Short time asymptotic behavior and large deviations for Brownian motion on scale irregular Sierpinski gaskets

（非正規なシェルピンスキーガスケット上のブラウン運動に対する熱核の短時間漸近挙動と大偏差原理）

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Notation

Let $\eta \in \{2,3\}^{\mathbb{N}}$.

- F^η : The scale irregular Sierpinski gasket with respect to η.
- F_0 : The set of vertices of a unit equilateral triangle in $\mathbb{R}^2$.
- $b(2) = 2$, $b(3) = 3$, $t(2) = 5$, $t(3) = 90/7$.
- $B_n(\eta) = \prod_{i=1}^{n} b(\eta_i)$, $T_n(\eta) = \prod_{i=1}^{n} t(\eta_i)$.
- $d_w^\eta(n) = \dfrac{\log T_n(\eta)}{\log B_n(\eta)}$.
- X : Brownian motion on F^η.
- d_η : The metric on a scale irregular Sierpinski gasket with respect to η.
- $W = \inf\{t \geq 0 : X_t \in F_0 \setminus \{X_0\}\}$.
- E_x^η : The expectation with respect to P_x^η.
- $g(s,\eta) = E_0^\eta[\exp(-sW)]$.
- $\Psi(s,\xi)$: The definition of Ψ is in (2.3.2).
- $\Psi^*(z,\xi) = \sup_{s>0}\{zs - \Psi(s,\xi)\}$, $\xi \in \{2,3\}^{\mathbb{Z}}$.
- $\{2,3\}_x^{\mathbb{N}} = \{\eta \in \{2,3\}^{\mathbb{N}} : x \in F^\eta\}$ for each $x \in \mathbb{R}^2$.
- $\Omega_x = C_x([0,1] \to \mathbb{R}^2) = \{\omega \in C([0,1] \to \mathbb{R}^2) : \omega(0) = x\}$.
- $\Omega_x^\eta = C_x([0,1] \to F^\eta) = \{\omega \in C([0,1] \to F^\eta) : \omega(0) = x\}$.

- $\|f-g\|_\eta = \sup_{0\le t\le 1} d_\eta(f(t),g(t))$ for each $f,g\in\Omega_x^\eta$.
- $\|f-g\| = \sup_{0\le t\le 1} |f(t)-g(t)|$ for each $f,g\in\Omega_x$.
- $B(f,r)=\{\omega\in\Omega_x : \|f-\omega\|<r\}$ for each $f\in\Omega_x$, $r>0$.
- $B_m(f,r)=\{\omega\in\Omega_x : |f(k/m)-\omega(k/m)|<r, 1\le k\le m\}$ for each $f\in\Omega_x$, $r>0$.
- $P_{x,\epsilon}^\eta$: The law of $X(\epsilon\,\cdot\,)$ starting from x.
- $\epsilon_n^z(\eta)=B_n(\eta)z/T_n(\eta)$.
- $P_{\epsilon_n^z}^\eta = P_{x,\epsilon_n^z(\eta)}^\eta$
- $D_\eta\phi(t)=\lim_{s\to 0}\dfrac{d_\eta(\phi(t+s),\phi(t))}{|t-s|}$.
- $I_{x,z}^\eta(\phi,\xi)=\begin{cases}\displaystyle\int_0^1 D_\eta\phi(t)\Psi^*\left(\frac{z}{D_\eta\phi(t)},\xi\right)dt & \phi\in\Omega_x^\eta \text{ and } \phi \text{ is absolutely continuous,}\\ \infty & \text{otherwise}\end{cases}$

本論文の概説

0.1　研究の背景

0.1.1　フラクタル上の確率解析

フラクタルという言葉はフランスの数学者 Mandelbrot によって 1970 年代に造られたが, いまだにフラクタルの明確な数学的な定義はない. Mandelbrot は海岸線や雲の形, 結晶の構造など自然界に存在するものの多くが, 直線や円などのような滑らかな図形ではなく, 不規則かつあらゆるスケールの細かい構造を含むという意味で複雑であるという視点から, それらの複雑な図形をフラクタルと呼んだ. ここではフラクタルとは何らかの自己相似性をもち, ユークリッド空間とは幾何学的性質が大きく異なるものという曖昧な意味で使うことにする.

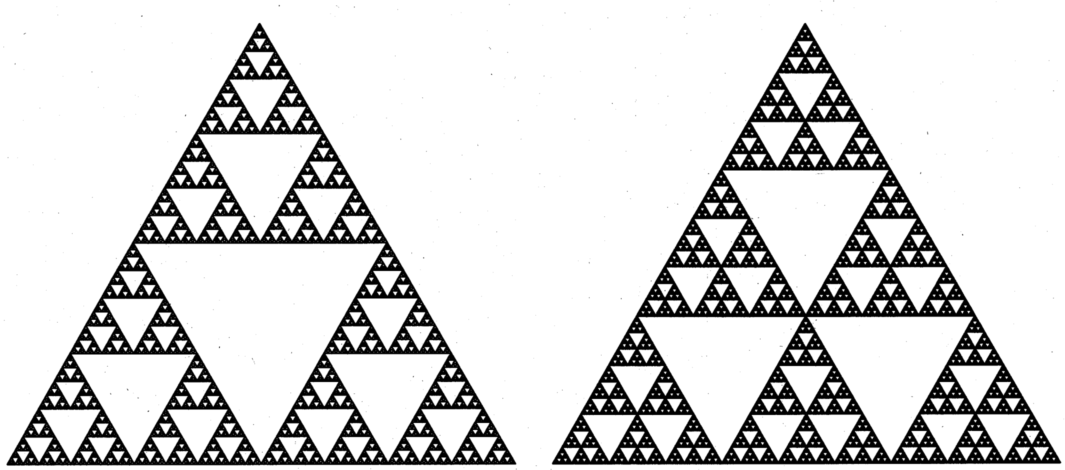

図 1: SG(2) と SG(3)

フラクタルの幾何学的性質に関する研究は古くから行われている. 一方, フラクタルの解析的な性質については, 物理学者らが興味を持っていたが, 本格的な数学的研究は 1980 年代である. 高分子やネットワークの構造, あるいはかびや結晶が成長していく様子といった disordered media (複雑な系) の性質を自己相似性を手がかりにして調べようという研究が物理学者により行われていたのを動機として, フラクタルを複雑な系の典型例と考え熱や波動がどのように伝導, 伝播するのかという解析的な性質に関心が寄せられ始めた. 滑らかさの欠如からフラクタルの上で微分計算を定義することはできない. そのためどのようにして厳密な意味で拡散現象を解析すればよいかが一つの大きな問題であったが, 確率論の研究者達はフラクタルの上に拡散過程を構成することによってこの問題の解決を試みた. Sierpinski gasket (以下 SG とも書く) と呼ばれる代表的なフラクタルの上での Brown 運動の構成とその性質を調べた Goldstein [G], Kusuoka [Kus], Barlow-Perkins [BP] による研究が確率論的アプローチによる萌芽的研究といえる. [BP] では Sierpinski gasket F の上の熱核 $p_t(x,y)$ に関して次のような大局的な両側評価を与えた: 任意の $x, y \in F$, $0 < t < \infty$ に対して

$$\begin{aligned} c_1 t^{-d_s/2} \exp\Big(-c_2 \big(\frac{|x-y|^{d_w}}{t}\big)^{1/(d_w-1)}\Big) &\le p_t(x,y) \\ &\le c_3 t^{-d_s/2} \exp\Big(-c_4 \big(\frac{|x-y|^{d_w}}{t}\big)^{1/(d_w-1)}\Big), \end{aligned}$$

ここで $d_w = 2d_f/d_s = \log 5/\log 2 = 2.32193\ldots$, また $d_f = \log 3/\log 2 = 1.58496\ldots$ はハウスドルフ次元, $d_s = 2\log 3/\log 5 = 1.36521\ldots$ はスペクトル次元と呼ばれる指数である. $c_1, \ldots, c_4$ は t, x, y に無関係な定数である. この評価式は劣ガウス型評価と呼ばれ, $d_w = 2$ のときが通常の Gauss 型評価であり, $d_w > 2$ であることは, 長時間挙動において粒子がゆっくりと拡散してくという, 劣拡散の状況を表している. [G], [Kus], [BP] での確率過程は SG に近づいていくようなグラフ上の random walk のスケール極限による非自明な拡散過程として構成されている. その後, Kigami [Ki], Fukushima-Shima [FS] らによって Dirichlet 形式の理論がフラクタル上の解析において大変有用であることが明らかとなり解析学的アプローチのさきがけとなった. Dirichlet 形式とはヒルベルト空間で定義された対称な 2 次形式の 1 つのクラスであり, 実際に用いられるのは H がある測度空間上の L^2 空間の場合が多い. 現在ではフラクタル上の拡散過程の構成は Dirichlet 形式を用いる方法が主流となっている. SG の上の Brown 運動の解析的諸性質は徹底的に詳しく研究され, 熱伝導や波動の挙動がユークリッド空間でのそれとまったく異なった性質を有することが分かった. その後, さらに様々なフラクタルの上に確率過程が構成され, その解析が進んできている.

そのようなフラクタルの一つとして本論文の対象となる Scale irregular Sierpinski gasket と呼ばれる図形の上の Brown 運動について 1992 年に Hambly [Ham] によって研究が行われた. この図形は空間的に一様ではあるが完全な自己相似性を持たないう点で SG に比べ (わずかではあるが) より現実的なモデルであるということができる. 1997 年 Barlow-Hambly [BH] は Dirichlet 形式を用いて SISG の上に Brown 運動を構成しその熱核に対して劣ガウス型的な評価を与えた (後述).

0.1.2 熱核の短時間漸近挙動

距離空間の上に拡散過程が構成されると, 拡散過程の性質とそのベースとなっている空間の性質との関係が興味の対象の一つとなる. 1967 年の Varadhan [V1] によるものがそのさきがけで, リーマン多様体 M の上の熱核 $p_t^M(x,y)$ の $t\to 0$ での次のような漸近挙動を与えた: 任意の $x,y\in M$ に対して

$$\lim_{t\to 0} t\log p_t^M(x,y) = -\frac{d_M(x,y)^2}{2}. \tag{0.1.1}$$

ここで d_M はリーマン距離である. その後, この結果は様々な状況で研究され一般化されており, Hino-Ramírez [HR] によって拡散過程が対称でありその Dirichlet 形式より定まるエネルギー測度と呼ばれるものが底空間の測度と絶対連続である場合に証明された. この結果は有限次元無限次元を問わず成立し, 著しく一般的である. Kumagai [Kum] は SG の上の Brown 運動の熱核について同様の事柄について考察を行い (0.1.1) の左辺の極限が存在しないことを示し, 次の結果を得た: 任意の $x,y\in F$ に対して

$$\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n z\right)^{1/(d_w-1)}\log p_{(\frac{2}{5})^n z}(x,y) = -d(x,y)^{d_w/(d_w-1)}G\left(\frac{z}{d(x,y)}\right). \tag{0.1.2}$$

ここで d は SG の上の固有の距離であり, $z\in[2/5,1]$ そして G は Brown 運動の, ある hitting time のラプラス変換から導かれる関数のルジャンドル変換として決まる $G(5s/2)=G(s)$ を満たす正の連続な周期関数であり定数ではない. Euclid 空間の Brown 運動をはじめとして有用な多くの拡散過程が先の Hino-Ramírez による設定に含まれるが, SG の上の Brown 運動に対してはその Dirichlet 形式より定まるエネルギー測度と底空間の測度が特異であることが Kusuoka [Kus1] により示されており上記の範疇には入らない. この特異性はより広いクラスの自己相似フラクタルについても示されている.

0.1.3 確率過程の大偏差原理

確率論における大偏差原理とはラフな言い方をすれば, 確率変数列の平均の挙動についての法則 (大数の法則), その平均からの “小さな” ずれについての法則 (中心極限定理) の次

に考察の対象となる平均からの“大きな”ずれについての法則である．これは次のように記述されることが多い．ここでは距離空間 E に限定して述べる．$I : E \to [0,\infty]$ は下半連続であるとき rate function と呼ばれ, さらに任意の $L > 0$ に対して $\{p \in E : I(p) \le L\}$ がコンパクトのとき good rate function と呼ばれる．E の上の測度の族 $\{Q_\epsilon\}_{\epsilon>0}$ が rate function I で大偏差原理を満たすとは任意の閉集合 $C \subset E$ と 開集合 G に対して

$$-\inf_{\phi\in G} I(\phi) \le \liminf_{\epsilon\to 0} \epsilon \log Q_\epsilon[G], \quad \limsup_{\epsilon\to 0} \epsilon \log Q_\epsilon[C] \le -\inf_{\phi\in C} I(\phi) \tag{0.1.3}$$

が成り立つことである．これが成立しているとき, 積分の族に対する漸近挙動について次の重要な変分公式が成立する．これは大偏差原理における最も重要な定理の一つであり Varadhan の定理として知られている: 任意の有界連続関数 $\Phi : E \to \mathbb{R}$ に対して

$$\lim_{\epsilon\to 0} \epsilon \log E^{Q_\epsilon}[e^{\Phi/\epsilon}] = \sup_{x\in E}\{\Phi(x) - I(x)\}. \tag{0.1.4}$$

逆に exponentially tight な測度の族 $\{Q_\epsilon\}_{\epsilon>0}$ があって, 任意の有界連続関数 $\Phi : E \to \mathbb{R}$ に対して (0.1.4) の左辺の極限が存在するとき good rate function I があって大偏差原理 (0.1.3) が成り立つ．ここで $\{Q_\epsilon\}_{\epsilon>0}$ が exponentially tight であるとは, 任意の $\alpha > 0$ に対してコンパクト集合 $K_\alpha \subset E$ があって

$$\limsup_{\epsilon\to 0} \epsilon \log Q_\epsilon[K_\alpha^c] < -\alpha$$

となることである．

以下では確率過程に対する大偏差原理について簡単に述べる．1966 年に Schilder [Sc] はユークリッド空間の上の Brown 運動 B に対して次の大偏差原理を証明した．P_x を $C_x([0,1] \to \mathbb{R}^d) = \{\phi \in C([0,1] \to \mathbb{R}^d) : \phi(0) = x\}$ 上の Wiener 測度とする．任意の閉集合 $C \subset C_x([0,1] \to \mathbb{R}^d)$ と開集合 G に対して

$$-\inf_{\phi\in G} I(\phi) \le \liminf_{\epsilon\to 0} \epsilon \log P_x[B_{\epsilon\cdot} \in G], \quad \limsup_{\epsilon\to 0} \epsilon \log P_x[B_{\epsilon\cdot} \in C] \le -\inf_{\phi\in C} I(\phi).$$

ただし

$$I(\phi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\phi'(t)|^2 dt & \left(\phi \in \left\{\displaystyle\int_0^1 f(t)dt : f \in L^2([0,1])\right\} \text{ のとき}\right) \\ \infty & (\text{その他のとき}). \end{cases}$$

この定理は Brown 運動に関する 1 つの大偏差原理であり, それは Strassen の関数型重複大数の法則に応用されたり, 確率微分方程式を通して一般の多次元拡散過程の大偏

差原理を導くことに用いられたりする重要な定理である．この定理の証明はいくつか知られているが Varadhan [V] による熱核の短時間漸近挙動 (0.1.1) が用いた方法があることをコメントしておく．Ben-Arous と Kumagai [BK] は (0.1.2) が与えられていたことから同様の方法を用いて Sierpinski gasket F の上の Brown 運動 X に対して Schilder 型の大偏差原理の考察を行い, 以下のような結果を得た．$x \in F$ として P_x を $C_x([0,1] \to F) = \{\phi \in C([0,1] \to F) : \phi(0) = x\}$ の上の測度とし, $\epsilon_{n,z} = (2/5)^n z$ とする．任意の閉集合 $C \subset C_x([0,1] \to F)$, 開集合 G と $z \in [2/5, 1]$ に対して

$$-\inf_{\phi \in G} I(\phi) \leq \liminf_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log P_x[X_{\epsilon_{n,z}\cdot} \in G],$$
$$\limsup_{n\to 0} \epsilon_{n,z}^{1/(d_w-1)} \log P_x[X_{\epsilon_{n,z}\cdot} \in C] \leq -\inf_{\phi \in C} I(\phi).$$

ただし

$$I(\phi) = \begin{cases} \displaystyle\int_0^1 (\dot{\phi}(t))^{d_w/(d_w-1)} G\left(\frac{z}{\dot{\phi}(t)}\right) dt & (\phi \text{ が絶対連続のとき}) \\ \infty & (\text{その他のとき}). \end{cases}$$

ここで $\dot{\phi}(t) = \lim_{s\to t} d(\phi(s), \phi(t))/|s-t|$. これは SG の上では Schilder 型の大偏差原理は $\epsilon \to 0$ については成立しないが, 各 z について列 $\epsilon_{n,z}$ で $n \to \infty$ とするときには成立することを意味している．さらに [BK] ではこの結果の一つとして SG の上の Brown 運動のパスの挙動が Strassen 型重複大数の法則を満たすことも示している．フラクタルの上の大偏差原理については, 弱い形の大偏差原理を定式化した [BaK] や より広いクラスを扱った [HK] 等がある．証明で使われる手法は, 基本的に熱核の短時間での評価を利用するものであり [V] と同様の議論であるといえる．

0.2 本論文の概説

本論文 では Scale irregular Sierpinski gasket (以下では SISG とも書く) の上の Brown 運動の熱核に対する短時間漸近挙動, Schilder 型の大偏差原理について考察を行った．

0.2.1 Scale irregular Sierpinski gasket の構成と既知の結果

ここでは最も簡単な場合の SISG について述べる．一般の SISG については [BH] を参照のこと．本論文では全ての結果を最も単純な場合で述べるが一般の場合に対しても同様の証明を適用することができる．

Scale irregular Sierpinski gasket の構成

図 2 にある三角形の一辺の分割数と分割した後の上向きの三角形の個数を表す量として $(b(2), m(2)) = (2,3)$, $(b(3), m(3)) = (3,6)$ とする．$F_0 = \{a_1, a_2, a_3\}$ を $\mathbb{R}^2$ の正三角形の頂点の集合とする．ただし $a_1 = 0$ とする．各 i $(1 \leq i \leq m(2))$ に対して写像 $\psi_i^{(2)} : \mathbb{R}^2 \to \mathbb{R}^2$ を

$$\psi_i^{(2)}(x) = \frac{1}{b(2)}(x - a_i) + a_i$$

によって定義する．続いて a_4, a_5, a_6 を正三角形の各辺の中点として, 各 i $(1 \leq i \leq m(3))$ に対して 写像 $\psi_i^{(3)} : \mathbb{R}^2 \to \mathbb{R}^2$ を

$$\psi_i^{(3)}(x) = \frac{1}{b(3)}(x - a_i) + a_i$$

で定義する．$\eta \in \{2,3\}^{\mathbb{N}}$ を *environment* と呼ぶことにする．特に全ての成分が 2 であるような environment を太字の **2**, 全ての成分が 3 であるような environment を太字の **3**, で表すことにする．任意の $k \in \mathbb{N}$ に対して射影 $\pi_k : \{2,3\}^{\mathbb{N}} \to \{2,3\}$ を $\pi_k \eta = \eta_k$, $\eta \in \{2,3\}^{\mathbb{N}}$ と定める．集合 $B \subset \mathbb{R}^2$ と $a = 2, 3$ に対して

$$\Phi^{(a)}(B) = \bigcup_{j=1}^{m(a)} \psi_j^{(a)}(B),$$

そして

$$\Phi_n^{(\eta)}(B) = \Phi^{(\pi_1 \eta)} \circ \cdots \circ \Phi^{(\pi_n \eta)}(B).$$

とする．以上の準備のもと

$$\bigcup_{n=1}^{\infty} \Phi_n^{(\eta)}(F_0).$$

の閉包を environment η に関する scale irregular Sierpinski gasket F^η と呼ぶことにする (図 3)．$F^{\mathbf{2}}$ は通常の Sierpinski gasket SG(2) (図 1 の左) であり $F^{\mathbf{3}}$ はその変種 SG(3) (図 1 の右) である．

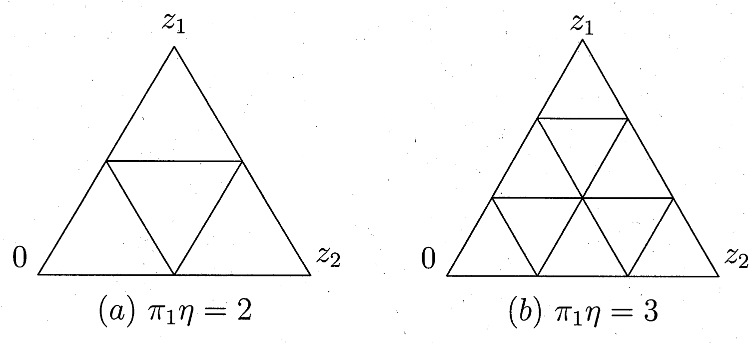

図 2: F_1^η

記号

長さ n の単語集合を $W_n^\eta = \{(w_1,\dots,w_n) : 1 \le w_i \le m(\pi_i\eta), 1 \le i \le n\}$ と書く．$w \in W_n^\eta$ に対して

$$\psi_w = \psi_{w_1}^{(\pi_1\eta)} \circ \cdots \circ \psi_{w_n}^{(\pi_n\eta)}. \tag{0.2.1}$$

と定め，$F_n^\eta = \cup_{w\in W_n^\eta}\psi_w(F_0)$ とする．また集合 $\psi_w(F_0)$ を n-cells と呼ぶ．x, y がともに同じ n-cell に入っているときに限り $\{x,y\}$ を辺とみなすことによって F_n^η の上に自然にグラフの構造を定義することができる．F_1^η を 図 2 のように設定する．$Y_\cdot^2$ を 図 2 (a) の上の 0 出発の simple random walk とし $Y_\cdot^3$ を 図 2 (b) の上の 0 出発の simple random walk とする．$Y_\cdot^2$ が $\{z_1, z_2\}$ へ最初に到達する時間を $W(Y_\cdot^2)$, $Y_\cdot^3$ が $\{z_1, z_2\}$ へ最初に到達する時間を $W(Y_\cdot^3)$ をとかくことにする．このとき SG(2) の time scaling factor $t(2)$ を 0 出発の $Y_\cdot^2$ が $\{z_1, z_2\}$ へ最初に到達する時間の平均 $E_0[W(Y_\cdot^2)]$ で定義する．SG(3) の time scaling factor $t(3)$ についても同様に定義する．simple random walk のマルコフ性を使った簡単な計算より $t(2) = 5$, $t(3) = 90/7$ を得る．

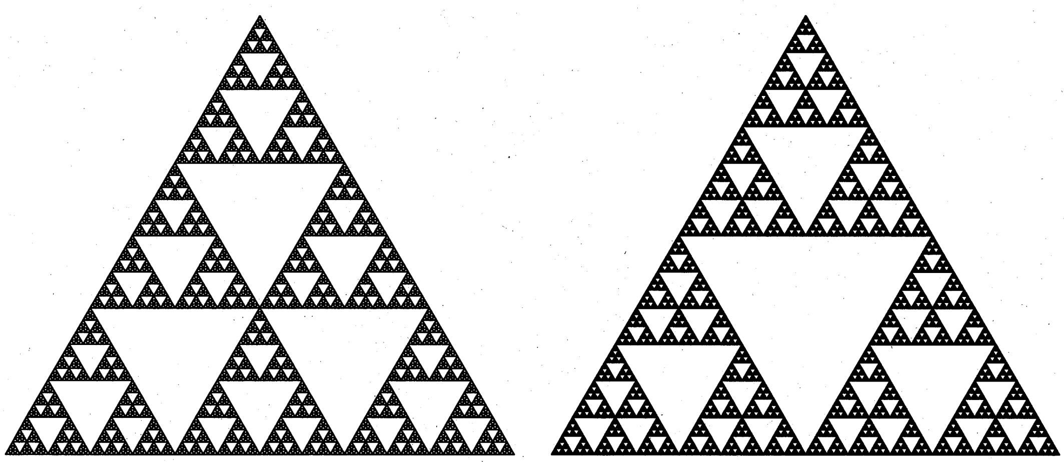

図 3: $\eta = \{3,2,2,3,3,3,\dots\}$ と $\eta = \{2,2,3,2,3,2,\dots\}$ の SISG

さらにいくつかの記号を用意する．各 $n \in \mathbb{N}$ に対して $B_n : \{2,3\}^{\mathbb{N}} \to (0,\infty)$, $M_n : \{2,3\}^{\mathbb{N}} \to (0,\infty)$ そして $T_n : \{2,3\}^{\mathbb{N}} \to (0,\infty)$ を

$$B_0(\eta) = M_0(\eta) = T_0(\eta) = 1,$$

$$B_n(\eta) = \prod_{i=1}^n b(\pi_i\eta), \quad M_n(\eta) = \prod_{i=1}^n m(\pi_i\eta), \quad T_n(\eta) = \prod_{i=1}^n t(\pi_i\eta) \tag{0.2.2}$$

によって定義する．また $d_w^\eta(n)$ と $d_s^\eta(n)$, $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$ を

$$d_w^\eta(n) = \frac{\log T_n(\eta)}{\log B_n(\eta)} = \frac{\sum_{i=1}^n \log t(\pi_i \eta)}{\sum_{i=1}^n \log b(\pi_i \eta)}, \quad d_s^\eta(n) = 2\frac{\log M_n(\eta)}{\log T_n(\eta)} = 2\frac{\sum_{i=1}^n \log m(\pi_i \eta)}{\sum_{i=1}^n \log t(\pi_i \eta)}$$

で定義する．任意の $n \in \mathbb{N}$ に対して

$$d_w^{\mathbf{2}}(n) = \frac{\log t(2)}{\log b(2)} = 2.321928\ldots, \quad d_w^{\mathbf{3}}(n) = \frac{\log t(3)}{\log b(3)} = 2.324660\ldots.$$

であることに注意．$d_w^{\mathbf{2}} = \log t(2)/\log b(2)$, $d_w^{\mathbf{3}} = \log t(3)/\log b(3)$ とおく．

F^η の上の距離と測度

F_n^η の上のグラフ距離を $\rho_n(x,y)$ で表すことにすると F^η の距離 d_η で次のような性質を持つものを定義することができる [BH]: 任意の $n \geq 1$ と $x, y \in F_n^\eta$ に対して

$$d_\eta(x,y) = B_n(\eta)^{-1}\rho_n(x,y)$$

であり, ある定数 $c > 0$ があって任意の $x, y \in F^\eta$ に対して

$$|x-y| \leq d_\eta(x,y) \leq c|x-y| \tag{0.2.3}$$

が成り立つ．また長さが $B_n(\eta)^{-1}$ である F^η の小三角形に測度 $M_n(\eta)^{-1}$ を与えるような "平坦" な測度 μ^η がある．

熱核の評価

各 $\eta \in \{2,3\}^{\mathbb{N}}$ に対して $L^2(F^\eta, \mu^\eta)$ 上の Dirichlet 形式を定義することができ Dirichlet 形式の理論より Feller 拡散過程 $(X_t,\ t \geq 0,\ P_x^\eta,\ x \in F^\eta)$ が存在することがわかる．これを F^η の上の Brown 運動と呼ぶことにする．[BH] による結果を述べる．$\eta \in \{2,3\}^{\mathbb{N}}$, $m, n \in \mathbb{N}$ に対して

$$k_\eta(m,n) = \inf\{j \geq 0 : T_{m+j}(\eta)/B_{m+j}(\eta) \geq T_n(\eta)/B_m(\eta)\}$$

とする．このとき次が成り立つ．

Theorem 0.2.1. Barlow-Hambly (1997)
$\eta \in \{2,3\}^{\mathbb{N}}$ とする．

(a) P_t^η は μ^η に関して連続な推移確率密度関数 $p_t^\eta(x,y)$ を持つ．

(b) $\eta \in \{2,3\}^{\mathbb{N}}$ によらない c_1, c_2, c_3, c_4 があって次を満たす．任意の $x, y \in F^\eta$, $t \in [0,1]$ が $B_m(\eta)^{-1} \le d_\eta(x,y) < B_{m-1}(\eta)^{-1}$, $T_n(\eta)^{-1} \le t < T_{n-1}(\eta)^{-1}$ をみたすとき

$$c_3 t^{-d_s^\eta(n)/2} \exp\Big(-c_4\big(\frac{d_\eta(x,y)^{d_w^\eta(m+k)}}{t}\big)^{1/(d_w^\eta(m+k)-1)}\Big) \le p_t^\eta(x,y)$$
$$\le c_1 t^{-d_s^\eta(n)/2} \exp\Big(-c_2\big(\frac{d_\eta(x,y)^{d_w^\eta(m+k)}}{t}\big)^{1/(d_w^\eta(m+k)-1)}\Big),$$

が成立．ただし $k = k_\eta(m,n)$.

0.2.2 本論文で得られた結果 - 熱核の短時間漸近挙動

本論文で得られた結果を述べるためにいくつか記号を用意する．まず最初にずらし作用素 $\theta : \{2,3\}^{\mathbb{N}} \to \{2,3\}^{\mathbb{N}}$ を $\pi_k\theta\eta = \pi_{k+1}\eta, \eta \in \{2,3\}^{\mathbb{N}}$ で定義する．また任意の $k \in \mathbb{Z}$ に対して $\pi_k^{\mathbb{Z}} : \{2,3\}^{\mathbb{Z}} \to \{2,3\}$ と $\theta_{\mathbb{Z}} : \{2,3\}^{\mathbb{Z}} \to \{2,3\}^{\mathbb{Z}}$ を $\mathbb{N}$ の場合と同様に定義する．混乱の恐れがないときは簡単のため添字 $\mathbb{Z}$ を省略して単に θ, π_k と書くことにする．また各 $\eta \in \{2,3\}^{\mathbb{N}}$ に対して関数 $\varphi : [0,1] \times \{2,3\} \to \mathbb{R}$ を

$$\varphi(u,k) = E_0^\eta[u^{W(Y^k)}]$$

で定義し, $h : [0,1] \times \{2,3\} \to [0,\infty)$ を

$$h(u,k) = -\log\frac{\varphi(u,k)}{u^{b(k)}}.$$

で定義する．ここで E_0^η は P_0^η に関する期待値を表す．最初に天下り的に次の定理を与える．$\delta > 0$ に対して $D_\delta = \{z \in \mathbb{C} : \mathrm{Re}(z) > -\delta\}$ とおく．

Theorem 0.2.2. 次を満たす関数 $g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ がただ一つ存在する．

(a) $g(z,\eta) = \varphi(g(z/T_1(\eta), \theta\eta), \pi_1\eta), \quad g(0,\eta) = 1, \quad g'(0,\eta) = -1,$

(b) $g(z,\eta)$ は各 $\eta \in \{2,3\}^{\mathbb{N}}$ に対して D_ϵ で正則.

(c) $g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ は連続.

(0.2.2) と同様に $B_{-n} : \{2,3\}^{\mathbb{Z}} \to \mathbb{R}$ と $T_{-n} : \{2,3\}^{\mathbb{Z}} \to \mathbb{R}$ を $n \ge 0$ に対して

$$B_{-n}(\xi) = \Big\{\prod_{i=0}^n b(\pi_{-i}\xi)\Big\}^{-1}, \quad T_{-n}(\xi) = \Big\{\prod_{i=0}^n t(\pi_{-i}\xi)\Big\}^{-1},$$

$n \geq 1$ に対して

$$B_n(\xi) = B_n(P\xi), \quad T_n(\xi) = T_n(P\xi),$$

で定義する．ここで $P : \{2,3\}^{\mathbb{Z}} \to \{2,3\}^{\mathbb{N}}$ は $\pi_k P(\xi) = \pi_k \xi$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $k \in \mathbb{N}$ で定義される射影を表す．また $d_w^{\xi}(n)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $n \in \mathbb{Z}$ を $n \geq 1$ に対して $d_w^{\xi}(n) = d_w^{P\xi}(n)$, $n \geq 0$ に対して

$$d_w^{\xi}(-n) = \frac{\log T_{-n}(\xi)}{\log B_{-n}(\xi)}$$

で定義する．関数 g を用いて関数 $\Psi : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ を次で定義する：

$$\Psi(s,\xi) = -\log g(s, P(\xi)) + \sum_{k=0}^{\infty} F_k(s,\xi). \tag{0.2.4}$$

ここで $F_k : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$, $k \geq 0$ は

$$F_k(s,\xi) = B_{-k}(\xi) h\big(g(s/T_{-(k-1)}(\xi), P(\theta^{-k}\xi)), \pi_{-k}\xi\big), \quad k \geq 1,$$

$$F_0(s,\xi) = \frac{1}{b(\pi_0\xi)} h\big(g(s, P(\xi)), \pi_0\xi\big)$$

である．更に各 $\xi \in \{2,3\}^{\mathbb{Z}}$ に対して Ψ の Legendre 変換 $\Psi^* : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ を

$$\Psi^*(z,\xi) = \sup_{s>0} \{\Psi(s,\xi) - zs\}$$

で定義する．Ψ^* は次の性質をもつ．

Theorem 0.2.3. (1) $\Psi^* : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ は連続．

(2) 次を満たす定数 $c, c' > 0$ があって，任意の $z > 0$ と $\xi \in \{2,3\}^{\mathbb{Z}}$ に対して

$$c z^{-1/(d_w^{\xi}(n)-1)} \leq \Psi^*(z,\xi) \leq c' z^{-1/(d_w^{\xi}(n)-1)}$$

が成り立つ．ここで n は $B_n(\xi)/T_n(\xi) \leq z < B_{n-1}(\xi)/T_{n-1}(\xi)$ を満たす整数．

(3) 任意の $\xi \in \{2,3\}^{\mathbb{Z}}$ と $z > 0$ に対して

$$\Psi^*(z,\xi) = \begin{cases} B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi), \theta^n\xi) & (n \geq 1 \text{ のとき}), \\ B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi), \theta^{n-1}\xi) & (n < 0 \text{ のとき}) \end{cases} \tag{0.2.5}$$

が成立．

各 $\eta \in \{2,3\}^{\mathbb{N}}$ に関して $\chi_\eta \in \{2,3\}^{\mathbb{Z}}$ を $k \geq 1$ に対して $\pi_k\chi_\eta = \pi_k\eta$, $k \leq 0$ に対して $\pi_k\chi_\eta = 2$ となるものとする．このとき関数 Ψ^* を使って熱核の短時間漸近挙動を次のように述べることができる．

Theorem 0.2.4. 任意のコンパクト集合 $K \subset (0,\infty)$ に対して

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y\in F^\eta \\ z\in K}} \Biggl| \left(\frac{B_n(\eta)}{T_n(\eta)}\right)^{1/(d_w^\eta(n)-1)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y)\Psi^*\left(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\right)\Biggr| = 0.$$

次の系がただちに従う．

Corollary 0.2.5. $\eta \in \{2,3\}^{\mathbb{N}}$ とする．$\{\theta^n\chi_\eta\}_{n\in\mathbb{N}}$ の 部分列 $\{\theta^{n_k}\chi_\eta\}_{k\in\mathbb{N}}$ が $\xi_0 \in \{2,3\}^{\mathbb{Z}}$ に収束するとき

$$\lim_{k\to\infty} \left(\frac{B_{n_k}(\eta)}{T_{n_k}(\eta)}\right)^{1/(d_w^\eta(n_k)-1)} \log p^\eta_{\frac{B_{n_k}(\eta)}{T_{n_k}(\eta)}z}(x,y) = -d_\eta(x,y)\Psi^*\left(\frac{z}{d_\eta(x,y)}, \xi_0\right)$$

が成り立つ．この収束は $x, y \in F^\eta$ と $z \in K$ に関して一様である．

定理から導かれる系をいくつか挙げる．

Corollary 0.2.6. 任意の $\delta_0 > 0$ に対して

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y)\geq\delta_0} \left| \frac{1}{\Psi^*\Bigl(t/d_\eta(x,y), \chi_\eta\Bigr)} \log p^\eta_t(x,y) + d_\eta(x,y) \right| = 0$$

次の表示は SG の熱核の短時間での挙動 (0.1.2) に近いものである．

Corollary 0.2.7. $K \subset (0,\infty)$ をコンパクト集合とする．次を満たす各 $z \in K$ に依存する関数 $V_z : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ と K に依存する定数 $c = c(K), c' = c'(K) > 0$ がある: 任意の $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z \in K$ に対して $c \leq V_z(s,\xi) \leq c'$ であり

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y\in F^\eta \\ z\in K}} \Biggl| \left(\frac{B_n(\eta)}{T_n(\eta)}\right)^{1/(d_w^\eta(n)-1)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y)^{d_w^{\theta^n\chi_\eta}(m)/(d_w^{\theta^n\chi_\eta}(m)-1)} V_z\left(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\right)\Biggr| = 0.$$

ここで $m = m_{\eta,n,z,x,y}$ は

$$B_m(\theta^n\chi_\eta)/T_m(\theta^n\chi_\eta) \leq z/d_\eta(x,y) < B_{m-1}(\theta^n\chi_\eta)/T_{m-1}(\theta^n\chi_\eta)$$

を満たす整数．

最後に (0.1.1) との比較として次を挙げる.

Corollary 0.2.8. 次を満たす関数 $G:(0,\infty)\times\{2,3\}^{\mathbb{Z}}\to(0,\infty)$ と定数 $c,c'>0$ がある: 任意の $s\in(0,\infty)$, $\xi\in\{2,3\}^{\mathbb{Z}}$ に対して $c\le G(s,\xi)\le c'$ であり

$$\lim_{t\to0}\sup_{\eta\in\{2,3\}^{\mathbb{N}}}\sup_{x,y\in F^\eta}\Bigg|t^{1/(d_w^{\chi_\eta}(m)-1)}\log p_t^\eta(x,y) + d_\eta(x,y)^{d_w^{\chi_\eta}(m)/(d_w^{\chi_\eta}(m)-1)}G\Big(\frac{t}{d_\eta(x,y)},\chi_\eta\Big)\Bigg|=0.$$

ここで $m=m_{\eta,t,x,y}$ は $B_m(\chi_\eta)/T_m(\chi_\eta)\le t/d_\eta(x,y)<B_{m-1}(\chi_\eta)/T_{m-1}(\chi_\eta)$ を満たす整数.

0.2.3 本論文で得られた結果 - 大偏差原理

ここでは $x\in F^\eta$ となる $\eta\in\{2,3\}^{\mathbb{N}}$ が少なくとも一つは存在する $x\in\mathbb{R}^2$ のみを考える. 以下ではそのような $x\in\mathbb{R}^2$ を一つ固定する. $\{2,3\}^{\mathbb{N}}_x=\{\eta\in\{2,3\}^{\mathbb{N}}:x\in F^\eta\}$ とおく. $\eta\in\{2,3\}^{\mathbb{N}}$ に対して $C_x([0,1]\to F^\eta)=\{\phi\in C([0,1]\to F^\eta):\phi(0)=x\}$ とおく. 同様に $C_x([0,1]\to\mathbb{R}^2)=\{\phi\in C([0,1]\to\mathbb{R}^2):\phi(0)=x\}$ とおく. $\phi\in C_x([0,1]\to F^\eta)$ が距離 d_η に関して絶対連続であるとは任意の $\epsilon>0$ に対して, $\delta>0$ が存在して $[0,1]$ に含まれてどの 2 つも交わらないような区間の有限列 $\{(t_{i-1},t_i)\}_{i=1}^n$ が $\sum_i^n(t_i-t_{i-1})<\delta$ を満たすかぎりいつも $\sum_{i=1}^n d_\eta(\phi(t_i),\phi(t_{i-1}))<\epsilon$ となることである. このとき $[0,1]$ 上ほとんどいたるところで $\lim_{s\to t}d_\eta(\phi(s),\phi(t))/|s-t|$ が存在する. これを d_η に関する微分として $D_\eta\phi(t)$ で表すことにする:

$$D_\eta\phi(t)=\lim_{s\to t}d_\eta(\phi(s),\phi(t))/|s-t|.$$

ユークリッド距離との同値性 (0.2.3) より $\phi\in C_x([0,1]\to F^\eta)$ に対しては, ユークリッド距離に関する絶対連続性と d_η に関する絶対連続性は同値である. $z>0$, $\eta\in\{2,3\}^{\mathbb{N}}_x$ に対して関数 $I^\eta_{x,z}:C_x([0,1]\to\mathbb{R}^2)\times\{2,3\}^{\mathbb{Z}}\to[0,\infty]$ を

$$I^\eta_{x,z}(\phi,\xi)=\begin{cases}\displaystyle\int_0^1 D_\eta\phi(t)\Psi^*\left(\frac{z}{D_\eta\phi(t)},\xi\right)dt & \phi\in C_x([0,1]\to F^\eta)\ \text{かつ絶対連続のとき},\\ \infty & \text{その他のとき}\end{cases}$$

で定義する. $\eta\notin\{2,3\}^{\mathbb{N}}_x$, $z>0$ に対しては任意の $\xi\in\{2,3\}^{\mathbb{Z}}$, $\phi\in C_x([0,1]\to\mathbb{R}^2)$ に対して $I^\eta_{x,z}(\phi,\xi)=\infty$ と定義しておく. 次の性質を満たすことからこれは good rate function と呼ぶにふさわしいものであることがわかる.

Lemma 0.2.9. 各 $\eta \in \{2,3\}^{\mathbb{N}}$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z > 0$ に対して $I^{\eta}_{x,z}(\cdot,\xi) : C_x([0,1] \to \mathbb{R}^2) \to [0,\infty]$ は下半連続であり, 各 $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z > 0$ に対して任意の $N \geq 0$ で集合 $\{\phi \in C_x([0,1] \to \mathbb{R}^2) : I^{\eta}_{x,z}(\phi,\xi) \leq N\}$ はコンパクトである.

また $A = \{f \in C_x([0,1] \to \mathbb{R}^2) : f(1) = y\}$ とするとき $\inf_{\phi \in A} I^{\eta}_{x,z}(\phi,\xi)$ を達成するのは $z > 0$ にも $\xi \in \{2,3\}^{\mathbb{Z}}$ によらない x から y に一定速度で動く geodesic であることが分かる. もしもこの rate function で大偏差原理のようなものが成り立つとすると "最も起こりやすいパス" は geodesic であるが, そのエネルギーは z, ξ に依存すると, 標語的にいうことができる.
以下簡単のため $\epsilon^z_n(\eta) = B_n(\eta)z/T_n(\eta)$ とおく. また以下では K を $(0,\infty)$ のコンパクト集合, $\mathcal{B}(C_x([0,1] \to \mathbb{R}^2)$ を $C_x([0,1] \to \mathbb{R}^2)$ の上の Borel σ-algebra とする. $(C_x([0,1] \to \mathbb{R}^2), \mathcal{B}(C_x([0,1] \to \mathbb{R}^2))$ 上の測度の族 $\{P^{\eta}_x \circ X^{-1}_{\epsilon^z_n(\eta)} .\ \}_{n \in \mathbb{N}}$ に対して次の弱い大偏差型定理が成り立つ. まず上からの評価を述べる. $\alpha > 0$ に対して

$$\mathcal{F}_{\alpha,K} = \left\{ C \in \mathcal{B}(C_x([0,1] \to \mathbb{R}^2)) : \sup_{\substack{\xi \in \{2,3\}^{\mathbb{Z}}, z \in K \\ \eta \in \{2,3\}^{\mathbb{N}}}} \inf_{\phi \in C} I^{\eta}_{x,z}(\phi,\xi) \leq \alpha \right\}$$

とおく. また $B(f,r) = \{\phi \in C_x([0,1] \to \mathbb{R}^2) : \sup_{0 \leq t \leq 1} |g(t) - f(t)| < r\}$ とする.

Proposition 0.2.10. $\delta > 0$ とする. 任意の $\epsilon > 0$ に対してある $N \in \mathbb{N}$ があって $n \geq N$ のとき任意の $z \in K, \eta \in \{2,3\}^{\mathbb{N}}, C \in \mathcal{F}_{\alpha,K}$ に対して

$$\frac{1}{B_n(\eta)} \log P^{\eta}_x[\omega(\epsilon^z_n(\eta)\ \cdot\ ) \in C] \leq -\inf_{\phi \in C_\delta} I^{\eta}_{x,z}(\phi, \theta^n \chi_\eta) + \epsilon.$$

ただし $C_\delta = \bigcup_{\psi \in C} B(\psi,\delta)$.

次に下からの評価を述べる. 各 $\eta \in \{2,3\}^{\mathbb{N}}$ に対して集合 $C_\eta \subset C_x([0,1] \to F^\eta) \subset C_x([0,1] \to \mathbb{R}^2)$ をとって

$$\bigcup_{\eta \in \{2,3\}^{\mathbb{N}}} C_\eta \quad \text{は同程度連続を仮定する.}$$

このとき次が成り立つ.

Proposition 0.2.11. $r > \delta > 0$ とする. 任意の $\epsilon > 0$ に対してある $N \in \mathbb{N}$ があって $n \geq N$ のとき任意の $z \in K, \eta \in \{2,3\}^{\mathbb{N}}, f \in C_\eta$ に対して

$$\frac{1}{B_n(\eta)} \log P^{\eta}_x[\omega(\epsilon^z_n(\eta)\ \cdot\ ) \in B(f,r)] \geq -\inf_{\phi \in B(f,r-\delta)} I^{\eta}_{x,z}(\phi, \theta^n \chi_\eta) + \epsilon.$$

上記のようにある性質を満たす集合に対しての一様性のある形で大偏差原理を示したのは以下で示す Varadhan の定理を導くためである. その前に exponentially tightness について述べておく. これは上からの評価より示すことができる.

Proposition 0.2.12. 任意の $\alpha > 0$ に対してコンパクト集合 $K_\alpha \subset C_x([0,1] \to \mathbb{R}^2)$ があって

$$\limsup_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}, z\in K} \frac{1}{B_n(\eta)} \log P_x^\eta[\omega(\epsilon_n^z(\eta)\,\cdot\,) \in K_\alpha^c] < -\alpha$$

が成り立つ.

Proposition 0.2.10 と Proposition 0.2.11 を使って次の Varadhan の定理にあたるものを示すことができる. 確率測度 $P_x^\eta \circ X_{\epsilon_n^z(\eta)}^{-1}$ に関する期待値を $E^{P_{\epsilon_n^z}^\eta}$ で表すことにする.

Theorem 0.2.13. 任意の有界連続関数 $\Psi : C_x([0,1] \to \mathbb{R}^2) \to \mathbb{R}$ に対して

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}_x^{\mathbb{N}}, z\in K} \left| \frac{1}{B_n(\eta)} \log E^{P_{\epsilon_n^z}^\eta}[\exp(B_n(\eta)\Psi)] - \sup_{\psi\in C_x([0,1]\to\mathbb{R}^2)} (\Psi(\psi) - I_{x,z}^\eta(\psi, \theta^n\chi_\eta)) \right| = 0$$

が成り立つ.

Proposition 0.2.10 と Proposition 0.2.11 で見られるように rate function $I_{x,z}^\eta(\psi, \theta^n\chi_\eta)$ が n とともに変化することが障害となって今のところ上で挙げたような大偏差原理しか得られていない. 0.1.3 節で紹介した一般論では exponentially tight と Varadhan の定理が成り立つことと大偏差原理が成り立つことは同値である. 本論文で扱っているケースはその一般論には含まれないがそれらに対応する Proposition 0.2.12 と Theorem 0.2.13 が言えていることから, よりよい大偏差原理が導けるのではないかと期待している. また ξ が変化するときに $I_{x,z}^\eta(\phi, \xi)$ の挙動が問題となることが多いことから, $I_{x,z}^\eta(\phi, \xi)$ の ξ に関するふるまいをより詳しく調べる必要があると考えている.

Chapter 1

Introduction

Fractals are ideal examples of the disordered media. We cannot define differential calculus on fractals because of the lack of smoothness. So a big problem is how to analyze diffusion phenomena in a rigorous way. Probabilists try to solve the problem by constructing a diffusion processes on fractal. The first work was the construction of Brownian motion on the Sierpinski gasket done by Goldstein [G] and Kusuoka [Kus]. Then Barlow-Perkins [BP] shows the existence of the transition probability density of Brownian Motion and the estimation for them on the Sierpinski gasket. Let F be the Sierpinski gasket on $\mathbb{R}^2$ with an intrinsic geodesic metric d, called a shortest path metric. There exist constants $c_1, c_2 > 0$ such that if $x, y \in F$ and $t > 0$ then

$$p_t(x,y) \leq \frac{c_1}{t^{d_s/2}} \exp\Big(- c_2\big(\frac{d(x,y)^{d_w}}{t}\big)^{1/(d_w-1)}\Big)$$

with a lower bound of the same form but different constants, where $d_w = 2.321928\ldots$ is the walk dimension and $d_s = 1.365212\ldots$ is the spectral dimension of F.

Since in this paper we study Varadhan type short time asymptotic estimates of heat kernels and Schilder type large deviation for Brownian motion on scale irregular Sierpinski gasket, We look back on the history concerning them a little. Varadhan's result [V1] is famous about the study of short time asymptotic behavior of heat kernel $p_t^M(x,y)$ on Riemannian manifold M:

$$\lim_{t\to 0} t \log p_t^M(x,y) = \frac{\rho(x,y)}{2},$$

where ρ is Riemannian metric. Since then, we have tried generalization of this result in various direction. In particular the result by Hino and Ramirez [HR] is remarkably general. However, many diffusions on fractals are not contained in the framework of [HR]. Kumagai [Kum] gives that Varadhan's type formula for short time asymptotic with respect to the transition probability density $p_t(x,y)$ of Brownian Motion on the Sierpinski gasket: for $x,y\in F$ and $z\in[2/5,1]$,

$$\lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n z\right)^{1/(d_w-1)}\log p_{(\frac{2}{5})^n z}(x,y)=-d(x,y)^{d_w/(d_w-1)}G\Big(\frac{z}{d(x,y)}\Big),$$

where d is an intrinsic geodesic metric called a shortest path metric on the Sierpinski gasket and G is the same periodic non-constant positive continuous function with $G(5s/2)=G(s)$. This G is defined as a Legendre transform of some limiting function of a Laplace transform of some hitting time of Brownian motion. Also Kumagai shows that

$$\lim_{t\to 0}H(t)t^{1/(d_w-1)}\log p_t(x,y)$$

does not exist for any choice of bounded functions H. Schilder [Sc] surveys on large deviations for Brownian motion B in $\mathbb{R}^d$. Let P_x be Wiener measures starting x on $C([0,1]\to\mathbb{R}^d)$. For any closed set $C\subset C([0,1]\to\mathbb{R}^d)$ and open set G,

$$-\inf_{\phi\in G}I(\phi)\le\liminf_{\epsilon\to 0}\epsilon\log P_x[B_{\epsilon\cdot}\in G]$$
$$\limsup_{\epsilon\to 0}\epsilon\log P_x[B_{\epsilon\cdot}\in C]\le-\inf_{\phi\in C}I(\phi),$$

where

$$I(\phi)=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^1|\phi'(t)|^2dt & \text{if } \phi \text{ is absolutely continuous,}\\ \infty & \text{otherwise.}\end{cases}\tag{1.0.1}$$

On the other hand, Ben-Arous and Kumagai [BK] study about large deviation on the path space of the Sierpinski gasket. Let $C_x([0,1]\to F)=\{\omega\in C([0,1]\to F):\omega(0)=x\}$ with uniformly continuous topology. Let X be Brownian motion on F. Then the following holds. For each $z\in[2/5,1]$, closed set C and open set $G\subset C_x([0,1]\to F)$,

$$-\inf_{\phi\in G}I_{x,z}(\phi)\le\liminf_{n\to\infty}\epsilon_{n,z}^{1/(d_w-1)}\log P_x[X_{\epsilon_{n,z}\cdot}\in G],\tag{1.0.2}$$
$$\limsup_{n\to 0}\epsilon_{n,z}^{1/(d_w-1)}\log P_x[X_{\epsilon_{n,z}\cdot}\in C]\le-\inf_{\phi\in C}I_{x,z}(\phi).$$

Here $\epsilon_{n,z} = (2/5)^n z$ and $\{I_{x,z}\}_{z\in[2/5/1]}$ is a sequence of rate functions defined as follows for each $\phi \in C_x([0,1] \to F)$,

$$I_{x,z}(\phi) = \begin{cases} \displaystyle\int_0^1 D\phi(t)^{d_w/(d_w-1)} G\left(\frac{z}{D\phi(t)}\right) dt & \text{if } \phi \text{ is absolutely continuous,} \\ \infty & \text{otherwise,} \end{cases}$$

where $D\phi(t) = \lim_{s\to t} d(\phi(s), \phi(t))/|s-t|$ for $t \in [0,1]$. This result tells us that the classical Schilder-type large deviation does not hold when $\epsilon \to 0$.

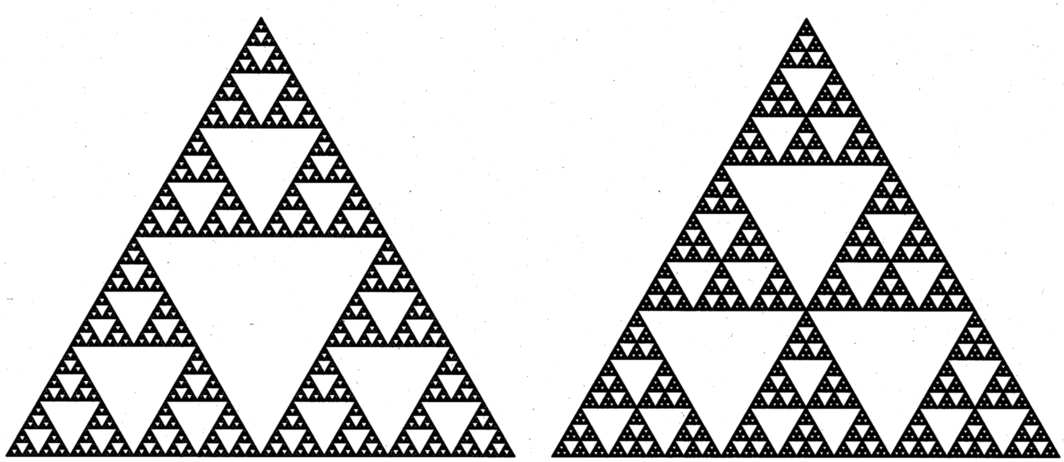

Figure 1.1: The standard Sierpinski gasket SG(2) and a variant SG(3)

Barlow-Hambly [BH] introduced scale irregular Sierpinski gaskets and showed the existence of Brownian Motion on them and that of the transition probability density. They also gave the estimation for transition probability density functions as we will state in subsection 2.1.2. In the present paper, we study short time asymptotic behaviors for them and large deviations on the path space of scale irregular Sierpinski gaskets. However we are obtaining only a weak result of large deviation at present. Instead, we show the one corresponding to Varadhan's theorem. This theorem for the case Sierpinski gasket is the following: $\Phi : C_x([0,1] \to F) \to \mathbb{R}$ be a bounded continuous function. Then

$$\lim_{n\to\infty} \epsilon_{n,z}^{1/(d_w-1)} \log E_x\left[\exp\left(\frac{\Phi(\omega(\epsilon_{n,z}\,\cdot\,))}{\epsilon_{n,z}^{1/(d_w-1)}}\right)\right] = \sup_{\phi\in C_x([0,1]\to F)} (\Phi(\phi) - I_{x,z}(\phi)).$$

The above result is immediately obtained from (1.0.2) by general argument of large deviation theory. See chapter 4 in [DZ] for example. Note that our settings are not contained in a general frame of large deviation theory.

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Chapter 2

Short time asymptotic behavior

2.1 Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets

2.1.1 Scale irregular Sierpinski gaskets

We describe the construction of the simplest scale irregular Sierpinski gaskets. This paper is restricted to the situation, see [BH] for a more detail account of the general setting. We set $(b(2), m(2)) = (2, 3)$ and $(b(3), m(3)) = (3, 6)$. Note that $b(i)$ and $m(i)$ are the length and mass scaling factors on SG(i) for $i = 2, 3$, see Figure 1.1. Let $a_1 = 0$ and $F_0 = \{a_1, a_2, a_3\}$ be the set of vertices of a unit equilateral triangle in $\mathbb{R}^2$. We define $\psi_i^{(2)} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_i^{(2)}(x) = \frac{1}{b(2)}(x - a_i) + a_i \quad \text{for each} \quad 1 \leq i \leq m(2).$$

Let a_4, a_5, a_6 be the midpoints of the 3 sides of F_0 and define $\psi_i^{(3)} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi_i^{(3)}(x) = \frac{1}{b(3)}(x - a_i) + a_i \quad \text{for each} \quad 1 \leq i \leq m(3).$$

We call $\eta \in \{2, 3\}^{\mathbb{N}}$ an *environment*. In particular we denote the element in $\{2, 3\}^{\mathbb{N}}$ whose all components are 2 (resp. 3) by $\mathbf{2}$ (resp. $\mathbf{3}$). Define the projection $\pi_k : \{2, 3\}^{\mathbb{N}} \to \{2, 3\}$ by $\pi_k \eta = \eta_k, \eta \in \{2, 3\}^{\mathbb{N}}$ for each $k \in \mathbb{N}$ and the left shift operator $\theta : \{2, 3\}^{\mathbb{N}} \to \{2, 3\}^{\mathbb{N}}$ by $\pi_k \theta \eta = \pi_{k+1} \eta, \eta \in \{2, 3\}^{\mathbb{N}}, k \in \mathbb{N}$. We denote by χ_η an element of $\{2, 3\}^{\mathbb{Z}}$ such that

$\pi_k \chi_\eta = \eta_k$ if $k \in \mathbb{N}$ and $\pi_k \chi_\eta = 2$ otherwise for $\eta \in \{2,3\}^{\mathbb{N}}$. For $B \subset \mathbb{R}^2$ set

$$\Phi^{(a)}(B) = \bigcup_{j=1}^{m(a)} \psi_j^{(a)}(B) \quad \text{for each } a = 2,3 \tag{2.1.1}$$

and

$$\Phi_n^{(\eta)}(B) = \Phi^{(\pi_1\eta)} \circ \cdots \circ \Phi^{(\pi_n\eta)}(B).$$

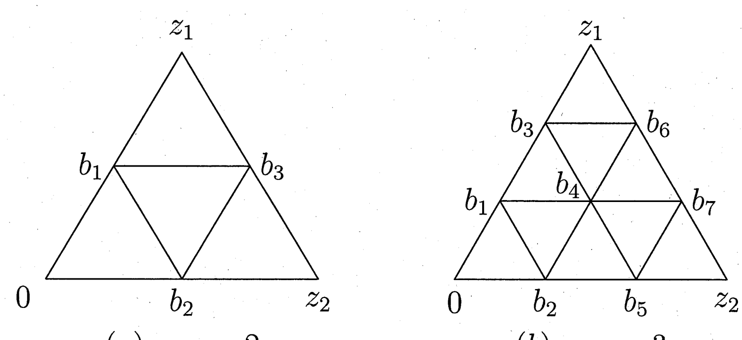
Then the scale irregular Sierpinski gasket F^η associated with the environment sequence η is defined by the closure of

$$\bigcup_{n=1}^{\infty} \Phi_n^{(\eta)}(F_0).$$

Note that $F^{\mathbf{2}}$ is the standard Sierpinski gasket SG(2) and $F^{\mathbf{3}}$ is a variant SG(3), see Figure 1.1 . We write $W_n^\eta = \{(w_1, \dots, w_n) : 1 \le w_i \le m(\pi_i\eta), 1 \le i \le n\}$ for the set of words of length n. For $w \in W_n^\eta$ we define

$$\psi_w = \psi_{w_1}^{(\pi_1\eta)} \circ \cdots \circ \psi_{w_n}^{(\pi_n\eta)}. \tag{2.1.2}$$

We define $F_n^\eta = \cup_{w \in W_n^\eta} \psi_w(F_0)$, and call sets of the form $\psi_w(F_0)$ n-cells for $w \in W_n^\eta$. We define a natural graph structure on F_n^η by letting $\{x,y\}$ be an edge if and only if x,y both belong to the small n-cell. This graph is connected. Write $\rho_n(x,y)$ for the graph distance in F_n^η.



(a) $\pi_1\eta = 2$ (b) $\pi_1\eta = 3$

Figure 2.1: F_1^η

Let F_1^η be as in Figure 2.1. Also let $Y^2_\cdot$ be a simple random walk on Figure 2.1 (a), starting at 0 and $Y^3_\cdot$ a simple random walk on Figure 2.1 (b), starting at 0. Then the time scaling factor $t(2)$ associated SG(2) is defined by the expectation of the first hitting time to $\{z_1, z_2\}$ by $Y^2_\cdot$ starting 0. The time scaling factor $t(3)$ associated SG(3) is defined similarly. By easy calculations we see that $t(2) = 5$ and $t(3) = 90/7$. Let $B_n : \{2,3\}^{\mathbb{N}} \to [1,\infty)$, $M_n : \{2,3\}^{\mathbb{N}} \to [1,\infty)$ and $T_n : \{2,3\}^{\mathbb{N}} \to [1,\infty)$ be given by $B_0(\eta) = M_0(\eta) = T_0(\eta) = 1$ and

$$B_n(\eta) = \prod_{i=1}^{n} b(\pi_i\eta), \quad M_n(\eta) = \prod_{i=1}^{n} m(\pi_i\eta), \quad T_n(\eta) = \prod_{i=1}^{n} t(\pi_i\eta) \tag{2.1.3}$$

for each $n \in \mathbb{N}$.

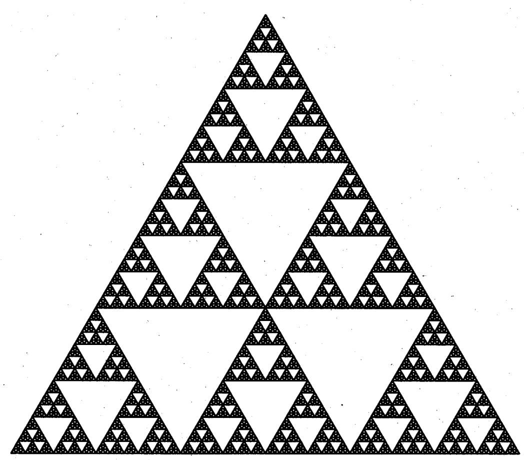


Figure 2.2: The scale irregular Sierpinski gasket about $\eta = \{3, 2, 2, 3, 3, \cdots\}$

In [BH], Barlow-Hambly have defined a metric d_η on F^η which have following properties:

$$d_\eta(x,y) = B_n(\eta)^{-1}\rho_n(x,y) \quad \text{for all } \ x,y \in F_n^\eta \ \text{ and } \ n \geq 0. \tag{2.1.4}$$

$$\text{There exists a constant } \ c > 0 \ \text{ such that} \tag{2.1.5}$$
$$|x-y| \leq d_\eta(x,y) \leq c|x-y| \quad \text{for any } \ x,y \in F^\eta.$$

Let $B^\eta(x,r) = \{y \in F^\eta : d_\eta(x,y) < r\}$ for each $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $r > 0$. There is a natural 'flat' measure μ^η on F^η which is characterized by the property that it assigns mass $M_n(\eta)^{-1}$ to each triangle in F^η of side $B_n(\eta)^{-1}$. In addition the following property is stated in [BH]: there are constants c, c' such that if $1/B_n(\eta) \leq r < 1/B_{n-1}(\eta)$, then

$$c' r^{d_f^\eta(n)} \leq \mu^\eta(B^\eta(x,r)) \leq c r^{d_f^\eta(n)}, \ x \in F^\eta. \tag{2.1.6}$$

We now define the approximate walk and spectral dimensions by

$$d_w^\eta(n) = \frac{\log T_n(\eta)}{\log B_n(\eta)} = \frac{\sum_{i=1}^n \log t(\pi_i\eta)}{\sum_{i=1}^n \log b(\pi_i\eta)}, \quad d_s^\eta(n) = 2\frac{\log M_n(\eta)}{\log T_n(\eta)} = 2\frac{\sum_{i=1}^n \log m(\pi_i\eta)}{\sum_{i=1}^n \log t(\pi_i\eta)}$$

for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Note that

$$d_w^{\mathbf{2}}(n) = \frac{\log t(2)}{\log b(2)} = 2.321928\ldots, \quad d_w^{\mathbf{3}}(n) = \frac{\log t(3)}{\log b(3)} = 2.324660\ldots.$$

for each $n \in \mathbb{N}$. We set $d_w^{\mathbf{2}} = \log t(2)/\log b(2)$ and $d_w^{\mathbf{3}} = \log t(3)/\log b(3)$. It is easy to see that we have $d_w^{\mathbf{2}} \le d_w^\eta(n) \le d_w^{\mathbf{3}}$ for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $k_\eta(n,m)$, $\eta \in \{2,3\}^{\mathbb{N}}$, $m, n \in \mathbb{N}$ be given by

$$k_\eta(m,n) = \inf\{j \ge 0 : T_{m+j}(\eta)/B_{m+j}(\eta) \ge T_n(\eta)/B_m(\eta)\}. \tag{2.1.7}$$

2.1.2 Dirichlet form and Brownian motion

We now construct a Dirichlet form $\mathcal{E}^\eta$ on $L^2(F^\eta,\mu)$. See [BH] for details. Note that as F_n^η is a discrete set, the space $C(F_n^\eta)$ of continuous functions on F_n^η is just the space of all functions on F_n^η. For $f \in C(F_0)$ define

$$\mathcal{E}_0(f,g) = \frac{1}{2}\sum_{x,y\in F_0}(f(x)-f(y))(g(x)-g(y)).$$

Set $r(a) = t(a)/m(a)$. We call $r(a)$ the resistance scaling factor of SG(a). Set

$$R_n(\eta) = \prod_{i=1}^n r(\pi_i\eta),$$

$$\mathcal{E}_n^\eta(f,g) = R_n(\eta)\sum_{w\in W_n^\eta}\mathcal{E}_0(f\circ\psi_w, g\circ\psi_w).$$

The choice of $R_n(\eta)$ above ensures that Dirichlet forms $\mathcal{E}_n^\eta$ have the decimation property

$$\mathcal{E}_{n-1}^\eta(g,g) = \inf\{\mathcal{E}_n^\eta : f|_{F_{n-1}^\eta} = g\} \qquad \text{for } g \in C(F_{n-1}^\eta).$$

This property implies that if $f : F^\eta \to \mathbb{R}$ then $\mathcal{E}_n^\eta(f|_{F_n^\eta}, f|_{F_n^\eta})$ is non decreasing in n. This enables us to define a limit bilinear form $(\mathcal{E}^\eta,\mathcal{F})$ by

$$\mathcal{F}^\eta = \{f \in C(F^\eta) : \lim_{n\to\infty}\mathcal{E}_n^\eta(f,f) < \infty\},$$

and

$$\mathcal{E}^\eta(f,f) = \lim_{n\to\infty}\mathcal{E}_n^\eta(f,f), \qquad f \in \mathcal{F}^\eta.$$

Theorem 2.1.1. (Theorem 3.3 in [BH])

(a) The bilinear form $(\mathcal{E}^\eta, \mathcal{F}^\eta)$ is a regular local Dirichlet form on $L^2(F^\eta, \mu^\eta)$.

(b) There is a constant $c > 0$ such that $|f(x) - f(y)|^2 \leq c\mathcal{E}^\eta(f, f)$ for all $f \in \mathcal{F}^\eta$.

Let $\{P_t^\eta\}_{t\geq 0}$ be the semigroup of Markov operators associated with the Dirichlet form $(\mathcal{E}^\eta, \mathcal{F}^\eta)$ on $L^2(F^\eta, \mu^\eta)$. As $(\mathcal{E}^\eta, \mathcal{F}^\eta)$ is regular and local, there exists a Feller diffusion $(X_t,\ t \geq 0,\ P_x^\eta,\ x \in F^\eta)$ with semigroup $\{P_t^\eta\}_{t\geq 0}$, which is called *Brownian motion* on F^η in [BH]. Besides they remark that $G_\lambda^\eta = \int e^{\lambda t} P_t^\eta dt$ has a bounded symmetric density $u_\lambda^\eta(x, y)$ with respect to μ^η and $u_\lambda^\eta(x, \cdot)$ is continuous for each $x \in F^\eta$. They proved the following.

Theorem 2.1.2. *Let $\eta \in \{2, 3\}^{\mathbb{N}}$.*

(a) *P_t^η has a continuous transition probability density $p_t^\eta(x, y)$ with respect to μ^η.*

(b) *There exist constants c_1, c_2, c_3, c_4 (not depending on $\eta \in \{2, 3\}^{\mathbb{N}}$) such that if $B_m(\eta)^{-1} \leq d_\eta(x, y) < B_{m-1}(\eta)^{-1}$, $T_n(\eta)^{-1} \leq t < T_{n-1}(\eta)^{-1}$, then*

$$c_3 t^{-d_s^\eta(n)/2} \exp\Big(- c_4 \big(\frac{d_\eta(x, y)^{d_w^\eta(m+k)}}{t}\big)^{1/(d_w^\eta(m+k)-1)}\Big) \leq p_t^\eta(x, y)$$
$$\leq c_1 t^{-d_s^\eta(n)/2} \exp\Big(- c_2 \big(\frac{d_\eta(x, y)^{d_w^\eta(m+k)}}{t}\big)^{1/(d_w^\eta(m+k)-1)}\Big),$$

where $k = k_\eta(m, n)$.

2.2 Properties of moment generating function of W

For Brownian motion $X.$ on F^η, define the stopping times S^k and S_i^k by $S^k = S_0^k = \inf\{t \geq 0 : X_t \in F_k^\eta\}$ and

$$S_i^k = \inf\{t > S_{i-1}^k : X_t \in F_k^\eta \setminus \{X_{S_{i-1}^k}\}\} \quad \text{for} \quad i \in \mathbb{N}.$$

These are the times of the successive visits to F_k^η by $X.$. Also for $X.$, let $W = \inf\{t \geq 0 : X_t \in F_0^\eta \setminus \{X_0\}\}$. Let $Y_i^m = X_{S_i^m}$, then $Y_\cdot^m$ is a simple random walk on F_m^η. Similarly define stopping times $S^k(Y_\cdot^m)$, $S_i^k(Y_\cdot^m)$, $i \in \mathbb{N}$ and $W(Y_\cdot^m)$ by $S^k(Y_\cdot^m) = S_0^k(Y_\cdot^m) = \inf\{n \in \mathbb{Z}_+ : Y_n^m \in F_k^\eta\}$, $S_i^k(Y_\cdot^m) = \inf\{n > S_{i-1}^k(Y_\cdot^m) : Y_n^m \in F_k^\eta \setminus \{Y_{S_{i-1}^k(Y_\cdot^m)}\}\}$ for

$i \in \mathbb{N}$ and $W(Y^m_\cdot) = \inf\{n \in \mathbb{Z}_+ : Y^m_n \in F^\eta_0 \setminus \{Y^m_0\}\}$. Let us define the probability generating function of $W(Y^1_\cdot)$ with respect to P^η_0, $\varphi : [0,1] \times \{2,3\} \to \mathbb{R}$ by

$$\varphi(u, \pi_1\eta) = E^\eta_0[u^{W(Y^1_\cdot)}].$$

Note that $\pi_1\eta$ relates to only $\pi_1\eta$.

Remark 2.2.1. *By easy calculations we have*

$$\varphi(u,2) = \frac{u^2}{4-3u} \quad \textit{and} \quad \varphi(u,3) = \frac{u^3(u+6)}{u^4+12u^3-6u^2-96u+96}.$$

Also let $f_n(u,\eta)$ be a probability generating function of $W(Y^n_\cdot)$ with respect to P^η_0:

$$f_n(u,\eta) = E^\eta_0[u^{W(Y^n_\cdot)}].$$

Since we have $W(Y^n_\cdot) = \sum_{k=1}^{W(Y^1_\cdot)}(S^1_k(Y^n_\cdot) - S^1_{k-1}(Y^n_\cdot))$ with P^η_0-probability 1 and

$$E^\eta_0[u^{S^1_k(Y^n_\cdot)-S^1_{k-1}(Y^n_\cdot)}] = E^{\theta\eta}_0[u^{S^0_k(Y^{n-1}_\cdot)-S^0_{k-1}(Y^{n-1}_\cdot)}] = f_{n-1}(u,\theta\eta)$$

for each $k \in \mathbb{N}$, we have

$$f_n(u,\eta) = \varphi(f_{n-1}(u,\theta\eta), \pi_1\eta). \tag{2.2.1}$$

Lemma 2.2.1. *It follows that $E^\eta_0[W(Y^n_\cdot)] = T_n(\eta)$ and*

$$E^\eta_0[W(Y^n_\cdot)^2] = T_n(\eta)^2 \sum_{i=0}^{n-1} \frac{E^{\theta^i\eta}_0[W(Y^1_\cdot)^2]}{t(\pi_{i+1}\eta)T_{i+1}(\eta)}$$

for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Proof. By (2.2.1), we see that $f'_n(u,\eta) = f'_{n-1}(u,\theta\eta)\varphi'(f_{n-1}(u,\theta\eta),\pi_1\eta)$ for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Letting $u=1$, it follows that

$$f'_n(1,\eta) = E^\eta_0[W(Y^n_\cdot)] = t(\pi_1\eta)E^{\theta\eta}_0[W(Y^{n-1}_\cdot)].$$

Inductively we obtain $E^\eta_0[W(Y^n_\cdot)] = T_n(\eta)$. Similarly we have

$$f''_n(1,\eta) = f''_{n-1}(1,\theta\eta)\varphi'(1,\pi_1\eta) + f''_{n-1}(1,\theta\eta)^2\varphi''(1,\pi_1\eta)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \geq 2$. From this we obtain

$$E^\eta_0[W(Y^n_\cdot)^2] = t(\pi_1\eta)E^{\theta\eta}_0[W(Y^{n-1}_\cdot)^2] + T_{n-1}(\theta\eta)^2E^\eta_0[W(Y^1_\cdot)^2]$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \geq 2$. By easy inductive argument, we have

$$E_0^\eta[W(Y_\cdot^n)^2] = \sum_{i=0}^{n-1} T_i(\eta) T_{n-i-1}(\theta^{i+1}\eta)^2 E_0^{\theta^i \eta}[W(Y_\cdot^1)^2].$$

The relation $T_{n-i-1}(\theta^{i+1}\eta) = T_n(\eta)/T_{i+1}(\eta)$ implies our assertion. $\square$

We have the following theorem in [BH].

Theorem 2.2.2. (i) *$E_0^\eta[W] = 1$ for all $\eta \in \{2,3\}^{\mathbb{N}}$.* (ii) *$W(Y_\cdot^n)/T_n(\eta) \to W$ P_0^η-a.s. for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} E_0^\eta[|W(Y_\cdot^n)/T_n(\eta) - W|^2] \to 0$ as $n \to \infty$.*

Proof. All but the uniformity in L^2-convergence are stated in [BH]. Since we see that $\{W(Y_\cdot^n)/T_n(\eta)\}_{n\in\mathbb{N}}$ is the martingale relative to σ-algebra generated $W(Y_\cdot^n)/T_n(\eta)$ as the proof of Theorem 8.2 in [H], for any $m > n$

$$\begin{aligned} E_0^\eta\Big[\Big|\frac{W(Y_\cdot^m)}{T_m(\eta)} - \frac{W(Y_\cdot^n)}{T_n(\eta)}\Big|^2\Big] &= E_0^\eta\Big[\frac{W(Y_\cdot^m)^2}{T_m(\eta)^2}\Big] - E_0^\eta\Big[\frac{W(Y_\cdot^n)^2}{T_n(\eta)^2}\Big] \\ &= \sum_{i=n}^{m-1} \frac{E_0^{\theta^i\eta}[W(Y_\cdot^1)^2]}{t(\pi_{i+1}\eta)T_{i+1}(\eta)} \leq c\sum_{i=n}^{m-1}\frac{1}{t(2)^{i+2}} = \frac{c'}{t(2)^{n+2}}\Big(1 - \frac{1}{t(2)^{m-n}}\Big) \end{aligned}$$

by Lemma 2.2.1, where $c = \max_{\eta\in\{2,3\}^{\mathbb{N}}} E_0^\eta[W(Y_\cdot^1)^2] < \infty$. This implies the uniformity in L^2-convergence. $\square$

In addition we define functions $g(z,\eta)$, $g_n(z,\eta) : \{z \in \mathbb{C} : Re(z) \geq 0\} \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$, $n = 0, 1, 2, \ldots$ to be

$$g_n(z,\eta) = E_0^\eta\Big[\exp\Big(-z\frac{W(Y_\cdot^n)}{T_n(\eta)}\Big)\Big] \quad \text{and} \quad g(z,\eta) = E_0^\eta[\exp(-zW)].$$

Lemma 2.2.3. *For all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \geq 0$ and $n \geq 1$,*

$$g_{n+1}(T_1(\eta)s,\eta) = \varphi(g_n(s,\theta\eta),\pi_1\eta). \tag{2.2.2}$$

Proof. We see that

$$\begin{aligned} g_{n+1}(T_1(\eta)s,\eta) &= f_{n+1}\Big(\exp\Big(-\frac{s}{T_n(\theta\eta)}\Big),\eta\Big) \\ &= \varphi\Big(f_n\Big(\exp\Big(-\frac{s}{T_n(\theta\eta)}\Big),\theta\eta\Big),\pi_1\eta\Big) = \varphi(g_n(s,\theta\eta),\pi_1\eta) \end{aligned}$$

from (2.2.1). This completes the proof. $\square$

Let $G_n(s,\eta) = -\log g_n(s,\eta)$ for $s \geq 0$ and $\eta \in \{2,3\}^{\mathbb{N}}$. Noting that $\varphi(u, \pi_1\eta) = g_1(-T_1(\eta)\log u, \eta)$ for all $u \in (0,1]$ and $\eta \in \{2,3\}^{\mathbb{N}}$, by Lemma 2.2.3 we have

$$G_{n+1}(s,\eta) = G_1\Big(T_1(\eta)G_n\Big(\frac{s}{T_1(\eta)}, \theta\eta\Big), \eta\Big) \tag{2.2.3}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \geq 0$ and $n \geq 1$. The function $G_1(\cdot, \eta)$ is extensible to a holomorphic function in a neighborhood of the origin. Let $B(0,\delta) = \{z \in \mathbb{C} : |z| < \delta\}$ for $\delta > 0$. By easy calculation we have $G_1(0,\eta)=0$, $G_1'(0,\eta) = 1$. Therefore there are $\epsilon_0 > 0$ and $M > 0$ such that

$$|G_1(z,\eta)| \leq M|z| \quad \text{for any} \quad z \in B(0,\epsilon_0) \quad \text{and} \quad \eta \in \{2,3\}^{\mathbb{N}}.$$

Constants ϵ_0 and M are not depending on $\eta \in \{2,3\}^{\mathbb{N}}$, because G_1 depend on $\pi_1\eta = \eta_1 \in \{2,3\}$ only. Let $D_\delta = \{z \in \mathbb{C} : \mathrm{Re}(z) > -\delta\}$ for $\delta > 0$. We can show the following in the same way as Proposition 3.7 of [Kus].

Proposition 2.2.4. *Let $\epsilon_1 = (\epsilon_0/2) \wedge (1/(2M))$. Then for each $n \in \mathbb{N}$, there are holomorphic functions $H_n(z,\eta)$ defined in $B(0,\epsilon_1)$ such that*

$$G_n(z,\eta) = z(1 + H_n(z,\eta)), \quad |H_n(z,\eta)| \leq 2M|z| \quad \textit{for any} \quad \eta \in \{2,3\}^{\mathbb{N}}.$$

In particular for $\epsilon_2 = \epsilon_1/4$

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}, n \in \mathbb{N}} |g_n(-\epsilon_2, \eta)| \vee |g(-\epsilon_2, \eta)| < \infty.$$

Proof. We will prove this by induction. Assume that our assertion is true for n. Then we see that

$$\begin{aligned}
\Big|T_1(\eta)G_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big)\Big| &= |z|\Big(1 + \Big|H_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big)\Big|\Big) \\
&\leq |z|\big(1 + \frac{2M|z|}{T_1(\eta)}\big) \leq \epsilon_1\big(1 + \frac{2M\epsilon_1}{T_1(\eta)}\big) \leq \frac{\epsilon_0}{2}\big(1 + \frac{1}{T_1(\eta)}\big) \leq \epsilon_0
\end{aligned}$$

for $z \in B(0,\epsilon_1)$. Thus

$$\begin{aligned}
&H_{n+1}(z,\eta) \\
&= H_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big) + \Big(1 + H_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big)\Big)H_1\Big(T_1(\eta)G_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big), \eta\Big)
\end{aligned}$$

is well defined for $z \in B(0, \epsilon_1)$. Therefore it is easy to see that $G_{n+1}(\cdot, \eta)$ is extensible to a holomorphic function in $B(0, \epsilon_1)$ for each $\eta \in \{2,3\}^{\mathbb{N}}$ and

$$G_{n+1}(z, \eta) = G_1\Big(T_1(\eta) G_n\Big(\frac{z}{T_1(\eta)}, \theta\eta\Big), \eta\Big) = z(1 + H_{n+1}(z, \eta))$$

for all $z \in B(0, \epsilon_1)$ by (2.2.3). Also since $2M|z| < 1$ for $z \in B(0, \epsilon_1)$, we obtain

$$\begin{aligned}
&|H_{n+1}(z, \eta)| \\
&\qquad \le \frac{2M|z|}{T_1(\eta)} + \big(1 + \frac{2M|z|}{T_1(\eta)}\big) M|z| \le M|z| \big(\frac{4}{T_1(\eta)} + 1\big) \le 2M|z|.
\end{aligned}$$

This completes the induction. In particular we have

$$\sup_{n\in\mathbb{N}, \eta\in\{2,3\}^{\mathbb{N}}} g_n(-2\epsilon_2, \eta) = \sup_{n\in\mathbb{N}, \eta\in\{2,3\}^{\mathbb{N}}} E_0^{\eta}[\exp(2\epsilon_2 W(Y_\cdot^n)/T_n(\eta))] < \infty,$$

where $\epsilon_2 = \epsilon_1/4$. So the sequence $\{\exp(\epsilon_2 W(Y_\cdot^n)/T_n(\eta))\}_{n=1}^{\infty}$ is uniformly integrable. By the way we have already known that $W(Y_\cdot^n)/T_n(\eta) \to W$ almost surely as $n \to \infty$ from Theorem 2.2.2. These imply $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} E_0^{\eta}[\exp(\epsilon_2 W)] < \infty$. This completes our assertion. □

Let $\epsilon = \epsilon_2/4$. Then we have the following lemma.

Lemma 2.2.5. (1) *Functions $g_n(\cdot, \eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $g_n : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous for each $n \in \mathbb{N}$.*

(2) *For each compact set $K \subset D_\epsilon$,*

$$\sup_{z\in K, \eta\in\{2,3\}^{\mathbb{N}}} |g(z, \eta) - g_n(z, \eta)| \to 0 \quad \text{as} \quad n \to \infty.$$

In particular from (1) and (2), $g(\cdot, \eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous.

Proof. (1) It is easy to see that $g_n(\cdot, \eta) : D_\epsilon \to \mathbb{C}$ is holomorphic for each $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ from Proposition 2.2.4. Also noting that $g_n(z, \eta)$ relates to only first n components of η, the continuity of g_n is obvious.

(2) Since there is a constant $c = c(\epsilon) > 0$ such that

$$\begin{aligned}|g(z,\eta) - g_n(z,\eta)| &\le E_0^\eta\Big[\Big|\int_{W(Y^n)/T(\eta)}^{W} z\exp(-zt)dt\Big|\Big] \\ &\le |z|E_0^\eta\Big[\exp\Big(\epsilon W + \epsilon\frac{W(Y^n)}{T_n(\eta)}\Big)\Big|W - \frac{W(Y^n)}{T(\eta)}\Big|\Big] \\ &\le |z|cE_0^\eta\Big[\Big|W - \frac{W(Y^n)}{T(\eta)}\Big|^2\Big]^{1/2}\end{aligned}$$

for all $z \in D_\epsilon$ by the Schwarz inequality and Proposition 2.2.4, Theorem 2.2.2 (ii) implies our assertion. □

Corollary 2.2.6. *$g^{(k)} : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous for each $k \ge 1$, where $g^{(k)}$ denotes k-th derivative of $g(z,\eta)$ with respect to z.*

Proof. Since $g_n^{(k)}(z,\eta) : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous for each $n, k \in \mathbb{N}$, Lemma 2.2.5 and Cauchy's theorem imply our assertion. □

Theorem 2.2.7. *The moment-generating function $g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ satisfies following properties:*

(a) $g(z,\eta) = \varphi(g(z/T_1(\eta),\theta\eta),\pi_1\eta),\quad g(0,\eta) = 1,\quad g'(0,\eta) = -1,$

(b) *$g(z,\eta)$ is holomorphic in D_ϵ for each $\eta \in \{2,3\}^{\mathbb{N}}$,*

(c) *$g : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ is continuous.*

Moreover $g(z,\eta)$ is the unique solution of functional equation (a) *satisfying* (b) *and* (c).

Proof. Since (2.2.2) holds for $s \in D_\epsilon$, continuity of $\varphi(\cdot,\pi_1\eta)$ and Lemma 2.2.5 (2) imply (a). (b) and (c) are obvious from Lemma 2.2.5.
Next we prove the uniqueness. Though we follow the proof of Theorem 8.2 in [H], we need some improvements. Let $U_1(z,\eta)$ and $U_2(z,\eta)$ be functions satisfying (a), (b) and (c). We choose $\epsilon_1 > 0$ such that $\epsilon_1 < \epsilon$. By assumption for any $\eta \in \{2,3\}^{\mathbb{N}}$ there are $a_k(\eta), b_k(\eta) \in \mathbb{C}, k \ge 2$ such that

$$U_1(z,\eta) = 1 - z + \sum_{k=2}^{\infty} a_k(\eta)z^k \quad\text{and}\quad U_2(z,\eta) = 1 - z + \sum_{k=2}^{\infty} b_k(\eta)z^k$$

for any $z \in B(0, \epsilon_1)$. Setting $M_i(r) = \sup\{|U_i(z, \eta)| : |z| = r, \eta \in \{2,3\}^{\mathbb{N}}\}$ for $i = 1, 2$ and $r > 0$, we have

$$|a_n(\eta)| \leq \frac{M_1(\epsilon_1)}{\epsilon_1^n}, \quad |b_n(\eta)| \leq \frac{M_2(\epsilon_1)}{\epsilon_1^n}.$$

Note that by the continuity of U_1 and U_2, M_1 and M_2 exist. Define the function $\gamma : D_\epsilon \times \{2,3\}^{\mathbb{N}} \to \mathbb{C}$ by

$$\gamma(z, \eta) = z \sum_{k=2}^{\infty} (a_k(\eta) - b_k(\eta)) z^{k-2}.$$

Then since there is a constant $c = c(\epsilon_1)$ such that

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} |\gamma(z, \eta)| \leq |z| \frac{1}{\epsilon_1^2} \sum_{k=2}^{\infty} \left(\left| \frac{M_1(\epsilon_1)}{\epsilon_1^{k-2}} \right| + \left| \frac{M_2(\epsilon_1)}{\epsilon_1^{k-2}} \right| \right) |z|^{k-2} \leq c|z| \sum_{k=0}^{\infty} \left| \frac{z}{\epsilon_1} \right|^k,$$

we get $\lim_{z \to 0} \sup_{\eta \in \{2,3\}^{\mathbb{N}}} |\gamma(z, \eta)| = 0$ for all $z \in B(0, \epsilon_1)$. From the definition $\gamma(z, \eta)$ note that $U_1(z, \eta) - U_2(z, \eta) = z\gamma(z, \eta)$. By the way there are constants $s_0 = s_0(\epsilon_1) > 0$ and $c' = c'(\epsilon_1)$ such that

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}, i=1,2} |U_i(s, \eta)| \leq 1 - s + \frac{c'(\epsilon_1)}{\epsilon_1 - s} s^2 < 1$$

for any $s > 0$ with $s < s_0$. Also we have $|\varphi'(z, \pi_1 \eta)| \leq T_1(\eta)$ for $z \in B(0, 1)$. Therefore since

$$\begin{aligned} &|T_1(\eta) s \gamma(T_1(\eta) s, \eta)| \\ &\qquad = |\varphi(U_1(s, \theta\eta), \pi_1(\eta)) - \varphi(U_2(s, \theta\eta), \pi_1(\eta))| \leq T_1(\eta) |s \gamma(s, \theta\eta)| \end{aligned}$$

for any $s < s_0$ and $\eta \in \{2,3\}^{\mathbb{N}}$ from (a), we get $|\gamma(s, \eta)| \leq |\gamma(s/T_1(\eta), \theta\eta)|$. Hence we obtain

$$|\gamma(s, \eta)| \leq \lim_{n \to \infty} |\gamma(\frac{s}{T_n(\eta)}, \theta^n \eta)| = 0 \quad \text{for all} \quad s < s_0 \quad \text{and} \quad \eta \in \{2,3\}^{\mathbb{N}}.$$

Uniqueness theorem implies our assertion. □

Lemma 2.2.8. *It follows that*

$$4\left(\frac{u}{4}\right)^{B_n(\eta)} \leq f_n(u, \eta) \leq u^{B_n(\eta)}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $u \in [0, 1]$.

Proof. The definition of $f_n(u,\eta)$ implies $f_n(u,\eta) \geq P_0^\eta[W(Y^n) = B_n(\eta)]u^{B_n(\eta)}$. Also from $W(Y^n) \geq B_n(\eta)$ with P_0^η-probability 1, we obtain $f_n(u,\eta) \leq u^{B_n(\eta)}$. Also the other inequality follows from $P_0^\eta[W(Y^n) = B_n(\eta)] = (1/4)^{B_n(\eta)-1}$. □

Lemma 2.2.9. *There exist constants $c, c' > 0$ such that*

$$\exp(-cs^{1/d_w^\eta(n)}) \leq g(s,\eta) \leq \exp(-c's^{1/d_w^\eta(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \geq 0$, $s \in [T_n(\eta), T_{n+1}(\eta)]$.

Proof. We follow the argument of Proposition 3.2 in [Kum1].
(i) Proof of the lower bounds: By Jensen's inequality $g(s,\eta) = E_0^\eta[e^{-sW}] \geq e^{-s}$ for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [0,\infty)$. We can choose $c > 0$ such that

$$e^{-s} \geq e^{-t(3)} \geq 4\exp(-cs^{1/d_w^{\mathbf{3}}}) \quad \text{for any} \quad s \in [1, t(3)].$$

Hence we obtain

$$g(s,\tilde{\eta}) \geq e^{-s} \geq 4\exp(-cs^{1/d_w^{\mathbf{3}}}) \geq 4\exp(-cs^{1/d_w^{\overline{\eta}}(n)})$$

for all $n \geq 1$, $\tilde{\eta}, \overline{\eta} \in \{2,3\}^{\mathbb{N}}$, $s \in [1, t(3)]$. Therefore by using Lemma 2.2.8 and Theorem 2.2.7 (a) we get

$$\begin{aligned} g(T_n(\eta)s,\eta) = f_n(g(s,\theta^n\eta),\eta) &\geq 4\Big(\frac{g(s,\theta^n\eta)}{4}\Big)^{B_n(\eta)} \\ &\geq 4\exp(-cB_n(\eta)s^{1/d_w^\eta(n)}) = 4\exp(-c(T_n(\eta)s)^{1/d_w^\eta(n)}) \end{aligned}$$

for any $s \in [1, t(3)]$, $\eta \in \{2,3\}^{\mathbb{N}}$. Thus we see that

$$g(s,\eta) \geq 4\exp(-cs^{1/d_w^\eta(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [T_n(\eta), T_{n+1}(\eta)]$.
(ii) Proof of the upper bounds: It is easy to see that $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} g(1,\eta) < 1$ by Theorem 2.2.7 (c). So there exists $c' > 0$ such that

$$g(s,\eta) \leq \exp(-c't(3)^{1/d_w^{\mathbf{2}}}) \leq \exp(-c's^{1/d_w^{\mathbf{2}}})$$

for any $s \in [1, t(3)]$ and $\eta \in \{2,3\}^{\mathbb{N}}$. So we have

$$g(s,\tilde{\eta}) \leq \exp(-c's^{1/d_w^{\mathbf{2}}}) \leq \exp(-c's^{1/d_w^{\overline{\eta}}(n)})$$

for any $\tilde{\eta},\overline{\eta} \in \{2,3\}^{\mathbb{N}}, s \in [1,t(3)]$. Therefore by using Lemma 2.2.8 and Theorem 2.2.7 (a),

$$\begin{aligned} g(T_n(\eta)s,\eta) &= f_n(g(s,\theta^n\eta),\eta) \le g(s,\theta^n\eta)^{B_n(\eta)} \\ &\le \exp(-c'B_n(\eta)s^{1/d_w^{\eta}(n)}) = \exp(-c'(T_n(\eta)s)^{1/d_w^{\eta}(n)}) \end{aligned}$$

for any $s \in [1,t(3)]$. As a result we deduce that

$$g(s,\eta) \le \exp(-c's^{1/d_w^{\eta}(n)})$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $s \in [T_n(\eta),T_{n+1}(\eta)]$. This completes the proof of lemma. $\square$

Lemma 2.2.10. *2-th derivative of $-\log g(s,\eta)$ is strictly negative for all $s \in (0,\infty)$ and $\eta \in \{2,3\}^{\mathbb{N}}$.*

Proof. First, we have

$$\begin{aligned} &\frac{d^2}{ds^2}\big(-\log g(s,\eta)\big) \\ &= \frac{E_0^{\eta}[W\exp(-sW)]^2 - E_0^{\eta}[\exp(-sW)]E_0^{\eta}[W^2\exp(-sW)]}{g(s,\eta)^2} \end{aligned}$$

for each $\eta \in \{2,3\}^{\mathbb{N}}$. By the Schwarz inequality for each $s \in (0,\infty)$

$$E_0^{\eta}[W\exp(-sW)]^2 \le E_0^{\eta}[\exp(-sW)]E_0^{\eta}[W^2\exp(-sW)]$$

with equality if and only if $W\exp(-sW/2) = c\exp(-sW/2)$ P_0^{η}-a.s. with some constant $c = c(s)$. But as W is not a constant P_0^{η}-a.s. we have our assertion. $\square$

2.3 Properties of $\Psi(s,\xi)$

First we define $\pi_k^{\mathbb{Z}} : \{2,3\}^{\mathbb{Z}} \to \{2,3\}$ for each $k \in \mathbb{Z}$ and $\theta_{\mathbb{Z}} : \{2,3\}^{\mathbb{Z}} \to \{2,3\}^{\mathbb{Z}}$ in the same manner as π_k and θ. When there is no possibility confusion, we simply write θ (resp. π_k) in place of $\theta_{\mathbb{Z}}$ (resp. $\pi_k^{\mathbb{Z}}$). Also in like manner (2.1.3) let $B_{-n} : \{2,3\}^{\mathbb{Z}} \to \mathbb{R}$, and $T_{-n} : \{2,3\}^{\mathbb{Z}} \to \mathbb{R}$ be given by

$$B_{-n}(\xi) = \Big\{\prod_{i=0}^{n} b(\pi_{-i}\xi)\Big\}^{-1} \quad \text{and} \quad T_{-n}(\xi) = \Big\{\prod_{i=0}^{n} t(\pi_{-i}\xi)\Big\}^{-1}$$

for each $n \geq 0$ and

$$B_n(\xi) = B_n(P\xi) \text{ and } T_n(\xi) = T_n(P\xi) \text{ for each } n \geq 1,$$

where $P : \{2,3\}^{\mathbb{Z}} \to \{2,3\}^{\mathbb{N}}$ is the projection defined by $\pi_k P(\xi) = \pi_k \xi$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $k \in \mathbb{N}$.

From the definition of $\varphi(u, \pi_1\eta)$, we have $\varphi(u, \pi_1\eta) = P_0^\eta[W(Y^1_\cdot) = b(\pi_1\eta)]u^{b(\pi_1\eta)} + \sum_{j=b(\pi_1\eta)+1}^{\infty} P_0^\eta[W(Y^1_\cdot) = j]u^j$. Let us define $h : [0,1] \times \{2,3\} \to [0,\infty)$ by

$$h(u,k) = -\log \frac{\varphi(u,k)}{u^{b(k)}}.$$

Note that $h(\cdot, k) : [0,1] \to [0,\infty)$ is continuous for each $k \in \{2,3\}$ and there exist constants $c, c' > 0$ such that

$$0 \leq h(u,k) \leq c \text{ and } -c \leq h'(u,k) \leq -c' \tag{2.3.1}$$

for any $u \in [0,1]$ and $k \in \{2,3\}$ by the definition.

By using Theorem 2.2.7 repeatedly, we have

$$g(T_n(\eta)s, \eta) = g(s, \theta^n\eta)^{B_n(\eta)} \prod_{k=1}^{n} \exp\big(-B_{k-1}(\eta) h\big(g\big(\frac{T_n(\eta)}{T_k(\eta)}s, \theta^k\eta\big), \pi_k\eta\big)\big)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s \in [0,\infty)$ and $n \in \mathbb{N}$. Here we define $L_n : [0,\infty) \times \{2,3\}^{\mathbb{N}} \to [0,\infty)$ by

$$\begin{aligned}
L_n(s,\eta) &= -\frac{\log g(T_n(\eta)s, \eta)}{B_n(\eta)} \\
&= -\log g(s, \theta^n\eta) + \frac{1}{B_n(\eta)} \sum_{k=1}^{n} B_{k-1}(\eta) h\big(g\big(\frac{T_n(\eta)}{T_k(\eta)}s, \theta^k\eta\big), \pi_k\eta\big) \\
&= -\log g(s, \theta^n\eta) + \frac{1}{B_n(\eta)} \sum_{k=1}^{n-1} B_{k-1}(\eta) h\big(g\big(\frac{T_n(\eta)}{T_k(\eta)}s, \theta^k\eta\big), \pi_k\eta\big) \\
&\qquad\qquad + \frac{1}{b(\pi_n\eta)} h\big(g(s, \theta^n\eta), \pi_n\eta\big) \\
&= -\log g(s, \theta^n\eta) + \frac{1}{b(\pi_0(\theta^n\chi_\eta))} h\big(g(s, \theta^n\eta), \pi_0(\theta^n\chi_\eta)\big) \\
&\quad + \sum_{k=1}^{n-1} B_{-(n-k)}(\theta^n\eta) h\big(g\big(\frac{s}{T_{-(n-k-1)}(\theta^n\eta)}, \theta^{-(n-k)}\theta^n\eta\big), \pi_{-(n-k)}(\theta^n\eta)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&= -\log g(s,\theta^n\eta) + \frac{1}{b(\pi_0(\theta^n\chi_\eta))} h\big(g(s,\theta^n\eta),\pi_0(\theta^n\chi_\eta)\big) \\
&\quad + \sum_{j=1}^{n-1} B_{-j}(\theta^n\eta) h\big(g(\frac{s}{T_{-(j-1)}(\theta^n\eta)},\theta^{-j}\theta^n\eta),\pi_{-j}(\theta^n\eta))\big) \\
&\le -\log g(s,\theta^n\eta) + \frac{1}{b(\pi_0(\theta^n\chi_\eta))} h\big(g(s,\theta^n\eta),\pi_0(\theta^n\chi_\eta)\big) \\
&\quad + \sum_{j=1}^{\infty} B_{-j}(\theta^n\chi_\eta) h\big(g(\frac{s}{T_{-(j-1)}(\theta^n\chi_\eta)},\theta^{-j}\theta^n\chi_\eta),\pi_{-j}(\theta^n\chi_\eta))\big)
\end{aligned}
$$

Note that $L_n(s,\eta)$ is a concave function with respect to s for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. In the case $\eta = \mathbf{2}$ (i.e. usual Sierpinski gasket) $\lim_{n\to\infty} L_n(s,\mathbf{2})$ exists for all $s \ge 0$. But $\lim_{n\to\infty} L_n(s,\eta)$ does not necessarily exist for general $\eta \in \{2,3\}^{\mathbb{N}}$. Let $F_k : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ for $k \ge 0$ be given by

$$F_k(s,\xi) = B_{-k}(\xi) h\big(g(s/T_{-(k-1)}(\xi), P(\theta^{-k}\xi)),\pi_{-k}\xi\big), \quad k \ge 1$$

and

$$F_0(s,\xi) = \frac{1}{b(\pi_0\xi)} h\big(g(s,P(\xi)),\pi_0\xi\big).$$

By (2.3.1), we see that $\sup_{s\in[0,\infty),\xi\in\{2,3\}^{\mathbb{Z}}} \sum_{k=0}^{\infty} F_k(s,\xi) < \infty$. So we can define the function $\Psi : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ to be

$$\Psi(s,\xi) = -\log g(s,P(\xi)) + \sum_{k=0}^{\infty} F_k(s,\xi). \tag{2.3.2}$$

Also we have

$$
\begin{aligned}
F_k'(s,\xi) &= -\frac{B_{-k}(\xi)}{T_{-(k-1)}(\xi)} \\
&\quad \times h'\big(g(\frac{s}{T_{-(k-1)}(\xi)},P(\theta^{-k}\xi)),\pi_{-k}\xi\big) E_0^{P(\theta^{-k}\xi)}\big[W\exp(-\frac{sW}{T_{-(k-1)}(\xi)})\big]
\end{aligned}
$$

for each $k \ge 1$ and

$$F_0'(s,\xi) = -\frac{1}{b(\pi_0\xi)} h'\big(g(s,P(\xi)),\pi_0\xi\big) E_0^{P(\xi)}\big[W\exp(-sW)\big].$$

By Lemma 2.2.9 and the Schwarz inequality, there are constants $c, c' > 0$ such that

$$
\begin{aligned}
E_0^{P(\theta^{-k}\xi)}\big[W\exp(-\frac{sW}{T_{-(k-1)}(\xi)})\big] &\le E_0^{P(\theta^{-k}\xi)}[W^2] E_0^{P(\theta^{-k}\xi)}\big[\exp(-\frac{2sW}{T_{-(k-1)}(\xi)})\big]^2 \\
&\le c\exp(-c'(t(2)^k s)^{1/d_w^3})
\end{aligned} \tag{2.3.3}
$$

for any $s > 0$ and $k \geq K$, where $K = K(s)$ is a non-negative integer with $t(2)^K s \geq 1$. For each $s_0 \in (0, \infty)$, let $(a, b) \subset (0, \infty)$ be an open interval containing the point s_0. Then from (2.3.1) and (2.3.3), we see that

$$\sum_{k=K}^{\infty} |F_k'(s, \xi)| \leq c'' \sum_{k=K}^{\infty} \Big(\frac{t(3)}{b(3)}\Big)^k \exp(-c'(t(2)^k a)^{1/d_w^3}) < \infty$$

for any $s \in (a, b)$. So since $\Psi(s, \xi)$ is differentiable with respect to s for each $\xi \in \{2, 3\}^{\mathbb{Z}}$, we obtain

$$\Psi'(s, \xi) = -\frac{g'(s, P(\xi))}{g(s, P(\xi))} + \sum_{k=0}^{\infty} F_k'(s, \xi). \tag{2.3.4}$$

$\Psi(s, \xi)$ has following properties and approximates $L_n(s, \eta)$ in the following sense.

Lemma 2.3.1. (1) $\Psi : [0, \infty) \times \{2, 3\}^{\mathbb{Z}} \to [0, \infty)$ is continuous.

(2) It follows that

$$\sup_{s \in [0,\infty), \xi \in \{2,3\}^{\mathbb{Z}}} |L_n(s, P(\xi)) - \Psi(s, \theta^n \xi)| \to 0 \quad \text{as } n \to \infty.$$

(3) The function Ψ satisfies the following functional equation:

$$\Psi(T_1(\xi)s, \xi) = B_1(\xi)\Psi(s, \theta\xi) \quad \text{and} \quad \Psi(0, \xi) = 0$$

for all $s \in [0, \infty)$ and $\xi \in \{2, 3\}^{\mathbb{Z}}$.

Proof. (1) From Theorem 2.2.7 (c), it suffices to show the continuity of the second term of the right hand side of (2.3.2). It is easy to see that $F_k : [0, \infty) \times \{2, 3\}^{\mathbb{N}} \to [0, \infty)$ is continuous by Theorem 2.2.7 (c) and continuity of $h(\cdot, \pi_{-k}\xi)$ for each $k \geq 0$. Since by (2.3.1) there is some constant $c > 0$ such that

$$\sup_{\substack{s \in [0,\infty) \\ \xi \in \{2,3\}^{\mathbb{Z}}}} \Big| \sum_{k=1}^{\infty} F_k(s, \xi) - \sum_{k=1}^{m} F_k(s, \xi) \Big| \leq c \sum_{k=m+1}^{\infty} \frac{1}{b(2)^{k+1}} = \frac{c}{b(2)^m},$$

for any $m \in \mathbb{N}$, the partial sum $\sum_{k=1}^{m} F_k(s, \xi)$ converges uniformly with respect to $s \in [0, \infty)$ and $\xi \in \{2, 3\}^{\mathbb{Z}}$ as $m \to \infty$. This implies our assertion.

(2) By easy consideration, we have $B_{-(n-k)}(\theta^n\xi) = B_{k-1}(\xi)/B_n(\xi)$, $T_{-(n-k-1)}(\theta^n\xi) = T_k(\xi)/T_n(\xi)$, $\pi_{-(n-k)}\theta^n\xi = \pi_k P\xi$ and $P(\theta^n\xi) = \theta^n(P\xi)$. These imply that

$$F_{n-k}(s,\theta^n\xi) = \frac{B_{k-1}(\xi)}{B_n(\xi)} h\big(g\big(\frac{T_n(\xi)}{T_k(\xi)}s, \theta^k P\xi\big), \pi_k P\xi\big)$$

for all $k \in \mathbb{N}$ with $1 \leq k \leq n-1$ and

$$F_0(s,\theta^n\xi) = \frac{1}{b(\pi_n P\xi)} h\big(g(s,\theta^n P\xi), \pi_n P\xi\big).$$

Therefore it follows that

$$\Big|\frac{\log g(T_n(\xi)s, P\xi)}{B_n(\xi)} - \Psi(s,\theta^n\xi)\Big| = \Big|\sum_{k=n}^{\infty} F_k(s,\theta^n\xi)\Big| \leq \frac{c}{b(2)^n}$$

for some constant $c > 0$ by (2.3.1).
(3) By simple calculations we deduce that

$$\begin{aligned}
&B_1(\xi)F_k(s,\theta\xi) = F_{k-1}(T_1(\xi)s,\xi) \quad \text{for all} \quad k \in \mathbb{N},\\
&B_1(\xi)(-\log g(s,P(\theta\xi)) + F_0(s,\theta\xi)) = -\log g(T_1(\xi)s, P(\xi)),
\end{aligned}$$

where we use Theorem 2.2.7 (a) in the second equation. This implies our assertion. □

Next we state some properties of Ψ'.

Lemma 2.3.2. (1) $\Psi' : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ is continuous.

(2) $\Psi'(s,\xi)$ is strictly decreasing with respect to $s \in (0,\infty)$ for each $\xi \in \{2,3\}^{\mathbb{Z}}$.

(3) It follows that

$$\min_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi'(s,\xi) \to \infty \quad \text{as} \quad s \downarrow 0 \quad \text{and} \quad \max_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi'(s,\xi) \to 0 \quad \text{as} \quad s \to \infty.$$

Proof. (1) This is the same as that of Lemma 2.3.1 except for obvious modifications.
(2) Let η be an element of $\{2,3\}^{\mathbb{N}}$ whose the first component $\pi_1\eta$ is $\pi_{-k}\xi$. Then note that by Hölder's inequality

$$\begin{aligned}
&h\big(g(s/T_{-(k-1)}(\xi), P(\theta^{-k}\xi)), \pi_{-k}\xi\big)\\
&\quad = -\log E_0^{\eta}\Big[E_0^{P(\theta^{-k}\xi)}\big[\exp\big(-\frac{sW}{T_{-(k-1)}(\xi)}\big)\big]^{W(Y^1)-b(\pi_1\eta)}\Big]
\end{aligned}$$

is a concave function with respect to $s > 0$ for each $k \geq 1$, $\xi \in \{2,3\}^{\mathbb{Z}}$. Hence the second term in the right hand side of (2.3.4) is monotone decreasing with respect to s for each $\xi \in \{2,3\}^{\mathbb{Z}}$. By Lemma 2.2.10 the first term in the right hand side of (2.3.4) is strictly monotone decreasing. This implies our assertion.

(3) Fix $\xi \in \{2,3\}^{\mathbb{Z}}$. As we stated above, the first term in the right hand side of (2.3.4) is strictly monotone decreasing. We assume that this term converges to $c > 0$ as $s \to \infty$. Then by easy calculation there is a constant c' such that $-\log g(s, P(\xi)) \geq c's$ for large enough s which contradicts Lemma 2.2.9. So the first term in the right hand side of (2.3.4) converges to 0 as $s \to \infty$. Also by (2.3.1) and Monotone convergence theorem we see that

$$\sum_{k=1}^{\infty} F_k'(s,\xi) \geq c' \sum_{k=1}^{\infty} \frac{B_{-k}(\xi)}{T_{-(k-1)}(\xi)} E_0^{P(\theta^{-k}\xi)}[W \exp(-\frac{sW}{T_{-(k-1)}(\xi)})] \to \infty$$

as $s \downarrow 0$. Similarly we have

$$\sum_{k=1}^{\infty} F_k'(s,\xi) \to 0 \quad \text{as} \quad s \to \infty$$

by (2.3.1) and Lebesgue convergence theorem. As a result we have $\Psi'(s,\xi) \to \infty$ as $s \downarrow 0$ and $\Psi'(s,\xi) \to 0$ as $s \to \infty$ for each $\xi \in \{2,3\}^{\mathbb{Z}}$. Since $\Psi'(\cdot,\xi)$ is strictly monotone decreasing from (2), Dini's theorem implies our assertion. □

From Lemma 2.3.2 (2), $\Psi'(\cdot,\xi)$ is strictly decreasing for each $\xi \in \{2,3\}^{\mathbb{Z}}$. Therefore $\Psi'(\cdot,\xi)$ has the inverse function $(\Psi')^{-1}(\cdot,\xi)$ for each $\xi \in \{2,3\}^{\mathbb{Z}}$.

Lemma 2.3.3. (1) For each compact set $K \subset (0,\infty)$ there exists a compact set $K_1 \subset (0,\infty)$ such that

$$\bigcup_{\xi \in \{2,3\}^{\mathbb{Z}}} (\Psi')^{-1}(K,\xi) \subset K_1. \tag{2.3.5}$$

(2) $(\Psi')^{-1} : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ is continuous.

Proof. (1) Let us denote by K_2 the left hand side of (2.3.5). Assume that $\inf K_2 = 0$. Then there is $a_n \in K_2$ such that $0 < a_n < 1/n$ for each $n \in \mathbb{N}$. Further for each a_n there exist $\xi^n \in \{2,3\}^{\mathbb{Z}}$ and $z_n \in K$ such that $\Psi'(a_n,\xi^n) = z_n \in K$. This contradicts Lemma 2.3.2 (3). Next assume that $\sup K_2 = \infty$. Then there exist $b_n \in K_2$ and $\xi^n \in \{2,3\}^{\mathbb{Z}}$ such that $b_n > n$ and $\Psi'(b_n,\xi^n) \in K$ for each $n \in \mathbb{N}$. This contradicts Lemma 2.3.2 (3).

(2) It is obvious that $(\Psi')^{-1}(\cdot,\xi) : (0,\infty) \to \mathbb{R}$ is continuous for each $\xi \in \{2,3\}^{\mathbb{Z}}$. By Lemma 2.3.2 (1), $\Psi'(s,\xi')$ converges to $\Psi'(s,\xi)$ pointwise as $\xi' \to \xi$ for each $s \in (0,\infty)$. So does the inverse function. Since $(\Psi')^{-1}(\cdot,\xi)$ is monotone decreasing, this convergence is compact uniform on $(0,\infty)$. This completes our assertion. □

Define the Legendre transform $\Psi^*(z,\xi)$ by $\Psi^*(z,\xi) = \sup_{s>0}\{\Psi(s,\xi) - zs\}$ for $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$. Since we have

$$\begin{aligned}\Psi^*(z,\xi) &= \sup_{s>0}\{\Psi(T_1(\xi)s,\xi) - zT_1(\xi)s\} \\ &= \sup_{s>0}\{B_1(\xi)\Psi(s,\theta\xi) - B_1(\xi)\frac{T_1(\xi)}{B_1(\xi)}zs\} = B_1(\xi)\Psi^*(\frac{T_1(\xi)}{B_1(\xi)}z,\theta\xi)\end{aligned}$$

by Lemma 2.3.1 (5), the function Ψ^* satisfies the following functional equation:

$$\Psi^*\Big(\frac{B_1(\xi)}{T_1(\xi)}z,\xi\Big) = B_1(\xi)\Psi^*(z,\theta\xi).$$

From $T_{-n}(\xi) = T_{n+1}(\theta^{-(n+1)}\xi)^{-1}$ and $B_{-n}(\xi) = B_{n+1}(\theta^{-(n+1)}\xi)^{-1}$, we obtain

$$\Psi^*(z,\xi) = \begin{cases} B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi),\theta^n\xi) & \text{if } n \geq 1, \\ B_n(\xi)\Psi^*(T_n(\xi)z/B_n(\xi),\theta^{n-1}\xi) & \text{if } n < 0 \end{cases} \tag{2.3.6}$$

for all $\xi \in \{2,3\}^{\mathbb{Z}}$ and $z > 0$. Let $d_w^{\xi}(n)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $n \in \mathbb{Z}$ be given by

$$d_w^{\xi}(n) = d_w^{P\xi}(n) \text{ if } n \geq 1 \quad \text{and} \quad d_w^{\xi}(-n) = \frac{\log T_{-n}(\xi)}{\log B_{-n}(\xi)} \quad \text{if } \; n \geq 0.$$

Then we have the following.

Lemma 2.3.4. (1) $\Psi^* : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ is continuous.

(2) There exist constants $c, c' > 0$ such that

$$cz^{-1/(d_w^{\xi}(n)-1)} \leq \Psi^*(z,\xi) \leq c'z^{-1/(d_w^{\xi}(n)-1)}$$

for any $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$, where n is an integer with $B_n(\xi)/T_n(\xi) \leq z < B_{n-1}(\xi)/T_{n-1}(\xi)$.

(3) For any $a > 0$

$$\lim_{\delta\to 0} \sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \sup_{\substack{x_1,x_2\in(0,a] \\ |x_1-x_2|<\delta}} |x_1\Psi^*(1/x_1,\xi) - x_2\Psi^*(1/x_2,\xi)| = 0.$$

Proof. (1) Since $\Psi'(\cdot,\xi)$ is strictly decreasing for each $\xi \in \{2,3\}^{\mathbb{Z}}$, $(\Psi')^{-1}(z,\xi)$ is a unique point such that $\Psi^*(z,\xi) = \Psi((\Psi')^{-1}(z,\xi),\xi) - z(\Psi')^{-1}(z,\xi)$ for each $z \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$. By Lemma 2.3.1 (1) and Lemma 2.3.3 (2), $\Psi : [0,\infty) \times \{2,3\}^{\mathbb{Z}} \to [0,\infty)$ and $(\Psi')^{-1} : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ are continuous. These imply our assertion.
(2) Since $\sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi^*(1,\xi) < \infty$, from (2.3.6) there is a constant $c' > 0$ such that

$$\Psi^*(z,\xi) \le \Psi^*\Big(\frac{B_n(\xi)}{T_n(\xi)},\xi\Big) = B_n(\xi) \sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \Psi^*(1,\xi) \le c' z^{-1/(d_w^{\xi}(n)-1)}$$

for any $n \in \mathbb{Z}$, $\xi \in \{2,3\}^{\mathbb{Z}}$ and $z \in [B_n(\xi)/T_n(\xi), B_{n-1}(\xi)/T_{n-1}(\xi))$. The lower bound is proved in exactly the same way.
(3) For any $\epsilon > 0$ there is $\delta_0 = \delta_0(\epsilon) > 0$ such that if $x_1, x_2 < \delta_0$ then

$$\sup_{\xi\in\{2,3\}^{\mathbb{Z}}} |x_1\Psi^*(1/x_1,\xi) - x_2\Psi^*(1/x_2,\xi)| < \epsilon$$

from (2). Let $K = [\delta_0/2, a]$. By (1) there is $\delta_1 = \delta_1(\epsilon, K) > 0$ such that

$$\sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \sup_{\substack{x_1,x_2\in K \\ |x_1-x_2|<\delta}} |x_1\Psi^*(1/x_1,\xi) - x_2\Psi^*(1/x_2,\xi)| < \epsilon$$

for all $\delta < \delta_1$. So if $\delta < \delta_0/2 \wedge \delta_1$ then

$$\sup_{\xi\in\{2,3\}^{\mathbb{Z}}} \sup_{\substack{x_1,x_2\in (0,a] \\ |x_1-x_2|<\delta}} |x_1\Psi^*(1/x_1,\xi) - x_2\Psi^*(1/x_2,\xi)| < \epsilon.$$

This completes the proof. □

2.4 Hitting time and distance

In this section we shall prove the following proposition.

Proposition 2.4.1. *The following holds*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ s\in K}} \left| \frac{\log E_x^{\eta}[\exp(-T_n(\eta)s\tau_y)]}{\log g(T_n(\eta)s,\eta)} - d_\eta(x,y) \right| = 0$$

for any $\delta_0 > 0$ and compact set $K \subset (0,\infty)$, where $\tau_y = \inf\{t \ge 0 : X_t = y\}$.

In the case $\eta = \mathbf{2}$, this proposition corresponds Lemma 3.4 in [Kum]. We make some preparations to prove this proposition. As the following lemma is standard, we omit the proof.

Lemma 2.4.2. (1) *It follows that* $E_0^\eta[u^W] = E_0^\eta[E_0^\eta[u^{S_1^n}]^{W(Y^n_\cdot)}]$ *for all* $\eta \in \{2,3\}^{\mathbb{N}}$, $n \geq 1$ *and* $u \in [0,1]$.

(2) *It follows that* $E_x^\eta[u^{S^0}] = E_x^\eta[E_0^\eta[u^{S_1^n}]^{S^0(Y^n_\cdot)}]$ *for all* $\eta \in \{2,3\}^{\mathbb{N}}$, $n \geq 1$, $x \in F_n^\eta$ *and* $u \in [0,1]$.

(3) *It follows that* $E_x^\eta[u^{S^0(Y^n_\cdot)}] = E_x^\eta[E_0^\eta[u^{S_1^1(Y^n_\cdot)}]^{S^0(Y^1_\cdot)}]$ *for all* $\eta \in \{2,3\}^{\mathbb{N}}$, $n \geq 1$, $x \in F_1^\eta$ *and* $u \in [0,1]$.

Let $c_1 = \max_{\eta \in \{2,3\}^{\mathbb{N}}} \max_{x \in F_1^\eta} E_x^\eta[S^0(Y^1_\cdot)]$.

Lemma 2.4.3. *It follows that*

$$E_x^\eta[u^{S^0(Y^n_\cdot)}] \geq E_0^\eta[u^{W(Y^n_\cdot)}]^{c_1} \tag{2.4.1}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $x \in F_n^\eta$ *and* $u \in [0,1]$.

Proof. Let $z_1, z_2 \in F_0^\eta$ and $b_1 \in F_1^\eta$ be as in Figure 2.1. Then By Jensen's inequality we have

$$E_0^\eta[u^{W(Y^1_\cdot)}]^{c_1} \vee E_{b_1}^\eta[u^{S^0(Y^1_\cdot)}]^{c_1} \leq u^{c_1} \leq E_x^\eta[u^{S^0(Y^1_\cdot)}] \tag{2.4.2}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F_1^\eta$ and $u \in [0,1]$. Thus by Lemma 2.4.2 (3) we see that

$$\begin{aligned} E_{b_1}^\eta[u^{S^0(Y^m_\cdot)}]^{c_1} &= E_{b_1}^\eta\left[E_0^\eta[u^{S_1^1(Y^m_\cdot)}]^{S^0(Y^1_\cdot)}\right]^{c_1} \\ &\leq E_x^\eta\left[E_0^\eta[u^{S_1^1(Y^m_\cdot)}]^{S^0(Y^1_\cdot)}\right] = E_x^\eta[u^{S^0(Y^m_\cdot)}] \end{aligned} \tag{2.4.3}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F_1^\eta$, $m \in \mathbb{N}$ and $u \in [0,1]$. Now we will prove (2.4.1) by induction on n. Assume that our assertion is true for $n-1$. Then we have

$$E_0^\eta[u^{S_1^1(Y^n_\cdot)}]^{c_1} \leq E_x^\eta[u^{S^1(Y^n_\cdot)}] \tag{2.4.4}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$ and $x \in F_n^\eta$. Note that $S_1^1(Y^n_\cdot)$ is the first hitting time of b_1, b_2 under P_0^η. Hence by (2.4.3) and (2.4.4) we obtain

$$\begin{aligned} E_x^\eta[u^{S^0(Y^n_\cdot)}] &= \sum_{y \in C_1(x)} E_x^\eta[1_{\{X_{S^1(Y^n_\cdot)}=y\}} u^{S^1(Y^n_\cdot)}] E_y^\eta[u^{S^0(Y^n_\cdot)}] \\ &\geq E_x^\eta[u^{S^1(Y^n_\cdot)}] E_{b_1}^\eta[u^{S^0(Y^n_\cdot)}]^{c_1} \geq E_0^\eta[u^{S_1^1(Y^n_\cdot)}]^{c_1} E_{b_1}^\eta[u^{S^0(Y^n_\cdot)}]^{c_1} \end{aligned}$$

for all $x \in F_n^\eta \setminus F_1^\eta$, where $C_1(x)$ is the 1-cell which contains x. Note that this is true for all $x \in F_1^\eta$ from (2.4.3). On the other hand by the strong Markov property

$$E_0^\eta[u^{W(Y^n)}] = E_0^\eta[u^{S_1^1(Y^n)}]E_{b_1}^\eta[u^{\tau_{1,2}(Y^n)}] \le E_0^\eta[u^{S_1^1(Y^n)}]E_{b_1}^\eta[u^{S^0(Y^n)}],$$

where $\tau_{1,2}(Y^n) = \inf\{i \in \mathbb{N} : Y_i^n \in \{z_1, z_2\}\}$. This implies (2.4.1) is true for n. $\square$

Next we prove the continuous version of Lemma 2.4.3.

Lemma 2.4.4. *It holds that*

$$E_x^\eta[u^{S^0}] \ge E_0^\eta[u^W]^{c_1} \quad \textit{for all } \eta \in \{2,3\}^{\mathbb{N}},\ x \in F^\eta \textit{ and } u \in [0,1].$$

Proof. Firstly from Lemma (2.4.2) (1), (2) and Lemma (2.4.3) we see that

$$E_0^\eta[u^W]^{c_1} = E_0^\eta\left[E_0^\eta[u^{S_1^n}]^{W(Y^n)}\right]^{c_1} \le E_x^\eta\left[E_0^\eta[u^{S_1^n}]^{S^0(Y^n)}\right] = E_x^\eta[u^{S^0}] \tag{2.4.5}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $x \in F_n^\eta$ and $u \in [0,1]$. Next we consider in case $x \in F^\eta \setminus F_\infty^\eta$. We have by the strong Markov property

$$E_x^\eta[u^{S^0}] = \sum_{i=0}^{2} E_{w_i^n}^\eta[u^{S^0}]E_x^\eta[u^{S^n}1_{\{X_{S^n}=w_i^n\}}] \ge E_0^\eta[u^W]^{c_1}E_x^\eta[u^{S^n}] \tag{2.4.6}$$

for all $n \in \mathbb{N}$, where $\{w_0^n, w_1^n, w_2^n\}$ is the boundary of n-complex which contains x. Since $P_x^\eta[\lim_{n\to\infty} S^n = 0] = 1$, the Dominated Convergence Theorem implies our assertion. $\square$

Let $\eta \in \{2,3\}^{\mathbb{N}}$ and remind the function

$$u_s^\eta(x,y) = \int_0^\infty e^{-st}p_t^\eta(x,y)\,dt, \quad (s,x,y) \in (0,\infty) \times F^\eta \times F^\eta.$$

Lemma 2.4.5. (1) *Let $\eta \in \{2,3\}^{\mathbb{N}}$. It follows that*

$$E_x^\eta[e^{-s\tau_y}] = \frac{u_s^\eta(x,y)}{u_s^\eta(x,x)} \quad \textit{for all } x,y \in F^\eta \textit{ and } s > 0,$$

where $\tau_y = \inf\{t \ge 0 : X_t = y\}$.

(2) *There exist constants $c, c' > 0$ such that*

$$c\frac{M_n(\eta)}{T_n(\eta)} \le u_s^\eta(x,x) \le c'\frac{M_n(\eta)}{T_n(\eta)}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \ge 1$, where n is a positive integer $T_n(\eta) \le s \le T_{n+1}(\eta)$.

Proof. (1) This follows from Lemma 3.4.2, Theorem 3.6.3 and Theorem 3.6.5 in [MR].
(2) Firstly We shall prove the lower bounds. The function $p_t^\eta(x,x)$ is decreasing in t for each $x \in F^\eta$ (this is a general fact about symmetric processes). So

$$u_s^\eta(x,x) \geq \int_0^{1/s} e^{-st} p_t^\eta(x,x)dt \geq \frac{1-e^{-1}}{s} p_{1/s}^\eta(x,x).$$

By Lemma 5.1 in [BH] we have the lower bounds.

Next we shall prove the upper bounds. From the proof of Lemma 4.1 in [BH] there exists $c_1 > 0$ such that $p_t^\eta(x,y) \leq c_1 M_n(\eta)$ if $1/T_{n+1}(\eta) \leq t \leq 1/T_n(\eta)$ and $p_t^\eta(x,y) \leq c_1$ if $1 \leq t$. We divide $u_s^\eta(x,x)$ into three parts:

$$\begin{aligned} u_s^\eta(x,x) &= \sum_{j=0}^{n} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t^\eta(x,x)dt \\ &\quad + \sum_{j=n+1}^{\infty} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t^\eta(x,x)dt + \int_1^\infty e^{-st} p_t^\eta(x,x)dt. \end{aligned}$$

Clearly the third part is smaller than $c_1/T_n(\eta)$. In the first part there is a constant $c_2 > 0$ such that

$$\begin{aligned} \sum_{j=0}^{n} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t(x,x)dt &\leq \sum_{j=0}^{n} \frac{M_j(\eta)}{s} (e^{-s/T_{j+1}(\eta)} - e^{-s/T_j(\eta)}) \\ &\leq \frac{M_n(\eta)}{T_n(\eta)} \sum_{j=1}^{n} \frac{M_j(\eta)}{M_n(\eta)} \leq c_2 \frac{M_n(\eta)}{T_n(\eta)} \end{aligned}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$, $s \geq 1$ and $n \in \mathbb{N}$ with $T_n(\eta) \leq s \leq T_{n+1}(\eta)$. Finally we estimate the second term. Note that $e^{-s/T_{j+1}(\eta)} \leq e^{-s/(t(3)T_j(\eta))}$. Since $e^{-x/t(3)} - e^{-x} \leq (1 - 1/t(3))x$ for all $x > 0$, there are constants $c_3, c_4 > 0$ such that

$$\sum_{j=n+1}^{\infty} \int_{1/T_{j+1}(\eta)}^{1/T_j(\eta)} e^{-st} p_t(x,x)dt \leq c_3 \sum_{j=n+1}^{\infty} \frac{M_j(\eta)}{T_j(\eta)} \leq c_4 \frac{M_n(\eta)}{T_n(\eta)}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$, $s \geq 1$ and $n \in \mathbb{N}$ with $T_n(\eta) \leq s \leq T_{n+1}(\eta)$. This completes our assertion. $\square$

By using above Lemma 2.4.5 (1) and (2) there is a constant $C > 0$ such that

$$E_x^\eta[\exp(-s\tau_y)] = \frac{u_s^\eta(x,y)}{u_s^\eta(x,x)} \leq \frac{c'}{c} \frac{u_s^\eta(y,x)}{u_s^\eta(y,y)} = CE_y^\eta[\exp(-s\tau_x)] \tag{2.4.7}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^{\eta}$ and $s \geq 1$.

Proof of Proposition 2.4.1.

Let $\eta \in \{2,3\}^{\mathbb{N}}$, $m \in \mathbb{N}$ and a shortest F^{η}_m-path be $\pi = \{x_0, \cdots, x_l\}$ connecting $x(= x_0)$ and $y(= x_l)$ for each $x, y \in F^{\eta}_m$. Then

$$\begin{aligned} &E^{\eta}_x[\exp(-s\tau_y)1_{\{X_{S^m_k}=x_k, 1\leq k\leq l\}}] \\ &\qquad \leq E^{\eta}_x[\exp(-s\tau_y)] \leq E^{\eta}_0\Big[\exp(-s\sum_{i=1}^{l}(S^m_i - S^m_{i-1}))\Big]. \end{aligned}$$

By Theorem 2.2.7 (1) and Lemma 2.4.2 (1) we see that

$$g\Big(\frac{s}{T_m(\eta)}, \theta^m\eta\Big) = E^{\eta}_0\Big[\exp(-sS^m_1)\Big] \tag{2.4.8}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $m \geq 1$ and $s > 0$. Hence we get

$$\Big(\frac{1}{6}\Big)^l g\Big(\frac{s}{T_m(\eta)}, \theta^m\eta\Big)^l \leq E^{\eta}_x[\exp(-s\tau_y)] \leq g\Big(\frac{s}{T_m(\eta)}, \theta^m\eta\Big)^l.$$

Substituting $T_n(\eta)s$ for s, we get

$$\begin{aligned} &\frac{c}{\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} \log\frac{1}{6} + d_{\eta}(x,y) \\ &\qquad \geq \frac{\log E^{\eta}_x[\exp(-T_n(\eta)s\tau_y)]}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} \geq d_{\eta}(x,y), \end{aligned} \tag{2.4.9}$$

where c is the constant in (2.1.5). Note that $d_{\eta}(x,y) = l/B_m(\eta)$. In particular considering in the case $x, y \in F^{\eta}_0$, we have

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{s\in K} \left| \frac{\log g(T_n(\eta)s, \eta)}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} - 1 \right| = 0 \tag{2.4.10}$$

for each $m \in \mathbb{N}$. By adding (2.4.9) and (2.4.10) we see that

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y\in F^{\eta}_m \\ s\in K}} \left| \frac{\log E^{\eta}_x[\exp(-T_n(\eta)s\tau_y)]}{\log g(T_n(\eta)s, \eta)} - d_{\eta}(x,y) \right| = 0 \tag{2.4.11}$$

for each $m \in \mathbb{N}$. Next we consider in the the case $x, y \in F^{\eta} \setminus F^{\eta}_m$. Let $A_m(x)$ be a m-complex which contains x. For any $\delta_0 > 0$ there exists $M = M(\delta_0) \in \mathbb{N}$ such that if

$m \geq M$ then $A_m(x) \cap A_m(y) = \phi$ for any $x, y \in F^\eta \setminus F^\eta_m$ with $d_\eta(x,y) \geq \delta_0$. By (2.4.7) and the strong Markov property we have

$$\begin{aligned} E^\eta_x[\exp(-s\tau_y)] &= \sum_{i=0}^{2} E^\eta_x[\exp(-sS^m_0)1_{\{X_{S^m_0}=z^m_i\}}]E^\eta_{z^m_i}[\exp(-s\tau_y)] \\ &\geq \frac{1}{C}\sum_{i=0}^{2} E^\eta_x[\exp(-sS^m_0)1_{\{X_{S^m_0}=z^m_i\}}]E^\eta_y[\exp(-s\tau_{z^m_i})] \\ &\geq \frac{1}{C}E^\eta_x[\exp(-sS^m_0)]E^\eta_y[\exp(-sS^m_0)]\min_{i,j} E^\eta_{w^m_j}[\exp(-s\tau_{z^m_i})] \end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \geq 1$, where $\partial A_m(x) = \{z^m_0, z^m_1, z^m_2\}$, $\partial A_m(y) = \{w^m_0, w^m_1, w^m_2\}$. In the same way we have

$$E^\eta_x[\exp(-s\tau_y)] \leq C\max_{i,j} E^\eta_{w^m_j}[\exp(-s\tau_{z^m_i})]$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta$ and $s \geq 1$. By the way we have

$$\frac{\log E^\eta_x[\exp(-sS^m_0)]}{\log g(s/T_m(\eta), \theta^m\eta)} = \frac{\log E^\eta_x[\exp(-sS^m_0)]}{\log E^\eta_0[\exp(-sS^m_1)]} \leq c_1$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x \in F^\eta, m \in \mathbb{N}$ and $s > 0$ from (2.4.8) and Lemma 2.4.4. Therefore we get

$$\begin{aligned} &\frac{\log C}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} + \frac{\max_{i,j}\log E^\eta_{w^m_i}[\exp(-T_n(\eta)s\tau_{z^m_j})]}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} \\ &\leq \frac{\log E^\eta_x[\exp(-T_n(\eta)s\tau_y)]}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} \\ &\leq \frac{\min_{i,j}\log E^\eta_{w^m_i}[\exp(-T_n(\eta)s\tau_{z^m_j})] - \log C}{B_m(\eta)\log g(T_n(\eta)s/T_m(\eta), \theta^m\eta)} + \frac{2c_1}{B_m(\eta)} \end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $m \geq M$, $x, y \in F^\eta \setminus F^\eta_m$ with $d_\eta(x,y) \geq \delta_0$, $s \in K$ and $n \in \mathbb{N}$ with $T_n(\eta)s \geq 1$. Since $\sup_{\eta\in\{2,3\}^{\mathbb{N}}}\sup_{x,y\in F^\eta}\sup_{i,j}|d_\eta(x,y) - d_\eta(z^m_i, w^m_j)| \to 0$ as $m \to \infty$, from (2.4.10) and (2.4.11) we have completed the proof.

Define the function $k^{x,y}_n(s,\eta) : [0,\infty) \times \{2,3\}^{\mathbb{N}} \to \mathbb{R}$ by

$$k^{x,y}_n(s,\eta) = -\frac{\log E^\eta_x[\exp(-T_n(\eta)\tau_y s)]}{B_n(\eta)}$$

for each $x, y \in F^\eta$ and $n \in \mathbb{N}$. Set $c_K = \sup_{s\in K, \eta\in\{2,3\}^{\mathbb{N}}, n\in\mathbb{N}} |L_n(s,\eta)|$ for a compact set $K \subset (0,\infty)$. Then Theorem 2.2.7 (c) implies $0 < c_K < \infty$. For any $\epsilon > 0$ there exists $N = N(\epsilon, K) \in \mathbb{N}$ such that if $n \geq N$ then

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} \left| \frac{k_n^{x,y}(s,\eta)}{L_n(s,\eta)} - d_\eta(x,y) \right| < \frac{\epsilon}{c_K}$$

from Lemma 2.4.1. Therefore we deduce that

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} |k_n^{x,y}(s,\xi) - d_\eta(x,y) L_n(s,\xi)| = 0 \tag{2.4.12}$$

for any compact set $K \subset (0,\infty)$.

Lemma 2.4.6. *It follows that*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{s\in K \\ d_\eta(x,y)\geq\delta_0}} |k_n^{x,y}(s,\eta) - d_\eta(x,y)\Psi(s,\theta^n\chi_\eta)| = 0$$

for each compact set $K \subset (0,\infty)$ and $\delta_0 > 0$.

Proof. Note that

$$\begin{aligned} &|k_n^{x,y}(s,\eta) - d_\eta(x,y)\Psi(s,\theta^n\chi_\eta)| \\ &\leq |k_n^{x,y}(s,\eta) - d_\eta(x,y)L_n(s,\eta)| + d_\eta(x,y)|L_n(s,\eta) - \Psi(s,\theta^n\chi_\eta)| \end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$. $n \in \mathbb{N}$ and $s > 0$. By using Lemma 2.3.1 (2) and (2.4.12) we have completed the proof. $\square$

2.5 Short time asymptotic behavior

First we prove the following.

Lemma 2.5.1. Let K be a compact set on $(0,\infty)$. There exists a compact set $\Gamma = \Gamma(K) \subset (0,\infty)$ such that

$$\Psi^*(z,\xi) = \sup_{s\in\Gamma}\{\Psi(s,\xi) - zs\} \quad \text{for any} \quad z \in K \quad \text{and} \quad \xi \in \{2,3\}^{\mathbb{Z}}.$$

Proof. $(\Psi')^{-1}(z,\xi)$ is a unique point such that

$$\Psi^*(z,\xi) = \Psi((\Psi')^{-1}(z,\xi),\xi) - z(\Psi')^{-1}(z,\xi)$$

for each $z \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$ by Lemma 2.3.2 (2). Now $z \in K$ is bounded from above and below. Hence Lemma 2.3.3 (2) implies our assertion. □

Proposition 2.5.2. *It follows that*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\geq\delta_0 \\ z\in K}} \Big| \frac{1}{B_n(\eta)} \log P_x^\eta[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z] + d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\Big)\Big| = 0$$

for any compact set $K \subset (0,\infty)$, $\delta_0 > 0$.

Proof. We follow the proof of Large deviation in VII.3. of [E].
(i) From Chebyshev's inequality we have

$$P_x^\eta\Big[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z\Big] \leq E_x^\eta[e^{s(B_n(\eta)z - T_n(\eta)\tau_y)}] = e^{B_n(\eta)(sz - k_n^{x,y}(s,\eta))}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $s > 0$, $z > 0$, $x, y \in F^\eta$ and $n \in \mathbb{N}$. We can choose a compact set $\Gamma(K,\delta_0) \subset (0,\infty)$ such that

$$\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\Big) = \sup_{s\in\Gamma(K,\delta_0)} \Big\{\Psi(s,\theta^n\chi_\eta) - \frac{z}{d_\eta(x,y)} s\Big\} \tag{2.5.1}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \geq \delta_0$, $n \in \mathbb{N}$ and $z \in K$ from Lemma 2.5.1. By Lemma 2.4.6 for any $\epsilon > 0$ there exists $N = N(\epsilon, \Gamma(K,\delta_0), \delta_0)$ such that if $n \geq N$ then

$$\frac{1}{B_n(\eta)} \log P_x^\eta\Big[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z\Big] \leq -(d_\eta(x,y)\Psi(s,\theta^n\chi_\eta) - zs) + \epsilon$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $s \in \Gamma(K,\delta_0)$, $z \in K$, $x, y \in F^\eta$ with $d_\eta(x,y) \geq \delta_0$. Since $s \in \Gamma(K,\delta_0)$ is arbitrary, (2.5.1) implies that for any $\epsilon > 0$ if $n \geq N$ then

$$\frac{1}{B_n(\eta)} \log P_x^\eta\Big[\tau_y \leq \frac{B_n(\eta)}{T_n(\eta)} z\Big] \leq -d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\Big) + \epsilon$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $x, y \in F^\eta$ with $d_\eta(x,y) \geq \delta_0$.

(ii) Without loss of generality, we can assume that K is a closed interval $[p,q]$ for $p,q \in (0,\infty)$ with $p<q$. Let $Q_n^{\eta,x,y}$ be the distribution function of $T_n(\eta)\tau_y/B_n(\eta)$ under P_x^η. Define probability measures

$$dQ_{n,t}^{\eta,x,y}(v) = \frac{\exp(-B_n(\eta)tv)}{\exp(-B_n(\eta)k_n^{x,y}(t,\eta))} dQ_n^{\eta,x,y}(v) \quad \text{for each} \quad t \geq 0.$$

To simplify the notation, we will drop the subscript x,y,η and refer to $Q_n^{\eta,x,y}$ (resp. $Q_{n,t}^{\eta,x,y}$) as Q_n (resp. $Q_{n,t}$). By virtue of continuity of Ψ^*, for any $\epsilon>0$ we can choose $\gamma_\epsilon>0$ such that $p>2\gamma_\epsilon$ and

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\geq\delta_0 \\ z\in K, n\in\mathbb{N}}} \left|\Psi^*\left(\frac{z-\gamma_\epsilon}{d_\eta(x,y)}, \theta^n\chi_\eta\right) - \Psi^*\left(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\right)\right| < \epsilon. \tag{2.5.2}$$

Let us abbreviate $z-\gamma_\epsilon$ by z_{γ_ϵ}. Then we have

$$\begin{aligned} Q_n[(0,z]] &\geq Q_n[(z_{\gamma_\epsilon}-\beta, z_{\gamma_\epsilon}+\beta)] \\ &= \exp(-B_n(\eta)k_n^{x,y}(t,\eta)) \int_{z_{\gamma_\epsilon}-\beta}^{z_{\gamma_\epsilon}+\beta} \exp(B_n(\eta)tv)dQ_{n,t}(v) \end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $x,y\in F^\eta$, $z\in K$ and $\beta>0$ with $\beta<\gamma_\epsilon$. If v is a point in $(z_{\gamma_\epsilon}-\beta, z_{\gamma_\epsilon}+\beta)$, then $tz_{\gamma_\epsilon}-t\beta < tv$ for all $t>0$. Hence we have

$$\begin{aligned} \log Q_n[(0,z]] \geq &- B_n(\eta)k_n^{x,y}(t,\eta) \\ &+ B_n(\eta)(tz_{\gamma_\epsilon}-t\beta) + \log Q_{n,t}[(z_{\gamma_\epsilon}-\beta, z_{\gamma_\epsilon}+\beta)]. \end{aligned}$$

Set $K_\epsilon = [p-\gamma_\epsilon, q-\gamma_\epsilon]$. Note that $z_{\gamma_\epsilon} \in [p-\gamma_\epsilon, q-\gamma_\epsilon]$ and $K_\epsilon \subset [p/2, q]$ for small enough $\epsilon>0$. Then we can choose a compact set $\Gamma(K_\epsilon,\delta_0) \subset (0,\infty)$ such that

$$\Psi^*\left(\frac{z_{\gamma_\epsilon}}{d_\eta(x,y)}, \theta^n\chi_\eta\right) = \sup_{s\in\Gamma(K_\epsilon,\delta_0)} \left\{\Psi(s,\theta^n\chi_\eta) - \frac{z_{\gamma_\epsilon}}{d_\eta(x,y)}s\right\}$$

for any $\eta\in\{2,3\}^{\mathbb{N}}$, $x,y\in F^\eta$ with $d_\eta(x,y)\geq\delta_0$, $n\in\mathbb{N}$ and $z\in K$ from Lemma 2.5.1. Let $t=t(\eta,x,y,z_{\gamma_\epsilon},n) = (\Psi')^{-1}(z_{\gamma_\epsilon}/d_\eta(x,y), \theta^n\chi_\eta) > 0$ for each $\eta\in\{2,3\}^{\mathbb{N}}$, $x,y\in F^\eta$ with $d_\eta(x,y)\geq\delta_0$, $n\in\mathbb{N}$ and $z_{\gamma_\epsilon}\in K_\epsilon$. Note that $z_{\gamma_\epsilon} = d_\eta(x,y)\Psi'(t,\theta^n\chi_\eta)$ and $t\in\Gamma(K_\epsilon,\delta_0)$. Also there exists a compact set $C\subset(0,\infty)$ such that $\Gamma(K_\epsilon,\delta_0)\subset C$ for small enough $\epsilon>0$. From (2.5.2) and Lemma 2.4.6, for any $\epsilon>0$ there exists

$N_1 = N_1(\Gamma(K_\epsilon, \delta_0), \delta_0, \epsilon)$ such that if $n \geq N_1$ then

$$\begin{aligned}\frac{\log Q_n[(0,z]]}{B_n(\eta)} - \frac{\log Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]}{B_n(\eta)} &\geq -\Big(k_n^{x,y}(t,\eta) - z_{\gamma_\epsilon} t\Big) - \beta t \\ &\geq - \sup_{s \in \Gamma(K_\epsilon, \delta_0)} \{d_\eta(x,y)\Psi(s, \theta^n \chi_\eta) - z_{\gamma_\epsilon} s\} - \beta t - \epsilon \qquad (2.5.3) \\ &\geq -d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big) - \beta t - (c+1)\epsilon\end{aligned}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \geq \delta_0$, $z \in K$ and $\beta > 0$ with $\beta < \gamma_\epsilon$, where c is the constant in (2.1.5). Now we consider $Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]$. Let us fix a $\epsilon > 0$. First, we have

$$\begin{aligned}&Q_{n,t}[\{v \in [0,\infty) : |v - z_{\gamma_\epsilon}| > \beta\}] \\ &= \int_{\{v \leq z_{\gamma_\epsilon} - \beta\}} dQ_{n,t}(v) + \int_{\{v \geq z_{\gamma_\epsilon} + \beta\}} dQ_{n,t}(v) \\ &\leq \exp(-\beta B_n(\eta) s)\Big(\int_0^\infty \exp(-sB_n(\eta)(v - z_{\gamma_\epsilon}))dQ_{n,t}(v) \\ &\qquad + \int_0^\infty \exp(sB_n(\eta)(v - z_{\gamma_\epsilon}))dQ_{n,t}(v)\Big).\end{aligned}$$

By the way for $s > 0$ we have

$$\begin{aligned}&\frac{1}{B_n(\eta)} \log \int_0^\infty \exp(-B_n(\eta)s(v - z_{\gamma_\epsilon}))dQ_{n,t}(v) \\ &\qquad = -\big(k_n^{x,y}(t+s,\eta) - k_n^{x,y}(t,\eta) - d_\eta(x,y)\Psi'(t, \theta^n\chi_\eta)s\big). \qquad (2.5.4)\end{aligned}$$

Let $s_0 \in (0, \min \Gamma(K_\epsilon, \delta_0)/2)$ and $K_{\epsilon,\delta_0} = [\min \Gamma(K_\epsilon, \delta_0) - s_0, \max \Gamma(K_\epsilon, \delta_0) + s_0] \supset \Gamma(K_\epsilon, \delta_0)$. Then by easy calculations we see that

$$\begin{aligned}&|\Psi(t+s, \theta^n\chi_\eta) - \Psi(t, \theta^n\chi_\eta) - \Psi'(t, \theta^n\chi_\eta)s| \\ &\qquad \leq s \sup_{\substack{u,v \in K_{\epsilon,\delta_0}, |u-v|<s \\ \xi \in \{2,3\}^{\mathbb{Z}}}} |\Psi'(u,\xi) - \Psi'(v,\xi)|\end{aligned}$$

for any $t \in \Gamma(K_\epsilon, \delta_0)$, $s \leq s_0$. Let $r(s), s \leq s_0$ denote the right hand side of above inequality. Then from Lemma 2.4.6 for any $\epsilon' > 0$ there exists $N_2 = N_2(K_{\epsilon,\delta_0}, \delta_0, \epsilon')$ such that if $n \geq N_2$ then

$$\begin{aligned}&|k_n^{x,y}(t+s,\eta) - k_n^{x,y}(t,\eta) - d_\eta(x,y)\Psi'(t,\theta^n\chi_\eta)s| \\ &\leq |k_n^{x,y}(t+s,\eta) - d_\eta(x,y)\Psi(t+s, \theta^n\chi_\eta) - (k_n^{x,y}(t,\eta) - d_\eta(x,y)\Psi(t,\theta^n\chi_\eta))| \\ &+ d_\eta(x,y)|\Psi(t+s,\theta^n\chi_\eta) - \Psi(t,\theta^n\chi_\eta) - \Psi'(t,\theta^n\chi_\eta)s| \leq 2\epsilon' + cr(s)\end{aligned}$$

for any $t \in \Gamma(K_\epsilon, \delta_0)$, $s \le s_0$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$. Hence by adding (2.5.4) we obtain

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K, t\in\Gamma(K_\epsilon,\delta_0)}} \frac{1}{B_n(\eta)} \log \int_0^\infty \exp(-B_n(\eta)s(v - z_{\gamma_\epsilon}))dQ_{n,t}(v) \le 2\epsilon' + cr(s)$$

for any $s \le s_0$ and $n \ge N_2$. By replacing s with $-s$ ($s > 0$), we deduce the following in exactly the same way. For any $\epsilon' > 0$ there exists $N_3 = N_3(K_{\epsilon,\delta_0}, \delta_0, \epsilon') > 0$ such that if $n \ge N_3$ then

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K, t\in\Gamma(K_\epsilon,\delta_0)}} \frac{1}{B_n(\eta)} \log \int_0^\infty \exp(B_n(\eta)s(v - z_{\gamma_\epsilon}))dQ_{n,t}(v) \le 2\epsilon' + cr(s)$$

for any $s \le s_0$ and $n \ge N_3$. Therefore it follows that for any $\epsilon' > 0$ there is $N_4(K_{\epsilon,\delta_0}, \delta_0, \epsilon') = N_2(K_{\epsilon,\delta_0}, \delta_0, \epsilon') \vee N_3(K_{\epsilon,\delta_0}, \delta_0, \epsilon')$ such that if $n \ge N_4$ then

$$\begin{aligned}\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K, t\in\Gamma(K_\epsilon,\delta_0)}} & \frac{1}{B_n(\eta)} \log Q_{n,t}[\{v \in [0,\infty) : |v - z_{\gamma_\epsilon}| > \beta\}] \\ & \le -\beta s + 2\epsilon' + cr(s) + \frac{\log 2}{B_n(\eta)}\end{aligned}$$

for any $\beta > 0$ with $\beta < \gamma_\epsilon$ and $s < s_0$. From Lemma 2.3.2 (1) for any $\beta > 0$ with $\beta < \gamma_\epsilon$ there exist $\epsilon' = \epsilon'(\beta) > 0$, $s_1 = s_1(\beta) > 0, \delta = \delta(\beta) > 0, N_5 = N_5(K_{\epsilon,\delta_0}, \delta_0, \beta) > 0$ such that if $n > N_5$ then $-\beta s_1 + 2\epsilon' + cr(s_1) + \log 2/B_n(\eta) < -\delta$. Consequently for any $\beta > 0$ with $\beta < \gamma_\epsilon$ there is $\delta = \delta(\beta) > 0$ such that

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K, t\in\Gamma(K_\epsilon,\delta_0)}} Q_{n,t}[\{v \in [0,\infty) : |v - z_{\gamma_\epsilon}| > \beta\}] \le \exp(-\delta B_n(\eta))$$

for any $n \ge N_4(K_{\epsilon,\delta_0}, \delta_0, \epsilon'(\beta)) \vee N_5(K_{\epsilon,\delta_0}, \delta_0, \beta)$. Therefore for any $\epsilon'' > 0$ there exists $N_6 = N_6(\epsilon'', K_{\epsilon,\delta_0}, \delta_0, \beta)$ such that if $n \ge N_6$ then

$$\inf_{\eta\in\{2,3\}^{\mathbb{N}}} \inf_{\substack{d_\eta(x,y)\ge\delta_0 \\ z\in K, t\in\Gamma(K_\epsilon,\delta_0)}} \log Q_{n,t}[(z_{\gamma_\epsilon} - \beta, z_{\gamma_\epsilon} + \beta)]/B_n(\eta) > -\epsilon''.$$

As a result for any $\epsilon, \epsilon'' > 0$, $\beta < \gamma_\epsilon$ there is $N = N(\epsilon, \epsilon'', \beta, \delta_0, K) = N_1 \vee N_6$ such that if $n \ge N$ then

$$\frac{\log Q_n[(0,z]]}{B_n(\eta)} + d(x,y)\Psi^*\Big(\frac{z}{d(x,y)}, \theta^n \chi_\eta\Big) \qquad \ge -\beta t - (c+1)\epsilon - \epsilon''$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$. (i) and (ii) imply our assertion. □

From this proposition we have the following.

Proposition 2.5.3. *It follows that*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y)\geq\delta_0\\ z\in K}} \Big| \frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\Big)\Big| = 0$$

for any compact set $K \subset (0,\infty)$ and $\delta_0 > 0$.

Proof. We follow the proof of Theorem 1.2 in [Kum]. Let $A^\eta_t(y) = \min_{0\leq s\leq t} p^\eta_{t-s}(y,y)$ for each $\eta \in \{2,3\}^{\mathbb{N}}$, $t \leq 1$ and $y \in F^\eta$. By Theorem 2.1.2, there exists $l = l_{t,s} \in \mathbb{N}$ such that $c_3/(t-s)^{d^\eta_s(l)/2} \leq p^\eta_{t-s}(y,y) \leq c_1/(t-s)^{d^\eta_s(l)/2}$. Then we have

$$\frac{c_3}{t^{d^{\mathbf{2}}_s/2}} \leq A^\eta_t(y) \leq \frac{c_1}{t^{d^{\mathbf{3}}_s/2}} \tag{2.5.5}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $t \leq 1$ and $y \in F^\eta$, where $d^{\mathbf{2}}_s/2 = \log m(2)/\log t(2)$ and $d^{\mathbf{3}}_s/2 = \log m(3)/\log t(3)$. This follows because $d^{\mathbf{2}}_s/2 \leq d^\eta_s(l)/2 \leq d^{\mathbf{3}}_s/2 < 1$ and $(t-s)^{d^{\mathbf{3}}_s/2} \leq (t-s)^{d^\eta_s(l)/2} \leq (t-s)^{d^{\mathbf{2}}_s/2}$ for all $\eta \in \{2,3\}^{\mathbb{N}}$, $l \in \mathbb{N}$ and s,t with $0 \leq s \leq t \leq 1$. Since $p^\eta_t(x,y) = \int_0^t p^\eta_{t-s}(y,y)P^\eta_x[\tau_y \in ds]$, we see that $p^\eta_t(x,y) \geq A^\eta_t(y)P^\eta_x[\tau_y \leq t]$. Note that by (2.5.5)

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{z\in K, y\in F^\eta} \Big|\frac{1}{B_n(\eta)} \log A^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(y)\Big| = 0.$$

Hence for any $\epsilon > 0$ there exists a $N = N(\epsilon, K, \delta_0)$ such that if $n \geq N$ then

$$\frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n\chi_\eta\Big) \geq -\epsilon \tag{2.5.6}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$ and $x, y \in F^\eta$ with $d_\eta(x,y) > \delta_0$ by Proposition 2.5.2.
Next let $D_l(y) = \{C : C \text{ is an } l\text{-complex which contains } y\}$ and $D^1_l(y) = D_l(y) \cup \{C : C$ is an l-complex which connected to $D_l(y)\}$ for each $y \in F^\eta$ and $l \in \mathbb{N}$. Note that there is $i_0 > 0$ such that $\sup_{y\in F^\eta, l\in\mathbb{N}} \sharp\{p \in F^\eta : p \text{ is the boundary of } D^1_l(y)\} \leq i_0$. Denote $b^l_i(y), i \in \mathbb{N}$ the boundary of $D^1_l(y)$ for each $y \in F^\eta$ and $l \in \mathbb{N}$. There exists $L = L(\delta_0) \in \mathbb{N}$ such that $\min_i d_\eta(x, b^L_i(y)) \geq \delta_0/2$ and $x \notin D^1_L(y)$ for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \geq \delta_0$. Letting $M^\eta_{L,t}(y) = \max_{i, 0\leq s\leq t} p^\eta_{t-s}(b^L_i(y), y)$, there is $t_0 = t_0(L) > 0$ such that $\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{0<t\leq t_0} M^\eta_{L,t}(y) \leq 1$ by using Theorem 2.1.2. By

the way since

$$p_t^\eta(x,y) \le \sum_i \int_0^t p_{t-s}^\eta(b_i^L(y),y) P_x^\eta[\tau_{b_i^L(y)} \in ds]$$
$$\le M_{L,t}^\eta(y) i_0 \max_i P_x^\eta[\tau_{b_i^L(y)} \le t],$$

there exists $N = N(L,K)$ such that if $n \ge N$ then

$$\log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) \le \log i_0 + \max_i \log P_x^\eta[\tau_{b_i^L(y)} \le \frac{B_n(\eta)}{T_n(\eta)} z]$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$ and $z \in K$. By virtue of Proposition 2.5.2 for any $\epsilon > 0$ there exists $N' = N'(\epsilon, L, \delta_0, K)$ such that if $n \ge N'$ then

$$\frac{1}{B_n(\eta)} \log P_x^\eta[\tau_{b_i^L(y)} \le \frac{B_n(\eta)}{T_n(\eta)} z]$$
$$\le -d_\eta(x, b_i^L(y)) \Psi^*\Big(\frac{z}{d_\eta(x, b_i^L(y))}, \theta^n \chi_\eta\Big) + \frac{\epsilon}{2}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$, $i \le i_0$ and $z \in K$. Since we have

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{x,y \in F^\eta, i} |d_\eta(x,y) - d_\eta(x, b_i^l(y))| \le \sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{y \in F^\eta, i} d_\eta(y, b_i^l(y)) \to 0$$

as $l \to \infty$, Lemma 2.3.4 (1) implies that for any $\epsilon > 0$ there is $L_1 = L_1(\epsilon, \delta_0, K) \in \mathbb{N}$ such that

$$\Big| d_\eta(x, b_i^{L_1}(y)) \Psi^*(\frac{z}{d_\eta(x, b_i^{L_1}(y))}, \theta^n \chi_\eta) - d_\eta(x,y) \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big) \Big| < \frac{\epsilon}{2}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$, $z \in K$, $i \le i_0$ and $n \in \mathbb{N}$. As a consequence for any $\epsilon > 0$ there exists $N'' = N(L \vee L_1, K) \vee N'(\epsilon, L \vee L', \delta_0, K)$ such that if $n \ge N''$ then

$$\frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y) \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big) \le \epsilon \tag{2.5.7}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $x, y \in F^\eta$ with $d_\eta(x,y) \ge \delta_0$ and $z \in K$. (2.5.6) and (2.5.7) imply our assertion. □

Theorem 2.5.4. *There exists a continuous function $\Psi^* : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ such that*

$$\lim_{n \to \infty} \sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{\substack{x,y \in F^\eta \\ z \in K}} \Big| \Big(\frac{B_n(\eta)}{T_n(\eta)}\Big)^{1/(d_w^\eta(n)-1)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y)$$
$$+ d_\eta(x,y) \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big) \Big| = 0$$

for any compact set $K \subset (0,\infty)$.

Proof. Recall that the upper bound of the heat kernel of Theorem 2.1.2 (b) is written in the following form (4.21) in [BH]. There exist constants $c_1, c_2 > 0$ such that if $1/B_m(\eta) \le d_\eta(x,y) < 1/B_{m-1}(\eta), 1/T_n(\eta) \le t < 1/T_{n-1}(\eta)$ then

$$p_t(x,y) \le c_1 M_n(\eta) \exp\Big(- c_2 \frac{B_{m+k_\eta(m,n)}(\eta)}{B_m(\eta)}\Big). \tag{2.5.8}$$

Because of this we have

$$\frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) \le \frac{\log(c_1 + M_{l(n,z,\eta)})}{B_n(\eta)} - \frac{c_2}{B_n(\eta)} \frac{B_{m+k_\eta(m,l(n,z,\eta))}(\eta)}{B_m(\eta)},$$

where $m \in \mathbb{N}$ with $1/B_m(\eta) \le d_\eta(x,y) \le 1/B_{m-1}(\eta)$, $l = l(n,z,\eta) \in \mathbb{N}$ with $1/T_l(\eta) \le B_n(\eta)z/T_n(\eta) < 1/T_{l-1}(\eta)$. There exists a constant $c = c(K) \in \mathbb{Z}_+$ such that $B_{n+c}(\eta)/T_{n+c}(\eta) \le B_n(\eta)z/T_n(\eta)$ for any $z \in K \subset (0,\infty)$. Then we have $T_{l-1}(\eta) \le T_{n+c}(\eta)/B_{n+c}(\eta)$. Since if $m \ge 4$ then $t(3) = 90/7 \le 2^4 = b(2)^4 \le B_m(\eta)$, we see that

$$\frac{T_{l(n,z,\eta)}(\eta)}{B_m(\eta)} \le \frac{T_{l(n,z,\eta)-1}(\eta)t(3)}{B_m(\eta)} \le \frac{T_{n+c}(\eta)}{B_{n+c}(\eta)}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$, $z \in K$ and $m \ge 4$. Recalling the definition (2.1.7) of k_η, we deduce $k_\eta(m, l(n,z,\eta)) + m \le n + c$. Thus we obtain

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \frac{1}{B_n(\eta)} \sup_{m \ge M, z \in K} \frac{B_{m+k_\eta(m,l(n,z,\eta))}(\eta)}{B_m(\eta)} \le \frac{b(3)^c}{b(2)^M}$$

for any $M \ge 4$, $n \ge 1$. By the way from Lemma 2.3.4 (1) for any $\epsilon > 0$ we choose $M = M(\epsilon, K) \ge 4$, $\delta = \delta(\epsilon, K) > 0$ and $N = N(\epsilon, K) \in \mathbb{N}$ such that

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y) \le \delta, n \in \mathbb{N}} d_\eta(x,y) \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big) \le \frac{\epsilon}{3},$$

$$c_2 \frac{b(3)^c}{b(2)^M} \le \frac{\epsilon}{3} \quad \text{and} \quad \sup_{\eta \in \{2,3\}^{\mathbb{N}}, z \in K, n \ge N} \frac{\log(c_1 + M_{l(n,z,\eta)})}{B_n(\eta)} \le \frac{\epsilon}{3}.$$

As a result for any $\epsilon > 0$ there are $\delta_0(\epsilon, K) = \delta(\epsilon, K) \wedge b(3)^{-M(\epsilon,K)}$ and $N = N(\epsilon, K) > 0$ such that

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{\substack{d_\eta(x,y) \le \delta_0 \\ z \in K}} \Big| \frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y) \Psi^*\Big(\frac{z}{d_\eta(x,y)}, \theta^n \chi_\eta\Big)\Big| \le \epsilon$$

for any $n \ge N$. By adding Proposition 2.5.3 this completes the proof. $\square$

Let us state some corollaries.

Corollary 2.5.5. *Let $\eta \in \{2,3\}^{\mathbb{N}}$. If a subsequence $\{\theta^{n_k}\chi_\eta\}_{k\in\mathbb{N}}$ of $\{\theta^n\chi_\eta\}_{n\in\mathbb{N}}$ converges to $\xi_0 \in \{2,3\}^{\mathbb{Z}}$ as $k \to \infty$, then*

$$\lim_{k\to\infty}\Big(\frac{B_{n_k}(\eta)}{T_{n_k}(\eta)}\Big)^{1/(d_w^\eta(n_k)-1)} \log p^\eta_{\frac{B_{n_k}(\eta)}{T_{n_k}(\eta)}z}(x,y) = -d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)},\xi_0\Big).$$

This convergence is uniform with respect to $x, y \in F^\eta$ and $z \in K$.

Proof. Let $\eta \in \{2,3\}^{\mathbb{N}}$. For any $\epsilon > 0$ there is $\delta_0 > 0$ such that

$$\sup_{\substack{d_\eta(x,y)<\delta_0, z\in K\\ \xi\in\{2,3\}^{\mathbb{Z}}}} d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)},\xi_0\Big) < \frac{\epsilon}{2}$$

from Lemma 2.3.4 (2). Therefore we get

$$\lim_{k\to\infty}\sup_{x,y\in F^\eta, z\in K}\Big|d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)},\theta^{n_k}\chi_\eta\Big) - d_\eta(x,y)\Psi^*\Big(\frac{z}{d_\eta(x,y)},\xi_0\Big)\Big| = 0$$

from Lemma 2.3.4 (1). This implies our assertion. □

Corollary 2.5.6. *Let $K \subset (0,\infty)$ be a compact set. There exists a function $V_z : (0,\infty)\times\{2,3\}^{\mathbb{Z}} \to (0,\infty)$ for each $z \in K$ and constants $c = c(K), c' = c'(K) > 0$ such that $c \le V_z(s,\xi) \le c'$ for all $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z \in K$ and*

$$\lim_{n\to\infty}\sup_{\eta\in\{2,3\}^{\mathbb{N}}}\sup_{\substack{x,y\in F^\eta\\ z\in K}}\Big|\Big(\frac{B_n(\eta)}{T_n(\eta)}\Big)^{1/(d_w^\eta(n)-1)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) \\ + d_\eta(x,y)^{d_w^{\theta^n\chi_\eta}(m)/(d_w^{\theta^n\chi_\eta}(m)-1)} V_z\Big(\frac{z}{d_\eta(x,y)},\theta^n\chi_\eta\Big)\Big| = 0,$$

where $m = m_{\eta,n,z,x,y}$ is an integer with

$$B_m(\theta^n\chi_\eta)/T_m(\theta^n\chi_\eta) \le z/d_\eta(x,y) < B_{m-1}(\theta^n\chi_\eta)/T_{m-1}(\theta^n\chi_\eta).$$

Proof. For each $z \in K$ we define $V_z : (0,\infty)\times\{2,3\}^{\mathbb{Z}} \to (0,\infty)$ by

$$V_z(t,\xi) = t^{1/(d_w^\xi(m)-1)}\Psi^*(t,\xi)z^{-1/(d_w^\xi(m)-1)},$$

where m is an integer with $B_m(\xi)/T_m(\xi) \le t < B_{m-1}(\xi)/T_{m-1}(\xi)$. Then Lemma 2.3.4 implies our assertion. □

Finally we state the other corollary. Let $G : (0,\infty) \times \{2,3\}^{\mathbb{Z}} \to (0,\infty)$ be given by

$$G(s,\xi) = s^{1/(d_w^{\xi}(m)-1)} \Psi^*(s,\xi)$$

where $B_m(\xi)/T_m(\xi) \le s < B_{m-1}(\xi)/T_{m-1}(\xi)$. Then there exist constants $c, c' > 0$ such that $c \le G(s,\xi) \le c'$ for any $s \in (0,\infty)$, $\xi \in \{2,3\}^{\mathbb{Z}}$ by Lemma 2.3.4 (2). Then we have the following.

Corollary 2.5.7. *It follows that*

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{x,y\in F^\eta} \Big| t^{1/(d_w^{\chi_\eta}(m)-1)} \log p_t^\eta(x,y) \\ + d_\eta(x,y)^{d_w^{\chi_\eta}(m)/(d_w^{\chi_\eta}(m)-1)} G\Big(\frac{t}{d_\eta(x,y)}, \chi_\eta\Big)\Big| = 0,$$

where $m = m_{\eta,t,x,y}$ is an integer with

$$B_m(\chi_\eta)/T_m(\chi_\eta) \le t/d_\eta(x,y) < B_{m-1}(\chi_\eta)/T_{m-1}(\chi_\eta).$$

Proof. For any $\epsilon > 0$, $\delta_0 > 0$ there exists $N = N(\epsilon, \delta_0, K) > 0$ such that if $n \ge N$ then

$$\sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{\substack{z\in K \\ d_\eta(x,y)\ge\delta_0}} \left| \frac{1}{\Psi^*\Big(z/d_\eta(x,y), \theta^n\chi_\eta\Big)} \frac{1}{B_n(\eta)} \log p^\eta_{\frac{B_n(\eta)}{T_n(\eta)}z}(x,y) + d_\eta(x,y)\right| < \epsilon$$

by Proposition 2.5.3, where $K = [1, t(3)/b(3)]$. Let $n = n(\eta,t)$ be an integer satisfying $B_n(\eta)/T_n(\eta) \le t < B_{n-1}(\eta)/T_{n-1}(\eta)$ for each $\eta \in \{2,3\}^{\mathbb{N}}$ and $t > 0$. Note that setting $z = z(\eta,t) = tT_{n(\eta,t)}(\eta)/B_{n(\eta,t)}(\eta)$, we have $z \in K$. Since we see that $\inf_{\eta\in\{2,3\}^{\mathbb{N}}} n(\eta,t) \ge n(\mathbf{3},t) > N$ for small enough $t > 0$, it follows that

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y)\ge\delta_0} \left| \frac{1}{\Psi^*\Big(t/d_\eta(x,y), \chi_\eta\Big)} \log p_t^\eta(x,y) + d_\eta(x,y)\right| = 0$$

for any $\delta_0 > 0$ from (2.3.6). By the definition G we can easily see that

$$\lim_{t\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \sup_{d_\eta(x,y)\ge\delta_0} \Big| t^{1/(d_w^{\chi_\eta}(m)-1)} \log p_t^\eta(x,y) \\ + d_\eta(x,y)^{d_w^{\chi_\eta}(m)/(d_w^{\chi_\eta}(m)-1)} G\Big(\frac{t}{d_\eta(x,y)}, \chi_\eta\Big)\Big| = 0$$

for any $\delta_0 > 0$, where $m = m_{\eta,t,x,y}$ is an integer with $B_m(\chi_\eta)/T_m(\chi_\eta) \le t/d_\eta(x,y) < B_{m-1}(\chi_\eta)/T_{m-1}(\chi_\eta)$.

Next we consider when x and y are near enough. If $B_m(\eta)^{-1} \leq d_\eta(x,y) < B_{m-1}(\eta)^{-1}$ and $T_l(\eta)^{-1} \leq t < T_{l-1}(\eta)^{-1}$, by (2.5.8) and Theorem 2.1.2 (2) there exist constants $c_1, c_2 > 0$ such that

$$\log p_t^\eta(x,y) \leq \log c_1 - \frac{d_s^\eta(l)}{2}\log t - c_2 \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)}.$$

It is enough to prove that for any ϵ there is a positive constant $\delta_0 = \delta_0(\epsilon) < 1$ such that

$$\sup_{\eta \in \{2,3\}^{\mathbb{N}}} \sup_{\substack{t<\delta_0 \\ d_\eta(x,y)<\delta_0}} t^{1/(d_w^{\chi_\eta}(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} < \epsilon, \tag{2.5.9}$$

where n is an integer with $B_n(\chi_\eta)/T_n(\chi_\eta) \leq t/d_\eta(x,y) < B_{n-1}(\chi_\eta)/T_{n-1}(\chi_\eta)$. For $c \in \mathbb{N}$ with $(b(2)/t(2))^c \leq 1/t(3)$, we choose $\delta_0 = \delta_0(\epsilon, c) > 0$ such that

$$b(3)^c \Big(\frac{t(3)}{b(3)}\Big)^{1/(d_w^2-1)} \delta_0^{1/(d_w^3-1)+1} < \epsilon.$$

Now there are following three cases : (i) $d_\eta(x,y) > t$ and $m < l$, (ii) $d_\eta(x,y) > t$ and $m \geq l$, (iii) $d_\eta(x,y) \leq t$ and $m \geq l$. Since $k_\eta(m,l) = 0$ in case (ii) and (iii), it is easy to see that (2.5.9) holds. So we consider the case (i). In this case since $n \in \mathbb{N}$, we have $B_n(\chi_\eta) = B_n(\eta)$ and $T_n(\chi_\eta) = T_n(\eta)$. From the definition of c we see that

$$\frac{B_{n+c}(\eta)}{T_{n+c}(\eta)} \leq \frac{B_n(\eta)}{T_n(\eta)} \frac{b(2)^c}{t(2)^c} \leq \frac{1}{t(3)} \frac{t}{d_\eta(x,y)} \leq \frac{B_m(\eta)}{T_l(\eta)}.$$

This implies $m + k_\eta(m,l) \leq n + c$. Since $(B_n(\eta)/T_n(\eta))^{1/(d_w^\eta(n)-1)} = 1/B_n(\eta)$ we obtain

$$\begin{aligned} t^{1/(d_w^{\chi_\eta}(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} &\leq \Big(\frac{t(3)}{b(3)} \frac{B_n(\eta)}{T_n(\eta)} \delta_0\Big)^{1/(d_w^\eta(n)-1)} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_m(\eta)} \\ &\leq b(3)^c \Big(\frac{t(3)}{b(3)}\Big)^{1/(d_w^2-1)} \delta_0^{1/(d_w^3-1)+1} \frac{B_{m+k_\eta(m,l)}(\eta)}{B_{n+c}(\chi_\eta)} \leq \epsilon \end{aligned}$$

for any $\eta \in \{2,3\}^{\mathbb{N}}$, $t < \delta_0$, and $d_\eta(x,y) < \delta_0$. This completes our assertion. □

Chapter 3

Large Deviation

In this chapter we study large deviations for Brownian motion on the scale irregular Sierpinski gasket in the short time limit. Let $\{2,3\}^{\mathbb{N}}_x = \{\eta \in \{2,3\}^{\mathbb{N}} : x \in F^\eta\}$ for each $x \in \mathbb{R}^2$. We fix some point $x \in \mathbb{R}^2$ such that $\{2,3\}^{\mathbb{N}}_x$ is not empty through this chapter.

3.1 Preparations

We define some notations. For $\eta \in \{2,3\}^{\mathbb{N}}_x$, let $\Omega^\eta_x = C_x([0,1] \to F^\eta) = \{\omega \in C([0,1] \to F^\eta) : \omega(0) = x\}$ and $\Omega_x = C_x([0,1] \to \mathbb{R}^2) = \{\omega \in C([0,1] \to \mathbb{R}^2) : \omega(0) = x\}$. For $\omega_1, \omega_2 \in \Omega^\eta_x$, define $\|\omega_1 - \omega_2\|_\eta = \sup_{0 \le t \le 1} d_\eta(\omega_1(t), \omega_2(t))$. Also define $\|\omega_1 - \omega_2\| = \sup_{0 \le t \le 1} |\omega_1(t) - \omega_2(t)|$ for $\omega_1, \omega_2 \in \Omega_x$. The space Ω^η_x is metric space with $\|\cdot\|_\eta$ for each $\eta \in \{2,3\}^{\mathbb{N}}_x$ and so is space Ω_x with $\|\cdot\|$. Note that $\|\cdot\|_\eta$ and $\|\cdot\|$ are equivalent from (2.1.5). For $\phi \in \Omega^\eta_x$, we say ϕ is absolutely continuous if for each $\epsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n d_\eta(\phi(t_i), \phi(t_{i-1})) < \epsilon$ for any n and any disjoint collection of intervals $\{(t_{i-1}, t_i)\}_{i=1}^n$ in $[0,1]$ whose lengths satisfy $\sum_i (t_i - t_{i-1}) < \delta$. It can be proved by routine arguments that if ϕ is absolutely continuous, then $D_\eta \phi(t) = \lim_{s \to t} d_\eta(\phi(s), \phi(t))/|s-t|$ exists almost everywhere in $[0,1]$ and $D_\eta \phi \in L^1([0,1], dt)$ for each $\eta \in \{2,3\}^{\mathbb{N}}_x$ with and $\phi \in \Omega^\eta_x$. Define $I^\eta_{x,z} : \Omega_x \times \{2,3\}^{\mathbb{Z}} \to [0,\infty]$ by

$$I^\eta_{x,z}(\phi,\xi) = \begin{cases} \displaystyle\int_0^1 D_\eta\phi(t)\Psi^*\left(\frac{z}{D_\eta\phi(t)}, \xi\right) dt & \phi \in \Omega^\eta_x \text{ and } \phi \text{ is absolutely continuous,} \\ \infty & \text{otherwise} \end{cases}$$

for each $z > 0$, $\eta \in \{2,3\}^{\mathbb{N}}_x$. For each $\eta \notin \{2,3\}^{\mathbb{N}}_x$ and $z > 0$, define $I^{\eta}_{x,z}(\phi, \xi) = \infty$ for any $\xi \in \{2,3\}^{\mathbb{Z}}$ and $\phi \in \Omega_x$. Note that if $\phi \in C_x([0,1] \to \mathbb{R}^2)$ satisfies $I^{\eta}_{x,z}(\phi, \xi) < \infty$ for some $\eta \in \{2,3\}^{\mathbb{N}}$, $\xi \in \{2,3\}^{\mathbb{Z}}$ and $z > 0$ then $x \in F^{\eta}$ and $\phi \in C_x([0,1] \to F^{\eta})$. The following lemma is an easy consequence of Lemma 2.3.4 (2).

Lemma 3.1.1. *Let $K \subset (0, \infty)$ be a compact set. There exist constants $c = c(K), c' = c'(K) > 0$ such that*

$$c \int_0^1 D_{\eta}\phi(t)^{d^{\xi}_w(m)/(d^{\xi}_w(m)-1)} dt \leq I^{\eta}_{x,z}(\phi, \xi) \leq c' \int_0^1 D_{\eta}\phi(t)^{d^{\xi}_w(m)/(d^{\xi}_w(m)-1)} dt$$

for any $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\phi \in C_x([0,1] \to F^{\eta})$, $\xi \in \{2,3\}^Z$ and $z \in K$ where m is an integer with $B_m(\xi)/T_m(\xi) \leq z/D_{\eta}\phi(t) < B_{m-1}(\xi)/T_{m-1}(\xi)$.

Remark 3.1.1. *Let $K \subset (0, \infty)$ be a compact set. There exist constants $c = c(K), c' = c'(K) > 0$ such that*

$$c \int_0^1 T_m(\xi) dt \leq \int_0^1 D_{\eta}\phi(t)^{d^{\xi}_w(m)/(d^{\xi}_w(m)-1)} dt \leq c' \int_0^1 T_m(\xi) dt$$

for any $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\phi \in C_x([0,1] \to F^{\eta})$, $z \in K$ and $\xi \in \{2,3\}^Z$ where m is an integer with $B_m(\xi)/T_m(\xi) \leq z/D_{\eta}\phi(t) < B_{m-1}(\xi)/T_{m-1}(\xi)$.

3.2 Geodesic path and properties of $I^{\eta}_{x,z}$

The following result is fundamental.

Proposition 3.2.1. *Let $\eta \in \{2,3\}^{\mathbb{N}}_x$. For each $y, z \in F^{\eta}$ and $\alpha, \beta \in [0,1]$, there exists a geodesic path $\psi(t) \in \Omega^{\eta}_x, \alpha \leq t \leq \beta$ such that $\psi(\alpha) = y$, $\psi(\beta) = z$ and*

$$d_{\eta}(\psi(s), \psi(t)) = \frac{|t-s|}{\beta - \alpha} d_{\eta}(y, z)$$

for any $s, t \in [\alpha, \beta]$.

Proof. For Lemma 3.1 in [B], it is enough to show that the metric space (F^{η}, d_{η}) has the midpoint property: for each $y, z \in F^{\eta}$, there exists $v \in F^{\eta}$ such that $d_{\eta}(y, v) = d_{\eta}(v, z) = d_{\eta}(y, z)/2$. But this is obvious by the method of constructions of d_{η}. $\square$

Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be some time points. We denote by $\triangle$ this partition of the interval $[0,1]$. For $\phi \in \Omega^{\eta}_x$, we set $\prod_{\triangle} \phi = \{\phi(t_0), \phi(t_1), \cdots, \phi(t_m)\}$. Also define $\triangle\phi \in \Omega^{\eta}_x$ by taking points $\{\omega(t_j)\}$ and joining the successive ones by geodesic with natural parameterization. If there are more than one geodesics between two points, it is not important which one is chosen. Thus $\triangle\phi$ is piecewise geodesic and $\triangle\phi(t_j) = \phi(t_j)$ $(0 \le j \le m)$. Then we have the following.

Lemma 3.2.2. (1) *For each $\xi \in \{2,3\}^{\mathbb{Z}}$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\alpha, \beta \in [0,1]$, $z > 0$ and $a, b \in F^{\eta}$, it holds that*

$$\inf_{\substack{\phi \in \Omega_x \\ \phi(\alpha)=a, \phi(\beta)=b}} I^{\eta}_{x,z}(\phi, \xi) = d_{\eta}(a,b)\Psi^*\Big(\frac{z(\beta-\alpha)}{d_{\eta}(a,b)}, \xi\Big)$$

where the infimum is attained by the geodesic path $\psi \in \Omega^{\eta}_x$ which satisfies

$$d_{\eta}(\psi(s), \psi(t)) = \frac{|t-s|}{\beta-\alpha} d_{\eta}(a,b).$$

(2) *For each $\xi \in \{2,3\}^{\mathbb{Z}}$, $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $z > 0$ it holds that*

$$\inf_{\substack{\phi(t_i)=x_i \\ i=1,\ldots,m}} I^{\eta}_{x,z}(\phi, \xi) = \sum_{i=1}^{m} d_{\eta}(x_i, x_{i-1})\Psi^*\Big(\frac{z(t_i - t_{i-1})}{d_{\eta}(x_i, x_{i-1})}, \xi\Big),$$

where $x_0 = x$, $x_i \in F^{\eta}$, $1 \le i \le m$. In particular for $\phi \in C_x([0,1] \to F^{\eta})$,

$$I^{\eta}_{x,z}(\triangle\phi, \xi) = \inf_{\substack{\psi(t_j)=\phi(t_i) \\ i=1,\ldots,m}} I^{\eta}_{x,z}(\triangle\psi, \xi) \le I^{\eta}_{x,z}(\phi, \xi),$$

where $\triangle$ is the partition $t_0 = 0 < t_1 < \cdots < t_m = 1$.

Proof. This can be proved in the same way as [BK] Lemma 3.2. Note that (2) is an obvious extension of (1). For (1) it is enough to prove for the case $\alpha = 0$, $\beta = 1$ as otherwise the infimum is attained by the path which does not move in the intervals $[0,\alpha]$ and $[\beta,1]$. Now for each $\phi \in C_x([0,1] \to F^{\eta})$ which is absolutely continuous, set $L_{\eta}(\phi) = \int_0^1 D_{\eta}\phi(t)dt$. Then

$$\begin{aligned} I^{\eta}_{x,z}(\phi,\xi) &= \int_0^1 D_{\eta}\phi(t)\Psi^*\left(\frac{z}{D_{\eta}\phi(t)}, \xi\right) dt = L_{\eta}(\phi)\int_0^1 \Psi^*\left(\frac{z}{D_{\eta}\phi(t)}, \xi\right) D_{\eta}\phi(t)\frac{dt}{L_{\eta}(\phi)} \\ &\ge L_{\eta}(\phi)\Psi^*\left(\frac{1}{L_{\eta}(\phi)}\int_0^1 \frac{z}{D_{\eta}\phi(t)} D_{\eta}\phi(t)dt, \xi\right) = L_{\eta}(\phi)\Psi^*\left(\frac{z}{L_{\eta}(\phi)}, \xi\right) \\ &\ge d_{\eta}(a,b)\Psi^*\left(\frac{z}{d_{\eta}(a,b)}, \xi\right). \end{aligned}$$

Here we use Jensen's inequality in the first inequality and the second inequality is because $L_\eta(\phi) \ge d_\eta(a,b)$ and Ψ^* is monotone decreasing. As Ψ^* is strictly convex, the equalities hold if and only if $D_\eta\phi$ =constant and $L_\eta(\phi) = d_\eta(a,b)$. That is the geodesic with natural parametrization. We thus obtain the result. □

Using the results as $m=2,\ 0=t_0 \le t_1 = \alpha \le t_2 = \beta$ we see that

$$I^\eta_{x,z}(\phi,\xi) \ge d_\eta(\phi(\alpha),x)\Psi^*\Big(\frac{z\alpha}{d_\eta(\phi(\alpha),x)},\xi\Big)+d_\eta(\phi(\alpha),\phi(\beta))\Psi^*\Big(\frac{z(\beta-\alpha)}{d_\eta(\phi(\alpha),\phi(\beta))},\xi\Big) \tag{3.2.1}$$

for $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $\phi \in C_x([0,1] \to F^\eta)$.

Lemma 3.2.3. *For every $N>0$ and compact set $K \subset (0,\infty)$, the function family*

$$D = \bigcup_{\substack{z\in K,\ \xi\in\{2,3\}^{\mathbb{Z}}\\ \eta\in\{2,3\}^{\mathbb{N}}}} \{\phi \in \Omega_x : I^\eta_{x,z}(\phi,\xi) \le N\} \subset \bigcup_{\eta\in\{2,3\}^{\mathbb{N}}} \Omega^\eta_x$$

is equi-continuous.

Proof. Let $\phi \in D$. Then there are $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\xi \in \{2,3\}^{\mathbb{Z}}$, $z>0$ such that $\phi \in C_x([0,1] \to F^\eta)$ and $I^\eta_{x,z}(\phi,\xi) \le N$. Let n be an integer with

$$\frac{B_n(\xi)}{T_n(\xi)} \le \frac{z(\beta-\alpha)}{d_\eta(\phi(\alpha),\phi(\beta))} < \frac{B_{n-1}(\xi)}{T_{n-1}(\xi)}$$

for each $\alpha,\beta \in [0,1]$, $\phi \in D$, $z \in K$, $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $\xi \in \{2,3\}^{\mathbb{Z}}$. Then we have

$$\begin{aligned}(z(\beta-\alpha))^{1/(d^\xi_w(n)-1)} & I^\eta_x(\phi,\xi) \\ &\ge d_\eta(\phi(\alpha),\phi(\beta))^{d^\xi_w(n)/(d^\xi_w(n)-1)}\Big(\frac{z(\beta-\alpha)}{d_\eta(\phi(\alpha),\phi(\beta))}\Big)^{1/(d^\xi_w(n)-1)}\Psi^*\Big(\frac{z(\beta-\alpha)}{d_\eta(\phi(\alpha),\phi(\beta))},\xi\Big)\end{aligned}$$

from (3.2.1). By using (2.1.5) and Lemma 2.3.4 (2), there is a constant $c=c(N,K)$ such that

$$|\phi(\alpha)-\phi(\beta)| \le d_\eta(\phi(\alpha),\phi(\beta)) \le c(\beta-\alpha)^{1/d^3_w}$$

for $\beta-\alpha$ sufficiently small. □

Let $B(f,r) = \{\omega \in \Omega_x : \|f-\omega\| < r\}$ for each $f \in \Omega_x$. Also define $C_\delta = \{\phi \in \Omega_x : \|\phi-\psi\| < \delta \text{ for some } \psi \in C\} = \bigcup_{\psi\in C} B(\psi,\delta)$ for $C \in \Omega_x$ and $\delta > 0$.

Lemma 3.2.4. (1) *The function $I^{\eta}_{x,z}(\cdot,\xi):\Omega_x\to[0,\infty]$ is lower semi-continuous for each $\eta\in\{2,3\}^{\mathbb{N}}$, $z\in K$ and $\xi\in\{2,3\}^{\mathbb{Z}}$. Further, for every $N>0$ the set $\{\phi\in C_x([0,1]\to\mathbb{R}^2): I^{\eta}_{x,z}(\phi,\xi)\le N\}$ is compact for each $\eta\in\{2,3\}^{\mathbb{N}}_x$, $z>0$ and $\xi\in\{2,3\}^{\mathbb{Z}}$.*

(2) *If C is closed subset in Ω_x, then*

$$\lim_{\delta\to 0}\inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\xi)=\inf_{\phi\in C} I^{\eta}_{x,z}(\phi,\xi)$$

for each $\eta\in\{2,3\}^{\mathbb{N}}$, $\xi\in\{2,3\}^{\mathbb{Z}}$ and $z\in K$.

Proof. (1) Since $I^{\eta}_{x,z}\equiv\infty$ for $\eta\notin\{2,3\}^{\mathbb{N}}_x$, we can assume that $\eta\in\{2,3\}^{\mathbb{N}}_x$. For the lower semi-continuity, it is enough to show that if $I^{\eta}_{x,z}(\phi_m,\xi)\le N$ and $\|\phi_m-\phi\|\to 0$, then $I^{\eta}_{x,z}(\phi,\xi)\le N$. First note that there is a constant $c>0$ such that

$$\int_0^1 D_\eta\phi_n(t)^{d_w^{\mathbf{3}}/(d_w^{\mathbf{3}}-1)}dt\le 1+\int_{\{D_\eta\phi_n(t)\ge 1\}} D_\eta\phi_n(t)^{d_w^{\mathbf{3}}/(d_w^{\mathbf{3}}-1)}dt\le 1+cI^{\eta}_{x,z}(\phi_n,\xi)\le 1+cN$$

by Lemma 3.1.1. Now we proof that $\phi\in F^{\eta}$ is absolutely continuous. Let $\{(a_i,b_i)\}_{i=1}^n$ be a collection of mutually disjoint intervals in $[0,1]$. Then by Hölder's inequality

$$\begin{aligned}
&\sum_{i=1}^n d_\eta(\phi_m(a_i),\phi_m(b_i))\\
&\quad\le\sum_{i=1}^n\int_{a_i}^{b_i} D_\eta\phi_m(t)dt\le\sum_{i=1}^n\Big\{\Big(\int_{a_i}^{b_i} D_\eta\phi_m(t)^{d_w^{\mathbf{3}}/(d_w^{\mathbf{3}}-1)}dt\Big)^{(d_w^{\mathbf{3}}-1)/d_w^{\mathbf{3}}}|b_i-a_i|^{1/d_w^{\mathbf{3}}}\Big\}\\
&\quad\le\Big(\sum_{i=1}^n\int_{a_i}^{b_i} D_\eta\phi_m(t)^{d_w^{\mathbf{3}}/(d_w^{\mathbf{3}}-1)}dt\Big)^{(d_w^{\mathbf{3}}-1)/d_w^{\mathbf{3}}}\Big(\sum_{i=1}^n|b_i-a_i|\Big)^{1/d_w^{\mathbf{3}}}\\
&\quad=\Big(\int_0^1 D_\eta\phi_m(t)^{d_w^{\mathbf{3}}/(d_w^{\mathbf{3}}-1)}dt\Big)^{(d_w^{\mathbf{3}}-1)/d_w^{\mathbf{3}}}\Big(\sum_{i=1}^n|b_i-a_i|\Big)^{1/d_w^{\mathbf{3}}}\le c'\Big(\sum_{i=1}^n|b_i-a_i|,\Big)^{1/d_w^{\mathbf{3}}}.
\end{aligned}$$

where $c'=1+cN$. Hence ϕ is absolutely continuous. Next letting $f_z(x,\xi)=x\Psi^*(z/x,\xi)=\sup_{s>0}\{x\Psi(s,\xi)-zs\}$, it it obvious that $f_z(x,\xi)$ is monotone increasing, strictly convex by easy calculation. Note that $I^{\eta}_{x,z}(\phi,\xi)=\int_0^1 f_z(D_\eta\phi(t),\xi)dt$. By Jensen's inequality

$$\begin{aligned}
f_z\Big(\frac{d_\eta(\phi_m(t+h),\phi_m(t))}{h},\xi\Big)&\le f_z\Big(\int_t^{t+h} D_\eta\phi_m(s)\frac{ds}{h},\xi\Big)\\
&\le\int_t^{t+h} f_z(D_\eta\phi_m(s),\xi_m)\frac{ds}{h}=\frac{1}{h}\int_t^{t+h} f_z(D_\eta\phi_m(s),\xi)ds.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_0^{1-h} f_z\Big(\frac{d_\eta(\phi_m(t+h),\phi_m(t))}{h},\xi\Big)dt \\
&\quad\le \frac{1}{h}\int_0^{1-h} dt\int_t^{t+h} f_z(D_\eta\phi_m(s),\xi)ds = \frac{1}{h}\int_0^{1-h} dt\int_0^{h} f_z(D_\eta\phi_m(s+t),\xi)ds \\
&\quad= \frac{1}{h}\int_0^{h} ds\int_0^{1-h} f_z(D_\eta\phi_m(s+t),\xi)dt \le \frac{N}{h}\int_0^h ds = N.
\end{aligned}$$

Hence by Fatou's lemma we have

$$\int_0^{1-h} f_z\Big(\frac{d_\eta(\phi(t+h),\phi(t))}{h},\xi\Big)dt \le \liminf_{m\to\infty}\int_0^{1-h} f_z\Big(\frac{d_\eta(\phi_m(t+h),\phi_m(t))}{h},\xi\Big)dt \le N.$$

By Fatou's Lemma again, let $h \to 0$ to get $I^\eta_{x,z}(\phi,\xi) \le N$, completing the proof of the lower semi-continuity. Next Lemma 3.2.3 shows that $\{\phi \in \Omega_x : I^\eta_{x,z}(\phi,\xi) \le N\}$ is uniformly bounded and equi-continuous for each $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$. As it is closed by lower semi-continuity of $I^\eta_{x,z}(\cdot,\xi)$, the compactness follows Ascoli-Arzéla's theorem.

(2) Note that $\inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi,\xi)$ is decreasing with respect to δ. So we see that

$$\lim_{\delta\to 0}\inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi,\xi) \le \inf_{\phi\in C} I^\eta_{x,z}(\phi,\xi).$$

Let $\lambda = \lim_{\delta\to 0}\inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi,\xi)$. If $\lambda = \infty$ then $\inf_{\phi\in C} I^\eta_{x,z}(\phi,\xi) = \infty$. Assume that $\lambda < \infty$. Since there exists $\phi_\delta \in C_\delta$ such that $I^\eta_{x,z}(\phi_\delta,\xi) \to \lambda$, the family $\{\phi_\delta\} \subset \Omega^\eta_x$ is equi-continuous from Lemma 3.2.3. Hence Ascoli-Arzéla's theorem implies that there is a convergent subsequence $\{\phi_{\delta_n}\}_{n\in\mathbb{N}}$ with the limit ϕ_0. It is clear $\phi_0 \in C$. By the lower semi-continuity of $I^\eta_{x,z}(\cdot,\xi)$,

$$\lim_{\delta\to 0}\inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi,\xi) \ge \inf_{\phi\in C} I^\eta_{x,z}(\phi,\xi).$$

This completes the proof. □

From above lemma, $I^\eta_{x,z}(\cdot,\xi)$ is a rate function for each $\eta \in \{2,3\}^{\mathbb{N}}$, $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$. For $m \in \mathbb{N}$, let $\triangle_m : 0 = t_0 < t_1 < t_2 \cdots < t_m = 1$ be an equally spaced partition, i.e. $t_j = j/m (0 \le j \le m)$. Also Let $P^\eta_{x,\epsilon}$ be the law for $X^x(\epsilon\,\cdot\,)$ where X^x is the process stating x. We can regard $\{P^\eta_{x,\epsilon} : \epsilon > 0, \eta \in \{2,3\}^{\mathbb{N}}\}$ as the family of measures on $(\Omega_x, \mathcal{B}(\Omega_x))$, where $\mathcal{B}(\Omega_x)$ is Borel σ-field on Ω_x. For simplicity, we sometimes denote $P^\eta_{\epsilon^z_n}$ as $P^\eta_{x,\epsilon^z_n(\eta)}$. Set $\epsilon^z_n(\eta) = B_n(\eta)z/T_n(\eta)$. Then we have the following.

Lemma 3.2.5. *For every $\delta > 0$ and compact set $K \subset (0,\infty)$*

$$\lim_{l\to\infty} \sup_{\substack{n\in\mathbb{N},z\in K\\ \eta\in\{2,3\}^{\mathbb{N}}}} \frac{1}{B_n(\eta)} \log P^{\eta}_{x}\Big[\|\omega(\epsilon^z_n(\eta)\,\cdot\,) - \triangle_l\omega(\epsilon^z_n(\eta)\,\cdot\,)\|_{\eta} \geq \delta\Big] = -\infty.$$

In particular

$$\lim_{l\to\infty} \sup_{\substack{n\in\mathbb{N},z\in K\\ \eta\in\{2,3\}^{\mathbb{N}}}} \frac{1}{B_n(\eta)} \log P^{\eta}_{x}\Big[\|\omega(\epsilon^z_n(\eta)\,\cdot\,) - \triangle_l\omega(\epsilon^z_n(\eta)\,\cdot\,)\| \geq \delta\Big] = -\infty.$$

Proof. Let $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\omega \in \Omega^{\eta}_x$ and $l \in \mathbb{N}$. Since $\omega(t_j) = \triangle_l\omega(t_j)$ for $j = 0,\dots,l$ and $\triangle_l\omega$ is piecewise geodesic,

$$\begin{aligned} d_{\eta}(\triangle_l\omega(t), \triangle_l\omega(t_j)) &\leq d_{\eta}(\triangle_l\omega(t_{j+1}), \triangle_l\omega(t_j)) \\ &= d_{\eta}(\omega(t_{j+1}), \omega(t_j)) \leq \sup_{t_j\leq t\leq t_{j+1}} d_{\eta}(\omega(t), \omega(t_j)). \end{aligned}$$

for t in (t_j, t_{j+1}). Therefore we have

$$\begin{aligned} \sup_{0\leq t\leq 1} d_{\eta}(\omega(t),\triangle_l\omega(t)) &= \sup_{0\leq j\leq l-1} \sup_{t_j\leq t\leq t_{j+1}} d_{\eta}(\omega(t), \triangle_l\omega(t)) \\ &\leq \sup_{0\leq j\leq l-1} \sup_{t_j\leq t\leq t_{j+1}} \big(d_{\eta}(\omega(t), \omega(t_j)) + d_{\eta}(\triangle_l\omega(t_j), \triangle_l\omega(t))\big) \\ &\leq 2 \sup_{0\leq j\leq l-1} \sup_{t_j\leq t\leq t_{j+1}} d_{\eta}(\omega(t), \omega(t_j)). \end{aligned}$$

For $\delta > 0$ we choose $m \in \mathbb{N}$ such that $\delta > 2/2^m \geq 2/B_m(\eta)$. By Lemma 4.4 in [BH], we see that

$$\begin{aligned} &P^{\eta}_{\epsilon^z_n}\Big[\|\omega - \triangle_l\omega\|_{\eta} \geq \delta\Big] \leq P^{\eta}_{\epsilon^z_n}\Big[\|\omega - \triangle_l\|_{\eta} \geq \frac{2}{B_m(\eta)}\Big] \\ &= P^{\eta}_{\epsilon^z_n}\Big[\sup_{0\leq t\leq 1} d_{\eta}(\omega(t), \triangle_l\omega(t)) \geq \frac{2}{B_m(\eta)}\Big] \leq P^{\eta}_{\epsilon^z_n}\Big[\sup_{0\leq j\leq l-1} \sup_{t_j\leq t\leq t_{j+1}} d_{\eta}(\omega(t), \omega(t_j)) \geq \frac{1}{B_m(\eta)}\Big] \\ &\leq \sum_{j=0}^{l-1} P^{\eta}_{\epsilon^z_n}\Big[\sup_{t_j\leq t\leq t_{j+1}} d_{\eta}(\omega(t), \omega(t_j)) \geq \frac{1}{B_m(\eta)}\Big] \leq lP^{\eta}_{\epsilon^z_n}\Big[\sup_{0\leq t\leq 1/l} d_{\eta}(\omega(t), \omega(0)) \geq \frac{1}{B_m(\eta)}\Big] \\ &= lP^{\eta}_{x}\Big[\sup_{0\leq t\leq \epsilon^z_n(\eta)/l} d_{\eta}(\omega(t), \omega(0)) \geq \frac{1}{B_m(\eta)}\Big] \leq cl\exp\Big(-c'\frac{B_{m+k'}(\eta)}{B_m(\eta)}\Big), \end{aligned}$$

where $k' = k_{\eta}(m,n')$, $n' = n'(n,z,l,\eta)$ with $1/T_{n'}(\eta) \leq \epsilon^z_n(\eta)/l < 1/T_{n'-1}(\eta)$ and c, $c' > 0$ are constants in Lemma 4.4 in [BH] . Therefore we have

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon^z_n}\Big[\|\omega - \triangle_l\omega\|_{\eta} \geq \delta\Big] \leq \frac{\log(c_{11}l)}{B_n(\eta)} - \frac{cB_{m+k'}(\eta)}{B_m(\eta)B_n(\eta)}. \tag{3.2.2}$$

Next let

$$C = C_{m,l} = \max\{j \geq 0 : \frac{l}{3^m \max K} > \Big(\frac{t(3)}{b(3)}\Big)^j\}.$$

Note that

$$C_{m,l} \geq \frac{\log l - \log(3^m \max K)}{\log t(3) - \log b(3)} - 1 \quad \text{and} \quad \frac{l}{zB_m(\eta)} \geq \Big(\frac{t(3)}{b(3)}\Big)^{C_{m,l}} \tag{3.2.3}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$ and $z \in K$. Also Since the definition of n' yields

$$\frac{T_{n'}(\eta)}{B_m(\eta)} \geq \frac{T_n(\eta)}{B_n(\eta)} \frac{l}{zB_m(\eta)} \geq \frac{T_n(\eta)}{B_n(\eta)}\Big(\frac{t(3)}{b(3)}\Big)^C \geq \frac{T_{n+C}(\eta)}{B_{n+C}(\eta)} = \frac{T_{m+(n+C-m)}(\eta)}{B_{m+(n+C-m)}(\eta)},$$

by adding the definition of $k' = k_\eta(m, n')$, we get $n + C_{m,l} - m < k'$. Hence $B_{m+k'}(\eta)/B_n(\eta) > B_{n+C}(\eta)/B_n(\eta) > b(2)^C$. So we obtain

$$\begin{aligned} \sup_{\substack{n\in\mathbb{N}, z\in K \\ \eta\in\{2,3\}^{\mathbb{N}}}} \frac{1}{B_n(\eta)} \log P^x_{\epsilon^z_n}\Big[\|\omega - \triangle_l \omega\|_\eta > \delta\Big] &\leq \sup_{\substack{n\in\mathbb{N}, z\in K \\ \eta\in\{2,3\}^{\mathbb{N}}}} \Big(\frac{\log(cl)}{B_n(\eta)} - \frac{c'}{B_m(\eta)} b(3)^C\Big) \\ &\leq \log(cl) - \frac{c'}{b(3)^m} b(3)^{C_{m,l}} \end{aligned}$$

by (3.2.2). Then (3.2.3) implies our assertion. □

From Corollary 2.5.5, Lemma 3.2.2, 3.2.4 and 3.2.5 , we can deduce the following theorem in exactly the same way as [V] or [BK].

Theorem 3.2.6. *Let $\eta \in \{2,3\}^{\mathbb{N}}_x$. Let $C \subset \Omega^\eta_x$ be closed and $G \subset \Omega^\eta_x$ open. If there exists a sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\theta^{n_k}\chi \to \xi_0$, then*

$$\begin{aligned} -\inf_{\phi\in G} I^\eta_{x,z}(\phi, \xi_0) \leq \liminf_{k\to\infty} \frac{1}{B_{n_k}(\eta)} \log P^\eta_x[\omega(\epsilon^z_{n_k} \cdot\ ) \in G], \\ \limsup_{k\to\infty} \frac{1}{B_{n_k}(\eta)} \log P^\eta_x[\omega(\epsilon^z_{n_k} \cdot\ ) \in C] \leq -\inf_{\phi\in C} I^\eta_{x,z}(\phi, \xi_0). \end{aligned}$$

3.3 Upper bound

First, we prove the following.

Lemma 3.3.1. *Let $K \subset (0, \infty)$ be a compact set, $m \in \mathbb{N}$ and $\triangle$ be the partition $0 = t_0 < t_1 < t_2 \cdots < t_m = 1$. For any $\epsilon > 0$ there is $N = N(\epsilon, m, K) \in \mathbb{N}$ such that*

$$\frac{1}{B_n(\eta)} \log P^\eta_x[\omega(\epsilon^z_n(\eta)\ \cdot\ ) \in \textstyle\prod_\triangle^{-1} A] \leq \epsilon - \inf_{\phi\in\prod_\triangle^{-1} A} I^\eta_{x,z}(\phi, \theta^n\chi_\eta)$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$ and measurable set $A \subset \{x\} \times (\mathbb{R}^2)^m$.

Proof. It is enough to consider the case in which $\eta \in \{2,3\}^{\mathbb{N}}_x$. Let $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $A' \subset (\mathbb{R}^2)^m$ be $A = \{x\} \times A'$. First we have

$$\begin{aligned} P^\eta_x[\omega(\epsilon^z_n(\eta)\,\cdot\,) \in & \textstyle\prod_\triangle^{-1} A] = \int_{A'\cap(F^\eta)^m} \prod_{j=1}^m p^\eta(\epsilon^z_n(\eta)(t_j - t_{j-1}), y_{j-1}, y_j)\mu^\eta(dy_j) \\ &\le \mu^\eta_m(A' \cap (F^\eta)^m) \sup_{(y_1,\dots,y_m)\in A'\cap(F^\eta)^m} \prod_{j=1}^m p^\eta(\epsilon^z_n(\eta)(t_j - t_{j-1}), y_{j-1}, y_j), \end{aligned}$$

where $y_0 = x$. Note that $\mu^\eta_m(A' \cap (F^\eta)^m) \le \mu^\eta_m((F^\eta)^m) = \mu^\eta(F^\eta)^m \le 1$. By Theorem 2.5.4 and Lemma 3.2.2 (2), for $\epsilon > 0$ there exists $N > 0$ such that

$$\begin{aligned} &\frac{1}{B_n(\eta)} \log P^\eta_x[\omega(\epsilon^z_n(\eta)\,\cdot\,) \in \textstyle\prod_\triangle^{-1} A] \\ &\qquad \le \frac{1}{B_n(\eta)} \sup_{(y_1,\dots,y_m)\in A'\cap(F^\eta)^m} \sum_{j=1}^m \log p^\eta(\epsilon^z_n(\eta)(t_j - t_{j-1}), y_{j-1}, y_j) \\ &\qquad \le - \inf_{(y_1,\dots,y_m)\in A'\cap(F^\eta)^m} \sum_{j=1}^m d_\eta(y_{j-1}, y_j)\Psi^*\Big(\frac{z(t_j - t_{j-1})}{d_\eta(y_{j-1}, y_j)}, \theta^n\chi_\eta\Big) + m\epsilon \\ &\qquad = -\inf_{\phi\in\prod_\triangle^{-1} A} I^\eta_{x,z}(\phi, \theta^n\chi_\eta) + m\epsilon \end{aligned}$$

for all $n \ge N$, $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$ and measurable set $A \subset \{x\} \times (\mathbb{R}^2)^m$. This implies our assertion. $\square$

Let

$$\mathcal{F}_{\alpha,K} = \Big\{ C \in \mathcal{B}(\Omega_x) : \sup_{\substack{\xi\in\{2,3\}^{\mathbb{Z}}, z\in K \\ \eta\in\{2,3\}^{\mathbb{N}}_x}} \inf_{\phi\in C} I^\eta_{x,z}(\phi, \xi) \le \alpha \Big\}$$

for each compact set $K \subset (0,\infty)$ and $\alpha > 0$.

Proposition 3.3.2. *Let $\delta > 0$, $\alpha > 0$ and $K \subset (0,\infty)$ be a compact set. For any $\epsilon > 0$ there is $N = N(\epsilon, \delta, \alpha, K)$ such that*

$$\frac{1}{B_n(\eta)} \log P^\eta_x[\omega(\epsilon^z_n(\eta)\,\cdot\,) \in C] \le \epsilon - \inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi, \theta^n\chi_\eta)$$

for all $n \ge N$, $z \in K$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $C \in \mathcal{F}_{\alpha,K}$.

Proof. It is enough to consider the case in which $\eta \in \{2,3\}^{\mathbb{N}}_x$. Let $I^{\eta,\delta}_{x,z}(\omega,\xi) = \inf_{\phi:\|\phi-\omega\|<\delta} I^\eta_{x,z}(\phi,\xi)$. If $\omega \in C$ then $I^{\eta,\delta}_{x,z}(\omega,\xi) \ge \inf_{\phi\in C_\delta} I^\eta_{x,z}(\phi,\xi)$ for any $\xi \in \{2,3\}^{\mathbb{Z}}$

and therefore

$$\begin{aligned}
P^{\eta}_{\epsilon^z_n}[C] &\le P^{\eta}_{\epsilon^z_n}\Big[\bigcap_{j=1}^{\infty}\{I^{\eta,\delta}_{x,z}(\omega,\theta^j\chi_\eta) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta)\Big] \\
&\le P^{\eta}_{\epsilon^z_n}[\|\omega-\triangle_l\omega\|\ge\delta] + P^{\eta}_{\epsilon^z_n}\Big[\bigcap_{j=1}^{\infty}\{I^{\eta,\delta}_{x,z}(\omega,\theta^j\chi_\eta) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta), \|\omega-\triangle_l\omega\|<\delta\}\Big] \\
&\le P^{\eta}_{\epsilon^z_n}\Big[\|\omega-\triangle_l\omega\|\ge\delta\Big] + P^{\eta}_{\epsilon^z_n}\Big[\bigcap_{j=1}^{\infty}\{I^{\eta}_{x,z}(\triangle_l\omega,\theta^j\chi_\eta) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta)\}\Big]
\end{aligned}$$

for any $l\in\mathbb{N}$. Let

$$\begin{aligned}
C^l_{\eta,z} &= \bigcap_{j=1}^{\infty}\{\omega\in\Omega_x : I^{\eta}_{x,z}(\triangle_l\omega,\theta^j\chi_\eta) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta)\} \\
&= \bigcap_{j=1}^{\infty}\{\omega\in\Omega_x : \sum_{i=1}^{l} d_\eta(\omega(t_i),\omega(t_{i-1}))\Psi^*\Big(\frac{z(t_i-t_{i-1})}{d_\eta(\omega(t_i),\omega(t_{i-1}))},\theta^j\chi_\eta\Big) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta)\}
\end{aligned}$$

and

$$A^l_{\eta,z} = \bigcap_{j=1}^{\infty}\{y\in\{x\}\times(\mathbb{R}^2)^m : \sum_{i=1}^{l} d_\eta(y_i,y_{i-1})\Psi^*\Big(\frac{z(t_i-t_{i-1})}{d_\eta(y_i,y_{i-1})},\theta^j\chi_\eta\Big) \ge \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^j\chi_\eta)\}$$

for any $\eta\in\{2,3\}^{\mathbb{N}}_x$, $z\in K$ and $l\in\mathbb{N}$, where $t_i=i/l$, $i=0,1,\dots,l$. Then we have $C^l_{\eta,z}=\prod_{\triangle_l}^{-1}A^l_{\eta,z}$. There is $L=L(\alpha)\in\mathbb{N}$ such that

$$\frac{1}{B_n(\eta)}\log P^{\eta}_{\epsilon^z_n}[\|\omega-\triangle_L\omega\|\ge\delta] \le -\alpha$$

for all $n\in\mathbb{N}$, $z\in K$ and $\eta\in\{2,3\}^{\mathbb{N}}$ by Lemma 3.2.5. For any $\epsilon>0$ there is $N=N(\epsilon,L)\in\mathbb{N}$ such that

$$\frac{1}{B_n(\eta)}\log P^{\eta}_{\epsilon^z_n}[C^L_{\eta,z}] \le \frac{\epsilon}{2} - \inf_{\phi\in C^L_{\eta,z}} I^{\eta}_{x,z}(\phi,\theta^n\chi_\eta) \quad\text{and}\quad \frac{\log 2}{B_n(\eta)} < \frac{\epsilon}{2}$$

for all $n\ge N$, $\eta\in\{2,3\}^{\mathbb{N}}$ and $z\in K$ by Lemma 3.3.1. Hence we have

$$\begin{aligned}
\frac{1}{B_n(\eta)}\log P^{\eta}_{\epsilon^z_n}[C] &\le \frac{\log 2}{B_n(\eta)} + \frac{1}{B_n(\eta)}\log P^{\eta}_{\epsilon^z_n}[C^L_{\eta,z}] \vee \frac{1}{B_n(\eta)}\log P^{\eta}_{\epsilon^z_n}[\|\omega-\triangle_L\omega\|\ge\delta] \\
&\le \frac{\epsilon}{2} + \Big(\frac{\epsilon}{2} - \inf_{\phi\in C_\delta} I^{\eta}_{x,z}(\phi,\theta^n\chi_\eta)\Big) \vee (-\alpha)
\end{aligned}$$

for all $n \geq N$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $z \in K$ because

$$\inf_{\phi \in C_\delta} I_{x,z}^\eta(\phi, \theta^n \chi_\eta) \leq \inf_{\phi \in C_{\eta,z}^L} I_{x,z}^\eta(\phi, \theta^n \chi_\eta)$$

for all $n \geq N$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $z \in K$. By the way for $C \in \mathcal{F}_{\alpha,K}$

$$-\alpha \leq \frac{\epsilon}{2} - \alpha \leq \frac{\epsilon}{2} - \inf_{\phi \in C} I_{x,z}^\eta(\phi, \theta^n \chi_\eta) \leq \frac{\epsilon}{2} - \inf_{\phi \in C_\delta} I_{x,z}^\eta(\phi, \theta^n \chi_\eta)$$

for all $n \geq N$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $z \in K$. This completes the proof. □

Also we obtain the following.

Proposition 3.3.3. *Let $K \subset (0,\infty)$ be a compact set. For any $C \in \mathcal{B}(\Omega_x)$ and $\delta > 0$*

$$\limsup_{n\to\infty} \sup_{\eta \in \{2,3\}_x^{\mathbb{N}}, z \in K} \left\{ \frac{1}{B_n(\eta)} \log P_x^\eta[\omega(\epsilon_n^z \,\cdot\,) \in C] + \inf_{\phi \in C_\delta} I_{x,z}^\eta(\phi, \theta^n \chi_\eta) \right\} \leq 0.$$

Proof. This can be proved in exactly the same way as Theorem 3.3.2. □

3.4 Exponentially tightness

Let $K \subset (0,\infty)$ be a compact set. First, we prove the following.

Lemma 3.4.1. *The family of probability measures on $(\Omega_x, \mathcal{B}(\Omega_x))$, $\{P_{x,\epsilon_n^z(\eta)}^\eta : \eta \in \{2,3\}^{\mathbb{N}}, n \in \mathbb{N}, z \in K\}$ is tight.*

Proof. It is enough to show that

$$\lim_{\delta\to 0} \sup_{\substack{n\in\mathbb{N}, z\in K \\ \eta\in\{2,3\}^{\mathbb{N}}}} P_{x,\epsilon_n^z(\eta)}^\eta \left[\sup_{\substack{0\leq s,t\leq 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon \right] = 0$$

for any $\epsilon > 0$. Since

$$P_{x,\epsilon'}^\eta \left[\sup_{\substack{0\leq s,t\leq 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon \right] \leq P_x^\eta \left[\sup_{\substack{0\leq s,t\leq 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon \right]$$

for each $0 < \epsilon' \leq 1$ and $\eta \in \{2,3\}^{\mathbb{N}}$, it is enough to show that

$$\lim_{\delta\to 0} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} P_x^\eta \left[\sup_{\substack{0\leq s,t\leq 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon \right] = 0 \tag{3.4.1}$$

for any $\epsilon > 0$. Let us take $n = n_\delta \in \mathbb{N}$ such that $n = \max\{m \in \mathbb{N} : \delta \le 1/m\}$. Also Let $A_i = [i/n, (i+2)/n]$ for $i = 0, \dots, n-2$. Note that if $s, t \in [0,1]$ with $|s-t| < \delta$, then $|s-t| < 1/n$ and $s, t \in A_i$ for some $i = 0, \dots, n-2$. Hence we see that

$$\Big\{ \sup_{\substack{0 \le s,t \le 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon \Big\} \subset \bigcup_{i=0}^{n-2} \Big\{ \sup_{\substack{s,t \in A_i \\ |s-t|<1/n}} d_\eta(\omega(s), \omega(t)) > \epsilon \Big\}.$$

Since

$$\begin{aligned} &\sup_{s,t \in A_i} d_\eta(\omega(s), \omega(t)) \\ &\le \sup_{s,t \in A_i} d_\eta(\omega(s), \omega(i/n)) + \sup_{s,t \in A_i} d_\eta(\omega(i/n), \omega(t)) \le 2 \sup_{s \in A_i} d_\eta(\omega(i/n), \omega(s)), \end{aligned}$$

we have

$$\begin{aligned} P_x^\eta\big[\sup_{s,t \in A_i} d_\eta(\omega(s), \omega(t)) > \epsilon\big] &\le P_x^\eta\Big[\sup_{\frac{i}{n} \le s \le \frac{i+2}{n}} d_\eta(\omega(i/n), \omega(s)) > \frac{\epsilon}{2}\Big] \\ = E_x^\eta\Big[P^\eta_{X_{i/n}}\Big[\sup_{0 \le s \le \frac{2}{n}} d_\eta(\omega(0), \omega(s)) > \frac{\epsilon}{2}\Big]\Big] &\le \sup_{a \in F_\eta} P_a^\eta\Big[\sup_{0 \le s \le \frac{2}{n}} d_\eta(\omega(0), \omega(s)) > \frac{\epsilon}{2}\Big] \end{aligned}$$

for each $i = 0, \dots, n-2$. So by Lemma 4.4 in [BH] we get

$$P_x^\eta\big[\sup_{s,t \in A_i} d_\eta(\omega(s), \omega(t)) > \epsilon\big] \le c \exp\big(-c'(\epsilon^{d_w^3} n)^{1/(d_w^2-1)}\big)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$ and small enough $\epsilon > 0$ and $\delta > 0$ with $\epsilon^{d_w^3} n_\delta \ge 1$, where $c, c' > 0$ are constants in Lemma 4.4 in [BH]. Therefore we obtain

$$P_x^\eta\Big[\sup_{\substack{0 \le s,t \le 1 \\ |s-t|<\delta}} d_\eta(\omega(s), \omega(t)) > \epsilon\Big] \le c(n-1) \exp\big(-c'(\epsilon^{d_w^3} n)^{1/(d_w^2-1)}\big)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$. Letting $\delta \to 0$, we obtain (3.4.1). □

Proposition 3.4.2. *The family of probability measures $\{P^\eta_{x, \epsilon_n^z(\eta)} : \eta \in \{2,3\}^{\mathbb{N}}, n \in \mathbb{N}, z \in K\}$ is exponentially tight. That is, for every $L < \infty$, there exists a compact set $H = H(L) \subset \Omega_x$ such that*

$$\lim_{n \to \infty} \sup_{\eta \in \{2,3\}^{\mathbb{N}}, z \in K} \frac{1}{B_n(\eta)} \log P_x^\eta[\omega(\epsilon_n^z(\eta) \cdot) \in H^c] < -L.$$

Proof. We follow the proof of Lemma 2.6 in [LS]. Let $\{\omega_i, i \in \mathbb{N}\}$ be a countable dense set in $C_x([0,1] \to \mathbb{R}^2)$. For each $k \in \mathbb{N}$, we have $\bigcup_{i\in\mathbb{N}} B(\omega_i, 1/k) = C_x([0,1] \to \mathbb{R}^2)$. Fix $M > 0$ and $k \in \mathbb{N}$. Setting $\Gamma = \Gamma_{k,M,K} = \bigcup_{\eta\in\{2,3\}^{\mathbb{N}},\xi\in\{2,3\}^{\mathbb{Z}},z\in K}\{\phi \in \Omega_x : I^{\eta}_{x,z}(\phi,\xi) \leq 2kM\}$, this is relatively compact from Lemma 3.2.3. Hence there exists a finite open covering

$$\overline{\Gamma} \subset \bigcup_{i=1}^{n_k} B(\omega_i, \frac{1}{k}),$$

where $n_k = n_{k,M,K} \in \mathbb{N}$ and $\{\omega_i\}_{i\in\mathbb{N}} \subset \Omega_x$. Let

$$S_k = S_{k,M,K} = \bigcup_{i=1}^{n_k} B(\omega_i, \frac{1}{k}), \quad T_k = T_{k,M,K} = \bigcup_{i=1}^{n_k} B(\omega_i, \frac{2}{k}).$$

Then for $\delta_k = 1/k > 0$,

$$(T_k^c)_{\delta_k} \subset S_k^c. \tag{3.4.2}$$

In fact if $f \in \bigcup_{w'\in T_k^c} B(w', \delta_k)$, there exists $\omega' \in T_k^c$ such that $f \in B(w', \delta_k)$. Hence we have

$$\|f - \omega_i\| \geq \|\omega_i - \omega'\| - \|\omega' - f\| > \frac{2}{k} - \frac{1}{k} = \frac{1}{k}$$

for all i with $1 \leq i \leq n$. This implies (3.4.2). For any $\epsilon > 0$ there is $N = N(k, M, K)$ such that

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[T_k^c] \leq -\inf_{\phi\in S_k^c} I^{\eta}_{x,z}(\phi, \theta^n\chi_\eta) + \epsilon$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $z \in K$ by Proposition 3.3.3. Since

$$2kM \leq \inf_{\phi\in\Gamma^c} I^{\eta}_{x,z}(\phi, \theta^n\chi_\eta) \leq \inf_{\phi\in S_k^c} I^{\eta}_{x,z}(\phi, \theta^n\chi_\eta)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $z \in K$, we see that

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[T_k^c] \leq -2kM + \epsilon.$$

Hence if we take $\epsilon = kM$,

$$P^{\eta}_{\epsilon_n^z}[T_k^c] \leq e^{-kMb(2)^n}$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $z \in K$. We can find a larger finite union

$$U_k = U_{k,M,K} = \bigcup_{i=1}^{m_k} B(\omega_i, \frac{2}{k})$$

with $m_k = m_{k,M,K} \geq n_k$ such that

$$P^{\eta}_{\epsilon^z_n}[U^c_k] \leq e^{-kMb(2)^n}$$

for all $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $z \in K$ by Lemma 3.4.1. The set $H = \bigcap_{k=1}^{\infty} \overline{U_k}$ is totally bounded and closed, and hence is compact. Furthermore,

$$P^{\eta}_{\epsilon^z_n}[H^c] = P^{x}_{\epsilon^z_n}\Big[\bigcup_{k=1}^{\infty} \overline{U_k}^c\Big] \leq \sum_{k=1}^{\infty} P^{\eta}_{\epsilon^z_n}[U^c_k] = \frac{e^{-Mb(2)^n}}{1-e^{-Mb(2)^n}} \leq \frac{e^{-Mb(2)^n}}{1-e^{-M}}$$

for all $n \in \mathbb{N}$, $\eta \in \{2,3\}^{\mathbb{N}}$ and $z \in K$. This completes the proof of proposition. □

3.5 Lower bound

Let $B_m(f,r) = \{\omega \in \Omega_x : |\omega(k/m) - f(k/m)| < r, 1 \leq k \leq m\}$ and $\overline{B}_m(f,r) = \{\omega \in \Omega_x : |\omega(k/m) - f(k/m)) \leq r, 1 \leq k \leq m\}$ for each $m \in \mathbb{N}$ and $f \in \Omega_x$. First we prove the following.

Lemma 3.5.1. *Let $r > \delta > 0$, $l \in \mathbb{N}$ and $K \subset (0,\infty)$ be a compact set. For any $\epsilon > 0$ there is $N = N(\epsilon, r, \delta, l, K) \in \mathbb{N}$*

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{x}[\omega(\epsilon^z_n(\eta)\ \cdot\ ) \in B_l(f,r)] \geq -I^{\eta}_{x,z}(\phi, \theta^n \chi_\eta) - \epsilon$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $f \in \Omega_x$ and $\phi \in B_l(f, r-\delta)$.

Proof. This is obvious for $\eta \notin \{2,3\}^{\mathbb{N}}_x$. Let $\eta \in \{2,3\}^{\mathbb{N}}_x$, $f \in \Omega_x$ and $\delta < r$. Note that if $B_l(f, r-\delta) \cap \Omega^{\eta}_x$ is empty set, $\inf_{\phi \in B_l(f,r-\delta)} I^{\eta}_{x,z}(\phi, \theta^n \chi_\eta) = \infty$ by the definition of I-function. We assume that $B_l(f, r-\delta) \cap \Omega^{\eta}_x$ is not empty. For any $g \in B_l(f, r-\delta) \cap \Omega^{\eta}_x$ and $0 < \delta' < \delta$, we have

$$\begin{aligned} P^{\eta}_{\epsilon^z_n}[B_l(f,r)] \geq P^{\eta}_{\epsilon^z_n}[B_l(g,\delta')] &= \int_{A_l(g,\delta') \cap (F^\eta)^l} \prod_{j=1}^{l} p^{\eta}(\epsilon^z_n(\eta)/l, y_{j-1}, y_j)\mu(dy_j) \\ &\geq \mu^{\eta}_l(A_l(g,\delta') \cap (F^\eta)^l) \inf_{(y_1,\dots y_l) \in A_l(g,\delta') \cap (F^\eta)^l} \prod_{j=1}^{l} p^{\eta}(\epsilon^z_n(\eta)/l, y_{j-1}, y_j) \end{aligned}$$

where $y_0 = x$ and $A_l(g,\delta')$ is the set $\{y \in (\mathbb{R}^2)^l : |y_j - g(j/l)| < \delta' \text{ for } j = 1, \cdots, l\}$. From Theorem 2.5.4, for any $\epsilon > 0$ there is $N_1 = N_1(\epsilon, r, \delta, l, K) \in \mathbb{N}$ such that

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[B_l(f,r)] \geq \frac{\log \mu_l^{\eta}(A_l(g,\delta') \cap (F^{\eta})^l)}{B_n(\eta)} - l\epsilon$$
$$- \sup_{(y_1,\dots y_l)\in A_l(g,\delta')} \sum_{j=1}^{l} d_\eta(y_{j-1}, y_j)\Psi^*\Big(\frac{z/l}{d_\eta(y_{j-1},y_j)}, \theta^n \chi_\eta\Big)$$

for all $n \geq N_1$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\delta' < \delta$, $z \in K$, $f \in \Omega_x$ and $g \in B_l(f, r-\delta) \cap \Omega^{\eta}_x$. Moreover by Lemma 2.3.4 (3), there is $\delta_0 = \delta_0(\epsilon, \delta, l, K) < \delta \wedge 1$ such that then

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[B_l(f,r)] \geq \frac{\log \mu_l^{\eta}(A_l(g,\delta_0) \cap (F^{\eta})^l)}{B_n(\eta)} - 2l\epsilon$$
$$- \sum_{j=1}^{l} d_\eta(g((j-1)/l), g(j/l))\Psi^*\Big(\frac{z/l}{d_\eta(g((j-1)/l), g(j/l))}, \theta^n \chi_\eta\Big) \qquad (3.5.1)$$

for all $n \geq N_1$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$, $f \in \Omega_x$ and $g \in B_l(f, r-\delta) \cap \Omega^{\eta}_x$. Since the third term of right hand side in (3.5.1) equals $I^{\eta}_{x,z}(\triangle_l g, \theta^n\chi_\eta)$, (3.5.1) implies

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[B_l(f,r)] \geq \frac{\log \mu_l^{\eta}(A_l(g,\delta_0) \cap (F^{\eta})^l)}{B_n(\eta)} - 2l\epsilon - I^{\eta}_{x,z}(g, \theta^n\chi_\eta)$$

from Lemma 3.2.2 (2). By the way we have

$$\log \mu_l^{\eta}(A_l(h,\delta_0) \cap (F^{\eta})^l) \geq \log c' + l\frac{m(3)}{b(3)} \log \delta_0$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $h \in \Omega^{\eta}_x$ by (2.1.6). Therefore any $\epsilon > 0$ there is $N_2 = N_2(\epsilon, l, \delta_0) \in \mathbb{N}$ such that

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon_n^z}[B_l(f,r)] \geq -(2l+1)\epsilon - I^{\eta}_{x,z}(g, \theta^n\chi_\eta)$$

for all $n \geq N_1 \vee N_2$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$, $f \in \Omega_x$ and $g \in B_l(f, r-\delta) \cap \Omega^{\eta}_x$. This completes the proof. □

Let $C_\eta \subset C_x([0,1] \to F^\eta) \subset C_x([0,1] \to \mathbb{R}^2)$ for each $\eta \in \{2,3\}^{\mathbb{N}}$. Supposed that

$$\bigcup_{\eta \in \{2,3\}^{\mathbb{N}}} C_\eta \text{ is equi-continuous.} \qquad (3.5.2)$$

Note that C_η is empty set for $\eta \notin \{2,3\}^{\mathbb{N}}_x$.

Lemma 3.5.2. *Let $r > \delta > 0$ be sufficiently small.*

(i) *For each $l \in \mathbb{N}$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $f \in \Omega_x^\eta$, there is $\phi = \phi_{\eta,f,l} \in (\overline{B}_l(f,r) \setminus B(f,r+\delta)) \cap \Omega_x^\eta$ such that*

$$\sup_{0 \le t \le 1} D_\eta \phi(t) < 2cl$$

where c is the constant in (2.1.5).

(ii) *For each $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in C_\eta$, there is $\phi = \phi_{\eta,f} \in B(f, r-\delta)$ such that*

$$\sup_{0 \le t \le 1} D_\eta \phi(t) < cm$$

where c is the constant in (2.1.5) and the constant m is independent of $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in C_\eta$.

Proof. (i) Let $l \in \mathbb{N}$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $f \in \Omega_x^\eta$. We choose some integer $k_0 \in \{0, 1, \ldots, l-1\}$. Note that there is $a \in F_0 = \{a_1, a_2, a_3\}$ such that

$$d_\eta(f(\frac{k_0 + 1/2}{l}), a) \ge c(r+\delta),$$

where c is the constant in (2.1.5). Recall that F_0 is the set of vertices of a unit equilateral triangle in $\mathbb{R}^2$. Let $B^\eta(f,r) = \{\omega \in \Omega_x^\eta : \|\omega - f\|_\eta < r\}$ and $\overline{B}_m^\eta(f,r) = \{\omega \in \Omega_x^\eta : d_\eta(\omega(k/m), f(k/m)) \le r, 1 \le k \le m\}$ for each $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in \Omega_x^\eta$ and $m \in \mathbb{N}$. For any $x_k \in F^\eta$, $1 \le k \le l$ with $d_\eta(f(k/l), x_k) \le r$, We consider the following piecewise geodesic path $\phi \in \overline{B}_l^\eta(f,r) \setminus B^\eta(f, c(r+\delta))$ such that

$$\phi(k/l) = x_k, \ 1 \le k \le l \ \text{ and } \ \phi(\frac{k_0 + 1/2}{l}) = a.$$

By easy calculations we obtain $D_\eta\phi(t) \le 2cl$ from (2.1.5). Since $\phi \in \overline{B}_l^\eta(f,r) \setminus B^\eta(f, c(r+\delta)) \subset \overline{B}_l(f,r) \setminus B(f, r+\delta)$, this completes the proof.

(ii) Let $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $f \in C_\eta$. We can choose $m = m(r,\delta) \in \mathbb{N}$ (not depending on $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in C_\eta$) such that $d_\eta(f(s), f(t)) < (r-\delta)/2$ for all $s, t \in [0,1]$ with $|s-t| \le 1/m$. Let $\psi = \triangle_m f$. That is,

$$d_\eta(\psi(s), \psi(t)) = \frac{|s-t|}{1/m} d_\eta(f(\frac{i}{m}), f(\frac{i+1}{m})), \quad \frac{i}{m} \le |s-t| \le \frac{i+1}{m},$$

for $i = 0, 1, \ldots, m-1$. Note that $\sup_{0 \le t \le 1} D_\eta \psi(t) < cm$ from (2.1.5). Since $\psi(i/m) = f(i/m)$ for $i = 0, 1, \ldots, m$, we have

$$d_\eta(f(s), \psi(s)) \le d_\eta(\psi(s), \psi(i/m)) + d_\eta(f(i/m), f(s)) \le r - \delta.$$

for any $s \in [i/m, (i+1)/m]$. As $B^\eta(f, r-\delta) \subset B(f, r-\delta)$, this implies our assertion. $\square$

Set $K_l(f,r,\delta) = \overline{B}_l(f,r) \setminus B(f,r+\delta)$ for $f \in \Omega_x$.

Lemma 3.5.3. *Let $r > 0$, $\delta > 0$ and $K \subset (0,\infty)$ be a compact set. For any $M > 0$, there are $L = L(M,K,\delta) \in \mathbb{N}$ and $N = N(L,r) \in \mathbb{N}$*

$$\frac{1}{B_n(\eta)} \log P_x^\eta[\omega(\epsilon_n^z(\eta)\,\cdot\,) \in K_L(f,r,\delta)] \leq -M \tag{3.5.3}$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$, $f \in C_\eta$ and $z \in K$.

Proof. (3.5.3) is obvious for $\eta \notin \{2,3\}_x^{\mathbb{N}}$. So we consider the case in which $\eta \in \{2,3\}_x^{\mathbb{N}}$. By Lemma 3.5.2 (1), we see that $K_l(f,r,\delta) \cap \Omega_x^\eta \neq \emptyset$ for all $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in C_\eta$, $l \in \mathbb{N}$ and

$$\sup_{\eta \in \{2,3\}_x^{\mathbb{N}}} \sup_{\substack{f \in C^\eta \\ z \in K, n \in \mathbb{N}}} \inf_{\phi \in K_l(f,r,\delta)} I_{x,z}^\eta(\phi, \theta^n \chi_\eta) < \infty \tag{3.5.4}$$

for each $l \in \mathbb{N}$. There is $L = L(\delta) \in \mathbb{N}$

$$|f(s) - f(t)| < \frac{\delta}{4}$$

for all $\eta \in \{2,3\}_x^{\mathbb{N}}$, $f \in C_\eta$, $l \geq L$ and $|s-t| \leq 1/l$ with $s,t \in [0,1]$. Note that if $\omega \in K_l(f,r,\delta)_{\delta/4} \cap \Omega_x^\eta$ then $\omega \in (\overline{B}_l(f,r+\delta/4) \setminus B(f,r+3\delta/4)) \cap \Omega_x^\eta$. Let $t_j = j/l$, $j = 0,1,\ldots l$. Then for every j it follows that $|\omega(t_j) - f(t_j)| \leq r + \delta/4$. By the way we can choose $j \in \mathbb{N}$ and $s \in (t_j, t_{j+1})$ such that

$$|\omega(t_j) - f(t_j)| \leq r + \delta/4, \quad |\omega(t_{j+1}) - f(t_{j+1})| \leq r + \delta/4, \quad |\omega(s) - f(s)| \geq r + 3\delta/4$$

from $\|\omega - f\| \geq r + 3\delta/4$. Therefore as $|\omega(t_j) - f(s)| \leq |\omega(t_j) - f(t_j)| + |f(t_j) - f(s)| \leq r + \delta/2$, we obtain

$$d_\eta(\omega(s), \omega(t_j)) \geq |\omega(s) - \omega(t_j)| \geq |\omega(s) - f(s)| - |f(s) - \omega(t_j)| \geq \frac{\delta}{4}.$$

Hence

$$\begin{aligned} I_{x,z}^\eta(\omega, \theta^n \chi_\eta) &\geq d_\eta(\omega(s),\omega(t_j)) \Psi^*\Big(\frac{z(s-t_j)}{d_\eta(\omega(s),\omega(t_j))}, \theta^n \chi_\eta\Big) \\ &\geq \frac{\delta}{4} \Psi^*\Big(\frac{4z(t_{j+1}-t_j)}{\delta}, \theta^n \chi_\eta\Big) = \frac{\delta}{4} \Psi^*\Big(\frac{4z}{l\delta}, \theta^n \chi_\eta\Big) \end{aligned}$$

by Lemma 3.2.2. For any $M > 0$ there is $L = L(M,\delta,K)$ such that

$$I_{x,z}^\eta(\omega, \theta^n \chi_\eta) \geq \frac{\delta}{4} \Psi^*\Big(\frac{4z}{L\delta}, \theta^n \chi_\eta\Big) \geq \frac{\delta}{4} \inf_{\xi \in \{2,3\}^{\mathbb{Z}}} \Psi^*\Big(\frac{4z}{L\delta}, \xi\Big) \geq M + 1$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$, $f \in C_\eta$, $\omega \in K_L(f,r,\delta)_{\delta/4} \cap \Omega^\eta_x$ and $z \in K$ from Lemma 2.3.4 (2). Note that this is trivial for $\omega \notin \Omega^\eta_x$. Since $K_L(f,r,\delta)$ is closed subset, there is $N = N(L,r,\delta,K)$ such that

$$\frac{1}{B_n(\eta)} \log P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)] \leq 1 - \inf_{\phi \in K_L(f,r,\delta)_{\delta/4}} I^\eta_{x,z}(\omega, \theta^n \chi_\eta)$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $f \in C_\eta$, $z \in K$ from (3.5.4) and Theorem 3.3.2. As a consequence it follows that

$$\frac{1}{B_n(\eta)} \log P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)] \leq -M$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $f \in C_\eta$ and $z \in K$. □

Proposition 3.5.4. *Let $r > \delta > 0$ and $K \subset (0,\infty)$ be a compact set. For any $\epsilon > 0$ there is $N = N(\epsilon, r, \delta, K) \in \mathbb{N}$*

$$\frac{1}{B_n(\eta)} \log P^\eta_x[\omega(\epsilon^z_n(\eta)\ \cdot\ ) \in B(f,r)] \geq -I^\eta_{x,z}(\phi, \theta^n \chi_\eta) - \epsilon \tag{3.5.5}$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}$, $z \in K$, $f \in C_\eta$ and $\phi \in B(f, r-\delta)$.

Proof. (3.5.5) is obvious for $\eta \notin \{2,3\}^{\mathbb{N}}_x$. So we consider the case in which $\eta \in \{2,3\}^{\mathbb{N}}_x$ only. First note that $P^\eta_{\epsilon^z_n}[K_l(f,r,\delta)] \leq P^\eta_{\epsilon^z_n}[B_l(f,r)]$ for all $l \in \mathbb{N}$ and there is some constant $c > 0$ such that

$$\inf_{\phi \in B_l(f,r-\delta)} I^\eta_{x,z}(\phi, \theta^n \chi_\eta) \leq \inf_{\phi \in B(f,r-\delta)} I^\eta_{x,z}(\phi, \theta^n \chi_\eta) < c$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$, $f \in C_\eta$, $l \in \mathbb{N}$ and $n \in \mathbb{N}$ by Lemma 3.5.2 (2). Therefore for any $\epsilon > 0$ there is $M = M(\epsilon) > 0$ such that

$$\exp\Big(\inf_{\phi \in B_l(f,r-\delta)} I^\eta_{x,z}(\phi, \theta^n \chi_\eta) - M\Big) < \epsilon$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$, $f \in C_\eta$, $l \in \mathbb{N}$ and $n \in \mathbb{N}$. For this M, there are $L = L(M,\delta,K) \in \mathbb{N}$ and $N = N(L,r,\delta,K) \in \mathbb{N}$ such that

$$\frac{P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)]^{1/B_n(\eta)}}{P^\eta_{\epsilon^z_n}[B_L(f,r)]^{1/B_n(\eta)}} \leq \exp\Big(1 + \inf_{\phi \in B_L(f,r-\delta)} I^\eta_{x,z}(\phi, \theta^n \chi_\eta) - (M+1)\Big) < \epsilon$$

for all $n \geq N$, $\eta \in \{2,3\}_x^{\mathbb{N}}$, $z \in K$, $f \in C_\eta$, $\phi \in B_L(f, r-\delta)$ by Lemma 3.5.1 and Lemma 3.5.3. In other words for any $\epsilon > 0$, there are $L = L(\epsilon, \delta, K)$ and $N = N(L, r, \delta, K)$ such that

$$1 - \epsilon < 1 - \frac{P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)]^{1/B_n(\eta)}}{P^\eta_{\epsilon^z_n}[B_L(f,r)]^{1/B_n(\eta)}} \leq 1 \tag{3.5.6}$$

for all $n \geq N$, $\eta \in \{2,3\}_x^{\mathbb{N}}$, $z \in K$ and $f \in C_\eta$. Since $P^\eta_{\epsilon^z_n}[B(f, r+\delta)] \geq P^\eta_{\epsilon^z_n}[B_L(f,r)] - P^\eta_{\epsilon^z_n}[B_L(f,r) \setminus B(f, r+\delta)]$, we see that

$$\begin{aligned}
\frac{1}{B_n(\eta)} \log P^\eta_{\epsilon^z_n}[B(f, r+\delta)] &\geq \frac{1}{B_n(\eta)} \log \left(P^\eta_{\epsilon^z_n}[B_L(f,r)] - P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)]\right) \\
&\geq \log \left(P^\eta_{\epsilon^z_n}[B_L(f,r)]^{1/B_n(\eta)} - P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)]^{1/B_n(\eta)}\right) \\
&\geq \frac{1}{B_n(\eta)} \log P^\eta_{\epsilon^z_n}[B_L(f,r)] + \log \left(1 - \frac{P^\eta_{\epsilon^z_n}[K_L(f,r,\delta)]^{1/B_n(\eta)}}{P^\eta_{\epsilon^z_n}[B_L(f,r)]^{1/B_n(\eta)}}\right),
\end{aligned}$$

where we use $a^\epsilon = (b + (a-b))^\epsilon \leq b^\epsilon + (a-b)^\epsilon$ for $a, b > 0, a > b, 0 < \epsilon < 1$. As a consequence, for any $\epsilon > 0$ there is $N_1 = N_1(L, r, \delta, K)$ such that

$$\frac{1}{B_n(\eta)} \log P^\eta_{\epsilon^z_n}[B(f, r+\delta)] \geq -I^\eta_{x,z}(\phi, \theta^n \chi_\eta) - \epsilon$$

for all $n \geq N_1$, $\eta \in \{2,3\}_x^{\mathbb{N}}$, $z \in K$, $f \in C_\eta$ and $\phi \in B_L(f, r-\delta)$. Since $B(f, r-\delta) \subset B_L(f, r-\delta)$, this implies our assertion. $\square$

Also we have the following.

Proposition 3.5.5. *Let $K \subset (0,\infty)$ be a compact set. For any $r > 0$, $\delta > 0$ with $r > \delta$ and $f \in \Omega_x$*

$$\liminf_{n\to\infty} \inf_{\eta \in \{2,3\}_x^{\mathbb{N}}, z \in K} \left\{\frac{1}{B_n(\eta)} \log P^\eta_x[\omega(\epsilon^z_n \,\cdot\,) \in B(f,r)] + \inf_{\phi \in B(f, r-\delta)} I^\eta_{x,z}(\phi, \theta^n \chi_\eta)\right\} \geq 0.$$

Proof. This can be proved in the same way as Proposition 3.5.4. $\square$

3.6 Varadhan's theorem

In this section let fix a compact set $K \subset (0,\infty)$. Let $\Phi : C_x([0,1] \to F^\eta) \to \mathbb{R}$ be any continuous function. We denote by $E^{P^\eta_{x,\epsilon^z_n(\eta)}}$ or $E^{P^\eta_{\epsilon^z_n}}$ the expectation with respect to the measure $P^\eta_{x,\epsilon^z_n(\eta)}$. Assume further either the tail condition

$$\lim_{M\to\infty} \limsup_{n\to\infty} \sup_{\eta \in \{2,3\}^{\mathbb{N}}, z \in K} \frac{1}{B_n(\eta)} \log E^{P^\eta_{x,\epsilon^z_n(\eta)}}[\exp(B_n(\eta)\Phi(\omega)) 1_{\{\Phi(\omega) \geq M\}}] = -\infty \tag{3.6.1}$$

or the following moment condition for some $\gamma > 1$

$$\limsup_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}, z\in K} \frac{1}{B_n(\eta)} \log E^{P^\eta_{x,\epsilon^z_n(\eta)}}[\exp(\gamma B_n(\eta)\Phi(\omega))] < \infty. \tag{3.6.2}$$

Lemma 3.6.1. *Condition (3.6.2) implies the tail condition (3.6.1).*

Proof. This can be proved in the same way as Lemma 4.3.8 of [DZ]. For $n \in \mathbb{N}$, define $Y_n(\omega) = \exp(B_n(\eta)(\Phi(\omega) - M))$ and let $\gamma > 1$ be the constant given in the moment condition (3.6.2). Then

$$\begin{aligned} e^{-B_n(\eta)M} E^{P^\eta_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega)\geq M\}}] &= E^{P^\eta_{\epsilon^z_n}}[Y_n 1_{\{Y_n\geq 1\}}] \\ &\leq E^{P^\eta_{\epsilon^z_n}}[Y_n^\gamma] = e^{-\gamma B_n(\eta)M} E^{P^\eta_{\epsilon^z_n}}[\exp(\gamma B_n(\eta)\Phi(\omega)]. \end{aligned}$$

Therefore

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{B_n(\eta)} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \log E^{P^\eta_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega)\geq M\}}] \\ &\qquad \leq -(\gamma-1)M + \limsup_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}} \frac{1}{B_n(\eta)} \log E^{P^\eta_{\epsilon^z_n}}[\exp(\gamma B_n(\eta)\Phi(\omega))]. \end{aligned}$$

The right side of this inequality is finite by the moment condition (3.6.2). In the limit $M \to \infty$, it yields the tail condition (3.6.1). □

Proposition 3.6.2. *If $\Phi : \Omega_x \to \mathbb{R}$ is a continuous function for which the tail condition (3.6.1) holds, then*

$$\limsup_{n\to\infty} \sup_{\substack{\eta\in\{2,3\}^{\mathbb{N}}_x \\ z\in K}} \left\{ \frac{1}{B_n(\eta)} \log E^{P^\eta_{x,\epsilon^z_n(\eta)}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi\in\Omega_x} (\Phi(\psi) - I^\eta_{x,z}(\psi, \theta^n\chi_\eta)) \right\} \leq 0.$$

Proof. Consider first a function Φ bounded above. Set

$$\ell^\eta_{n,z} = \sup_{\psi\in\Omega_x} (\Phi(\psi) - I^\eta_{x,z}(\psi, \theta^n\chi_\eta))$$

for each $n \in \mathbb{N}$, $z > 0$, $\eta \in \{2,3\}^{\mathbb{N}}_x$ and define

$$\Gamma = \bigcup_{\substack{\xi\in\{2,3\}^{\mathbb{Z}}, z\in K \\ \eta\in\{2,3\}^{\mathbb{N}}}} \{\psi \in \Omega_x : I^\eta_{x,z}(\psi, \xi) \leq \sup_{\psi\in\Omega_x} \Phi(\psi) + \sup_{m\in\mathbb{N}, \eta\in\{2,3\}^{\mathbb{N}}_x} |\ell^\eta_{m,z}|\}.$$

We use same notation x to express a constant function starting x on Ω_x. Note that $-\infty < \Phi(x) \leq \ell^\eta_{m,z} \leq \sup_{\psi\in\Omega_x} \Phi(\psi) < \infty$ for all $m \in \mathbb{N}$, $z \in K$ and $\eta \in \{2,3\}^{\mathbb{N}}_x$. Then

Γ is relatively compact by Lemma 3.2.3. Given $\epsilon > 0$, there are $N = N(\epsilon) \in \mathbb{N}$, $r_i > 0$ and $\omega_i \in C_x([0,1] \to \mathbb{R}^2), 1 \leq i \leq N$ such that

$$\overline{\Gamma} \subset \bigcup_{i=1}^{N} B(\omega_i, r_i) \quad \text{and} \quad \sup_{f,g \in \overline{B}(\omega_i, 4r_i)} |\Phi(f) - \Phi(g)| < \epsilon$$

for all $1 \leq i \leq N$, where $\overline{B}(\omega, r) = \{\omega' \in \Omega_x : \|\omega' - \omega\| \leq r\}$ for each $\omega \in \Omega_x$ and $r > 0$. In other words it follows that

$$\Phi(\omega) \leq \inf_{f \in \overline{B}(\omega_i, 4r_i)} \Phi(f) + \epsilon \tag{3.6.3}$$

for any $\omega \in \overline{B}(\omega_i, 4r_i)$. Set $r = \min_{1 \leq i \leq N} r_i$ and

$$F = \Big(\bigcup_{i=1}^{N} B(\omega_i, 2r_i)\Big)^c = \bigcap_{i=1}^{N} B(\omega_i, 2r_i)^c \subset \bigcap_{i=1}^{N} B(\omega_i, r_i)^c \subset \Gamma^c.$$

From Proposition 3.3.3, there is $N_1 = N_1(\epsilon) \in \mathbb{N}$ such that

$$P^{\eta}_{\epsilon_n^z}[\overline{B}(\omega_i, 2r_i)] \leq \exp\Big(-B_n(\eta)\big(\inf_{\psi \in \overline{B}(\omega_i, 4r_i)} I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta) - \epsilon\big)\Big)$$

for all $1 \leq i \leq N$ and

$$P^{\eta}_{\epsilon_n^z}[F] \leq \exp\Big(-B_n(\eta)\big(\inf_{\psi \in F_r} I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta) - \epsilon\big)\Big)$$

for all $n \geq N_1$, $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $z \in K$. Recall that $C_\delta = \cup_{\psi \in C} B(\psi, \delta)$ for $C \subset \Omega_x$, $\delta > 0$. Since $F_r \subset \bigcap_{i=1}^{N} B(\omega_i, r_i)^c \subset \Gamma^c$, we have

$$\inf_{\psi \in F_r} I^{\eta}_{x,z}(\psi, \xi) \geq \sup_{\psi \in \Omega_x} |\Phi(\psi)| + \sup_{n \in \mathbb{N}, \eta \in \{2,3\}^{\mathbb{N}}_x} |\ell^{\eta}_{n,z}| \geq \sup_{\psi \in \Omega_x} |\Phi(\psi)| + |\ell^{\eta}_{m,z}|$$

for all $\xi \in \{2,3\}^{\mathbb{Z}}$, $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$ and $m \in \mathbb{N}$. Hence it follows that

$$\Phi(\omega) \leq \inf_{\psi \in F_r} I^{\eta}_{x,z}(\psi, \xi) - |\ell^{\eta}_{m,z}|$$

for all $\omega \in F_r$, $\eta \in \{2,3\}_x^{\mathbb{N}}$, $z \in K$ and $m \in \mathbb{N}$. Therefore by this and (3.6.3), we obtain

$$\begin{aligned}
E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi)] &\le \sum_{i=1}^{N} E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi), \overline{B}(x_i, 2r_i)] + E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi), F] \\
&\le \sum_{i=1}^{N} \exp\Big(B_n(\eta)\big(\inf_{\omega\in\overline{B}(\omega_i,4r_i)} \Phi(\omega) + \epsilon\big)\Big) P^{\eta}_{\epsilon^z_n}[\overline{B}(x_i, 2r_i)] \\
&\qquad + \exp\Big(B_n(\eta)\big(\inf_{\psi\in F_r} I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta) - |\ell^{\eta}_{n,z}|\big)\Big) P^{\eta}_{\epsilon^z_n}[F] \\
&\le \sum_{i=1}^{N} \exp\Big(B_n(\eta)\big(\inf_{\omega\in\overline{B}(\omega_i,4r_i)} \Phi(\omega) + 2\epsilon - \inf_{\psi\in\overline{B}(\omega_i,4r_i)} I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta)\big)\Big) \\
&\qquad + \exp\big(B_n(\eta)(\epsilon - |\ell^{\eta}_{n,z}|)\big) \\
&= \exp\big(B_n(\eta)(\ell^{\eta}_{n,z} + 2\epsilon)\big) \\
&\qquad \times \Big(\sum_{i=1}^{N} \exp\Big(B_n(\eta)\big(\inf_{\omega\in\overline{B}(\omega_i,4r_i)} \Phi(\omega) - \inf_{\psi\in\overline{B}(\omega_i,4r_i)} I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta) - \ell^{\eta}_{n,z}\big)\Big) \\
&\qquad + \exp\big(B_n(\eta)(-\epsilon - |\ell^{\eta}_{n,z}| - \ell^{\eta}_{n,z})\big)\Big)
\end{aligned}$$

for all $n \ge N_1$, $\eta \in \{2,3\}_x^{\mathbb{N}}$ and $z \in K$. Note that

$$\inf_{\omega\in\overline{B}(\omega_i,4r_i)} \Phi(\omega) - \inf_{\psi\in\overline{B}(\omega_i,4r_i)} I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta) \le \sup_{\psi\in\overline{B}(\omega_i,4r_i)} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta)) \le \ell^{\eta}_{n,z}.$$

So this proves that

$$\limsup_{n\to\infty} \sup_{\substack{\eta\in\{2,3\}_x^{\mathbb{N}} \\ z\in K}} \Big\{\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi\in\Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta))\Big\} \le 2\epsilon.$$

To treat the general case, set $\Phi_M(\omega) = \Phi(\omega) \wedge M$. Since we have

$$\begin{aligned}
&E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega)\le M\}}] \\
&\qquad \le E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega)\le M\}} + \exp(B_n(\eta)M)1_{\{\Phi(\omega)\ge M\}}] \\
&\qquad\qquad \le E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi_M(\omega))],
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi)] \le \frac{\log 2}{B_n(\eta)} \\
&+ \Big(\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega)\ge M\}}]\Big) \vee \Big(\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi_M(\omega))]\Big).
\end{aligned}$$

Hence for any $\epsilon >$ there is $N = N(\epsilon, M)$ such that if $n \geq N$ then

$$
\begin{aligned}
&\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon \frac{z}{n}}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi \in \Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta)) \\
&\quad \leq \frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon \frac{z}{n}}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi \in \Omega_x} (\Phi_M(\psi) - I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta)) \\
&\quad \leq \Big(\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon \frac{z}{n}}}[\exp(B_n(\eta)\Phi(\omega))1_{\{\Phi(\omega) \geq M\}}] - \sup_{\psi \in \Omega_x} (\Phi_M(\psi) - I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta))\Big) \\
&\qquad \vee \Big(\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon \frac{z}{n}}}[\exp(B_n(\eta)\Phi_M(\omega))] - \sup_{\psi \in \Omega_x} (\Phi_M(\psi) - I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta))\Big) \\
&\quad \leq \epsilon
\end{aligned}
$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $z \in K$, where we use the tail condition (3.6.1). $\square$

Next we prepare two easy lemmas for the lower bound.

Lemma 3.6.3. *If $\Phi : \Omega_x \to \mathbb{R}$ is continuous, then*

$$
\inf_{\substack{\xi \in \{2,3\}^{\mathbb{Z}}, z \in K \\ \eta \in \{2,3\}^{\mathbb{N}}_x}} \sup_{\psi \in \Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi)) > -\infty.
$$

Proof. We consider the function $x : [0,1] \to F^{\eta}$ such that $x(t) = x$. It follows that

$$
\sup_{\psi \in \Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi)) \geq \Phi(x) - I^{\eta}_{x,z}(x, \xi) = \Phi(x) > -\infty
$$

for any $\eta \in \{2,3\}^{\mathbb{N}}_x$, $\xi \in \{2,3\}^{\mathbb{Z}}$ and $z \in K$. This implies our assertion. $\square$

Lemma 3.6.4. *Let $C \subset \Omega_x$ be closed set and $\Phi : \Omega_x \to [-\infty, \infty]$ be bounded above and upper semi-continuous function. Then there is $\psi_0 \in C$ such that $\Phi(\psi_0) - I^{\eta}_{x,z}(\psi_0, \xi) = \sup_{\psi \in C}(\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi))$ for each $\eta \in \{2,3\}^{\mathbb{N}}$, $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$.*

Proof. This lemma is obvious for $\eta \notin \{2,3\}^{\mathbb{N}}_x$. Let $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z > 0$ and $\xi \in \{2,3\}^{\mathbb{Z}}$. If $\sup_{\psi \in C}(\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi)) = -\infty$, then $\Phi(\omega) - I^{\eta}_{x,z}(\omega, \xi) = -\infty$ for every $\omega \in C$. Assume that $\sup_{\psi \in C}(\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi)) > -\infty$. For any $m \in \mathbb{N}$, there is $\psi_m \in C$ such that $\sup_{\psi \in C}(\Phi(\psi) - I^{\eta}_{x,z}(\psi, \xi)) \leq \Phi(\psi_m) - I^{\eta}_{x,z}(\psi_m, \xi) + 1/m$. Since $\{\psi_m\}_{m \in \mathbb{N}}$ is equi-continuous by Lemma 3.2.3 and C is closed, there is a convergent subsequence $\{\psi_{m_k}\}_{k \in \mathbb{N}}$ with the limit $\psi_0 \in C$. The upper semi-continuity of $\Phi(\cdot) - I^{\eta}_{x,z}(\cdot, \xi)$ implies our assertion. $\square$

Then we have the following.

Proposition 3.6.5. *If $\Phi : \Omega_x \to \mathbb{R}$ is bounded continuous, it follows that*

$$\liminf_{n\to\infty} \inf_{\substack{\eta\in\{2,3\}^{\mathbb{N}}_x \\ z\in K}} \left\{ \frac{1}{B_n(\eta)} \log E^{P^{\eta}_{x,\epsilon^z_n(\eta)}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi\in\Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta)) \right\} \geq 0.$$

Proof. There is a function $f^{\eta,z}_n \in \Omega_x$ such that $\sup_{\psi\in\Omega_x}(\Phi(\psi) - I^{\eta}_{x,z}(\psi,\theta^n\chi_\eta)) = \Phi(f^{\eta,z}_n)$ $- I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta)$ for each $\eta \in \{2,3\}^{\mathbb{N}}_x$, $n \in \mathbb{N}$ and $z \in K$ by Lemma 3.6.4. Also there is some constant $c > 0$ such that

$$\sup_{\substack{\eta\in\{2,3\}^{\mathbb{N}} \\ z\in K, n\in\mathbb{N}}} I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta) \leq \sup_{\substack{\eta\in\{2,3\}^{\mathbb{N}} \\ z\in K, n\in\mathbb{N}}} \Phi(f^{\eta,z}_n) + c < \infty$$

by Lemma 3.6.3. Note that $\{f^{\eta,z}_n\}_{n\in\mathbb{N}, z\in K} \subset \Omega^{\eta}_x$ for each $\eta \in \{2,3\}^{\mathbb{N}}_x$. The set $E = \{f^{\eta,z}_n \in \Omega_x : n \in \mathbb{N}, \eta \in \{2,3\}^{\mathbb{N}}_x, z \in K\}$ is equi-continuous from Lemma 3.2.3. Hence since $\overline{E}$ is compact, for any $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\inf_{\omega\in B(f^{\eta,z}_n, \delta)} \Phi(\omega) + \epsilon \geq \Phi(f^{\eta,z}_n)$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$ and $n \in \mathbb{N}$. Therefore we have

$$\begin{aligned}
&\frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi\in\Omega_x} \{\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta)\} \\
&\geq \frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi) : B(f^{\eta,z}_n, \delta)] - (\Phi(f^{\eta,z}_n) - I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta)) \\
&\geq \frac{1}{B_n(\eta)} \log E^{P^{\eta}_{\epsilon^z_n}}[\exp(B_n(\eta)\Phi) : B(f^{\eta,z}_n, \delta)] - (\Phi(f^{\eta,z}_n) - I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta)) \\
&\geq \frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon^z_n}[B(f^{\eta,z}_n, \delta)] + \Phi(f^{\eta,z}_n) - (\Phi(f^{\eta,z}_n) - I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta)) - \epsilon \\
&= \frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon^z_n}[B(f^{\eta,z}_n, \delta)] + I^{\eta}_{x,z}(f^{\eta,z}_n, \theta^n\chi_\eta) - \epsilon
\end{aligned}$$

for all $\eta \in \{2,3\}^{\mathbb{N}}_x$, $z \in K$ and $n \in \mathbb{N}$. Considering the set $\{f^{\eta,z}_n\}_{n\in\mathbb{N}, z\in K}$ to be the set C_η in (3.5.2) for each $\eta \in \{2,3\}^{\mathbb{N}}_x$, there is $N = N(\epsilon)$ such that

$$\frac{1}{B_n(\eta)} \log P^{\eta}_{\epsilon^z_n}[B(f^{\eta,z}_n, \delta)] + \inf_{\psi\in B(f^{\eta,z}_n, \delta/2)} I^{\eta}_{x,z}(\psi, \theta^n\chi_\eta) \geq -\epsilon$$

for all $n \geq N$, $\eta \in \{2,3\}^{\mathbb{N}}_x$ and $z \in K$ by Proposition 3.5.4. This completes the proof. □

Now the following theorem is an easy consequence of Proposition 3.6.2 and Proposition 3.6.5.

Theorem 3.6.6. *Let $\Phi : \Omega_x \to \mathbb{R}$ a bounded continuous function. Then*

$$\lim_{n\to\infty} \sup_{\eta\in\{2,3\}^{\mathbb{N}}_x, z\in K} \left| \frac{1}{B_n(\eta)} \log E^{P^{\eta}_{x,\epsilon^z_n(\eta)}}[\exp(B_n(\eta)\Phi)] - \sup_{\psi\in\Omega_x} (\Phi(\psi) - I^{\eta}_{x,z}(\psi, \theta^n \chi_\eta)) \right| = 0.$$

The rate function $I^{\eta}_{x,z}$ changes with n as seen in Proposition 3.3.3 and Proposition 3.5.5. This fact becomes a trouble, then we obtain only weak large deviations stated above. However we show that $\{P^{\eta}_x \circ X^{-1}_{\epsilon^z_n(\eta)\cdot}\}_{\eta\in\{2,3\}^{\mathbb{N}}, z\in K, n\in\mathbb{N}}$ is an exponentially tight family of probability measure on $C_x([0,1] \to \mathbb{R}^2)$ and satisfies Varadhan's theorem. Hence we can expect that better large deviations hold though our settings is not included in general framework. Also we think that we should examine the behavior of $I^{\eta}_{x,z}(\phi, \xi)$ with respect to ξ more in detail because this often causes the problem.

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