n respectively, then $\psi \circ \varphi : A \rightarrow C$ becomes algebraic of degree $\leq mn$.

Proof. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \\ & & \downarrow \kappa_B & \circlearrowleft & \downarrow \kappa_C \\ & & K_0B & \xrightarrow{\tilde{\psi}} & K_0C \end{array} ,$$

where $\tilde{\psi} : K_0B \rightarrow K_0C$ is the unique extension of ψ as an algebraic map, and satisfies $\deg \tilde{\psi} = \deg \psi = n$. Since by definition $\kappa_B \circ \varphi$ is algebraic of degree $\leq m$, the composition $\tilde{\psi} \circ (\kappa_B \circ \varphi)$ becomes algebraic of degree $\leq mn$ by Remark 18.4. Thus $\kappa_C \circ (\psi \circ \varphi) = \tilde{\psi} \circ \kappa_B \circ \varphi$ is algebraic of degree $\leq mn$, i.e. $\psi \circ \varphi : A \rightarrow C$ is algebraic of degree $\leq mn$. \square

Proposition 18.10. Let A, B be semi-rings, let $\varphi : A \rightarrow B$ be an algebraic map with respect to the additive structures of A and B , and let $\tilde{\varphi} : K_0A \rightarrow K_0B$ denote the unique extension of φ as an algebraic map. If φ is moreover a multiplicative homomorphism, then $\tilde{\varphi}$ also becomes multiplicative.

Proof. For any $\alpha \in K_0A$, consider the maps

$$\begin{array}{ccc} \tilde{\varphi}_\alpha : K_0A & \longrightarrow & K_0B \\ \cup & & \cup \\ x \mapsto & \longrightarrow & \tilde{\varphi}(\alpha x) \end{array}$$

and

$$\begin{array}{ccc} \tilde{\varphi}^\alpha : K_0A & \longrightarrow & K_0B \\ \cup & & \cup \\ x \mapsto & \longrightarrow & \tilde{\varphi}(\alpha)\tilde{\varphi}(x). \end{array}$$

These are algebraic by Remark 18.6 and Remark 18.9. If $\alpha = a \in A$, then each of $\tilde{\varphi}_a$ and $\tilde{\varphi}^a$ is an extension as an algebraic map of

$$\begin{array}{ccc} A & \longrightarrow & K_0B \\ \cup & & \cup \\ x \mapsto & \longrightarrow & \varphi(ax) = \varphi(a)\varphi(x) , \end{array}$$

which is also algebraic. By the uniqueness of the extension, we obtain $\tilde{\varphi}_a = \tilde{\varphi}^a$, i.e.

$$(18.1) \quad \tilde{\varphi}(ax) = \tilde{\varphi}(a)\tilde{\varphi}(x) \quad (\forall a \in A, x \in K_0A)$$

Then for any $\alpha \in K_0A$, each of the maps

$$\tilde{\varphi}_\alpha, \tilde{\varphi}^\alpha : K_0A \rightarrow K_0B$$

is an extension as an algebraic map of

$$\begin{array}{ccc} A & \longrightarrow & K_0B \\ \cup & & \cup \\ x \mapsto & \longrightarrow & \tilde{\varphi}(x\alpha) \stackrel{(18.1)}{=} \tilde{\varphi}(x)\tilde{\varphi}(\alpha) , \end{array}$$

which is also algebraic. Thus by the uniqueness, we obtain $\tilde{\varphi}^\alpha = \tilde{\varphi}_\alpha$, i.e.

$$\tilde{\varphi}(\alpha x) = \tilde{\varphi}(\alpha)\tilde{\varphi}(x) \quad (\forall \alpha, x \in K_0 A).$$

This means $\tilde{\varphi}$ is multiplicative. \square

Let $f \in {}_G\text{Set}_{\mathcal{C}}(X, Y)$ be a morphism with finite fibers and assume Y consists of one orbit. We define the degree of f by $\deg f = \#f^{-1}(y)$. Here y is a point of Y , and $\deg f$ does not depend on the choice of $y \in Y$.

Proposition 18.11. *Let S be a Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$, and let $X, Y \in \text{Ob}({}_G\text{Set}_{\mathcal{C}})$, $\eta \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$. If Y consists of one orbit, then $\eta_\bullet : S(X) \rightarrow S(Y)$ is algebraic, and satisfies $\deg(\eta_\bullet) \leq \deg(\eta)$.*

Proof. Let

$$\begin{array}{ccc} X & \xleftarrow{\gamma} & U \\ \eta \downarrow & \circlearrowleft & \downarrow \tau \\ Y & \xleftarrow{\sigma} & V \end{array}$$

be the T-diagram of η as in Definition 17.9. Put $V_k := \{(y, C) \in V \mid \#C = k\}$ ($k \in \mathbb{N}_{\geq 0}$). If we put $\deg \eta = n$, we have $V_k = \emptyset$ ($k > n$), and

$$V = \coprod_{0 \leq k \leq n} V_k.$$

Thus if $\iota_k : V_k \hookrightarrow V$ denotes the inclusion, we have an isomorphism

$$(18.2) \quad (\iota_k^*)_{0 \leq k \leq n} : S(V) \xrightarrow{\cong} \bigoplus_{0 \leq k \leq n} S(V_k).$$

Similarly, put $U_k := \{(x, C) \mid \#C = k\}$ and let $\iota'_k : U_k \hookrightarrow U$ be the inclusion ($0 \leq k \leq n$).

$\sigma \circ \iota_n : V_n \rightarrow Y$ is an isomorphism, and the inverse is

$$\begin{array}{ccc} \rho_n : Y & \xrightarrow{\quad} & V_n \\ \Downarrow & & \Downarrow \\ y & \xrightarrow{\quad} & (y, \eta^{-1}(y)). \end{array}$$

Put $\rho := \iota_n \circ \rho_n : Y \rightarrow V$ and $\chi := \tau_\bullet \circ \gamma^* : S(X) \rightarrow S(V)$. Let χ_k denote the k -th component of χ :

$$\chi_k := \iota_k^* \circ \chi : S(X) \rightarrow S(V_k) \quad (0 \leq k \leq n).$$

Lemma 18.12. $\eta_\bullet = \rho^* \circ \chi = \rho_n^* \circ \chi_n$.

Proof. (Proof of Lemma 18.12) Put $\tau_n := \tau|_{U_n} : U_n \rightarrow V_n$. Since

$$\begin{array}{ccc} U & \xleftarrow{\iota'_n} & U_n \\ \tau \downarrow & \square & \downarrow \tau_n \\ V & \xleftarrow{\iota_n} & V_n \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xleftarrow{\gamma \circ \iota'_n} & U_n \\ \eta \downarrow & \square & \downarrow \tau_n \\ Y & \xleftarrow{\sigma \circ \iota_n} & V_n \end{array}$$

are the pull-back diagrams, we have

$$\begin{aligned}
\eta_{\bullet} &= (\sigma \circ \iota_n)^{* -1} \tau_{n\bullet} (\gamma \circ \iota'_n)^* = \rho_n^* \tau_{n\bullet} (\gamma \circ \iota'_n)^* \\
&= \rho_n^* \iota_n^* \tau_{\bullet} \gamma^* = \rho_n^* \iota_n^* \chi \\
&= \rho_n^* \chi_n = \rho^* \chi.
\end{aligned}$$

□

By this lemma, to show that η_{\bullet} is algebraic of degree $\leq n$, it suffices to show χ_n is algebraic of degree $\leq n$. (Remark that ρ_n^* is algebraic of degree ≤ 1 .) In fact, we show the following by induction:

Lemma 18.13. χ_k is algebraic of degree $\leq k$ ($0 \leq \forall k \leq n$).

Since

$$\begin{array}{ccc}
U & \longleftarrow & \emptyset \\
\tau \downarrow & \square & \downarrow \\
V & \xleftarrow{\iota_0} & V_0
\end{array}$$

is a pull-back diagram, we have

$$(18.3) \quad \chi_0(x) = \iota_0^* \tau_{\bullet} \gamma^*(x) = 1 \quad (\forall x \in S(X)).$$

Assume χ_s is algebraic of degree $\leq s$ ($\forall s \leq k-1$). The T- and F-diagrams of τ are naturally isomorphic to

$$\begin{array}{ccc}
U \xleftarrow{\ell_1} U_1^{(2)} & & U \xleftarrow{\ell_2} U_2^{(2)} \\
\tau \downarrow \quad \circ \quad \downarrow \nu_1 & \text{and} & \tau \downarrow \quad \circ \quad \downarrow \nu_2 \\
V \xleftarrow{\mu} V^{(2)} & & V \xleftarrow{\mu} V^{(2)}
\end{array}$$

respectively, where

$$\begin{aligned}
V^{(2)} &:= \{(y, C_1, C_2) \mid y \in Y, C_1, C_2 \subset \eta^{-1}(y), C_1 \cap C_2 = \emptyset\} \\
U_1^{(2)} &:= \{(x, C_1, C_2) \mid (\eta(x), C_1, C_2) \in V^{(2)}, x \in C_1\} \\
U_2^{(2)} &:= \{(x, C_1, C_2) \mid (\eta(x), C_1, C_2) \in V^{(2)}, x \in C_2\} \\
\nu_i(x, C_1, C_2) &:= (\eta(x), C_1, C_2) \quad (i = 1, 2) \\
\ell_i(x, C_1, C_2) &:= (x, C_1 \amalg C_2) \quad (i = 1, 2) \\
\mu(y, C_1, C_2) &:= (y, C_1 \amalg C_2).
\end{aligned}$$

By Proposition 17.12, we have

$$\tau_{\bullet}(u_1 + u_2) = \mu_+(\nu_{1\bullet} \ell_1^*(u_1) \cdot \nu_{2\bullet} \ell_2^*(u_2)) \quad (\forall u_1, u_2 \in S(U)).$$

If we put

$$\begin{aligned}
\pi_i &: V^{(2)} \rightarrow V, \quad \pi_i(y, C_1, C_2) = (y, C_i) \quad (i = 1, 2) \\
\varpi_i &: U_i^{(2)} \rightarrow U, \quad \varpi_i(x, C_1, C_2) = (x, C_i) \quad (i = 1, 2),
\end{aligned}$$

then

$$\begin{array}{ccc} U & \xleftarrow{\varpi_i} & U_i^{(2)} \\ \tau \downarrow & \square & \downarrow \nu_i \\ V & \xleftarrow{\pi_i} & V^{(2)} \end{array}$$

is a pull-back diagram for each $i = 1, 2$. So for any $x \in S(X)$, we have

$$\nu_{i\bullet} \ell_i^*(\gamma^*(x)) = \nu_{i\bullet} \varpi_i^*(\gamma^*(x)) = \pi_i^* \tau_{\bullet} \gamma^*(x) = \pi_i^* \chi(x) \quad (i = 1, 2).$$

So, we obtain

$$\chi(x+a) = \tau_{\bullet}(\gamma^*(x) + \gamma^*(a)) = \mu_+(\pi_1^* \chi(x) \cdot \pi_2^* \chi(a)) \quad (\forall x, a \in S(X)).$$

For any integers $0 \leq s, t \leq k$ with $s+t=k$, put

$$\begin{aligned} V_{s,t}^{(2)} &:= \{(y, C_1, C_2) \in V^{(2)} \mid \#C_1 = s, \#C_2 = t\} \\ \mu_{s,t} &: V_{s,t}^{(2)} \rightarrow V_k, \quad (y, C_1, C_2) \mapsto (y, C_1 \amalg C_2) \\ \iota_{s,t}^{(2)} &: V_{s,t}^{(2)} \hookrightarrow V^{(2)}, \quad \text{inclusion.} \end{aligned}$$

Then since

$$\begin{array}{ccc} \coprod_{s+t=k} V_{s,t}^{(2)} & \xrightarrow{\bigcup_{s+t=k} \mu_{s,t}} & V_k \\ \downarrow \bigcup_{s+t=k} \iota_{s,t}^{(2)} & \square & \downarrow \iota_k \\ V^{(2)} & \xrightarrow{\mu} & V \end{array}$$

is a pull-back diagram, we have

$$\begin{aligned} \iota_k^* \mu_+(\pi_1^* \chi(x) \cdot \pi_2^* \chi(a)) &= \sum_{s+t=k} (\mu_{s,t})_+ (\iota_{s,t}^{(2)})^* (\pi_1^* \chi(x) \cdot \pi_2^* \chi(a)) \\ &= \sum_{s+t=k} (\mu_{s,t})_+ ((\pi_{1,s,t})^* \chi_s(x) \cdot (\pi_{2,s,t})^* \chi_t(a)). \end{aligned}$$

Here $\pi_{1,s,t}$ and $\pi_{2,s,t}$ are defined by

$$\begin{aligned} \pi_{1,s,t} &: V_{s,t}^{(2)} \rightarrow V_s, \quad (y, C_1, C_2) \mapsto (y, C_1) \\ \pi_{2,s,t} &: V_{s,t}^{(2)} \rightarrow V_t, \quad (y, C_1, C_2) \mapsto (y, C_2). \end{aligned}$$

$$\begin{array}{ccc} V_{s,t}^{(2)} & \xrightarrow{\pi_{1,s,t}} & V_s \\ \iota_{s,t}^{(2)} \downarrow & \circlearrowleft & \downarrow \iota_s \\ V^{(2)} & \xrightarrow{\pi_1} & V \end{array} \quad \text{and} \quad \begin{array}{ccc} V_{s,t}^{(2)} & \xrightarrow{\pi_{2,s,t}} & V_t \\ \iota_{s,t}^{(2)} \downarrow & \circlearrowleft & \downarrow \iota_t \\ V^{(2)} & \xrightarrow{\pi_2} & V \end{array}$$

Thus we obtain

$$\chi_k(x+a) = \sum_{s+t=k} (\mu_{s,t})_+ ((\pi_{1,s,t})^* \chi_s(x) \cdot (\pi_{2,s,t})^* \chi_t(a)).$$

Since $\pi_{1,k,0} = \mu_{k,0}$ is an isomorphism, we have

$$\begin{aligned} & (\mu_{k,0})_+((\pi_{1,k,0})^* \chi_k(x) \cdot (\pi_{2,k,0})^* \chi_0(a)) \\ & \stackrel{(18.3)}{=} (\mu_{k,0})_+((\mu_{k,0})^* \chi_k(x) \cdot 1) \\ & = (\mu_{k,0})_+(\mu_{k,0})^* \chi_k(x) \stackrel{\text{Prop 15.16}}{=} \chi_k(x). \end{aligned}$$

So for any $a \in S(X)$ we have

$$\begin{aligned} D_a \chi_k(x) & = \chi_k(x+a) - \chi_k(x) \\ & = \sum_{s=0}^{k-1} (\mu_{s,k-s})_+((\pi_{1,s,k-s})^* \chi_s(x) \cdot A_{k,s}), \end{aligned}$$

where

$$A_{k,s} := (\pi_{2,s,k-s})^* \chi_{k-s}(a) \in S(V_{s,k-s}^{(2)})$$

is a constant independent of x for each $0 \leq s \leq k$. By assumption χ_s is algebraic of degree $\leq s$ ($0 \leq \forall s \leq k-1$). So $D_a \chi_k$ becomes algebraic of degree $\leq k-1$, i.e. χ_k is algebraic of degree $\leq k$. \square

Corollary 18.14. *For any $X, Y \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})$ and any $\eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}}(X, Y)$, there exists a unique extension $\tilde{\eta}_{\bullet} : K_0 S(X) \rightarrow K_0 S(Y)$ of η_{\bullet} such that*

(i) *for any orbit Y_{λ} in Y , $j_{\lambda}^* \circ \tilde{\eta}_{\bullet}$ is the extension of $j_{\lambda}^* \circ \eta_{\bullet}$ as an algebraic map, where $j_{\lambda} : Y_{\lambda} \hookrightarrow Y$ is the inclusion.*

$$K_0(S(X)) \xrightarrow{\tilde{\eta}_{\bullet}} K_0(S(Y)) \xrightarrow{j_{\lambda}^*} K_0(S(Y_{\lambda}))$$

(ii) $\tilde{\eta}_{\bullet}$ is a homomorphism of multiplicative monoids.

Proof. Let $Y = \coprod_{\lambda \in \Lambda} Y_{\lambda}$ be the orbit decomposition and $j_{\lambda} : Y_{\lambda} \hookrightarrow Y$ ($\lambda \in \Lambda$) be the inclusions. For any $\lambda \in \Lambda$, since $\eta_{\bullet, \lambda} := j_{\lambda}^* \circ \eta_{\bullet}$ is algebraic by Proposition 18.11 (via the Mackey condition), there exists a unique extension $\tilde{\eta}_{\bullet, \lambda} : K_0 S(X) \rightarrow K_0 S(Y_{\lambda})$ of $\eta_{\bullet, \lambda}$ as an algebraic map. Moreover, since $\eta_{\bullet, \lambda}$ is a homomorphism of multiplicative monoids, so is $\tilde{\eta}_{\bullet, \lambda}$.

By the isomorphism

$$(j_{\lambda}^*)_{\lambda \in \Lambda} : K_0 S(Y) \xrightarrow{\cong} \prod_{\lambda \in \Lambda} K_0 S(Y_{\lambda}) = K_0(\prod_{\lambda \in \Lambda} S(Y_{\lambda})),$$

we define $\tilde{\eta}_{\bullet} := (j_{\lambda}^*)_{\lambda \in \Lambda}^{-1} \circ (\tilde{\eta}_{\bullet, \lambda})_{\lambda \in \Lambda}$.

$$\begin{array}{ccc} & K_0 S(X) & \\ \tilde{\eta}_{\bullet} \swarrow & & \searrow (\tilde{\eta}_{\bullet, \lambda})_{\lambda \in \Lambda} \\ K_0 S(Y) & \xrightarrow{(j_{\lambda}^*)_{\lambda \in \Lambda}} & \prod_{\lambda \in \Lambda} K_0 S(Y_{\lambda}) \end{array}$$

This satisfies (i) and (ii), and $\tilde{\eta}_{\bullet}$ is unique by the uniqueness of $\tilde{\eta}_{\bullet, \lambda}$ ($\lambda \in \Lambda$). \square

Remark that for any finite subset $\Lambda' \subset \Lambda$, if we put $Y' := \coprod_{\lambda \in \Lambda'} Y_{\lambda} \subset Y$, then the homomorphism of multiplicative monoids $j'^* \circ \eta_{\bullet} : S(X) \rightarrow S(Y')$ is algebraic, where $j' : Y' \hookrightarrow Y$ is the inclusion. This follows from the fact

$$(1) (j_{\lambda}^*)_{\lambda \in \Lambda'} : S(Y') \xrightarrow{\cong} \prod_{\lambda \in \Lambda'} S(Y_{\lambda}), \quad (\text{here, } j_{\lambda} : Y_{\lambda} \hookrightarrow Y')$$

- (2) $j'_\lambda \circ j'^* \circ \eta_\bullet = j'_\lambda \circ \eta_\bullet$ is algebraic for any $\lambda \in \Lambda'$, and
 (3) finite product of algebraic maps is algebraic.

By the same reason, $\widetilde{j}'^* \circ \widetilde{\eta}_\bullet : K_0S(X) \rightarrow K_0S(Y')$ is also algebraic, and thus we have the following:

Lemma 18.15. *Let $\widetilde{\eta}_\bullet$ be the map constructed in Corollary 18.14. For any $Y' \hookrightarrow Y$ as above, $\widetilde{j}'^* \circ \widetilde{\eta}_\bullet : K_0S(X) \rightarrow K_0S(Y')$ is the unique extension of $j'^* \circ \eta_\bullet$ as an algebraic map.*

Theorem 18.16. *Let S be a semi-Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$.*

With $(\gamma S)(X) = K_0S(X)$ and $\widetilde{\xi}_+$, $\widetilde{\eta}_\bullet$, $\widetilde{\zeta}^$ constructed above, γS becomes a Tambara functor.*

Proof. Condition (i), (ii), (iii) in Definition 17.10 concerning $\widetilde{\xi}_+$, $\widetilde{\zeta}^*$ are satisfied by Remark 15.15. So it suffices to show the following:

- (A) For any $\eta \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(X, Y)$ and $\omega \in {}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}(Y, Z)$, we have

$$\widetilde{(\omega \circ \eta)}_\bullet = \widetilde{\omega}_\bullet \circ \widetilde{\eta}_\bullet \quad \text{and} \quad \widetilde{(\text{id}_X)}_\bullet = \text{id}_{K_0(S(X))}.$$

- (B) If

$$\begin{array}{ccc} X' & \xrightarrow{\eta'} & Y' \\ \zeta' \downarrow & \square & \downarrow \zeta \\ X & \xrightarrow{\eta} & Y \end{array}$$

is a pull-back diagram in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}}$ where η is a morphism in ${}_{\mathcal{C}}\text{Set}_{\mathcal{C}, \mathcal{O}_\bullet}$, then $\widetilde{\zeta}^* \circ \widetilde{\eta}_\bullet = \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}'^*$.

- (C) If

$$\begin{array}{ccccc} X & \xleftarrow{\xi} & Z & \xleftarrow{\zeta} & X' \\ \eta \downarrow & & \text{exp} & & \downarrow \eta' \\ Y & \xleftarrow{\pi} & & & Y' \end{array}$$

is an exponential diagram, then $\widetilde{\eta}_\bullet \circ \widetilde{\xi}_+ = \widetilde{\pi}_+ \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}^*$.

proof of (A)

By the direct product decomposition of $S(Z)$, it suffices to show for any orbit $\ell : Z' \hookrightarrow Z$,

$$\widetilde{\ell}^* \circ \widetilde{(\omega \circ \eta)}_\bullet = \widetilde{\ell}^* \circ (\widetilde{\omega}_\bullet \circ \widetilde{\eta}_\bullet).$$

Remark that $\ell^* \circ (\omega \circ \eta)_\bullet = \ell^* \circ (\omega_\bullet \circ \eta_\bullet)$ is algebraic by Proposition 18.11 (via the Mackey condition). If we take the pull-back of ω and ℓ

$$\begin{array}{ccc} Y' & \xrightarrow{\omega'} & Z' \\ \ell' \downarrow & \square & \downarrow \ell \\ Y & \xrightarrow{\omega} & Z \end{array},$$

then (since $\omega \in {}_G\text{Set}_{\mathcal{C}, \circ_\bullet}(Y, Z)$,) $Y' = \omega^{-1}(Z')$ consists of finite number of orbits. By the definition of $\widetilde{\omega}_\bullet$, we have

$$(18.4) \quad \widetilde{\ell}^* \circ \widetilde{\omega}_\bullet = \widetilde{\omega}'_\bullet \circ \widetilde{\ell}'^*$$

$$\begin{array}{ccc} K_0S(X) & & \\ \searrow \widetilde{\eta}_\bullet & \xrightarrow{(\widetilde{\omega \circ \eta})_\bullet} & \\ K_0S(Y) & \xrightarrow{\widetilde{\omega}_\bullet} & K_0S(Z) \\ \downarrow \widetilde{\ell}'^* & \circ & \downarrow \widetilde{\ell}^* \\ K_0S(Y') & \xrightarrow{\widetilde{\omega}'_\bullet} & K_0S(Z') \end{array}$$

By Lemma 18.15, $\widetilde{\ell}^* \circ (\widetilde{\omega \circ \eta})_\bullet$ is the extension of $\ell^* \circ (\omega \circ \eta)_\bullet$ as an algebraic map. On the other hand, by Lemma 18.15 (and Proposition 18.11), $\widetilde{\ell}'^* \circ \widetilde{\eta}_\bullet$ and $\widetilde{\omega}'_\bullet$ are the extensions of $\ell'^* \circ \eta_\bullet$ and ω'_\bullet as algebraic maps, and thus $\widetilde{\omega}'_\bullet \circ (\widetilde{\ell}'^* \circ \widetilde{\eta}_\bullet) \stackrel{(18.4)}{=} \widetilde{\ell}^* \circ \widetilde{\omega}_\bullet \circ \widetilde{\eta}_\bullet$ is the extension of $\ell^* \circ \omega_\bullet \circ \eta_\bullet$ as an algebraic map. So, by the uniqueness of the extension, we obtain

$$\widetilde{\ell}^* \circ (\widetilde{\omega \circ \eta})_\bullet = \widetilde{\ell}^* \circ (\widetilde{\omega}_\bullet \circ \widetilde{\eta}_\bullet).$$

$(\widetilde{\text{id}_X})_\bullet = \text{id}_{K_0(S(X))}$ can be shown in the same way, or directly from the definition of $(\widetilde{\text{id}_X})_\bullet$.
proof of (B)

It suffices to show for any orbit $j'_0 : Y'_0 \hookrightarrow Y'$,

$$\widetilde{j}'_0 \circ (\widetilde{\zeta}^* \circ \widetilde{\eta}_\bullet) = \widetilde{j}'_0 \circ (\widetilde{\eta}'_\bullet \circ \widetilde{\zeta}'^*)$$

Remark that $j'_0 \circ (\zeta^* \circ \eta_\bullet) = j'_0 \circ (\eta'_\bullet \circ \zeta'^*)$ is algebraic by Proposition 18.11. There is an orbit $j_0 : Y_0 \hookrightarrow Y$ containing the image of Y'_0 by ζ . We put $\zeta_0 := \zeta|_{Y'_0} : Y'_0 \rightarrow Y_0$ and take the fiber product of η' and j'_0 :

$$\begin{array}{ccccc} & & X'_0 & \xrightarrow{\eta'_0} & Y'_0 \\ & \swarrow i'_0 & \square & \searrow j'_0 & \downarrow \zeta_0 \\ X' & \xrightarrow{\eta'} & Y' & \downarrow \zeta & Y_0 \\ \downarrow \zeta' & & \square & \downarrow j_0 & \\ X & \xrightarrow{\eta} & Y & & \end{array}$$

Then $\widetilde{j}'_0 \circ \widetilde{\eta}'_\bullet = \widetilde{\eta}'_0 \circ \widetilde{i}'_0$ by (18.4). Since Y'_0 consists of one orbit, the map $\widetilde{\eta}'_0$ is algebraic. Thus $\widetilde{j}'_0 \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}'^* = \widetilde{\eta}'_0 \circ \widetilde{i}'_0 \circ \widetilde{\zeta}'^*$ is algebraic. On the other hand, since

$\widetilde{j}_0^* \circ \widetilde{\eta}_\bullet$ is algebraic by Lemma 18.15, the map $\widetilde{j}_0^* \circ \widetilde{\zeta}^* \circ \widetilde{\eta}_\bullet = \widetilde{\zeta}_0^* \circ \widetilde{j}_0^* \circ \widetilde{\eta}_\bullet$ is algebraic.

$$\begin{array}{ccccc}
 & & K_0S(X'_0) & \xrightarrow{\widetilde{\eta}'_0} & K_0S(Y'_0) \\
 & \nearrow \widetilde{i}'_0 & & & \nearrow \widetilde{j}'_0 \\
 K_0S(X') & \xrightarrow{\widetilde{\eta}'_\bullet} & K_0S(Y') & & K_0S(Y_0) \\
 \uparrow \widetilde{\zeta}'^* & & \uparrow \widetilde{\zeta}^* & & \uparrow \widetilde{\zeta}_0^* \\
 K_0S(X) & \xrightarrow{\widetilde{\eta}_\bullet} & K_0S(Y) & & K_0S(Y_0) \\
 & & & \nearrow \widetilde{j}_0^* &
 \end{array}$$

Thus, each of $\widetilde{j}_0^* \circ \widetilde{\zeta}^* \circ \widetilde{\eta}_\bullet$ and $\widetilde{j}_0^* \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}'^*$ is the extension of $j_0^* \circ \zeta^* \circ \eta_\bullet = j_0^* \circ \eta'_\bullet \circ \zeta'^*$ as an algebraic map, and we obtain $\widetilde{j}_0^* \circ (\widetilde{\zeta}^* \circ \widetilde{\eta}_\bullet) = \widetilde{j}_0^* \circ (\widetilde{\eta}'_\bullet \circ \widetilde{\zeta}'^*)$ by the uniqueness of the extension.

proof of (C)

This can be shown in the same way as (A) and (B). For any orbit $j : Y_0 \hookrightarrow Y$, if we pull it back by π

$$\begin{array}{ccc}
 Y & \xleftarrow{\pi} & Y' \\
 j \uparrow & \square & \uparrow j' \\
 Y_0 & \xleftarrow{\pi_0} & Y'_0
 \end{array} ,$$

then Y'_0 consists of finite number of orbits, and $\widetilde{j}^* \circ \widetilde{\pi}_+ \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}^* = \widetilde{\pi}_{0+} \circ \widetilde{j}'^* \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}^*$ becomes algebraic since $\widetilde{j}'^* \circ \widetilde{\eta}'_\bullet$ is algebraic by Lemma 18.15. Thus $\widetilde{j}^* \circ \widetilde{\eta}_\bullet \circ \widetilde{\xi}_+$ and $\widetilde{j}^* \circ \widetilde{\pi}_+ \circ \widetilde{\eta}'_\bullet \circ \widetilde{\zeta}^*$ are extensions of $j^* \circ \eta_\bullet \circ \xi_+ = j^* \circ \pi_+ \circ \eta'_\bullet \circ \zeta^*$ as algebraic maps, and must agree by the uniqueness of the extension. \square

Thus we obtained a Tambara functor γS . By the construction of γS , the set of completion maps $\kappa = (\kappa_X : S(X) \rightarrow K_0S(X))_{X \in \text{Ob}(\mathcal{C}\text{Set}_c)}$ gives a morphism of semi-Tambara functors $\kappa : S \rightarrow \gamma S$.

Proposition 18.17. *Let S be a semi-Tambara functor and T be a Tambara functor. For any morphism $\varphi : S \rightarrow T$ of semi-Tambara functors, there exists a unique morphism of Tambara functors $\widetilde{\varphi} : \gamma S \rightarrow T$ such that $\widetilde{\varphi} \circ \kappa = \varphi$.*

$$\begin{array}{ccc}
 S & \xrightarrow{\varphi} & T \\
 \searrow \kappa & & \nearrow \widetilde{\varphi} \\
 & \gamma S &
 \end{array}$$

Proof. $\widetilde{\varphi}$ is obviously unique, since the morphism of rings $\widetilde{\varphi}_X : K_0S(X) \rightarrow T(X)$ satisfying $\widetilde{\varphi}_X \circ \kappa_X = \varphi_X$ is unique for each $X \in \text{Ob}(\mathcal{C}\text{Set}_c)$, i.e. $\widetilde{\varphi}_X = K_0(\varphi_X)$. So it suffices to show that $(\widetilde{\varphi}_X)_{X \in \text{Ob}(\mathcal{C}\text{Set}_c)}$ is compatible with all $\widetilde{\xi}_+$, $\widetilde{\eta}_\bullet$, $\widetilde{\zeta}^*$. Since $(\widetilde{\varphi}_X)_{X \in \text{Ob}(\mathcal{C}\text{Set}_c)}$ is indeed compatible with $\widetilde{\xi}_+ = K_0(\xi_+)$ and $\widetilde{\zeta}^* = K_0(\zeta^*)$, it remains to show the compatibility with $\widetilde{\eta}_\bullet$. But this immediately follows from the

fact that for each orbit $j : Y' \hookrightarrow Y$, the following diagram is commutative:

$$\begin{array}{ccccc}
 S(X) & \xrightarrow{\eta_\bullet} & S(Y) & \xrightarrow{j^*} & S(Y') \\
 \kappa_X \downarrow & & \circ & & \downarrow \kappa_{Y'} \\
 K_0 S(X) & \xrightarrow{\tilde{\eta}_\bullet} & K_0 S(Y) & \xrightarrow{\tilde{j}^*} & K_0 S(Y') \\
 & \searrow & \circ & \nearrow & \\
 & & \widetilde{j^* \circ \eta_\bullet} & &
 \end{array}$$

□

By virtue of Theorem 18.16 and Proposition 18.17, with the same proof as in the case of a finite group [30], we can show the following theorem. By Theorem 17.17, we identify $\mathcal{S}\text{Tam}(c, \mathcal{O}_+, \mathcal{O}_\bullet)$ with $[\mathcal{U}, (\text{Set})]_0$. For example, for any $X \in \text{Ob}(\mathcal{U})$, $\mathcal{U}(X, -)$ is regarded as a semi-Tambara functor. Remark also that if $\Sigma \in \text{Ob}([\mathcal{U}, (\text{Set})]_0)$ is ring-valued, Σ can be regarded as a Tambara functor.

Theorem 18.18. *There exists a unique pair (\mathcal{V}, κ) of a category \mathcal{V} with arbitrary products and a functor $\kappa : \mathcal{U} \rightarrow \mathcal{V}$ such that the following conditions are satisfied.*

- (i) $\text{Ob}(\mathcal{U}) = \text{Ob}(\mathcal{V})$.
- (ii) $\mathcal{V}(X, Y) = K_0 \mathcal{U}(X, Y)$, where K_0 denotes the Grothendieck ring of the semi-ring $\mathcal{U}(X, Y)$, and any object in \mathcal{V} is a ring object.
- (iii) κ is identity on objects, and for any $X, Y \in \text{Ob}(\mathcal{U})$, the component of κ

$$\kappa_{X, Y} : \mathcal{U}(X, Y) \rightarrow \mathcal{V}(X, Y)$$

is the completion map.

- (iv) κ preserves arbitrary products.

Proof. Put $\text{Ob}(\mathcal{V}) := \text{Ob}(\mathcal{U})$. For any $X \in \text{Ob}(\mathcal{V})$, define a Tambara functor T^X by

$$T^X := \gamma(\mathcal{U}(X, -)).$$

(Remark that the Hom-functor $\mathcal{U}(X, -)$ is a semi-Tambara functor.) As already shown, there exists a morphism of semi-Tambara functors

$$\kappa^X : \mathcal{U}(X, -) \rightarrow T^X$$

which satisfies the universality of Proposition 18.17. We define the morphism of \mathcal{V} by

$$\mathcal{V}(X, Y) := T^X(Y) = K_0 \mathcal{U}(X, Y)$$

for any $X, Y \in \text{Ob}(\mathcal{V})$. For each $\alpha \in \mathcal{V}(X, Y)$, by Yoneda's lemma

$$T^X(Y) \xrightarrow{\cong} \mathcal{S}\text{Tam}(\mathcal{U}(Y, -), T^X),$$

there exists a corresponding morphism $\alpha^\natural \in \mathcal{S}\text{Tam}(\mathcal{U}(Y, -), T^X)$. This α^\natural is the unique morphism satisfying

$$(18.5) \quad \alpha_Y^\natural(\text{id}_Y) = \alpha.$$

By Proposition 18.17, there exists a unique morphism of Tambara functors $\widetilde{\alpha}^\natural : T^Y \rightarrow T^X$ such that $\widetilde{\alpha}^\natural \circ \kappa^Y = \alpha^\natural$.

$$(18.6) \quad \begin{array}{ccc} \mathcal{U}(Y, -) & \xrightarrow{\alpha^\natural} & T^X \\ & \searrow \kappa^Y & \nearrow \widetilde{\alpha}^\natural \\ & & T^Y \end{array} \quad \circ$$

We define the composition law

$$\begin{array}{ccc} \mathcal{V}(X, Y) \times \mathcal{V}(Y, Z) & \longrightarrow & \mathcal{V}(X, Z) \\ \Downarrow & & \Downarrow \\ (\alpha, \beta) & \longmapsto & \beta \circ \alpha \end{array}$$

by $\beta \circ \alpha := \widetilde{\alpha}^\natural_Z(\beta)$. The identity morphism in $\mathcal{V}(X, X)$ is $\kappa_X^X(\text{id}_X)$.

Associativity of the composition

For any $\alpha \in \mathcal{V}(X, Y)$ and $\beta \in \mathcal{V}(Y, Z)$, there exists a commutative diagram

$$\begin{array}{ccc} T^Y(Z) & \xrightarrow{\widetilde{\alpha}^\natural_Z} & T^X(Z) \\ \cong \downarrow & \circ & \downarrow \cong \\ \mathcal{STam}(\mathcal{U}(Z, -), T^Y) & \xrightarrow{\widetilde{\alpha}^\natural_{\circ -}} & \mathcal{STam}(\mathcal{U}(Z, -), T^X) \end{array}$$

where the vertical arrows are Yoneda isomorphisms. So we have

$$\widetilde{\alpha}^\natural \circ \beta^\natural = (\widetilde{\alpha}^\natural_Z(\beta))^\natural : \mathcal{U}(Z, -) \rightarrow T^X.$$

Thus, for any $\gamma \in \mathcal{V}(Z, W)$,

$$\gamma \circ (\beta \circ \alpha) = (\widetilde{\beta \circ \alpha})^\natural_W(\gamma) = (\widetilde{\alpha}^\natural_Z(\beta))^\natural_W(\gamma) = ((\widetilde{\alpha}^\natural \circ \beta^\natural))^\natural_W(\gamma)$$

while

$$(\gamma \circ \beta) \circ \alpha = \widetilde{\alpha}^\natural_W(\gamma \circ \beta) = \widetilde{\alpha}^\natural_W(\widetilde{\beta}^\natural_W(\gamma)).$$

Since $\widetilde{\alpha}^\natural \circ \widetilde{\beta}^\natural$ makes the following diagram commutative;

$$\begin{array}{ccc} \mathcal{U}(Z, -) & \xrightarrow{\widetilde{\alpha}^\natural \circ \widetilde{\beta}^\natural} & T^X \\ & \searrow \kappa^Z & \nearrow \widetilde{\alpha}^\natural \circ \widetilde{\beta}^\natural \\ & & T^Z \end{array} \quad \circ$$

it must agree with $(\widetilde{\alpha}^\natural \circ \beta^\natural)$ by the uniqueness. Thus we obtain

$$\gamma \circ (\beta \circ \alpha) = (\widetilde{\alpha}^\natural \circ \beta^\natural)^\natural_W(\gamma) = \widetilde{\alpha}^\natural_W(\widetilde{\beta}^\natural_W(\gamma)) = (\gamma \circ \beta) \circ \alpha.$$

Unit law

For any $\alpha \in \mathcal{V}(X, Y)$, we have

$$\kappa_Y^Y(\text{id}_Y) \circ \alpha = \widetilde{\alpha}^{\natural}_Y(\kappa_Y^Y(\text{id}_Y)) \stackrel{(18.6)}{=} \alpha^{\natural}_Y(\text{id}_Y) \stackrel{(18.5)}{=} \alpha.$$

On the other hand, by the uniqueness of the morphism satisfying (18.5), we have $(\kappa_X^X(\text{id}_X))^{\natural} = \kappa^X$. Since $\widetilde{\kappa^X} = \text{id}_{T^X}$, it follows that

$$\alpha \circ \kappa_X^X(\text{id}_X) = \kappa_X^X(\widetilde{\kappa^X}(\text{id}_X))^{\natural}(\alpha) = \alpha.$$

Thus \mathcal{V} becomes in fact a category.

Functor $\kappa : \mathcal{U} \rightarrow \mathcal{V}$ is defined by

$$\begin{aligned} \kappa(X) &= X \\ \kappa_{X,Y} &:= \kappa_Y^X : \mathcal{U}(X, Y) \rightarrow \mathcal{V}(X, Y) = T^X(Y) \quad (\forall X, Y \in \text{Ob}(\mathcal{U})). \end{aligned}$$

Remark that $\kappa_{X,Y}$ is the completion map of the semi-ring $\mathcal{U}(X, Y)$. We show that κ is in fact a functor. Obviously κ preserves identities. So, it remains to show

$$\kappa(\beta \circ \alpha) = \kappa(\beta) \circ \kappa(\alpha) \quad (\forall \alpha \in \mathcal{U}(X, Y), \forall \beta \in \mathcal{U}(Y, Z)).$$

Since $\kappa^X : \mathcal{U}(X, -) \rightarrow T^X$ is a natural transformation, we have

$$\begin{array}{ccc} \mathcal{U}(X, Y) & \xrightarrow{\beta \circ -} & \mathcal{U}(X, Z) \\ \kappa_Y^X \downarrow & \circlearrowleft & \downarrow \kappa_Z^X \\ T^X(Y) & \xrightarrow{T^X(\beta)} & T^X(Z). \end{array}$$

By the naturality of the Yoneda isomorphisms, we have

$$\begin{array}{ccc} \mathcal{U}(Y, Y) & \xrightarrow{(\kappa_Y^X(\alpha))^{\natural}_Y} & T^X(Y) \\ \beta \circ - \downarrow & \circlearrowleft & \downarrow T^X(\beta) \\ \mathcal{U}(Y, Z) & \xrightarrow{(\kappa_Y^X(\alpha))^{\natural}_Z} & T^X(Z). \end{array}$$

By the definition of $(\widetilde{\kappa_Y^X(\alpha)})^{\natural}$, we have

$$\begin{array}{ccc} \mathcal{U}(Y, Z) & \xrightarrow{(\kappa_Y^X(\alpha))^{\natural}_Z} & T^X(Z) \\ \searrow \kappa_Z^Y & \circlearrowleft & \nearrow (\widetilde{\kappa_Y^X(\alpha)})^{\natural}_Z \\ & T^Y(Z) & \end{array}$$

So, we have

$$\begin{aligned}
\kappa(\beta) \circ \kappa(\alpha) &= \kappa_Z^Y(\beta) \circ \kappa_Y^X(\alpha) = (\widetilde{\kappa_Y^X(\alpha)})_{\sharp_Z}(\kappa_Z^Y(\beta)) \\
&= (\kappa_Y^X(\alpha))_{\sharp_Z}(\beta) = (\kappa_Y^X(\alpha))_{\sharp_Z}(\beta \circ \text{id}_Y) \\
&= T^X(\beta) \circ (\kappa_Y^X(\alpha))_{\sharp_Y}(\text{id}_Y) \stackrel{(18.5)}{=} T^X(\beta)(\kappa_Y^X(\alpha)) \\
&= \kappa_Z^X(\beta \circ \alpha) = \kappa(\beta \circ \alpha).
\end{aligned}$$

Thus $\kappa : \mathcal{U} \rightarrow \mathcal{V}$ is in fact a functor. Remark that this (\mathcal{V}, κ) satisfies conditions (i),(ii),(iii) in the statement of the Theorem. To show (iv), let

$$(p_\lambda : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

be a product in \mathcal{U} . This is equivalent to the fact that

$$(p_\lambda \circ -)_{\lambda \in \Lambda} : \mathcal{U}(A, X) \rightarrow \prod_{\lambda \in \Lambda} \mathcal{U}(A, X_\lambda)$$

is an isomorphism for each $A \in \text{Ob}(\mathcal{U})$. Since we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{U}(A, X) & \xrightarrow{p_\lambda \circ -} & \mathcal{U}(A, X_\lambda) \\
\kappa_{A, X} \downarrow & \circlearrowleft & \downarrow \kappa_{A, X_\lambda} \\
\mathcal{V}(A, X) & \xrightarrow{\kappa(p_\lambda) \circ -} & \mathcal{V}(A, X_\lambda)
\end{array}$$

for each $\lambda \in \Lambda$, thus we have

$$\begin{array}{ccc}
\mathcal{U}(A, X) & \xrightarrow[\cong]{(p_\lambda \circ -)_{\lambda \in \Lambda}} & \prod_{\lambda \in \Lambda} \mathcal{U}(A, X_\lambda) \\
\kappa_{A, X} \downarrow & \circlearrowleft & \downarrow \prod_{\lambda \in \Lambda} \kappa_{A, X_\lambda} \\
\mathcal{V}(A, X) & \xrightarrow{(\kappa(p_\lambda) \circ -)_{\lambda \in \Lambda}} & \prod_{\lambda \in \Lambda} \mathcal{V}(A, X_\lambda) \xrightarrow{=} K_0(\prod_{\lambda \in \Lambda} \mathcal{U}(A, X))
\end{array}$$

completion

Thus

$$(\kappa(p_\lambda) \circ -)_{\lambda \in \Lambda} : \mathcal{V}(A, X) \rightarrow \prod_{\lambda \in \Lambda} \mathcal{V}(A, X_\lambda)$$

is an isomorphism, which means that

$$(\kappa(p_\lambda) : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

is a product in \mathcal{V} . Thus (\mathcal{V}, κ) satisfies all of the conditions in the statement of the theorem. If any other (\mathcal{V}', κ') satisfies these conditions, then by Proposition 18.17, for each $X \in \text{Ob}(\mathcal{V})$ there exists a unique isomorphism $\widetilde{\kappa'^X} : T^X \xrightarrow{\cong} \mathcal{V}'(X, \kappa'(-))$ which satisfies

$$\begin{array}{ccc}
& \widetilde{\kappa'^X} \circ \kappa^X = \kappa'^X & \\
& \kappa^X \searrow & \nearrow \widetilde{\kappa'^X} \\
\mathcal{U}(X, -) & \xrightarrow{\kappa'^X} & \mathcal{V}'(X, \kappa'(-)) \\
& \circlearrowleft & \\
& \kappa^X \searrow & \nearrow \widetilde{\kappa'^X} \\
& T^X &
\end{array}$$

By using (the uniqueness of) the universality, we can show that the following diagram is commutative for any $\alpha \in \mathcal{V}(X', X)$ and $Y \in \text{Ob}(\mathcal{V})$:

$$\begin{array}{ccc} \mathcal{V}(X, Y) & \xrightarrow{\widetilde{\alpha^h_Y}} & \mathcal{V}(X', Y) \\ \widetilde{\kappa'_{X,Y}} \downarrow & \circlearrowleft & \downarrow \widetilde{\kappa'_{X',Y}} \\ \mathcal{V}'(X, Y) & \xrightarrow{-\circ \widetilde{\kappa'_{X',X}(\alpha)}} & \mathcal{V}'(X', Y) \end{array}$$

(See also the proof of Claim 18.21.) From this, we can see that $\widetilde{\kappa'_{X,Y}}$ ($X, Y \in \text{Ob}(\mathcal{V})$) constitute an isomorphism $\widetilde{\kappa'} : \mathcal{V} \xrightarrow{\cong} \mathcal{V}'$, compatible with κ and κ' . \square

In the proof of Theorem 18.18, the following was shown:

Remark 18.19. For any set of objects $(X_\lambda)_{\lambda \in \Lambda}$ in \mathcal{V} , their product in \mathcal{V} can be written in the form

$$(\kappa(p_\lambda) : X \rightarrow X_\lambda)_{\lambda \in \Lambda},$$

where

$$(p_\lambda : X \rightarrow X_\lambda)_{\lambda \in \Lambda}$$

is the product of $(X_\lambda)_{\lambda \in \Lambda}$ in \mathcal{U} .

Theorem 18.20. Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)$ be a Tambara system as in Proposition 17.5. There is an equivalence of categories

$$\text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)} \xrightarrow{\cong} [\mathcal{V}, (\text{Set})]_0,$$

compatible with the isomorphism of Theorem 17.17, i.e. the following diagram is commutative;

$$\begin{array}{ccc} [\mathcal{V}, (\text{Set})]_0 & \xrightarrow{\cong} & \text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)} \\ \kappa^\# \downarrow & \circlearrowleft & \downarrow \text{fully faithful} \\ [\mathcal{U}, (\text{Set})]_0 & \xrightarrow{\cong} & \mathcal{S}\text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)} \end{array}$$

where $\kappa^\#$ denotes the pull-back by κ .

Proof. Let $\tau : [\mathcal{U}, (\text{Set})]_0 \xrightarrow{\cong} \mathcal{S}\text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)}$ be the isomorphism in Theorem 17.17. As in Theorem 16.14, there exists a functor $\bar{\tau} : [\mathcal{V}, (\text{Set})]_0 \rightarrow \text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)}$, which makes the following diagram commutative:

$$\begin{array}{ccc} [\mathcal{V}, (\text{Set})]_0 & \xrightarrow{\bar{\tau}} & \text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)} \\ \kappa^\# \downarrow & \circlearrowleft & \downarrow \text{fully faithful} \\ [\mathcal{U}, (\text{Set})]_0 & \xrightarrow[\cong]{\tau} & \mathcal{S}\text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_\bullet)} \end{array}$$

In the following, we show $\bar{\tau}$ is an equivalence. We use the notation in the proof of Theorem 18.18.

To show $\bar{\tau}$ is fully faithful, it suffices to show $\kappa^\#$ is fully faithful. So for each $\Sigma_1, \Sigma_2 \in \text{Ob}([\mathcal{V}, (\text{Set})]_0)$, we show the map

$$(18.7) \quad \begin{array}{ccc} [\mathcal{V}, (\text{Set})]_0(\Sigma_1, \Sigma_2) & \longrightarrow & [\mathcal{U}, (\text{Set})]_0(\kappa^\#\Sigma_1, \kappa^\#\Sigma_2) \\ \cup & & \cup \\ \varphi = (\varphi_X)_{X \in \text{Ob}(\mathcal{V})} & \longmapsto & \varphi = (\varphi_{\kappa(X)})_{X \in \text{Ob}(\mathcal{U})} \end{array}$$

is bijective. Injectivity is trivial. To show the surjectivity, take an element $\psi \in [\mathcal{U}, (\text{Set})]_0(\kappa^\#\Sigma_1, \kappa^\#\Sigma_2)$. Remark that

$$\Sigma_{iX,Y} : \mathcal{V}(X, Y) \rightarrow (\text{Set})(\Sigma_i(X), \Sigma_i(Y)) \quad (i = 1, 2)$$

is a ring homomorphism (cf. Proposition 16.10).

For any $\alpha \in \mathcal{V}(X, Y)$, since $\mathcal{V}(X, Y) = K_0\mathcal{U}(X, Y)$, there are $a, b \in \mathcal{U}(X, Y)$ such that $\alpha = \kappa_Y^X(a) - \kappa_Y^X(b)$. Thus we have

$$\begin{aligned} \Sigma_2(\alpha) \circ \psi_X &= (\Sigma_2(\kappa_Y^X(a)) - \Sigma_2(\kappa_Y^X(b))) \circ \psi_X \\ &= \kappa^\#\Sigma_2(a) \circ \psi_X - \kappa^\#\Sigma_2(b) \circ \psi_X \\ &= \psi_Y \circ \kappa^\#\Sigma_1(a) - \psi_Y \circ \kappa^\#\Sigma_1(b) \\ &= \psi_Y \circ (\Sigma_1(\alpha)), \end{aligned}$$

which means $\psi \in [\mathcal{V}, (\text{Set})]_0(\Sigma_1, \Sigma_2)$. Thus (18.7) is bijective, and $\kappa^\#$ becomes fully faithful.

To show that $\bar{\tau}$ is essentially surjective, for any $T \in \text{Ob}(\text{Tam}_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_*)})$, take the corresponding object $\Sigma \in \text{Ob}([\mathcal{U}, (\text{Set})]_0)$ such that $T = \tau(\Sigma)$. It suffices to show the existence of $\tilde{\Sigma} \in \text{Ob}([\mathcal{V}, (\text{Set})]_0)$ such that $\kappa^\#\tilde{\Sigma} = \Sigma$. For each $X \in \text{Ob}(\mathcal{U})$, we have a morphism of semi-Tambara functors

$$\Sigma^X : \mathcal{U}(X, -) \rightarrow (\text{Set})(\Sigma(X), \Sigma(-))$$

induced from Σ . Remark that $(\text{Set})(\Sigma(X), \Sigma(-))$ is a Tambara functor (under the identification by τ). So by Proposition 18.17, we obtain a unique morphism of Tambara functors

$$\widetilde{\Sigma^X} : T^X \rightarrow (\text{Set})(\Sigma(X), \Sigma(-))$$

such that

$$(18.8) \quad \begin{array}{ccc} \mathcal{U}(X, -) & \xrightarrow{\Sigma^X} & (\text{Set})(\Sigma(X), \Sigma(-)) \\ & \searrow \kappa^X & \nearrow \widetilde{\Sigma^X} \\ & T^X & \end{array}$$

Claim 18.21. For any $\alpha \in \mathcal{V}(X, Y) = T^X(Y)$,

$$(18.9) \quad \begin{array}{ccc} T^Y & \xrightarrow{\widetilde{\alpha^\#}} & T^X \\ \widetilde{\Sigma^Y} \downarrow & \circ & \downarrow \widetilde{\Sigma^X} \\ (\text{Set})(\Sigma(Y), \Sigma(-)) & \xrightarrow{-\circ \widetilde{\Sigma^X}_Y(\alpha)} & (\text{Set})(\Sigma(X), \Sigma(-)) \end{array}$$

is commutative.

Proof. Since

$$\mathcal{U}(Y, -) \xrightarrow{\alpha^{\natural}} T^X \xrightarrow{\widetilde{\Sigma^X}} (\text{Set})(\Sigma(X), \Sigma(-))$$

are morphisms in $[\mathcal{U}, (\text{Set})]_0$, for any $a \in \mathcal{U}(Y, Z)$ we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}(Y, Y) & \xrightarrow{\alpha_Y^{\natural}} & T^X(Y) & \xrightarrow{\widetilde{\Sigma^X}_Y} & (\text{Set})(\Sigma(X), \Sigma(Y)) \\ \alpha \circ - \downarrow & \circlearrowleft & T^X(a) \downarrow & \circlearrowleft & \downarrow \Sigma(a) \circ - \\ \mathcal{U}(Y, Z) & \xrightarrow{\alpha_Z^{\natural}} & T^X(Z) & \xrightarrow{\widetilde{\Sigma^X}_Z} & (\text{Set})(\Sigma(X), \Sigma(Z)) \end{array}$$

In particular, we have

$$\begin{aligned} \Sigma(a) \circ \widetilde{\Sigma^X}_Y(\alpha) &= \Sigma(a) \circ \widetilde{\Sigma^X}_Y(\alpha_Y^{\natural}(\text{id}_Y)) \\ &= \widetilde{\Sigma^X}_Z(\alpha_Z^{\natural}(a)) \\ &= \widetilde{\Sigma^X}_Z(\alpha_Z^{\natural}(\kappa_Z^Y(a))). \end{aligned}$$

Thus the following diagram in $[\mathcal{U}, (\text{Set})]_0$ is commutative:

$$\begin{array}{ccccc} \mathcal{U}(Y, -) & \xrightarrow{\Sigma^Y} & (\text{Set})(\Sigma(Y), \Sigma(-)) & \xrightarrow{- \circ \widetilde{\Sigma^X}_Y(\alpha)} & (\text{Set})(\Sigma(X), \Sigma(-)) \\ & \searrow \kappa^Y & \circlearrowleft & \nearrow \widetilde{\Sigma^X} & \\ & & T^Y & \xrightarrow{\widetilde{\alpha^{\natural}}} & T^X \end{array}$$

Since $\widetilde{\Sigma^Y} \circ \kappa^Y = \Sigma^Y$, we obtain the commutativity of (18.9) by the universality of κ^Y . \square

By Claim 18.21, we can show easily

$$\begin{aligned} \widetilde{\Sigma^Y}_Z(\beta) \circ \widetilde{\Sigma^X}_Y(\alpha) &= \widetilde{\Sigma^X}_Z(\beta \circ \alpha) \quad (\forall \alpha \in \mathcal{V}(X, Y), \forall \beta \in \mathcal{V}(Y, Z)) \\ &\text{(and } \widetilde{\Sigma^X}_X(\kappa_X^X(\text{id}_X)) = \text{id} \text{ by definition).} \end{aligned}$$

Thus if we define $\widetilde{\Sigma}$ by

$$\begin{aligned} \widetilde{\Sigma}(X) &:= \Sigma(X) \quad (\forall X \in \text{Ob}(\mathcal{V})) \\ \widetilde{\Sigma}(\alpha) &:= \widetilde{\Sigma^X}_Y(\alpha) \quad (\forall \alpha \in \mathcal{V}(X, Y)), \end{aligned}$$

then $\widetilde{\Sigma}$ becomes a functor from \mathcal{V} to (Set) . By (18.8), $\widetilde{\Sigma}$ satisfies

$$\widetilde{\Sigma} \circ \kappa = \Sigma.$$

In particular, by Remark 18.19 (and the fact that Σ preserves arbitrary products), it follows that $\widetilde{\Sigma}$ also preserves arbitrary products. Thus we obtain $\widetilde{\Sigma} \in \text{Ob}([\mathcal{V}, (\text{Set})]_0)$ satisfying $\kappa^{\sharp} \widetilde{\Sigma} = \Sigma$. This is what we wanted to show. \square

As a corollary of Theorem 18.20, there exists a Tambara functor $\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})}$ on any Tambara system $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$ as in Proposition 17.5, corresponding to the Hom-functor

$$\mathcal{V}(\emptyset, -) \in [\mathcal{V}, (\text{Set})]_0.$$

For any $X \in \text{Ob}(\mathcal{V}) = \text{Ob}({}_G\text{Set}_{\mathcal{C}})$, we have

$$\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})}(X) = \mathcal{V}(\emptyset, X) = K_0\mathcal{U}(\emptyset, X).$$

Since

$$\begin{aligned} & \mathcal{U}(\emptyset, X) \\ &= \{(\emptyset \leftarrow \emptyset \xrightarrow{q} B \xrightarrow{\xi} X) \mid B \in \text{Ob}({}_G\text{Set}_{\mathcal{C}}), \xi \in {}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}(B, X)\} / \sim_{\text{equiv.}} \\ &\cong \text{cl}({}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}/X), \end{aligned}$$

we obtain $\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})}(X) = K_0({}_G\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}/X)$.

Definition 18.22. We call $\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})}$ the Burnside Tambara functor on the Tambara system $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$.

Remark 18.23. Definition 18.22 gives a Tambara functor structure on Burnside functor $\Omega_{(\mathcal{C}, \mathcal{O}_{\mathcal{C}})}$ defined in Definition 16.16.

For the rest of this section, we show that Elliott's functor \mathbf{V}_M becomes a Tambara functor on a profinite group G . According to Elliott, an almost finite G -set is a discrete G -space which satisfies

$$\varphi_U(X) < \infty$$

for any open subgroup $U \leq G$, where $\varphi_U(X)$ is defined by $\varphi_U(X) = \#(X^U) = \#\{a \in X \mid u \cdot a = a \ (\forall u \in U)\}$. For any multiplicative monoid M , an M -valued G -set is an almost finite G -set with a map $\chi_X : X \rightarrow M$. M -valued G -sets form a category $M\text{-Alm}_G$ with finite products and finite sums as defined in [10], whose Grothendieck category is denoted by $\mathbf{V}_M(G)$. If $M = 0$, then $\mathbf{V}_0(G)$ agrees with the completed Burnside ring defined in [8]. Since these rings have Mackey-functorial properties, we can expect them to be Tambara functors. Indeed, we give a Tambara functor structure to \mathbf{V}_M in the following.

Let $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$ be the open-natural Tambara system on G (see Example 17.6). Remark that in this case, each of $\mathcal{C}(H)$, $\mathcal{O}_{\mathcal{C}}(H)$, $\mathcal{O}_{\bullet}(H)$ is simply the set of all open subgroups of H , for each $H \in \mathcal{C}$. For any $p \in {}_G\text{Set}_{\mathcal{C}}(A, X)$, we say p has almost finite fibers when p satisfies

$$\varphi_U(p^{-1}(x)) < \infty \quad (\forall x \in X, \forall U \in \mathcal{C}(G_x)).$$

Morphisms $A \xrightarrow{p} X$ with almost finite fibers form a full subcategory Alm/X of ${}_G\text{Set}_{\mathcal{C}}/X$. It can be easily seen that, for each X , Alm/X has finite products (i.e. fiber products over X) and finite sums.

For any multiplicative monoid M , a category $M\text{-Alm}/X$ is defined as follows:

- An object of $M\text{-Alm}/X$ is a pair $(A \xrightarrow{p} X, \chi_A)$, where $(A \xrightarrow{p} X)$ is an object in $\text{Ob}(\text{Alm}/X)$ and $\chi_A : A \rightarrow M$ is a map of sets.
- A morphism from $(A \xrightarrow{p} X, \chi_A)$ to $(B \xrightarrow{q} X, \chi_B)$ is a morphism $f : (A \xrightarrow{p} X) \rightarrow (B \xrightarrow{q} X)$ in Alm/X which satisfies $\chi_B \circ f = \chi_A$.

Then $M\text{-Alm}/X$ also has finite products and finite sums, defined by

$$\begin{aligned} (A \xrightarrow{p} X, \chi_A) \times (B \xrightarrow{q} X, \chi_B) &:= (A \times_X B \rightarrow X, \chi_A \cdot \chi_B), \\ (A \xrightarrow{p} X, \chi_A) \coprod (B \xrightarrow{q} X, \chi_B) &:= (A \coprod B \rightarrow X, \chi_A \cup \chi_B), \end{aligned}$$

where $(\chi_A \cdot \chi_B)(a, b) := \chi(a) \cdot \chi(b)$ for any $(a, b) \in A \times_X B$.

Remark that when X is a transitive G -set $X = G/H$ ($H \in \mathcal{C}$), then $M\text{-Alm}/X$ is equivalent to the category $M\text{-Alm}_H$ of M -valued H -sets. Thus we have $\mathbf{V}_M(H) \cong K_0(M\text{-Alm}/(G/H))$. So, we define abbreviately

$$\mathbf{V}_M(X) := K_0(M\text{-Alm}/X)$$

for any $X \in \text{Ob}(\mathcal{G}\text{Set}_{\mathcal{C}})$.

Theorem 18.24. \mathbf{V}_M is a Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$.

Proof. By Theorem 18.16, it suffices to show that the correspondence $X \mapsto S(X) := \text{cl}(M\text{-Alm}/X)$ is a semi-Tambara functor.

First, we describe the structure maps of S . Let $(A \xrightarrow{p} X, \chi_A)$ be an object in $M\text{-Alm}/X$, and $(B \xrightarrow{q} Y, \chi_B)$ be an object in $M\text{-Alm}/Y$.

- For any $\xi \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\mathcal{C}}}(X, Y)$, define $\xi_+ : M\text{-Alm}/X \rightarrow M\text{-Alm}/Y$ by $\xi_+(A \xrightarrow{p} X, \chi_A) = (A \xrightarrow{\xi \circ p} Y, \chi_A)$.

- For any $\eta \in \mathcal{G}\text{Set}_{\mathcal{C}, \mathcal{O}_{\bullet}}(X, Y)$, define $\eta_{\bullet} : M\text{-Alm}/X \rightarrow M\text{-Alm}/Y$ by $\eta_{\bullet}(A \xrightarrow{p} X, \chi_A) = (\Pi_{\eta}(A) \xrightarrow{\pi = \pi_{\eta}(p)} Y, \chi_{(A, \eta)})$, where $\chi_{(A, \eta)}$ is defined by

$$\chi_{(A, \eta)}(y, \sigma) := \prod_{x \in \eta^{-1}(y)} \chi_A(\sigma(x)).$$

- For any $\zeta \in \mathcal{G}\text{Set}_{\mathcal{C}}(X, Y)$, define $\zeta^* : M\text{-Alm}/Y \rightarrow M\text{-Alm}/X$ by $\zeta^*(B \xrightarrow{q} Y, \chi_B) = (B' \xrightarrow{q'} X, \chi_B \circ \zeta')$, where

$$\begin{array}{ccc} B' & \xrightarrow{\zeta'} & B \\ q' \downarrow & \square & \downarrow q \\ X & \xrightarrow{\zeta} & Y \end{array}$$

is a pull-back diagram.

These are well-defined, i.e., we have

$$\begin{aligned} (A \xrightarrow{\xi \circ p} Y, \chi_A) &\in \text{Ob}(M\text{-Alm}/Y), \\ (\Pi_{\eta}(A) \xrightarrow{\pi} Y, \chi_{(A, \eta)}) &\in \text{Ob}(M\text{-Alm}/Y), \\ (B' \xrightarrow{q'} X, \chi_B \circ \zeta') &\in \text{Ob}(M\text{-Alm}/X), \end{aligned}$$

for each ξ, η, ζ as above.

We only show $(\Pi_{\eta}(A) \xrightarrow{\pi} Y) \in \text{Ob}(M\text{-Alm}/Y)$. The rest can be shown easily. For any $y \in Y$, put $\eta^{-1}(y) = \{x_1, \dots, x_n\}$. Then, $\pi^{-1}(y)$ is bijective to $\prod_{1 \leq i \leq n} p^{-1}(x_i)$

by the map

$$\begin{array}{ccc} \pi^{-1}(y) & \xrightarrow{\quad} & \prod_{1 \leq i \leq n} p^{-1}(x_i) \\ \cup & & \cup \\ \sigma & \xrightarrow{\quad} & (\sigma(x_1), \dots, \sigma(x_n)). \end{array}$$

By this bijection, we can give a G_y -action on $\prod_{1 \leq i \leq n} p^{-1}(x_i)$. When $g \in G_{x_i} (\leq G_y)$,

this action satisfies

$$(g \cdot (a_1, \dots, a_n))_i = g \cdot a_i$$

for any $(a_1, \dots, a_n) \in \prod_{1 \leq i \leq n} p^{-1}(x_i)$, where the left hand side denotes the i -th component of $g \cdot (a_1, \dots, a_n)$. Thus, under this bijection, we have

$$\pi^{-1}(y)^U \subseteq p^{-1}(x_1) \times \cdots \times p^{-1}(x_{i-1}) \times p^{-1}(x_i)^U \times p^{-1}(x_{i+1}) \times \cdots \times p^{-1}(x_n)$$

for any $U \in \mathcal{C}(G_{x_i})$.

Thus for any $V \in \mathcal{C}(G_y)$, we have

$$\begin{aligned} \pi^{-1}(y)^V &\subseteq \bigcap_{1 \leq i \leq n} p^{-1}(x_1) \times \cdots \times p^{-1}(x_i)^{V \cap G_{x_i}} \times \cdots \times p^{-1}(x_n) \\ &= \prod_{1 \leq i \leq n} p^{-1}(x_i)^{V \cap G_{x_i}}, \end{aligned}$$

since $V \cap G_{x_i} \in \mathcal{C}(G_{x_i})$. So we obtain

$$\begin{aligned} \varphi_V(\pi^{-1}(y)) &= \#(\pi^{-1}(y)^V) \\ &\leq \prod_{1 \leq i \leq n} \#(p^{-1}(x_i)^{V \cap G_{x_i}}) \\ &= \prod_{1 \leq i \leq n} \varphi_{V \cap G_{x_i}}(p^{-1}(x_i)) \\ &< \infty, \end{aligned}$$

since $\varphi_{V \cap G_{x_i}}(p^{-1}(x_i)) < \infty$ for each i .

We confirm the conditions in Definition 17.10. By definition $S(X)$ is a semi-ring. Since ξ_+ preserves finite sums, the induced map $\xi_+ : S(X) \rightarrow S(Y)$ becomes an additive homomorphism. Similarly ζ^* becomes a semi-ring homomorphism. We demonstrate how η_\bullet becomes a multiplicative homomorphism. Let $(A \xrightarrow{p} X, \chi_A), (B \xrightarrow{q} X, \chi_B)$ be two elements in $M\text{-Alm}/X$. We show that $(\Pi_\eta(A) \times_Y \Pi_\eta(B) \rightarrow Y, \chi_{(A,\eta)} \cdot \chi_{(B,\eta)})$ is isomorphic to $(\Pi_\eta(A \times_X B) \rightarrow Y, \chi_{(A \times_X B, \eta)})$. Remark that $\chi_{A \times_X B} = \chi_A \cdot \chi_B$. Since there is an isomorphism

$$\begin{array}{ccc} f : \Pi_\eta(A) \times_Y \Pi_\eta(B) & \xrightarrow{\cong} & \Pi_\eta(A \times_X B) \\ \cup & & \cup \\ ((y, \sigma), (y, \tau)) & \longmapsto & (y, (\sigma, \tau)) \end{array}$$

in Alm/Y , it suffices to show $\chi_{(A \times_X B, \eta)} \circ f = \chi_{(A, \eta)} \cdot \chi_{(B, \eta)}$. This is satisfied, since

$$\begin{aligned} \chi_{(A \times_X B, \eta)} \circ f((y, \sigma), (y, \tau)) &= \prod_{x \in \eta^{-1}(y)} (\chi_A \cdot \chi_B)((\sigma, \tau)(x)) \\ &= \prod_{x \in \eta^{-1}(y)} \chi_A(\sigma(x)) \chi_B(\tau(x)) \\ &= \prod_{x \in \eta^{-1}(y)} \chi_A(\sigma(x)) \prod_{x \in \eta^{-1}(y)} \chi_B(\tau(x)) \\ &= \chi_{(A, \eta)}(y, \sigma) \cdot \chi_{(B, \eta)}(y, \tau) \\ &= \chi_{(A, \eta)} \cdot \chi_{(B, \eta)}((y, \sigma), (y, \tau)). \end{aligned}$$

Since conditions from (i) to (iv) in Definition 17.10 are shown in almost the same way, we only show condition (v).

Let

$$\begin{array}{ccccc}
 X & \xleftarrow{\xi} & Z & \xleftarrow{\zeta} & X' = X \times_Y \Pi_\eta(Z) \\
 \eta \downarrow & & & \circ & \downarrow \eta' \\
 Y & \xleftarrow{\pi_\eta(\xi)} & & & Y' = \Pi_\eta(Z)
 \end{array}$$

be an exponential diagram. For any object $(A \xrightarrow{p} Z, \chi_A)$ in $M\text{-Alm}/Z$, we have

$$\begin{aligned}
 \eta_\bullet \circ \xi_+((A \xrightarrow{p} Z, \chi_A)) &= (\Pi_\eta(A) \xrightarrow{\pi_\eta(\xi \circ p)} Y, \chi_{(A, \eta)}), \\
 \pi_\eta(\xi)_+ \circ \eta'_\bullet \circ \zeta^*((A \xrightarrow{p} Z, \chi_A)) &= \pi_\eta(\xi)_+ \circ \eta'_\bullet((A' \xrightarrow{p'} X', \chi_{A'} = \chi_A \circ \zeta')) \\
 &= (\Pi_{\eta'}(A') \xrightarrow{\pi_\eta(\xi) \circ \pi_{\eta'}(p')} Y, \chi_{(A', \eta')}),
 \end{aligned}$$

where

$$\begin{array}{ccc}
 A' & \xrightarrow{\zeta'} & A \\
 p' \downarrow & \square & \downarrow p \\
 X' & \xrightarrow{\zeta} & Z
 \end{array}$$

is a pull-back diagram.

So it suffices to construct an isomorphism between $(\Pi_\eta(A) \xrightarrow{\pi_\eta(\xi \circ p)} Y, \chi_{(A, \eta)})$ and $(\Pi_{\eta'}(A') \xrightarrow{\pi_\eta(\xi) \circ \pi_{\eta'}(p')} Y, \chi_{(A', \eta')})$.

Let (y', σ') be an element of $\Pi_{\eta'}(A')$, where $y' \in Y'$ and σ' is a map from $\eta'^{-1}(y')$ to A' . If we put $y := \pi_\eta(\xi)(y')$ then $\eta'^{-1}(y') = \eta^{-1}(y) \times \{y'\}$, and we define $\sigma \in \text{Map}(\eta^{-1}(y), A)$ by

$$\sigma(x) = \zeta' \circ \sigma'((x, y')) \quad (\forall x \in \eta^{-1}(y)).$$

In this notation, we define $f : \Pi_{\eta'}(A') \rightarrow \Pi_\eta(A)$ by $f((y', \sigma')) := (y, \sigma)$. It is easily seen that this becomes an isomorphism in Alm/Y . It remains to show

$$(18.10) \quad \chi_{(A, \eta)} \circ f = \chi_{(A', \eta')}.$$

For any $(y', \sigma') \in \Pi_{\eta'}(A')$, the left hand side can be calculated as

$$\begin{aligned}
 \chi_{(A, \eta)} \circ f(y', \sigma') &= \chi_{(A, \eta)}(y, \sigma) \\
 &= \prod_{x \in \eta^{-1}(y)} \chi_A(\sigma(x)) \\
 &= \prod_{x \in \eta^{-1}(y)} \chi_A(\zeta' \circ \sigma'(x, y')),
 \end{aligned}$$

while the right hand side is

$$\begin{aligned}
 \chi_{(A', \eta')}(y', \sigma') &= \prod_{x' \in \eta'^{-1}(y')} \chi_{A'}(\sigma'(x')) \\
 &= \prod_{x \in \eta^{-1}(y)} \chi_{A'}(\sigma'(x, y')) \\
 &= \prod_{x \in \eta^{-1}(y)} \chi_A \circ \zeta'(\sigma'(x, y')).
 \end{aligned}$$

Thus (18.10) is satisfied, and \mathbf{V}_M becomes a Tambara functor on $(\mathcal{C}, \mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\bullet})$. \square

Part 4. Cohomology theory in 2-categories

19. INTRODUCTION FOR PART 4

In 1970s, B. Pareigis started his study on the Brauer groups of symmetric monoidal categories in [28]. Around 2000, the notion of symmetric categorical groups are introduced to this study by E. M. Vitale in [32] (see also [31]). By definition, a symmetric categorical group is a categorification of an abelian group, and in this sense the 2-category of symmetric categorical groups SCG can be regarded as a 2-dimensional analogue of the category Ab of abelian groups. As such, SCG and its variants (e.g. 2-category \mathbb{G} -SMod of symmetric categorical groups with \mathbb{G} -action where \mathbb{G} is a fixed categorical group) admit a 2-dimensional analogue of the homological algebra in Ab.

For example, E. M. Vitale constructed for any monoidal functor $F : \mathbf{C} \rightarrow \mathbf{D}$ between symmetric monoidal categories \mathbf{C} and \mathbf{D} , a 2-exact sequence of Picard and Brauer categorical groups

$$\mathcal{P}(\mathbf{C}) \rightarrow \mathcal{P}(\mathbf{D}) \rightarrow \overline{\mathcal{F}} \rightarrow \mathcal{B}(\mathbf{C}) \rightarrow \mathcal{B}(\mathbf{C}).$$

By taking π_0 and π_1 , we can induce the well-known Picard-Brauer and Unit-Picard exact sequences of abelian groups respectively. In [29], A. del Río, J. Martínez-Moreno and E. M. Vitale defined a more subtle notion of the relative 2-exactness, and succeeded in constructing a cohomology long 2-exact sequence from any short relatively 2-exact sequence of complexes in SCG. In this part, we consider a 2-categorical analogue of an abelian category, in such a way that it contains SCG as an example, so as to treat SCG and their variants in a more abstract, unified way.

In section 20, we review general definitions in a 2-category and properties of SCG, with simple comments. In section 21, we define the notion of a relatively exact 2-category as a generalization of SCG, also as a 2-dimensional analogue of an abelian category. We try to make the homological algebra in SCG ([29]) work well in this general 2-category. It will be worthy to note that our definition of a relatively exact 2-category is self-dual.

	category	2-category
general theory	abelian category	relatively exact 2-category
example	Ab	SCG

In section 22, we show the existence of *proper factorization systems* in any relatively exact 2-category, which will make several diagram lemmas more easy to handle. In any abelian category, any morphism f can be written in the form $f = e \circ m$ (uniquely up to an isomorphism), where e is epimorphic and m is monomorphic. As a 2-dimensional analogue, we show that any 1-cell f in a relatively exact 2-category \mathbf{S} admits the following two ways of factorization:

- (1) $i \circ m \implies f$ where i is fully cofaithful and m is faithful.
- (2) $e \circ j \implies f$ where e is cofaithful and j is fully faithful.

(In the case of SCG, see [17].) In section 23, complexes in \mathbf{S} and the relative 2-exactness are defined, generalizing those in SCG ([29]). Since we start from the self-dual definition, we can make good use of duality in the proofs. In section 24, as a main theorem, we construct a long cohomology 2-exact sequence from any short relatively 2-exact sequence (i.e. an extension) of complexes. Our proof is purely diagrammatic, and is an analogy of that for an abelian category. In section 23 and 24, several 2-dimensional diagram lemmas are shown. Most of them have

1-dimensional analogues in an abelian category, so we only have to be careful about the compatibility of 2-cells.

Since SCG is an example of a relatively exact 2-category, we expect some other 2-categories constructed from SCG will be a relatively exact 2-category. For example, $\mathbb{G}\text{-SMod}$, $\text{SCG} \times \text{SCG}$ and the 2-category of categories fibered in SCG are candidates.

20. PRELIMINARIES

Definitions in a 2-category

Notation 20.1. Throughout this part, \mathbf{S} denotes a 2-category (in the strict sense). We use the following notation.

$\mathbf{S}^0, \mathbf{S}^1, \mathbf{S}^2$: class of 0-cells, 1-cells, and 2-cells in \mathbf{S} , respectively.

$\mathbf{S}^1(A, B)$: 1-cells from A to B , where $A, B \in \mathbf{S}^0$.

$\mathbf{S}^2(f, g)$: 2-cells from f to g , where $f, g \in \mathbf{S}^1(A, B)$ for certain $A, B \in \mathbf{S}^0$.

$\mathbf{S}(A, B)$: Hom-category between A and B (i.e. $\text{Ob}(\mathbf{S}(A, B)) = \mathbf{S}^1(A, B)$ and $\mathbf{S}(A, B)(f, g) = \mathbf{S}^2(f, g)$).

In diagrams, \longrightarrow represents a 1-cell, \Longrightarrow represents a 2-cell, \circ represents a horizontal composition, and \cdot represents a vertical composition. We use capital letters A, B, \dots for 0-cells, small letters f, g, \dots for 1-cells, and Greek symbols α, β, \dots for 2-cells.

For example, one of the conditions in the definition of a 2-category can be written as follows (see for example [20]):

Remark 20.2. For any diagram in \mathbf{S}

$$\begin{array}{ccccc} & & f_1 & & g_1 \\ & & \curvearrowright & & \curvearrowright \\ A & & & B & & C \\ & & \alpha \Downarrow & & \beta \Downarrow \\ & & \curvearrowleft & & \curvearrowleft \\ & & f_2 & & g_2 \end{array},$$

we have

$$(20.1) \quad (f_1 \circ \beta) \cdot (\alpha \circ g_2) = (\alpha \circ g_1) \cdot (f_2 \circ \beta).$$

(Note: composition is always written diagrammatically.)

This equality is frequently used in later arguments.

Products, pullbacks, difference kernels and their duals are defined by the universality.

Definition 20.3. For any A_1 and $A_2 \in \mathbf{S}^0$, their product $(A_1 \times A_2, p_1, p_2)$ is defined as follows:

(a) $A_1 \times A_2 \in \mathbf{S}^0$, $p_i \in \mathbf{S}^1(A_1 \times A_2, A_i)$ ($i = 1, 2$).

(b1) (existence of a factorization)

For any $X \in \mathbf{S}^0$ and $q_i \in \mathbf{S}^1(X, A_i)$ ($i = 1, 2$), there exist $q \in \mathbf{S}^1(X, A_1 \times A_2)$ and $\xi_i \in \mathbf{S}^2(q \circ p_i, q_i)$ ($i = 1, 2$).

$$\begin{array}{ccccc} & & X & & \\ & q_1 \swarrow & \downarrow q & \searrow q_2 & \\ & \xi_1 \swarrow & & \searrow \xi_2 & \\ A_1 & \xleftarrow{p_1} & A_1 \times A_2 & \xrightarrow{p_2} & A_2 \end{array}$$

(b2) (uniqueness of the factorization)

For any factorizations (q, ξ_1, ξ_2) and (q', ξ'_1, ξ'_2) which satisfy (b1), there exists a unique 2-cell $\eta \in \mathbf{S}^2(q, q')$ such that $(\eta \circ p_i) \cdot \xi'_i = \xi_i$ ($i = 1, 2$).

$$\begin{array}{ccc} q \circ p_i & \xrightarrow{\eta \circ p_i} & q' \circ p_i \\ & \searrow \xi_i & \swarrow \xi'_i \\ & q_i & \end{array}$$

The coproduct of A_1 and A_2 is defined dually.

Definition 20.4. For any $A_1, A_2, B \in \mathbf{S}^0$ and $f_i \in \mathbf{S}^1(A_i, B)$ ($i = 1, 2$), the pullback $(A_1 \times_B A_2, f'_1, f'_2, \xi)$ of f_1 and f_2 is defined as follows:

(a) $A_1 \times_B A_2 \in \mathbf{S}^0$, $f'_1 \in \mathbf{S}^1(A_1 \times_B A_2, A_2)$, $f'_2 \in \mathbf{S}^1(A_1 \times_B A_2, A_1)$, $\xi \in \mathbf{S}^2(f'_1 \circ f_2, f'_2 \circ f_1)$.

$$\begin{array}{ccccc} & & A_2 & & \\ & f'_1 \nearrow & & f_2 \searrow & \\ A_1 \times_B A_2 & & & & B \\ & f'_2 \searrow & & f_1 \nearrow & \\ & & A_1 & & \end{array}$$

(b1) (existence of a factorization)

For any $X \in \mathbf{S}^0$, $g_1 \in \mathbf{S}^1(X, A_2)$, $g_2 \in \mathbf{S}^1(X, A_1)$ and $\eta \in \mathbf{S}^2(g_1 \circ f_2, g_2 \circ f_1)$, there exist $g \in \mathbf{S}^1(X, A_1 \times_B A_2)$, $\xi_i \in \mathbf{S}^2(g \circ f'_i, g_i)$ ($i = 1, 2$) such that $(\xi_1 \circ f_2) \cdot \eta = (g \circ \xi) \cdot (\xi_2 \circ f_1)$.

$$\begin{array}{ccc} \begin{array}{ccccc} & & A_2 & & \\ & g_1 \nearrow & & f_2 \searrow & \\ X & \xrightarrow{g} & A_1 \times_B A_2 & & B \\ & g_2 \searrow & & f_1 \nearrow & \\ & & A_1 & & \end{array} & \begin{array}{ccc} g \circ f'_1 \circ f_2 & \xrightarrow{\xi_1 \circ f_2} & g_1 \circ f_2 \\ g \circ \xi \Downarrow & \circ & \Downarrow \eta \\ g \circ f'_2 \circ f_1 & \xrightarrow{\xi_2 \circ f_1} & g_2 \circ f_1 \end{array} \end{array}$$

(b2) (uniqueness of the factorization)

For any factorizations (g, ξ_1, ξ_2) and (g', ξ'_1, ξ'_2) which satisfy (b1), there exists a unique 2-cell $\zeta \in \mathbf{S}^2(g, g')$ such that $(\zeta \circ f'_i) \cdot \xi'_i = \xi_i$ ($i = 1, 2$).

The pushout of $f_i \in \mathbf{S}^1(A, B_i)$ ($i = 1, 2$) is defined dually.

Definition 20.5. For any $A, B \in \mathbf{S}^0$ and $f, g \in \mathbf{S}^1(A, B)$, the difference kernel

$$(\text{DK}(f, g), d_{(f, g)}, \varphi_{(f, g)})$$

of f and g is defined as follows:

(a) $\text{DK}(f, g) \in \mathbf{S}^0$, $d_{(f, g)} \in \mathbf{S}^1(\text{DK}(f, g), A)$, $\varphi_{(f, g)} \in \mathbf{S}^2(d_{(f, g)} \circ f, d_{(f, g)} \circ g)$.

$$\begin{array}{ccc} \text{DK}(f, g) & \xrightarrow{d_{(f, g)}} & A \begin{array}{ccc} \xrightarrow{f} & & B \\ \xleftarrow{g} & & \end{array} \\ & & \downarrow \varphi_{(f, g)} \\ & & \text{DK}(f, g) \begin{array}{ccc} \xrightarrow{d_{(f, g)} \circ f} & & B \\ \xleftarrow{d_{(f, g)} \circ g} & & \end{array} \end{array}$$

(b1) (existence of a factorization)

For any $X \in \mathbf{S}^0$, $d \in \mathbf{S}^1(X, A)$, $\varphi \in \mathbf{S}^2(d \circ f, d \circ g)$, there exist $\underline{d} \in \mathbf{S}^1(X, \text{DK}(f, g))$ and $\underline{\varphi} \in \mathbf{S}^2(\underline{d} \circ d_{(f, g)}, d)$ such that $(\underline{d} \circ \varphi_{(f, g)}) \cdot (\underline{\varphi} \circ g) = (\underline{\varphi} \circ f) \cdot \varphi$.

(b2) (uniqueness of the factorization)

For any factorizations $(\underline{d}, \underline{\varphi})$ and $(\underline{d}', \underline{\varphi}')$ which satisfy (b1), there exists a unique 2-cell $\eta \in \mathbf{S}^2(\underline{d}, \underline{d}')$ such that $(\eta \circ d_{(f,g)}) \cdot \underline{\varphi}' = \underline{\varphi}$.

The difference cokernel of f and g is defined dually.

The following definition is from [9].

Definition 20.6. Let $f \in \mathbf{S}^1(A, B)$.

(1) f is said to be faithful if $f^\flat := - \circ f : \mathbf{S}(C, A) \rightarrow \mathbf{S}(C, B)$ is faithful for any $C \in \mathbf{S}^0$.

(2) f is said to be fully faithful if f^\flat is fully faithful for any $C \in \mathbf{S}^0$.

(3) f is said to be cofaithful if $f^\sharp := f \circ - : \mathbf{S}(B, C) \rightarrow \mathbf{S}(A, C)$ is faithful for any $C \in \mathbf{S}^0$.

(4) f is said to be fully cofaithful if f^\sharp is fully faithful for any $C \in \mathbf{S}^0$.

Properties of SCG

By definition, a symmetric categorical group is a symmetric monoidal category $(\mathbb{G}, \otimes, 0)$, in which each arrow is an isomorphism and each object has an inverse up to an equivalence with respect to the tensor \otimes . More precisely;

Definition 20.7. A symmetric categorical group $(\mathbb{G}, \otimes, 0)$ consists of

- (a1) a category \mathbb{G}
- (a2) a tensor functor $\otimes : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$
- (a3) a unit object $0 \in \text{Ob}(\mathbb{G})$
- (a4) natural isomorphisms

$$\begin{aligned} \alpha_{A,B,C} : A \otimes (B \otimes C) &\rightarrow (A \otimes B) \otimes C, \\ \lambda_A : 0 \otimes A &\rightarrow A, \quad \rho_A : A \otimes 0 \rightarrow A, \quad \gamma_{A,B} : A \otimes B \rightarrow B \otimes A \end{aligned}$$

which satisfy certain compatibility conditions (cf. [27]), and the following two conditions are satisfied:

(b1) For any $A, B \in \text{Ob}(\mathbb{G})$ and $f \in \mathbb{G}(A, B)$, there exists $g \in \mathbb{G}(B, A)$ such that $f \circ g = \text{id}_A$, $g \circ f = \text{id}_B$.

(b2) For any $A \in \text{Ob}(\mathbb{G})$, there exist $A^* \in \text{Ob}(\mathbb{G})$ and $\eta_A \in \mathbb{G}(0, A \otimes A^*)$.

In particular, there is a ‘zero categorical group’ 0 , which consists of only one object 0 and one morphism id_0 .

Definition 20.8. For symmetric categorical groups \mathbb{G} and \mathbb{H} , a monoidal functor F from \mathbb{G} to \mathbb{H} consists of

- (a1) a functor $F : \mathbb{G} \rightarrow \mathbb{H}$
- (a2) natural isomorphisms

$$F_{A,B} : F(A \otimes B) \rightarrow F(A) \otimes F(B) \text{ and } F_I : F(0) \rightarrow 0$$

which satisfy certain compatibilities with $\alpha, \lambda, \rho, \gamma$. (cf. [27])

Remark 20.9. For any monoidal functors $F : \mathbb{G} \rightarrow \mathbb{H}$ and $G : \mathbb{H} \rightarrow \mathbb{K}$, their composition $F \circ G : \mathbb{G} \rightarrow \mathbb{K}$ is defined by

$$(20.2) \quad (F \circ G)_{A,B} := G(F_{A,B}) \circ G_{F(A),F(B)}$$

$$(20.3) \quad (F \circ G)_I := G(F_I) \circ G_I.$$

In particular, there is a ‘zero monoidal functor’ $0_{\mathbb{G}, \mathbb{H}} : \mathbb{G} \rightarrow \mathbb{H}$ for each \mathbb{G} and \mathbb{H} , which sends every object in \mathbb{G} to $0_{\mathbb{H}}$, every arrow in \mathbb{G} to $\text{id}_{0_{\mathbb{H}}}$, and $(0_{\mathbb{G}, \mathbb{H}})_{A, B} = \lambda_0^{-1} = \rho_0^{-1}$, $(0_{\mathbb{G}, \mathbb{H}})_I = \text{id}_0$. It is easy to see that $0_{\mathbb{G}, \mathbb{H}} \circ 0_{\mathbb{H}, \mathbb{K}} = 0_{\mathbb{G}, \mathbb{K}}$ ($\forall \mathbb{G}, \mathbb{H}, \mathbb{K}$).

Remark 20.10. *Our notion of a monoidal functor is equal to that of a ‘ γ -monoidal functor’ in [29].*

Definition 20.11. For monoidal functors $F, G : \mathbb{G} \rightarrow \mathbb{H}$, a natural transformation φ from F to G is said to be a monoidal transformation if it satisfies

$$(20.4) \quad \begin{aligned} \varphi_{A \otimes B} \circ G_{A, B} &= F_{A, B} \circ (\varphi_A \otimes \varphi_B) \\ F_I &= \varphi_0 \circ G_I. \end{aligned}$$

The following remark is from [32].

Remark 20.12. *By condition (b2), it is shown that there exists a 2-cell $\varepsilon_A \in \mathbb{G}(A^* \otimes A, 0)$ for each object A , such that the following compositions are identities:*

$$\begin{array}{ccccccccccc} A & \xrightarrow{\lambda_A^{-1}} & 0 \otimes A & \xrightarrow{\eta_{A \otimes 1}} & (A \otimes A^*) \otimes A & \xrightarrow{\alpha^{-1}} & A \otimes (A^* \otimes A) & \xrightarrow{1 \otimes \varepsilon_A} & A \otimes 0 & \xrightarrow{\rho_A} & A \\ A^* & \xrightarrow{\rho_{A^*}^{-1}} & A^* \otimes 0 & \xrightarrow{1 \otimes \eta_A} & A^* \otimes (A \otimes A^*) & \xrightarrow{\alpha} & (A^* \otimes A) \otimes A^* & \xrightarrow{\varepsilon_{A \otimes 1}} & 0 \otimes A^* & \xrightarrow{\lambda_{A^*}} & A^* \end{array}$$

For each monoidal functor $F : \mathbb{G} \rightarrow \mathbb{H}$, there exists a natural morphism $\iota_{F, A} : F(A^*) \rightarrow F(A)^*$.

Definition 20.13. SCG is defined to be the 2-category whose 0-cells are symmetric categorical groups, 1-cells are monoidal functors, and 2-cells are monoidal transformations.

The following two propositions are satisfied in SCG (see for example [5]).

Proposition 20.14. *For any symmetric categorical groups \mathbb{G} and \mathbb{H} , if we define a monoidal functor $F \otimes_{\mathbb{G}, \mathbb{H}} G : \mathbb{G} \rightarrow \mathbb{H}$ by*

$$\begin{aligned} F \otimes_{\mathbb{G}, \mathbb{H}} G(A) &:= F(A) \otimes_{\mathbb{H}} G(A) \\ (F \otimes_{\mathbb{G}, \mathbb{H}} G)_{A, B} &:= (F(A \otimes B) \otimes G(A \otimes B)) \\ &\xrightarrow{F_{A, B} \otimes G_{A, B}} F(A) \otimes F(B) \otimes G(A) \otimes G(B) \\ &\xrightarrow{\simeq} F(A) \otimes G(A) \otimes F(B) \otimes G(B) \\ (F \otimes_{\mathbb{G}, \mathbb{H}} G)_I &:= (F(I) \otimes G(I) \xrightarrow{F_I \otimes G_I} I \otimes I \xrightarrow{\simeq} I), \end{aligned}$$

then $(\text{SCG}(\mathbb{G}, \mathbb{H}), \otimes_{\mathbb{G}, \mathbb{H}}, 0_{\mathbb{G}, \mathbb{H}})$ becomes again a symmetric categorical group with appropriately defined $\alpha, \lambda, \rho, \gamma$, and

$$\text{Hom} = \text{SCG}(-, -) : \text{SCG} \times \text{SCG} \rightarrow \text{SCG}$$

becomes a 2-functor (cf. section 6 in [5]).

In SCG, by definition of the zero categorical group we have $\mathbf{S}^1(\mathbb{G}, 0) = \{0_{\mathbb{G}, 0}\}$, while $\mathbf{S}^1(0, \mathbb{G})$ may have more than one objects. In this point SCG might be said to have ‘non self-dual’ structure, but $\mathbf{S}^1(\mathbb{G}, 0)$ and $\mathbf{S}^1(0, \mathbb{G})$ have the following ‘self-dual’ property.

Remark 20.15. (1) *For any symmetric categorical group \mathbb{G} and any monoidal functor $F : \mathbb{G} \rightarrow 0$, there exists a unique 2-cell $\varphi : F \Rightarrow 0_{\mathbb{G}, 0}$.*

(2) *For any symmetric categorical group \mathbb{G} and any monoidal functor $F : 0 \rightarrow \mathbb{G}$, there exists a unique 2-cell $\varphi : F \Rightarrow 0_{0, \mathbb{G}}$.*

Proof. (1) follows from the fact that the zero categorical group has only one morphism id_0 . (2) follows from condition (20.4) in Definition 20.11. \square

The usual compatibility arguments show the following Lemma.

Lemma 20.16. *Let $F : \mathbb{G} \rightarrow \mathbb{H}$ be a monoidal functor. For any $A, B \in \text{Ob}(\mathbb{G})$,*

$$\begin{array}{ccc} \Phi_{A,B} : \mathbb{G}(A, B) & \longrightarrow & \mathbb{G}(A \otimes B^*, 0) \\ \downarrow \Psi & & \downarrow \Psi \\ f & \longmapsto & (f \otimes 1_{B^*}) \circ \eta_B^{-1} \end{array}$$

and

$$\begin{array}{ccc} \Psi_{A,B} : \mathbb{G}(A \otimes B^*, 0) & \longrightarrow & \mathbb{G}(A, B) \\ \downarrow \Psi & & \downarrow \Psi \\ g & \longmapsto & \rho_A^{-1} \circ (1_A \otimes \varepsilon_B^{-1}) \circ \alpha_{A, B^*, B} \circ (g \otimes 1_B) \circ \lambda_B \end{array}$$

are mutually inverse, and the following diagram is commutative;

$$\begin{array}{ccc} \mathbb{G}(A, B) & \xrightarrow{\Phi_{A,B}} & \mathbb{G}(A \otimes B^*, 0) \\ \downarrow F & & \downarrow F \\ \mathbb{H}(F(A), F(B)) & \circ & \mathbb{H}(F(A \otimes B^*), F(0)) \\ \downarrow \Phi_{F(A), F(B)} & & \downarrow \Theta_{A,B}^F \\ & \searrow & \swarrow \\ & \mathbb{H}(F(A) \otimes F(B)^*, 0) & \end{array}$$

where $\Theta_{A,B}^F$ is defined by

$$\begin{array}{ccc} \Theta_{A,B}^F : \mathbb{H}(F(A \otimes B^*), F(0)) & \longrightarrow & \mathbb{H}(F(A) \otimes F(B)^*, 0) \\ \downarrow \Psi & & \downarrow \Psi \\ h & \longmapsto & (1_{F(A)} \otimes (\iota_B^F)^{-1}) \circ (F_{A, B^*})^{-1} \circ h \circ F_I. \end{array}$$

21. DEFINITION OF A RELATIVELY EXACT 2-CATEGORY

Locally SCG 2-category

We define a locally SCG 2-category not only as a category enriched by SCG, but with some more conditions, in order to let it be a 2-dimensional analogue of that of an additive category.

Definition 21.1. A locally small 2-category \mathbf{S} is said to be locally SCG if the following conditions are satisfied:

(A1) For every $A, B \in \mathbf{S}^0$, there is a given functor $\otimes_{A,B} : \mathbf{S}(A, B) \times \mathbf{S}(A, B) \rightarrow \mathbf{S}(A, B)$, and an object $0_{A,B} \in \text{Ob}(\mathbf{S}(A, B))$ such that $(\mathbf{S}(A, B), \otimes_{A,B}, 0_{A,B})$ becomes a symmetric categorical group, and the following naturality conditions are satisfied:

$$0_{A,B} \circ 0_{B,C} = 0_{A,C} \quad (\forall A, B, C \in \mathbf{S}^0)$$

(A2) $\text{Hom} = \mathbf{S}(-, -) : \mathbf{S} \times \mathbf{S} \rightarrow \text{SCG}$ is a 2-functor which satisfies for any $A, B, C \in \mathbf{S}^0$,

$$(21.1) \quad (0_{A,B})_I^\# = \text{id}_{0_{A,C}} \in \mathbf{S}^2(0_{A,C}, 0_{A,C})$$

$$(21.2) \quad (0_{A,B})_I^b = \text{id}_{0_{C,B}} \in \mathbf{S}^2(0_{C,B}, 0_{C,B}).$$

(A3) There is a 0-cell $0 \in \mathbf{S}^0$ called a zero object, which satisfy the following conditions:

(a3-1) $\mathbf{S}(0, 0)$ is the zero categorical group.

(a3-2) For any $A \in \mathbf{S}^0$ and $f \in \mathbf{S}^1(0, A)$, there exists a unique 2-cell $\theta_f \in \mathbf{S}^2(f, 0_{0,A})$.

(a3-3) For any $A \in \mathbf{S}^0$ and $f \in \mathbf{S}^1(A, 0)$, there exists a unique 2-cell $\tau_f \in \mathbf{S}^2(f, 0_{A,0})$.

(A4) For any $A, B \in \mathbf{S}^0$, their product and coproduct exist.

Let us explain about these conditions.

Remark 21.2. By condition (A1) of Definition 21.1; every 2-cell in a locally SCG 2-category becomes invertible, as in the case of SCG (cf. [32]). This helps us to avoid being fussy about the directions of 2-cells in many propositions and lemmas, and we use the word ‘dual’ simply to reverse 1-cells.

Remark 21.3. By condition (A2) in Definition 21.1,

$$\begin{aligned} f^\# &:= f \circ - : \mathbf{S}(B, C) \rightarrow \mathbf{S}(A, C) \\ f^b &:= - \circ f : \mathbf{S}(C, A) \rightarrow \mathbf{S}(C, B) \end{aligned}$$

are monoidal functors ($\forall C \in \mathbf{S}^0$) for any $f \in \mathbf{S}^1(A, B)$, and the following naturality conditions are satisfied:

(a2-1) For any $f \in \mathbf{S}^1(A, B)$, $g \in \mathbf{S}^1(B, C)$ and $D \in \mathbf{S}^0$, we have $(f \circ g)^\# = g^\# \circ f^\#$ as monoidal functors.

$$\begin{array}{ccc} A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow D & & \begin{array}{ccc} \mathbf{S}(C, D) & \xrightarrow{g^\#} & \mathbf{S}(B, D) \\ & \searrow (f \circ g)^\# & \swarrow f^\# \\ & \circ & \\ & \swarrow & \searrow \\ & \mathbf{S}(A, D) & \end{array} \end{array}$$

(a2-2) The dual of (a2-1) for $-^b$.

(a2-3) For any $f \in \mathbf{S}^1(A, B)$, $g \in \mathbf{S}^1(C, D)$, we have $f^\# \circ g^b = g^b \circ f^\#$ as monoidal functors.

$$\begin{array}{ccc} A \xrightarrow{f} B \longrightarrow C \xrightarrow{g} D & & \begin{array}{ccc} \mathbf{S}(B, C) & \xrightarrow{f^\#} & \mathbf{S}(A, C) \\ g^b \downarrow & \circ & \downarrow g^b \\ \mathbf{S}(B, D) & \xrightarrow{f^\#} & \mathbf{S}(A, D) \end{array} \end{array}$$

Since already $(f \circ g)^\# = g^\# \circ f^\#$ as functors, (a2-1) means $(f \circ g)_I^\# = (g^\# \circ f^\#)_I$, and by (20.3) in Remark 20.9, this is equivalent to

$$(f \circ g)_I^\# = f^\#(g_I^\#) \cdot f_I^\# = (f \circ g_I^\#) \cdot f_I^\#.$$

Similarly, we obtain

$$(21.3) \quad (f \circ g)_I^b = (f_I^b \circ g) \cdot g_I^b,$$

$$(21.4) \quad (f_I^\# \circ g) \cdot g_I^b = (f \circ g_I^b) \cdot f_I^\#.$$

Remark 21.4. By condition (A2), for any $f, g \in \mathbf{S}^1(A, B)$ and any $\alpha \in \mathbf{S}^2(f, g)$, $\alpha \circ - : f^\# \Rightarrow g^\#$ becomes a monoidal transformation. So, the diagrams

$$\begin{array}{ccc} f \circ (k \otimes h) & \xrightarrow{\alpha \circ (k \otimes h)} & g \circ (k \otimes h) \\ f_{k,h}^\# \downarrow & \circlearrowleft & \downarrow g_{k,h}^\# \\ (f \circ k) \otimes (f \circ h) & \xrightarrow{(\alpha \circ k) \otimes (\alpha \circ h)} & (g \circ k) \otimes (g \circ h) \end{array} \quad \text{and} \quad \begin{array}{ccc} f \circ 0_{B,C} & \xrightarrow{\alpha \circ 0_{B,C}} & g \circ 0_{B,C} \\ f_I^\# \swarrow & \circlearrowleft & \searrow g_I^\# \\ & 0_{A,C} & \end{array}$$

are commutative for any $C \in \mathbf{S}^0$ and $k, h \in \mathbf{S}^1(B, C)$. Similar statement also holds for $- \circ \alpha : f^\flat \Rightarrow g^\flat$.

Corollary 21.5. In a locally SCG 2-category \mathbf{S} , the following are satisfied:

(1) For any diagram in \mathbf{S}

$$\begin{array}{ccccc} & h & & f & \\ C & \xrightarrow{\quad} & A & \xrightarrow{\quad} & B \\ & \Downarrow \varepsilon & & \Downarrow \alpha & \\ & 0_{C,A} & & g & \end{array}$$

we have

$$(21.5) \quad h \circ \alpha = (\varepsilon \circ f) \cdot f_I^\flat \cdot g_I^{b-1} \cdot (\varepsilon^{-1} \circ g).$$

(2) For any diagram in \mathbf{S}

$$\begin{array}{ccccc} & f & & h & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\ & \Downarrow \alpha & & \Downarrow \varepsilon & \\ & g & & 0_{B,C} & \end{array},$$

we have

$$(21.6) \quad \alpha \circ h = (f \circ \varepsilon) \cdot f_I^\# \cdot g_I^{\#-1} \cdot (g \circ \varepsilon^{-1}).$$

(3) For any diagram in \mathbf{S}

$$\begin{array}{ccccc} & f & & g & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\ & \Downarrow \alpha & & \Downarrow \beta & \\ & 0_{A,B} & & 0_{B,C} & \end{array},$$

we have

$$(21.7) \quad (f \circ \beta) \cdot f_I^\# = (\alpha \circ g) \cdot g_I^\flat.$$

Proof. (1) $(h \circ \alpha) \stackrel{20.1}{=} (\varepsilon \circ f) \cdot (0_{C,A} \circ \alpha) \cdot (\varepsilon^{-1} \circ g) = (\varepsilon \circ f) \cdot f_I^\flat \cdot g_I^{b-1} \cdot (\varepsilon^{-1} \circ g)$. (2) is the dual of (1). And (3) follows from (21.1), (21.2), (21.5), (21.6). \square

Remark 21.6. We don't require a locally SCG 2-category to satisfy $\mathbf{S}^1(A, 0) = \{0_{A,0}\}$, for the sake of duality (see the comments before Remark 20.15).

Relatively exact 2-category

Definition 21.7. Let \mathbf{S} be a locally SCG 2-category. \mathbf{S} is said to be relatively exact if the following conditions are satisfied:

(B1) For any 1-cell $f \in \mathbf{S}^1(A, B)$, its kernel and cokernel exist.

(B2) For any 1-cell $f \in \mathbf{S}^1(A, B)$, f is faithful if and only if $f = \ker(\text{cok}(f))$.

(B3) For any 1-cell $f \in \mathbf{S}^1(A, B)$, f is cofaithful if and only if $f = \text{cok}(\ker(g))$.

It is shown in [32] that SCG satisfies these conditions.

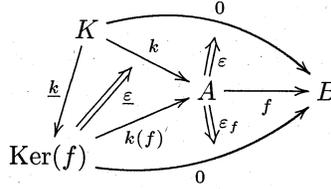
Let us explain about these conditions.

Definition 21.8. Let \mathbf{S} be a locally SCG 2-category. For any $f \in \mathbf{S}^1(A, B)$, its kernel $(\text{Ker}(f), \text{ker}(f), \varepsilon_f)$ is defined by universality as follows (we abbreviate $\text{ker}(f)$ to $k(f)$):

(a) $\text{Ker}(f) \in \mathbf{S}^0$, $k(f) \in \mathbf{S}^1(\text{Ker}(f), A)$, $\varepsilon_f \in \mathbf{S}^2(k(f) \circ f, 0)$.

(b1) (existence of a factorization)

For any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$ and $\varepsilon \in \mathbf{S}^2(k \circ f, 0)$, there exist $\underline{k} \in \mathbf{S}^1(K, \text{Ker}(f))$ and $\underline{\varepsilon} \in \mathbf{S}^2(\underline{k} \circ k(f), k)$ such that $(\underline{\varepsilon} \circ f) \cdot \varepsilon = (\underline{k} \circ \varepsilon_f) \cdot (\underline{k})_I^\#$.



(b2) (uniqueness of the factorization)

For any factorizations $(\underline{k}, \underline{\varepsilon})$ and $(\underline{k}', \underline{\varepsilon}')$ which satisfy (b1), there exists a unique 2-cell $\xi \in \mathbf{S}^2(\underline{k}, \underline{k}')$ such that $(\xi \circ k(f)) \cdot \underline{\varepsilon}' = \underline{\varepsilon}$.

Remark 21.9. (1) *By its universality, the kernel of f is unique up to an equivalence. We write this equivalence class again $\text{Ker}(f) = [\text{Ker}(f), k(f), \varepsilon_f]$.*

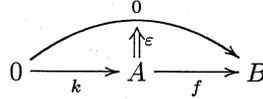
(2) *It is also easy to see that if f and f' are equivalent, then*

$$[\text{Ker}(f), k(f), \varepsilon_f] = [\text{Ker}(f'), k(f'), \varepsilon_{f'}].$$

For any f , its cokernel $\text{Cok}(f) = [\text{Cok}(f), c(f), \pi_f]$ is defined dually, and the dual statements also hold for the cokernel.

Remark 21.10. *Let \mathbf{S} be a locally SCG 2-category, and let f be in $\mathbf{S}^1(A, B)$.*

For any pair (k, ε) with $k \in \mathbf{S}^1(0, A)$, $\varepsilon \in \mathbf{S}^2(k \circ f, 0)$



and for any pair (k', ε') with $k' \in \mathbf{S}^1(0, A)$, $\varepsilon' \in \mathbf{S}^2(k' \circ f, 0)$, there exists a unique 2-cell $\xi \in \mathbf{S}^2(k, k')$ such that $(\xi \circ f) \cdot \varepsilon' = \varepsilon$.

Proof. By condition (a3-2) of Definition 21.1, $\varepsilon \in \mathbf{S}^2(k \circ f, 0)$ must be equal to the unique 2-cell $(\theta_k \circ f) \cdot f_I^b$. Similarly we have $\varepsilon' = (\theta_{k'} \circ f) \cdot f_I^b$, and, ξ should be the unique 2-cell $\theta_k \cdot \theta_{k'}^{-1} \in \mathbf{S}^2(k, k')$, which satisfies $(\xi \circ f) \cdot \varepsilon' = \varepsilon$. \square

From this, it makes no ambiguity if we abbreviate $\text{Ker}(f) = [0, 0_{0,A}, f_I^b]$ to $\text{Ker}(f) = 0$, because $[0, k, \varepsilon] = [0, k', \varepsilon']$ for any (k, ε) and (k', ε') . Dually, we abbreviate $\text{Cok}(f) = [0, 0_{A,0}, f_I^\#]$ to $\text{Cok}(f) = 0$.

By using condition (A3) of Definition 21.1, we can show the following easily:

Example 21.11. (1) For any $A \in \mathbf{S}^0$, $\text{Ker}(0_{A,0} : A \rightarrow 0) = [A, \text{id}_A, \text{id}_0]$.

(2) For any $A \in \mathbf{S}^0$, $\text{Cok}(0_{0,A} : 0 \rightarrow A) = [A, \text{id}_A, \text{id}_0]$.

Caution 21.12. (1) $\text{Ker}(0_{0,A} : 0 \rightarrow A)$ need not be equivalent to 0. Indeed, in the case of SCG, for any symmetric categorical group \mathbb{G} , $\text{Ker}(0_{0,\mathbb{G}} : 0 \rightarrow \mathbb{G})$ is equivalent to an important invariant $\pi_1(\mathbb{G})[0]$.

(2) $\text{Cok}(0_{A,0} : A \rightarrow 0)$ need not be equivalent to 0 either. In the case of SCG, $\text{Cok}(0_{\mathbb{G},0} : \mathbb{G} \rightarrow 0)$ is equivalent to $\pi_0(\mathbb{G})[1]$.

Remark 21.13. *The precise meaning of condition (B2) in Definition 21.7 is that, for any 1-cell $f \in \mathbf{S}^1(A, B)$ and its cokernel $[\text{Cok}(f), \text{cok}(f), \pi_f]$, f is faithful if and only if $\text{Ker}(\text{cok}(f)) = [A, f, \pi_f]$. Similarly for condition (B3).*

Relative (co-)kernel and first properties of a relatively exact 2-category

In the rest of this section, \mathbf{S} is a relatively exact 2-category.

Definition 21.14. For any diagram in \mathbf{S}

$$(21.8) \quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow \varphi & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C, \\ & & \downarrow & & \end{array}$$

its relative kernel $(\text{Ker}(f, \varphi), \text{ker}(f, \varphi), \varepsilon_{(f, \varphi)})$ is defined as follows (we abbreviate $\text{ker}(f, \varphi)$ to $k(f, \varphi)$):

(a) $\text{Ker}(f, \varphi) \in \mathbf{S}^0$, $k(f, \varphi) \in \mathbf{S}^1(\text{Ker}(f, \varphi), A)$, $\varepsilon_{(f, \varphi)} \in \mathbf{S}^2(k(f, \varphi) \circ f, 0)$.

(b0) (compatibility of the 2-cells)

$\varepsilon_{(f, \varphi)}$ is compatible with φ i.e. $(k(f, \varphi) \circ \varphi) \cdot k(f, \varphi)_I^\# = (\varepsilon_{(f, \varphi)} \circ g) \cdot g_I^\#$.

(b1) (existence of a factorization)

For any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$ and $\varepsilon \in \mathbf{S}^2(k \circ f, 0)$ which are compatible with φ , i.e. $(k \circ \varphi) \cdot k_I^\# = (\varepsilon \circ g) \cdot g_I^\#$, there exist $\underline{k} \in \mathbf{S}^1(K, \text{Ker}(f, \varphi))$ and $\underline{\varepsilon} \in \mathbf{S}^2(\underline{k} \circ k(f, \varphi), k)$ such that $(\underline{\varepsilon} \circ f) \cdot \varepsilon = (\underline{k} \circ \varepsilon_{(f, \varphi)}) \cdot (k)_I^\#$.

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \uparrow \varphi & & & \\ K & \xrightarrow{k} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \underline{k} & \nearrow \underline{\varepsilon} & \uparrow \varepsilon & \downarrow \varepsilon_{(f, \varphi)} & & & \\ \text{Ker}(f, \varphi) & & & & & & \end{array}$$

(b2) (uniqueness of the factorization)

For any factorizations $(\underline{k}, \underline{\varepsilon})$ and $(\underline{k}', \underline{\varepsilon}')$ which satisfy (b1), there exists a unique 2-cell $\xi \in \mathbf{S}^2(\underline{k}, \underline{k}')$ such that $(\xi \circ k(f, \varphi)) \cdot \underline{\varepsilon}' = \underline{\varepsilon}$.

Remark 21.15. *By its universality, the relative kernel of (f, φ) is unique up to an equivalence. We write this equivalence class $[\text{Ker}(f, \varphi), k(f, \varphi), \varepsilon_{(f, \varphi)}]$.*

Definition 21.16. Let \mathbf{S} be a relatively exact 2-category. For any diagram (21.8) in \mathbf{S} , its relative cokernel $(\text{Cok}(g, \varphi), \text{cok}(g, \varphi), \pi_{(g, \varphi)})$ is defined dually by universality. We abbreviate $\text{cok}(g, \varphi)$ to $c(g, \varphi)$, and write the equivalence class of the relative cokernel $[\text{Cok}(g, \varphi), c(g, \varphi), \pi_{(g, \varphi)}]$.

Caution 21.17. In the rest of this part, \mathbf{S} denotes a relatively exact 2-category, unless otherwise specified. In the following propositions and lemmas, we often omit the statement and the proof of their duals. Each term should be replaced by its dual. For example, kernel and cokernel, faithfulness and cofaithfulness are mutually dual.

Remark 21.18. *By using condition (A3) of Definition 21.1, we can show the following easily. (These are also corollaries of Proposition 21.20.)*

(1) $\text{Ker}(f, f_I^\#) = \text{Ker}(f)$ (and thus the ordinary kernel can be regarded as a relative kernel).

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow f_I^\# & \searrow & \\ A & \xrightarrow{f} & B & \xrightarrow{0} & 0 \end{array}$$

(2) $\text{ker}(f, \varphi)$ is faithful.

Lemma 21.19. Let $f \in \mathbf{S}^1(A, B)$ and take its kernel $[\text{Ker}(f), k(f), \varepsilon_f]$. If $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, \text{Ker}(f))$ and $\sigma \in \mathbf{S}^2(k \circ k(f), 0)$

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow \varepsilon_f & & \\ K & \xrightarrow{k} & \text{Ker}(f) & \xrightarrow{k(f)} & A & \xrightarrow{f} & B \\ & \searrow & \sigma \downarrow & \nearrow & & & \\ & & 0 & & & & \end{array}$$

is compatible with ε_f , i.e. if σ satisfies

$$(21.9) \quad (\sigma \circ f) \cdot f_I^\flat = (k \circ \varepsilon_f) \cdot k_I^\#,$$

then there exists a unique 2-cell $\zeta \in \mathbf{S}^2(k, 0)$ such that $\sigma = (\zeta \circ k(f)) \cdot k(f)_I^\flat$.

Proof. By (21.9), $\sigma : k \circ k(f) \Rightarrow 0$ is a factorization compatible with ε_f and f_I^\flat . On the other hand, by Remark 21.4, $k(f)_I^\flat : 0 \circ k(f) \Rightarrow 0$ is also a factorization compatible with ε_f, f_I^\flat . So, by the universality of the kernel, there exists a unique 2-cell $\zeta \in \mathbf{S}^2(k, 0)$ such that $\sigma = (\zeta \circ k(f)) \cdot k(f)_I^\flat$. \square

It is easy to see that the same statement also holds for relative (co-)kernels. In any relatively exact 2-category, the relative (co-)kernel always exist. More precisely, the following proposition holds.

Proposition 21.20. Consider diagram (21.8). By the universality of $\text{Ker}(g) = [\text{Ker}(g), \ell, \varepsilon]$, f factors through ℓ uniquely up to an equivalence as $\underline{\varphi} : \underline{f} \circ \ell \Rightarrow f$, where $\underline{f} \in \mathbf{S}^1(A, \text{Ker}(g))$ and $\underline{\varphi} \in \mathbf{S}^2(\underline{f} \circ \ell, f)$:

$$(\underline{f} \circ \varepsilon) \cdot (\underline{f})_I^\# = (\underline{\varphi} \circ g) \cdot \varphi$$

$$\begin{array}{ccccc} & & \text{Ker}(g) & & \\ & \searrow & \uparrow \ell & \searrow & \\ & & \downarrow \varepsilon & & \\ & \searrow & \downarrow \varepsilon & \searrow & \\ 0 & \xrightarrow{\underline{f}} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \nearrow \varepsilon_f & \downarrow \varphi & \nearrow & & & \\ & & \text{Ker}(f) & & & & \end{array}$$

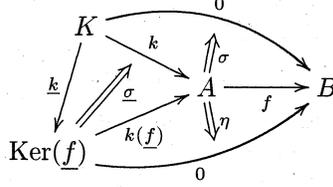
Then we have $\text{Ker}(f, \varphi) = [\text{Ker}(\underline{f}), k(\underline{f}), \eta]$, where $\eta := (k(\underline{f}) \circ \underline{\varphi}^{-1}) \cdot (\varepsilon_f \circ \ell) \cdot \ell_I^\flat \in \mathbf{S}^2(k(\underline{f}) \circ \underline{f}, 0)$. We abbreviate this to $\text{Ker}(f, \varphi) = \text{Ker}(\underline{f})$.

Proof. For any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$ and $\sigma \in \mathbf{S}^2(k \circ f, 0)$ which are compatible with φ , i.e. $(\sigma \circ g) \cdot g_I^\flat = (k \circ \varphi) \cdot k_I^\#$, if we put

$$\rho := (k \circ \underline{\varphi}) \cdot \sigma \in \mathbf{S}^2(k \circ \underline{f} \circ \ell, 0),$$

then ρ is compatible with ε . By Lemma 21.19, there exists a 2-cell $\zeta : k \circ \underline{f} \Rightarrow 0$ such that $\rho = (\zeta \circ \ell) \cdot \ell_I^\flat$. So, by the universality of $\text{Ker}(\underline{f})$, there exist $\underline{k} \in \mathbf{S}^1(K, \text{Ker}(\underline{f}))$

and $\underline{\sigma} \in \mathbf{S}^2(\underline{k} \circ k(f), k)$ such that $(\underline{\sigma} \circ f) \cdot \zeta = (\underline{k} \circ \underline{\varepsilon}_f) \cdot (k)_I^\sharp$. Then, $\underline{\sigma}$ is compatible with σ and η ,



and the existence of a factorization is shown. To show the uniqueness of the factorization, let $(\underline{k}', \underline{\sigma}')$ be another factorization which is compatible with σ , η , i.e. $(\underline{\sigma}' \circ f) \cdot \sigma = (\underline{k}' \circ \eta) \cdot (k)_I^\sharp$. Then, by using $\eta = (k(f) \circ \underline{\varepsilon}^{-1}) \cdot (\underline{\varepsilon}_f \circ \ell) \cdot \ell_I^\flat$ and $\zeta \circ \ell = \rho \cdot \ell_I^{\flat-1} = (k \circ \underline{\varepsilon}) \cdot \sigma \cdot \ell_I^{\flat-1}$, we can show $((\underline{\sigma}' \circ f) \cdot \zeta) \circ \ell = ((\underline{k}' \circ \underline{\varepsilon}_f) \cdot (k)_I^\sharp) \circ \ell$. Since ℓ is faithful, we obtain $((\underline{\sigma}' \circ f) \cdot \zeta) = (\underline{k}' \circ \underline{\varepsilon}_f) \cdot (k)_I^\sharp$. Thus, $\underline{\sigma}'$ is compatible with ζ and $\underline{\varepsilon}_f$. By the universality of $\text{Ker}(f)$, there exists a 2-cell $\xi \in \mathbf{S}^2(\underline{k}, \underline{k}')$ such that $(\xi \circ k(f)) \cdot \underline{\sigma}' = \underline{\sigma}$. Uniqueness of such $\xi \in \mathbf{S}^2(\underline{k}, \underline{k}')$ follows from the faithfulness of $k(f)$. \square

Proposition 21.21. *Let $f \in \mathbf{S}^1(A, B)$, $g \in \mathbf{S}^1(B, C)$ and suppose g is fully faithful. Then, $\text{Ker}(f \circ g) = [\text{Ker}(f), k(f), (\underline{\varepsilon}_f \circ g) \cdot g_I^\flat]$. We abbreviate this to $\text{Ker}(f \circ g) = \text{Ker}(f)$.*

Proof. Since g is fully faithful, for any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$ and $\sigma \in \mathbf{S}^2(k \circ f \circ g, 0)$, there exists $\rho \in \mathbf{S}^2(k \circ f, 0)$ such that $\sigma = (\rho \circ g) \cdot g_I^\flat$. And by the universality of $\text{Ker}(f)$, there are $\underline{k} \in \mathbf{S}^1(K, \text{Ker}(f))$ and $\underline{\sigma} \in \mathbf{S}^2(\underline{k} \circ k(f), k)$ such that $(\underline{\sigma} \circ f) \cdot \rho = (\underline{k} \circ \underline{\varepsilon}_f) \cdot (k)_I^\sharp$. Then, it can be easily seen that $\underline{\sigma}$ is compatible with σ and $(\underline{\varepsilon}_f \circ g) \cdot g_I^\flat$:

$$(\underline{\sigma} \circ f \circ g) \cdot \sigma = (\underline{k} \circ ((\underline{\varepsilon}_f \circ g) \cdot g_I^\flat)) \cdot (k)_I^\sharp$$

Thus we obtain a desired factorization. To show the uniqueness of the factorization, let $(\underline{k}', \underline{\sigma}')$ be another factorization of k which satisfies

$$(\underline{\sigma}' \circ f \circ g) \cdot \sigma = (\underline{k}' \circ ((\underline{\varepsilon}_f \circ g) \cdot g_I^\flat)) \cdot (k)_I^\sharp$$

Then, we can show $\underline{\sigma}'$ is compatible with ρ and $\underline{\varepsilon}_f$. By the universality of $\text{Ker}(f)$, there exists a 2-cell $\xi \in \mathbf{S}^2(\underline{k}, \underline{k}')$ such that $(\xi \circ k(f)) \cdot \underline{\sigma}' = \underline{\sigma}$. Uniqueness of such ξ follows from the faithfulness of $k(f)$. \square

By definition, $f \in \mathbf{S}^1(A, B)$ is faithful (resp. fully faithful) if and only if $- \circ f : \mathbf{S}^2(g, h) \rightarrow \mathbf{S}^2(g \circ f, h \circ f)$ is injective (resp. bijective) for any $K \in \mathbf{S}^0$ and any $g, h \in \mathbf{S}^1(K, A)$. Concerning this, we have the following lemma.

Lemma 21.22. *Let $f \in \mathbf{S}^1(A, B)$.*

(1) *f is faithful if and only if for any $K \in \mathbf{S}^0$ and $k \in \mathbf{S}^1(K, A)$,*

$$- \circ f : \mathbf{S}^2(k, 0) \rightarrow \mathbf{S}^2(k \circ f, 0 \circ f) \text{ is injective.}$$

- (2) f is fully faithful if and only if for any $K \in \mathbf{S}^0$ and $k \in \mathbf{S}^1(K, A)$,
 $-\circ f : \mathbf{S}^2(k, 0) \rightarrow \mathbf{S}^2(k \circ f, 0 \circ f)$ is bijective.

Proof. By Lemma 20.16, we have the following commutative diagram for any $g, h \in \mathbf{S}^1(K, A)$:

$$\begin{array}{ccc}
 \mathbf{S}^2(g, h) & \xrightarrow[\text{bij.}]{\Phi_{g,h}} & \mathbf{S}^2(g \otimes h^*, 0) \\
 \swarrow -\circ f & & \searrow -\circ f \\
 \mathbf{S}^2(g \circ f, h \circ f) & & \mathbf{S}^2((g \otimes h^*) \circ f, 0 \circ f) \\
 \searrow \Phi_{g \circ f, h \circ f} & \circlearrowleft & \swarrow \Theta_{g,h}^b \\
 \text{bij.} & & \text{bij.} \\
 \mathbf{S}^2((g \circ f) \otimes (h \circ f)^*, 0) & &
 \end{array}$$

So we have

$$\begin{aligned}
 &-\circ f : \mathbf{S}^2(g, h) \rightarrow \mathbf{S}^2(g \circ f, h \circ f) \text{ is injective (resp. bijective)} \\
 \Leftrightarrow &-\circ f : \mathbf{S}^2(g \otimes h^*, 0) \rightarrow \mathbf{S}^2((g \otimes h^*) \circ f, 0 \circ f) \text{ is injective (resp. bijective)}.
 \end{aligned}$$

□

Corollary 21.23. For any $f \in \mathbf{S}^1(A, B)$, f is faithful if and only if the following condition is satisfied:

$$(21.10) \quad \alpha \circ f = \text{id}_{0 \circ f} \Rightarrow \alpha = \text{id}_0 \quad (\forall K \in \mathbf{S}^0, \forall \alpha \in \mathbf{S}^2(0_{K,A}, 0_{K,A}))$$

Proof. If f is faithful, (21.10) is trivially satisfied, since we have $\text{id}_{0 \circ f} = \text{id}_0 \circ f$. To show the other implication, by Lemma 21.22, it suffices to show that $-\circ f : \mathbf{S}^2(k, 0) \rightarrow \mathbf{S}^2(k \circ f, 0 \circ f)$ is injective. For any $\alpha_1, \alpha_2 \in \mathbf{S}^2(k, 0)$ which satisfy $\alpha_1 \circ f = \alpha_2 \circ f$, we have $(\alpha_1^{-1} \cdot \alpha_2) \circ f = (\alpha_1 \circ f)^{-1} \cdot (\alpha_2 \circ f) = \text{id}_{0 \circ f}$. From the assumption we obtain $\alpha_1^{-1} \cdot \alpha_2 = \text{id}_0$, i.e. $\alpha_1 = \alpha_2$. □

The next corollary immediately follows from Lemma 21.22.

Corollary 21.24. For any $f \in \mathbf{S}^1(A, B)$, f is fully faithful if and only if for any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$, and any $\sigma \in \mathbf{S}^2(k \circ f, 0)$, there exists unique $\tau \in \mathbf{S}^2(k, 0)$ such that $\sigma = (\tau \circ f) \cdot f_I^\sharp$.

Corollary 21.25. For any $f \in \mathbf{S}^1(A, B)$, the following are equivalent:

- (1) f is fully faithful.
- (2) $\text{Ker}(f) = 0$.

Proof. (1) \Rightarrow (2)

Since f is fully faithful, for any $k \in \mathbf{S}^1(K, A)$ and $\varepsilon \in \mathbf{S}^2(k \circ f, 0)$, there exists a 2-cell $\underline{\varepsilon} \in \mathbf{S}^2(0_{K,A}, k)$ such that $(\underline{\varepsilon} \circ f) = (0 \circ f_I^\sharp) \cdot 0_I^\sharp \cdot \varepsilon^{-1} = (0 \circ f_I^\sharp) \cdot \varepsilon^{-1}$, and the existence of a factorization is shown. To show the uniqueness of the factorization, it suffices to show that for any other factorization $(\underline{k}', \underline{\varepsilon}')$ with $(\underline{\varepsilon}' \circ f) \cdot \varepsilon = (\underline{k}' \circ f_I^\sharp) \cdot (\underline{k}')_I^\sharp$, the unique 2-cell $\tau \in \mathbf{S}^2(\underline{k}', 0)$ (see condition (a3-2) in Definition 21.1) satisfies $(\tau \circ 0) \cdot \underline{\varepsilon} = \underline{\varepsilon}'$. Since f is faithful, this is equivalent to $(\tau \circ 0 \circ f) \cdot (\underline{\varepsilon} \circ f) \cdot \varepsilon = (\underline{\varepsilon}' \circ f) \cdot \varepsilon$, and this follows easily from $\tau \circ 0 = (\tau \circ 0) \cdot 0_I^\sharp = (\underline{k}')_I^\sharp$ and $(\tau \circ 0 \circ f) \cdot (0 \circ f_I^\sharp) = (\underline{k}' \circ f_I^\sharp) \cdot (\tau \circ 0)$.

(see Corollary 21.5.)

(2) \Rightarrow (1) Since $\text{Ker}(f) = [0, 0, f_I^b]$, for any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A)$ and any $\sigma \in \mathbf{S}^2(k \circ f, 0)$, there exist $\underline{k} \in \mathbf{S}^1(K, 0)$ and $\underline{\sigma} \in \mathbf{S}^2(\underline{k} \circ 0, k)$ such that $(\underline{\sigma} \circ f) \cdot \sigma = (\underline{k} \circ f_I^b) \cdot (\underline{k})_I^\sharp$. Thus $\tau := \underline{\sigma}^{-1} \cdot \underline{k}_I^\sharp$ satisfies $\sigma = (\tau \circ f) \cdot f_I^b$. If there exists another $\tau' \in \mathbf{S}^2(k, 0)$ satisfying $\sigma = (\tau' \circ f) \cdot f_I^b$, then by the universality of the kernel, there exists $v \in \mathbf{S}^2(\underline{k}, 0)$ such that $(v \circ 0) \cdot \tau'^{-1} = \underline{\tau}$. Since $v \circ 0 = \underline{k}_I^\sharp$ by (21.7), we obtain $\tau = \tau'$. Thus τ is uniquely determined. \square

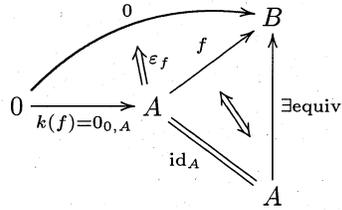
Proposition 21.26. *For any $f \in \mathbf{S}^1(A, B)$, the following are equivalent.*

- (1) f is an equivalence.
- (2) f is cofaithful and fully faithful.
- (3) f is faithful and fully cofaithful.

Proof. Since (1) \Leftrightarrow (3) is the dual of (1) \Leftrightarrow (2), we show only (1) \Leftrightarrow (2).

(1) \Rightarrow (2) : trivial.

(2) \Rightarrow (1) : Since f is cofaithful, we have $f = \text{cok}(\text{ker}(f))$, $\text{Cok}(k(f)) = [B, f, \varepsilon_f]$. On the other hand, since f is fully faithful, we have $\text{Ker}(f) = [0, 0, f_I^b]$, and so we have $\text{Cok}(k(f)) = [A, \text{id}_A, \text{id}_0]$. And by the uniqueness (up to an equivalence) of the cokernel, there is an equivalence from A to B , which is equivalent to f . Thus, f becomes an equivalence. \square



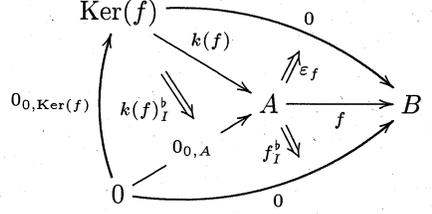
Lemma 21.27. *Let $f : A \rightarrow B$ be a faithful 1-cell in \mathbf{S} . Then, for any $K \in \mathbf{S}^0$ and $k \in \mathbf{S}^1(K, 0)$, we have $\mathbf{S}^2(k \circ 0_{0, \text{Ker}(f)}, 0_{K, \text{Ker}(f)}) = \{k_I^\sharp\}$.*

$$\begin{array}{ccc}
 & 0_{K, \text{Ker}(f)} & \\
 & \uparrow k_I^\sharp & \\
 K & \xrightarrow{k} 0 & \xrightarrow{0_{0, \text{Ker}(f)}} \text{Ker}(f)
 \end{array}$$

Proof. For any $\sigma \in \mathbf{S}^2(k \circ 0_{0, \text{Ker}(f)}, 0_{K, \text{Ker}(f)})$, we can show $((\sigma \circ k(f)) \cdot k(f)_I^b) \circ f = ((k \circ k(f)_I^b) \cdot k_I^\sharp) \circ f$. By the faithfulness of f , we have $(\sigma \circ k(f)) \cdot k(f)_I^b = (k \circ k(f)_I^b) \cdot k_I^\sharp$. Thus, we have $\sigma \circ k(f) = k_I^\sharp \circ k(f)$. By the faithfulness of $k(f)$, we obtain $\sigma = k_I^\sharp$. \square

Corollary 21.28. *$f : A \rightarrow B$ is faithful if and only if $\text{Ker}(0_{0,A}, f_I^b) = 0$.*

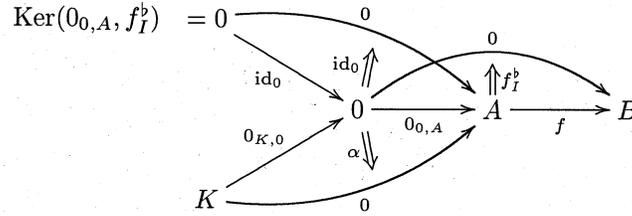
Proof. Since there is a factorization diagram with $(0_{0, \text{Ker}(f)} \circ \varepsilon_f) \cdot (0_{0, \text{Ker}(f)})_I^\sharp = (k(f)_I^b \circ f) \cdot f_I^b$



(see (a3-2) in Definition 21.1) we have $\text{Ker}(0_{0, A}, f_I^b) = \text{Ker}(0_{0, \text{Ker}(f)})$ by Proposition 21.20. So, it suffices to show $\text{Ker}(0_{0, \text{Ker}(f)}) = 0$. For any $K \in \mathbf{S}^0$ and $k \in \mathbf{S}^1(K, 0)$, we have $\mathbf{S}^2(k \circ 0_{0, \text{Ker}(f)}, 0_{K, \text{Ker}(f)}) = \{k_I^\sharp\}$ by the Lemma 21.27. So $0_{0, \text{Ker}(f)}$ becomes fully faithful, and thus $\text{Ker}(0_{0, \text{Ker}(f)}) = 0$.

Conversely, assume $\text{Ker}(0_{0, A}, f_I^b) = 0$. For any $K \in \mathbf{S}^0$ and $\alpha \in \mathbf{S}^2(0_{K, A}, 0_{K, A})$ satisfying $\alpha \circ f = \text{id}_{0 \circ f}$, we show $\alpha = \text{id}_0$ (Corollary 21.23).

By $\alpha \circ f = \text{id}_{0 \circ f}$, α is compatible with f_I^b :



So there exist $k \in \mathbf{S}^1(K, 0)$ and $\varepsilon \in \mathbf{S}^2(k \circ \text{id}_0, 0_{K, 0})$ satisfying

$$(\varepsilon \circ 0_{0, A}) \cdot \alpha = (k \circ \text{id}_0) \cdot k_I^\sharp.$$

Since $\varepsilon \circ 0_{0, A} = k_I^\sharp$ by (21.1) and (21.6), we obtain $\alpha = \text{id}_0$. \square

In any relatively exact 2-category \mathbf{S} , the difference kernel of any pair of 1-cells $g, h : A \rightarrow B$ always exists. More precisely, we have the following proposition:

Proposition 21.29. *For any $g, h \in \mathbf{S}^1(A, B)$, if we take the kernel $\text{Ker}(g \otimes h^*) = [\text{Ker}(g \otimes h^*), k, \varepsilon]$ of $g \otimes h^*$ and put $\tilde{\varepsilon} := \Psi_{k \circ g, k \circ h}(\Theta_{g, h}^{k_I^\sharp}(\varepsilon \cdot k_I^{\sharp-1})) \in \mathbf{S}^2(k \circ g, k \circ h)$, then $(\text{Ker}(g \otimes h^*), k, \tilde{\varepsilon})$ is the difference kernel of g and h .*

Proof. For any $K \in \mathbf{S}^0$ and $\ell \in \mathbf{S}^1(K, A)$, there exists a natural isomorphism (Lemma 20.16)

$$\begin{array}{ccc} \mathbf{S}^2(\ell \circ (g \otimes h^*), 0) & \xrightarrow{\quad} & \mathbf{S}^2(\ell \circ g, \ell \circ h) \\ \Downarrow \Psi & & \Downarrow \Psi \\ \sigma & \xrightarrow{\quad} & \tilde{\sigma} := \Psi_{\ell \circ g, \ell \circ h}(\Theta_{g, h}^{\ell_I^\sharp}(\sigma \cdot \ell_I^{\sharp-1})). \end{array}$$

So, to give a 2-cell $\sigma \in \mathbf{S}^2(\ell \circ (g \otimes h^*), 0)$ is equivalent to give a 2-cell $\tilde{\sigma} \in \mathbf{S}^2(\ell \circ g, \ell \circ h)$. And, by using Remark 21.4 and Corollary 21.5, the usual compatibility argument shows the proposition. \square

In any relatively exact 2-category \mathbf{S} , the pullback of any pair of morphisms $f_i : A_i \rightarrow B$ ($i = 1, 2$) always exists. More precisely, we have the following proposition:

Proposition 21.30. For any $f_i \in \mathbf{S}^1(A_i, B)$ ($i = 1, 2$), if we take the product of A_1 and A_2 ($A_1 \times A_2, p_1, p_2$), and take the difference kernel (D, d, φ) of $p_1 \circ f_1$ and $p_2 \circ f_2$

$$D \xrightarrow{d} A_1 \times A_2 \begin{array}{c} \xrightarrow{p_1 \circ f_1} \\ \xrightarrow{p_2 \circ f_2} \end{array} B \quad \begin{array}{ccc} D & \begin{array}{c} \xrightarrow{d \circ p_1} \\ \xrightarrow{d \circ p_2} \end{array} & \begin{array}{c} A_1 \\ \Downarrow \varphi \\ A_2 \end{array} & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} & B \end{array},$$

then, $(D, d \circ p_1, d \circ p_2, \varphi)$ is the pullback of f_1 and f_2 .

Proof.

proof of condition (b1) (in Definition 20.4)

For any $X \in \mathbf{S}^0$, $g_i \in \mathbf{S}^1(X, A_i)$ ($i = 1, 2$) and $\eta \in \mathbf{S}^2(g_1 \circ f_1, g_2 \circ f_2)$, by the universality of $A_1 \times A_2$, there exist $g \in \mathbf{S}^1(X, A_1 \times A_2)$ and $\xi_i \in \mathbf{S}^2(d \circ p_i, g_i)$ ($i = 1, 2$). Applying the universality of the difference kernel to the 2-cell

$$(21.11) \quad \zeta := (\xi_1 \circ f_1) \cdot \eta \cdot (\xi_2^{-1} \circ f_2) \in \mathbf{S}^2(g \circ p_1 \circ f_1, g \circ p_2 \circ f_2),$$

we see there exist $\underline{g} \in \mathbf{S}^1(X, D)$ and $\underline{\zeta} \in \mathbf{S}^2(\underline{g} \circ d, g)$

$$(21.12) \quad \begin{array}{ccccc} & X & & & \\ & \swarrow g & & \searrow g & \\ D & \xrightarrow{d} & A_1 \times A_2 & \xrightarrow[p_2 \circ f_2]{p_1 \circ f_1} & B \\ & \nearrow \underline{\zeta} & & & \end{array}$$

such that

$$(21.13) \quad (\underline{g} \circ \varphi) \cdot (\underline{\zeta} \circ p_2 \circ f_2) = (\underline{\zeta} \circ p_1 \circ f_1) \cdot \zeta.$$

By (21.11) and (21.13), we have $(\underline{g} \circ \varphi) \cdot (((\underline{\zeta} \circ p_2) \cdot \xi_2) \circ f_2) = (((\underline{\zeta} \circ p_1) \cdot \xi_1) \circ f_1) \cdot \eta$, and thus condition (b1) is satisfied.

$$\begin{array}{ccccc} & & & & (\underline{\zeta} \circ p_1) \cdot \xi_1 \\ & & & & \vdots \\ & & & & \dots \\ X & \xrightarrow{g} & D & \begin{array}{c} \xrightarrow{d \circ p_1} \\ \xrightarrow{d \circ p_2} \end{array} & \begin{array}{c} A_1 \\ \Downarrow \varphi \\ A_2 \end{array} & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} & B \\ & \nearrow g_1 & \nearrow g_2 & & & & \\ & & & & & & (\underline{\zeta} \circ p_2) \cdot \xi_2 \end{array}$$

proof of condition (b2)

If we take $h \in \mathbf{S}^1(X, D)$ and $\eta_i \in \mathbf{S}^2(h \circ d \circ p_i, g_i)$ ($i = 1, 2$) which satisfy $(h \circ \varphi) \cdot (\eta_2 \circ f_2) = (\eta_1 \circ f_1) \cdot \eta$, then by the universality of $A_1 \times A_2$, there exists a unique 2-cell $\kappa \in \mathbf{S}^2(h \circ d, g)$ such that

$$(21.14) \quad (\kappa \circ p_i) \cdot \xi_i = \eta_i \quad (i = 1, 2).$$

Then, κ becomes compatible with φ and ζ , i.e. $(h \circ \varphi) \cdot (\kappa \circ p_2 \circ f_2) = (\kappa \circ p_1 \circ f_1) \cdot \zeta$. So, comparing this with factorization (21.12), by the universality of the difference

kernel, we see there exists a unique 2-cell $\chi \in \mathbf{S}^2(h, \underline{g})$ which satisfies

$$(21.15) \quad (\chi \circ d) \cdot \underline{\zeta} = \kappa$$

Then we have $(\chi \circ d \circ p_i) \cdot (\underline{\zeta} \circ p_i) \cdot \xi_i = (\kappa \circ p_i) \cdot \xi_i = \eta_i$ ($i = 1, 2$). Thus χ is the desired 2-cell in condition (b2), and the uniqueness of such a χ follows from the uniqueness of κ and χ which satisfy (21.14) and (21.15). \square

By the universality of the pullback, we have the following remark:

Remark 21.31. *Let*

$$(21.16) \quad \begin{array}{ccccc} & & A_2 & & \\ & f'_1 \nearrow & & f_2 \searrow & \\ A_1 \times_B A_2 & & A_2 & & B \\ & f'_2 \searrow & & f_1 \nearrow & \\ & & A_1 & & \end{array} \quad \Downarrow \xi$$

be a pull-back diagram. Then, for any $K \in \mathbf{S}^0$, $g, h \in \mathbf{S}^1(K, A_1 \times_B A_2)$ and $\alpha, \beta \in \mathbf{S}^2(g, h)$, we have

$$\alpha \circ f'_i = \beta \circ f'_i \quad (i = 1, 2) \implies \alpha = \beta.$$

Proof. To the diagram

$$\begin{array}{ccccc} & & A_2 & & \\ & g \circ f'_1 \nearrow & & f_2 \searrow & \\ K & & A_2 & & B \\ & g \circ f'_2 \searrow & & f_1 \nearrow & \\ & & A_1 & & \end{array} \quad \Downarrow g \circ \xi$$

the following diagram gives a factorization which satisfies condition (b1) in Definition 20.4.

$$\begin{array}{ccccc} & & A_2 & & \\ & g \circ f'_1 \nearrow & & f_2 \searrow & \\ K & \xrightarrow{g} & A_1 \times_B A_2 & & B \\ & g \circ f'_2 \searrow & & f_1 \nearrow & \\ & & A_1 & & \end{array} \quad \Downarrow \xi$$

Since each of $\text{id}_g : g \implies g$ and $\alpha \circ \beta^{-1} : g \implies g$ gives a 2-cell which satisfies condition (b2), we have $\alpha \circ \beta^{-1} = \text{id}$ by the uniqueness. Thus $\alpha = \beta$. \square

Proposition 21.32. (See also Proposition 23.12.) *Let (21.16) be a pull-back diagram. We have*

- (1) f_1 : faithful $\implies f'_1$: faithful.
- (2) f_1 : fully faithful $\implies f'_1$: fully faithful.
- (3) f_1 : cofaithful $\implies f'_1$: cofaithful.

Proof. proof of (1) By Corollary 21.23, it suffices to show $\alpha \circ f'_1 = \text{id}_{0 \circ f'_1} \implies \alpha = \text{id}_0$ for any $K \in \mathbf{S}^0$ and $\alpha \in \mathbf{S}^2(0_{K, A_1 \times_B A_2}, 0_{K, A_1 \times_B A_2})$. Since $(0 \circ \xi) \cdot (\alpha \circ f'_2 \circ f_1) = (\alpha \circ f'_1 \circ f_2) \cdot (0 \circ \xi) = \text{id}_{0 \circ f'_1 \circ f_2} \cdot (0 \circ \xi) = 0 \circ \xi$, we have $\alpha \circ f'_2 \circ f_1 = \text{id}_{0 \circ f'_2 \circ f_1} = \text{id}_{0 \circ f'_2} \circ f_1$. Since f_1 is faithful, we obtain $\alpha \circ f'_2 = \text{id}_{0 \circ f'_2}$. Thus, we have $\alpha \circ f'_i = \text{id}_{0 \circ f'_i} = \text{id}_0 \circ f'_i$ ($i = 1, 2$). By Remark 21.31, this implies $\alpha = \text{id}_0$.

proof of (2) By (1), f'_1 is already faithful. By Corollary 21.23, it suffices to show that for any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A_1 \times_B A_2)$ and any $\sigma \in \mathbf{S}^2(k \circ f'_1, 0)$, there exists

a unique 2-cell $\kappa \in \mathbf{S}^2(k, 0)$ such that $\sigma = (\kappa \circ f'_1) \cdot (f'_1)_I^\flat$. Since f_1 is fully faithful, for any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, A_1 \times_B A_2)$ and any $\sigma \in \mathbf{S}^2(k \circ f'_1, 0)$, there exists $\tau \in \mathbf{S}^2(k \circ f'_2, 0)$ such that $(\tau \circ f_1) \cdot (f_1)_I^\flat = (k \circ \xi^{-1}) \cdot (\sigma \circ f_2) \cdot (f_2)_I^\flat$. Then, for the diagram

$$\begin{array}{ccc} & A_1 & \\ 0 \nearrow & & \searrow f_1 \\ K & & B \\ 0 \searrow & & \nearrow f_2 \\ & A_2 & \end{array} \quad \begin{array}{c} \parallel \\ (f_1)_I^\flat \cdot (f_2)_I^{\flat^{-1}} \\ \Downarrow \end{array}$$

each of the factorizations

$$\begin{array}{ccc} & A_2 & \\ \overset{0}{\curvearrowright} & & \searrow f_2 \\ K & \xrightarrow{k} & A_1 \times_B A_2 \\ \underset{0}{\curvearrowleft} & & \nearrow f_1 \\ & A_1 & \end{array} \quad \begin{array}{c} \Downarrow \xi \\ \Downarrow \end{array} \quad \begin{array}{ccc} & A_2 & \\ \overset{0}{\curvearrowright} & & \searrow f_2 \\ K & \xrightarrow{k} & A_1 \times_B A_2 \\ \underset{0}{\curvearrowleft} & & \nearrow f_1 \\ & A_1 & \end{array} \quad \begin{array}{c} \Downarrow \xi \\ \Downarrow \end{array}$$

satisfies condition (b1) in Definition 20.4. So there exists a 2-cell $\kappa \in \mathbf{S}^2(k, 0)$ such that $\sigma = (\kappa \circ f'_1) \cdot (f'_1)_I^\flat$. Uniqueness of such κ follows from the faithfulness of f'_1 .
proof of (3) Let $(A_1 \times A_2, p_1, p_2)$ be the product of A_1 and A_2 . For $\text{id}_{A_1} \in \mathbf{S}^1(A_1, A_1)$ and $0 \in \mathbf{S}^1(A_1, A_2)$, by the universality of $A_1 \times A_2$, there exist $i_1 \in \mathbf{S}^1(A_1, A_1 \times A_2)$, $\xi_1 \in \mathbf{S}^2(i_1 \circ p_1, \text{id}_{A_1})$ and $\xi_2 \in \mathbf{S}^2(i_2 \circ p_2, 0)$. Similarly, there is a 1-cell $i_2 \in \mathbf{S}^1(A_2, A_1 \times A_2)$ such that there are equivalences $i_2 \circ p_2 \simeq \text{id}_{A_2}$, $i_2 \circ p_1 \simeq 0$. If we put $t := (p_1 \circ f_1) \otimes (p_2 \circ f_2)^*$, then by Proposition 21.29 and 21.30, we have $A_1 \times_B A_2 = \text{Ker}(t)$. So we may assume $\text{Ker}(t) = [A_1 \times_B A_2, d, \varepsilon_t]$ and $f'_1 = d \circ p_2$.

$$\begin{array}{ccc} & 0 & \\ & \uparrow \varepsilon_t & \\ A_1 \times_B A_2 & \xrightarrow{d} & A_1 \times A_2 \xrightarrow{t} B \end{array}$$

Since $i_1 \circ t$ and f_1 are equivalent;

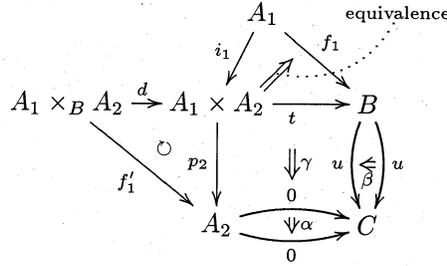
$$i_1 \circ t \simeq (i_1 \circ p_1 \circ f_1) \otimes (i_1 \circ p_2 \circ f_2^*) \simeq (\text{id}_{A_1} \circ f_1) \otimes (0 \circ f_2^*) \simeq f_1,$$

by the cofaithfulness of f_1 , it follows that t is cofaithful. Thus, we have $B = \text{Cok}(\text{ker}(t))$, i.e. $\text{Cok}(d) = [B, t, \varepsilon_t]$. By (the dual of) Corollary 21.23, it suffices to show $f'_1 \circ \alpha = \text{id}_{f'_1 \circ 0} \Rightarrow \alpha = \text{id}_0$ for any $C \in \mathbf{S}^0$ and any $\alpha \in \mathbf{S}^2(0_{A_2, C}, 0_{A_2, C})$. For the 2-cell $(d \circ p_2)_I^\sharp \in \mathbf{S}^2(d \circ p_2 \circ 0_{A_2, C}, 0)$ (see the following diagram), by the universality of $\text{Cok}(d)$, there exist $u \in \mathbf{S}^1(B, C)$ and $\gamma \in \mathbf{S}^2(t \circ u, p_2 \circ 0)$ such that $(d \circ \gamma) \cdot (d \circ p_2)_I^\sharp = (\varepsilon_t \circ u) \cdot u_I^\flat$. Thus, if we put $\gamma' := \gamma \cdot (p_2 \circ \alpha)$, we have

$$\begin{aligned} (d \circ \gamma') \cdot (d \circ p_2)_I^\sharp &= (d \circ \gamma) \cdot (d \circ p_2 \circ \alpha) \cdot (d \circ p_2)_I^\sharp \\ &= (d \circ \gamma) \cdot (f'_1 \circ \alpha) \cdot (d \circ p_2)_I^\sharp = (\varepsilon_t \circ u) \cdot u_I^\flat. \end{aligned}$$

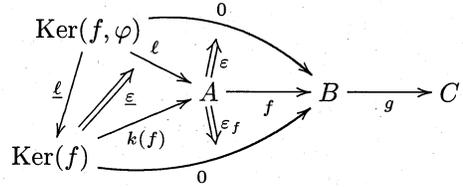
So, γ and $\gamma' \in \mathbf{S}^2(t \circ u, p_2 \circ 0)$ give two factorization of $p_2 \circ 0$ compatible with ε_t and $(d \circ p_2)_I^\sharp$. By the universality of $\text{Cok}(d) = [B, t, \varepsilon_t]$, there exists a unique 2-cell $\beta \in \mathbf{S}^2(u, u)$ such that

$$(21.17) \quad (t \circ \beta) \cdot \gamma = \gamma'.$$



Then we have $(i_1 \circ t \circ \beta) \cdot (i_1 \circ \gamma) = i_1 \circ \gamma' = (i_1 \circ \gamma) \cdot (i_1 \circ p_2 \circ \alpha) = (i_1 \circ \gamma) \cdot (\xi_2 \circ 0) \cdot (0 \circ \alpha) \cdot (\xi_2^{-1} \circ 0) = (i_1 \circ \gamma)$, and thus, $(i_1 \circ t) \circ \beta = \text{id}_{i_1 \circ t \circ u}$. Since $i_1 \circ t \simeq f_1$ is cofaithful, we obtain $\beta = \text{id}_u$. So, by (21.17), we have $\gamma = \gamma' = \gamma \cdot (p_2 \circ \alpha)$, and consequently $p_2 \circ \alpha = \text{id}_{p_2 \circ 0}$. Since p_2 is cofaithful (because $i_2 \circ p_2 \simeq \text{id}_{A_2}$ is cofaithful), we obtain $\alpha = \text{id}_0$. \square

Proposition 21.33. Consider diagram (21.8) in \mathbf{S} . If we take the relative kernel $\text{Ker}(f, \varphi) = [\text{Ker}(f, \varphi), \ell, \varepsilon]$, then by the universality of $\text{Ker}(f) = [\text{Ker}(f), k(f), \varepsilon_f]$, ℓ factors uniquely up to an equivalence as



where $(\underline{\varepsilon} \circ f) \cdot \varepsilon = (\underline{\ell} \circ \varepsilon_f) \cdot (\underline{\ell})_I^\sharp$. Then, $\underline{\ell}$ becomes fully faithful.

Proof. Since $\underline{\ell} \circ k(f)$ is equivalent to a faithful 1-cell ℓ , so $\underline{\ell}$ becomes faithful. For any $K \in \mathbf{S}^0$, $k \in \mathbf{S}^1(K, \text{Ker}(f, \varphi))$ and $\sigma \in \mathbf{S}^2(k \circ \underline{\ell}, 0)$, if we put $\sigma' := (k \circ \underline{\varepsilon}^{-1}) \cdot (\sigma \circ k(f)) \cdot k(f)_I^\sharp \in \mathbf{S}^2(k \circ \ell, 0)$, then σ' becomes compatible with ε . So, by Lemma 21.19, there exists $\tau \in \mathbf{S}^2(k, 0)$ such that $\sigma' = (\tau \circ \ell) \cdot \ell_I^\sharp$, i.e.

$$(21.18) \quad (k \circ \underline{\varepsilon}^{-1}) \cdot (\sigma \circ k(f)) \cdot (k(f))_I^\sharp = (\tau \circ \ell) \cdot \ell_I^\sharp.$$

Now, since $(k \circ \underline{\varepsilon}) \cdot (\tau \circ \ell) \cdot \ell_I^\sharp = (\tau \circ \underline{\ell} \circ k(f)) \cdot (\underline{\ell} \circ k(f))_I^\sharp$ by Corollary 21.5, (21.18) is equivalent to $(\sigma \circ k(f)) \cdot (k(f))_I^\sharp = (\tau \circ \underline{\ell} \circ k(f)) \cdot (\underline{\ell}_I^\sharp \circ k(f)) \cdot (k(f))_I^\sharp$.

Thus, we obtain $\sigma \circ k(f) = ((\tau \circ \underline{\ell}) \cdot \underline{\ell}_I^\sharp) \circ k(f)$. Since $k(f)$ is faithful, it follows that $\sigma = (\tau \circ \underline{\ell}) \cdot \underline{\ell}_I^\sharp$. Uniqueness of such τ follows from the faithfulness of $\underline{\ell}$. Thus $\underline{\ell}$ becomes fully faithful by Corollary 21.24. \square

22. EXISTENCE OF PROPER FACTORIZATION SYSTEMS

Definition 22.1. For any $A, B \in \mathbf{S}^0$ and $f \in \mathbf{S}^1(A, B)$, we define its image as $\text{Ker}(\text{cok}(f))$.

Remark 22.2. By the universality of the kernel, there exist $i(f) \in \mathbf{S}^1(A, \text{Im}(f))$ and $\iota \in \mathbf{S}^2(i(f) \circ k(c(f)), f)$ such that $(\iota \circ c(f)) \cdot \pi_f = (i(f) \circ \varepsilon_{c(f)}) \cdot i(f)_I^\sharp$. Coimage of f is defined dually, and we obtain a factorization through $\text{Coim}(f)$.

Proposition 22.3 (1st factorization). *For any $f \in \mathbf{S}^1(A, B)$, the factorization $\iota : i(f) \circ k(c(f)) \Rightarrow f$ through $\text{Im}(f)$*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow i(f) & \nearrow k(c(f)) \\ & \text{Im}(f) & \end{array}$$

satisfies the following properties:

- (A) $i(f)$ is fully cofaithful and $k(c(f))$ is faithful.
- (B) For any factorization $\eta : i \circ m \Rightarrow f$ where m is faithful, following (b1) and (b2) hold:
 - (b1) There exist $t \in \mathbf{S}^1(\text{Im}(f), C)$, $\zeta_m \in \mathbf{S}^2(t \circ m, k(c(f)))$, $\zeta_i \in \mathbf{S}^2(i(f) \circ t, i)$

$$\begin{array}{ccc} & C & \\ & \nearrow i & \searrow m \\ A & \xrightarrow{\zeta_i} t & \xrightarrow{\zeta_m} B \\ & \searrow i(f) & \nearrow k(c(f)) \\ & \text{Im}(f) & \end{array}$$

such that $(i(f) \circ \zeta_m) \cdot \iota = (\zeta_i \circ m) \cdot \eta$.

- (b2) If both (t, ζ_m, ζ_i) and (t', ζ'_m, ζ'_i) satisfy (b1), then there is a unique 2-cell $\kappa \in \mathbf{S}^2(t, t')$ such that $(i(f) \circ \kappa) \cdot \zeta'_i = \zeta_i$ and $(\kappa \circ m) \cdot \zeta'_m = \zeta_m$.

Dually, we obtain the following proposition for the coimage factorization.

Proposition 22.4 (2nd factorization). *For any $f \in \mathbf{S}^1(A, B)$, the factorization $\mu : c(k(f)) \circ j(f) \Rightarrow f$ through $\text{Coim}(f)$*

$$\begin{array}{ccc} & \text{Coim}(f) & \\ c(k(f)) \nearrow & \mu \Downarrow & \searrow j(f) \\ A & \xrightarrow{f} & B \end{array}$$

satisfies the following properties:

- (A) $j(f)$ is fully faithful and $c(k(f))$ is cofaithful.
- (B) For any factorization $\nu : e \circ j \Rightarrow f$ where e is cofaithful, following (b1) and (b2) (the dual of the conditions in Proposition 22.3) hold:
 - (b1) There exists $s \in \mathbf{S}^1(C, \text{Coim}(f))$, $\zeta_e \in \mathbf{S}^2(e \circ s, c(k(f)))$, and $\zeta_j \in \mathbf{S}^2(s \circ j(f), j)$

$$\begin{array}{ccc} & \text{Coim}(f) & \\ c(k(f)) \nearrow & \uparrow s & \searrow j(f) \\ A & \xrightarrow{\zeta_e} & \xrightarrow{\zeta_j} B \\ & \searrow e & \nearrow j \\ & C & \end{array}$$

such that $(e \circ \zeta_j) \cdot \nu = (\zeta_e \circ j(f)) \cdot \mu$.

- (b2) If both (s, ζ_e, ζ_j) and (s', ζ'_e, ζ'_j) satisfy (b1), then there is a unique 2-cell $\lambda \in \mathbf{S}^2(s, s')$ such that $(\lambda \circ j(f)) \cdot \zeta'_j = \zeta_j$ and $(e \circ \lambda) \cdot \zeta'_e = \zeta_e$.

In the rest of this section, we show Proposition 22.3.

Lemma 22.5. *For any $f \in \mathbf{S}^1(A, B)$, $i(f)$ is cofaithful.*

Proof. It suffices to show that for any $C \in \mathbf{S}^0$ and $\alpha \in \mathbf{S}^2(0_{\text{Im}(f), C}, 0_{\text{Im}(f), C})$

$$A \xrightarrow{i(f)} \text{Im}(f) \begin{array}{c} \xrightarrow{0} \\ \Downarrow \alpha \\ \xrightarrow{0} \end{array} C,$$

we have $i(f) \circ \alpha = \text{id}_{i(f) \circ 0} \implies \alpha = \text{id}_0$. Take the pushout of $k(c(f))$ and $0_{\text{Im}(f), C}$

$$\begin{array}{ccc} k(c(f)) & \nearrow & B \\ \text{Im}(f) & \searrow & C \\ 0 & \searrow & C \end{array} \quad \begin{array}{c} i_B \\ \Downarrow \xi \\ i_C \end{array} \quad \begin{array}{c} C \\ \amalg \\ \text{Im}(f) \\ B \end{array}$$

and put

$$\xi_1 := \xi \cdot (\xi_1 \circ f_2) \cdot \eta = (g \circ \xi) \cdot (\xi_2 \circ f_1) \cdot (i_C)_I^b = (k(c(f)) \circ i_B \xrightarrow{\xi} 0 \circ i_C \xrightarrow{(i_C)_I^b} 0)$$

$$\xi_2 := \xi \cdot (\alpha \circ i_C) \cdot (i_C)_I^b = (k(c(f)) \circ i_B \xrightarrow{\xi} 0 \circ i_C \xrightarrow{\alpha \circ i_C} 0 \circ i_C \xrightarrow{(i_C)_I^b} 0).$$

Then, since i_C is faithful by (the dual of) Lemma 21.32, we have

$$\alpha = \text{id}_0 \iff \alpha \circ i_C = \text{id}_{0 \circ i_C} \iff \xi \cdot (\alpha \circ i_C) \cdot (i_C)_I^b = \xi \cdot \text{id}_{0 \circ i_C} \cdot (i_C)_I^b \iff \xi_1 = \xi_2.$$

So, it suffices to show $\xi_1 = \xi_2$. For each $i = 1, 2$, since we have $\text{Cok}(k(c(f))) = [\text{Cok}(f), c(f), \varepsilon_{c(f)}]$, there exist $e_i \in \mathbf{S}^1(\text{Cok}(f), C \amalg_{\text{Im}(f)} B)$ and $\varepsilon_i \in \mathbf{S}^2(c(f) \circ e_i, i_B)$

such that

$$(22.1) \quad (k(c(f)) \circ \varepsilon_i) \cdot \xi_i = (\varepsilon_{c(f)} \circ e_i) \cdot (e_i)_I^b.$$

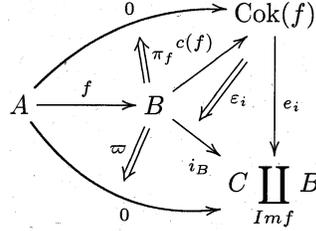
$$\begin{array}{ccc} & & \text{Cok}(f) \\ & \nearrow & \uparrow \\ & \varepsilon_{c(f)} & c(f) \\ & \nearrow & \uparrow \\ \text{Im}(f) & \xrightarrow{k(c(f))} & B \\ & \searrow & \downarrow \varepsilon_i \\ & 0 & C \amalg_{\text{Im}(f)} B \\ & \searrow & \uparrow e_i \end{array}$$

Since by assumption $i(f) \circ \alpha = \text{id}_{i(f) \circ 0}$, we have

$$\begin{aligned} i(f) \circ \xi_2 &= (i(f) \circ \xi) \cdot (i(f) \circ \alpha \circ i_C) \cdot (i(f) \circ (i_C)_I^b) \\ &= (i(f) \circ \xi) \cdot (\text{id}_{i(f) \circ 0 \circ i_C}) \cdot (i(f) \circ (i_C)_I^b) = i(f) \circ \xi_1. \end{aligned}$$

So, if we put $\varpi := (\iota^{-1} \circ i_B) \cdot (i(f) \circ \xi_i) \cdot (i(f))_I^\# \in \mathbf{S}^2(f \circ i_B, 0)$, this doesn't depend on $i = 1, 2$. We can show easily $(f \circ \varepsilon_i) \cdot \varpi = (\pi_f \circ e_i) \cdot (e_i)_I^b$ ($i = 1, 2$). Thus (e_1, ε_1)

and (e_2, ε_2) are two factorizations of i_B compatible with ϖ and π_f .



By the universality of $\text{Cok}(f)$, there exists a 2-cell $\beta \in \mathbf{S}^2(e_1, e_2)$ such that $(c(f) \circ \beta) \cdot \varepsilon_2 = \varepsilon_1$, and thus we have $\varepsilon_1^{-1} = \varepsilon_2^{-1} \cdot (c(f) \circ \beta^{-1})$. So, by (22.1), we have

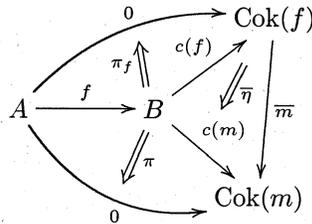
$$\begin{aligned} \xi_1 &= (k(c(f)) \circ \varepsilon_1^{-1}) \cdot (\varepsilon_{c(f)} \circ e_1) \cdot (e_1)_I^b \\ &= (k(c(f)) \circ \varepsilon_2^{-1}) \cdot (k(c(f)) \circ c(f) \circ \beta^{-1}) \cdot (\varepsilon_{c(f)} \circ e_1) \cdot (e_1)_I^b \\ &\stackrel{21.5}{=} (k(c(f)) \circ \varepsilon_2^{-1}) \cdot (\varepsilon_{c(f)} \circ e_2) \cdot (e_2)_I^b = \xi_2. \end{aligned}$$

□

Lemma 22.6. *Let $f \in \mathbf{S}^1(A, B)$. Let $\iota : i(f) \circ k(c(f)) \Rightarrow f$ be the factorization of f through $\text{Im}(f)$ as before. If we are given a factorization $\eta : i \circ m \Rightarrow f$ of f where $i \in \mathbf{S}^1(A, C)$, $m \in \mathbf{S}^1(C, B)$ and m is faithful, then there exist $t \in \mathbf{S}^1(\text{Im}(f), C)$, $\zeta_i \in \mathbf{S}^2(i(f) \circ t, i)$ and $\zeta_m \in \mathbf{S}^2(t \circ m, k(c(f)))$ such that $(\zeta_i \circ m) \cdot \eta = (i(f) \circ \zeta_m) \cdot \iota$.*

Proof. By the universality of $\text{Cok}(f)$, for $\pi := (\eta^{-1} \circ c(m)) \cdot (i \circ \pi_m) \cdot i_I^\# \in \mathbf{S}^2(f \circ c(m), 0)$, there exist $\bar{m} \in \mathbf{S}^1(\text{Cok}(f), \text{Cok}(m))$ and $\bar{\eta} \in \mathbf{S}^2(c(f) \circ \bar{m}, c(m))$ such that

$$(22.2) \quad (f \circ \bar{\eta}) \cdot \pi = (\pi_f \circ \bar{m}) \cdot (\bar{m})_I^b.$$



Since m is faithful by assumption, it follows $\text{Ker}(c(m)) = [C, m, \pi_m]$. By the universality of $\text{Ker}(c(m))$, for the 2-cell

$$(22.3) \quad \zeta := (k(c(f)) \circ \bar{\eta}^{-1}) \cdot (\varepsilon_{c(f)} \circ \bar{m}) \cdot (\bar{m})_I^b \in \mathbf{S}^2(k(c(f)) \circ c(m), 0),$$

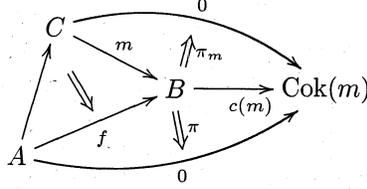
there exist $t \in \mathbf{S}^1(\text{Im}(f), C)$ and $\zeta_m \in \mathbf{S}^2(t \circ m, k(c(f)))$ such that $(\zeta_m \circ c(m)) \cdot \zeta = (t \circ \pi_m) \cdot t_I^\#$.

If we put $\bar{\zeta} := (i(f) \circ \zeta_m) \cdot \iota$, then the following claim holds:

Claim 22.7. *Each of the two factorizations of f through $\text{Ker}(c(m))$*

$$\eta : i \circ m \Rightarrow f \quad \text{and} \quad \bar{\zeta} : i(f) \circ t \circ m \Rightarrow f$$

is compatible with π_m and π .



If the above claim is proven, then by the universality of $\text{Ker}(c(m)) = [C, m, \pi_m]$, there exists a unique 2-cell $\zeta_i \in \mathbf{S}^2(i(f) \circ t, i)$ such that $(\zeta_i \circ m) \cdot \eta = \bar{\zeta}$. Thus we obtain (t, ζ_m, ζ_i) which satisfies $(\zeta_i \circ m) \cdot \eta = \bar{\zeta} = (i(f) \circ \zeta_m) \cdot \iota$, and the lemma is proven. So, we show Claim 22.7.

(a) compatibility of η with π_m, π

This follows immediately from the definition of π .

(b) compatibility of $\bar{\zeta}$ with π_m, π

We have

$$\begin{aligned} i(f) \circ \zeta &\stackrel{22.3}{=} (\iota \circ c(m)) \cdot (f \circ \bar{\eta}^{-1}) \cdot (\iota^{-1} \circ c(f) \circ \bar{m}) \\ &\quad \cdot (i(f) \circ \varepsilon_{c(f)} \circ \bar{m}) \cdot (i(f) \circ (\bar{m})_I^\flat) \\ &\stackrel{22.2}{=} (\iota \circ c(m)) \cdot \pi \cdot i(f)_I^{\sharp-1}. \end{aligned}$$

From this, we obtain $(i(f) \circ t \circ \pi_m) \cdot (i(f) \circ t_I^\sharp) = (\bar{\zeta} \circ c(m)) \cdot \pi \cdot i(f)_I^{\sharp-1}$. So we have

$$(\bar{\zeta} \circ c(m)) \cdot \pi = (i(f) \circ t \circ \pi_m) \cdot (i(f) \circ t_I^\sharp) \cdot i(f)_I^{\sharp-1} = (i(f) \circ t \circ \pi_m) \cdot (i(f) \circ t_I^\sharp).$$

□

Lemma 22.8. Let $A, B, C \in \mathbf{S}^0$, $f, m, i \in \mathbf{S}^1$, $\eta \in \mathbf{S}^2$ be as in Lemma 22.6. If a triplet (t', ζ'_m, ζ'_i) (where $t' \in \mathbf{S}^1(\text{Im}(f), C)$, $\zeta'_m \in \mathbf{S}^2(t' \circ m, k(c(f)))$, $\zeta'_i \in \mathbf{S}^2(i(f) \circ t', i)$) satisfies

$$(22.4) \quad (i(f) \circ \zeta'_m) \cdot \iota = (\zeta'_i \circ m) \cdot \eta,$$

then ζ'_m becomes compatible with ζ and π_m (in the notation of the proof of Lemma 22.6), i.e. we have $(\zeta'_m \circ c(m)) \cdot \zeta = (t' \circ \pi_m) \cdot (t')_I^\sharp$.

Remark 22.9. Since m is faithful, ζ'_m which satisfies (22.4) is uniquely determined by t' and ζ'_i if it exists.

Proof. (Proof of Lemma 22.8) Since we have

$$\begin{aligned} i(f) \circ ((\zeta'_m \circ c(m)) \cdot \zeta) &\stackrel{22.4, 20.1}{=} (\zeta'_i \circ m \circ c(m)) \cdot (\eta \circ c(m)) \cdot (f \circ \bar{\eta}^{-1}) \cdot (\iota^{-1} \circ c(f) \circ \bar{m}) \\ &\quad \cdot (i(f) \circ \varepsilon_{c(f)} \circ \bar{m}) \cdot (i(f) \circ (\bar{m})_I^\flat) \\ &\stackrel{22.2}{=} ((i(f) \circ t' \circ \pi_m) \cdot (i(f) \circ (t')_I^\sharp)), \end{aligned}$$

we obtain $(\zeta'_m \circ c(m)) \cdot \zeta = (t' \circ \pi_m) \cdot (t')_I^\sharp$ by the cofaithfulness of $i(f)$. □

Corollary 22.10. Let A, B, C, f, m, i, η as in Proposition 22.3. If both (t, ζ_m, ζ_i) and (t', ζ'_m, ζ'_i) satisfy (b1), then there exists a unique 2-cell $\kappa \in \mathbf{S}^2(t, t')$ such that $(i(f) \circ \kappa) \cdot \zeta'_i = \zeta_i$ and $(\kappa \circ m) \cdot \zeta'_m = \zeta_m$.

Proof. By Lemma 22.8, there exists a 2-cell $\kappa \in \mathbf{S}^2(t, t')$ such that $(\kappa \circ m) \cdot \zeta'_m = \zeta_m$ by the universality of $\text{Ker}(c(m)) = [C, m, \pi_m]$. This κ also satisfies $\zeta_i = (i(f) \circ \kappa) \cdot \zeta'_i$, and unique by the cofaithfulness of $i(f)$. \square

Considering the case of $C = \text{Im}(f)$, we obtain the following corollary.

Corollary 22.11. *For any $t \in \mathbf{S}^1(\text{Im}(f), \text{Im}(f))$, $\zeta_m \in \mathbf{S}^2(t \circ k(c(f)), k(c(f)))$ and $\zeta_i \in \mathbf{S}^2(i(f) \circ t, i(f))$ satisfying $(\zeta_i \circ k(c(f))) \cdot \iota = (i(f) \circ \zeta_m) \cdot \iota$, there exists a unique 2-cell $\kappa \in \mathbf{S}^2(t, \text{id}_{\text{Im}(f)})$ such that $i(f) \circ \kappa = \zeta_i$ and $\kappa \circ k(c(f)) = \zeta_m$.*

Now, we can prove Proposition 22.3.

Proof. (Proof of Proposition 22.3)

Since all the other is already shown, it suffices to show the following:

Claim 22.12. *For any $C \in \mathbf{S}^0$ and any $g, h \in \mathbf{S}^1(\text{Im}(f), C)$,*

$$i(f) \circ - : \mathbf{S}^2(g, h) \longrightarrow \mathbf{S}^2(i(f) \circ g, i(f) \circ h)$$

is surjective.

So, we show Claim 22.12. If we take the difference kernel of g and h ,

$$d_{(g,h)} : \text{DK}(g, h) \longrightarrow \text{Im}(f), \quad \varphi_{(g,h)} : d_{(g,h)} \circ g \Longrightarrow d_{(g,h)} \circ h,$$

then by the universality of the difference kernel, for any $\beta \in \mathbf{S}^2(i(f) \circ g, i(f) \circ h)$ there exist $i \in \mathbf{S}^1(A, \text{DK}(g, h))$ and $\lambda \in \mathbf{S}^2(i \circ d_{(g,h)}, i(f))$

$$\begin{array}{ccccc} & & A & & \\ & i \swarrow & & \searrow i(f) & \\ \text{DK}(g, h) & \xrightarrow{\lambda} & \text{Im}(f) & \xrightarrow{g} & C \\ & \xrightarrow{d_{(g,h)}} & & \xrightarrow{h} & \end{array}$$

such that $(i \circ \varphi_{(g,h)}) \cdot (\lambda \circ h) = (\lambda \circ g) \cdot \beta$.

If we put $m := d_{(g,h)} \circ k(c(f))$, then m becomes faithful since $d_{(g,h)}$ and $k(c(f))$ are faithful. Applying Lemma 22.6 to the factorization $\eta := (\lambda \circ k(c(f))) \cdot \iota : i \circ m \Longrightarrow f$, we obtain $t \in \mathbf{S}^1(\text{Im}(f), \text{DK}(g, h))$, $\zeta_m \in \mathbf{S}^2(t \circ m, k(c(f)))$ and $\zeta_i \in \mathbf{S}^2(i(f) \circ t, i)$ such that $(\zeta_i \circ m) \cdot \eta = (i(f) \circ \zeta_m) \cdot \iota$. Thus we have

$$(\zeta_i \circ d_{(g,h)} \circ k(c(f))) \cdot (\lambda \circ k(c(f))) \cdot \iota = (i(f) \circ \zeta_m) \cdot \iota.$$

So, if we put $\bar{\zeta}_i := (\zeta_i \circ d_{(g,h)}) \cdot \lambda \in \mathbf{S}^2(i(f) \circ t \circ d_{(g,h)}, i(f))$, then we have

$$(\bar{\zeta}_i \circ k(c(f))) \cdot \iota = (i(f) \circ \zeta_m) \cdot \iota.$$

By Corollary 22.11, there exists a 2-cell $\kappa \in \mathbf{S}^2(t \circ d_{(g,h)}, \text{id}_{\text{Im}(f)})$ such that $\kappa \circ k(c(f)) = \zeta_m$ and $i(f) \circ \kappa = \bar{\zeta}_i$. If we put $\alpha := (\kappa^{-1} \circ g) \cdot (t \circ \varphi_{(g,h)}) \cdot (\kappa \circ h) \in \mathbf{S}^2(g, h)$, we can show that α satisfies $i(f) \circ \alpha = \beta$. Thus $i(f) \circ - : \mathbf{S}^2(g, h) \longrightarrow \mathbf{S}^2(i(f) \circ g, i(f) \circ h)$ is surjective. \square

Remark 22.13. *In condition (B) of Proposition 22.3, if moreover i is fully cofaithful, then t becomes fully cofaithful since i and $i(f)$ are fully cofaithful. On the other hand, t is faithful since $k(c(f))$ is faithful. So, in this case t becomes an equivalence by Proposition 21.26.*

Together with Corollary 22.11, we can show easily the following corollary:

Corollary 22.14. For any $f \in \mathbf{S}^1(A, B)$, the following (b1) and (b2) hold:
 (b1) If in the factorizations

$$\begin{array}{ccc} & C & \\ i \nearrow & \eta \Downarrow & \searrow m \\ A & \xrightarrow{f} & B \end{array} \quad \begin{array}{ccc} & C' & \\ i' \nearrow & \eta' \Downarrow & \searrow m' \\ A & \xrightarrow{f} & B, \end{array}$$

m, m' are faithful and i, i' are fully cofaithful, then there exist $t \in \mathbf{S}^1(C, C')$, $\zeta_m \in \mathbf{S}^2(t \circ m', m)$, and $\zeta_i \in \mathbf{S}^2(i \circ t, i')$ such that $(i \circ \zeta_m) \cdot \eta = (\zeta_i \circ m') \cdot \eta'$.
 (b2) If both (t, ζ_m, ζ_i) and (t', ζ'_m, ζ'_i) satisfy (b1), then there is a unique 2-cell $\kappa \in \mathbf{S}^2(t, t')$ such that $(i \circ \kappa) \cdot \zeta'_i = \zeta_i$ and $(\kappa \circ m') \cdot \zeta'_m = \zeta_m$.

Remark 22.15. Proposition 22.3 and Proposition 22.4 implies respectively the existence of (2,1)-proper factorization system and (1,2)-proper factorization system in any relatively exact 2-category, in the sense of [9].

In the notation of this section, condition (B2) and (B3) in Definition 21.7 can be written as follows:

Corollary 22.16. For any $f \in \mathbf{S}^1(A, B)$, we have;

- (1) f is faithful iff $i(f) : A \rightarrow \text{Im}(f)$ is an equivalence.
- (2) f is cofaithful iff $j(f) : \text{Coim}(f) \rightarrow B$ is an equivalence.

Proof. Since (1) is the dual of (2), we show only (2).

In the coimage factorization diagram

$$\begin{array}{ccc} & \text{Coim}(f) & \\ c(k(f)) \nearrow & \mu_f \Downarrow & \searrow j(f) \\ A & \xrightarrow{f} & B, \end{array}$$

since $c(k(f))$ is cofaithful and $j(f)$ is fully faithful, we have
 f is cofaithful $\iff j(f)$ is cofaithful $\xleftrightarrow{\text{Prop. 21.26}}$ $j(f)$ is an equivalence. \square

23. DEFINITION OF RELATIVE 2-EXACTNESS

Diagram lemmas (1)

Definition 23.1. A complex $A_\bullet = (A_n, d_n^A, \delta_n^A)$ is a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow \delta_{n-1}^A & & \uparrow \delta_{n+1}^A & & \\ \cdots & A_{n-2} & \xrightarrow{d_{n-2}^A} & A_{n-1} & \xrightarrow{d_{n-1}^A} & A_n & \xrightarrow{d_n^A} & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_{n+2} & \cdots \\ & & \downarrow d_{n-2}^A & & \downarrow d_n^A & & \downarrow d_{n+1}^A & & & & \\ & & 0 & & 0 & & 0 & & & & \end{array}$$

where $A_n \in \mathbf{S}^0$, $d_n^A \in \mathbf{S}^1(A_n, A_{n+1})$, $\delta_n^A \in \mathbf{S}^2(d_{n-1}^A \circ d_n^A, 0)$, and satisfies the following compatibility condition for each $n \in \mathbb{Z}$:

$$(d_{n-1}^A \circ \delta_{n+1}^A) \cdot (d_{n-1}^A)_I^\sharp = (\delta_n^A \circ d_{n+1}^A) \circ (d_{n+1}^A)_I^\flat$$

Remark 23.2. We consider a bounded complex as a particular case of a complex, by adding zeroes.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow (d_0^A)_I^\# & & \uparrow \delta_2^A & & \\
 \cdots & 0 & \xrightarrow{0} & 0 & \xrightarrow{d_0^A} & A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 & \cdots \\
 & & \downarrow \text{id} & & \downarrow \delta_1^A & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
 \end{array}$$

Definition 23.3. For any complexes $A_\bullet = (A_n, d_n^A, \delta_n^A)$ and $B_\bullet = (B_n, d_n^B, \delta_n^B)$, a complex morphism $f_\bullet = (f_n, \lambda_n) : A_\bullet \rightarrow B_\bullet$ consists of $f_n \in \mathbf{S}^1(A_n, B_n)$ and $\lambda_n \in \mathbf{S}^2(d_n^A \circ f_{n+1}, f_n \circ d_n^B)$ for each n , satisfying

$$(\delta_n^A \circ f_{n+1}) \cdot (f_{n+1})_I^\flat = (d_{n-1}^A \circ \lambda_n) \cdot (\lambda_{n-1} \circ d_n^B) \cdot (f_{n-1} \circ \delta_n^B) \cdot (f_{n-1})_I^\sharp.$$

$$\begin{array}{ccccccc}
 \cdots & A_{n-2} & \xrightarrow{d_{n-2}^A} & A_{n-1} & \xrightarrow{d_{n-1}^A} & A_n & \xrightarrow{d_n^A} & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_{n+2} & \cdots \\
 & \downarrow f_{n-2} & \lambda_{n-2} \parallel & \downarrow f_{n-1} & \lambda_{n-1} \parallel & \downarrow f_n & \lambda_n \parallel & \downarrow f_{n+1} & \lambda_{n+1} \parallel & \downarrow f_{n+2} & \\
 \cdots & B_{n-2} & \xrightarrow{d_{n-2}^B} & B_{n-1} & \xrightarrow{d_{n-1}^B} & B_n & \xrightarrow{d_n^B} & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_{n+2} & \cdots
 \end{array}$$

Proposition 23.4. Consider the following diagram in \mathbf{S} .

$$(23.1) \quad \begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ a \downarrow & \lambda \Downarrow & \downarrow b \\ A_2 & \xrightarrow{f_2} & B_2 \end{array}$$

If we take the cokernels of f_1 and f_2 , then there exist $\bar{b} \in \mathbf{S}^1(\text{Cok}(f_1), \text{Cok}(f_2))$ and $\bar{\lambda} \in \mathbf{S}^2(c(f_1) \circ \bar{b}, b \circ c(f_2))$ such that

$$(\pi_{f_1} \circ \bar{b}) \cdot (\bar{b})_I^\flat = (f_1 \circ \bar{\lambda}) \cdot (\lambda \circ c(f_2)) \cdot (a \circ \pi_{f_2}) \cdot a_I^\sharp.$$

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow \pi_{f_1} & & \\
 A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\
 a \downarrow & \lambda \Downarrow & \downarrow b & \Downarrow \bar{\lambda} & \downarrow \bar{b} \\
 A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2) \\
 & & \downarrow \pi_{f_2} & & \\
 & & 0 & &
 \end{array}$$

If $(\bar{b}', \bar{\lambda}')$ also satisfies this condition, there exists a unique 2-cell $\xi \in \mathbf{S}^2(\bar{b}, \bar{b}')$ such that $(c(f_1) \circ \xi) \cdot \bar{\lambda}' = \bar{\lambda}$.

Proof. This follows immediately if we apply the universality of $\text{Cok}(f_1)$ to $(\lambda \circ c(f_2)) \cdot (a \circ \pi_{f_2}) \cdot a_I^\sharp \in \mathbf{S}^2(f_1 \circ b \circ c(f_2), 0)$. \square

Proposition 23.5. Consider the following diagrams in \mathbf{S} ,

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow a_1 & \lambda_1 \Downarrow & \downarrow b_1 \\ \alpha \Leftarrow A_2 & \xrightarrow{f_2} & B_2 \Rightarrow \beta \\ \downarrow a_2 & \lambda_2 \Downarrow & \downarrow b_2 \\ A_3 & \xrightarrow{f_3} & B_3 \end{array} \quad \begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow a & \lambda \Downarrow & \downarrow b \\ A_3 & \xrightarrow{f_3} & B_3 \end{array}$$

which satisfy $(f_1 \circ \beta) \cdot \lambda = (\lambda_1 \circ b_2) \cdot (a_1 \circ \lambda_2) \cdot (\alpha \circ f_3)$. Applying Proposition 23.4, we obtain diagrams

$$\begin{array}{ccc} B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\ \downarrow b & \Downarrow \bar{\lambda} & \downarrow \bar{b} \\ B_3 & \xrightarrow{c(f_3)} & \text{Cok}(f_3) \end{array} \quad \begin{array}{ccc} B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\ \downarrow b_1 & \Downarrow \bar{\lambda}_1 & \downarrow \bar{b}_1 \\ B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2) \end{array} \quad \begin{array}{ccc} B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2) \\ \downarrow b_2 & \Downarrow \bar{\lambda}_2 & \downarrow \bar{b}_2 \\ B_3 & \xrightarrow{c(f_3)} & \text{Cok}(f_3) \end{array}$$

with

$$(23.2) \quad \begin{aligned} (\pi_{f_1} \circ \bar{b}) \cdot (\bar{b})_I^b &= (f_1 \circ \bar{\lambda}) \cdot (\lambda \circ c(f_3)) \cdot (a \circ \pi_{f_3}) \cdot a_I^\# \\ (\pi_{f_1} \circ \bar{b}_1) \cdot (\bar{b}_1)_I^b &= (f_1 \circ \bar{\lambda}_1) \cdot (\lambda_1 \circ c(f_2)) \cdot (a_1 \circ \pi_{f_2}) \cdot (a_1)_I^\# \\ (\pi_{f_2} \circ \bar{b}_2) \cdot (\bar{b}_2)_I^b &= (f_2 \circ \bar{\lambda}_2) \cdot (\lambda_2 \circ c(f_3)) \cdot (a_2 \circ \pi_{f_3}) \cdot (a_2)_I^\# \end{aligned}$$

Then, there exists a unique 2-cell $\bar{\beta} \in \mathbf{S}^2(\bar{b}_1 \circ \bar{b}_2, \bar{b})$ such that

$$(c(f_1) \circ \bar{\beta}) \cdot \bar{\lambda} = (\bar{\lambda}_1 \circ \bar{b}_2) \cdot (b_1 \circ \bar{\lambda}_2) \cdot (\beta \circ c(f_3)).$$

Proof. By (23.2), $\bar{\lambda}$ is compatible with π_{f_1} and $(\lambda \circ c(f_3)) \cdot (a \circ \pi_{f_3}) \cdot a_I^\#$.

$$\begin{array}{ccc} & & \text{Cok}(f_1) \\ & \nearrow 0 & \uparrow c(f_1) \\ A_1 & \xrightarrow{f_1} & B_1 \\ & \searrow 0 & \downarrow \bar{b} \\ & & \text{Cok}(f_3) \end{array}$$

$(\lambda \circ c(f_3)) \cdot (a \circ \pi_{f_3}) \cdot a_I^\#$

On the other hand, $\bar{\lambda}' := (\bar{\lambda}_1 \circ \bar{b}_2) \cdot (b_1 \circ \bar{\lambda}_2) \cdot (\beta \circ c(f_3))$ is also compatible with π_{f_1} and $(\lambda \circ c(f_3)) \cdot (a \circ \pi_{f_3}) \cdot a_I^\#$. So, by the universality of the $\text{Cok}(f_1)$, there exists a unique 2-cell $\bar{\beta} \in \mathbf{S}^2(\bar{b}_1 \circ \bar{b}_2, \bar{b})$ such that $(c(f_1) \circ \bar{\beta}) \cdot \bar{\lambda} = \bar{\lambda}'$. \square

Corollary 23.6. Let $(f_n, \lambda_n) : (A_n, d_n^A, \delta_n^A) \rightarrow (B_n, d_n^B, \delta_n^B)$ be a complex morphism. Then, by taking the cokernels, we obtain a complex morphism $(c(f_n), \bar{\lambda}_n) : (B_n, d_n^B, \delta_n^B) \rightarrow (\text{Cok}(f_n), \bar{d}_n^B, \bar{\delta}_n^B)$ which satisfies

$$(23.3) \quad (d_n^A \circ \pi_{f_{n+1}}) \cdot (d_n^A)_I^\# = (\lambda_n \circ c(f_{n+1})) \cdot (f_n \circ \bar{\lambda}_n) \cdot (\pi_{f_n} \circ \bar{d}_n^B) \cdot (\bar{d}_n^B)_I^b$$

for each n .

Proof. By Proposition 23.4, we obtain \bar{d}_n^B and $\bar{\lambda}_n$ which satisfy (23.3). And by Proposition 23.5, for each n , there exists a unique 2-cell $\bar{\delta}_n^B \in \mathbf{S}^2(\bar{d}_{n-1}^B \circ \bar{d}_n^B, 0)$ such that $((\bar{\delta}_n^B \circ c(f_{n+1})) \cdot c(f_{n+1}))_I^\flat = (d_{n-1}^B \circ \bar{\lambda}_n) \cdot (\bar{\lambda}_{n-1} \circ \bar{d}_n^B) \cdot (c(f_{n-1}) \circ \bar{\delta}_n^B) \cdot c(f_{n-1})_I^\sharp$. By the uniqueness of $\bar{\beta}$ in Proposition 23.5, it is easy to see that

$$(\bar{\delta}_n^B \circ \bar{d}_{n+1}^B) \cdot (\bar{d}_{n+1}^B)_I^\flat = (\bar{d}_{n-1}^B \circ \bar{\delta}_{n+1}^B) \cdot (\bar{d}_{n-1}^B)_I^\sharp.$$

These are saying that $(\text{Cok}(f_n), \bar{d}_n^B, \bar{\delta}_n^B)$ is a complex and $(c(f_n), \bar{\lambda}_n)$ is a complex morphism. \square

Proposition 23.7. *Consider the following diagram in \mathbf{S} .*

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ a \downarrow & \lambda \Downarrow & \downarrow b \\ A_3 & \xrightarrow{f_3} & B_3 \end{array}$$

By taking the cokernels of f_1 and f_2 , we obtain

$$\begin{array}{ccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\ a \downarrow & \lambda \Downarrow & \downarrow b & \Downarrow \lambda' & \downarrow \bar{b} \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2), \end{array}$$

and from this diagram, by taking the cokernels of a, b, \bar{b} , we obtain

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow \pi_{f_2} & & \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2) \\ c(a) \downarrow & \bar{\lambda} \Downarrow & \downarrow c(b) & \Downarrow \bar{\lambda}' & \downarrow c(\bar{b}) \\ \text{Cok}(a) & \xrightarrow{\bar{f}_2} & \text{Cok}(b) & \xrightarrow{c(\bar{f}_2)} & \text{Cok}(\bar{b}) \\ & & \downarrow \bar{\pi}_{f_2} & & \\ & & 0 & & \end{array}$$

Then we have $\text{Cok}(\bar{f}_2) = [\text{Cok}(\bar{b}), c(\bar{f}_2), \bar{\pi}_{f_2}]$. We abbreviate this to $\text{Cok}(\bar{f}_2) = \text{Cok}(\bar{b})$.

Proof. Left to the reader. \square

Proposition 23.8. *In the following diagram, assume $f_\bullet : A_\bullet \rightarrow B_\bullet$ is a complex morphism.*

$$(23.4) \quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow \delta_2^A & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \\ f_1 \downarrow & \lambda_1 \Downarrow & \downarrow f_2 & \lambda_2 \Downarrow & \downarrow f_3 \\ B_1 & \xrightarrow{d_1^B} & B_2 & \xrightarrow{d_2^B} & B_3 \\ & & \downarrow \delta_2^B & & \\ & & 0 & & \end{array}$$

If we take the cokernels of d_1^A and d_1^B ,

$$\begin{array}{ccccc} A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{c(d_1^A)} & \text{Cok}(d_1^A) \\ f_1 \downarrow & \lambda_1 \Downarrow & f_2 \downarrow & \bar{\lambda}_1 \Downarrow & \downarrow \bar{f}_2 \\ B_1 & \xrightarrow{d_1^B} & B_2 & \xrightarrow{c(d_1^B)} & \text{Cok}(d_1^B) \end{array}$$

then by the universality of cokernel, we obtain $\bar{d}_2^A \in \mathbf{S}^1(\text{Cok}(d_1^A), A_3)$ and $\bar{\delta}_2^A \in \mathbf{S}^2(c(d_1^A) \circ \bar{d}_2^A, d_2^A)$ such that $(d_1^A \circ \bar{\delta}_2^A) \cdot \delta_2^A = (\pi_{d_1^A} \circ \bar{d}_2^A) \cdot (\bar{d}_2^A)_I^\flat$. Similarly, we obtain $\bar{d}_2^B \in \mathbf{S}^1(\text{Cok}(d_1^B), B_3)$, $\bar{\delta}_2^B \in \mathbf{S}^2(c(d_1^B) \circ \bar{d}_2^B, d_2^B)$ with $(d_1^B \circ \bar{\delta}_2^B) \cdot \delta_2^B = (\pi_{d_1^B} \circ \bar{d}_2^B) \cdot (\bar{d}_2^B)_I^\flat$. Then, there exists a unique 2-cell $\bar{\lambda}_2 \in \mathbf{S}^2(\bar{d}_2^A \circ f_3, \bar{f}_2 \circ \bar{d}_2^B)$ such that $(c(d_1^A) \circ \bar{\lambda}_2) \cdot (\bar{\lambda}_1 \circ \bar{d}_2^B) \cdot (f_2 \circ \bar{\delta}_2^B) = (\bar{\delta}_2^A \circ f_3) \cdot \lambda_2$.

Proof. If we put $\delta := (d_1^A \circ \lambda_2^{-1}) \cdot (\delta_2^A \circ f_3) \cdot (f_3)_I^\flat$, then both the factorizations

$$\begin{aligned} (\bar{\delta}_2^A \circ f_3) \cdot \lambda_2 & : c(d_1^A) \circ (\bar{d}_2^A \circ f_3) \implies f_2 \circ d_2^B \\ (\bar{\lambda}_1 \circ \bar{d}_2^B) \cdot (f_2 \circ \bar{\delta}_2^B) & : c(d_1^A) \circ (\bar{f}_2 \circ \bar{d}_2^B) \implies f_2 \circ d_2^B \end{aligned}$$

are compatible with $\pi_{d_1^A}$ and δ . So the proposition follows from the universality of $\text{Cok}(d_2^A)$. \square

Proposition 23.9. *In diagram (23.1), if we take the coimage factorizations*

$$\begin{aligned} \mu_a & : c(k(a)) \circ j(a) \implies a \\ \mu_b & : c(k(b)) \circ j(b) \implies b, \end{aligned}$$

then there exist $f \in \mathbf{S}^1(\text{Coim}(a), \text{Coim}(b))$, $\lambda_1 \in \mathbf{S}^2(f_1 \circ c(k(b)), c(k(a)) \circ f)$ and $\lambda_2 \in \mathbf{S}^2(f \circ j(b), j(a) \circ f_2)$ such that $(f_1 \circ \mu_b) \cdot \lambda = (\lambda_1 \circ j(b)) \cdot (c(k(a)) \circ \lambda_2) \cdot (\mu_a \circ f_2)$.

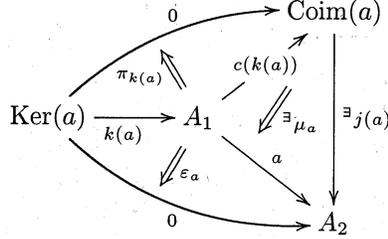
$$(23.5) \quad \begin{array}{ccccc} & A_1 & \xrightarrow{f_1} & B_1 & \\ & \downarrow c(k(a)) & & \downarrow c(k(b)) & \\ a & \xleftarrow{\mu_a} & \text{Coim}(a) & \xrightarrow{f} & \text{Coim}(b) & \xrightarrow{\mu_b} & b \\ & \downarrow j(a) & & \downarrow j(b) & \\ & A_2 & \xrightarrow{f_2} & B_2 & \end{array}$$

$\Downarrow \lambda_1$ $\Downarrow \lambda_2$

Moreover, for any other $(f', \lambda'_1, \lambda'_2)$ with this property, there exists a unique 2-cell $\xi \in \mathbf{S}^2(f, f')$ such that $\lambda_1 \cdot (c(k(a)) \circ \xi) = \lambda'_1$ and $(\xi \circ j(b)) \cdot \lambda'_2 = \lambda_2$.

Proof. Since the coimage factorization is unique up to an equivalence and is obtained by the factorization which fills in the following diagram, we may assume $\text{Ker}(a) = [\text{Ker}(a), k(a), \varepsilon_a]$, $\text{Cok}(k(a)) = [\text{Coim}(a), c(k(a)), \pi_{k(a)}]$, and $(k(a) \circ \mu_a) \cdot$

$$\varepsilon_a = (\pi_{k(a)} \circ j(a)) \cdot j(a)_I^b.$$



Similarly, we may assume

$$\begin{aligned} \text{Ker}(b) &= [\text{Ker}(b), k(b), \varepsilon_b], \\ \text{Cok}(k(b)) &= [\text{Coim}(b), c(k(b)), \pi_{k(b)}] \end{aligned}$$

and $(k(b) \circ \mu_b) \cdot \varepsilon_b = (\pi_{k(b)} \circ j(b)) \cdot j(b)_I^b$. By (the dual of) Proposition 23.4, there are $\underline{f}_1 \in \mathbf{S}^1(\text{Ker}(a), \text{Ker}(b))$ and $\underline{\lambda} \in \mathbf{S}^2(\underline{f}_1 \circ k(b), k(a) \circ f_1)$ such that $(\underline{\lambda} \circ b) \cdot (k(a) \circ \lambda) \cdot (\varepsilon_a \circ f_2) \cdot (f_2)_I^b = (\underline{f}_1 \circ \varepsilon_b) \cdot (\underline{f}_1)_I^\#$. Applying Proposition 23.8, we can show the existence of $(f, \lambda_1, \lambda_2)$. To show the uniqueness (up to an equivalence), let $(f', \lambda'_1, \lambda'_2)$ satisfy

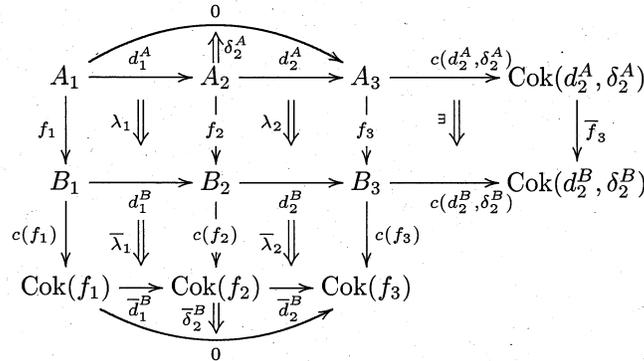
$$(f_1 \circ \mu_b) \cdot \lambda = (\lambda'_1 \circ j(b)) \cdot (c(k(a)) \circ \lambda'_2) \cdot (\mu_a \circ f_2).$$

From this, we can obtain

$$(\underline{f}_1 \circ \pi_{k(b)}) \cdot (\underline{f}_1)_I^\# = (\underline{\lambda} \circ c(k(b))) \cdot (k(a) \circ \lambda'_1) \cdot (\pi_{k(a)} \circ f') \cdot f_I^b.$$

And the uniqueness follows from the uniqueness of 2-cells in Proposition 23.4 and Proposition 23.8. \square

Proposition 23.10. *Let $f_\bullet : A_\bullet \rightarrow B_\bullet$ be a complex morphism as in diagram (23.4). If we take the cokernels of f_1, f_2, f_3 and relative cokernels of the complex A_\bullet and B_\bullet as in the following diagram, then we have $\text{Cok}(\bar{f}_3) = \text{Cok}(\bar{d}_2^B, \bar{\delta}_2^B)$.*



Proof. Immediately follows from Proposition 23.7, Proposition 23.8 and (the dual of) Proposition 21.20. \square

Proposition 23.11. *In diagram (23.1), if a is fully cofaithful, then the following diagram obtained in Proposition 23.4 is a pushout diagram.*

$$\begin{array}{ccc} B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\ b \downarrow & \Downarrow \bar{\lambda} & \downarrow \bar{b} \\ B_2 & \xrightarrow{c(f_2)} & \text{Cok}(f_2) \end{array}$$

Proof. Left to the reader. □

Concerning Proposition 21.32, we have the following proposition.

Proposition 23.12. *Let*

$$\begin{array}{ccc} A_1 \times_B A_2 & \xrightarrow{f'_1} & A_2 \\ f'_2 \downarrow & \xi \Downarrow & \downarrow f_2 \\ A_1 & \xrightarrow{f_1} & B \end{array}$$

be a pullback diagram in \mathbf{S} . If f_1 is fully cofaithful, then f'_1 is fully cofaithful.

Proof. Since f_1 is cofaithful, in the notation of the proof of Proposition 21.32, $\text{Cok}(i_1) = [A_2, p_2, \xi_2]$ and $\text{Cok}(d) = [B, t, \varepsilon_t]$. Applying Proposition 23.7 to the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{0} & A_1 \\ 0 \downarrow & \Downarrow & \downarrow i_1 \\ A_1 \times_B A_2 & \xrightarrow{d} & A_1 \times A_2, \end{array}$$

we obtain

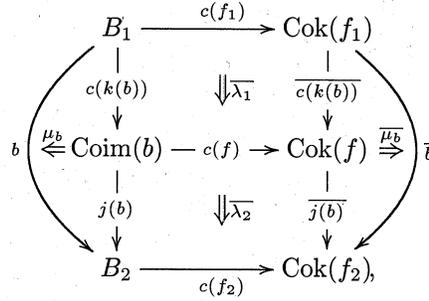
$$\text{Cok}(f_1) = 0 \iff \text{Cok}(f'_1) = 0.$$

$$\begin{array}{ccccccc} 0 & \xrightarrow{0} & A_1 & \xlongequal{\text{id}} & A_1 & & \\ 0 \downarrow & & \Downarrow & \downarrow i_1 & \Downarrow & \downarrow f_1 & \\ A_1 \times_B A_2 & \xrightarrow{d} & A_1 \times A_2 & \xrightarrow{t} & B & & \\ \text{id} \parallel & & \circlearrowleft & \downarrow p_2 & \Downarrow & \downarrow 0 & \\ A_1 \times_B A_2 & \xrightarrow{f'_1} & A_2 & \xrightarrow{0} & 0 & & \end{array}$$

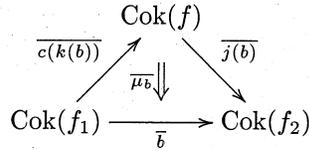
□

Proposition 23.13. *In diagram (23.1), assume a is cofaithful. By Proposition 23.9, we obtain a coimage factorization diagram as (23.5). If we take the cokernel*

of this diagram as

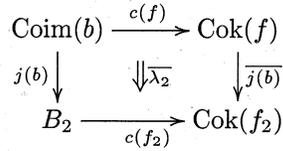


then the factorization



becomes again a coimage factorization.

Proof. It suffices to show that $\overline{c(k(b))}$ is cofaithful and $\overline{j(b)}$ is fully faithful. Since $c(k(b))$ and $c(f)$ are cofaithful, it follows that $\overline{c(k(b))}$ is cofaithful. Since $j(a)$ is an equivalence,



is a pushout diagram by Proposition 23.11. By (the dual of) Proposition 23.12, $\overline{j(b)}$ becomes fully faithful. \square

Definition of the relative 2-exactness

Lemma 23.14. Consider the following diagram in \mathbf{S} .

$$(23.6) \quad \begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & & \varphi \downarrow & & \\
 & & 0 & &
 \end{array}$$

If we factor it as

$$(23.7) \quad \begin{array}{ccccc}
 & \text{Ker}(g) & & \text{Cok}(f) & \\
 \downarrow \varphi & \downarrow k(g) & & \downarrow \overline{\varphi} & \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & & \varphi \downarrow & & \\
 & & 0 & &
 \end{array}$$

with

$$\begin{aligned}(\underline{\varphi} \circ g) \cdot \varphi &= (\underline{f} \circ \varepsilon_g) \cdot (f)_I^\# \\ (f \circ \bar{\varphi}) \cdot \varphi &= (\pi_f \circ \bar{g}) \cdot (\bar{g})_I^b,\end{aligned}$$

then $\text{Cok}(\underline{f}) = 0$ if and only if $\text{Ker}(\bar{g}) = 0$.

Proof. We show only $\text{Cok}(\underline{f}) = 0 \Rightarrow \text{Ker}(\bar{g}) = 0$, since the other implication can be shown dually. If $\text{Cok}(\underline{f}) = 0$, i.e. if \underline{f} is fully cofaithful, then we have

$$\text{Cok}(f) = \text{Cok}(\underline{f} \circ k(g)) = \text{Cok}(k(g)) = \text{Coim}(g).$$

Thus the following diagram is a coimage factorization, and \bar{g} becomes fully faithful.

$$\begin{array}{ccc} & \text{Cok}(f) & \\ c(f) \nearrow & \Downarrow \bar{\varphi} & \searrow \bar{g} \\ B & \xrightarrow{g} & C \end{array}$$

□

Definition 23.15. Diagram (23.6) is said to be 2-exact in B , if $\text{Cok}(\underline{f}) = 0$ (or equivalently $\text{Ker}(\bar{g}) = 0$).

Remark 23.16. In the notation of Lemma 23.14, the following are equivalent :

- (i) (23.6) is 2-exact in B .
- (ii) \underline{f} is fully cofaithful.
- (iii) \bar{g} is fully faithful.
- (iv) $c(f) = \text{cok}(k(g))$ (i.e. $\text{Cok}(f) = \text{Coim}(g)$).
- (v) $k(g) = \text{ker}(c(f))$ (i.e. $\text{Ker}(g) = \text{Im}(f)$).

Proof. By the duality, we only show (i) \Leftrightarrow (iii) \Leftrightarrow (v).

- (i) \Leftrightarrow (iii) follows from Corollary 21.25.
- (iii) \Rightarrow (v) follows from Proposition 21.21.
- (v) \Rightarrow (iii) follows from Proposition 22.3.

□

Let us fix the notation for relative (co-)kernels of a complex.

Definition 23.17. For any complex $A_\bullet = (A_n, d_n, \delta_n)$ in \mathbf{S} , we put

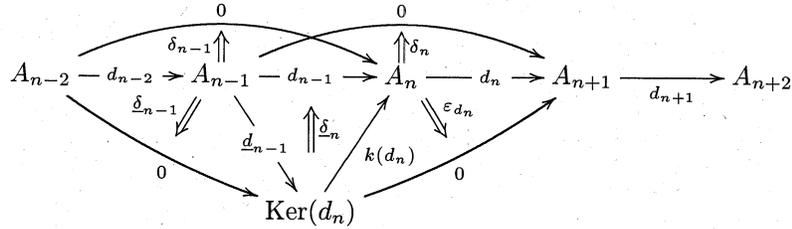
- (1) $[Z^n(A_\bullet), z_n^A, \zeta_n^A] := \text{Ker}(d_n, \delta_{n+1})$.
- (2) $[Q^n(A_\bullet), q_n^A, \rho_n^A] := \text{Cok}(d_{n-1}, \delta_{n-1})$.

Remark 23.18. By the universality of $\text{Ker}(d_n, \delta_{n+1})$ and Lemma 21.19, there exist $k_n \in \mathbf{S}^1(A_{n-1}, Z^n(A_\bullet))$, $\nu_{n,1} \in \mathbf{S}^2(k_n \circ z_n, d_{n-1})$ and $\nu_{n,2} \in \mathbf{S}^2(d_{n-2} \circ k_n, 0)$ such that

$$\begin{aligned}(\nu_{n,1} \circ d_n) \cdot \delta_n &= (k_n \circ \zeta_n) \cdot (k_n)_I^\# \\ (d_{n-2} \circ \nu_{n,1}) \cdot \delta_{n-1} &= (\nu_{n,2} \circ z_n) \cdot (z_n)_I^b.\end{aligned}$$

$$\begin{array}{ccccccc} & & & Z^n(A_\bullet) & & & \\ & & & \uparrow & \searrow & & \\ & & & \nu_{n,1} \Downarrow & \zeta_n \nearrow & & \\ & & & \nu_{n,2} \nearrow & \nu_{n,1} \Downarrow & & \\ & & & \delta_{n-1} \Downarrow & \delta_n \Downarrow & & \\ A_{n-2} & \xrightarrow{d_{n-2}} & A_{n-1} & \xrightarrow{d_{n-1}} & A_n & \xrightarrow{d_n} & A_{n+1} \xrightarrow{d_{n+1}} A_{n+2} \\ & & & \delta_{n-1} \Downarrow & \delta_n \Downarrow & & \\ & & & 0 & 0 & & \end{array}$$

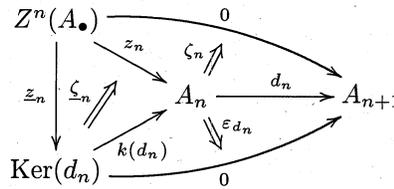
On the other hand, by the universality of $\text{Ker}(d_n)$, we obtain a factorization diagram



which satisfy

$$\begin{aligned}
 (\underline{\delta}_n \circ d_n) \cdot \delta_n &= (\underline{d}_{n-1} \circ \varepsilon_{d_n}) \cdot (\underline{d}_{n-1})_I^\sharp \\
 (d_{n-2} \circ \underline{\delta}_n) \cdot \delta_{n-1} &= (\underline{\delta}_{n-1} \circ k(d_n)) \cdot (k(d_n))_I^\flat
 \end{aligned}$$

By Proposition 21.20, there exists a factorization of z_n through $\text{Ker}(d_n)$



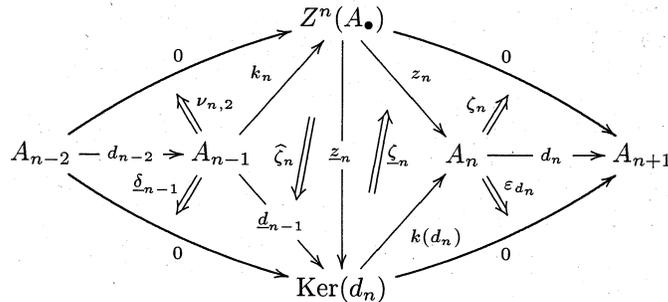
which satisfies

$$(\underline{\zeta}_n \circ d_n) \cdot \zeta_n = (\underline{z}_n \circ \varepsilon_{d_n}) \cdot (\underline{z}_n)_I^\sharp.$$

Moreover \underline{z}_n is fully faithful by Proposition 21.33.

By the universality of $\text{Ker}(d_n)$, we can show easily the following claim.

Claim 23.19. *There exists a unique 2-cell $\widehat{\zeta}_n \in S(k_n \circ z_n, \underline{d}_{n-1})$*



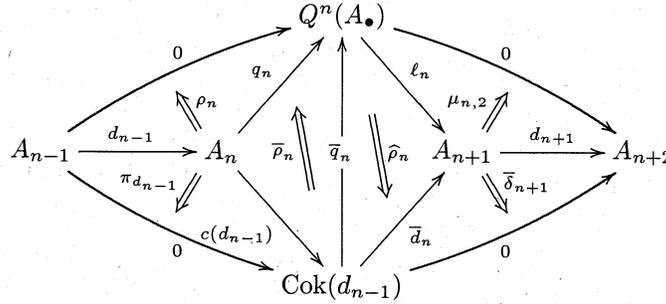
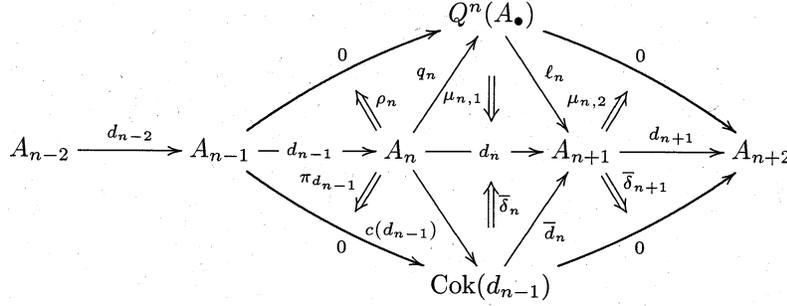
such that

$$(\widehat{\zeta}_n \circ k(d_n)) \cdot \underline{\delta}_n = (k_n \circ \underline{\zeta}_n) \cdot \nu_{n,1}.$$

This $\widehat{\zeta}_n$ also satisfies

$$(d_{n-2} \circ \widehat{\zeta}_n) \cdot \underline{\delta}_{n-1} = (\nu_{n,2} \circ \underline{z}_n) \cdot (\underline{z}_n)_I^\flat.$$

Remark 23.20. *Dually, by the universality of the cokernels, we obtain the following two factorization diagrams, where \bar{q}_n is fully cofaithful.*



We define relative 2-cohomology in the following two ways, which will be shown to be equivalent later.

Definition 23.21.

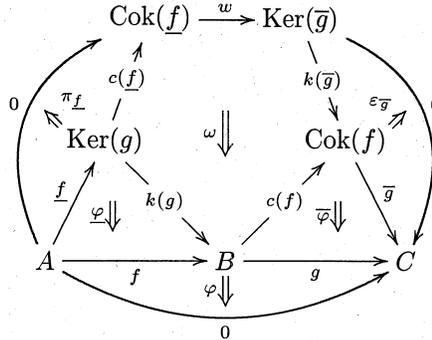
$$H_1^n(A_\bullet) := \text{Cok}(k_n, \nu_{n,2})$$

$$H_2^n(A_\bullet) := \text{Ker}(\ell_n, \mu_{n,2})$$

Lemma 23.22. *In the factorization diagram (23.7) in Lemma 23.14, if we take the cokernel of \underline{f} and the kernel of \bar{g} , then there exist $w \in \mathbf{S}^1(\text{Cok}(\underline{f}), \text{Ker}(\bar{g}))$ and $\omega \in \mathbf{S}^2(c(\underline{f}) \circ w \circ k(\bar{g}), k(g) \circ c(f))$ such that*

$$(23.8) \quad (\underline{f} \circ \omega) \cdot (\varphi \circ c(f)) \cdot \pi_f = (\pi_f \circ w \circ k(\bar{g})) \cdot (w \circ k(\bar{g}))_I^b$$

$$(23.9) \quad (\omega \circ \bar{g}) \cdot (k(g) \circ \bar{\varphi}) \cdot \varepsilon_g = (c(\underline{f}) \circ w \circ \varepsilon_{\bar{g}}) \cdot (c(\underline{f}) \circ w)_I^\sharp.$$



Moreover, for any other factorization (w', ω') with these properties, there exists a unique 2-cell $\kappa \in \mathbf{S}^2(w, w')$ such that $(c(\underline{f}) \circ \kappa \circ k(\underline{g})) \cdot \omega' = \omega$.

Proof. Applying Proposition 23.4 to

$$(23.10) \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \underline{f} \downarrow & \varphi^{-1} \Downarrow & \downarrow f \\ \text{Ker}(g) & \xrightarrow{k(g)} & B, \end{array}$$

we obtain $w_1 \in \mathbf{S}^1(\text{Cok}(\underline{f}), \text{Cok}(f))$ and $\omega_1 \in \mathbf{S}^2(c(\underline{f}) \circ w_1, k(g) \circ c(f))$ which satisfy

$$(23.11) \quad (\underline{f} \circ \omega_1) \cdot (\varphi \circ c(f)) \cdot \pi_f = (\pi_f \circ w_1) \cdot (w_1)_I^b.$$

Then $(\omega_1 \circ \bar{g}) \cdot (k(g) \circ \bar{\varphi}) \cdot \varepsilon_g \in \mathbf{S}^2(c(\underline{f}) \circ w_1 \circ \bar{g}, 0)$ becomes compatible with π_f .

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \pi_f & \searrow & \\ A & \xrightarrow{\underline{f}} & \text{Ker}(g) & \xrightarrow{c(f)} & \text{Cok}(\underline{f}) & \xrightarrow{w_1 \circ \bar{g}} & C \\ & & & & \downarrow & & \\ & & & & 0 & \xrightarrow{(\omega_1 \circ \bar{g}) \cdot (k(g) \circ \bar{\varphi}) \cdot \varepsilon_g} & \end{array}$$

By Lemma 21.19, there exists a 2-cell $\delta \in S(w_1 \circ \bar{g}, 0)$ such that

$$(c(\underline{f}) \circ \delta) \cdot c(\underline{f})_I^\sharp = (\omega_1 \circ \bar{g}) \cdot (k(g) \circ \bar{\varphi}) \cdot \varepsilon_g.$$

So, if we take the cokernels of $k(g)$ and w_1 , then by Proposition 23.8, we obtain the following diagram:

$$\begin{array}{ccccccc} & & & & g & & \\ & & & & \uparrow \mu_g & & \\ \text{Ker}(g) & \xrightarrow{k(g)} & B & \xrightarrow{c(k(g))} & \text{Coim}(g) & \xrightarrow{j(g)} & C \\ \downarrow c(f) & \xRightarrow{\omega_1} & \downarrow c(f) & \xRightarrow{\exists \bar{\varphi}_1} & \downarrow \exists \bar{c} & \xRightarrow{\exists \bar{\varphi}_2} & \downarrow \text{id}_C \\ \text{Cok}(\underline{f}) & \xrightarrow{w_1} & \text{Cok}(f) & \xrightarrow{c(w_1)} & \text{Cok}(w_1) & \xrightarrow{\exists g^\dagger} & C \\ & & & & \downarrow \exists & & \\ & & & & 0 & & \bar{g} \end{array}$$

Applying Proposition 23.7 to (23.10), we obtain

$$[\text{Cok}(w_1), \bar{c}, (\bar{c})_I^b] = \text{Cok}(0 \xrightarrow{0} \text{Coim}(g)).$$

Thus \bar{c} is an equivalence. Since $j(g)$ is fully faithful, g^\dagger becomes fully faithful. Thus the following diagram is 2-exact in $\text{Cok}(f)$.

$$(23.12) \quad \begin{array}{ccc} \text{Cok}(\underline{f}) & \xrightarrow{w_1} & \text{Cok}(f) & \xrightarrow{\bar{g}} & C \\ & & \downarrow \delta & \searrow & \\ & & 0 & & \end{array}$$

So if we factor (23.12) by $w \in \mathbf{S}^1(\text{Cok}(\underline{f}), \text{Ker}(\overline{g}))$ and $\omega_2 \in \mathbf{S}^2(w \circ k(\overline{g}), w_1)$ as in the diagram

$$(23.13) \quad \begin{array}{ccccc} & & \text{Ker}(\overline{g}) & & \\ & \nearrow w & \downarrow \omega_2 & \searrow k(\overline{g}) & \\ & & \text{Cok}(\underline{f}) & \xrightarrow{w_1} & \text{Cok}(\underline{f}) & \xrightarrow{\overline{g}} & C \\ & & \delta \downarrow & & \nearrow \varepsilon_{\overline{g}} & & \\ & & 0 & & & & \end{array}$$

which satisfies

$$(w \circ \varepsilon_{\overline{g}}) \cdot w_I^\# = (\omega_2 \circ \overline{g}) \cdot \delta,$$

then w becomes fully cofaithful by Lemma 23.14. If we put $\omega := (c(\underline{f}) \circ \omega_2) \cdot \omega_1$, then (w, ω) satisfies conditions (23.8) and (23.9).

If (w', ω') satisfies

$$(23.14) \quad \begin{aligned} (\underline{f} \circ \omega') \cdot (\varphi \circ c(\underline{f})) \cdot \pi_f &= (\pi_f \circ w' \circ k(\overline{g})) \cdot (w' \circ k(\overline{g}))_I^\flat \\ (\omega' \circ \overline{g}) \cdot (k(\underline{g}) \circ \overline{\varphi}) \cdot \varepsilon_g &= (c(\underline{f}) \circ w' \circ \varepsilon_{\overline{g}}) \cdot (c(\underline{f}) \circ w')_I^\# \end{aligned}$$

then, since both the factorization of $k(\underline{g}) \circ c(\underline{f})$ through $\text{Cok}(\underline{f})$

$$\begin{aligned} \omega' &: c(\underline{f}) \circ w' \circ k(\overline{g}) \implies k(\underline{g}) \circ c(\underline{f}) \\ \omega_1 &: c(\underline{f}) \circ w_1 \implies k(\underline{g}) \circ c(\underline{f}) \end{aligned}$$

are compatible with π_f and $(\varphi \circ c(\underline{f})) \cdot \pi_f$ by (23.11) and (23.14), there exists $\omega'_2 \in \mathbf{S}^2(w' \circ k(\overline{g}), w_1)$ such that

$$(c(\underline{f}) \circ \omega'_2) \cdot \omega_1 = \omega'.$$

Then we can see ω'_2 is compatible with $\varepsilon_{\overline{g}}$ and δ . So, comparing this with the factorization (23.13), by the universality of $\text{Ker}(\overline{g})$, we see there exists a unique 2-cell $\kappa \in \mathbf{S}^2(w, w')$ such that $(\kappa \circ k(\overline{g})) \cdot \omega'_2 = \omega_2$. Then κ satisfies $(c(\underline{f}) \circ \kappa \circ k(\overline{g})) \cdot \omega' = \omega$. Uniqueness of such κ follows from the fact that $c(\underline{f})$ is cofaithful and $k(\overline{g})$ is faithful. \square

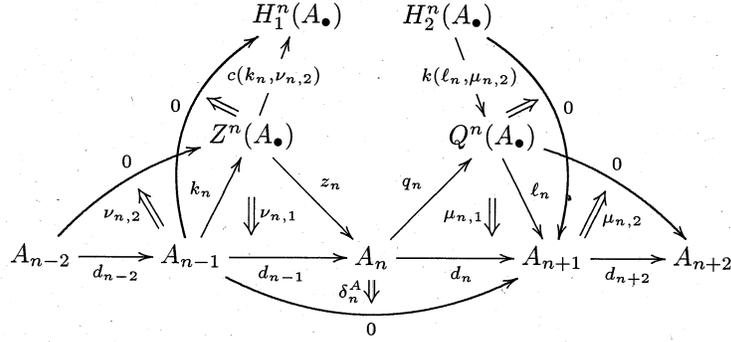
Proposition 23.23. *In Lemma 23.22, w is an equivalence.*

Proof. We showed Lemma 23.22 by taking the cokernel first and the kernel second, but we obtain the same (w, ω) if we take the kernel first and the cokernel second, because of the symmetricity of the statement (and the uniqueness of (w, ω) up to an equivalence) of Lemma 23.22. As shown in the proof, since (23.12) is 2-exact in $\text{Cok}(\underline{f})$, w becomes fully cofaithful in the factorization (23.13). By the above remark, similarly w can be obtained also by the factorization

$$\begin{array}{ccccc} & & \text{Cok}(\underline{f}) & & \\ & \nearrow & \downarrow & \searrow w & \\ A & \longrightarrow & \text{Ker}(g) & \longrightarrow & \text{Ker}(\overline{g}) \\ & \searrow & \downarrow & \nearrow & \\ & & 0 & & \end{array}$$

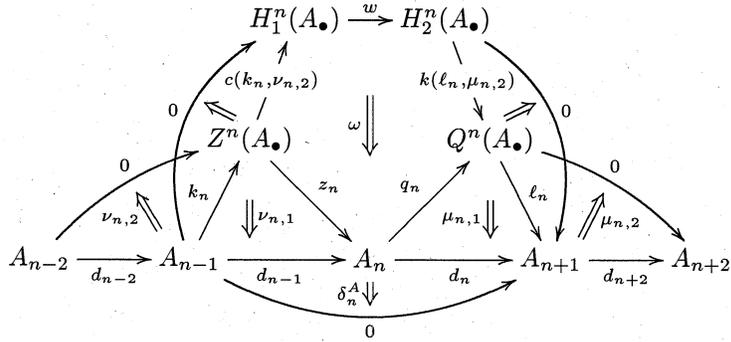
where the bottom row is 2-exact in $\text{Ker}(g)$. So w becomes fully faithful. Thus, w is fully cofaithful and fully faithful, i.e. an equivalence. \square

Corollary 23.24. *For any complex $A_\bullet = (A_n, d_n, \delta_n)$, if we factor it as*



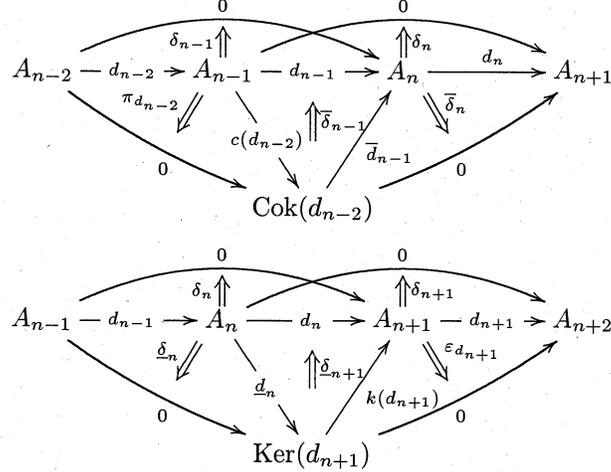
(in the notation of Definition 23.17, Remark 23.18 and Remark 23.20), then there exist $w \in \mathbf{S}^1(H_1^n(A_\bullet), H_2^n(A_\bullet))$ and $\omega \in \mathbf{S}^2(c(k_n, \nu_{n,2}) \circ w \circ k(l_n, \mu_{n,2}), z_n \circ q_n)$ such that

$$\begin{aligned} (k_n \circ \omega) \cdot (\nu_{n,1} \circ q_n) \cdot \rho_n &= (\pi(k_n, \nu_{n,2}) \circ w \circ k(l_n, \mu_{n,2})) \cdot (w \circ k(l_n, \mu_{n,2}))_I^\flat \\ (\omega \circ l_n) \cdot (z_n \circ \mu_{n,1}) \cdot \zeta_n &= (c(k_n, \nu_{n,2}) \circ w \circ \varepsilon(l_n, \mu_{n,2})) \cdot (c(k_n, \nu_{n,2}) \circ w)_I^\sharp. \end{aligned}$$



For any other factorization (w', ω') with these conditions, there exists a unique 2-cell $\kappa \in \mathbf{S}^2(\omega, \omega')$ such that $(c(k_n, \nu_{n,2}) \circ \kappa \circ k(l_n, \mu_{n,2})) \cdot \omega' = \omega$. Moreover, this w becomes an equivalence.

Proof. For the factorization diagrams



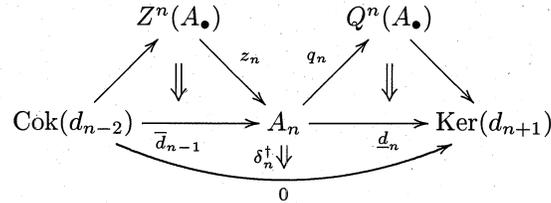
which satisfy

$$\begin{aligned} (d_{n-2} \circ \bar{d}_{n-1}) \cdot \delta_{n-1} &= (\pi_{d_{n-2}} \circ \bar{d}_{n-1}) \cdot (\bar{d}_{n-1})_I^b \\ (\bar{d}_{n-1} \circ d_n) \cdot \delta_n &= (c(d_{n-2}) \circ \bar{d}_n) \cdot c(d_{n-2})_I^\# \\ (\underline{d}_{n+1} \circ d_{n+1}) \cdot \delta_{n+1} &= (\underline{d}_n \circ \varepsilon_{d_{n+1}}) \cdot (\underline{d}_n)_I^\# \\ (d_{n-1} \circ \underline{d}_{n+1}) \cdot \delta_n &= (\underline{d}_n \circ k(d_{n+1})) \cdot k(d_{n+1})_I^b, \end{aligned}$$

there exists a unique 2-cell $\delta_n^\dagger \in \mathbf{S}^2(\bar{d}_{n-1} \circ \underline{d}_n, 0)$ such that

$$\begin{aligned} (\delta_n^\dagger \circ k(d_{n+1})) \cdot k(d_{n+1})_I^b &= (\bar{d}_{n-1} \circ \underline{d}_{n+1}) \cdot \bar{d}_n \\ (c(d_{n-2}) \circ \delta_n^\dagger) \cdot c(d_{n-2})_I^\# &= (\bar{d}_{n-1} \circ \underline{d}_n) \cdot \underline{d}_n. \end{aligned}$$

By Proposition 21.20, applying Lemma 23.22 and Proposition 23.23 to the following diagram, we can obtain Corollary 23.24.

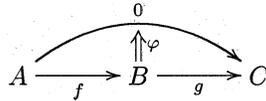


□

Thus $H_1^n(A_\bullet)$ and $H_2^n(A_\bullet)$ are equivalent. We abbreviate this to $H^n(A_\bullet)$.

Definition 23.25. A complex A_\bullet is said to be relatively 2-exact in A_n if $H^n(A_\bullet)$ is equivalent to zero.

Remark 23.26. If the complex is bounded, we consider the relative 2-exactness after adding zeroes as in Remark 23.2. For example, a bounded complex



is relatively 2-exact in B if and only if

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow \uparrow \varphi & & & \\
 0 & \xrightarrow{0} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \xrightarrow{0} 0 \\
 & \searrow 0 & \downarrow f_I^\# & & \downarrow g_I^\# & \searrow 0 & \\
 & & 0 & & 0 & &
 \end{array}$$

is relatively 2-exact in B , and this is equivalent to the 2-exactness in B by Remark 21.18.

24. LONG COHOMOLOGY SEQUENCE IN A RELATIVELY EXACT 2-CATEGORY

Diagram lemmas (2)

Lemma 24.1. Let A_\bullet be a complex in \mathbf{S} , in which $A_5 = 0$ and $d_4 = 0$:

$$(24.1) \quad \begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow \uparrow \delta_2^A & & & \\
 A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 & \xrightarrow{d_3^A} & A_4 \xrightarrow{0} 0 \\
 & \searrow & \downarrow \delta_3^A & & \downarrow \delta_3^A & \searrow & \\
 & & 0 & & 0 & &
 \end{array}$$

Then, (24.1) is relatively 2-exact in A_3 and A_4 if and only if $\text{Cok}(d_2, \delta_2) = A_4$, i.e. $[Q^3(A_\bullet), q_3, \rho_3] = [A_4, d_3, \delta_3]$.

Proof. As in Remark 23.20, we have two factorization diagrams

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & Q^3(A_\bullet) & & \\
 & \nearrow 0 & \nearrow q_3 & \searrow \ell_3 & \\
 A_2 & \xrightarrow{d_2} & A_3 & \xrightarrow{d_3} & A_4 \\
 & \searrow \pi_{d_2} & \downarrow \mu_{3,1} & \downarrow \bar{\delta}_3 & \searrow \bar{d}_3 \\
 & & \text{Cok}(d_2) & &
 \end{array} & &
 \begin{array}{ccccc}
 & & Q^3(A_\bullet) & & \\
 & \nearrow 0 & \nearrow q_3 & \searrow \ell_3 & \\
 A_2 & \xrightarrow{d_2} & A_3 & \xrightarrow{d_3} & A_4 \\
 & \searrow \pi_{d_2} & \downarrow \bar{\rho}_3 & \downarrow \bar{q}_3 & \searrow \bar{\rho}_3 \\
 & & \text{Cok}(d_2) & &
 \end{array}
 \end{array}$$

where \bar{q}_3 is fully cofaithful. We have

$$\begin{aligned}
 (24.1) \text{ is relatively 2-exact in } A_4 & \Leftrightarrow \text{Cok}(d_3, \delta_3) = 0 \\
 & \Leftrightarrow \text{Cok}(\bar{d}_3) = 0 \Leftrightarrow \text{Cok}(\bar{q}_3 \circ \ell_3) = 0 \\
 & \stackrel{\text{Prop. 21.20}}{\Leftrightarrow} \\
 & \stackrel{\text{Prop. 21.21}}{\Leftrightarrow} \text{Cok}(\ell_3) = 0 \Leftrightarrow \ell_3 \text{ is fully cofaithful}
 \end{aligned}$$

and

$$\begin{aligned}
 (24.1) \text{ is relatively 2-exact in } A_3 & \Leftrightarrow \text{Ker}(\ell_3, (\ell_3)_I^\#) = 0 \\
 & \stackrel{\text{Rem. 21.18}}{\Leftrightarrow} \text{Ker}(\ell_3) = 0 \Leftrightarrow \ell_3 \text{ is fully faithful.}
 \end{aligned}$$

Thus, (24.1) is relatively 2-exact in A_3 and A_4 if and only if ℓ is fully cofaithful and fully faithful, i.e. ℓ is an equivalence. \square

By Remark 21.18, we have the following corollary:

Uniqueness (up to an equivalence) of (x_n, ξ_n, η_n) follows from the universality of the relative kernel $Z^{n+1}(A_\bullet)$ and the uniqueness of η_n . \square

Lemma 24.4. *Consider the following complex diagram in \mathbf{S} .*

$$(24.4) \quad \begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \varphi & \curvearrowleft & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

If (24.4) is 2-exact in B and g is cofaithful, then we have $\text{Cok}(f) = [C, g, \varphi]$.

Proof. If we factor (24.4) as

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \pi_f & \curvearrowleft & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & \text{Cok}(f) \\ & \curvearrowleft & \downarrow \varphi & \downarrow \bar{g} & \\ & & 0 & & C \end{array}$$

then, since (24.4) is 2-exact in B , \bar{g} becomes fully faithful. On the other hand, since g is cofaithful, \bar{g} is also cofaithful. Thus \bar{g} becomes an equivalence. \square

Lemma 24.5. *Consider the following complex morphism in \mathbf{S} .*

$$\begin{array}{ccccccc} & & 0 & & & & \\ & \curvearrowright & \uparrow \delta_2^A & \curvearrowleft & & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 & \xrightarrow{0} & 0 \\ \text{id} \parallel & \lambda \Downarrow & f_2 \downarrow & \kappa \Downarrow & f_3 \downarrow & & \\ A_1 & \xrightarrow{d_1^B} & B_2 & \xrightarrow{d_2^B} & B_3 & \xrightarrow{0} & 0 \\ & \curvearrowleft & \delta_2^B \Downarrow & \curvearrowright & & & \\ & & 0 & & & & \end{array}$$

If the complexes are relatively 2-exact in A_2, A_3 and B_2, B_3 respectively, i.e. they satisfy $\text{Cok}(d_1^A) = [A_3, d_2^A, \delta_2^A]$ and $\text{Cok}(d_1^B) = [B_3, d_2^B, \delta_2^B]$ (see Corollary 24.2), then the following diagram obtained by taking the kernel of f_2 becomes 2-exact in A_3 .

$$(24.5) \quad \begin{array}{ccc} & & 0 \\ & \curvearrowright & \uparrow \\ \text{Ker}(f_2) & \xrightarrow{k(f_2) \circ d_2^A} & A_3 \xrightarrow{f_3} B_3 \\ & & \downarrow (k(f_2) \circ \kappa) \cdot (\varepsilon_{f_2} \circ d_2^B) \cdot (d_2^B)_1 \end{array}$$

Proof. By taking the kernels of id_{A_1} and f_2 in the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{d_1^A} & A_2 \\ \text{id} \parallel & \lambda \Downarrow & \downarrow f_2 \\ A_1 & \xrightarrow{d_1^B} & B_2 \end{array}$$

and taking the cokernels of $0_{0,A_1}$ and $k(f_2)$, we obtain the following diagram by Proposition 23.8, where $\theta = k(f_2)_I^b \cdot (d_1^A)_I^{b-1}$:

$$(24.6) \quad \begin{array}{ccccccc} 0 & \xrightarrow{0} & A_1 & \xlongequal{\text{id}} & A_1 & \xlongequal{\text{id}} & A_1 \\ \downarrow 0 & \Rightarrow \theta & \downarrow d_1^A & \Rightarrow \lambda_1 & \downarrow \bar{d}_1^A & \Rightarrow \lambda_2 & \downarrow d_1^B \\ \text{Ker}(f_2) & \xrightarrow{k(f_2)} & A_2 & \xrightarrow{c(k(f_2))} & \text{Coim}(f_2) & \xrightarrow{j(f_2)} & B_2 \\ & & & \searrow \mu_{f_2} \Downarrow & & \nearrow & \\ & & & & & & f_2 \end{array}$$

By taking the cokernels of $0_{0,\text{Ker}(f_2)}$, d_1^A and d_1^B in (24.6), we obtain the left of the following diagrams, while by Proposition 23.13 we obtain the right as a coimage factorization if we take the cokernels of d_1^A , \bar{d}_1^A and d_1^B in (24.6):

$$\begin{array}{ccc} \begin{array}{ccc} 0 & \xrightarrow{\varepsilon_{f_2} \uparrow} & B_2 \\ \text{Ker}(f_2) \xrightarrow{k(f_2)} A_2 \xrightarrow{f_2} B_2 \\ \text{id} \parallel \circlearrowleft d_2^A \downarrow \cong \downarrow d_2^B \\ \text{Ker}(f_2) \xrightarrow{k(f_2) \circ d_2^A} A_3 \xrightarrow{f_3} B_3 \\ \downarrow \text{dotted} \\ 0 \end{array} & \begin{array}{ccc} f_2 & \xrightarrow{\mu_{f_2} \uparrow} & B_2 \\ A_2 \xrightarrow{c(k(f_2))} \text{Coim}(f_2) \xrightarrow{j(f_2)} B_2 \\ d_2^A \downarrow \cong \bar{d}_2^A \downarrow \cong \downarrow d_2^B \\ A_3 \xrightarrow{c(k(f_3))} \text{Coim}(f_3) \xrightarrow{j(f_3)} B_3 \\ \downarrow \mu_{f_3} \\ f_3 \end{array} \\ (k(f_2) \circ \kappa) \cdot (\varepsilon_{f_2} \circ d_2^B) \cdot (d_2^B)_I^b & & \end{array}$$

On the other hand by Proposition 23.7, if we take the compatible 2-cell $v = (k(f_2) \circ \kappa_1) \cdot (\pi_{k(f_2)} \circ \bar{d}_2^A) \cdot (\bar{d}_2^A)_I^b \in \mathbf{S}^2(k(f_2) \circ d_2^A \circ c(k(f_3)), 0)$,

$$\begin{array}{ccc} 0 & \xrightarrow{\uparrow \pi_{k(f_2)}} & \text{Coim}(f_2) \\ \text{Ker}(f_2) \xrightarrow{k(f_2)} A_2 \xrightarrow{c(k(f_2))} \text{Coim}(f_2) \\ \text{id} \parallel \circlearrowleft d_2^A \downarrow \cong \bar{d}_2^A \downarrow \\ \text{Ker}(f_2) \xrightarrow{k(f_2) \circ d_2^A} A_3 \xrightarrow{c(k(f_3))} \text{Coim}(f_3) \\ \downarrow v \\ 0 \end{array}$$

then we have $\text{Cok}(k(f_2) \circ d_2^A) = [\text{Coim}(f_3), c(k(f_3)), v]$. It can be easily shown that v is compatible with μ_{f_3} and $(k(f_2) \circ \kappa) \cdot (\varepsilon_{f_2} \circ d_2^B) \cdot (d_2^B)_I^b$.

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & \text{Coim}(f_3) \\ \text{Ker}(f_2) \xrightarrow{k(f_2) \circ d_2^A} A_3 \xrightarrow{c(k(f_3))} \text{Coim}(f_3) \\ \downarrow \mu_{f_3} \swarrow \searrow j(f_3) \\ \text{Ker}(f_2) \xrightarrow{k(f_2) \circ d_2^A} A_3 \xrightarrow{f_3} B_3 \\ \downarrow \text{dotted} \\ 0 \end{array}$$

Since $\text{Cok}(k(f_2) \circ d_2^A) = [\text{Coim}(f_3), c(k(f_3)), v]$ and $j(f_3)$ is fully faithful by Proposition 22.4, this means (24.5) is 2-exact in A_3 . \square

Lemma 24.6. Consider the following complex morphism in \mathbf{S} .

$$(24.7) \quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow \delta_2^A & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & 0 & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \\ f_1 \downarrow & \lambda_1 \Downarrow & f_2 \downarrow & \lambda_2 \Downarrow & \downarrow f_3 \\ 0 & \xrightarrow{0} & B_1 & \xrightarrow{d_1^B} & B_2 & \xrightarrow{d_2^B} & B_3 \\ & & & & \delta_2^B \Downarrow & & \\ & & & & 0 & & \end{array}$$

If the complexes are relatively 2-exact in A_2 and B_1, B_2 respectively, then the following diagram obtained by taking the kernels

$$(24.8) \quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow \delta_2^A & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & 0 & & \\ \text{Ker}(f_1) & \xrightarrow{d_1^A} & \text{Ker}(f_2) & \xrightarrow{d_2^A} & \text{Ker}(f_3) \end{array}$$

is 2-exact in $\text{Ker}(f_2)$.

Proof. If we decompose (24.7) into

$$\begin{array}{ccc} A_1 \xrightarrow{d_1^{A^\dagger}} \text{Ker}(d_2^A) & & \text{Ker}(d_2^A) \xrightarrow{k(d_2^A)} A_2 \xrightarrow{d_2^A} A_3 \\ f_1 \downarrow \quad \lambda_1^\dagger \Downarrow \quad \downarrow f_2 & \text{and} & f_2 \downarrow \quad \lambda_1 \Downarrow \quad f_2 \downarrow \quad \lambda_2 \Downarrow \quad \downarrow f_3 \\ 0 \xrightarrow{0} B_1 \xrightarrow{d_1^{B^\dagger}} \text{Ker}(d_2^B) & & \text{Ker}(d_2^B) \xrightarrow{k(d_2^B)} B_2 \xrightarrow{d_2^B} B_3 \end{array}$$

then by (the dual of) Proposition 23.7, we have $\text{Ker}(d_2^A) = \text{Ker}(f_2)$. Since $d_1^{B^\dagger}$ is an equivalence by (the dual of) Corollary 24.2, the diagram obtained by taking the kernels of f_1 and f_2

$$\begin{array}{ccc} \text{Ker}(f_1) & \xrightarrow{d_1^{A^\dagger}} & \text{Ker}(f_2) \\ k(f_1) \downarrow & \lambda_1^\dagger \Downarrow & \downarrow k(f_2) \\ A_1 & \xrightarrow{d_1^{A^\dagger}} & \text{Ker}(d_2^A) \end{array}$$

becomes a pullback diagram by (the dual of) Proposition 23.11. Since $d_1^{A^\dagger}$ is fully cofaithful, $d_1^{A^\dagger}$ becomes also fully cofaithful by Proposition 23.12. This means (24.8) is 2-exact in $\text{Ker}(f_2)$. \square

Lemma 24.7. Consider the following complex morphism in \mathbf{S} .

$$(24.9) \quad \begin{array}{ccccc} & & 0 & & \\ & & \uparrow \delta_2^A & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & \delta_2^A & & \\ & & \uparrow & & \\ & & 0 & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \\ \text{id} \parallel & \lambda_1 \Downarrow & \text{id} \parallel & \lambda_2 \Downarrow & \downarrow f_3 \\ A_1 & \xrightarrow{d_1^B} & A_2 & \xrightarrow{d_2^B} & B_3 \\ & & & & \delta_2^B \Downarrow & \\ & & & & 0 & & \end{array}$$

If f_3 is faithful and the bottom row is 2-exact in A_2 , then the top row is also 2-exact in A_2 .

Proof. By taking the cokernels of d_1^A and d_1^B in (24.9), we obtain (by Proposition 23.8)

$$\begin{array}{ccc} \text{Cok}(d_1^A) & \xrightarrow{\bar{d}_2^A} & A_3 \\ \text{id} \parallel & \bar{\lambda}_2 \Downarrow & \downarrow f_3 \\ \text{Cok}(d_1^B) & \xrightarrow{\bar{d}_2^B} & B_3. \end{array}$$

Since \bar{d}_2^B is fully faithful, by taking the kernels in this diagram, we obtain the following diagram.

$$\begin{array}{ccccc} & & 0 & \xrightarrow{0} & \text{Ker}(f_3) \\ & & 0 \downarrow & \exists \Downarrow & \downarrow k(f_3) \\ \text{Ker}(\bar{d}_2^A) & \xrightarrow{k(\bar{d}_2^A)} & \text{Cok}(d_1^A) & \xrightarrow{\bar{d}_2^A} & A_3 \\ 0 \downarrow & \exists \Downarrow & \text{id} \parallel & \bar{\lambda}_2 \Downarrow & \downarrow f_3 \\ 0 & \xrightarrow{0} & \text{Cok}(d_1^B) & \xrightarrow{\bar{d}_2^B} & B_3 \end{array}$$

In this diagram, we have

$$\text{Ker}(\bar{d}_2^A) = \text{Ker}(\text{Ker}(\bar{d}_2^A) \rightarrow 0) \underset{\text{Prop. 23.7}}{=} \text{Ker}(0 \rightarrow \text{Ker}(f_3)) \underset{\text{Cor. 21.28}}{=} 0.$$

This means that the top row in (24.9) is 2-exact in A_2 . \square

Corollary 24.8. *Let*

$$\begin{array}{c} 0 \\ \delta_2^A \Uparrow \\ A_1 \xrightarrow{d_1^A} A_2 \xrightarrow{d_2^A} A_3 \\ \delta_2^B \Uparrow \\ B_1 \xrightarrow{d_1^B} A_2 \xrightarrow{d_2^B} B_3 \end{array} \quad \text{and} \quad \begin{array}{c} 0 \\ \delta_2^B \Uparrow \\ B_1 \xrightarrow{d_1^B} A_2 \xrightarrow{d_2^B} B_3 \\ \delta_2^A \Uparrow \\ A_1 \xrightarrow{d_1^A} A_2 \xrightarrow{d_2^A} A_3 \end{array}$$

be two complexes, and assume that there exist 1-cells f_1, f_3 and 2-cells $\lambda_1, \lambda_2, \sigma$ as in the following diagram

$$\begin{array}{ccccc} B_1 & & & & A_3 \\ & \searrow^{d_1^B} & & \searrow^{d_2^A} & \downarrow f_3 \\ & & A_2 & & \\ & \nearrow_{\lambda_1} & & \searrow_{\lambda_2} & \\ f_1 \downarrow & & & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^B} & B_3 \\ & \searrow_{\sigma} & & \searrow_{\sigma} & \\ & & & & 0 \end{array}$$

where f_1 is cofaithful and f_3 is faithful. Assume they satisfy

$$\begin{aligned} \text{(d1)} \quad & (\lambda_1 \circ d_2^B) \cdot \delta_2^B = (f_1 \circ \sigma) \cdot (f_1)_I^\# \\ \text{(d2)} \quad & (d_1^A \circ \lambda_2) \cdot \sigma = (\delta_2^A \circ f_3) \cdot (f_3)_I^b. \end{aligned}$$

Then, if the diagram

$$\begin{array}{c} 0 \\ \delta_2^B \Uparrow \\ B_1 \xrightarrow{d_1^B} A_2 \xrightarrow{d_2^B} B_3 \end{array}$$

is 2-exact in A_2 , then the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \delta_2^A & \nearrow & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \end{array}$$

is also 2-exact in A_2 .

Proof. This follows if we apply Lemma 24.7 and its dual to the following diagrams:

$$\begin{array}{ccc} \begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \delta_2^B & \nearrow & \\ B_1 & \xrightarrow{d_1^B} & A_2 & \xrightarrow{d_2^B} & B_3 \\ f_1 \downarrow \lambda_1^{-1} \Downarrow & & \parallel \text{id} & \circ & \parallel \text{id} \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^B} & B_3, \\ & & \sigma \Downarrow & & \\ & & 0 & & \end{array} & , & \begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \delta_2^A & \nearrow & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \\ \text{id} \parallel & & \circ & \text{id} \parallel & \lambda_2 \Downarrow \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^B} & B_3 \\ & & \sigma \Downarrow & & \downarrow f_3 \\ & & 0 & & \end{array} \end{array}$$

□

By Corollary 24.8, it can be shown that the 2-exactness plus compatibility implies the relative 2-exactness (see [29] in the case of SCG):

Corollary 24.9. *Let $A_\bullet = (A_n, d_n, \delta_n)$ be a complex in \mathbf{S} . If*

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \delta_n^A & \nearrow & \\ A_{n-1} & \xrightarrow{d_{n-1}^A} & A_n & \xrightarrow{d_n^A} & A_{n+1} \end{array}$$

is 2-exact in A_n , then A_\bullet is relatively 2-exact in A_n .

Proof. This follows immediately if we apply Corollary 24.8 to the following diagram (see the proof of Corollary 23.24):

$$\begin{array}{ccccc} A_{n-1} & \xrightarrow{d_{n-1}} & A_n & \xrightarrow{d_n} & \text{Ker}(d_{n+1}) \\ c(d_{n-2}) \downarrow & \nearrow \bar{\delta}_{n-1} & & \searrow \underline{\delta}_{n+1} & \downarrow k(d_{n+1}) \\ \text{Cok}(d_{n-2}) & \xrightarrow{\bar{d}_{n-1}} & A_n & \xrightarrow{d_n} & A_{n+1} \\ & & \bar{\delta}_n \Downarrow & & \\ & & 0 & & \end{array}$$

□

Construction of the long cohomology sequence

Definition 24.10. A complex in \mathbf{S}

$$(24.10) \quad \begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \varphi & \nearrow & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

is called an extension if it is relatively 2-exact in every 0-cell.

Remark 24.11. *By Corollary 24.2 (and its dual), (24.10) is an extension if and only if $\text{Ker}(g) = [A, f, \varphi]$ and $\text{Cok}(f) = [C, g, \varphi]$.*

Definition 24.12. Let $(f_\bullet, \lambda_\bullet) : A_\bullet \rightarrow B_\bullet$ and $(g_\bullet, \kappa_\bullet) : B_\bullet \rightarrow C_\bullet$ be complex morphisms and $\varphi_\bullet = \{\varphi_n : f_n \circ g_n \Rightarrow 0\}$ be 2-cells. Then,

$$(24.11) \quad \begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \varphi_\bullet & \curvearrowleft & \\ A_\bullet & \xrightarrow{f_\bullet} & B_\bullet & \xrightarrow{g_\bullet} & C_\bullet \end{array}$$

is said to be an extension of complexes if it satisfies the following properties:

(e1) For every n , the following complex is an extension:

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \varphi_n & \curvearrowleft & \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \end{array}$$

(e2) φ_\bullet satisfies

$$(\lambda_n \circ g_{n+1}) \cdot (f_n \circ \kappa_n) \cdot (\varphi_n \circ d_n^C) \cdot (d_n^C)_I^b = (d_n^A \circ \varphi_{n+1}) \cdot (d_n^A)_I^\#.$$

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \varphi_n & \curvearrowleft & \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\ d_n^A \downarrow & \xrightarrow{\lambda_n} & d_n^B & \xrightarrow{\kappa_n} & d_n^C \\ A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ & \curvearrowright & \downarrow \varphi_{n+1} & \curvearrowleft & \\ & & 0 & & \end{array}$$

Our main theorem is the following:

Theorem 24.13. For any extension of complexes in \mathbf{S}

$$\begin{array}{ccccc} & & 0 & & \\ & \curvearrowright & \uparrow \varphi_\bullet & \curvearrowleft & \\ A_\bullet & \xrightarrow{f_\bullet} & B_\bullet & \xrightarrow{g_\bullet} & C_\bullet, \end{array}$$

we can construct a long 2-exact sequence:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \curvearrowright & \uparrow & \curvearrowleft & \uparrow & \curvearrowleft & \\ \cdots & \rightarrow & H^n(B_\bullet) & \rightarrow & H^n(C_\bullet) & \rightarrow & H^{n+1}(A_\bullet) & \rightarrow & H^{n+1}(B_\bullet) & \rightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

Caution 24.14. This sequence is not necessarily a complex. (See Remark 24.19.)

We prove this theorem in the rest of this section.

Lemma 24.15. In the notation of Lemma 24.3, we have

- (1) $\text{Ker}(x_n) = H^n(A_\bullet)$,
- (2) $\text{Cok}(x_n) = H^{n+1}(A_\bullet)$.

Proof. We only show (1), since (2) can be shown in the same way. In the notation of Lemma 24.3 and Remark 23.18, we can show that the factorization

$$(x_n \circ \zeta_{n+1}) \cdot \xi_n : (x_n \circ \zeta_{n+1}) \circ k(d_{n+1}) \Rightarrow \ell_n$$

is compatible with $\varepsilon_{d_{n+1}}$ and $\mu_{n,2}$.

$$\begin{array}{ccccc}
 Q^n(A_\bullet) & \xrightarrow{\quad 0 \quad} & & & \\
 \downarrow & \nearrow \ell_n & \mu_{n,2} \Uparrow & & \\
 & & A_{n+1} & \xrightarrow{d_{n+1}} & A_{n+2} \\
 x_n \circ \underline{z}_{n+1} \dashrightarrow & & \downarrow \varepsilon_{d_{n+1}} & & \\
 \text{Ker}(d_{n+1}) & \xrightarrow{k(d_{n+1})} & & & \\
 (x_n \circ \underline{z}_{n+1}) \cdot \xi_n \dashrightarrow & & & & \\
 & \searrow & & & \\
 & & & & 0
 \end{array}$$

So, by Proposition 21.20, Proposition 21.21 and the fact that \underline{z}_{n+1} is fully faithful, we have $H^n(A_\bullet) = \text{Ker}(\ell_n, \mu_{n,2}) = \text{Ker}(x_n \circ \underline{z}_{n+1}) = \text{Ker}(x_n)$. \square

Lemma 24.16. *For any extension (24.11) of complexes in \mathbf{S} , we can construct a complex morphism*

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow Q^n(\varphi_\bullet) & & & \\
 Q^n(A_\bullet) & \xrightarrow{Q^n(f_\bullet)} & Q^n(B_\bullet) & \xrightarrow{Q^n(g_\bullet)} & Q^n(C_\bullet) & \xrightarrow{0} & 0 \\
 \downarrow x_n^A & & \tilde{\lambda}_n \Downarrow & \downarrow x_n^B & \Downarrow \tilde{\kappa}_n & \downarrow x_n^C & \\
 0 \longrightarrow & Z^{n+1}(A_\bullet) & \xrightarrow{Z^{n+1}(f_\bullet)} & Z^{n+1}(B_\bullet) & \xrightarrow{Z^{n+1}(g_\bullet)} & Z^{n+1}(C_\bullet) & \\
 & & & \downarrow Z^{n+1}(\varphi_\bullet) & & & \\
 & & & 0 & & &
 \end{array}$$

where the top line is a complex which is relatively 2-exact in $Q^n(B_\bullet)$, $Q^n(C_\bullet)$, and the bottom line is a complex which is relatively 2-exact in $Z^{n+1}(A_\bullet)$, $Z^{n+1}(B_\bullet)$.

Proof. If we take the relative cokernels $Q^n(A_\bullet)$, $Q^n(B_\bullet)$ and $Q^n(C_\bullet)$ of the complex diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow \delta_{n-1}^A & & & \\
 A_{n-2} & \xrightarrow{d_{n-2}^A} & A_{n-1} & \xrightarrow{d_{n-1}^A} & A_n & \xrightarrow{d_n^A} & A_{n+1} \\
 f_{n-2} \downarrow & \lambda_{n-2} \Downarrow & \downarrow f_{n-1} & \Downarrow \lambda_{n-1} & \downarrow f_n & \Downarrow \lambda_n & \downarrow f_{n+1} \\
 B_{n-2} & \xrightarrow{d_{n-2}^B} & B_{n-1} & \xrightarrow{d_{n-1}^B} & B_n & \xrightarrow{d_n^B} & B_{n+1} \\
 g_{n-2} \downarrow & \kappa_{n-2} \Downarrow & \downarrow g_{n-1} & \Downarrow \kappa_{n-1} & \downarrow g_n & \Downarrow \kappa_n & \downarrow g_{n+1} \\
 C_{n-2} & \xrightarrow{d_{n-2}^C} & C_{n-1} & \xrightarrow{d_{n-1}^C} & C_n & \xrightarrow{d_n^C} & C_{n+1} \\
 & & & \downarrow \delta_{n-1}^C & & & \\
 & & & 0 & & &
 \end{array}$$

then by (the dual of) Proposition 21.20, Proposition 23.4 and Proposition 23.5, we obtain a factorization diagram

$$\begin{array}{ccccccc}
A_{n-1} & \xrightarrow{d_{n-1}^A} & A_n & \xrightarrow{q_n^A} & Q^n(A_\bullet) & \xrightarrow{\ell_n^A} & A_{n+1} \\
f_{n-1} \downarrow & \lambda_{n-1} \Downarrow & f_n \downarrow & \bar{\lambda}_{n-1,1} \Downarrow & Q^n(f_\bullet) \downarrow & \Downarrow \bar{\lambda}_{n-1,2} & f_{n+1} \downarrow \\
B_{n-1} & \xrightarrow{d_{n-1}^B} & B_n & \xrightarrow{q_n^B} & Q^n(B_\bullet) & \xrightarrow{\ell_n^B} & B_{n+1} \\
g_{n-1} \downarrow & \kappa_{n-1} \Downarrow & g_n \downarrow & \bar{\kappa}_{n-1,1} \Downarrow & Q^n(g_\bullet) \downarrow & \Downarrow \bar{\kappa}_{n-1,2} & g_{n+1} \downarrow \\
C_{n-1} & \xrightarrow{d_{n-1}^C} & C_n & \xrightarrow{q_n^C} & Q^n(C_\bullet) & \xrightarrow{\ell_n^C} & C_{n+1}
\end{array}$$

and a 2-cell $Q^n(\varphi_\bullet) \in \mathbf{S}^2(Q^n(f_\bullet) \circ Q^n(g_\bullet), 0)$, which satisfy compatibility conditions in Proposition 23.4 and Proposition 23.5. It is also easy to see by the universality of the relative cokernels that

$$(\ell_n^A \circ \lambda_{n+1}) \cdot (\bar{\lambda}_{n-1,2} \circ d_{n+1}^B) \cdot (Q^n(f_\bullet) \circ \mu_{n,2}^B) \cdot (Q^n(f_\bullet))_I^\# = (\mu_{n,2}^A \circ f_{n+2}) \cdot (f_{n+2})_I^b.$$

Now, since

$$\begin{array}{ccccc}
& & 0 & & \\
& & \uparrow \varphi_n & & \\
A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \xrightarrow{0} 0
\end{array}$$

is relatively 2-exact in B_n and C_n , we have $\text{Cok}(f_n) = [C_n, g_n, \varphi_n]$. So, from $\text{Cok}(f_n) = [C_n, g_n, \varphi_n]$ and $\text{Cok}(f_{n-1}) = [C_{n-1}, g_{n-1}, \varphi_{n-1}]$, by Proposition 23.10 we obtain

$$\text{Cok}(Q^n(f_\bullet)) = [Q^n(C_\bullet), Q^n(g_\bullet), Q^n(\varphi_\bullet)],$$

i.e. the complex

$$\begin{array}{ccccccc}
& & 0 & & & & \\
& & \uparrow Q^n(\varphi_\bullet) & & & & \\
Q^n(A_\bullet) & \xrightarrow{Q^n(f_\bullet)} & Q^n(B_\bullet) & \xrightarrow{Q^n(g_\bullet)} & Q^n(C_\bullet) & \xrightarrow{0} & 0
\end{array}$$

is relatively 2-exact in $Q^n(B_\bullet), Q^n(C_\bullet)$. Dually, we obtain a factorization diagram

$$\begin{array}{ccccccc}
A_n & \xrightarrow{k_{n+1}^A} & Z^{n+1}(A_\bullet) & \xrightarrow{z_{n+1}^A} & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_{n+2} \\
f_n \downarrow & \lambda_{n+1,2} \Downarrow & Z^{n+1}(f_\bullet) \downarrow & \Downarrow \lambda_{n+1,1} & f_{n+1} \downarrow & \Downarrow \lambda_{n+1} & f_{n+2} \downarrow \\
B_n & \xrightarrow{k_{n+1}^B} & Z^{n+1}(B_\bullet) & \xrightarrow{z_{n+1}^B} & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_{n+2} \\
g_n \downarrow & \kappa_{n+1,2} \Downarrow & Z^{n+1}(g_\bullet) \downarrow & \Downarrow \kappa_{n+1,1} & g_{n+1} \downarrow & \Downarrow \kappa_{n+1} & g_{n+2} \downarrow \\
C_n & \xrightarrow{k_{n+1}^C} & Z^{n+1}(C_\bullet) & \xrightarrow{z_{n+1}^C} & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_{n+2}
\end{array}$$

such that

$$\begin{aligned}
(z_{n+1}^A \circ \lambda_{n+1}) \cdot (\lambda_{n+1,1} \circ d_{n+1}^B) \cdot (Z^{n+1}(f_\bullet) \circ \zeta_{n+1}^B) \cdot Z^{n+1}(f_\bullet)_I^\# \\
= (\zeta_{n+1}^A \circ f_{n+2}) \cdot (f_{n+2})_I^b.
\end{aligned}$$

Then, it can be shown that each of the factorizations

$$\begin{aligned} Q^n(f_\bullet) \circ \xi_n^B & : Q^n(f_\bullet) \circ x_n^B \circ z_{n+1}^B \implies Q^n(f_\bullet) \circ \ell_n^B \\ (x_n^A \circ \underline{\lambda}_{n+1,1}^{-1}) \cdot (\xi_n^A \circ f_{n+1}) \cdot \bar{\lambda}_{n-1,2} & : x_n^A \circ Z^{n+1}(f_\bullet) \circ z_{n+1}^B \implies Q^n(f_\bullet) \circ \ell_n^B \end{aligned}$$

are compatible with ζ_{n+1}^B and $(Q^n(f_\bullet) \circ \mu_{n,2}^B) \cdot (Q^n(f_\bullet))_I^\sharp$.

So, by the universality of the relative kernel, there exists a unique 2-cell $\tilde{\lambda}_n \in \mathbf{S}^2(Q^n(f_\bullet) \circ x_n^B, x_n^A \circ Z^{n+1}(f_\bullet))$ such that

$$(\tilde{\lambda}_n \circ z_{n+1}^B) \cdot (x_n^A \circ \underline{\lambda}_{n+1,1}^{-1}) \cdot (\xi_n^A \circ f_{n+1}) \cdot \bar{\lambda}_{n-1,2} = Q^n(f_\bullet) \circ \xi_n^B.$$

This $\tilde{\lambda}_n$ also satisfies $(q_n^A \circ \tilde{\lambda}_n) \cdot (\eta_n^A \circ Z^{n+1}(f_\bullet)) \cdot \underline{\lambda}_{n+1,2} = (\bar{\lambda}_{n-1,1} \circ x_n^B) \cdot (f_n \circ \eta_n^B)$ (see Remark 24.17). Similarly, we obtain a 2-cell $\tilde{\kappa}_n \in \mathbf{S}^2(Q^n(g_\bullet) \circ x_n^C, x_n^B \circ Z^{n+1}(g_\bullet))$ such that

$$(\tilde{\kappa}_n \circ z_{n+1}^C) \cdot (x_n^B \circ \underline{\kappa}_{n+1,1}^{-1}) \cdot (\xi_n^B \circ g_{n+1}) \cdot \bar{\kappa}_{n-1,2} = Q^n(g_\bullet) \circ \xi_n^C.$$

In the rest, we show the following:

(24.12)

$$(Q^n(f_\bullet) \circ \tilde{\kappa}_n) \cdot (\tilde{\lambda}_n \circ Z^{n+1}(g_\bullet)) \cdot (x_n^A \circ Z^{n+1}(\varphi_\bullet)) \cdot (x_n^A)_I^\sharp = (Q^n(\varphi_\bullet) \circ x_n^C) \cdot (x_n^C)_I^\flat.$$

We have the following equalities:

$$\begin{aligned} & (Q^n(f_\bullet) \circ \tilde{\kappa}_n \circ z_{n+1}^C) \cdot (\tilde{\lambda}_n \circ Z^{n+1}(g_\bullet) \circ z_{n+1}^C) \\ & = (Q^n(f_\bullet) \circ Q^n(g_\bullet) \circ \xi_n^C) \cdot (Q^n(f_\bullet) \circ \bar{\kappa}_{n-1,2}^{-1}) \cdot (\bar{\lambda}_{n-1,2}^{-1} \circ g_{n+1}) \\ & \quad \cdot ((\xi_n^A)^{-1} \circ f_{n+1} \circ g_{n+1}) \cdot (x_n^A \circ \underline{\lambda}_{n+1,1} \circ g_{n+1}) \cdot (x_n^A \circ Z^{n+1}(f_\bullet) \circ \underline{\kappa}_{n+1,1}), \\ & (Q^n(\varphi_\bullet) \circ x_n^C \circ z_{n+1}^C) \cdot (x_n^C \circ z_{n+1}^C)_I^\flat \cdot (x_n^A \circ z_{n+1}^A)_I^{\sharp-1} \cdot (x_n^A \circ z_{n+1}^A \circ \varphi_{n+1}^{-1}) \\ & = (Q^n(f_\bullet) \circ Q^n(g_\bullet) \circ \xi_n^C) \cdot (Q^n(f_\bullet) \circ \bar{\kappa}_{n-1,2}^{-1}) \cdot (\bar{\lambda}_{n-1,2}^{-1} \circ g_{n+1}) \\ & \quad \cdot ((\xi_n^A)^{-1} \circ f_{n+1} \circ g_{n+1}), \end{aligned}$$

$$\begin{aligned} (z_{n+1}^C)_I^{\flat-1} & = (x_n^A \circ z_{n+1}^A)_I^{\sharp-1} \cdot (x_n^A \circ z_{n+1}^A \circ \varphi_{n+1}^{-1}) \cdot (x_n^A \circ \underline{\lambda}_{n+1,1} \circ g_{n+1}) \\ & \quad \cdot (x_n^A \circ Z^{n+1}(f_\bullet) \circ \underline{\kappa}_{n+1,1}) \cdot (x_n^A \circ Z^{n+1}(\varphi_\bullet) \circ z_{n+1}^C) \cdot ((x_n^A)_I^\sharp \circ z_{n+1}^C). \end{aligned}$$

From these equalities and the faithfulness of z_{n+1}^C , we obtain (24.12). \square

Remark 24.17. It can be also shown that $\tilde{\lambda}_n$ in the proof of Lemma 24.16 satisfies

$$(q_n^A \circ \tilde{\lambda}_n) \cdot (\eta_n^A \circ Z^{n+1}(f_\bullet)) \cdot \underline{\lambda}_{n+1,2} = (\bar{\lambda}_{n-1,1} \circ x_n^B) \cdot (f_n \circ \eta_n^B).$$

which satisfies

$$\begin{aligned}\eta_0 &= (f_1)_I^{\sharp-1} \cdot (f_1 \circ \delta_2^{B-1}) \\ \rho &= (f_1 \circ \eta_1) \cdot (\pi_{f_1} \circ q_1) \circ (q_1)_I^b \\ \text{id}_0 &= \eta_0 \cdot (f_1 \circ d_1^B \circ \eta_2) \cdot (\rho \circ q_2) \cdot (q_2)_I^b \\ \delta_2^B &= (d_1^B \circ \eta_2) \cdot (\eta_1 \circ q_2) \cdot (c(f_1) \circ \beta_2) \cdot c(f_1)_I^\sharp,\end{aligned}$$

and we have $\text{Ker}(q_2) = [\text{Cok}(f_1), q_1, \beta_2]$. (The “un-duality” in appearance is simply because of the direction of the 2-cells.) Thus, we obtain complex morphisms:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\text{id}} & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{c(f_1)} & \text{Cok}(f_1) \\ k_1 \downarrow & \xRightarrow{\xi_1} & \downarrow d_1^A & \xRightarrow{\lambda_1} & \downarrow d_1^B & \xRightarrow{\eta_1} & \downarrow q_1 \\ K & \xrightarrow{k} & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{q} & Q \\ k_2 \downarrow & \xRightarrow{\xi_2} & \downarrow d_2^A & \xRightarrow{\lambda_2} & \downarrow d_2^B & \xRightarrow{\eta_2} & \downarrow q_2 \\ \text{Ker}(f_3) & \xrightarrow{k(f_3)} & A_3 & \xrightarrow{f_3} & B_3 & \xrightarrow{\text{id}} & B_3 \end{array}$$

If we put

$$\begin{aligned}c &:= k \circ f_2 \circ q \\ \alpha_K &:= (\xi_1 \circ f_2 \circ q) \cdot (\lambda_1 \circ q) \cdot \rho \\ \beta_Q &:= (k \circ f_2 \circ \eta_2^{-1}) \cdot (k \circ \lambda_2^{-1}) \cdot \zeta,\end{aligned}$$

then, it can be shown that the following diagram is a complex.

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \alpha_K \uparrow \uparrow & \searrow & \\ A_1 & \xrightarrow{k_1} & K & \xrightarrow{c} & Q & \xrightarrow{q_2} & B_3 \\ & & \downarrow \beta_Q & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

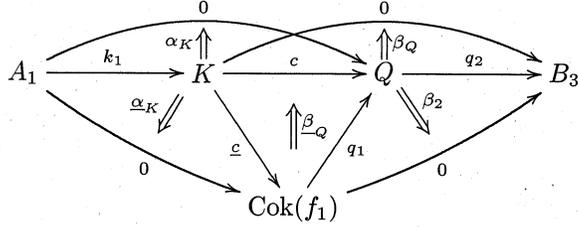
Since $\text{Cok}(k_1) = [\text{Ker}(f_3), k_2, \alpha_2]$ as already shown, we have a factorization diagram

$$\begin{array}{ccccc} & & \text{Ker}(f_3) & & \\ & \searrow & \downarrow \bar{\alpha}_K & \searrow & \\ A_1 & \xrightarrow{k_1} & K & \xrightarrow{c} & Q & \xrightarrow{q_2} & B_3 \\ & & \downarrow \alpha_K & & \downarrow \beta_Q & & \\ & & 0 & & 0 & & \end{array}$$

which satisfies

$$\begin{aligned}(k_1 \circ \bar{\alpha}_K) \cdot \alpha_K &= (\alpha_2 \circ \bar{c}) \cdot (\bar{c})_I^b \\ (\bar{\alpha}_K \circ q_2) \cdot \beta_Q &= (k_2 \circ \bar{\beta}_Q) \cdot (k_2)_I^\sharp.\end{aligned}$$

Similarly, since $\text{Ker}(q_2) = [\text{Cok}(f_1), q_1, \beta_2]$, we have a factorization diagram



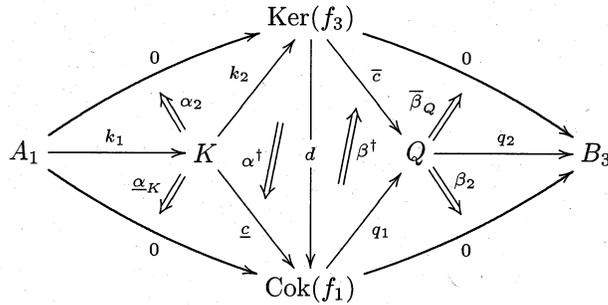
which satisfies

$$\begin{aligned}
 (\underline{\beta}_Q \circ q_2) \cdot \beta_Q &= (\underline{c} \circ \beta_2) \cdot (\underline{c})_I^\sharp \\
 (k_1 \circ \underline{\beta}_Q) \cdot \alpha_K &= (\underline{\alpha}_K \circ q_1) \cdot (q_1)_I^\flat.
 \end{aligned}$$

Then, there exist $d \in \mathbf{S}^1(\text{Ker}(f_3), \text{Cok}(f_1))$, $\alpha^\dagger \in \mathbf{S}^2(k_2 \circ d, \underline{c})$ and $\beta^\dagger \in \mathbf{S}^2(d \circ q_1, \bar{c})$ such that

$$\begin{aligned}
 (k_1 \circ \alpha^\dagger) \cdot \underline{\alpha}_K &= (\alpha_2 \circ d) \cdot d_I^\flat \\
 (\beta^\dagger \circ q_2) \cdot \bar{\beta}_Q &= (d \circ \beta_2) \cdot d_I^\sharp \\
 (k_2 \circ \beta^\dagger) \cdot \bar{\alpha}_K &= (\alpha^\dagger \circ q_1) \cdot \underline{\beta}_Q
 \end{aligned}$$

(note that $\text{Cok}(k_1) = [\text{Ker}(f_3), k_2, \alpha_2]$ and $\text{Ker}(q_2) = [\text{Cok}(f_1), q_1, \beta_2]$ (cf. Lemma 24.3)):



Applying (the dual of) Proposition 23.8 to the diagram

$$\begin{array}{ccccc}
 \text{Ker}(f_1) & \xrightarrow{d_1^A} & \text{Ker}(f_2) & \xrightarrow{d_2^A} & \text{Ker}(f_3) \\
 k(f_1) \downarrow & \Delta_1 \Downarrow & k(f_2) \downarrow & \Delta_2 \Downarrow & k(f_3) \downarrow \\
 A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 \\
 0 \downarrow & \epsilon_0 \Downarrow & d_2^A \circ f_3 \downarrow & \circ & f_3 \downarrow \\
 0 & \xrightarrow{0} & B_3 & \xrightarrow{\text{id}} & B_3,
 \end{array}$$

we see that there exist $k' \in \mathbf{S}^1(\text{Ker}(f_2), K)$, $\xi'_1 \in \mathbf{S}^2(\underline{d}_1^A \circ k', k(f_1) \circ k_1)$, $\xi'_2 \in \mathbf{S}^2(\underline{d}_2^A, k' \circ k_2)$ and $\xi \in \mathbf{S}^2(k' \circ k, k(f_2))$ such that

$$\begin{aligned} \delta_2^A &= (\underline{d}_1^A \circ \xi'_2) \cdot (\xi'_1 \circ k_2) \cdot (k(f_1) \circ \alpha_2) \cdot k(f_1)_I^\# \\ \lambda_2 &= (\xi'_2 \circ k(f_3)) \cdot (k' \circ \xi_2) \cdot (\xi \circ \underline{d}_2^A) \\ (\underline{d}_1^A \circ \xi) \cdot \lambda_1 &= (\xi'_1 \circ k) \cdot (k(f_1) \circ \xi_1). \end{aligned}$$

Similarly, there exist $q' \in \mathbf{S}^1(Q, \text{Cok}(f_2))$, $\eta'_1 \in \mathbf{S}^2(q_1 \circ q', \bar{d}_1^B)$, $\eta'_2 \in \mathbf{S}^2(q_2 \circ c(f_3), q' \circ \bar{d}_2^B)$ and $\eta \in \mathbf{S}^2(q \circ q', c(f_2))$ such that

$$\begin{aligned} (\beta_2 \circ c(f_3)) \cdot c(f_3)_I^\flat &= (q_1 \circ \eta'_2) \cdot (\eta'_1 \circ \bar{d}_2^B) \cdot \bar{\delta}_2^B \\ (\eta_1 \circ q') \cdot (c(f_1) \circ \eta'_1) &= (\underline{d}_1^B \circ \eta) \cdot \bar{\lambda}_1 \\ (\eta_2 \circ c(f_3)) \cdot (q \circ \eta'_2) \cdot (\eta \circ \bar{d}_2^B) &= \bar{\lambda}_2. \end{aligned}$$

If we put

$$\alpha_0 := (\underline{d}_2^A \circ \beta^{\dagger}) \cdot (\xi'_2 \circ \bar{c}) \cdot (k' \circ \bar{\alpha}_K) \cdot (\xi \circ f_2 \circ q) \cdot (\varepsilon_{f_2} \circ q) \cdot q_I^\flat,$$

then it can be shown that $\alpha_0 : \underline{d}_2^A \circ d \circ q_1 \implies 0$ is compatible with β_2 .

$$\begin{array}{ccccc} & & 0 & & \\ & & \alpha_0 \uparrow & & \\ \text{Ker}(f_2) & \xrightarrow{\underline{d}_2^A \circ d} & \text{Cok}(f_1) & \xrightarrow{q_1} & Q & \xrightarrow{q_2} & B_3 \\ & & & & \downarrow \beta_2 & & \\ & & & & 0 & & \end{array}$$

So by Lemma 21.19, there exists $\alpha \in \mathbf{S}^2(\underline{d}_2^A \circ d, 0)$ such that

$$(\alpha \circ q_1) \cdot (q_1)_I^\flat = \alpha_0.$$

Dually, if we put

$$\beta_0 := (\alpha^{\dagger} \circ \bar{d}_1^B) \cdot (\underline{c} \circ \eta_1'^{-1}) \cdot (\underline{c}_Q \circ q') \cdot (k \circ f_2 \circ \eta) \cdot (k \circ \pi_{f_2}) \cdot k_I^\#,$$

then $\beta_0 : k_2 \circ d \circ \bar{d}_1^B \implies 0$ is compatible with α_2 , and there exists $\beta \in \mathbf{S}^2(d \circ \bar{d}_1^B, 0)$ such that

$$(k_2 \circ \beta) \cdot (k_2)_I^\# = \beta_0.$$

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow \delta_2^A & & \uparrow \beta & & \\ \text{Ker}(f_1) & \xrightarrow{\underline{d}_1^A} & \text{Ker}(f_2) & \xrightarrow{\underline{d}_2^A} & \text{Ker}(f_3) & \xrightarrow{d} & \text{Cok}(f_1) & \xrightarrow{\bar{d}_1^B} & \text{Cok}(f_2) & \xrightarrow{\bar{d}_2^B} & \text{Cok}(f_3) \\ & & & & \downarrow \alpha & & & & \downarrow \bar{\delta}_2^B & & \\ & & & & 0 & & & & 0 & & \end{array}$$

In the rest, we show that this is 2-exact in $\text{Ker}(f_2), \text{Ker}(f_3), \text{Cok}(f_1), \text{Cok}(f_2)$. We show only the 2-exactness in $\text{Ker}(f_2)$ and $\text{Ker}(f_3)$, since the rest can be shown dually. The 2-exactness in $\text{Ker}(f_2)$ follows immediately from Lemma 24.6. So, we show the 2-exactness in $\text{Ker}(f_3)$. Since we have $\text{Cok}(d_1^A) = [A_3, d_2^A, \delta_2^A]$ and

$\text{Cok}(f_1 \circ d_2^B) = [Q, q, \rho]$, there exists a factorization (ℓ, ϖ_1)

$$(24.17) \quad \begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow \delta_2^A & & \searrow & & \\ A_1 & \xrightarrow{d_1^A} & A_2 & \xrightarrow{d_2^A} & A_3 & \xrightarrow{0} & 0 \\ \text{id} \parallel & \lambda_1 \Downarrow & f_2 \downarrow & \varpi_1 \Downarrow & \downarrow \ell & & \\ A_1 & \xrightarrow{f_1 \circ d_1^B} & B_2 & \xrightarrow{q} & Q & \xrightarrow{0} & 0 \\ & & \rho \Downarrow & & \nearrow & & \\ & & 0 & & & & \end{array}$$

such that

$$(d_1^A \circ \varpi_1) \cdot (\lambda_1 \circ q) \cdot \rho = (\delta_2^A \circ \ell) \cdot \ell_I^\flat.$$

Applying Lemma 24.5 to diagram (24.17), we see that the following diagram becomes 2-exact in A_3 :

$$(24.18) \quad \begin{array}{ccc} & 0 & \\ & \uparrow & \xrightarrow{(k(f_2) \circ \varpi_1) \cdot (\varepsilon_{f_2 \circ q}) \cdot q_I^\flat} \\ \text{Ker}(f_2) & \xrightarrow{k(f_2) \circ d_2^A} & A_3 \xrightarrow{\ell} Q \end{array}$$

Then it can be shown that $(\varpi_1 \circ q_2) \cdot (f_2 \circ \eta_2^{-1}) : d_2^A \circ \ell \circ q_2 \Rightarrow f_2 \circ d_2^B$ is compatible with δ_2^A and $(\lambda_1 \circ d_2^B) \cdot (f_1 \circ d_2^B) \cdot (f_1)_I^\sharp$. So, comparing the following two factorizations

$$\begin{array}{ccc} & (\varpi_1 \circ q_2) \cdot (f_2 \circ \eta_2^{-1}) & \\ & \nearrow & \\ A_1 & \xrightarrow{d_1^A} & A_2 \xrightarrow{d_2^A} & A_3 \\ & \searrow & \downarrow \ell \circ q_2 & \\ & & f_2 \circ d_2^B & \\ & & \downarrow & \\ & & B_3 & \\ & (\lambda_1 \circ d_2^B) \cdot (f_1 \circ d_2^B) \cdot (f_1)_I^\sharp & & \end{array} \quad \begin{array}{ccc} & & \\ & \nearrow & \\ A_1 & \xrightarrow{d_1^A} & A_2 \xrightarrow{d_2^A} & A_3 \\ & \searrow & \downarrow \lambda_2 \circ f_3 & \\ & & f_2 \circ d_2^B & \\ & & \downarrow & \\ & & B_3 & \\ & (\lambda_1 \circ d_2^B) \cdot (f_1 \circ d_2^B) \cdot (f_1)_I^\sharp & & \end{array}$$

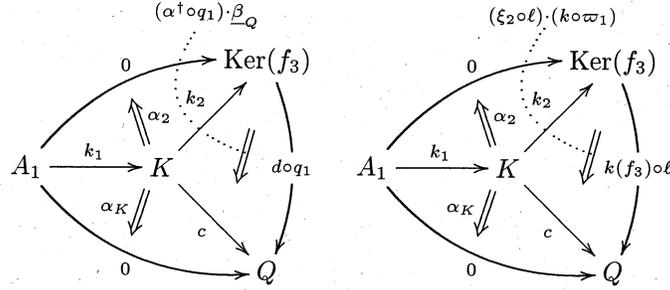
we see there exists a unique 2-cell $\varpi_2 \in \mathbf{S}^2(\ell \circ q_2, f_3)$ such that

$$(d_2^A \circ \varpi_2) \cdot \lambda_2 = (\varpi_1 \circ q_2) \cdot (f_2 \circ \eta_2^{-1}).$$

Then it can be shown that each of the two factorizations

- (1) $(\alpha^\dagger \circ q_1) \cdot \beta_Q : k_2 \circ d \circ q_1 \Rightarrow c$
- (2) $(\xi_2 \circ \ell) \cdot (k \circ \varpi_1) : k_2 \circ k(f_3) \circ \ell \Rightarrow k \circ f_2 \circ q = c$

is compatible with α_2 and α_K .



So there exists a unique 2-cell $\varpi_3 \in \mathbf{S}^2(d \circ q_1, k(f_3) \circ \ell)$ such that

$$(k_2 \circ \varpi_3) \cdot (\xi_2 \circ \ell) \cdot (k \circ \varpi_1) = (\alpha^\dagger \circ q_1) \cdot \underline{\beta}_Q$$

(recall that $\text{Cok}(k_1) = [\text{Ker}(f_3), k_2, \alpha_2]$). Then we have $(\varpi_3 \circ q_2) \cdot (k(f_3) \circ \varpi_2) \cdot \varepsilon_{f_3} = (d \circ \beta_2) \cdot d_1^\sharp$.

$$(24.19) \quad \begin{array}{ccccc} \text{Ker}(f_3) & \xrightarrow{d} & \text{Cok}(f_1) & & \\ \downarrow k(f_3) & \varpi_3 \Downarrow & \downarrow q_1 & & \\ 0 & \xleftarrow{\varepsilon_{f_3}} & A_3 & \xrightarrow{\ell} & Q \xrightarrow{\beta_2} 0 \\ \downarrow f_3 & \varpi_2 \Downarrow & \downarrow q_2 & & \\ B_3 & \xlongequal{\text{id}} & B_3 & & \end{array}$$

By taking kernels of d, ℓ and id_{B_3} in (24.19), we obtain the following diagram.

$$\begin{array}{ccccc} \text{Ker}(d) & \xrightarrow{k(d)} & \text{Ker}(f_3) & \xrightarrow{d} & \text{Cok}(f_1) \\ \downarrow k(f_3) & \varpi_3 \Downarrow & \downarrow k(f_3) & \Downarrow \varpi_3 & \downarrow q_1 \\ \text{Ker}(\ell) & \xrightarrow{k(\ell)} & A_3 & \xrightarrow{\ell} & Q \\ \downarrow 0 & \varpi_2 \Downarrow & \downarrow f_3 & \varpi_2 \Downarrow & \downarrow q_2 \\ 0 & \xrightarrow{0} & B_3 & \xlongequal{\text{id}} & B_3 \end{array}$$

Since $\text{Ker}(0 : \text{Ker}(\ell) \rightarrow 0) = \text{Ker}(d)$ by (the dual of) Proposition 23.7, so $k(f_3)$ becomes an equivalence. On the other hand, the following is a complex morphism, where $s := k(f_2) \circ d_2^A$.

$$(24.20) \quad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow \alpha & & \\ \text{Ker}(f_2) & \xrightarrow{d_2^A} & \text{Ker}(f_3) & \xrightarrow{d} & \text{Cok}(f_1) \\ \text{id} \parallel & \lambda_2 \Downarrow & \downarrow k(f_3) & \Downarrow \varpi_3 & \downarrow q_1 \\ \text{Ker}(f_2) & \xrightarrow{s} & A_3 & \xrightarrow{\ell} & Q \\ & & \downarrow \dots & & \\ & & 0 & & \downarrow (k(f_2) \circ \varpi_1) \cdot (\varepsilon_{f_2} \circ q) \cdot q_1^\sharp \end{array}$$

Thus by taking kernels of d and ℓ in diagram (24.20), we obtain the following factorization by (the dual of) Proposition 23.8.

$$\begin{array}{ccccc}
 & & d_2^A & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \text{Ker}(f_2) & \xrightarrow{\exists(d_2^A)^\dagger} & \text{Ker}(d) & \xrightarrow{k(d)} & \text{Ker}(f_3) \\
 \text{id} \parallel & & \uparrow \exists \alpha & & \\
 & \exists \Downarrow & \downarrow k(f_3) & \Downarrow \varpi_3 & \downarrow k(f_3) \\
 \text{Ker}(f_2) & \xrightarrow{\exists \underline{s}} & \text{Ker}(\ell) & \xrightarrow{k(\ell)} & A_3 \\
 & \exists \Downarrow & \exists \Downarrow & & \\
 & & s & &
 \end{array}$$

Since (24.18) is 2-exact in A_3 , so \underline{s} becomes fully cofaithful. Since $k(f_3)$ is an equivalence, this means $(\underline{d}_2^A)^\dagger$ is fully cofaithful, and

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \text{Ker}(f_2) & \xrightarrow{\underline{d}_2^A} & \text{Ker}(f_3) & \xrightarrow{d} & \text{Cok}(f_1) \\
 & & \uparrow \alpha & &
 \end{array}$$

becomes 2-exact in $\text{Ker}(f_3)$. \square

25. APPENDIX

In this section, concerning the 2-categorical version of condition (AB3) in Abelian category theory, we state how a pseudo-bicolimit is constructed from cokernels.

Throughout this section, \mathbf{S} denotes a 2-category with invertible 2-cells.

Definition 25.1. Let \mathcal{C} be a small category, \mathbf{S} be a 2-category, and $A : \mathcal{C} \rightarrow \mathbf{S}$ be a pseudo-functor. For any $S \in \mathbf{S}^0$, let $c_S : \mathcal{C} \rightarrow \mathbf{S}$ denote the constant 2-functor with value S . Let $\alpha_{f,g} : A(f) \circ A(g) \Rightarrow A(f \circ g)$ denote the composition 2-cell of A , for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} .

A pseudo-cocone on A with vertex S is a pseudo-natural transformation from A to c_S . We denote the category of pseudo-cocones by $\text{PsC}(A, S)$. Namely, $\text{PsC}(A, S) = \text{Ps-Nat}(A, c_S)$, i.e.,

- an object is a pseudo-natural transformation $f : A \rightarrow c_S$,
- morphisms are modifications.

A pseudo-bicolimit of A is a pair (L, ℓ) of

$$L \in \mathbf{S}^0, \ell \in \text{Ob}(\text{PsC}(A, L))$$

such that

$$\ell \circ - : \mathbf{S}(L, S) \rightarrow \text{PsC}(A, S)$$

gives an equivalence of categories.

Remark 25.2. The pseudo-bicolimit of A is determined up to an equivalence. We denote it by (L_A, ℓ_A) , or abbreviately $L = L_A = \varinjlim A$.

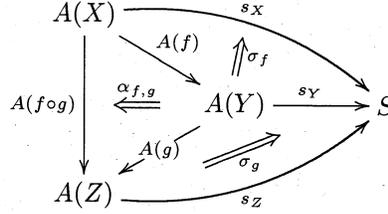
Remark 25.3. An object $s = (\{s_X\}, \{\sigma_f\})$ in $\text{PsC}(A, S)$ consists of the following data:

- $s_X \in \mathbf{S}^1(A(X), S)$ for each $X \in \text{Ob}(\mathcal{C})$,
- $\sigma_f \in \mathbf{S}^2(A(f) \circ s_Y, s_X)$ for each $f \in \mathcal{C}(X, Y)$ ($\forall X, Y \in \text{Ob}(\mathcal{C})$)

satisfying

$$(\alpha_{f,g} \circ s_Z) \cdot \sigma_{f \circ g} = (A(f) \circ \sigma_g) \cdot \sigma_f$$

for each $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} .



In this notation, the pseudo-bicolimit of A is nothing other than the universal one among those $(S, \{s_X\}, \{\sigma_f\})$ of

$$S \in \mathbf{S}^0, (\{s_X\}, \{\sigma_f\}) \in \text{Ob}(\text{PsC}(A, S)).$$

Namely, $(L, \ell) = (L, \{\ell_X\}, \{\gamma_f\})$ is the pseudo-bicolimit of A if it satisfies the following conditions for any such $(S, \{s_X\}, \{\sigma_f\})$:

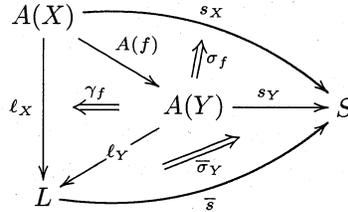
(U1) There exist a pair $(\bar{s}, \{\bar{\sigma}_X\})$ of

$$\begin{aligned} \bar{s} &\in \mathbf{S}^1(L, S) \\ \bar{\sigma}_X &\in \mathbf{S}^2(\ell_X \circ \bar{s}, s_X) \quad (\forall X \in \text{Ob}(\mathcal{C})) \end{aligned}$$

such that

$$(\gamma_f \circ \bar{s}) \cdot \bar{\sigma}_X = (A(f) \circ \bar{\sigma}_Y) \cdot \sigma_f$$

for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} .



(U2) For each two pairs $(\bar{s}, \{\bar{\sigma}_X\})$ and $(\bar{s}', \{\bar{\sigma}'_X\})$ satisfying (U1), there exists a unique 2-cell $v \in \mathbf{S}(\bar{s}, \bar{s}')$ such that

$$(\ell_X \circ v) \cdot \bar{\sigma}'_X = \bar{\sigma}_X$$

holds for each $X \in \text{Ob}(\mathcal{C})$.

Proof. $\ell \circ - : \mathbf{S}(L, S) \rightarrow \text{PsC}(A, S)$ is essentially surjective if and only if (U1) is satisfied, and is fully faithful if and only if (U2) is satisfied. \square

Definition 25.4. Let Δ be the category of simplicial sets. Namely,

- an object is $[n] = \{1, \dots, n\}$, where n is a non-negative integer.
- a morphism $s : [m] \rightarrow [n]$ is a strictly increasing map.

Remark 25.5. Let $X, Y \in \mathbf{S}^0$ and $f, g \in \mathbf{S}^1(X, Y)$. By definition, the difference cokernel of

$$(25.1) \quad X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

is the universal one among those triplets (C, c, π) of $C \in \mathbf{S}^0$, $c \in \mathbf{S}^1(Y, C)$, and $\pi \in \mathbf{S}^2(g \circ c, f \circ c)$.

$$\begin{array}{ccccc}
& & d_{23} \circ d_2 \circ c & \xrightarrow{d_{23} \circ \pi} & d_{23} \circ d_1 \circ c \\
& \swarrow \delta_{2,23} \circ c & & & \searrow \delta_{1,23} \circ c \\
d_3 \circ c & & & & d_2 \circ c \\
\uparrow \delta_{2,13} \circ c & & & & \uparrow \delta_{2,12} \circ c \\
d_{13} \circ d_2 \circ c & & & & d_{12} \circ d_2 \circ c \\
\searrow d_{13} \circ \pi & & & & \swarrow d_{12} \circ \pi \\
d_{13} \circ d_1 \circ c & & & & d_{12} \circ d_1 \circ c \\
\searrow \delta_{1,13} \circ c & & & & \swarrow \delta_{1,12} \circ c \\
& & d_1 \circ c & &
\end{array}$$

Among those triplets (C, c, π) compatible with $\langle X, d, \delta \rangle$, we call the universal one “simplicial cokernel” of $\langle X, d, \delta \rangle$. (Namely, (C, c, π) satisfies (U1) and (U2) in Remark 25.5 for any other triplet (C', c', π') compatible with $\langle X, d, \delta \rangle$.)

Remark 25.8. *As in the case of the difference cokernel, the simplicial cokernel defined above is unique up to an equivalence.*

We denote this equivalence class abbreviately by $\text{Cok}(\langle X, d, \delta \rangle)$.

Remark 25.9. *Let $\langle X, d, \delta \rangle$ be a truncated cosimplicial diagram of type (25.2).*

$$\begin{array}{ccccc}
& & \xrightarrow{\quad} & & \\
X_3 & \xrightarrow{\quad} & X_2 & \xrightarrow{d_1^{(2)}} & X_1 \\
& \xrightarrow{\quad} & & \xrightarrow{d_2^{(2)}} & \\
& & & &
\end{array}$$

Then the simplicial cokernel of $\langle X, d, \delta \rangle$ factors through the difference cokernel of $d_1^{(2)}$ and $d_2^{(2)}$.

Definition 25.10. Let \mathcal{C} be a small category, and let $A : \mathcal{C} \rightarrow \mathbf{S}$ be a pseudo-functor. Assume \mathbf{S} admits any coproduct. For any family of 0-cells $\{A_\lambda\}_{\lambda \in \Lambda}$, we denote their coproduct by $\coprod_{\lambda} A_\lambda$ and the structure morphism by $i_\lambda : A_\lambda \rightarrow \coprod_{\lambda} A_\lambda$.

- (i) For any object $X \in \text{Ob}(\mathcal{C})$, put $A_X := A(X)$.
- (ii) For any morphism $f \in \mathcal{C}(X, Y)$, put $A_f := A(f)$.
- (iii) For any composable pair of morphisms $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, put $A_{f,g} := A(f \circ g)$.

In this notation, we have the following morphisms:

$$\begin{aligned}
(d_1^{(2)})_f &:= \text{id}_{A(X)} : A_f \rightarrow A_X, \\
(d_2^{(2)})_f &:= A(f) : A_f \rightarrow A_Y, \\
(d_{12}^{(3)})_{f,g} &:= \text{id}_{A(X)} : A_{f,g} \rightarrow A_f, \\
(d_{23}^{(3)})_{f,g} &:= A(f) : A_{f,g} \rightarrow A_g, \\
(d_{13}^{(3)})_{f,g} &:= \text{id}_{A(X)} : A_{f,g} \rightarrow A_{f \circ g}, \\
(d_1^{(3)})_{f,g} &:= \text{id}_{A(X)} : A_{f,g} \rightarrow A_X, \\
(d_2^{(3)})_{f,g} &:= A(f) : A_{f,g} \rightarrow A_Y, \\
(d_3^{(3)})_{f,g} &:= A(f \circ g) : A_{f,g} \rightarrow A_Z.
\end{aligned}$$

Remark we have a natural monoidal isomorphism

$$(\delta_{2,23})_{f,g} := \alpha_{f,g} : (d_{23}^{(3)})_{f,g} \circ (d_2^{(2)})_g \Longrightarrow (d_3^{(3)})_{f,g}.$$

For any other $s \in \Delta([1], [2])$ and $t \in \Delta([2], [3])$, put

$$(\delta_{s(1),t(1)t(2)})_{f,g} := \text{id}.$$

If we put

$$\begin{aligned} A_1 &:= \coprod_{X \in \text{Ob}(\mathcal{C})} A_X, \\ A_2 &:= \coprod_{f \in \mathcal{C}(X,Y)} A_f, \\ A_3 &:= \coprod_{\substack{X \rightarrow_f Y \rightarrow_g Z}} A_{f,g}, \end{aligned}$$

then by the universality of coproducts, we obtain induced monoidal functors

$$d_{s(1)\dots s(m)}^{(n)} : A_n \rightarrow A_m$$

correspondingly for any $1 \leq m < n \leq 3$ and any $s \in \Delta([m], [n])$. By construction, there are 2-cells

$$\begin{array}{ccc} A_f & \xrightarrow{\text{id}_{A(X)}} & A_X \\ i_f \downarrow & \xrightarrow{(\rho_1^{(2)})_f} & \downarrow i_X \\ A_2 & \xrightarrow{d_1^{(2)}} & A_1 \end{array} \quad \begin{array}{ccc} A_f & \xrightarrow{A(f)} & A_Y \\ i_f \downarrow & \xrightarrow{(\rho_2^{(2)})_f} & \downarrow i_Y \\ A_2 & \xrightarrow{d_2^{(2)}} & A_1 \end{array}$$

for any morphism $f \in \mathcal{C}(X, Y)$. Similarly, we have 2-cells

$$\begin{aligned} (\rho_{12}^{(3)})_{f,g} &: i_{f,g} \circ d_{12}^{(3)} \Longrightarrow (d_{12}^{(3)})_{f,g} \circ i_f = i_f, \\ (\rho_{23}^{(3)})_{f,g} &: i_{f,g} \circ d_{23}^{(3)} \Longrightarrow (d_{23}^{(3)})_{f,g} \circ i_g = A(f) \circ i_g, \\ (\rho_{13}^{(3)})_{f,g} &: i_{f,g} \circ d_{13}^{(3)} \Longrightarrow (d_{13}^{(3)})_{f,g} \circ i_{f \circ g} = i_{f \circ g}, \\ (\rho_1^{(3)})_{f,g} &: i_{f,g} \circ d_1^{(3)} \Longrightarrow (d_1^{(3)})_{f,g} \circ i_X = i_X, \\ (\rho_2^{(3)})_{f,g} &: i_{f,g} \circ d_2^{(3)} \Longrightarrow (d_2^{(3)})_{f,g} \circ i_Y = A(f) \circ i_Y, \\ (\rho_3^{(3)})_{f,g} &: i_{f,g} \circ d_3^{(3)} \Longrightarrow (d_3^{(3)})_{f,g} \circ i_Z = A(f \circ g) \circ i_Z. \end{aligned}$$

We often omit the subscripts f, g of ρ in the following.

Moreover, for each $s \in \Delta([1], [2])$ and $t \in \Delta([2], [3])$, there is a 2-cell

$$\delta_{s(1),t(1)t(2)} : d_{t(1)t(2)}^{(3)} \circ d_{s(1)}^{(2)} \Longrightarrow d_{t(s(1))}^{(3)}$$

induced from $(\delta_{s(1),t(1)t(2)})_{f,g}$.

By construction, $\delta_{s(1),t(1)t(2)}$ is compatible with ρ 's and $(\delta_{s(1),t(1)t(2)})_{f,g}$ for each $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} , i.e., we have

$$\begin{aligned} (i_{f,g} \circ \delta_{1,12}) \cdot \rho_1^{(3)} &= (\rho_{12}^{(3)} \circ d_1) \cdot (\rho_1^{(2)})_f, \\ (i_{f,g} \circ \delta_{1,23}) \cdot \rho_2^{(3)} &= (\rho_{23}^{(3)} \circ d_1) \cdot (A(f) \circ (\rho_1^{(2)})_g), \\ (i_{f,g} \circ \delta_{1,13}) \cdot \rho_1^{(3)} &= (\rho_{13}^{(3)} \circ d_1) \cdot (\rho_1^{(2)})_{f \circ g}, \\ (i_{f,g} \circ \delta_{2,12}) \cdot \rho_2^{(3)} &= (\rho_{12}^{(3)} \circ d_2) \cdot (\rho_2^{(2)})_f, \\ (i_{f,g} \circ \delta_{2,23}) \cdot \rho_3^{(3)} &= (\rho_{23}^{(3)} \circ d_2) \cdot (A(f) \circ (\rho_2^{(2)})_g) \cdot (\alpha_{f,g} \circ i_Z), \\ (i_{f,g} \circ \delta_{2,13}) \cdot \rho_3^{(3)} &= (\rho_{13}^{(3)} \circ d_2) \cdot (\rho_2^{(2)})_{f \circ g}. \end{aligned}$$

$$\begin{array}{ccc} \begin{array}{ccccc} & & d_1 & & \\ & \nearrow^{d_{12}} & \uparrow \delta_{1,12} & \searrow^{d_1} & \\ A_3 & \xrightarrow{d_{12}} & A_2 & \xrightarrow{d_1} & A_1 \\ \uparrow i_{f,g} & \Downarrow \rho_{12}^{(3)} & \uparrow i_f & \Downarrow (\rho_1^{(2)})_f & \uparrow i_X \\ A_{f,g} & \xrightarrow{\text{id}} & A_f & \xrightarrow{\text{id}} & A_X \\ & \searrow^{\text{id}} & \circ & \nearrow^{\text{id}} & \\ & & \text{id} & & \end{array} & \begin{array}{ccccc} & & d_3 & & \\ & \nearrow^{d_{23}} & \uparrow \delta_{2,23} & \searrow^{d_2} & \\ A_3 & \xrightarrow{d_{23}} & A_2 & \xrightarrow{d_2} & A_1 \\ \uparrow i_{f,g} & \Downarrow \rho_{23}^{(3)} & \uparrow i_g & \Downarrow (\rho_2^{(2)})_g & \uparrow i_Z \\ A_{f,g} & \xrightarrow{A(f)} & A_g & \xrightarrow{A(g)} & A_Z \\ & \searrow^{A(f \circ g)} & \Downarrow \alpha_{f,g} & \nearrow^{A(f \circ g)} & \\ & & A(f \circ g) & & \end{array} \end{array}$$

Thus, for each pseudo-functor A , we obtain an associated truncated cosimplicial diagram $\langle A, d, \delta \rangle$.

Proposition 25.11. *Let $\langle A, d, \delta \rangle$ be the truncated cosimplicial diagram constructed above. If we denote the simplicial cokernel of $\langle A, d, \delta \rangle$ by $\langle C, c, \pi \rangle$, then we have $C \simeq \lim_{\rightarrow} A$.*

Proof. This can be easily reduced to the following:

Lemma 25.12. *Let $\langle A, d, \delta \rangle$ be the truncated cosimplicial diagram associated to pseudo-functor A as in Definition 25.10. For any $C \in \mathbf{S}^0$, we have the following:*
(1) *By the universality of the coproducts, a pair (c, π) of*

$$\begin{aligned} c &\in \mathbf{S}^1(A, C) \\ \pi &\in \mathbf{S}^2(d_2^{(2)} \circ c, d_1^{(2)} \circ c) \end{aligned}$$

and a pair $(\{c_X\}, \{\pi_f\})$ of families

$$\begin{aligned} c_X &\in \mathbf{S}^1(A_X, C) \quad (\forall X \in \text{Ob}(\mathcal{C})) \\ \pi_f &\in \mathbf{S}^2(A(f) \circ c_Y, c_X) \quad (\forall f \in \mathcal{C}(X, Y)). \end{aligned}$$

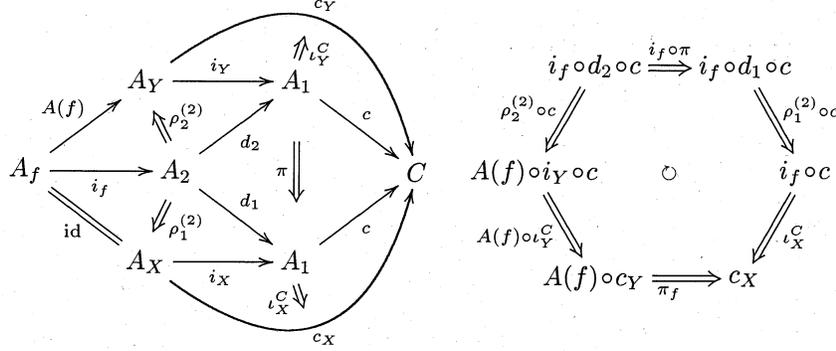
corresponds to each other. These are related by

$$\iota_X^C \in \mathbf{S}^2(i_X \circ c, c_X) \quad (X \in \text{Ob}(\mathcal{C})),$$

satisfying

$$(25.4) \quad ((\rho_2^{(2)})_f \circ c) \cdot (A(f) \circ \iota_Y^C) \cdot \pi_f = (i_f \circ \pi) \cdot ((\rho_1^{(2)})_f \circ c) \cdot \iota_X^C$$

for any $f \in \mathcal{C}(X, Y)$.



Under this correspondence, (C, c, π) is compatible with $\langle A, d, \delta \rangle$ if and only if $(\{c_X\}, \{\pi_f\})$ belongs to $\text{Ob}(\text{PsC}(A, C))$.

(2) Assume (C, c, π) corresponds to $(C, \{c_X\}, \{\pi_f\})$, and (S, s, σ) corresponds to $(S, \{s_X\}, \{\sigma_f\})$ as in (1). For any $\bar{s} \in \mathbf{S}^1(C, S)$, a 2-cell $\bar{\sigma} \in \mathbf{S}^2(c \circ \bar{s}, s)$ and a family of 2-cells $\{\bar{\sigma}_X\}$

$$\sigma_X \in \mathbf{S}^2(c_X \circ \bar{s}, s_X)$$

corresponds to each other, satisfying

$$(25.5) \quad (i_X \circ \bar{\sigma}) \cdot \iota_X^S = (\iota_X^C \circ \bar{s}) \cdot \bar{\sigma}_X \quad (\forall X \in \text{Ob}(C)).$$

Under this correspondence, $\bar{\sigma}$ satisfies

$$(25.6) \quad (\pi \circ \bar{s}) \cdot (d_1 \circ \bar{\sigma}) = (d_2 \circ \bar{\sigma}) \cdot \sigma$$

if and only if $\{\bar{\sigma}_X\}$ satisfies

$$(25.7) \quad (\pi_f \circ \bar{s}) \cdot \bar{\sigma}_X = (A(f) \circ \bar{\sigma}_Y) \cdot \sigma_f \quad (\forall f \in \mathcal{C}(X, Y)).$$

(3) Take the correspondings

$$\begin{aligned} (C, c, \pi) &\leftrightarrow (C, \{c_X\}, \{\pi_f\}) \\ (S, s, \sigma) &\leftrightarrow (S, \{s_X\}, \{\sigma_f\}) \end{aligned}$$

and $\bar{s}, \bar{s}' \in \mathbf{S}^1(C, S)$ as above. Assume $\bar{\sigma}$ corresponds to $\{\bar{\sigma}_X\}$, and $\bar{\sigma}'$ corresponds to $\{\bar{\sigma}'_X\}$ as in (2). Then for any $\theta \in \mathbf{S}^2(\bar{s}, \bar{s}')$, we have

$$(c \circ \theta) \cdot \bar{\sigma} = \bar{\sigma}'_X \iff (c_X \circ \theta) \cdot \bar{\sigma}_X = \bar{\sigma}'_X \quad (\forall X \in \text{Ob}(C)).$$

Proof. (1) By the universality of the coproduct, (25.3) is satisfied if and only if

$$\begin{aligned} (25.8) \quad & (i_{f,g} \circ d_{23} \circ \pi) \cdot (i_{f,g} \circ \delta_{1,23} \circ c) \cdot (i_{f,g} \circ \delta_{2,12}^{-1} \circ c) \\ & \cdot (i_{f,g} \circ d_{12} \circ \pi) \cdot (i_{f,g} \circ \delta_{1,12} \circ c) \\ & = (i_{f,g} \circ \delta_{2,23} \circ c) \cdot (i_{f,g} \circ \delta_{2,13}^{-1} \circ c) \cdot (i_{f,g} \circ d_{13} \circ \pi) \cdot (i_{f,g} \circ \delta_{1,13} \circ c) \end{aligned}$$

is satisfied for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} .

By the construction in Definition 25.10 and (25.4), we have

(LHS of (25.8))

$$\begin{aligned}
&= (i_{f,g} \circ d_{23} \circ \pi) \cdot (\rho_{23}^{(3)} \circ d_1 \circ c) \cdot (A(f) \circ (\rho_1^{(2)})_g \circ c) \cdot ((\rho_2^{(2)})_f^{-1} \circ c) \\
&\quad \cdot ((\rho_{12}^{(3)})^{-1} \circ d_2 \circ c) \cdot (i_{f,g} \circ d_{12} \circ \pi) \cdot (\rho_{12}^{(3)} \circ d_1 \circ c) \\
&\quad \cdot ((\rho_1^{(2)})_f \circ c) \cdot ((\rho_1^{(3)})^{-1} \circ c) \\
&= (\rho_{23}^{(3)} \circ d_2 \circ c) \cdot (A(f) \circ i_g \circ \pi) \cdot (A(f) \circ (\rho_1^{(2)})_g \circ c) \cdot ((\rho_1^{(2)})_f^{-1} \circ c) \\
&\quad \cdot (i_f \circ \pi) \cdot ((\rho_1^{(2)})_f \circ c) \cdot ((\rho_1^{(3)})^{-1} \circ c) \\
&\stackrel{(25.4)}{=} (\rho_{23}^{(3)} \circ d_2 \circ c) \cdot (A(f) \circ (\rho_2^{(2)})_g \circ c) \cdot (A(f) \circ A(g) \circ \iota_Z^C) \cdot (A(f) \circ \pi_g) \\
&\quad \cdot \pi_f \cdot (\iota_X^C)^{-1} \cdot ((\rho_1^{(3)})^{-1} \circ c),
\end{aligned}$$

and

(RHS of (25.8))

$$\begin{aligned}
&= (\rho_{23}^{(3)} \circ d_2 \circ c) \cdot (A(f) \circ (\rho_2^{(2)})_g \circ c) \cdot (\alpha_{f,g} \circ i_Z \circ c) \cdot ((\rho_2^{(2)})_{f \circ g}^{-1} \circ c) \\
&\quad \cdot ((\rho_{13}^{(3)})^{-1} \circ d_2 \circ c) \cdot (i_{f,g} \circ d_{13} \circ \pi) \cdot (\rho_{13}^{(3)} \circ d_1 \circ c) \\
&\quad \cdot ((\rho_1^{(2)})_{f \circ g} \circ c) \cdot ((\rho_1^{(3)})^{-1} \circ c) \\
&= (\rho_{23}^{(3)} \circ d_2 \circ c) \cdot (A(f) \circ (\rho_2^{(2)})_g \circ c) \cdot (\alpha_{f,g} \circ i_Z \circ c) \cdot ((\rho_2^{(2)})_{f \circ g}^{-1} \circ c) \\
&\quad \cdot (i_{f \circ g} \circ \pi) \cdot ((\rho_1^{(2)})_{f \circ g} \circ c) \cdot ((\rho_1^{(3)})^{-1} \circ c) \\
&= (\rho_{23}^{(3)} \circ d_2 \circ c) \cdot (A(f) \circ (\rho_2^{(2)})_g \circ c) \cdot (\alpha_{f,g} \circ i_Z \circ c) \\
&\quad \cdot \pi_{f \circ g} \cdot (\iota_X^C)^{-1} \cdot ((\rho_1^{(3)})^{-1} \circ c).
\end{aligned}$$

Thus we obtain

(25.8)

$$\begin{aligned}
&\iff (A(f) \circ A(g) \circ \iota_Z^C) \cdot (A(f) \circ \pi_g) \cdot \pi_f = (\alpha_{f,g} \circ i_Z \circ c) \cdot (A(f \circ g) \circ \iota_Z^C) \cdot \pi_{f \circ g} \\
&\iff (A(f) \circ \pi_g) \cdot \pi_f = (\alpha_{f,g} \circ c_Z) \cdot \pi_{f \circ g}.
\end{aligned}$$

(2) By the universality of the coproduct, (25.6) is satisfied if and only if

$$(25.9) \quad (i_f \circ \pi \circ \bar{s}) \cdot (i_f \circ d_1 \circ \bar{\sigma}) = (i_f \circ d_2 \circ \bar{\sigma}) \cdot (i_f \circ \sigma)$$

is satisfied for any morphism $f \in \mathcal{C}(X, Y)$. Thus (2) follows from

(25.7)

$$\begin{aligned}
&\stackrel{25.5}{\iff} (\pi_f \circ \bar{s}) \cdot ((\iota_X^C)^{-1} \circ \bar{s}) \cdot (i_X \circ \bar{\sigma}) \cdot \iota_X^S \\
&= (A(f) \circ (\iota_Y^C)^{-1} \circ \bar{s}) \cdot (A(f) \circ \bar{\sigma}) \cdot (A(f) \circ \iota_Y^S) \cdot \sigma_f \\
&\stackrel{25.4}{\iff} ((\rho_2^{(2)})_f^{-1} \circ c \circ \bar{s}) \cdot (i_f \circ \pi \circ \bar{s}) \cdot ((\rho_1^{(2)})_f \circ c \circ \bar{s}) \cdot (i_X \circ \bar{\sigma}) \\
&= (A(f) \circ \bar{\sigma}) \cdot ((\rho_2^{(2)})_f^{-1} \circ s) \cdot (i_f \circ \sigma) \cdot ((\rho_1^{(2)})_f \circ s) \\
&\iff (25.9).
\end{aligned}$$

(3) This immediately follows from (25.5). □

□

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO 3-8-1 KOMABA,
MEGURO, TOKYO, 153-8914 JAPAN

E-mail address: deutsche@ms.u-tokyo.ac.jp