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Laplace Approximations for Sums of Independent Random Vectors – The Degenerate Case –

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Abstract. Let $X_i, i \in \mathbf{N}$, be *i.i.d.* B-valued random variables, where B is a real separable Banach space. Let $\Phi : B \to \mathbf{R}$ be a mapping. The problem is to give an asymptotic evaluation of $Z_n = E\left(\exp\left(n\Phi(\sum_{i=1}^n X_i/n)\right)\right)$, up to a factor $(1 + o(1))$. Bolthausen [1] studied this problem in the case that there is a unique point maximizing $\Phi - h$, where h is the so-called entropy function, and the curvature at the maximum is nonvanishing, (these two will be called as *nondegenerate assumptions*), with some central limit theorem assumption. Kusuoka-Liang [5] studied the same problem, and succeeded in eliminating the central limit theorem assumption, but the nondegenerate assumptions are still left. In this paper, we study the same problem not assuming the central limit theorem assumption and the nondegenerate assumptions.

1. Introduction

Let B be a real separable Banach space with norm $||\cdot||$, μ be a probability measure on B. We assume that the smallest closed affined space that contains $\mathrm{supp}\mu$ is B. Moreover we assume

(A1) There exists a constant $C_1 > 0$, such that

$$\int_B \exp(C_1||x||^2)\mu(dx) < \infty.$$

Let $\Phi : B \to \mathbf{R}$ be a three times continuously Fréchet differentiable function satisfying the following:

(A2) There exist constants $C_2, C_3 > 0$, such that

$$\Phi(x) \leq C_2 + C_3||x||, \qquad \text{for any } x \in B.$$

Let X_n and S_n, $n \in \mathbf{N}$, be the random variables defined by $X_n(\underline{x}) = x_n$, $S_n(\underline{x}) = \sum_{k=1}^n x_k$, $\underline{x} = (x_1, x_2, x_3, \cdots) \in B^{\mathbf{N}}$. Let $Z_n = E^{\mu^{\otimes\infty}}\left[\exp(n\Phi(\frac{S_n}{n}))\right]$.

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By Donsker-Varadhan [3], we have that

$$\lim_{n\to\infty}\frac{1}{n}\log Z_n = \sup_{x\in B}\{\Phi(x)-h(x)\}\equiv\lambda,$$

where h is the entropy function of μ:

$$h(x)=\sup_{\phi\in B^*}\left\{\phi(x)-\log M(\phi)\right\},$$

B^* is the dual Banach space of B and $M(\phi)=\int_B e^{\phi(x)}\mu(dx)$, $\phi\in B^*$. Let

$$V=\{x\in B:\Phi(x)-h(x)=\lambda\}.$$

For each $x\in V$, let ν_x be the probability measure on B defined by

$$\nu_x(dy)=\exp(D\Phi(x)(y))\mu(dy)/M(D\Phi(x)). \tag{1.1}$$

Then by the fact that $x\in V$ maximizes $\Phi-h$, we have

$$\begin{aligned}&\int_B y\nu_x(dy)=x,\\ &h(x)=D\Phi(x)(x)-\log M(D\Phi(x)).\end{aligned} \tag{1.2}$$

Let $\nu_{x,0}$ be the 0-centered ν_x, that is,

$$d\nu_{x,0}(y)=d\nu_x(y+x),\qquad \forall y\in B,$$

and also, let Γ_x be the covariance on B^* defined by

$$\Gamma_x(\phi,\psi)=\int_B\phi(y)\psi(y)\nu_{x,0}(dy),\qquad\forall\phi,\psi\in B^*.$$

Then by the fact that $x\in V$ maximizes $\Phi-h$, we have

$$\Gamma_x(\phi,\phi)\geq D^2\Phi(x)(S_x\phi,S_x\phi),\qquad\forall\phi\in B^*,$$

where $S_x:B^*\to B$ is defined as $S_x\phi\equiv\int_B\phi(y)y\nu_{x,0}(dy),\forall\phi\in B^*$. Define

$$A_x\equiv\{\phi\in B^*:\Gamma_x(\phi,\phi)=D^2\Phi(x)(S_x\phi,S_x\phi)\}.$$

Let $H_x \equiv (\overline{B^*}^{\Gamma_x})^*$, where $\overline{B^*}^{\Gamma_x}$ means the completion of B^* with respect to Γ_x, denote by $(\cdot,\cdot)_x$ the inner product in H_x, and $||\cdot||_x$ the norm of it. Then $S_x(B^*) \subset H_x$, $\psi(S_x\phi) = \Gamma_x(\phi,\psi)$ for any $\phi,\psi \in B^*$, and

$$\begin{aligned} ||S_x\phi||_x^2 &= \sup\{\psi(S_x\phi)^2; \Gamma_x(\psi,\psi) \le 1\} \\ &= \sup\{\Gamma_x(\phi,\psi)^2; \Gamma_x(\psi,\psi) \le 1\} \\ &= \Gamma_x(\phi,\phi). \end{aligned}$$

So $(S_x\phi, S_x\psi)_x = \Gamma_x(\phi,\psi)$ for any $\phi,\psi \in B^*$.

Also, as it has been shown in Kusuoka-Liang [5, Proposition 2.1, Proposition 2.2], H_x can be regarded as a dense subset of B, and for any continuous bilinear function $A: B \times B \to \mathbf{R}$, $A\Big|_{H_x \times H_x}$ is a Hilbert-Schmidt function for any $x \in V$.

Moreover, we assume the following:

(A3) There exist constants $C_4 > 0$ and $\delta_0 > 0$, and a continuous bilinear symmetric function $K: B \times B \to \mathbf{R}$, such that

$$|D^3\Phi(x)(y,y,y)| \le C_4||y||K(y,y), \qquad \text{for any } y \in B \text{ and } x \in V_{\delta_0},$$

where V_{δ_0} denotes the δ_0-neighborhood of V in B.

Our result in this paper is the following:

Theorem 1.1. *Under the above assumptions, there exist an integer $d \ge 1$ and a d-dimensional manifold M embedded in B with Riemann metric such that $V \subset M$ and $S_x(A_x) \subset T_xM$ for any $x \in V$, and there exist continuous functions $x: M \to B$, $a: M \to [0,\infty)$ and $b: M \to (0,\infty)$ such that*

(1) *$x(\cdot) \in C^2(M)$, and $x(z) = z$ for any $z \in V$,*

(2) *$a(z) = 0$ if and only if $z \in V$, and*

(3) *for any bounded continuous function $f: B \to \mathbf{R}$,*

$$\begin{aligned} &E^{\mu^{\otimes\infty}}\left[f(\frac{S_n}{n})\exp(n\Phi(\frac{S_n}{n}))\right] \\ &\quad = e^{n\lambda}n^{\frac{d}{2}}\int_M f(x(z))b(z)e^{-n\cdot a(z)}v_M(dz)(1+o(1)) \end{aligned}$$

as $n \to \infty$, where v_M is the volume element on M.

See Lemma 5.2 for the precise expression of $a(z)$ and $b(z)$, $z \in M$.

As a corollary, we get the following:

COROLLARY 1.2. *Under the above assumptions, there exist an integer* $d \geq 1$ *and a d-dimensional manifold* M *embedded in* B *with Riemann metric such that* $V \subset M$ *and* $S_x(A_x) \subset T_xM$ *for any* $x \in V$, *and there exist continuous functions* $a : M \to [0, \infty)$ *and* $b : M \to (0, \infty)$ *such that*

(1) $a(z) = 0$ *if and only if* $z \in V$, *and*

(2)

$$Z_n = e^{n\lambda} n^{\frac{d}{2}} \int_M b(z) e^{-n \cdot a(z)} v_M(dz)(1 + o(1))$$

as $n \to \infty$, *where* v_M *is the volume element on* M.

Chiyonobu [2] studied the same problem under a certain central limit theorem assumption.

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2. Manifold Reflecting Singularities

In this section, we will show the existance of the manifold M with the properties described in our main theorem.

The following is well-known (*c.f.* Bolthausen [1]):

LEMMA 2.1. (1) *h is non-negative, lower semicontinuous and convex, and is strongly convex on* $\{x \in B : h(x) < \infty\}$.

(2) $h(x) = 0$ *if and only if* $x = \int y\mu(dy)$.

(3) *For all* $r \in [0, \infty)$, $\{x : h(x) \leq r\}$ *is compact in* B.

(4) $\lim_{r\to\infty} \inf_{|x|\geq r} h(x)/r = \infty$.

As in Kusuoka-Tamura [6] and Chiyonobu [2], we can show that

LEMMA 2.2. *V is a non-void compact set.*

PROOF. First take $z_n \in B, n = 1, 2, \cdots$, such that $\Phi(z_n) - h(z_n) \to \lambda$, where λ is the maximum of $\Phi - h$, as defined in section 1. Then z_n is bounded by assumption (A2) and Lemma 2.1 (4). So $h(z_n)$ is bounded. By

Lemma 2.1 (3), this implies that there exists a subsequence n_k and a $z \in B$, such that $z_{n_k} \to z$, as $k \to \infty$. So

$$\Phi(z) - h(z) \geq \limsup_{k\to\infty} (\Phi(z_{n_k}) - h(z_{n_k})) = \lambda,$$

which implies that $z \in V$. Therefore, V is non-void.

The same argument implies that V is compact. □

Definition 2.3. We say that M is a manifold reflecting singularities if M is a submanifold embedded in B, $V \subset M$, and $S_x(A_x) \subset T_xM$ for each $x \in V$.

In the rest of this section, we will show that a manifold reflecting singularities exists.

Define $\widetilde{\Gamma}(\phi,\psi) \equiv \int_B \phi(y)\psi(y)e^{\frac{1}{2}C_1||y||^2}\mu(dy)/\int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy)$, $\phi,\psi \in B^*$. Note that this is finite for all $\varphi,\psi \in B^*$ by the assumption (A1).

Proposition 2.4. *There exists a common constant $C > 0$ independent to $z \in V$, such that*

$$\Gamma_z(\phi,\phi) \leq C^2\widetilde{\Gamma}(\phi,\phi), \qquad \textit{for any } \phi \in B^* \textit{ and any } z \in V.$$

Proof. From the compactness of V and the continuity of $D\Phi$, there exist constants $K_1 > 0$ and $K_2 > 0$ such that $||z|| \leq K_1$ for all $z \in V$ and $|D\Phi(z)(y)| \leq K_2||y||$ for all $z \in V$ and all $y \in B$. So from the fact that $e^{K_2\cdot x - \frac{1}{2}C_1\cdot x^2} \leq e^{\frac{K_2^2}{2C_1}}$ for all $x \in \mathbf{R}$, we have that for all $z \in V$,

$$\int_B \phi(y)^2 e^{D\Phi(z)(y)}\mu(dy) \leq e^{\frac{K_2^2}{2C_1}} \int_B \phi(y)^2 e^{\frac{1}{2}C_1||y||^2}\mu(dy).$$

Therefore

$$\begin{aligned}\Gamma_z(\phi,\phi) &= \frac{\int_B \phi(y)^2 e^{D\Phi(z)(y)}\mu(dy)}{\int_B e^{D\Phi(z)(y)}\mu(dy)} - \phi(z)^2 \\ &\leq \frac{\int_B \phi(y)^2 e^{D\Phi(z)(y)}\mu(dy)}{\int_B e^{D\Phi(z)(y)}\mu(dy)}\end{aligned}$$

$$\leq \quad e^{\frac{K_2^2}{2C_1}} \cdot \frac{\int_B \phi(y)^2 e^{\frac{1}{2}C_1||y||^2}\mu(dy)}{\int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy)} \cdot \frac{\int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy)}{\int_B e^{D\Phi(z)(y)}\mu(dy)}$$

$$\leq \quad e^{\frac{K_2^2}{2C_1}} \cdot \frac{\int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy)}{\int_B e^{-K_2||y||}\mu(dy)} \cdot \widetilde{\Gamma}(\phi,\phi), \qquad \text{for all } \phi \in B^*.$$

This gives our assertion with $C \equiv \left(e^{\frac{K_2^2}{2C_1}} \cdot \int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy)/ \int_B e^{-K_2||y||}\mu(dy)\right)^{1/2}$. □

Let $\widetilde{H} \equiv (\overline{B^*}^{\tilde{\Gamma}})^*$, then $\widetilde{H}$ is a Hilbert space, Let $\widetilde{S} : B^* \to B$ be given by

$$\widetilde{S}\phi \equiv \int_B \phi(y) y e^{\frac{1}{2}C_1||y||^2}\mu(dy)/\int_B e^{\frac{1}{2}C_1||y||^2}\mu(dy), \qquad \phi \in B^*.$$

Then we have $(\widetilde{S}\phi, \widetilde{S}\psi)_{\widetilde{H}} = \widetilde{\Gamma}(\phi,\psi)$ for any $\phi, \psi \in B^*$. The norm of $\widetilde{H}$ will be denoted by $||\cdot||_{\widetilde{H}}$ in this paper. $\widetilde{H}$ is seperable, and by the same method as in Kusuoka-Liang [5, Proposition 2.1, Proposition 2.2], $\widetilde{H}$ can be considered as a dense subset of B, and for any continuous bilinear function $A : B \times B \to \mathbf{R}$, $A\Big|_{\widetilde{H}\times\widetilde{H}}$ is a Hilbert-Schmidt function.

From Proposition 2.4 and the definition of $||\cdot||_{\widetilde{H}}$,

$$(2.1) \qquad ||\varphi||_{H_z^*} \leq C||\varphi||_{\widetilde{H}^*}, \qquad \text{for any } \varphi \in \widetilde{H}^* \text{ and any } z \in V.$$

Therefore

$$(2.2) \qquad ||x||_{\widetilde{H}} \leq \frac{1}{C}||x||_{H_z}, \qquad \text{for any } x \in H_z \text{ and any } z \in V.$$

That is, H_z can be embedded into $\widetilde{H}$ naturally for each $z \in V$.

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal base of $\widetilde{H}^*$ with $\{e_n\} \subset B^*$. Then $\{\widetilde{S}e_n\}_{n=1}^{\infty}$ is the corresponding base of $\widetilde{H}$. Let $Q_n : B \to B$ be defined by $Q_n(z) = \sum_{i=1}^n e_i(z)\widetilde{S}e_i, \forall z \in B$, $\forall n \in \mathbf{N}$. Then $Q_n : B \to B, n \in \mathbf{N}$, is a sequence of bounded operators that satisfies the following: $dim(ImageQ_n) = n, Q_n^2 = Q_n$, $Q_{n+1}Q_n = Q_nQ_{n+1} = Q_n$ for any $n \in \mathbf{N}$ and $\cup_{n\in\mathbf{N}}Q_n(B)$ is dense in B.

Lemma 2.5. *There exists an integer d_1 such that for any $d \geq d_1$, $P_d D\Phi\Big|_V$ is injective, where $P_d : B^* \to B^*$ denotes the adjoint operator of Q_d.*

PROOF. It is obvious that the lemma can be seen if we can show that $P_{d_1} D\Phi\Big|_V$ is injective for $d_1 \in \mathbf{N}$ large enough. Suppose not. Then for each $d \in \mathbf{N}$, there exist $z_d^1, z_d^2 \in V$ such that $P_d D\Phi(z_d^1) = P_d D\Phi(z_d^2)$, and $z_d^1 \neq z_d^2$. Combining with (1.1) and (1.2), we see that $D\Phi(z_d^1) \neq D\Phi(z_d^2)$.

Since V is compact, we may assume that $z_d^1 \to z^1$ and $z_d^2 \to z^2$ as $d \to \infty$. Then $D\Phi(z^1)(Q_d u) = D\Phi(z^2)(Q_d u)$ for any $u \in B$ and any $d \in \mathbf{N}$. So $D\Phi(z^1) = D\Phi(z^2)$, hence $z^1 = z^2$. We write this same point by $z \in V$.

Let $f : B^* \to B^*$ be defined by $f(\phi) = D\Phi(\int_B y e^{\phi(y)} \mu(dy)/M(\phi)), \phi \in B^*$. Then f is continuous, Fréchet differenable, and $f(D\Phi(w)) = D\Phi(w)$ for any $w \in V$. Therefore,

$$\begin{aligned}
& D\Phi(z_d^1) - D\Phi(z_d^2) \\
= \ & f(D\Phi(z_d^1)) - f(D\Phi(z_d^2)) \\
= \ & Df(D\Phi(z))(D\Phi(z_d^1) - D\Phi(z_d^2)) \\
& + \int_0^1 \Big[Df(D\Phi(z_d^2) + t(D\Phi(z_d^1) - D\Phi(z_d^2))) - Df(D\Phi(z)) \Big] \\
& \qquad \qquad (D\Phi(z_d^1) - D\Phi(z_d^2))dt.
\end{aligned}$$

Let $\varphi_d = \frac{D\Phi(z_d^1) - D\Phi(z_d^2)}{\|D\Phi(z_d^1) - D\Phi(z_d^2)\|_{H_z^*}}$, which is well-defined since $D\Phi(z_d^1) \neq D\Phi(z_d^2)$. Then from the fact that $z_d^1 \to z, z_d^2 \to z$ as $d \to \infty$, we see from the equality above that $\varphi_d - Df(D\Phi(z))\varphi_d \to 0$ in H_z^* as $d \to \infty$. From the fact $z \in V$, we have that for any $\psi \in B^*$,

$$\begin{aligned}
& Df(D\Phi(z))(\psi) \\
= \ & \frac{d}{dt} \left(D\Phi \left(\frac{\int_B y e^{D\Phi(z)(y) + t\psi(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y) + t\psi(y)} \mu(dy)} \right) \right) \Bigg|_{t=0} \\
= \ & D^2\Phi \left(\frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right) \left(\frac{\int_B \psi(y) y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right. \\
& \qquad \left. - \frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \cdot \frac{\int_B \psi(y) e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)}, \quad \cdot \right) \\
= \ & D^2\Phi(z) \left(\int_B \psi(y) y \nu_z(dy) - \psi(z) z, \quad \cdot \right) \\
= \ & D^2\Phi(z) \left(\int_B \psi(y) y \nu_{z,0}(dy), \quad \cdot \right) \\
= \ & D^2\Phi(z)(S_z\psi, \cdot).
\end{aligned}$$

Since $D^2\Phi(z)\Big|_{H_z\times H_z}$ is a compact operator, we see from the above that there exists a $\varphi\in H_z^*$ such that $\varphi_d\to\varphi$ in H_z^*. Then $||\varphi||_{H_z^*}=1$ since $||\varphi_d||_{H_z^*}=1$. On the other hand, $\varphi_d(Q_dy)=0$ for all $y\in B$, which implies that $\varphi\equiv 0$. This makes a contradiction. □

Lemma 2.6. *There exists an integer d_2 large enough such that for any $d\geq d_2$, any $z\in V$ and any $x\in B$ with $x\neq 0$,*

$$x-\int_B\Big(D^2\Phi(z)(x,y)-D^2\Phi(z)(x,Q_dy)\Big)\,y\nu_{z,0}(dy)\neq 0.$$

Proof. If not, then for any $n\in\mathbf{N}$, there exist $d_n\in\mathbf{N}$, $z_n\in V$ and $x_n\neq 0$, such that

$$\begin{aligned}(2.3)\quad x_n &= \int_B\Big(D^2\Phi(z_n)(x_n,y)-D^2\Phi(z_n)(x_n,Q_{d_n}y)\Big)\,y\nu_{z_n,0}(dy)\\ &= S_{z_n}\Big((I-P_{d_n})D^2\Phi(z_n)(x_n,\cdot)\Big)\,.\end{aligned}$$

From the compactness of V, by taking subsequence if necessary, we may assume that z_n converge in V, *i.e.*, $z_n\to z\in V$ as $n\to\infty$. Note that $x_n\in H_{z_n}\subset\widetilde{H}$ for any $n\in\mathbf{N}$, so by dividing the both side by $||x_n||_{\widetilde{H}}$ if necessary, we may assume that $||x_n||_{\widetilde{H}}=1$. Therefore, from the fact that $D^2\Phi(z)\Big|_{\widetilde{H}\times\widetilde{H}}$ is a compact operator, by taking subsequence if necessary, we may assume that $D^2\Phi(z)(x_n,\cdot)$ converge in $\widetilde{H}^*$. On the same time, from the assumption that $||x_n||_{\widetilde{H}}=1$, we have that $||x_n||$ is bounded for $n\in\mathbf{N}$. So from the continuity of $D^2\Phi$, $D^2\Phi(z_n)(x_n,\cdot)$ is also convergent in $\widetilde{H}^*$. This implies that $(I-P_{d_n})D^2\Phi(z_n)(x_n,\cdot)\to 0$ in $\widetilde{H}^*$ as $n\to\infty$.

So from (2.3) and (2.1), $||x_n||_{H_{z_n}}=||(I-P_{d_n})D^2\Phi(z_n)(x_n,\cdot)||_{H_{z_n}^*}\to 0$ as $n\to\infty$. By (2.2), this implies that $||x_n||_{\widetilde{H}}\to 0$ as $n\to\infty$. This contradicts with the assumption that $||x_n||_{\widetilde{H}}=1$ for all $n\in\mathbf{N}$. □

Lemma 2.7. *There exists an integer $d_3\in\mathbf{N}$ large enough such that for any $d\geq d_3$, any $z\in V$, $\varphi\in A_z$ and $\varphi\neq 0$ imply $Q_dS_z\varphi\neq 0$.*

Proof. As the way of proof is similar to that of Lemma 2.6, we only give the sketch here.

Same as in the proof of Lemma 2.6, we only need to show that for $d_3 \in \mathbf{N}$ large enough, for any $z \in V$, $\varphi \in A_z$ and $\varphi \neq 0$ imply $Q_{d_3} S_z \varphi \neq 0$.

If not, then for any $d \in \mathbf{N}$, there exist $z_d \in V$ and $\varphi_d \in A_{z_d}$ with $\varphi_d \neq 0$ but $Q_d S_{z_d} \varphi_d = 0$. Without loss of generality, we can assume, by the compactness of V, that there exists a $z \in V$, such that $z_d \to z$. Also, we can assume that $||\varphi_d||_{\widetilde{H}^*} = 1$. Since $\varphi_d \in A_{z_d}$, it can be seen that $\varphi_d = D^2\Phi(z_d)(S_{z_d}\varphi_d, \cdot)$. By Proposition 2.4, $||S_{z_d}\varphi||_{\widetilde{H}} \leq 1$. So from the fact that $D^2\Phi(z)\Big|_{\widetilde{H}\times\widetilde{H}}$ is a compact operator and $D^2\Phi(\cdot)$ is continuous, $\varphi_d = D^2\Phi(z_d)(S_{z_d}\varphi_d, \cdot)$ converges in $\widetilde{H}^*$. Write the limit as φ_0. Then $||\varphi_0||_{\widetilde{H}^*} = 1$. On the other hand, $Q_d\varphi_0 = 0$ for any $d \in \mathbf{N}$, so $\varphi_0 = 0$. This is a contradiction. □

Let d be the maximum of d_1, d_2, and d_3, the integers chosen in Lemma 2.5, Lemma 2.6, and Lemma 2.7, respectively. Let $W \equiv \text{Image}P_d$, let $||\cdot||_W$ and $\text{dist}_W(\cdot,\cdot)$ denote the norm and the distance on it, respectively.

THEOREM 2.8. *There exist a neighborhood U of $\{P_d D\Phi(z), z \in V\}$ in W small enough, and a map $X(\cdot) : U \to B$, which is a C^2-diffeomorphism. In particular, there is a manifold reflecting singularities.*

PROOF. The proof will be divided into several steps.

Step 1. Let $f : B \times W \to B$ be defined by

$$f(z,\varphi) = z - \frac{\int y e^{D\Phi(z)(y) - D\Phi(z)(Q_d y) + \varphi(y)} \mu(dy)}{\int e^{D\Phi(z)(y) - D\Phi(z)(Q_d y) + \varphi(y)} \mu(dy)}, \qquad \forall z \in B, \quad \forall \varphi \in W.$$

Then f is twice continuously differentiable with respect to z, and $f(z, P_d D\Phi(z)) = 0$ for any $z \in V$. For any $z \in V$, let $\varphi_z \equiv P_d D\Phi(z)$. Then by Lemma 2.6, for any $x \in B$ not equal to 0,

$$\begin{aligned} & D_z f(z,\varphi)(x)\Big|_{\varphi=\varphi_z} \\ = \;& x - \left(\frac{\int_B \left(D^2\Phi(z)(x,y) - D^2\Phi(z)(x, Q_d y)\right) y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \right. \\ & - \frac{\int_B y e^{D\Phi(z)(y)} \mu(dy)}{\int_B e^{D\Phi(z)(y)} \mu(dy)} \end{aligned}$$

$$
\left. \cdot \frac{\int_B \left(D^2\Phi(z)(x,y) - D^2\Phi(z)(x,Q_d y)\right) e^{D\Phi(z)(y)}\mu(dy)}{\int_B e^{D\Phi(z)(y)}\mu(dy)} \right)
$$

$$
\begin{aligned}
&= x - \int_B \Big(D^2\Phi(z)(x,y) - D^2\Phi(z)(x,Q_d y)\Big)\, y\nu_{z,0}(dy) \\
&\neq 0.
\end{aligned}
$$

So by implicit function theorem, for any $z \in V$, there exists U_{φ_z}, a neighborhood of φ_z in W, and a unique twice continuously differentialbe function $G_z(\varphi)$ defined on U_{φ_z}, such that $f(G_z(\varphi),\varphi) = 0$ on $\varphi \in U_{\varphi_z}$.

Step 2. In this step, we will show that the functions $\{G_z\}_{z\in V}$ are consistent if the neighborhoods $\{U_{\varphi_z}\}_{z\in V}$ are taken small enough.

For any $z \in V$, let $U_{z,n} \equiv U_{\varphi_z} \cap B_W(P_d D\Phi(z), \frac{1}{n})$, where U_{φ_z} is the one chosen before, and $B_W(\varphi,\varepsilon)$ means the neighborhood of φ in W with radius ε. Let $U_n = \cup_{z\in V} U_{z,n}$. We only need to show that there exists an integer $n \in \mathbf{N}$ large enough, such that for any $z, w \in V$, G_z and G_w are consist on $U_{z,n} \cap U_{w,n}$. If not, then for any $n \in \mathbf{N}$, there exist $z_n, w_n \in V$ and $\varphi_n \in U_{z_n,n} \cap U_{w_n,n}$, such that $G_{z_n}(\varphi_n) \neq G_{w_n}(\varphi_n)$. From the compactness of V, by taking subsequence if necessary, we may suppose that there exist $z, w \in V$, such that $z_n \to z$, $w_n \to w$. If $z \neq w$, then $U_{z_n,n} \cap U_{w_n,n} = \emptyset$ for n large enough from Lemma 2.5 and the continuity of $D\Phi$. This is a condradiction. So we have $z = w$. From the definition of $U_{z_n,n}$, for any $\varepsilon > 0$, there exists a integer $N \in \mathbf{N}$, such that for any $n > N$, $\varphi_n \in B_W(P_d D\Phi(z),\varepsilon)$. As has been shown in step 1, if we take $\varepsilon > 0$ small enough, there exists only a unique $G_z(\varphi_n)$ that satisfies $f(G_z(\varphi_n),\varphi_n) = 0$. This makes a contradiction since $f(G_{z_n}(\varphi_n),\varphi_n) = 0$, $f(G_{w_n}(\varphi_n),\varphi_n) = 0$ and $G_{z_n}(\varphi_n) \neq G_{w_n}(\varphi_n)$ from the assumption.

Step 3. Now, we have shown that there exist U, a neighborhood of $\{P_d D\Phi(z); z \in V\}$ in W, and a twice continuously differentiable function $X(\varphi)$ on U, such that $f(X(\varphi),\varphi) = 0$ on $\varphi \in U$, *i.e.*

$$
X(\varphi) = \frac{\int_B y e^{D\Phi(X(\varphi))(y) - D\Phi(X(\varphi))(Q_d y) + \varphi(y)}\mu(dy)}{\int_B e^{D\Phi(X(\varphi))(y) - D\Phi(X(\varphi))(Q_d y) + \varphi(y)}\mu(dy)},
$$

and that $X(\varphi_z) = z$ for any $z \in V$. Differentiating the both side at φ_z, and we have for any $z \in V$ and any $\psi \in W$,

$$
DX(\varphi_z)(\psi) \tag{2.4}
$$

$$
\begin{aligned}
= & \int_B \Big(D^2\Phi(z)(DX(\varphi_z)(\psi), y) \\
& \qquad - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d y) + \psi(y) \Big) \, y\nu_z(dy) \\
& \quad - \Big(D^2\Phi(z)(DX(\varphi_z)(\psi), z) \\
& \qquad - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d z) + \psi(z) \Big) \, z \\
= & \int_B \Big(D^2\Phi(z)(DX(\varphi_z)(\psi), y) \\
& \qquad - D^2\Phi(z)(DX(\varphi_z)(\psi), Q_d y) + \psi(y) \Big) y\nu_{z,0}(dy).
\end{aligned}
$$

So if $DX(\varphi_z)(\psi) = 0$, then $\int_B \psi(y) y\nu_{z,0}(dy) = 0$, hence $\int_B \psi(y)^2 \nu_{z,0}(dy) = 0$. Therefore from the assumption that the smallest closed affined space that contains suppμ is B, we get $\psi = 0$. That is, $DX(\varphi_z)(\psi) \neq 0$ whenever $\psi \neq 0$. So we can take U small enough one more time again if needed, such that $\varphi \mapsto X(\varphi), \varphi \in U$, is a local diffeomorphism.

Step 4. In this step, we will show that we can take the neighborhood U small enough such that $\varphi \mapsto X(\varphi), \varphi \in U$, is an injective, which accompanying with the step 3 implies that $\varphi \mapsto X(\varphi), \varphi \in U$, is not only a local diffeomorphism, but also a diffeomorphism.

If not, for any $m \in \mathbf{N}$, there exist $\varphi_m, \psi_m \in W$, $\varphi_m \neq \psi_m$, $X(\varphi_m) = X(\psi_m)$, and $\mathrm{dist}_W(\varphi_m, \widetilde{V}) + \mathrm{dist}_W(\psi_m, \widetilde{V}) \to 0$ as $m \to \infty$, where $\widetilde{V} \equiv \{P_d D\Phi(z); z \in V\}$. $\widetilde{V}$ is compact in W, so by taking subsequence if necessary, we may assume that there exist $\varphi_\infty, \psi_\infty \in \widetilde{V}$, such that $\varphi_m \to \varphi_\infty$ and $\psi_m \to \psi_\infty$ in W as $m \to \infty$. So $X(\varphi_\infty) = X(\psi_\infty)$ from the continuity of the map $\varphi \mapsto X(\varphi)$. Accompanying with the fact that $\varphi_\infty, \psi_\infty \in V$, this implies that $\varphi_\infty = \psi_\infty$. From the fact that $\varphi \mapsto X(\varphi), \varphi \in U$ is a local diffeomorphism, this implies that there exists a $M \in \mathbf{N}$ large enough, such that for all $m \geq M$, $X(\varphi_m) \neq X(\psi_m)$. This makes a contradiction.

Step 5. Let $M = \{X(\varphi); \varphi \in U\}$. In the following, we will check that M satisfies all of the conditions in Definition 2.3. The first is obvious from the fact that $\varphi \mapsto X(\varphi), \varphi \in U$ is a diffeomorphism. The second is true since $z = X(P_d D\Phi(z))$ for any $z \in V$. For the third one, for any $u \in A_z$, we have $u = D^2\Phi(z)(S_z u, \cdot)$, where the operator S_z is defined in section 1. So

$$
S_z u = S_z(D^2\Phi(z)(S_z u, \cdot)) - S_z(P_d D^2\Phi(z)(S_z u, \cdot)) + S_z(P_d u).
$$

Combining this with (2.4), we see that both $S_z u$ and $DX(\varphi_z)(P_d u)$ (where $\varphi_z \equiv P_d D\Phi(z)$ as before) are solutions of

$$X = S_z((I - P_d)D^2\Phi(z)(X, \cdot)) + S_z(P_d u). \tag{2.5}$$

So from the uniqueness of the solution of the equatily (2.5), which comes from Lemma 2.6, $S_z u = DX(\varphi_z)(P_d u)$. Hence $S_z u \in T_z M$ for any $z \in V$ and any $u \in A_z$. This completes the proof of the fact that M is a manifold reflecting singularities.

This finishes the proof of the theorem. □

3. Resolution of Singularties

In this section, we construct a family of functions Φ_z defined on B, $z \in M \cap V_\delta$, for $\delta > 0$ small enough, such that Φ_z, $z \in M \cap V_\delta$, are not degenerate.

First, we show the following

Lemma 3.1. *There exist an integer $k \in \mathbf{N}$ large enough and a $\delta \in (0, \delta_0)$ such that $Q_k\Big|_{M\cap V_\delta}$ is injective, where δ_0 is the one in the assumption (A3), and for any $\varphi \in \{P_d D\Phi(z), z \in V\}$ and $\psi \in W$, $Q_k DX(\varphi)(\psi) = 0$ implies $\psi = 0$.*

Proof. If not, then for any $n \in \mathbf{N}$, there exist $x_n, y_n \in M \cap V_{1/n}$, such that $Q_n x_n = Q_n y_n$, but $x_n \neq y_n$. It is easy to see that by taking subsequence if necessary, we can assume that there exist $x, y \in M \cap V$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. $Q_n x_n = Q_n y_n$ for any $n \in \mathbf{N}$ implies that $x = y$. That is, x_n and y_n converge to a same limit as $n \to \infty$. As $x_n, y_n \in M$, by the definition of M, there exist $\varphi_n, \psi_n \in W$, such that $x_n = X(\varphi_n)$ and $y_n = X(\psi_n)$, $n \in \mathbf{N}$. $\frac{\varphi_n - \psi_n}{||\varphi_n - \psi_n||_W} \in W$ is bounded, so by taking subsequence if necessary, we can assume that it converges. Also, x_n and y_n converge as $n \to \infty$ implies that φ_n and ψ_n converge as $n \to \infty$, too. Note that $||x_n - y_n||_M = ||\varphi_n - \psi_n||_W$ for any $n \in \mathbf{N}$. Therefore,

$$w_n \equiv \frac{x_n - y_n}{||x_n - y_n||} = \int_0^1 DX(\psi_n + t(\varphi_n - \psi_n)) \left(\frac{\varphi_n - \psi_n}{||\varphi_n - \psi_n||} \right) dt$$

converges as $n \to \infty$. Write the limit as $w \in M$. Then as done before, from the assumption, $||w||_M = 1$, but $Q_n w = 0$ for any $n \in \mathbf{N}$, hence $w = 0$. This is a contradiction.

For the second part, we use the contradiction, too. If not, then for any $n \in \mathbf{N}$, there exist a $\varphi_n \in \{P_d D\Phi(z), z \in V\}$, and a $\psi_n \in W(\equiv ImP_d)$ with $\psi_n \neq 0$, such that $Q_n DX(\varphi_n)(\psi_n) = 0$. Without loss of generality, we may assume that $||\psi_n||_W = 1$, $n \in \mathbf{N}$. So, by taking subsequence if necessary, we may assume that ψ_n converges to a $\psi \in W$ in W. Hence, $||\psi||_W = 1$. On the same time, from the compactness of V, by taking subsequence if necessary, we may assume that there exists a $\varphi \in \{P_d D\Phi(z), z \in V\}$, such that $\varphi_n \to \varphi$. So, by the continuity, we get that $Q_n DX(\varphi)(\psi) = 0$ for every $n \in \mathbf{N}$. Therefore, $DX(\varphi)(\psi) = 0$, which implies that $\psi = 0$. This makes a contradiction. □

Obviously, we can assume that $k \geq d$. (Otherwise, just take $\max\{k, d\}$ as the new k.) For any $z \in M \cap V_\delta$, let

$$\Phi_z(y) = \Phi(y) - \frac{1}{2}||Q_k y - Q_k z||^2_{\mathrm{Im}Q_k}, \qquad \text{for any } y \in B,$$

where $||\cdot||_{\mathrm{Im}Q_k}$ means the norm of $\mathrm{Im}Q_k$ as considered as a subspace of $\widetilde{H}$. Let λ_z denote the supremum of $\Phi_z - h$. (Note that $\lambda_z \leq \lambda$ for all $z \in M \cap V_\delta$.) Then we have the following

Proposition 3.2. *The function $z \mapsto \lambda_z$, $z \in M \cap V_\delta$ is continuous.*

Proof. During and after the proof of this proposition, we will use, with a little abuse of the notation, $||Q_k V - Q_k z||_{\mathrm{Im}Q_k}$ to denote the distance between $Q_k V$ and $Q_k z$ under $||\cdot||_{\mathrm{Im}Q_k}$.

Take an arbitrary $\varepsilon > 0$ and fix it for a while. Now, note that for any $z \in M \cap V_\delta$, $\lambda_z \geq \lambda - \frac{1}{2}||Q_k V - Q_k z||^2_{\mathrm{Im}Q_k}$. So for any $y \in B$ with $||Q_k V - Q_k y||_{\mathrm{Im}Q_k} \geq 2||Q_k V - Q_k z||_{\mathrm{Im}Q_k} + \varepsilon$, which implies $||Q_k y - Q_k z||_{\mathrm{Im}Q_k} \geq ||Q_k V - Q_k z||_{\mathrm{Im}Q_k} + \varepsilon$, we have

$$\begin{aligned} & \Phi(y) - h(y) - \frac{1}{2}||Q_k y - Q_k z||^2_{\mathrm{Im}Q_k} \\ \leq \; & \lambda - \frac{1}{2}\left(||Q_k V - Q_k z||_{\mathrm{Im}Q_k} + \varepsilon\right)^2 \\ < \; & \lambda - \frac{1}{2}||Q_k V - Q_k z||^2_{\mathrm{Im}Q_k} \leq \lambda_z. \end{aligned}$$

Write $a_z \equiv 2||Q_k V - Q_k z||_{\mathrm{Im}Q_k} + \varepsilon$. Then we get

$$\lambda_z = \sup\{\Phi(y) - h(y) - \frac{1}{2}||Q_k y - Q_k z||^2_{\mathrm{Im}Q_k}; ||Q_k V - Q_k y||_{\mathrm{Im}Q_k} \leq a_z\}.$$

Therefore, for any $z_1, z_2 \in M \cap V_\delta$,

$$\begin{aligned}\lambda_{z_1} - \lambda_{z_2} &= \sup\Big\{\Phi(y) - h(y) - \frac{1}{2}||Q_k y - Q_k z_1||^2_{\mathrm{Im}Q_k};\\ &\qquad\qquad ||Q_k V - Q_k y||_{\mathrm{Im}Q_k} \le a_{z_1} \vee a_{z_2}\Big\}\\ &\quad - \sup\Big\{\Phi(y) - h(y) - \frac{1}{2}||Q_k y - Q_k z_2||^2_{\mathrm{Im}Q_k};\\ &\qquad\qquad ||Q_k V - Q_k y||_{\mathrm{Im}Q_k} \le a_{z_1} \vee a_{z_2}\Big\}\\ &\le \sup\Big\{ - \frac{1}{2}||Q_k y - Q_k z_1||^2_{\mathrm{Im}Q_k} + \frac{1}{2}||Q_k y - Q_k z_2||^2_{\mathrm{Im}Q_k};\\ &\qquad\qquad ||Q_k V - Q_k y||_{\mathrm{Im}Q_k} \le a_{z_1} \vee a_{z_2}\Big\}.\end{aligned}$$

Now, our proposition can be easily seen from the definition of $||\cdot||_{\mathrm{Im}Q_k}$. □

For any $z \in M \cap V_\delta$, let $K_z \equiv \{x; \Phi_z(x) - h(x) = \lambda_z\}$. (Note that if $z \in V$, then $K_z = \{z\}$ by Lemma 3.1.) As in Bolthausen [1], K_z is compact and non-empty. For any $x_z \in K_z$, the probability measure $\nu_z^{x_z}$ defined by

$$\nu_z^{x_z}(dy) = e^{D\Phi_z(x_z)(y)}\mu(dy)/M(D\Phi_z(x_z))$$

has mean x_z. Let $\nu_{z,0}^{x_z}$ be the 0-centered $\nu_z^{x_z}$, and let $\Gamma_z^{x_z}$ be the inner product on B^* defined by

$$\Gamma_z^{x_z}(\phi, \psi) = \int_B \phi(y)\psi(y)\nu_{z,0}^{x_z}(dy), \qquad \phi, \psi \in B^*.$$

Let $H_z^{x_z} \equiv (\overline{B^*}^{\Gamma_z^{x_z}})^*$, and $S_z^{x_z}\varphi \equiv \int \varphi(y) y \nu_{z,0}^{x_z}(dy)$, as done before. Then as in Kusuoka-Liang [5], we can show that $D^2\Phi_z(x_z)\Big|_{H_z^{x_z} \times H_z^{x_z}}$ is a Hilbert-Schmidt function for any $z \in M \cap V_\delta$ and any $x_z \in K_z$. Also, we can show the following

PROPOSITION 3.3. *Choose $z_n \in M \cap V_{1/n}$ with $z_n \to z \in V$. Then for any $x_n \in K_{z_n}$, $x_n \to z$.*

PROOF. Choose any subsequence of the natural numbers **N**, for the sake of simplicity, we write it as n, too. Since x_n maximizes $\Phi_{z_n} - h$,

$$\Phi(x_n) - h(x_n) - \frac{1}{2}||Q_k x_n - Q_k z_n||^2_{\mathrm{Im}Q_k} \ge \lambda - \frac{1}{2}||Q_k z - Q_k z_n||^2_{\mathrm{Im}Q_k}.$$

So $||Q_k x_n - Q_k z_n||_{\widetilde{H}} = ||Q_k x_n - Q_k z_n||_{\mathrm{Im}Q_k} \le ||Q_k z - Q_k z_n||_{\mathrm{Im}Q_k} \to 0$ as $n \to \infty$. Therefore, $Q_k x_n \to Q_k z$ in B as $n \to \infty$, and $\liminf_{n\to\infty} (\Phi(x_n) - h(x_n)) \ge \lambda$. By doing in the same way as in the proof of Lemma 2.2, there exist a subsequence x_{n_j} and a $x \in V$ such that $x_{n_j} \to x$. $Q_k x_{n_j} \to Q_k x$, too, hence $Q_k x = Q_k z$, this and Lemma 3.1 imply that $x = z$, *i.e.*, $x_{n_j} \to z$ as $j \to \infty$. This is true for any subsequence of $\mathbf{N}$.

This finishes the proof of our proposition. □

PROPOSITION 3.4. *All of the eigenvalues of $D^2\Phi_z(x_z)\big|_{H_z^{x_z} \times H_z^{x_z}}$ are smaller than 1, if $\delta > 0$ is small enough.*

PROOF. From the continuity showed in Proposition 3.3, we only need to show that for any $z \in V$ and any $\varphi \in B^*$ with $\varphi \neq 0$,

$$D^2\Phi(x_z)(S_z^{x_z}\varphi, S_z^{x_z}\varphi) - D^2\left(\frac{1}{2}||Q_k(\cdot) - Q_k z||^2_{\mathrm{Im}Q_k}\right)(x_z)(S_z^{x_z}\varphi, S_z^{x_z}\varphi) < \varphi(S_z^{x_z}\varphi). \tag{3.1}$$

But here, from the definition of $||\cdot||_{\mathrm{Im}Q_k}$,

$$||Q_k y - Q_k z||^2_{\mathrm{Im}Q_k} = \sum_{i=1}^{k} (e_i(y) - e_i(z))^2,$$

so

$$D\left(||Q_k(\cdot) - Q_k z||^2_{\mathrm{Im}Q_k}\right)(y)(u) = 2\sum_{i=1}^{k} (e_i(y) - e_i(z))\, e_i(u)$$

for any y and any u in B. For any $z \in V$, since $x_z = z$, the above implies that $D\Phi_z(x_z) = D\Phi(x_z) = D\Phi(z)$, so from the definition of $\nu_z^{x_z}$, $\nu_{z,0}^{x_z}$, $\Gamma_z^{x_z}$, $H_z^{x_z}$, and $S_z^{x_z}$ at the beginning of this section, we see that these quantities coincide with ν_z, $\nu_{z,0}$, Γ_z, H_z, and S_z, the ones defined in section 1, respectively. Moreover,

$$D^2\left(\frac{1}{2}||Q_k(\cdot) - Q_k z||^2_{\mathrm{Im}Q_k}\right)(z)(u,u) = \sum_{i=1}^{k} e_i(u)^2 = ||Q_k u||^2_{\mathrm{Im}Q_k} (\ge 0).$$

So (3.1) is equivalent to

$$D^2\Phi(z)(S_z\varphi, S_z\varphi) - ||Q_k S_z\varphi||^2_{\mathrm{Im}Q_k} < \varphi(S_z\varphi). \tag{3.2}$$

The inequality above is obviously if $\varphi \notin A_z$, from the definition of A_z. For $\varphi \in A_z$, by Lemma 2.7, $Q_d S_z \varphi \neq 0$, hence $Q_k S_z \varphi \neq 0$, too, so

$$\begin{aligned} & D^2\Phi(z)(S_z\varphi, S_z\varphi) - ||Q_k S_z \varphi||^2_{\mathrm{Im}Q_k} \\ = \; & \varphi(S_z\varphi) - ||Q_k S_z\varphi||^2_{\mathrm{Im}Q_k} \\ < \; & \varphi(S_z\varphi). \end{aligned}$$

That is, (3.2) still holds.

This gives our assertion. □

LEMMA 3.5. *$\delta > 0$ can be chosen small enough, such that for any $z \in M \cap V_\delta$, there is a unique x_z attains the maximum of $\Phi_z - h$. Moreover, the map $z \mapsto x_z, z \in M \cap V_\delta$ is in C^2.*

PROOF. First, we show the uniqueness. If not, for any $n \in \mathbf{N}$, there exist $z_n \in M \cap V_{1/n}$ and $x_n^1, x_n^2 \in B$, such that $x_n^1 \neq x_n^2$ and both of them maximize $\Phi_{z_n} - h$. By taking subsequence if necessary, we can assume that there exists a $z \in V$, such that $z_n \to z$ as $n \to \infty$. Also, by Proposition 3.3, we can assume that x_n^1 and x_n^2 converge to z, too, by taking subsequence if necesary.

From the definition of x_n^1 and x_n^2, we have that

$$\frac{\int_B y e^{D\Phi_{z_n}(x_n^1)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x_n^1)(y)}\mu(dy)} = x_n^1,$$

$$\frac{\int_B y e^{D\Phi_{z_n}(x_n^2)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x_n^2)(y)}\mu(dy)} = x_n^2.$$

Let $f_n(x) \equiv \frac{\int_B y e^{D\Phi_{z_n}(x)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)}\mu(dy)}$, then as before,

$$\begin{aligned} x_n^2 - x_n^1 &= f_n(x_n^2) - f_n(x_n^1) \\ &= Df_n(z)(x_n^2 - x_n^1) \\ &\quad + \int_0^1 \left[Df_n(x_n^1 + t(x_n^2 - x_n^1)) - Df_n(z)\right](x_n^2 - x_n^1)dt. \end{aligned}$$

But

$$Df_n(x)(u) = \frac{\int_B D^2\Phi_{z_n}(x)(u,y) y e^{D\Phi_{z_n}(x)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)}\mu(dy)}$$

$$-\frac{\int_B ye^{D\Phi_{z_n}(x)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)}\mu(dy)}\cdot\frac{\int_B D^2\Phi_{z_n}(x)(u,y)e^{D\Phi_{z_n}(x)(y)}\mu(dy)}{\int_B e^{D\Phi_{z_n}(x)(y)}\mu(dy)}.$$

So by a simple calculation and the fact that both x_n^1 and x_n^2 converge to z as $n \to \infty$, we get

$$\psi(\frac{x_n^2-x_n^1}{||x_n^2-x_n^1||}) - D^2\Phi_{z_n}(z)(\frac{x_n^2-x_n^1}{||x_n^2-x_n^1||}, S_z\psi) \to 0, \qquad \text{for any } \psi \in B^*,$$

therefore,

$$\psi(\frac{x_n^2-x_n^1}{||x_n^2-x_n^1||}) - D^2\Phi_{z}(z)(\frac{x_n^2-x_n^1}{||x_n^2-x_n^1||}, S_z\psi) \to 0, \qquad \text{for any } \psi \in B^*.$$

As in the proof of Lemma 2.5, we can get from this that $\frac{x_n^2-x_n^1}{||x_n^2-x_n^1||}$ converges as $n \to \infty$. Write the limit as x, then $\psi(x) = D^2\Phi_z(z)(x, S_z\psi)$ for any $\psi \in B^*$. This contradict with Proposition 3.4.

We just showed that for $z \in M \cap V_\delta$ with $\delta > 0$ small enough, x_z is the unique solution of the following equation with respect to x:

$$\frac{\int_B ye^{D\Phi_z(x)(y)}\mu(dy)}{\int_B e^{D\Phi_z(x)(y)}\mu(dy)} - x = 0.$$

Let the left hand side above be denoted by $f(x,z)$. Then f is twice continuously differentiable with respect to x, and by Proposition 3.4, $\frac{\partial f}{\partial x}(x_z, z)(u) \neq 0$ whenever $u \neq 0$. So by implicit function theorem, we get that $z \mapsto x_z$ is twice continuously differentiable.

This finishes the proof of our lemma. □

From the uniqueness of x_z from Lemma 3.5, from now on, we will abbreviate $\nu_z^{x_z}$, $\nu_{z,0}^{x_z}$, $\Gamma_z^{x_z}$, and $H_z^{x_z}$ as ν_z, $\nu_{z,0}$, Γ_z, and H_z, respectively.

4. Uniform Estimate

As in Kusuoka-Liang [5], for any $R > 2$, let $\tilde{\nu}_R$ be the probability measure of $\mathbf{R}$ given by

$$\tilde{\nu}_R(\{R\}) = \frac{3}{4R^2-1}, \quad \tilde{\nu}_R(\{\frac{1}{2}\}) = \frac{R-2}{2R-1}, \quad \tilde{\nu}_R(\{-\frac{1}{2}\}) = \frac{R+2}{2R+1}.$$

By a simple calculation, we have

$$E^{\tilde{\nu}_R}[Y]=0, \qquad E^{\tilde{\nu}_R}[Y^2]=1.$$

Let $\rho_a, a>0$, be the probability measures given by

$$\rho_a(dR)=C_a\exp(-\frac{aR^2}{2})dR, \quad R>2,$$

where C_a is the normalizing constant, *i.e.* $C_a=(\int_2^\infty e^{-\frac{aR^2}{2}}dR)^{-1}$. Let $\nu_a, a>0$ be the probability measure given by

$$\nu_a(dy)=\int\tilde{\nu}_R(dy)\rho_a(dR).$$

LEMMA 4.1. *For any $a>0$, there exists a constant D_a, depends only on a, such that for i.i.d. random variables $Y_i, i=1,2,\cdots$ with law ν_a,*

$$P(|\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i|\geq z)\leq 2\exp(-\frac{1}{4D_a}z^2), \qquad \forall z\geq 0. \tag{4.1}$$

PROPOSITION 4.2. *There exists a constant $C_1'>0$, independent to $z\in M\cap V_\delta$, such that*

$$C_5\equiv\sup_{z\in M\cap V_\delta}\int_B e^{C_1'||x||^2}\nu_{z,0}(dx)<\infty$$

Note. As mentioned above, the meaning of this proposition is that: there exist a $\delta>0$ and a $C_1'>0$, where C_1' is independent to δ, such that the expression holds. All of the lemmas that follows hold in the same meaning, and we will not emphasize it any more.

PROOF. First, from the definition of x_z, $||x_z||, z\in M\cap V_\delta$ is bounded. Write C_6 as its upper bound.

Also, since V is compact and $z\mapsto x_z, z\in M\cap V_\delta$ is continuous at V from Lemma 3.5, we see that if $\delta>0$ is small enough, $D\Phi_z(x_z), z\in$

$M \cap V_\delta$ is bounded in B^*. So $\int_B e^{2D\Phi_z(x_z)(x)}\mu(dx)$ is bounded above and $\int_B e^{D\Phi_z(x_z)(x)}\mu(dx)$ is bounded from 0 for $z \in M \cap V_\delta$. Therefore,

$$\begin{aligned}
& \int_B e^{C_1'||x||^2}\nu_{z,0}(dx) \\
= & \int_B e^{C_1'||x-x_z||^2} e^{D\Phi_z(x_z)(x)}\mu(dx)/M(D\Phi_z(x_z)) \\
\leq & (\int_B e^{2C_1'||x-x_z||^2}\mu(dx))^{1/2} \\
& \cdot (\int_B e^{2D\Phi_z(x_z)(x)}\mu(dx))^{1/2} / \int_B e^{D\Phi_z(x_z)(x)}\mu(dx) \\
\leq & (\int_B e^{4C_1'||x||^2}\mu(dx))^{1/2} e^{2C_1'C_6^2} \\
& \cdot (\int_B e^{2D\Phi_z(x_z)(x)}\mu(dx))^{1/2} / \int_B e^{D\Phi_z(x_z)(x)}\mu(dx) \\
< & \infty
\end{aligned}$$

for $C_1' \leq \frac{C_1}{4}$ by the assumption (A1). □

The following can be gotten from proposition 4.2, by using the same method as in Kusuoka-Liang [5], Lemma 3.2, Lemma 3.3, Lemma 3.5, Lemma 3.6, Lemma 3.7. We omit the proofs here.

LEMMA 4.3. *Under the assumption (A1) in section 1, for any $c > 0$, there exists a $a_0 > 0$ small enough, such that for any $a < a_0$, the following holds:*

$$c^n \left(\int_B ||x||^{2n}\nu_{z,0}(dx) \right)^{1/2} \leq \int_{\mathbf{R}} y^n \nu_a(dy), \qquad \forall n \geq 3, \forall z \in M \cap V_\delta. \tag{4.2}$$

LEMMA 4.4. *Let $\Psi_z, z \in M \cap V_\delta$ be a family of symmetric bilinear functions that satisfies:*

1. $\int_B \Psi_z(x,y)\nu_0(dy) = 0, \quad \forall x \in B, \forall z \in M.$

2. *There exists a constant $C_0 > 0$, independent to z, such that*

$$|\Psi_z(x,y)| \leq C_0||x|| \cdot ||y||, \qquad \forall x,y \in B, \quad \forall z \in M \cap V_\delta,$$

3. $\int_B \Psi_z(x,y)^2 \nu_{z,0}(dx)\nu_{z,0}(dy) = 1.$

Then, there exists an $a_0 > 0$, *depends only on* C_0 *and* $\sup_{z\in M\cap V_\delta}\int_B ||y||^2\nu_{z,0}(dy)$, *satisfying the following:*

$$E^{\nu_{z,0}^{\otimes\infty}}\left[\prod_{k=1}^{m}\Psi_z(X_{i_k},X_{j_k})\right]\le E^{\nu_a^{\otimes\infty}}\left[\prod_{k=1}^{m}Y_{i_k}Y_{j_k}\right],\tag{4.3}$$
$$\forall m\in\mathbf{N},1\le^{\forall} i_k<^{\forall} j_k\le n,k=1,\cdots,m,0<{}^{\forall}a<a_0,$$
$$\forall z\in M\cap V_\delta,$$

where $\{X_i\}_{i=1}^\infty$ *is the sequence of random variables defined in section 1, and* $\{Y_i\}_{i=1}^\infty$ *is defined by* $Y_n(\underline{y})=y_n,\forall\underline{y}=(y_1,y_2,\cdots)\in\mathbf{R}^{\mathbf{N}}$.

LEMMA 4.5. *Assume the same assumptions and use the same notations as in lemma 4.4. Then for* $\forall b<\frac{1}{2}$, *there exists* $\varepsilon>0$, *such that*

$$\sup_{z\in M\cap V_\delta}\sup_{n\in\mathbf{N}}E^{\nu_{z,0}^{\otimes\infty}}\left[\exp(b\cdot\frac{1}{n}\sum_{1\le i\ne j\le n}\Psi_z(X_i,X_j)),\right.$$
$$\left.|\frac{1}{n^2}\sum_{1\le i\ne j\le n}\Psi_z(X_i,X_j)|<\varepsilon\right]<\infty.\tag{4.4}$$

LEMMA 4.6. *Assume the same conditions as above. Then, for any* $\forall b<\frac{1}{2}$, *there exist constants* $\varepsilon_1>0$ *and* $\varepsilon_2>0$, *such that the following holds:*

$$\sup_{z\in M\cap V_\delta}\sup_{n\in\mathbf{N}}E^{\nu_{z,0}^{\otimes\infty}}\left[\exp\left(b\cdot n\Psi_z(\frac{S_n}{n},\frac{S_n}{n})\right),\right.$$
$$\left\{|\frac{1}{n^2}\sum_{i=1}^{n}\Psi_z(X_i,X_i)|<\varepsilon_1\right\}$$
$$\left.\cap\left\{||\frac{1}{n}\sum_{i=1}^{n}X_i||<\varepsilon_2\right\}\right]<\infty.$$

LEMMA 4.7. *Assume that* $\Psi_z, z\in M\cap V_\delta$ *is a family of symmetric, bilinear functions that satisfy the following conditions:*

1. $\int_B\Psi_z(x,y)\nu_{z,0}(dy)=0,\quad\forall x\in B,\forall z\in M\cap V_\delta.$

2. There exists a constant $C_0 > 0$*, such that*

$$|\Psi_z(x,y)| \le C_0||x|| \cdot ||y||, \qquad \forall x,y \in B, \quad \forall z \in M \cap V_\delta,$$

3. $\int_B \Psi_z(x,y)^2 \nu_{z,0}(dx)\nu_{z,0}(dy) \equiv b_z \le {}^{\exists} b < \frac{1}{2}$*.*

Then there exists a $\varepsilon > 0$*, such that*

$$\sup_{z\in M\cap V_\delta} \sup_{n\in\mathbf{N}} E^{\nu_{z,0}^{\otimes\infty}} \left[\exp(\frac{1}{n}\sum_{i,j=1}^n \Psi_z(X_i,X_j)), ||\frac{1}{n}\sum_{i=1}^n X_i|| < \varepsilon \right] < \infty. \tag{4.5}$$

5. Proof of the Theorem

First, note that by Donsker-Varadhan [3],

$$\limsup_{n\to\infty} \frac{1}{n} \log E^{\mu^{\otimes\infty}} \left[e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \notin V_{\delta/2} \right] < \lambda. \tag{5.1}$$

So we only need to do with the integration on the set $\{\frac{S_n}{n} \in V_{\delta/2}\}$ from now on.

Now, let $M \cap V_\delta$ be the new M. It is obvious that the new M still has the same property as the old one. Equip with M the Riemann metric, written as d_M, and use $v_M(dx)$ to denote the volume element on M.

Proposition 5.1. *There exists a continuous* $C : V_{\delta/2} \to \mathbf{R}$*, such that*

$$C(w)^{-1} n^{\frac{d}{2}} \int_{\{||z-w||\le\delta/2\}\cap M} \cdot \exp\left(-\frac{1}{2}n \cdot ||Q_k z - Q_k w||^2_{ImQ_k}\right) v_M(dz) \to 1 \tag{5.2}$$

uniformly in $w \in V_{\delta/2}$ *as* $n \to \infty$*.*

Proof. From the definition of M, for any $z \in M$, there exists a $\varphi \in U$ such that $z = X(\varphi)$. Let $\varphi_0 \equiv X^{-1}(w)$, then

$$\begin{aligned} & n^{\frac{d}{2}} \int_{\{||z-w||\le\delta/2\}\cap M} \exp\left(-\frac{1}{2}n \cdot ||Q_k z - Q_k w||^2_{\mathrm{Im}Q_k}\right) v_M(dz) \\ = \; & n^{\frac{d}{2}} \int_{\{||\varphi_0-\varphi||\le\delta'/2\}\cap U} \cdot \exp\left(-\frac{1}{2}n \cdot ||Q_k X(\varphi) - Q_k X(\varphi_0)||^2_{\mathrm{Im}Q_k}\right) v_U(d\varphi). \end{aligned} \tag{5.3}$$

But $X(\varphi) = X(\varphi_0) + DX(\varphi_0)(\varphi - \varphi_0) + r(\varphi)$, where $r(\varphi)$ is the 2nd Taylor remainder, and by the continuity, there exists a constant $K > 0$, such that $r(\varphi) \le K||\varphi - \varphi_0||^2$. Now, for any $w \in M \cap V_{\delta/2}$, let

$$B_w(\psi, \psi) = \sum_{i=1}^{k} \left[e_i \left(DX(X^{-1}(w))(\psi) \right) \right]^2, \qquad \psi \in W = ImP_d,$$

it is bilinear on $W \times W$, and by Lemma 3.1, if $\delta > 0$ is small enough, there exists a constant $C > 0$, such that $B_w(\psi, \psi) \ge C||\psi||^2$ for all $\psi \in W$. So by the definition of Q_k, if $\delta' > 0$ is small enough,

$$\begin{aligned} & n^{\frac{d}{2}} \int_{\{||\varphi_0 - \varphi|| \le \delta'/2\} \cap U} \exp\left(-\frac{1}{2} n \cdot ||Q_k DX(\varphi_0)(\varphi - \varphi_0)||^2_{\mathrm{Im}Q_k} \right) v_U(dz) \\ = \; & n^{\frac{d}{2}} \int_{\{||\varphi_0 - \varphi|| \le \delta'/2\} \cap ImP_d} \exp\left(-\frac{1}{2} n \cdot B_w(\varphi - \varphi_0, \varphi - \varphi_0) \right) v_{ImP_d}(dz), \end{aligned}$$

which, by the discussion above, converges to $(2\pi)^{\frac{d}{2}} (\det B_w)^{-\frac{1}{2}}$ as $n \to \infty$. As stated before, the $r(\varphi)$ is a high order of $||\varphi - \varphi_0||^2$, so (5.3) converges to the same limit. The uniformness can be gotten in the same way by the continuity. □

Now, we are ready to proof the following proposition, which certainly gives our main theorem. Write $C(w)$ as C_w from now on.

Lemma 5.2. *For any bounded continuous function $f : B \to \mathbf{R}$,*

$$\begin{aligned} & E^{\mu^{\otimes\infty}} \left[f(\frac{S_n}{n}) e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2} \right] \\ = \; & e^{n\lambda} n^{\frac{d}{2}} \int_M f(x(z)) b(z) e^{-n \cdot a(z)} v_M(dz)(1 + o(1)) \end{aligned}$$

as $n \to \infty$, where $b(z) = C_{x_z}^{-1} \exp(\frac{1}{2} \int_B D^2\Phi_z(x_z)(y, y)\nu_{z,0}(dy)) \times det_2(I - D^2\Phi_z(x_z))^{-1/2}$, $a(z) = \lambda - \lambda_z$, and $x(z) = x_z$, $z \in M$.

Note. From the definitions of $x(z)$, $a(z)$, and $b(z)$, and the discussions before, it is easy that they are continuous, and satisfy conditions (1), (2) of Theorem 1.1.

Proof. By Proposition 5.1,

$$C_w^{-1} n^{\frac{d}{2}} \int_M \exp\left(-\frac{1}{2} n \cdot ||Q_k z - Q_k w||^2_{\mathrm{Im}Q_k} \right) v_M(dz) \to 1$$

uniformly in $w \in V_{\delta/2}$, so

$$
\begin{aligned}
& E^{\mu^{\otimes\infty}}\left[f(\frac{S_n}{n})e^{n\Phi(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}\right] \\
\sim\ & E^{\mu^{\otimes\infty}}\Big[f(\frac{S_n}{n})e^{n\Phi(\frac{S_n}{n})}C^{-1}_{\frac{S_n}{n}} n^{\frac{d}{2}} \\
& \qquad \cdot \int_M \exp\left(-\frac{n}{2}||Q_k z - Q_k(\frac{S_n}{n})||^2_{\mathrm{Im}Q_k}\right) v_M(dz), \frac{S_n}{n} \in V_{\delta/2}\Big] \\
=\ & e^{n\lambda} n^{\frac{d}{2}} \int_M e^{-n(\lambda-\lambda_z)} \\
& \qquad \cdot e^{-n\cdot\lambda_z} E^{\mu^{\otimes\infty}}\left[C^{-1}_{\frac{S_n}{n}} f(\frac{S_n}{n}) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}\right] v_M(dz),
\end{aligned}
$$

where λ_z is the one defined in section 3.

Therefore, if we can show that $e^{-n\cdot\lambda_z} E^{\mu^{\otimes\infty}}[C^{-1}_{\frac{S_n}{n}} f(\frac{S_n}{n}) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}]$ is bounded for $z \in M(= M \cap V_\delta)$, and converges to $b(z)f(x(z))$ for each $z \in M$, it will complete the proof of the lemma.

The convergence to $b(z)f(x(z))$ for each $z \in M$ can be shown by using the same method as in Kusuoka-Liang [5]. In fact, as there, we have

$$
\begin{aligned}
& e^{-n\cdot\lambda_z} E^{\mu^{\otimes\infty}}[C^{-1}_{\frac{S_n}{n}} f(\frac{S_n}{n}) e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}] \\
=\ & E^{\nu^{\otimes\infty}_{z,0}}\Big[C^{-1}_{\{\frac{S_n}{n}+x_z\}} f(\frac{S_n}{n} + x_z) \\
& \qquad \cdot \exp\left(\frac{n}{2} D^2\Phi_z(x_z)(\frac{S_n}{n}, \frac{S_n}{n}) + nR(x_z, \frac{S_n}{n})\right), \frac{S_n}{n} + x_z \in V_{\delta/2}\Big],
\end{aligned}
$$

where $R(x_z, \frac{S_n}{n})$ is the 3rd remainder of the Taylor's formula. But $\frac{S_n}{n} \to 0$ almost surely under $\nu^{\otimes\infty}_{z,0}$. So by Kusuoka-Liang [5], we get the convergence here for each $z \in M$.

For the boundedness, since $C^{-1}_w, w \in V_{\delta/2}$ in bounded from the continuity of $C : V_{\delta/2} \to \mathbf{R}$, and f is bounded, we only need to show that $e^{-n\cdot\lambda_z} E^{\mu^{\otimes\infty}}[e^{n\Phi_z(\frac{S_n}{n})}, \frac{S_n}{n} \in V_{\delta/2}]$ is bounded for $z \in M$.

Here, for every $z \in M \cap V_\delta$, let a^z_l and f^z_l, $l \in \mathbf{N}$, be the eigenvalues and the corresponding eigenvectors of $D^2\Phi_z(x_z)\Big|_{H_z}$, where $|a_l|^z$ is decreasing with respect to l for each z. Let

$$
\Psi^{(N)}_{z,1}(x, y) = \sum_{l=1}^{N} a^z_l (f_l, x)_{H_z} (f_l, y)_{H_z},
$$

$$\Psi_{z,2}^{(N)}(x,y) = D^2\Phi_z(x_z)(x,y) - \Psi_{z,1}^{(N)}(x,y).$$

Since $D^2\Phi_z(x_z)\Big|_{H_z}$ is a Hilbert-Schmidt function, for any $\eta > 0$, there exists an $n_z \in \mathbf{N}$ large enough, such that

$$\int_B \int_B \Psi_{z,2}^{(n_z)}(x,y)^2 \nu_{z,0}(dx)\nu_{z,0}(dy) = \sum_{l=n_z+1}^{\infty} |a_l^z|^2 < \eta/2.$$

Also, since $D^2\Phi_z(x_z)$ is continuous with respect to z, we have that a_l^z is upper semi-continuous with respect to z, therefore, for each $z_0 \in M \cap V_\delta$, we can find a neighborhood U_{z_0} of z_0 in $M \cap V_\delta$, such that for every $z \in U_{z_0}$, we have

$$\sum_{l=n_{z_0}+1}^{\infty} |a_l^z|^2 < \eta.$$

Therefore, for any $\eta > 0$, we can find a N (independent to $z \in M \cap V_\delta$), such that

$$\int_B \int_B \Psi_{z,2}^{(N)}(x,y)^2 \nu_{z,0}(dx)\nu_{z,0}(dy) = \sum_{l=N+1}^{\infty} |a_l^z|^2 < \eta, \qquad \text{for all } z \in M \cap V_\delta.$$

Therefore, the boundness can be gotten from Lemma 4.7 and the Lemma 2.1 in Kusuoka-Tamura [6], since the boundness of $D^2\Phi_z, z \in M \cap V_\delta$ in $B^* \times B^*$ is easy from the fact that M is a manifold embedded in B.

This gives our assertion. □

6. Example

In this section, we will give an example, in which our conditions are satisfied, but the central limit theorem is not.

Example. Let B be the space $\mathcal{M}(\mathbf{T})$ of all signed measures on the torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, which is equal to the dual space of $C(\mathbf{T})$, with the norm induced by it. (B is not seperable now, but our argument still works.) Let

$$U(z) = 2\sum_{k=1}^{\infty} \left(\frac{\cos((4k+1)z)}{(4k+1)\log(4k+1)} - \frac{\cos((4k+3)z)}{(4k+3)\log(4k+3)} \right).$$

$U(z)$ is well-defined, *i.e.*, the series above converges for any $z \in [0, 2\pi)$, and $U(z)$ is continuous with respect to z. Actually, $F(z) = \sum_{n=4}^{\infty} \frac{\sin nz}{n \log n}$ is well-defined and absolutely continuous with respect to z by Edwards [4], and $U(z) = F(z + \frac{\pi}{2}) + F(-z + \frac{\pi}{2})$. Let $V(x, y) = CU(x - y)$, where the constant C is chosen so that $\int_0^{2\pi} \int_0^{2\pi} V(x, y)^2 dxdy \le \pi^2$. V is symmetric and continuous. Let

$$\Phi(R) = \int_0^{2\pi} \int_0^{2\pi} V(x, y) R(dx) R(dy)$$

for $R \in B$. Let $\mu(dx) = \frac{1}{2\pi} dx$, and consider

$$Z_n = E^{\mu^{\otimes \infty}} \left[\exp(n\Phi(n^{-1} \sum_{i=1}^{n} \delta_{X_i})) \right].$$

The entropy function now is

$$h(\nu) = \int_{\mathbf{T}} \left(\log \frac{2\pi d\nu(x)}{dx} \right) \nu(dx)$$

if $\nu(dx) << dx$ and $\log \frac{d\nu(x)}{dx}$ is intergable, and $h(\nu) = 0$ otherwise. So by the conditions above, $\nu_0(dx) = \frac{1}{2\pi} dx$ maximize $\Phi - h$. Therefore, the eigenvalues of $D^2\Phi(\nu_0)\Big|_{H \times H}$ are constant times $\frac{1}{(4k+1)\log(4k+1)}$, $-\frac{1}{(4k+3)\log(4k+3)}$, $k = 1, 2, \cdots$. So, although $D^2\Phi(\nu_0)$ is a Hilbert-Schmidt function, it is not a nuclear function, hence the central limit theorem does not hold.

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