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Some Remarks on $A_1^{(1)}$ *Soliton Cellular Automata*

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Abstract. We describe the $A_1^{(1)}$ soliton cellular automata as an evolution of a poset. This allows us to explain the conservation laws for the $A_1^{(1)}$ soliton cellular automata, one given by Torii, Takahashi and Satsuma, and the other given by Fukuda, Okado and Yamada, in terms of the stack permutations of states in a very natural manner. As a biproduct, we can prove a conjectured formula relating these laws.

1. Introduction

Several years ago, Torii, Takahashi and Satsuma [TTS] proved a conservation law for their box-ball system (soliton cellular automaton) using the Robinson-Schensted-Knuth correspondence: we associate a permutation to each state p, which we call the stack permutation of the state, then the shape of the P-symbols of these stack permutations is conserved. We denote this partition by $\lambda(p)$.

Recently, it was observed that there exists a crystal structure behind this system, and the identification of this box-ball system with a box-ball system arising from $A_1^{(1)}$-crystal was made in [HHIKTT]. In this crystal picture, the time evolution is described by combinatorial row-to-row transfer matrices, and the energy functions $E_l(p)$ ($l \in \mathbb{N}$) of this system naturally gives us another conservation law [FOY]. Further, it was conjectured how these laws were related. It is given by a simple formula:

$$E_l(p) - E_{l-1}(p) = \lambda_l(p)$$

where $\lambda_l(p)$ is the length of the l th row of the partition $\lambda(p)$.

In terms of the lengths of solitons $N_1, N_2, \ldots$, $\lambda(p)$ is the partition which has N_k columns of length k ($k \in \mathbb{N}$), and $E_l(p) = \sum_{k \in \mathbb{N}} \min(l, k) N_k$.

For example, if the state is an asymptotic soliton state,

$$\cdots 01^{k_1} 0^{l_1} 1^{k_2} 0^{l_2} \cdots \qquad (l_1, l_2, \cdots >> 0)$$

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it is straightforward to verify it.

A purpose of this short note is to prove the formula. It is done by supplying conceptual explanation about the appearence of stack permutations and their P-symbols.

Main idea is to interpret the box-ball system as a discrete dynamical system of a path on $\mathbb{Z} \times \mathbb{N}$, from which we naturally read off the evolution of the permutation poset of the stack permutation of the state of the original box-ball system. This gives us a natural explanation why the Torii-Takahashi-Satsuma law holds.

We then turn to the crystal picture, and describe the sites which contribute to the energy function by using stack permutations. This explains why these energy functions are related to stack permutations.

Our conclusion is that the depth of stacks explains both conservation laws, which proves the relation of these laws.

The author hopes that this explanation would be valid after modifications in the case of $A_r^{(1)}$ soliton cellular automata. In this case, Nagai's conserved quantities remain mysterious from a combinatorial point of view.

2. Fomin-Greene Theory on Posets

We start with the Fomin-Greene theory of posets. Good references are [BF] and [F]. Let $(\mathcal{P}, \leq)$ be a poset. A **chain** is a totally ordered subset of $\mathcal{P}$. An **antichain** is a subset of $\mathcal{P}$ on which no two elements are comparable.

Definition 1. Let $(\mathcal{P}, \leq)$ be a poset. We define $I_k(\mathcal{P})$, $D_k(\mathcal{P})$ for $k \in \mathbb{N}$ as follows.

$$I_k(\mathcal{P}) := \max\{\, |C_1 \sqcup \cdots \sqcup C_k| \mid C_i\text{:(possibly empty) chain}\,\}$$
$$D_k(\mathcal{P}) := \max\{\, |A_1 \sqcup \cdots \sqcup A_k| \mid A_i\text{:(possibly empty) antichain}\,\}$$

We also define $\lambda_k(\mathcal{P})$, $\lambda'_k(\mathcal{P})$ for $k \in \mathbb{N}$ by their differences:

$$\lambda_k(\mathcal{P}) := I_k(\mathcal{P}) - I_{k-1}(\mathcal{P}), \quad \lambda'_k(\mathcal{P}) := D_k(\mathcal{P}) - D_{k-1}(\mathcal{P})$$

We thus obtain two compositions

$$\lambda(\mathcal{P}) := (\lambda_1(\mathcal{P}), \lambda_2(\mathcal{P}), \ldots)$$
$$\lambda'(\mathcal{P}) := (\lambda'_1(\mathcal{P}), \lambda'_2(\mathcal{P}), \ldots)$$

The following theorem is due to Greene and Fomin, which justifies the use of the notation $\lambda'(\mathcal{P})$.

THEOREM 2. *Let $\mathcal{P}$ be a poset. Then $\lambda(\mathcal{P})$ and $\lambda'(\mathcal{P})$ are partitions. Further, $\lambda'(\mathcal{P})$ is the transpose of $\lambda(\mathcal{P})$.*

Let $x = x_1 \cdots x_n$ be a word in $[1,r]^n$, where $[1,r] := \{1, 2, \ldots, r\}$ is the set of alphabets. For a pair (T, k) of a semistandard tableau and $k \in [1,r]$, we have the (row) insertion algorithm which produces another semistandard tableau. We denote this semistandard tableau by $T \leftarrow k$.

DEFINITION 3. Let $x = x_1 \cdots x_n \in [1,r]^n$ be a word. The semistandard tableau $P(x)$ is defined by

$$P(x) = \emptyset \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n$$

and is called the P-symbol of x.

DEFINITION 4. Let S_n be the symmetric group of n letters acting on $[1,n]$. For $w \in S_n$, its permutation poset $(\mathcal{P}(w), \leq)$ is defined by

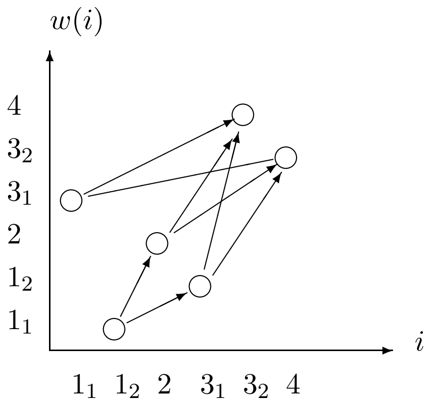
$$\mathcal{P}(w) = \{\, (i, w(i)) \,|\, i \in [1,n] \,\}$$

$$(i, w(i)) \leq (j, w(j)) \Leftrightarrow i \leq j,\ w(i) \leq w(j)$$

We identify $w \in S_n$ with the word $w(1)w(2)\cdots w(n) \in [1,n]^n$.

Let $x \in [1,r]^n$ be a word, and assume that k appears n_k times in x. Then we see x as a distinguished coset representative of $S_n/S_{n_1} \times \cdots \times S_{n_r}$. Thus we can consider permutation posets for arbitrary $x \in [1,r]^n$, which we denote by $\mathcal{P}(x)$.

Example 5. Let $x = 312143 \in [1,4]^6$. Then to see it as a distinguished coset representative (an element of S_6) is the same as seeing it as $3_1 1_1 2 1_2 4 3_2$. Here, we use $1_1 < 1_2 < 2 < 3_1 < 3_2 < 4$ instead of $1 < 2 < 3 < 4 < 5 < 6$. The permutation poset $\mathcal{P}(x)$ is as follows.



We have $I_1 = 3$, $I_2 = 5$, $I_3 = 6$, $I_4 = 6, \ldots$, and $\lambda(\mathcal{P}(x)) = (3, 2, 1)$.

For permutation posets, the following is well known.

Theorem 6. *Let $x \in [1, r]^n$ be a word, and $\mathcal{P}(x)$ be its permutation poset. Then $\lambda(\mathcal{P}(x))$ equals the shape of the P-symbol $P(x)$.*

3. Box-Ball System

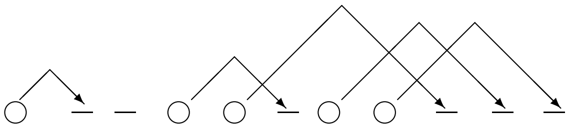
We now recall the box-ball system. Each state is given by an infinite sequence of $\{0, 1\}$ which has finitely many 1's. We denote by $1_1, \ldots, 1_N$ these 1 read from left to right. The description of one step time evolution is very simple: for $k = 1, \ldots, N$, we move 1_k to the leftmost 0 among those which sit on the right hand side of 1_k. We give an example.

Example 7.

$$t: \quad \cdots 0\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0\,0\,0 \cdots$$

$$t{+}1: \quad \cdots 0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,1\,1\,0 \cdots$$

It is visualized as follows, and in fact this is the original description of the rule.



For a state, we shall define a finite sequence consisting of "0", "(" and ")". We first choose subsequences of the form "1 0". These are called **pairs of stack depth** 1. We change these pairs "1 0" to "( )". We then delete these pairs from the original state, and choose subsequences "1 0" again. These pairs are called **pairs of stack depth** 2. We change these pairs "1 0" to "( )", and delete these pairs again. We continue this procedure repeatedly until all 1 are deleted. In the end, all 1 are made into pairs with ")" and their stack depths are defined. We say that an opening parenthesis and a closing parenthesis are **matched** if they belong to a same pair. We now define the stack permutation of the state.

Definition 8. For a state, we associate a finite sequence of "0", "(" and ")" as above. We read the opening parentheses from left to right, and number them $1, 2, \dots$ accordingly.

We number a closing parenthesis k if it makes a pair with the k th opening parenthesis. The permutation obtained by reading the numbering of closing parentheses from left to right is called the stack permutation of the state.

Example 9. For a state given by

$$\cdots 0\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0\,0\,0 \cdots,$$

we make pairs as follows.

$$\begin{array}{c}
\cdots 0\,0\,(\,)\,0\,1\,(\,)\,1\,(\,)\,0\,0\,0 \cdots \\
\cdots 0\,0\,(\,)\,0\,1\,(\,)\,(\,(\,)\,)\,0\,0 \cdots \\
\cdots 0\,0\,(\,)\,0\,(\,(\,)\,(\,(\,)\,)\,)\,0 \cdots
\end{array}$$

Thus the numbering of the closing parentheses is given by

$$\begin{array}{ccccccccccccccccc}
\cdots & 0 & 0 & (&) & 0 & (& (&) & (& (&) &) &) & 0 & \cdots \\
\cdots & & & & 1 & & & & 3 & & & 5 & 4 & 2 & & \cdots
\end{array}$$

That is, the stack permutation is $1\,3\,5\,4\,2$.

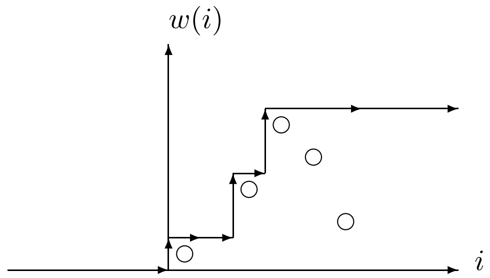
Note that an opening parenthesis does not move to the corresponding closing parenthesis, but the total set of the opening parentheses moves to the total set of the closing parentheses as a whole in each step of time evolution.

We now describe the box-ball system as a discrete dynamical system of a path on $\mathbb{Z}\times\mathbb{N}$. Recall that we have defined a finite sequence of "0", "(" and ")" for each state. We read the sequence from the first "(", and associate a walk on $\mathbb{Z}\times\mathbb{N}$ starting from $(0,0)$ by the rule that "(" corresponds to "$\uparrow$", and "0", ")" correspond to "$\rightarrow$". We see it as a walk from $(-\infty,0)$ to (∞,N), where N is the number of balls in the box-ball system, by adding infinitely many "$\rightarrow$" to both sides. We give an example.

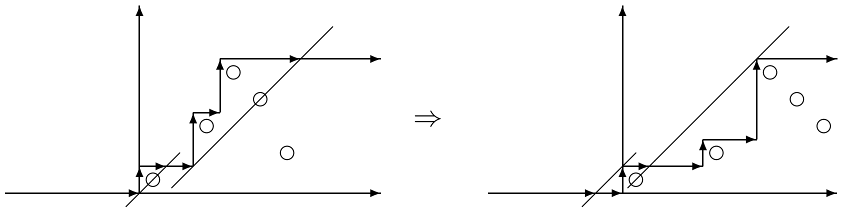
Example 10. For a state in the previous example, we have obtained the following sequence.

$$(\,)\,0\,(\,(\,)\,(\,(\,)\,)\,)$$

Thus we have a walk as follows. The permutation poset of the stack permutation $w = 13542$ is given by circles.



Let us think of a vertical edge corresponding to "(" of a subsequence "0 (". Since the left neighbor of the vertical edge is "0", we have that a positive number of "(" remain waiting to be matched as long as the number of "(" is greater than the number of ")". It implies that if the vertical edge starts at (x_0,y_0), and the walk is in the area $y–y_0 > x–x_0$, then all edges are "(" or ")", and no "0" appears. Using this, we can give a simple evolution rule of the path: Let us start walking with (x_0,y_0) and continue walking on the path until it hits the line $y-y_0 = x-x_0$ again. We say that these edges constitute a group. We partiton the edges of the walk into such groups. For each group with (x_0, y_0) as above, we reflect the edges in this group with respect to the line $y-y_0 = x-x_0$. We then move the zero of the (x, y)-plane to the lower vertex of the first vertical edge. These complete one step time evolution. This evolution rule is best understood by an example.



Since the closing parentheses of the pairs of "(" and ")" define the stack permutation, we have natural correspondence between the vertices of the permutation poset of the stack permutation and the matching pairs: for each vertical edge, we choose the horizontal edge in the matching pair. If the vertical edge is on the j th row and the horizontal edge is on the i th column, then (i, j) is an element of the permutation poset. In the above example, the leftmost three vertices correspond to the pairs of stack depth 1, and the left vertex of the remaining two corresponds to the pair of stack depth 2, and the rightmost vertex corresponds to the pair of stack depth 3.

To describe the poset structure, it is convenient to describe the matching pairs by framed boxes. For the left example of the above, we have

$$\boxed{(\ )}\ \boxed{(\ \boxed{(\ )}\ \boxed{(\ \boxed{(\ )}\ )}\ )}$$

We say that two pairs are in **outer relation** if each of the corresponding framed box does not contain the other. If one framed box contains the other, we say that these are in **inner relation**. Note that two pairs are either in outer relation or inner relation by the definition of the pairs.

Lemma 11. *Assume that two pairs of "(" and ")" are in outer (resp. inner) relation. Then the corresponding vertices in the permutation poset are comparable (resp. not comparable).*

Proof. If they are in outer relation, their positions in the sequence of

"(" and ")" are given by

$$\cdots(\ \cdots\ )\ \cdots\ (\ \cdots\ )\ \cdots.$$

If the left "(" is the i th opening parenthesis and the right "(" is the j th opening parenthsis, we have $i < j$ and $w(i) < w(j)$ by the definition of the stack permutation. Hence the corresponding vertices in the permutation poset are comparable. The argument for the inner case is similar. □

Proposition 12. *To each state, we associate the permutation poset of the stack permutation of the state as above. Then its vertices corresponding to pairs of stack depth k form a chain in the poset. We call it the depth k chain and denote it by C_k. We then have that $|C_1 \sqcup \cdots \sqcup C_k|$ gives the maximal number of vertices covered by k chains.*

Proof. By Lemma 11, C_k is a chain. We show that this permutation poset admits decomposition into disjoint union of antichains A_k such that each A_k has the form $\{v_1, \ldots, v_{l_k}\}$ where v_i is a vertex corresponding to a pair of stack depth i. Assume that we have already distributed vertices of stack depth smaller than k into such antichains. In the definition of the pairs, it corresponds to the stage that we have deleted "1 0"'s $k-1$ times. By the definition of the pairs, each framed box of stack depth k contains a framed box of stack depth $k-1$, and these framed boxes of stack depth k are in outer relation. The latter implies that we can choose distinct framed boxes of depth $k-1$ for framed boxes of depth k. Since the vertices of stack depth $k-1$ are distributed to distinct antichains, we can distribute the vertices of stack depth k to antichains without violating the required property.

We now assume that $C'_1 \cup \cdots \cup C'_k$ gives the maximal number of vertices covered by k chains. Since each antichain intersects $C'_1 \cup \cdots \cup C'_k$ at most k times, we can move these vertices into $C_1 \cup \cdots \cup C_k$ keeping them mutually distict. This is possible by the existence of the antichain decomposition we have just proved. Hence, $|C'_1 \cup \cdots \cup C'_k|$ can not exceed $|C_1 \cup \cdots \cup C_k|$. □

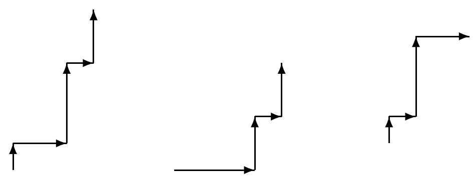
If we denote by $\mathcal{P}^t$ the permutation poset at time t, and by C^t_k the depth k chain of $\mathcal{P}^t$, we have $\lambda_k(\mathcal{P}^t) = |C^t_k|$ by Proposition 12. By Theorem 6, we have that $\lambda(\mathcal{P}^t)$ is nothing but the shape of the P-symbol of the stack permutation of the state at time t. Hence, the following theorem is almost obvious. It simply says that the length of depth k chain is conserved, which is easily seen from the evolution rule of the path as follows.

THEOREM 13 ([TTS]). *For each state, we compute its stack permutation. Then the shape of its P-symbol is conserved under time evolution.*

PROOF. We show that $|C_k^t|$ is conserved. For $k = 1$, the elements of the chain correspond to convex corners of the path. Hence it is obviously conserved by the evolution rule of the path. We then delete the depth 1 chain from the posets. This is the same as deleting convex corners from the original path and the reflected path.

To know that the deletion of convex corners from the original path and the reflected path gives a same walk, it is enough to see that deleting convex corners gives the same walk as deleting concave corners. To compare the location of 1's in the walks, We divide the cases by looking at vertical lines (the middle lines of the figures below). For the location of 0's, we divide the cases by looking at horizontal lines and the argument is entirely similar, which we omit.

The leftmost figure represents the case that we have vertical lines on both sides. One may subdivide the case into four by separating the case that there is exactly one 0 in the middle of 1's from the case that there are more than one 0's in the middle of 1's, if one wishs. The remaining two cases are the left end and the right end of the walk.



By comparing the results of the deletion of the concave corners and the convex corners, we know that the new walks are the same. In particular, deleting convex corners from the original path and the reflected path give a same walk. Since the depth 2 chain becomes the depth 1 chain of the new poset, we can apply the same argument to conclude that $|C_2^t|$ is conserved. By repeated use of the argument, we also have the conservation laws for all k. $\square$

4. Energy Functions and Stack Permutations

We now turn to the crystal description of the box-ball system. Let $B := \{\boxed{0}, \boxed{1}\}$ be the A_1 crystal associated with the vector representation whose highest weight vector is $\boxed{0}$. Its affinization is denoted by $\mathit{Aff}(B) := \mathbb{Z} \times B$. This is an $A_1^{(1)}$ crystal. Note that the numbering of $\boxed{0}$ and $\boxed{1}$ is different from the usual one. B is identified with the subset $\{0\} \times B$. For each state, we cut sufficiently remote 0's and consider it as an element in $B^{\otimes n}$.

To describe the time evolution rule, we take the crystal of the l th symmetric tensor B_l and its affinization $\mathit{Aff}(B_l)$ with l sufficiently large. The elements of B_l are nondecreasing sequences of length l whose entries are 0 and 1. We write $0^{m_1}1^{m_2}$ $(m_1 + m_2 = l)$ for these elements. We use combinatorial R matrices to get isomorphism of affine crystals as follows.

$$
\begin{tikzcd}
\otimes & \mathit{Aff}(B)\otimes \arrow[dd] & \mathit{Aff}(B)\otimes \arrow[dd] & \cdots \otimes & \mathit{Aff}(B) \arrow[dd] & \\
\mathit{Aff}(B_l) \arrow[rrrrr] & & & & & \mathit{Aff}(B_l) \\
 & \mathit{Aff}(B)\otimes & \mathit{Aff}(B)\otimes & \cdots \otimes & \mathit{Aff}(B) & \otimes
\end{tikzcd}
$$

After we embed $B^{\otimes n}$ to $\mathit{Aff}(B)^{\otimes n}$, we apply this combinatorial row-to-row transfer to the tensor product of $\boxed{0^l}$ with the upper $\mathit{Aff}(B)^{\otimes n}$ to get the lower $\mathit{Aff}(B)^{\otimes n}$ tensored by $\boxed{0^l}$. Then we forget the symmetric tensor part and the $\mathbb{Z}$ part of the affine crystal. The result is an element of $B^{\otimes n}$. This procedure gives one step time evolution of the box-ball system.

We consider the isomorphism for arbitrary l. Then for a state p, we have

$$
\left(0 \times \boxed{0^l}\right) \otimes p \mapsto p' \otimes \left(E_l(p) \times \boxed{0^{m_1}1^{m_2}}\right)
$$

for some $p' \in \mathit{Aff}(B)^{\otimes n}$ and $m_1, m_2 \in \mathbb{N}$. These $E_l(p)$ are called energy functions. It is known [NY] that if we set $E_l = 0$ and increase it by one at the sites of the following form, then the final value of E_l coincides with $E_l(p)$.

$$\begin{array}{ccc} & \boxed{0} & \\ & \downarrow & \\ \boxed{0^{m_1}1^{m_2}} & \longrightarrow & \boxed{0^{m_1+1}1^{m_2-1}} \\ & \downarrow & \\ & \boxed{1} & \end{array} \qquad (m_2 \geq 1,\ m_1+m_2 = l)$$

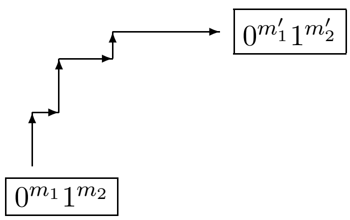
By using the fact that time evolution is obtained from a crystal isomorphism of affine crystals, Fukuda, Okado and Yamada [FOY, Theorem 3.2] have proved that these $E_l(p)$ are conserved quantities of this box-ball system.

The purpose of this section is to relate these quantities to the stack permutation of the state p.

THEOREM 14. *For each state p, we define the sequence of "0", "(" and ")" as in the previous section. Then E_l increases precisely at the sites corresponding to ")" whose stack depth are equal or less than l.*

PROOF. Assume that ")" corresponds to a pair of depth k. We shall show that if $l \geq k$, E_l does increase at this site.

Let "(" be the corresponding opening parenthesis, and $1^{k_1}0^{l_1} \cdots 1^{k_N}0^{l_N}$ be the walk starting from the vertical edge corresponding to the "(" and ending at the horizontal edge corresponding to the ")". We write the evolution of the symmetric tensor along the path as follows.



Assume that E_l does not increase at this site (the last end of the walk). Then the status of the symmetric tensor on the both ends of the last edge is $\boxed{0^l}$. We denote by $e_1, \ldots, e_N$ the last edges of $1^{k_1}, \ldots, 1^{k_N}$ respectively. Then we can prove the following by downward induction on i.

- The symmetric tensor on the upper end of e_i has the form $\boxed{0^{l-s}1^s}$

with $s < l$.

- Since the upper end of e_i is not saturated, we have steady increase of the number of 1 in the symmetric tensor during k_i vertical edges, and no saturation occurs during these edges.

- Since $l_{i-1} - k_i + \cdots - k_N + l_N \le k$ by the assumption on the stack depth, the symmetric tensor on the upper end of e_{i-1} has the form $\boxed{0^{l-s}1^s}$ with $s < l$.

We then have the following.

- Since $k_1 - l_1 + \cdots - l_i \ge 0$, we have steady decrease of the number of 1 in the symmetric tensor during l_i horizontal edges, and no saturation occurs.

Therefore, we conclude that the left end of the last edge of the walk has the symmetric tensor of the form $\boxed{0^{l-s}1^s}$ with $s > \sum k_i - \sum l_i \ge 0$, which contradicts the assumption at the beginning. (In particular, we have that $I_l(\mathcal{P}) \le E_l(p)$ where $\mathcal{P}$ is the permutation poset of the stack permutation of the state p.)

Next we show that if $l < k$, then E_l does not increase at this site. To prove this, we show for arbitrary k, l that the right end of the last edge of the walk has the symmetric tensor $\boxed{0^l}$ if the stack depth k is equal or greater than l, and $\boxed{0^{m_1}1^{m_2}}$ $(m_1 \ge k)$ if k is equal or smaller than l. We prove it by induction on k. If $k = 1$, the proof is obvious.

If $k \le l$, we choose the last closing parethesis of stack depth $k-1$. Then by the induction hypothesis, the symmetric tensor has the form $\boxed{0^{m_1}1^{m_2}}$ $(m_1 \ge k-1)$ at this site. Note that we have already proved that no saturation occurs during the walk if $k \le l$. Hence, if we start the walk with $\boxed{0^{m_1}1^{m_2}}$, we end the walk with $\boxed{0^{m_1}1^{m_2}}$. From this, we know that the symmetric tensor at the left end of the last edge of the walk is also $\boxed{0^{m_1}1^{m_2}}$ $(m_1 \ge k-1)$. Hence, the right end of the last edge has the form $\boxed{0^{m'_1}1^{m'_2}}$ $(m'_1 \ge k)$ if $m_2 > 0$ and $\boxed{0^l}$ if $m_2 = 0$. But we also have $l \ge k$ in the latter case.

We now assume that $k \ge l$. We choose the last closing parenthesis among those parentheses of stack depth equal or greater than l in the walk which are different from the last end of the walk. Since its stack depth is smaller than k, we can apply the induction hypothesis to know that the symmetric tensor has the form $\boxed{0^l}$ at this site. Further, since we have pairs of stack depth less than l during this site and the last edge of the walk, we have that the symmetric tensor at the left end of the last edge has $\boxed{0^l}$. Thus the same is true for the right end of the last edge. Hence we have proved the claim.

The claim implies that if $l < k$, E_l does not increase at the site in question (the last end of the walk). Therefore, we have proved that E_l increases precisely at the sites corresponding to ")". (In particular, we have also proved that $I_l(\mathcal{P}) = E_l(p)$.) $\square$

5. Conclusion

For a state p^t at time t, we denote by $\mathcal{P}^t$ the permutation poset of the stack permutation of the state p^t. Then the energy function $E_l(p^t)$ counts the vertices of $\mathcal{P}^t$ whose stack depth are equal or less than l. On the other hand, the l th row of the shape $\lambda(p^t)$ of the P-symbol of the stack permutation is equal to the number of vertices of $\mathcal{P}^t$ whose stack depth are l. Hence these quantities are naturally explained by the notion of stack depth, and we have $I_l(\mathcal{P}^t) = E_l(p^t)$.

Further, the evolution rule of a path naturally explains why these quantities are conserved.

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