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Laplace Approximations for Diffusion Processes on Torus: Nondegenerate Case

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Abstract. Let $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, and consider the family of probability measures $\{P_x\}_{x\in\mathbf{T}^d}$ on $C([0,\infty);\mathbf{T}^d)$ given by the infinitesimal generator $L_0 \equiv \frac{1}{2}\Delta + b\cdot\nabla$, where $b : \mathbf{T}^d \to \mathbf{R}^d$ is a continuous function. Let Φ be a mapping $\mathcal{M}(\mathbf{T}^d) \to \mathbf{R}$. Under a nuclearity assumption on the second Fréchet differential of Φ, an asymptotic evaluation of $Z_T^{x,y} \equiv E^{P_x}\left[\exp\left(T\Phi(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right)\Big|X_T = y\right]$, up to a factor $(1+o(1))$, has been gotten in Bolthausen-Deuschel-Tamura [2]. In this paper, we show that the same asymptotic evaluation holds without the nuclearity assumption.

1. Introduction

We consider the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$, which is a compact manifold. The tangent space $T(\mathbf{T}^d)$ can be identified with $\mathbf{R}^d$. Let $\mathcal{B}(\mathbf{T}^d)$ be the set of all Borel sets in $\mathbf{T}^d$.

Let $\mathcal{M}(\mathbf{T}^d)$ be the dual space of $C(\mathbf{T}^d)$. $\mathcal{M}(\mathbf{T}^d)$ is the set of all signed measures on $\mathbf{T}^d$ with finite total variation, and denote the norm derived by it, the total variation, by $||\cdot||$. We also think of the weak*-topology in $\mathcal{M}(\mathbf{T}^d)$. Let $\wp(\mathbf{T}^d)$ and $\mathcal{M}_0(\mathbf{T}^d)$ be the set of all probability measures on $\mathbf{T}^d$ and the set of all signed measures on $\mathbf{T}^d$ with total measure 0, respectively. Let $\mathrm{dist}(\cdot,\cdot)$ denote the Prohorov metric on $\wp(\mathbf{T})$. Note that the topology induced by the Prohorov metric and the weak*-topology coincide.

The path space $\Omega = C([0,\infty),\mathbf{T}^d)$ is the set of continuous functions $\omega : [0,\infty) \to \mathbf{T}^d$. Let $X_t(\omega) = \omega(t), t \geq 0$, let $\mathcal{F}_t = \sigma\{\omega(s); s \leq t\}$, and let $\mathcal{F} = \vee_t\mathcal{F}_t$.

Let $L_0 = \frac{1}{2}\Delta + b_0\cdot\nabla$, where $b_0 : \mathbf{T}^d \to \mathbf{R}^d$ is a C^∞ function. Let $\{P_x\}_{x\in\mathbf{T}^d}$ be the family of probability measures on Ω of the martingale

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problem L_0, *i.e.*, for any $f \in C^\infty(\mathbf{T}^d;\mathbf{R})$,

$$(1) \quad f(\omega_t) - f(\omega_0) - \int_0^t L_0 f(\omega_s)ds \text{ is a } (\Omega, \{\mathcal{F}_t\}, P_x) \text{ martingale,}$$

$$(2) \quad P_x(\omega_0 = x) = 1.$$

Denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$ by $\{P_t\}_{t\geq 0}$. $\{P_x\}$ has a unique invariant probability measure μ, which is absolutely continuous with respect to the Riemann volume on $\mathbf{T}^d$, and $\frac{d\mu}{dx}$ is a strictly positive smooth function. For any $T > 0$, the distribution law of $\{X_{T-t}(\omega)\}_{0\leq t\leq T}$ under $P_\mu(d\omega)$ is also a diffusion process. The infinitesimal generator of it is the adjoint operator of L_0 in $L^2(d\mu)$, and can be written as $L_0^{*\mu} = \frac{1}{2}\Delta + b_0^* \cdot \nabla$ for some $b_0^* \in C^\infty(\mathbf{T}^d;\mathbf{R}^d)$. Actually, $b_0^* = \nabla(\log\frac{d\mu}{dx}) - b_0$.

Also, for each $t > 0$, there exist transition probability densities $(p_t(x,y))_{x,y\in\mathbf{T}^d}$ of P_t with respect to μ, which satisfy $p_t \in C^\infty(\mathbf{T}^d \times \mathbf{T}^d)$ and p_t is strictly positive.

Let $\Phi : \mathcal{M}(\mathbf{T}^d) \to \mathbf{R}$ be a bounded and three times continuously Fréchet differentiable function satisfying the following:

A 1. *There exist functions* $\Phi^{(1)} \in C(\wp(\mathbf{T}^d)\times\mathbf{T}^d, \mathbf{R})$, $\Phi^{(2)} \in C(\wp(\mathbf{T}^d)\times \mathbf{T}^d \times \mathbf{T}^d, \mathbf{R})$, *and* $\Phi^{(3)} \in C(\wp(\mathbf{T}^d) \times (\mathbf{T}^d)^3, \mathbf{R})$, *such that for any* $\nu \in \wp(\mathbf{T}^d)$ *and any* $R_1, R_2, R_3 \in \mathcal{M}(\mathbf{T}^d)$,

$$D\Phi(\nu)(R_1) = \int_{\mathbf{T}^d} \Phi^{(1)}(\nu,x)R_1(dx),$$
$$D^2\Phi(\nu)(R_1,R_2) = \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} \Phi^{(2)}(\nu,x,y)R_1(dx)R_2(dy),$$
$$D^3\Phi(\nu)(R_1,R_2,R_3) = \int_{\mathbf{T}^d}\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} \Phi^{(3)}(\nu,x,y,z)R_1(dx)R_2(dy)R_3(dz).$$

Then by Donsker-Varadhan [4], we have (*c.f.* Lemma 4.4)

$$\frac{1}{T}\log E^{P_x}\left[\exp\left(T\Phi(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right)\Big|X_T = y\right] \to \lambda, \qquad T\to\infty$$

for every $x, y \in \mathbf{T}^d$, where $\lambda = \sup\{\Phi(\nu) - I(\nu); \nu \in \wp(\mathbf{T}^d)\}$ and I is given by

$$I(\nu) = \sup\left\{-\int_{\mathbf{T}^d}\frac{L_0 u}{u}d\nu; u \in C^\infty, u \geq 1\right\}, \qquad \nu \in \wp(\mathbf{T}^d).$$

The aim of this paper is to give a more precise evaluation of

$$Z_T^{x,y} \equiv E^{P_x}\left[\exp\left(T\Phi(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right)\Big|X_T = y\right]$$

up to order $1 + o(1)$ under some assumptions given below.

Define

$$K = \{\nu \in \wp(\mathbf{T}^d) : \Phi(\nu) - I(\nu) = \lambda\}.$$

We can easily see that K is not empty and is compact in $\wp(\mathbf{T}^d)$. In this paper, we assume that

A 2. *There exists only one element in K, say ν_0, that is, $K = \{\nu_0\}$.*

Now, let us construct a diffusion which has ν_0 as its invariant measure following Bolthausen-Deuschel-Tamura [2] and Bolthausen-Deuschel-Schmock [1]. For any $\varphi \in C(\mathbf{T}^d)$, let

$$P_t^\varphi(x, A) = E^{P_x}[\exp(\int_0^t \varphi(X_s)ds), X_t \in A], \qquad A \in \mathcal{B}(\mathbf{T}^d),$$

and

$$\Lambda(\varphi) = \sup\{\int_{\mathbf{T}^d} \varphi d\nu - I(\nu), \nu \in \wp(\mathbf{T}^d)\}.$$

Then P_t^φ has strictly positive right- and left-hand principal eigenfunctions h^φ and $l^\varphi \in C(\mathbf{T}^d)$, *i.e.*,

$$P_t^\varphi h^\varphi = \exp(\Lambda(\varphi)t)h^\varphi, \qquad t \geq 0,$$
$$\textstyle\int_{\mathbf{T}^d} \mu(dy)l^\varphi(y)P_t^\varphi(y, dz) = \exp(\Lambda(\varphi)t)l^\varphi(z)\mu(dz).$$

They are unique if they are appropriately normalized by

$$\int_{\mathbf{T}^d}(h^\varphi)^2 d\mu = 1, \qquad d\pi^\varphi \equiv l^\varphi h^\varphi d\mu \in \wp(\mathbf{T}^d).$$

Proposition 1.1. *π^φ is the stationary measure of the diffusion process whose transition probability $Q_t^\varphi(x, dy)$ is given by*

$$Q_t^\varphi(x, dy) \equiv e^{-\Lambda(\varphi)t}\frac{1}{h^\varphi(x)}P_t^\varphi(x, dy)h^\varphi(y).$$

Let

$$\begin{aligned}\phi^{\nu_0}(x) &= D\Phi(\nu_0)(\delta_x-\nu_0)+\Phi(\nu_0)\\ &= \Phi^{(1)}(\nu_0,x)-D\Phi(\nu_0)(\nu_0)+\Phi(\nu_0), \qquad x\in\mathbf{T}^d.\end{aligned}$$

Then we have $\lambda=\Lambda(\phi^{\nu_0})$. Denote $h^{\phi^{\nu_0}}$ by h, and $l^{\phi^{\nu_0}}$ by l.

Let $\{Q_x\}_{x\in\mathbf{T}^d}$ be the probability measures given by

$$\frac{dQ_x}{dP_x}(\omega)\Big|_{\mathcal{F}_t}=e^{-\lambda t}\frac{h(X_t(\omega))}{h(x)}\exp\Big(\int_0^t\phi^{\nu_0}(X_s(\omega))ds\Big).$$

$\{Q_x\}$ is a diffusion process. Denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$ by $\{Q_t\}$, and the infinitesimal generator of $\{Q_t\}$ by L. Actually, $h\in C^1(\mathbf{T}^d)$, and $L=L_0+\frac{\nabla h}{h}\cdot\nabla$. (*c.f.* Proposition 2.3). As has been shown in Bolthausen-Deushel-Tamura [2], $\pi^{\phi^{\nu_0}}=\nu_0$. So by proposition 1.1, we have

LEMMA 1.2. *$\{Q_x\}_{x\in\mathbf{T}^d}$ has ν_0 as its invariant measure.*

As a result, ν_0 is absoluately continuous with respect to μ, and $\frac{d\nu_0}{d\mu}>0$ is continuous, also, $\mathrm{supp}\nu_0=\mathbf{T}^d$.

Now, for any $t>0$ and any $x\in\mathbf{T}^d$, let $q_t(x,\cdot)$ be the density function of $Q_t(x,\cdot)$ with respect to ν_0 with $q_t\in C^+(\mathbf{T}^d\times\mathbf{T}^d)$. We will write it as $q(t,x,y)$ sometimes, too. By Boltuausen-Deuschel-Tamura [2] and Bolthausen-Deuschel-Schmock [1], $\sup_{x,y\in\mathbf{T}^d}|q_t(x,y)-1|\to 0$ exponentially fast as $t\to\infty$. So we can define

$$g(x,y)=\int_0^\infty(q_t(x,y)-1)dt. \tag{1.1}$$

Define $G:L^2(d\nu_0)\to L^2(d\nu_0)$ by

$$Gf(x)=\int_{\mathbf{T}^d}g(x,y)f(y)\nu_0(dy)=\int_0^\infty\Big(Q_tf(x)-\int_{\mathbf{T}^d}fd\nu_0\Big)dt.$$

Let G^* be the adjoint operator of it in $L^2(d\nu_0)$, *i.e.*, $G^*f(x)=\int_{\mathbf{T}^d}g(y,x)f(y)\nu_0(dy)$, and let $\overline{G}=G+G^*$.

In this paper, we will need the following operators: For $f_1,f_2\in L^2(d\nu_0)$, let $(\overline{G}\otimes\overline{G})(f_1\otimes f_2)(x,y)=(\overline{G}f_1)(x)(\overline{G}f_2)(y)$, and denote the continuous linear expansion of it on $L^2(d\nu_0)\otimes L^2(d\nu_0)$ as $\overline{G}\otimes\overline{G}$, too. Define $\overline{G}_x \equiv \overline{G}\otimes I$ and $\overline{G}_y \equiv I\otimes\overline{G}$ in the same way, where I means the identify operator on $L^2(d\nu_0)$. (So $\overline{G}_x\overline{G}_y = \overline{G}\otimes\overline{G}$.) G_x, G_x^*, G_y, G_y^* are defined similarly.

Let $\Gamma(f_1,f_2) \equiv \int_{\mathbf{T}^d} f_1\overline{G}f_2 d\nu_0$, $f_1, f_2 \in C(\mathbf{T}^d)$. Then it is easy to see (*c.f.* Proposition 2.5 below) that $\Gamma(f,f) = \int_{\mathbf{T}^d} ||\nabla(Gf)(x)||^2\nu_0(dx) \geq 0$, so $\Gamma(f,f)=0$ if and only if $f \equiv$ *constant*. Let us define a equivalent relation $\sim$ by $f\sim g \Leftrightarrow f-g \equiv$ *constant*, and let $\widetilde{C}(\mathbf{T}^d) \equiv C(\mathbf{T}^d)/\sim$. Then Γ is a inner product on $\widetilde{C}(\mathbf{T}^d)$. Let $H \equiv \left(\overline{\widetilde{C}(\mathbf{T}^d)}^{\Gamma}\right)^*$, where $\overline{\widetilde{C}(\mathbf{T}^d)}^{\Gamma}$ means the completion of $\widetilde{C}(\mathbf{T}^d)$ with respect to Γ. Since $\widetilde{C}(\mathbf{T}^d)^*$ is identified with $\mathcal{M}_0(\mathbf{T}^d)$, H can be regarded as a dense subset of $\mathcal{M}_0(\mathbf{T}^d)$, (see Proposition 2.6). H is a Hilbert space with norm $||\overline{G}fd\nu_0||_H^2 \equiv \int_{\mathbf{T}^d} f\overline{G}fd\nu_0$.

Also, as has been shown in Bolthausen-Deuschel-Tamura [2], for any $f\in C(\mathbf{T}^d)$,

$$(f,\overline{G}f)_{L^2(d\nu_0)} \geq D^2\Phi(\nu_0)(\overline{G}fd\nu_0, \overline{G}fd\nu_0),$$

which means that all of the eigenvalues of $D^2\Phi(\nu_0)\Big|_{H\times H}$ are less than or equal to 1. In addition, we assume the following

A 3. *All of the eigenvalues of* $D^2\Phi(\nu_0)\Big|_{H\times H}$ *are smaller than* 1.

A 4. *For any* $\delta>0$, *there exist a constant* $\varepsilon>0$ *and a symmetric continuous function* $K_\delta : \mathbf{T}^d\times\mathbf{T}^d \to \mathbf{R}$, *such that the function* $\widetilde{K_\delta}$ *given by* $\widetilde{K_\delta}(R_1,R_2) \equiv \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} K_\delta(x,y)R_1(dx)R_2(dy)$, $R_1,R_2\in\mathcal{M}_0(\mathbf{T}^d)$, *satisfies*

$$||\widetilde{K_\delta}\Big|_{H\times H}||_{H.S.} \leq \delta,$$

and

$$D^3\Phi(R)(\nu-\nu_0,\nu-\nu_0,\nu-\nu_0) \leq \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} K_\delta(x,y)(\nu-\nu_0)(dx)(\nu-\nu_0)(dy)$$

for any $R\in\wp(\mathbf{T}^d)$ *with* $dist(R,\nu_0)<\varepsilon$ *and any* $\nu\in\wp(\mathbf{T}^d)$ *with* $dist(\nu,\nu_0)<\varepsilon$.

Our main result is the following

THEOREM 1.3. *Under the assumptions above, for any* $x,y\in\mathbf{T}^d$,

$$\lim_{T\to\infty} e^{-T\lambda}Z_T^{x,y} = \frac{h(x)}{h(y)}\cdot\exp\left\{\frac{1}{2}\int_{\mathbf{T}^d}\overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)\Big|_{(u,u)}\nu_0(du)\right\} \times det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.$$

REMARK 1. The fact that $D^2\Phi(\nu_0)\Big|_{H\times H}$ is a Hilbert-Schmidt type function, which enssures that the factor $det_2(I_H - D^2\Phi(\nu_0))$ above is well-defined, can be seen from the Proposition 2.8.

2. Preparations

In this section, we will show in the first half an extended Ito's formula for Gf, where f is a continuous function. Also, we will give the proofs of the several facts claimed in section 1.

In general, consider a operator L given by $L \equiv \frac{1}{2}\Delta + b \cdot \nabla$, where $b \in C(\mathbf{T}^d; \mathbf{R}^d)$. For each $x \in \mathbf{T}^d$, let P_x^L denote the probability law of the diffusion process generated by L starting at x. Write the invariant measure of $\{P_x^L\}$ as μ_L. Let $\{P_t^L\}_{t\geq 0}$ denote the corresponding semigroup of linear operators in $C(\mathbf{T}^d)$. Also, let G_L be the corresponding Green operator, *i.e.*, $G_L f \equiv \int_0^\infty (P_t^L f - \int_{\mathbf{T}^d} f d\mu_L) dt$, $f \in C(\mathbf{T}^d)$. Let $||\cdot||_{op}$ denote the operator norm in $C(\mathbf{T}^d) \to C(\mathbf{T}^d)$. Then we have the following

PROPOSITION 2.1. *P_t^L is a compact operator on $C(\mathbf{T}^d)$ for any $t > 0$.*

PROOF. Let $L_B \equiv \frac{1}{2}\Delta$ and let P_t^0 be the semigroup of linear operators on $C(\mathbf{T}^d)$ corresponding to it. Then P_t^0 maps $C(\mathbf{T}^d)$ to $C^2(\mathbf{T}^d)$, and $||\nabla P_t^0||_{op} \leq \frac{2\sqrt{d}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{t}}$ for any $t > 0$. So P_t^0 is a compact operator for any $t > 0$. Also,

$$P_t^L = P_t^0 + \int_0^t P_s^L b \cdot \nabla P_{t-s}^0 ds,$$

where $b \cdot \nabla P_s^0$ is compact for any $s > 0$, Thus, P_t^L is compact for any $t > 0$. □

By Proposition 2.1, every number in the spectrum of P_t^L except 0 is a eigenvalue of it. Let $W_p^2(\mathbf{T}^d)$ denote the Sobolev space, *i.e.*, $W_p^2(\mathbf{T}^d) = \{(1-\Delta)^{-1} f; f \in L^p\}$. Then we have the following

LEMMA 2.2. *G_L maps $C(\mathbf{T}^d)$ into $W_p^2(\mathbf{T}^d)$ for any $p \in [1, \infty)$, and it is a bounded linear map. Also, for any $f \in C(\mathbf{T}^d)$ with $\int f d\mu_L = 0$, $u \equiv -G_L f$ is a solution of the equation $Lu = f$ in the sense of generalized functions. Also, if $v \in W_p^2(\mathbf{T}^d)$, $\int_{\mathbf{T}^d} v d\mu_L = 0$, and $Lv = f$ in L^p for some*

$p > 1$, then $u = v$ in $W_p^2(\mathbf{T}^d)$. Moreover, let $\{X_t\}$ be the diffusion process generated by L, and let $B_t = X_t - X_0 - \int_0^T b(X_s)ds$. Then $\{B_t\}_{t\geq 0}$ is a Brownian motion, and

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s)dB_s + \int_0^t f(X_s)ds. \tag{2.1}$$

PROOF. The fact that $\{B_t\}_{t\geq 0}$ is a Brownian motion is trival since by the definition of $\{X_t\}$, $g(X_t) - g(X_0) - \int_0^t (Lg)(X_s)ds$ is a $\mathcal{F}_t$-martingale for any $g \in C^2(\mathbf{T}^d)$.

Since $b \in C(\mathbf{T}^d; \mathbf{R}^d)$ and $f \in C(\mathbf{T}^d)$, we can find $b_n \in C^\infty(\mathbf{T}^d; \mathbf{R}^d)$ and $f_n \in C^\infty(\mathbf{T}^d)$, such that $b_n \to b \in C(\mathbf{T}^d; \mathbf{R}^d)$ and $f_n \to f$ in $C(\mathbf{T}^d)$ as $n \to \infty$, and $\int f_n d\mu_L = 0$. Let $L_n \equiv \frac{1}{2}\Delta + b_n \cdot \nabla$, and write the invariant probability measure, the semigroup of linear operators on $C(\mathbf{T}^d)$ and the Green operator corresponding to it as μ_n, P_t^n and G_n, respectively. Also, let $u_n \equiv -G_n f_n$. Then $u_n \in C^\infty(\mathbf{T}^d)$, and $L_n u_n = f_n - \int_{\mathbf{T}^d} f_n d\mu_n$. Therefore, by Ito's formula,

$$u_n(X_t) = u_n(X_0) + \int_0^t \nabla u_n(X_s)dB_s + \int_0^t Lu_n(X_s)ds. \tag{2.2}$$

By using Cameron-Martin-Maruyama-Girsanov formula, we get from the definition of P_t^n and P_t^L that $P_t^n \to P_t^L$ in the operator norm as $n \to \infty$ for any $t > 0$. Therefore, by Perron-Frobenious argument, it is not diffcult that $u_n \to u$ in $C(\mathbf{T}^d)$, and $\langle\cdot\rangle_{\mu_{L_n}} \to \langle\cdot\rangle_{\mu_L}$ as linear operators on $C(\mathbf{T}^d)$, as $n \to \infty$.

We show that $u_n \to u$ in $W_p^2(\mathbf{T}^d)$, too. We first show that u_n, $n \in \mathbf{N}$, is bounded in $W_p^2(\mathbf{T}^d)$. From the definition of u_n, $\Delta u_n = 2(f_n - b_n \cdot \nabla u_n)$. So, from the boundedness of b_n in $C(\mathbf{T}^d)$ for $n \in \mathbf{N}$, there exists a constant $C_3 > 0$, such that

$$||u_n||_{W_p^2} \leq C_3 \left(||f_n||_{L_p} + ||u_n||_{L_p} + ||\nabla u_n||_{L_p}\right).$$

By Friedman [5, Theorem 1.8.1], for any $\varepsilon > 0$, there exists a $C_{(\varepsilon)} > 0$, such that

$$||g||_{W_p^1} \leq \varepsilon ||g||_{W_p^2} + C_{(\varepsilon)}||g||_{L_p}, \qquad \text{for all } g \in W_p^2.$$

So we get

$$(1 - \varepsilon C_3)||u_n||_{W_p^2} \leq C_3(1 + C_{(\varepsilon)})(||f_n||_{L_p} + ||u_n||_{L_p}), \qquad n \geq 1, \tag{2.3}$$

for any $\varepsilon > 0$. Take $\varepsilon > 0$ small enough such that $1 - \varepsilon C_3 > 0$, and we see that $\sup_{n\in\mathbf{N}} ||u_n||_{W_p^2} < \infty$. Now, using the boundedness of u_n in W_p^2, in the same way, we can show that u_n, $n \in \mathbf{N}$, is a Cauchy sequence in W_p^2. Therefore, from the convergence on u_n to u in $C(\mathbf{T}^d)$ and the completeness of $W_p^2(\mathbf{T}^d)$, we see that $u \in W_p^2(\mathbf{T}^d)$ for any $p > 1$, and $u_n \to u$ in $W_p^2(\mathbf{T}^d)$ as $n \to \infty$.

Now, take $n \to \infty$ in (2.2), since $Lu_n = f_n - \int f_n d\mu_n + (b - b_n)\cdot\nabla u_n \to f$ in $C(\mathbf{T}^d)$ as $n \to \infty$, we get (2.1).

The linearity of $G_L : C(\mathbf{T}^d) \to W_p^2(\mathbf{T}^d)$ is trival. Also, from (2.3), there exists a constant $C > 0$ independent to f, such that

$$||u||_{W_p^2} \le C_5(||f||_{L_p} + ||u||_{L_p}). \tag{2.4}$$

So the boundedness of $G_L : C(\mathbf{T}^d) \to W_p^2(\mathbf{T}^d)$ follows from that of $G_L : C(\mathbf{T}^d) \to C(\mathbf{T}^d)$.

For the uniqueness of the solution of the equation $Lu = f$ in $W_p^2(\mathbf{T}^d)$ *s.t.* $\int_{\mathbf{T}^d} u d\mu_L = 0$, let $v \in W_p^2(\mathbf{T}^d)$ satisfies $Lv = f$ and $\int_{\mathbf{T}^d} v d\mu_L = 0$, we show that $v = u(= -G_L f)$ in $W_p^2(\mathbf{T}^d)$. Since $v \in W_p^2(\mathbf{T}^d)$, there exist $v_n \in C^\infty(\mathbf{T}^d)$ with $\int_{\mathbf{T}^d} v_n d\mu_L = 0$, such that $v_n \to v$ in $W_p^2(\mathbf{T}^d)$. Let $g_n = Lv_n$, then $v_n = \int_{\mathbf{T}^d} v_n d\mu_L - G_L g_n = -G_L g_n$. Therefore, from the completeness of W_p^2, we only need to show that $G_L g_n \to G_L f$ in L^p. But $g_n \to f$ in L^p from the definition, so this is easy to see from the definition of G and the fact that $\sup_{x,y\in\mathbf{T}^d} |\frac{P_t^L(x,dy)}{\mu_L(dy)}| \to 0$ exponentially as $t \to \infty$. □

Now, let us come back to our situation described in section 1, *i.e.*, let L be the infinitesimal generator corresponding to $\{Q_x\}$. Let $L^{*\nu_0}$ denote the adjoint operator of L in $L^2(d\nu_0)$. $L^{*\nu_0}$ is the infinitesimal generator of the diffusion process $\{X_{T-t}(\omega)\}_{0\le t\le T}$ under $Q_{\nu_0}(d\omega)$ for any $T > 0$. Note that the G^* defined in section 1 is nothing but the Green operator with respect to $L^{*\nu_0}$. We have the following

PROPOSITION 2.3. *$h \in C^1(\mathbf{T}^d)$, and $L = L_0 + \frac{\nabla h}{h}\cdot\nabla$. Also, $\ell \in C^1(\mathbf{T}^d)$, and $L^{*\nu_0} = L_0^{*\mu} + \frac{\nabla \ell}{\ell}\cdot\nabla$.*

PROOF. As the proof is the same, we only give the proof of the first assertion. By the definition of h, $h = h^{\phi^{\nu_0}}$, (and $\Lambda(\phi^{\nu_0}) = \lambda$), for any $x \in \mathbf{T}^d$,

$$E^{P_x}\left[\exp\left(\int_0^t \phi^{\nu_0}(X_s)ds\right) h(X_t)\right] = e^{\lambda t} h(x).$$

So we have $\lim_{t\to 0}\frac{1}{t}(P_t h - h) = \lambda h - \phi^{\nu_0}(x)h$ in $C(\mathbf{T}^d)$. Acting G_0 on the both side, since $G_0(P_t h - h) = t\int h d\mu - \int_0^t P_s h ds$, from the continuity of G_0 we get that

$$h - \int_{\mathbf{T}^d} h d\mu = G_0(\phi^{\nu_0}h - \lambda h).$$

Therefore, by Lemma 2.2 applied to L_0, $h \in W_p^2$ for any $p > 1$, which implies $h \in C^1(\mathbf{T}^d)$, and

$$h(X_t) = h(X_0) + \int_0^t \nabla h(X_s)dB_s + \int_0^t (\lambda h - \phi^{\nu_0}h)(X_s)ds.$$

Therefore, by Ito's formula, we have

$$\begin{aligned}\log h(X_t) &= \log h(X_0) + \int_0^t \frac{\nabla h}{h}(X_s)dB_s \\ &\quad +\lambda t - \int_0^t \phi^{\nu_0}(X_s)ds - \frac{1}{2}\int_0^t \left|\frac{\nabla h}{h}(X_s)\right|^2 ds,\end{aligned}$$

which implies that

$$\begin{aligned}& e^{-\lambda t}\frac{h(X_t)}{h(X_0)}\exp(\int_0^t \phi^{\nu_0}(X_s)ds) \\ &= \exp\left(\int_0^t \frac{\nabla h}{h}(X_s)dB_s - \frac{1}{2}\int_0^t \left|\frac{\nabla h}{h}(X_s)\right|^2 ds\right).\end{aligned}$$

The left hand side above is nothing but $\frac{dQ_{X_0}}{dP_{X_0}}(\omega)\Big|_{\mathcal{F}_t}$. This gives our assertion. □

From Lemma 2.2 and Proposition 2.3, we have the following

Corollary 2.4. *G maps $C(\mathbf{T}^d)$ into $W_p^2(\mathbf{T}^d)$ for any $p > 1$, and it is a bounded linear map. Also, for any $f \in C(\mathbf{T}^d)$, $u \equiv -Gf$ is the unique solution of the equation $Lu = f$ in the sense of generalized functions. Moreover, let $\{X_t\}$ be the diffusion process generated by L, and let $B_t \equiv X_t - X_0 - \int_0^t (b_0 + \frac{\nabla h}{h})(X_s)ds$, $t \geq 0$, then $\{B_t\}_{t\geq 0}$ is a Brownian motion, and*

$$u(X_t) = u(X_0) + \int_0^t \nabla u(X_s)dB_s + \int_0^t f(X_s)ds, \qquad a.s. \tag{2.5}$$

Proposition 2.5. *For any* $f \in C(\mathbf{T}^d)$,

$$\Gamma(f,f) = \int_{\mathbf{T}^d} ||\nabla(Gf)||^2 d\nu_0 = \int_{\mathbf{T}^d} ||\nabla(G^* f)||^2 d\nu_0.$$

Proof. We only give the proof of the first equality. The second is the same.

First, since ν_0 is $\{Q_t\}$ invariant, and L is the infinitesimal generator of it, we have that $\int_{\mathbf{T}^d} Lg d\nu_0 = 0$ for any $g \in C^2(\mathbf{T}^d)$. Also, by Proposition 2.3, for any $g \in C^2(\mathbf{T}^d)$,

$$gLg = \frac{1}{2}L(g^2) - \frac{1}{2}||\nabla g||^2,$$

so

$$-2\int_{\mathbf{T}^d} Lg \cdot g d\nu_0 = \int_{\mathbf{T}^d} ||\nabla g||^2 d\nu_0$$

for any $g \in C^2(\mathbf{T}^d)$. So the same is true for any $g \in \cap_{p>1} W_p^2(\mathbf{T}^d)$ (actually, with some $p > 1$ large enough, for any $g \in W_p^2(\mathbf{T}^d)$).

Now, for any $f \in C(\mathbf{T}^d)$, let $g \equiv Gf$. Then by Corollary 2.4, $g \in W_p^2(\mathbf{T}^d)$ for any $p > 1$. Also, $f = -Lg + a$ as generalized functions, where $a = \int_{\mathbf{T}^d} f d\nu_0$, and $\int_{\mathbf{T}^d} g d\nu_0 = 0$. Therefore,

$$\begin{aligned}
\int_{\mathbf{T}^d} f\overline{G}f d\nu_0 &= 2\int_{\mathbf{T}^d} fGf d\nu_0 \\
&= -2\int_{\mathbf{T}^d} Lg \cdot g d\nu_0 + 2a\int_{\mathbf{T}^d} g d\nu_0 \\
&= \int_{\mathbf{T}^d} ||\nabla g||^2 d\nu_0 = \int_{\mathbf{T}^d} ||\nabla Gf||^2 d\nu_0. \ \square
\end{aligned}$$

Proposition 2.6. *H can be regarded as a subset of $\mathcal{M}_0(\mathbf{T}^d)$, which is dense in $\mathcal{M}_0(\mathbf{T}^d)$ with respect to the weak*-topology.*

Proof. The fact that $H \subset \mathcal{M}_0(\mathbf{T}^d)$ is trival since $H = (\overline{\widetilde{C}(\mathbf{T}^d)}^{\Gamma})^*$ and $\mathcal{M}_0(\mathbf{T}^d) = \widetilde{C}(\mathbf{T}^d)^*$. So to finish the proof, we only need to show that $\mathcal{M}_0(\mathbf{T}^d) \subset \overline{H}^{weak*}$. If not, there will exist a $\nu \in \mathcal{M}_0(\mathbf{T}^d)$ satisfying $\nu \notin \overline{H}^{weak*}$. For the sake of simplicity, we denote the equivalent class which contains f by f, too. So there exists a function $f \in C(\mathbf{T}^d)$, such that

$(f,\nu)=1$, and $(f,h)=0$ for any $h\in H$. So $f=0\in H^*$, which means $\int_{\mathbf{T}^d} f\overline{G}f d\nu_0=0$. Therefore $f\equiv constant$, and so $\int_{\mathbf{T}^d} f d\nu=0$, which makes a contradiction. □

PROPOSITION 2.7. *For any symmetric continuous function $V:\mathbf{T}^d\times\mathbf{T}^d\to\mathbf{R}$, let $U_1(x,y)\equiv -G_xV(x,y)$, and let $U\equiv -G_y^*U_1$, then $U\in C^1(\mathbf{T}^d\times\mathbf{T}^d)$, $\nabla_xU(x,y)$ is continuously differentiable with respect to y, and $\nabla_y\nabla_xU(x,y)\in C(\mathbf{T}^d\times\mathbf{T}^d)$. Also,*

$$\begin{aligned}\text{(2.6)}\qquad &\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)\overline{G}_x\overline{G}_yV(x,y)\nu_0(dx)\nu_0(dy)\\ &=\int_{\mathbf{T}^d}\int_{\mathbf{T}^d}||\nabla_x\nabla_yU(x,y)||^2\nu_0(dx)\nu_0(dy).\end{aligned}$$

PROOF. From the compactness of $\mathbf{T}^d$ and the continuity of V, V is uniformly continuous, and the map $\mathbf{T}^d\to C(\mathbf{T}^d), y\mapsto V(\cdot,y)$, is continuous.

$U_1(x,y)=-G_xV(x,y)$, so by Corollary 2.4, $U_1(\cdot,y)\in C^1(\mathbf{T}^d)$ for any $y\in\mathbf{T}^d$, and $y\mapsto\nabla_xU_1(\cdot,y)\in C(\mathbf{T}^d)$ is continuous.

Now, from the definition of G^*, we see that G^* is continuous in $C(\mathbf{T}^d)$. So $\nabla_xU(x,y)=-\nabla_x(G_y^*U_1(x,y))=-G_y^*(\nabla_xU_1(x,y))$. Therefore, $\nabla_xU(x,\cdot)\in C^1(\mathbf{T}^d)$ for any $x\in\mathbf{T}^d$, and the function $x\mapsto\nabla_y\nabla_xU(x,\cdot)\in C(\mathbf{T}^d)$ is continuous. *i.e.*, $\nabla_y\nabla_xU(x,y)\in C(\mathbf{T}^d\times\mathbf{T}^d)$.

We show (2.6) now. First, if V can be expressed as $V(x,y)=\sum_{k=1}^n\varphi_k(x)\psi_k(y)$ for some $n\in\mathbf{N}$ and some $\varphi_k,\psi_k\in C(\mathbf{T}^d), k=1,\cdots,n$, then (2.6) is obvious by Proposition 2.5. For general V, by Weierstrass-Stone Theorem, there exist V_n, such that V_n has the expression above, and $V_n\to V$ in $C(\mathbf{T}^d\times\mathbf{T}^d)$. From the boundedness of $\overline{G}_x\overline{G}_y$ in $C(\mathbf{T}^d\times\mathbf{T}^d)$, the left hand side of (2.6) for V_n converges to that for V. For the right hand side, we have from Sobolev's inequality and (2.4) that for any $f\in C(\mathbf{T}^d)$, $u=-Gf$ is in $C^1(\mathbf{T}^d)$, and for $p>1$ large enough, we have

$$||\nabla u||_\infty\le C_6||\nabla u||_{W_p^1}\le C_7||u||_{W_p^2}\le C_8(||f||_{L^p}+||u||_{L^p})\le C_9||f||_\infty$$

for some proper constants C_6,C_7,C_8,C_9. So the right hand converges, too. Therefore, (2.6) is true for general V. □

PROPOSITION 2.8. *Given any continuous symmetric function $V:\mathbf{T}^d\times\mathbf{T}^d\to\mathbf{R}$, define a bilinear and continuous function $A_V:\mathcal{M}_0(\mathbf{T}^d)\times$*

$\mathcal{M}_0(\mathbf{T}^d) \to \mathbf{R}$ by $A_V(R_1, R_2) \equiv \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)R_1(dx)R_2(dy)$. Then $A_V\Big|_{H\times H}$ is a Hilbert-Schmidt type function.

PROOF. Let $\{f_n\}_{n=1}^{\infty}$ be a complete orthonormal base of H^* with $\{f_n\}_{n=1}^{\infty} \in \widetilde{C}(\mathbf{T}^d)$. Then by Proposition 2.5 and Proposition 2.7,

$$\begin{aligned}
||A_V||_{H.S.}^2 &= \sum_{n,m=1}^{\infty} A_V(\overline{G}f_n d\nu_0, \overline{G}f_m d\nu_0)^2 \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)\overline{G}f_n(x)\overline{G}f_m(y)\nu_0(dx)\nu_0(dy)\right)^2 \\
&= \sum_{k=1}^{d}\int_{\mathbf{T}^d}\sum_{m=1}^{\infty}\left(\frac{\partial}{\partial x_k}G_x V(x,\cdot), f_m\right)_{H^*}^2 \nu_0(dx) \\
&= \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} ||\nabla_x\nabla_y G_x G_y V(x,y)||^2 \nu_0(dx)\nu_0(dy) \\
&= \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)\overline{G}_x\overline{G}_y V(x,y)\nu_0(dx)\nu_0(dy) \\
&< \infty,
\end{aligned}$$

since V and $\overline{G}_x\overline{G}_y V$ are bounded. □

3. Lemmas

The following lemma is easy to see, from the definition of multiple integral.

LEMMA 3.1. *Let $\{W_t\}_{t\geq 0}$ be a Brownian motion. Then for any $T > 0$, and any symmetric function $h(\cdot,\cdot) : [0,T]\times[0,T] \to \mathbf{R}$ that satisfies*

$$\int_0^T\int_0^T h(t_1,t_2)^2 dt_1 dt_2 < \frac{1}{4},$$

we have

$$E^W\left[\exp\left(\int_0^T\int_0^T h(t_1,t_2)dW_{t_1}dW_{t_2}\right)\right] \leq \exp\left(\int_0^T\int_0^T h(t_1,t_2)^2 dt_1 dt_2\right).$$

PROOF. Let A be the symmetric operator on $L^2[0,T]$ given by $A : L^2[0,T] \to L^2[0,T]$,

$$Af(t) = \int_0^T h(t,s)f(s)ds.$$

A is a Hilbert-Schmidt operator. Therefore, it has discrete spectrum (except 0). So all of its spectrums except 0 are its eigenvalues. Write them as $\{\lambda_k\}_{k=1}^{\infty}$. By the assumption,

$$\sum_{k=1}^{\infty} \lambda_k^2 = ||A||_{H.S.}^2 = \int_0^T \int_0^T h(s,t)^2 dsdt < \frac{1}{4},$$

so $|\lambda_k| < 1/2$ for any $k \in \mathbf{N}$. Write the corresponding orthonormal eigenvectors as e_k, $k = 1, 2, \cdots$, so $h(s,t) = \sum_{k=1}^{\infty} \lambda_k e_k(s) e_k(t)$ in $L^2([0,T] \times [0,T])$. $\int_0^T e_k(s)dW_s, k = 1, 2, \cdots$, are *i. i. d.* normal distributed random variables. Note that $\frac{1}{\sqrt{1-2x}} e^{-x} \leq e^{x^2}$ for any $x < 1/2$, so we get

$$\begin{aligned}
& E^W \left[\exp\left(\int_0^T \int_0^T h(t_1,t_2) dW_{t_1} dW_{t_2} \right) \right] \\
= \; & E^W \left[\exp\left(\sum_{k=1}^{\infty} \lambda_k \left[(\int_0^T e_k(s)dW_s)^2 - 1 \right] \right) \right] \\
= \; & \prod_{i=1}^{\infty} \frac{1}{\sqrt{1-2\lambda_i}} e^{-\lambda_i} \leq \exp(\sum_{i=1}^{\infty} \lambda_i^2) \\
= \; & \exp\left(\int_0^T \int_0^T h(t_1,t_2)^2 dt_1 dt_2 \right). \; \square
\end{aligned}$$

LEMMA 3.2. *For any probability measure ν on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0})$, any continuous ν-local-martingale (M_t) with $M_0 = 0$, any pair of dual numbers $p_1, q_1 > 1$, i.e., $\frac{1}{p_1} + \frac{1}{q_1} = 1$, any $T > 0$, and any $A \in \mathcal{F}_T$,*

$$E^{\nu}\left[e^{M_T}, A \right] \leq E^{\nu}\left[\exp(\frac{p_1 q_1}{2} \langle M \rangle_T), A \right]^{1/q_1}.$$

PROOF. Since (M_t) is a continuous ν-local-martingale, $(p_1 M_t)$ is a continuous ν-local-martingale, too, so $\exp(p_1 M_t - \frac{p_1^2}{2} \langle M \rangle_t)$ is also a continuous

ν-local-martingale, and so a ν-super-martingale. Therefore,

$$\begin{aligned} E^{\nu}\left[e^{M_T}, A\right] &\le E^{\nu}\left[\exp(p_1\cdot(M_T-\frac{p_1}{2}\langle M\rangle_T))\right]^{1/p_1} \\ &\qquad \times E^{\nu}\left[\exp(q_1\cdot(\frac{p_1}{2}\langle M\rangle_T)), A\right]^{1/q_1} \\ &\le E^{\nu}\left[\exp(\frac{p_1 q_1}{2}\langle M\rangle_T), A\right]^{1/q_1}. \ \square \end{aligned}$$

Now, we are ready to proof the following:

Lemma 3.3. *Let $V : \mathbf{T}^d\times\mathbf{T}^d \to \mathbf{R}$ be a symmetric, continuous function that satisfies the following:*

1. $\int_{\mathbf{T}^d} V(x,y)\nu_0(dy) = 0$ *for any* $x \in \mathbf{T}^d$,
2. $\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)\overline{G}_x\overline{G}_y V(x,y)\nu_0(dx)\nu_0(dy) < \frac{1}{128}$.

Then there exists a constant $\varepsilon_0 > 0$, such that for any $x, y \in \mathbf{T}^d$, and any $\varepsilon \le \varepsilon_0$,

$$\sup_{T>0} E^{Q_x}\Big[\exp(\frac{1}{T}\int_0^T\int_0^T V(X_t,X_s)dsdt), \\ dist(\frac{1}{T}\int_0^T \delta_{X_t}dt, \nu_0) < \varepsilon\Big| X_T = y\Big] < \infty.$$

Proof. First, we have that for any $T > 1$,

$$\begin{aligned} &\left|\frac{1}{T}\int_0^T\int_0^T V(X_s,X_t)dsdt - \frac{1}{T}\int_1^{T-1}\int_1^{T-1} V(X_s,X_t)dsdt\right| \\ \le\ & \frac{4T-4}{T}||V||_\infty \le 4||V||_\infty. \end{aligned}$$

Let $C_{10} = \sup_{x,y\in\mathbf{T}^d}\{q(1,x,y), q^*(1,x,y)\} < \infty$, where $q^*(1,x,y) \equiv \frac{Q_1^*(x,dy)}{\nu_0(dy)} \in C(\mathbf{T}^d\times\mathbf{T}^d)$ and $q^*(1,x,y) > 0$. Then for any $A \in \mathcal{F}_T$,

$$E^{Q_x}\Big[\exp(\frac{1}{T}\int_0^T\int_0^T V(X_t,X_s)dsdt), A\Big| X_T = y\Big]$$

$$\begin{aligned}
&\leq E^{Q_{\nu_0}}\Big[q(1,x,X_1)q^*(1,y,X_{T-1}) \\
&\qquad \cdot \exp(\frac{1}{T}\int_1^{T-1}\int_1^{T-1} V(X_t,X_s)dsdt + 4||V||_\infty), A\Big] \\
&\leq C_{10}^2 e^{8||V||_\infty} E^{Q_{\nu_0}}\Big[\exp(\frac{1}{T}\int_0^T\int_0^T V(X_t,X_s)dsdt), A\Big].
\end{aligned}$$

Therefore, it is sufficient to prove that

$$\sup_{T>0} E^{Q_{\nu_0}}\Big[\exp(\frac{1}{T}\int_0^T\int_0^T V(X_t,X_s)dsdt), \mathrm{dist}(\frac{1}{T}\int_0^T \delta_{X_t}dt, \nu_0) < \varepsilon\Big] < \infty.$$

Since ν_0 is the invariant measure of (Q_x) as mentioned before, $(X_{T-t})_{t=0}^T$ under (Q_{ν_0}) is still a diffusion process for any $T > 0$, with the infinitesimal generator $L^{*\nu_0} = L_0^{*\mu} + \frac{\nabla \ell}{\ell} \cdot \nabla$. Let $U_1(x,y) \equiv -(G_x V)(x,y)$ and $U(x,y) \equiv -(G_y^* U_1)(x,y)$ as in Proposition 2.7. By condition, $\int_{\mathbf{T}^d} V(x,y)\nu_0(dy) = 0$ for any $x \in \mathbf{T}^d$, so

$$L_x L_y^{*\nu_0} U(x,y) = L_y^{*\nu_0} L_x U(x,y) = V(x,y), \qquad \text{for any } x,y \in \mathbf{T}^d$$

in the sense of generalized functions. From the condition (2) and Proposition 2.7, we have that $\nabla_x \nabla_y U$ exists, is continuous, and

$$\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} ||\nabla_x \nabla_y U(x,y)||^2 \nu_0(dx)\nu_0(dy) < \frac{1}{128}.$$

Let $\rho_T \equiv \frac{1}{T}\int_0^T \delta_{X_t}dt$ and $A_\varepsilon \equiv \{\mathrm{dist}(\frac{1}{T}\int_0^T \delta_{X_t}dt, \nu_0) < \varepsilon\}$. Then from the boundedness of $||\nabla_x \nabla_y U(x,y)||^2$, there exists a constant $\varepsilon_0 > 0$, such that for any $\varepsilon \leq \varepsilon_0$,

$$\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} ||\nabla_x \nabla_y U(x,y)||^2 \rho_T(dx)\rho_T(dy) < \frac{1}{128} \qquad \text{on } A_\varepsilon.$$

From the definition of U_1 and Corollary 2.4,

$$U_1(X_T, X_t) = U_1(X_t, X_t) + \int_t^T \nabla_x U_1(X_s, X_t)dB_s + \int_t^T V(X_s, X_t)ds,$$

where $(B_t)_{t\geq 0}$ is the Brownian motion defined in Corollary 2.4. Therefore,

$$\frac{1}{T}\int_0^T\int_0^T V(X_s,X_t)dsdt = \frac{2}{T}\int_0^T\int_t^T V(X_s,X_t)dsdt$$

$$
\begin{aligned}
&= \frac{2}{T}\left(\int_0^T (U_1(X_T, X_t) - U_1(X_t, X_t))dt\right) \\
&\qquad -\frac{2}{T}\int_0^T dt\left(\int_t^T \nabla_x U_1(X_s, X_t)dB_s\right).
\end{aligned}
$$

Here, $||U_1||_\infty < \infty$ from the continuity of U_1 and the compactness of $\mathbf{T}^d$, and the second term is equal to $-\frac{2}{T}\int_0^T(\int_0^s \nabla_x U_1(X_s, X_t)dt)dB_s$ by stochastic Fubini's theorem (*c.f.* Ikeda-Watanabe [6, Lemma 3.4.1]), hence a continuous Q_{ν_0}-martingale. So by Lemma 3.2 (with $p_1 = 2$ and $\nu = Q_{\nu_0}$),

$$
\begin{aligned}
&E^{Q_{\nu_0}}\left[\exp(\frac{1}{T}\int_0^T\int_0^T V(X_t, X_s)dsdt), A_\varepsilon\right] \\
\le\ & \exp(4||U_1||_\infty)\cdot E^{Q_{\nu_0}}\left[\exp(-\frac{2}{T}\int_0^T dB_s(\int_0^s \nabla_x U_1(X_s, X_t)dt)), A_\varepsilon\right] \\
\le\ & \exp(4||U_1||_\infty)\cdot E^{Q_{\nu_0}}\left[\exp(2\int_0^T |\frac{2}{T}\int_0^s \nabla_x U_1(X_s, X_t)dt|^2 ds), A_\varepsilon\right]^{1/2}.
\end{aligned}
$$

So, the problem now turns to show that

$$
\sup_{T>0} E^{Q_{\nu_0}}\left[\exp(\left(\frac{8}{T^2}\int_0^T ds|\int_0^s \nabla_x U_1(X_s, X_t)dt|^2\right), A_\varepsilon\right] < \infty
$$

for some $\varepsilon > 0$. Since $(X_{T-t})_{t=0}^T$ under Q_{ν_0} is a diffusion process for any $T > 0$, we have by Lemma 2.2 and the definition of U that $\hat{B}_t^T \equiv X_{T-t} - X_T - \int_0^t (b_0^* + \frac{\nabla \ell}{\ell}(X_{T-s})ds$, $t \in [0, T]$, is a Brownian motion, and for any $s' \in (0, T)$,

$$
\begin{aligned}
\nabla_x U(X_{T-s'}, X_0) &= \nabla_x U(X_{T-s'}, X_{T-s'}) + \int_{s'}^T \nabla_y \nabla_x U(X_{T-s'}, X_{T-t'})d\hat{B}_{t'}^T \\
&\qquad + \int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'})dt'.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\frac{1}{T^2}\int_0^T ds|\int_0^s \nabla_x U_1(X_s, X_t)dt|^2 \\
=\ & \frac{1}{T^2}\int_0^T ds'|\int_{s'}^T \nabla_x U_1(X_{T-s'}, X_{T-t'})dt'|^2
\end{aligned}
$$

$$
\begin{aligned}
\leq \ & \frac{2}{T^2}\int_0^T |\nabla_x U(X_{T-s'},X_0)-\nabla_x U(X_{T-s'},X_{T-s'})|^2 ds' \\
& +\frac{2}{T^2}\int_0^T |\int_{s'}^T \nabla_y\nabla_x U(X_{T-s'},X_{T-t'})d\hat{B}^T_{t'}|^2 ds'.
\end{aligned}
$$

Here the first term is bounded by the compactness of $\mathbf{T}^d$ and the continuity of $\nabla_x U$. So it is sufficient to show that for some $\varepsilon > 0$ small enough,

$$
\sup_{T>0} E^{Q_{\nu_0}}\left[\exp\left(\frac{16}{T^2}\int_0^T |\int_{s'}^T \nabla_y\nabla_x U(T_{T-s'},X_{T-t'})d\hat{B}^T_{t'}|^2 ds'\right), A_\varepsilon\right] < \infty.
$$

Let W_t be another d-dimension Brownian motion which is independent to $\{X_t\}_{t\in[0,\infty)}$. Write $g(t,s)\equiv \nabla_y\nabla_x U(X_{T-t},X_{T-s})$, then by Lemma 3.2,

$$
\begin{aligned}
& E^{Q_{\nu_0}}\left[\exp\left(\frac{16}{T^2}\int_0^T |\int_t^T \nabla_y\nabla_x U(X_{T-t},X_{T-s})d\hat{B}^T_s|^2 dt\right), A_\varepsilon\right] \\
= \ & E^{Q_{\nu_0}}\left[E^W\left[\exp\left(\frac{4\sqrt{2}}{T}\int_0^T (\int_t^T g(t,s)d\hat{B}^T_s)dW_t\right)\right], A_\varepsilon\right] \\
= \ & E^W\left[E^{Q_{\nu_0}}\left[\exp\left(\frac{4\sqrt{2}}{T}\int_0^T (\int_0^s g(t,s)dW_t)d\hat{B}^T_s\right), A_\varepsilon\right]\right] \\
\leq \ & E^W\left[E^{Q_{\nu_0}}\left[\exp\left(\frac{64}{T^2}\int_0^T |\int_0^s g(t,s)dW_t|^2 ds\right), A_\varepsilon\right]\right]^{1/2} \\
= \ & E^{Q_{\nu_0}}\left[E^W\left[\exp\left(\frac{64}{T^2}\int_0^T |\int_0^s g(t,s)dW_t|^2 ds\right)\right], A_\varepsilon\right]^{1/2}.
\end{aligned}
$$

Here,

$$
\begin{aligned}
& \frac{1}{T^2}\int_0^T |\int_0^s g(t,s)dW_t|^2 ds \\
= \ & \frac{1}{T^2}\int_0^T\int_0^T (\int_{t_1\vee t_2}^T g(t_1,s)\otimes g(t_2,s)ds)dW_{t_1}dW_{t_2} \\
& +\frac{1}{T^2}\int_0^T (\int_t^T |g(t,s)|^2 ds)dt.
\end{aligned}
$$

The second term is bounded from the compactness of $\mathbf{T}^d$ and Proposition 2.7. So we only need to show that

$$
\sup_{T>0} E^{Q_{\nu_0}}\Bigg[E^W\Bigg[\exp\Bigg(\frac{64}{T^2}\int_0^T\int_0^T
$$

$$(\int_{t_1\vee t_2}^T g(t_1,s)\otimes g(t_2,s)ds)dW_{t_1}dW_{t_2}\Big)\Big], A_\varepsilon\Big] < \infty.$$

On the other hand, as shown before, $\int_{\mathbf{T}^d}\int_{\mathbf{T}^d}||\nabla_x\nabla_y U(x,y)||^2\rho_T(dx)\rho_T(dy) < \frac{1}{128}$ on A_ε, so

$$\begin{aligned}
(3.1)\qquad & \frac{64^2}{T^4}\int_0^T\int_0^T dt_1dt_2||\int_{t_1\vee t_2}^T g(t_1,s)\otimes g(t_2,s)ds||^2 \\
\le\ & \frac{64^2}{T^4}\int_0^T dt_1\int_0^T dt_2(\int_{t_1}^T ||g(t_1,s)||^2ds)(\int_{t_2}^T ||g(t_2,s)||^2ds) \\
=\ & (64)^2\left\{\frac{1}{T^2}\int_0^T dt(\int_t^T ||g(t,s)||^2ds)\right\}^2 \\
\le\ & \left\{\frac{64}{T^2}\int_0^T\int_0^T ||g(t,s)||^2dtds\right\}^2 \\
=\ & \left\{64\int_{\mathbf{T}^d}\int_{\mathbf{T}^d}||\nabla_x\nabla_y U(x,y)||^2\rho_T(dx)\rho_T(dy)\right\}^2 \\
<\ & 64^2\cdot(\frac{1}{128})^2 = \frac{1}{4} \qquad \text{on } A_\varepsilon.
\end{aligned}$$

So from Lemma 3.1, we have

$$\begin{aligned}
& E^{Q_{\nu_0}}\Bigg[E^W\Bigg[\exp\Bigg(\frac{64}{T^2}\int_0^T\int_0^T(\int_{t_1\vee t_2}^T g(t_1,s)\otimes g(t_2,s)ds) \\
&\qquad\qquad \times dW_{t_1}dW_{t_2}\Bigg)\Bigg], A_\varepsilon\Bigg] \\
\le\ & E^{Q_{\nu_0}}\Bigg[\exp\Bigg(\frac{64^2}{T^4}\int_0^T\int_0^T dt_1dt_2 \\
&\qquad \times|\int_{t_1\vee t_2}^T g(t_1,s)\otimes g(t_2,s)ds|^2\Bigg), A_\varepsilon\Bigg] < e^{\frac{1}{4}}.
\end{aligned}$$

This completes the proof of the lemma. □

LEMMA 3.4. *For any* $e\in C(\mathbf{T}^d)$ *with* $\int_{\mathbf{T}^d} e(y)\nu_0(dy) = 0$ *and* $||e||_{H^*} = 1$*, and any* $a < 1$*, there exists a constant* $\varepsilon_0 > 0$*, such that for any* $\varepsilon \le \varepsilon_0$*,*

$$\sup_{T>0} E^{Q_x}\Big[\exp\left(\frac{a}{2T}(\int_0^T e(X_t)dt)^2\right), A_\varepsilon\Big|X_T = y\Big] < \infty,$$

where $A_\varepsilon = \{dist(\frac{1}{T}\int_0^T \delta_{X_t} dt, \nu_0) < \varepsilon\}$ *as in Lemma 3.3.*

PROOF. As in the proof of Lemma 3.3, we only need to show the assertion without the condition that $X_0 = x$ and $X_T = y$, *i.e.*, it is sufficient if we prove

$$\sup_{T>0} E^{Q_{\nu_0}}\Big[\exp\left(\frac{a}{2T}(\int_0^T e(X_t)dt)^2\right), A_\varepsilon\Big] < \infty.$$

Also, as there, since $\int_{\mathbf{T}^d} e(x)\nu_0(dx) = 0$, by Corollary 2.4, the function u defined by $u \equiv -Ge$ is in $W_p^2(\mathbf{T}^d)$ for any $p > 1$, hence in $C^1(\mathbf{T}^d)$, and

$$u(X_T) - u(X_0) = \int_0^T \nabla u(X_t)dB_t + \int_0^T e(X_t)dt.$$

So from the boundedness of u, it is sufficient if

$$\sup_{T>0} E^{Q_{\nu_0}}\Big[\exp(\frac{a}{2}\cdot\frac{1}{T}(\int_0^T \nabla u(X_t)dB_t)^2), A_\varepsilon\Big] < \infty$$

for $\varepsilon > 0$ small enough. Choose and fix a constant $\delta \in (0, \frac{1}{a} - 1)$ first. Since

$$\int_{\mathbf{T}^d} ||\nabla u(x)||^2 \nu_0(dx) = ||e||_{H^*}^2 = 1,$$

and $||\nabla u(x)||^2$ is bounded on $\mathbf{T}^d$, there exists an $\varepsilon_0 > 0$, such that for any $\varepsilon \le \varepsilon_0$, $\int_{\mathbf{T}^d} ||\nabla u(x)||^2 \rho_T(dx) \le 1 + \delta$ on A_ε. So, by Ikeda-Watanabe [6, Theorem II.7.2], there exists a standard Brownian motion $\widetilde{B}$, such that

$$\begin{aligned}(\int_0^T \nabla u(X_t)dB_t)^2 &= \left(\widetilde{B}([\int_0^\cdot \nabla u(X_t)dB_t, \int_0^\cdot \nabla u(X_t)dB_t]_T)\right)^2 \\ &= \widetilde{B}\left(\int_0^T ||\nabla u(X_t)||^2 dt\right)^2 \\ &= \widetilde{B}\left(T\cdot\int_{\mathbf{T}^d} ||\nabla u(x)||^2 \rho_T(dx)\right)^2 \\ &\le \sup_{0\le t\le(1+\delta)T} |\widetilde{B}(t)|^2 \qquad \text{on } A_\varepsilon.\end{aligned}$$

By the reflection principle, for any $T_0 > 0$ and any x,

$$P(\sup_{0\le t\le T_0} |\widetilde{B}(t)| \ge x) \le 2P(\sup_{0\le t\le T_0} \widetilde{B}(t) \ge x) = 2P(|\widetilde{B}(T_0)| \ge x).$$

Therefore, since $\delta \in (0, \frac{1}{a} - 1)$, we have

$$\begin{aligned}
& \sup_{T>0} E^{Q_{\nu_0}}\Big[\exp(\frac{a}{2}\cdot\frac{1}{T}(\int_0^T \nabla u(X_t)dB_t)^2), A_\varepsilon\Big] \\
\le\ & \sup_{T>0} E\left[\exp\left(\frac{a}{2}\cdot\frac{1}{T}\sup_{0\le t\le(1+\delta)T}|\widetilde{B}(t)|^2\right)\right] \\
=\ & \sup_{T>0}\int_0^\infty P(\sup_{0\le t\le(1+\delta)T}|\widetilde{B}(t)|\ge x)d(e^{\frac{a}{2T}x^2})+1 \\
\le\ & 2\sup_{T>0} E\left[\exp\left(\frac{a}{2}\cdot\frac{1}{T}|\widetilde{B}((1+\delta)T)|^2\right)\right]-1 \\
=\ & \frac{2}{\sqrt{1-a(1+\delta)}}-1<\infty.
\end{aligned}$$

This completes the proof of the lemma. □

Using the two lemmas above, we get the following:

LEMMA 3.5. *For any continuous symmetric function $V : \mathbf{T}^d\times\mathbf{T}^d \to \mathbf{R}$, which satisfies $\int_{\mathbf{T}^d} V(x,y)\nu_0(dy) = 0$ for any $x \in \mathbf{T}^d$, define a symmetric, bilinear, and continuous function $A_V : \mathcal{M}_0(\mathbf{T}^d) \times \mathcal{M}_0(\mathbf{T}^d) \to \mathbf{R}$ by $A_V(R_1,R_2) = \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} V(x,y)R_1(dx)R_2(dy)$. Suppose that all of the eigenvalues of $A_V\Big|_{H\times H}$ are smaller than 1. Then there exists a constant $\varepsilon > 0$ small enough, such that for any $x, y \in \mathbf{T}^d$,*

$$\sup_{T>0} E^{Q_x}\Big[\exp(\frac{1}{2T}\int_0^T\int_0^T V(X_t,X_s)dtds),\ dist(\frac{1}{T}\int_0^T \delta_{X_t}dt,\nu_0)<\varepsilon\Big|X_T=y\Big]<\infty.$$

PROOF. By Proposition 2.8, $A_V\Big|_{H\times H}$ is a Hilbert-Schmidt type function. Combining this with the condition, we see that the maximum of its eigenvalues, say a_0, is also smaller than 1. Choose and fix a $p > 1$ such that $a_0 p < 1$.

Write the eigenvalues of $A_V\Big|_{H\times H}$ as $\{a_n\}_{n\in\mathbf{N}}$ with $|a_1| \ge |a_2| \ge |a_3| \ge \cdots$, and the corresponding eigenvectors as $\{\overline{G}e_m d\nu_0\}_{m=1}^\infty$ with $\int_{\mathbf{T}^d} e_m(x)\overline{G}e_n(x)\nu_0(dx) = \delta_{mn}$. Then $A_V(\overline{G}e_m d\nu_0, R) = a_m\int_{\mathbf{T}^d} e_m(x)R(dx)$

for any $R \in \mathcal{M}_0(\mathbf{T}^d)$. So for any $m \in \mathbf{N}$ with $a_m \neq 0$, from the continuity of $V(x,y)$, we can assume that $e_m \in \widetilde{C}(\mathbf{T}^d)$.

Let q be the dual number of $p > 1$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Since $A_V\Big|_{H\times H}$ is a Hilbert-Schmidt function as claimed, there exists a $N \in \mathbf{N}$ large enough such that $\sum_{i=N+1}^{\infty} q^2 a_i^2 < \frac{1}{128}$. Apply lemma 3.3 to

$$V_1(x,y) := q\left(V(x,y) - \sum_{i=1}^{N} a_i e_i(x)\cdot e_i(y)\right), \qquad x,y \in \mathbf{T}^d,$$

and use Hölder's inequality, so it is sufficient if

$$\sup_{T>0} E^{Q_x}\Big[\exp(\sum_{i=1}^{N} \frac{p}{2T}\int_0^T\int_0^T a_i e_i(X_t)e_i(X_s)dsdt), A_\varepsilon \Big| X_T = y\Big] < \infty$$

for $\varepsilon > 0$ small enough, where A_ε is as before.

Obviously, we can assume that $a_1, \cdots, a_N \geq 0$, as if not, we can just omit the term corresponding to it. As in Kusuoka-Tamura [9], in general, we have that for any $\varepsilon_1 > 0$, there exists an integer $m > 0$ and $\xi_i = (\xi_i^1, \cdots, \xi_i^N) \in \mathbf{R}^N$, $i = 1, \cdots, m$, such that $||\xi_i||_{\mathbf{R}^N} = 1$, $i = 1, \cdots, m$, and

$$\bigcap_{i=1}^{m}\left\{x \in \mathbf{R}^N : (x,\xi_i) \leq \frac{1}{(1+\varepsilon_1)^{1/2}}\right\} \subset \left\{x \in \mathbf{R}^N : ||x|| < 1\right\},$$

so

$$||x||^2 \leq (1+\varepsilon_1)\max_{i=1,\cdots,m}(x,\xi_i)^2, \qquad x \in \mathbf{R}^N.$$

Replace ε_1 by $1 - pa_0$ in the above. Let $\widetilde{e}_i = \sum_{j=1}^{N}\xi_i^j e_j$, $i = 1, \cdots, m$. Then $(\overline{G}\widetilde{e}_i, \widetilde{e}_i)_{L^2(d\nu_0)} = 1$, $\int_{\mathbf{T}^d}\widetilde{e}_i(x)\nu_0(dx) = 0$, $i = 1, \cdots, m$, and

$$\begin{aligned}\sum_{j=1}^{N}\left(\int_0^T e_j(X_t)dt\right)^2 &\leq (1+\varepsilon_1)\max_{i=1,\cdots,m}\sum_{j=1}^{N}\left(\int_0^T e_j(X_t)dt\cdot\xi_i^j\right)^2 \\ &= (1+\varepsilon_1)\max_{i=1,\cdots,m}\left(\int_0^T \widetilde{e}_i(X_t)dt\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{T>0} E^{Q_x}\Big[\exp(\sum_{i=1}^{N}\frac{p}{2T}\int_0^T\int_0^T a_i e_i(X_t)e_i(X_s)dsdt), A_\varepsilon\Big|X_T = y\Big] \\ \leq\ &\sup_{T>0}\sum_{i=1}^{m} E^{Q_x}\Big[\exp(\frac{1-\varepsilon_1^2}{2}\cdot\frac{1}{T}(\int_0^T \widetilde{e}_i(X_t)dt)^2), A_\varepsilon\Big|X_T = y\Big],\end{aligned}$$

which is finite for $\varepsilon > 0$ small enough by Lemma 3.4.

This completes the proof of the lemma. □

4. Proof of the Theorem

In this section, we will give the proof of the main theorem. Let

$$\begin{aligned}\widetilde{\Phi}(\nu) &\equiv \Phi(\nu) - \int_{\mathbf{T}^d} \phi^{\nu_0}(y)\nu(dy), \\ &= \Phi(\nu) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu - \nu_0), \qquad \nu \in \mathcal{M}(\mathbf{T}^d).\end{aligned}$$

Also, let $A_\varepsilon = \{dist(\frac{1}{T}\int_0^T \delta_{X_t}dt, \nu_0) < \varepsilon\}$ as before. Since for any $A \in \mathcal{F}_T$,

$$\begin{aligned}& e^{-\lambda T} E^{P_x}\left[\exp\left(T\Phi(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right), A\Big|X_T = y\right] \\ = & \frac{h(x)}{h(y)} E^{Q_x}\left[\exp\left(T\widetilde{\Phi}(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right), A\Big|X_T = y\right],\end{aligned}$$

the theorem will be shown if we can show the following two lemmas.

LEMMA 4.1.

$$\limsup_{T\to\infty} \frac{1}{T}\log E^{Q_x}\left[\exp\left(T\widetilde{\Phi}(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right), A_\varepsilon^C\Big|X_T = y\right] < 0$$

for any $\varepsilon > 0$.

LEMMA 4.2.

$$\begin{aligned}& \lim_{\varepsilon\to 0}\lim_{T\to\infty} E^{Q_x}\Big[\exp\left(T\widetilde{\Phi}(\frac{1}{T}\int_0^T \delta_{X_t}dt)\right), A_\varepsilon\Big|X_T = y\Big] \\ = & \exp\left\{\frac{1}{2}\int_{\mathbf{T}^d} \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)\Big|_{(u,u)}\nu_0(du)\right\} \times det_2(I_H - D^2\Phi(\nu_0))^{-1/2}.\end{aligned}$$

We prove Lemma 4.1 in the first. By Donsker-Varahdan [4], we have the following

Proposition 4.3.

(1) *For any $x \in \mathbf{T}^d$ and any closed set $C \subset \wp(\mathbf{T}^d)$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x \left[\frac{1}{t} \int_0^t \delta_{X_s} ds \in C \right] \le -\inf\{I(\nu); \nu \in C\},$$

(2) *for any $x \in \mathbf{T}^d$ and any open set $G \subset \wp(\mathbf{T}^d)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x \left[\frac{1}{t} \int_0^t \delta_{X_s} ds \in G \right] \ge -\inf\{I(\nu); \nu \in G\}.$$

From this, we get the following

Lemma 4.4.

1. For any $x, y \in \mathbf{T}^d$ and any closed set $C \subset \wp(\mathbf{T}^d)$,

$$\limsup_{T\to\infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_s} ds \in C \Big| X_T = y \right] \le -\inf\{I(\nu); \nu \in C\}.$$

2. For any $x, y \in \mathbf{T}^d$ and any open set $G \subset \wp(\mathbf{T}^d)$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_x \left[\frac{1}{T} \int_0^T \delta_{X_s} ds \in G \Big| X_T = y \right] \ge -\inf\{I(\nu); \nu \in G\}.$$

Proof. We only give the proof of the first assertion, the second one can be proved in the same way.

First, for any path $\{X_t\}_{t\ge 0}$, $||\frac{1}{T}\int_0^T \delta_{X_t} dt - \frac{1}{T-1}\int_0^{T-1} \delta_{X_t} dt|| \le \frac{2}{T}$, therefore, for any $\varepsilon > 0$, there exists a $t_\varepsilon > 0$, such that for any $T > t_\varepsilon$ and any path $\{X_t\}_t$, $dist(\frac{1}{T}\int_0^T \delta_{X_t} dt, \frac{1}{T-1}\int_0^{T-1} \delta_{X_t} dt) \le \varepsilon$. Now, let C_ε be the ε-neighborhood of C in $\wp(\mathbf{T}^d)$, and let C_{10} be the constant defined in the proof of Lemma 3.3, *i.e.*, $q^*(1, x_1, x_2) \le C_{10}$ for any $x_1, x_2 \in \mathbf{T}^d$, then for any $T > t_\varepsilon$,

$$P_x \left[\frac{1}{T} \int_0^T \delta_{X_t} dt \in C \Big| X_T = y \right]$$

$$\begin{aligned}
&\le P_x\left[\frac{1}{T-1}\int_0^{T-1}\delta_{X_t}dt\in C_\varepsilon\Big|X_T=y\right]\\
&= E^{P_x}\left[1_{\{\frac{1}{T-1}\int_0^{T-1}\delta_{X_t}dt\in C_\varepsilon\}}q^*(1,y,X_{T-1})\right]\\
&\le C_{10}P_x\left[\frac{1}{T-1}\int_0^{T-1}\delta_{X_t}dt\in C_\varepsilon\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\log P_x\left[\frac{1}{T}\int_0^{T}\delta_{X_t}dt\in C\Big|X_T=y\right]\\
\le\ &\limsup_{T\to\infty}\frac{1}{T}\log P_x\left[\frac{1}{T-1}\int_0^{T-1}\delta_{X_t}dt\in C_\varepsilon\right]\\
\le\ &-\inf\{I(\nu);\nu\in C_\varepsilon\}
\end{aligned}$$

for any $\varepsilon>0$. The right hand side above converges to $-\inf\{I(\nu);\nu\in C\}$ as ε goes to 0. □

Lemma 4.1 can now be seen by the same method as used for the not pinned one.

For Lemma 4.2, we follow the way as used in Kusuoka-Tamura [9] and Kusuoka-Liang [8].

LEMMA 4.5. *There exist constants $p>1$ and $\varepsilon>0$, such that*

$$\sup_{T>0}E^{Q_x}\left[e^{pT\widetilde{\Phi}(\frac{1}{T}\int_0^T\delta_{X_t}dt)},A_\varepsilon|X_T=y\right]<\infty.$$

PROOF. The proof is similar with the one in Kusuoka-Liang [8]. Let $R(\nu_0,\cdot)$ be the 3rd remainder of the Taylor expansion around ν_0, *i.e.*, $R(\nu_0,\nu-\nu_0)=\widetilde{\Phi}(\nu)-D^2\Phi(\nu_0)(\nu-\nu_0,\nu-\nu_0)$. Then for any $p>1$ and any $r,s>1$ with $\frac{1}{r}+\frac{1}{s}=1$, by Hölder's inequality,

$$\begin{aligned}
&E^{Q_x}\left[e^{p\cdot T\widetilde{\Phi}(\frac{1}{T}\int_0^T\delta_{X_t}dt)},A_\varepsilon|X_T=y\right]\\
=\ &E^{Q_x}\Big[\exp\Big\{p\cdot\frac{T}{2}D^2\Phi(\nu_0)(\frac{1}{T}\int_0^T\delta_{X_t}dt-\nu_0,\frac{1}{T}\int_0^T\delta_{X_t}dt-\nu_0)
\end{aligned}$$

$$+p \cdot TR(\nu_0, \frac{1}{T}\int_0^T \delta_{X_t} dt - \nu_0)\Big\}, A_\varepsilon | X_T = y\Big]$$

$$\leq E^{Q_x}\Big[\exp\Big\{p \cdot \frac{T}{2} \cdot rD^2\Phi(\nu_0) \tag{4.1}$$

$$\times (\frac{1}{T}\int_0^T \delta_{X_t} dt - \nu_0, \frac{1}{T}\int_0^T \delta_{X_t} dt - \nu_0)\Big\}, A_\varepsilon | X_T = y\Big]^{1/r}$$

$$\times E^{Q_x}\Big[\exp\Big\{p \cdot T \cdot sR(\nu_0, \frac{1}{T}\int_0^T \delta_{X_t} dt - \nu_0)\Big\}, A_\varepsilon | X_T = y\Big]^{1/s}. \tag{4.2}$$

Now, for any function $U(\cdot,\cdot)$, define

$$\begin{aligned}\overline{U}(x,y) \ &\equiv \ U(x,y) - \int_{\mathbf{T}^d} U(x,y)\nu_0(dx) - \int_{\mathbf{T}^d} U(x,y)\nu_0(dy) \\ &\quad + \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} U(x,y)\nu_0(dx)\nu_0(dy),\end{aligned}$$

and

$$\widetilde{U}(R_1, R_2) = \int_{\mathbf{T}^d}\int_{\mathbf{T}^d} U(x,y) R_1(dx) R_2(dy),$$

then $\int \overline{U}(x,y)\nu_0(dx) = 0$ for any $x \in \mathbf{T}^d$, and $\widetilde{\overline{U}}(R_1, R_2) = \widetilde{U}(R_1, R_2)$ for any $R_1, R_2 \in \mathcal{M}_0(\mathbf{T}^d)$.

Since the maximum a_0 of the eigenvalues of $D^2\Phi(\nu_0)\Big|_{H\times H}$ is smaller than 1 by the assumption 4, we can find a $p > 1$ such that $a_0 \cdot p < 1$. For this p, there exists a $r > 1$ such that $a_0 \cdot p \cdot r < 1$. So since

$$\begin{aligned}&T \cdot D^2\Phi(\nu_0)(\frac{1}{T}\int_0^T \delta_{X_t} dy - \nu_0, \frac{1}{T}\int_0^T \delta_{X_t} dy - \nu_0) \\ = \ &\frac{1}{T}\int_0^T\int_0^T \overline{\Phi^{(2)}(\nu_0,\cdot,\cdot)}\Big|_{(X_t,X_s)} dt ds,\end{aligned}$$

we get by Lemma 3.5 that (4.1) is bounded for $T > 0$ if $\varepsilon > 0$ is small enough.

For (4.2), let s be the dual number of $r > 1$, choose a $\delta \in (0, \frac{1}{2ps})$ and fix it. By the assumption 4, for this $\delta > 0$, there exist a constant $\varepsilon' > 0$ and a K_δ, such that $||\widetilde{\overline{K_\delta}}\Big|_{H\times H}||_{H.S.} \leq \delta$, and

$$|TR(\nu_0, \frac{1}{T}\int_0^T \delta_{X_t} dt - \nu_0)|$$

$$\begin{aligned}
&\leq T\cdot\int_{\mathbf{T}^d}\int_{\mathbf{T}^d}K_\delta(x,y)(\frac{1}{T}\int_0^T\delta_{X_t}dy-\nu_0)^{\otimes 2}(dx\otimes dy)\\
&=\frac{1}{T}\cdot\int_0^T\int_0^T\overline{K_\delta}(X_t,X_s)dsdt\qquad\text{on } A_{\varepsilon'}.
\end{aligned}$$

So by using Lemma 3.5 again, we get that (4.2) is bounded for $T>0$ if $\varepsilon'>0$ is small enough.

This completes the proof of the lemma. □

Proof of Lemma 4.2. As in Kusuoka-Tamura [9], Q_x has the strong mixing property, so X_T and $\sqrt{T}(\frac{1}{T}\int_0^T\delta_{X_t}dt-\nu_0)$ are asymptotically independent as $T\to\infty$ under Q_x for any $x\in\mathbf{T}^d$, also,

$$\begin{aligned}
&E^{Q_x}\left[\exp\left(\sqrt{-1}\sqrt{T}\int_{\mathbf{T}^d}u(x)(\frac{1}{T}\int_0^T\delta_{X_t}dt-\nu_0)(dx)\right)\right]\\
\to\ &\exp\left(-\frac{1}{2}\int_{\mathbf{T}^d}u(y)\overline{G}u(y)\nu_0(dy)\right),\qquad\text{as } T\to\infty
\end{aligned}$$

for any $u\in L^2(\mathbf{T}^d,d\nu_0)$.

Take a seperable Hilbert space H_1 such that the set $\{\overline{G}ud\nu_0\big|\int_{\mathbf{T}^d}u\overline{G}ud\nu_0<\infty\}$ is a dense linear subspace of H_1, and the inclusion map is a Hilbert-Schmidt operator. Let W be an H_1-valued random variable with distribution γ such that

$$E\left[\exp(\sqrt{-1}(u,W))\right]=\exp\left(-\frac{1}{2}\int_{\mathbf{T}^d}u(y)\overline{G}u(y)\nu_0(dy)\right)$$

for any $u\in H_1^*$.

So from the central limit theorem for Hilbert space valued random variables, the distribution of $(X_T,\sqrt{T}(\frac{1}{T}\int_0^T\delta_{X_t}dt-\nu_0))$ under Q_x converges weakly to $\nu_0\otimes\gamma$ as $T\to\infty$ on $\mathbf{T}^d\times H_1$.

As before, $D^2\Phi(\nu_0)(\cdot,\cdot)\Big|_{H\times H}$ is a Hilbert-Schmidt function. Write the eigenvalues and the corresponding eigenvectors as a_m and $\overline{G}e_m d\nu_0$, $m=1,2,\cdots$. Then $\sum_{m=1}^N a_m\left((e_m,W)^2-1\right)$ converges in $L^2(d\gamma)$. Let $:D^2\Phi(\nu_0)(W,W):$ be the $L^2(d\gamma)$-limit of $\sum_{m=1}^N a_m\left((e_m,W)^2-1\right)$.

It is easy that

$$\frac{1}{T}\int_0^T\int_0^T\sum_{m=1}^N a_m e_m(X_s)e_m(X_t)dsdt$$

$$
\begin{aligned}
&-\frac{1}{T}\int_0^T \sum_{m=1}^N a_m e_m(X_s)\overline{G}e_m(X_s)ds \\
\to\quad & \sum_{m=1}^N a_m \Big((e_m, W)^2 - 1\Big)
\end{aligned}
$$

under Q_x in distribution as $T \to \infty$ for any $N \in \mathbf{N}$ and any $x \in \mathbf{T}^d$. Also,

$$
\begin{aligned}
&\sup_{T>0} E^{Q_x}\Big[\Big\{ \Big(\frac{1}{T}\int_0^T \int_0^T \Phi^{(2)}(\nu_0; X_t, X_s)dsdt \\
&\qquad -\frac{1}{T}\int_0^T \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)}ds\Big) \\
- &\quad \Big(\frac{1}{T}\int_0^T\int_0^T \sum_{m=1}^N a_m e_m(X_s)e_m(X_t)dsdt \\
&\qquad -\frac{1}{T}\int_0^T \sum_{m=1}^N a_m e_m(X_s)\overline{G}e_m(X_s)ds\Big)\Big\}^2\Big] \\
\to &\quad 0
\end{aligned}
$$

as $N \to \infty$. Therefore,

$$
\begin{aligned}
&\frac{1}{T}\int_0^T\int_0^T \Phi^{(2)}(\nu_0; X_t, X_s)dsdt - \frac{1}{T}\int_0^T \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)}ds \\
\to &\quad : D^2\Phi(\nu_0)(W,W) :
\end{aligned}
$$

in distribution as $T \to \infty$. Also,

$$
\frac{1}{T}\int_0^T \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(X_s,X_s)}ds \to \int_{\mathbf{T}^d} \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(u,u)}\nu_0(du)
$$

Q_x-almost surely as $T \to \infty$, and

$$
TR(\nu_0, \frac{1}{T}\int_0^T \delta_{X_t}dt) \to 0
$$

under Q_x in distribution as $T \to \infty$. Therefore, we have that

$$
T\widetilde{\Phi}(\frac{1}{T}\int_0^T \delta_{X_t}dt) \to : D^2\Phi(\nu_0)(W,W) : + \int_{\mathbf{T}^d} \overline{G}_x\Phi^{(2)}(\nu_0;\cdot,\cdot)\Big|_{(u,u)}\nu_0(du)
$$

in distribution as $T \to \infty$. This together with Lemma 4.5 gives our assertion. $\square$

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