J. Math. Sci. Univ. Tokyo
7 (2000), 241–295.

The Generalized Whittaker Functions for $\mathrm{Sp}(2,\mathbb{R})$ *and the Gamma Factor of the Andrianov* L*-function*

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Abstract. We study the archimedean generalized Whittaker functions for the generalized principal series and the large discrete series of the real symplectic group of degree 2. Using gradient type differential operators, which was introduced by Schmid, we give a system of differential equations which is satisfied by a Whittaker function. We study this system, and give the Mellin transform of its solution. We apply the result to a study of Andrianov's spinor L-function for a non-holomorphic Siegel modular form via Rankin-Selberg integral with an explicitly described archimedean factor.

Introduction

In this paper we study the generalized Whittaker functions associated with some admissible Hilbert representations of the real symplectic group of degree two $G = \mathrm{Sp}(2,\mathbb{R})$. Our motivation to study these functions is to obtain some basic material in the archimedean theory of the automorphic L-functions for the symplectic group of rank 2. Current works on the construction of the automorphic L-functions do not seem sufficiently detailed at archimedean places and ramified p-adic places. A more precise harmonic analysis of the generalized spherical functions on a real reductive group is indispensable to complete the analytic properties of an automorphic L-function. Here we want to support it by an investigation of the generalized Whittaker functions on G.

Let P be the Siegel maximal parabolic subgroup of G with the abelian unipotent radical N. Define a closed subgroup R of P by the semi direct product of N and the connected component of the stabilizer of a character of N in the Levi part. We consider the G-module induced from a non-degenerate unitary character of R. Then the generalized Whittaker functions are defined to be the generalized spherical functions that appear

1991 *Mathematics Subject Classification.* 22E46, 11F46.

in the images of G-intertwining maps from an admissible Hilbert G-module into the module considered above. These functions with specified K-types satisfy a holonomic systems of differential equations. We study these holonomic systems and their solutions.

Let π be a generalized principal series representations corresponding to the another maximal parabolic subgroup P_1 of G, or a large discrete series representations of G. Then main theorems are given on

(i) *Multiplicity free property of the space of the generalized Whittaker realizations of π that correspond to moderate growth functions on a split torus subgroup of G, Theorems 7.5, 7.14 for the generalized principal series, and 11.5 for the large discrete series;*

(ii) *Formulas of the Mellin transforms of the generalized Whittaker functions and its application to a study of Andrianov's spinor L-functions, Theorems 8.1, 8.2, 11.8, and 12.2.*

In this paper we take a definite character of N to give these theorems. In (i) the word "multiplicity free" means that "the dimension is less than or equal to one". We will also give existence theorems in some cases, 7.6, and 11.7.

To obtain the holonomic systems for the generalized Whittaker functions, we use the action of the Casimir operator in $Z(\mathfrak{g}_{\mathbb{C}})$ and also a differential operator of gradient type, which was introduced to characterize the discrete series representations by Schmid [S]. Yamashita [Y2, Y3] applies the later operators to study the realization of the discrete series representations into several types of induced representations.

Here is the organization of this article. From Sections 1 to 5, we collect fundamental ingredients in this paper. In Section 1 we define the generalized Whittaker functions for the admissible modules. Basic notation on Lie groups, Lie algebras is given in Section 2. We recall things about the representations of a maximal compact subgroup K of G in Section 3. We need the irreducible decompositions of the tensor products, which are given in 3.2. In Section 4 Schmid operator and shift operators are introduced. In Section 5, we give the explicit formula of the A-radial parts of the shift operators and the Casimir operator, Proposition 5.3 and Proposition 5.6.

From Sections 6 to 8, we study the generalized principal series representations. We recall its K-type decomposition in Section 6, and define the "corner" K-type. In Section 7, we give a system of differential equations, which are satisfied by the generalized Whittaker functions with the corner K-type of a generalized principal series. A solution with a good asymptotic behavior is obtained in Theorem 7.5 and 7.14. In Section 8 we calculate its generalized Mellin transform, Theorem 8.1 and 8.2.

From Sections 9 to 12, the large discrete series representations are studied. After recalling its K-type decomposition, Section 9, we give a holonomic system of rank 4, Section 10. In Section 11, we study about a formal power series solution with good analytic properties. The Mellin transform is obtained in Theorem 11.8. In Section 12, we apply our results to study of Andrianov's L-function via Rankin-Selberg convolution.

Niwa [Ni] studied the generalized Whittaker functions with trivial K-type for the spherical principal series of $\mathrm{Sp}(2,\mathbb{R})$, which was characterized by the action of generators in $Z(\mathfrak{g}_{\mathbb{C}})$.

The author would like to express his gratitude to Takayuki Oda for valuable advice on this work. Also he would like to thank Sinji Niwa, Takashi Sugano, Nobukazu Shimeno, and Takahiro Hayata for helpful discussions.

1. Definition of the Space of Generalized Whittaker Realizations

We recall a notion of a generalized Whittaker realizations of an irreducible admissible Hilbert representation $(\pi,\mathcal{H}_\pi)$ of a semisimple Lie group, [Y1]. In particular, we formulate it for the real symplectic group of rank 2.

1.1. Let $G=\mathrm{Sp}(2,\mathbb{R})=\left\{g\in \mathrm{SL}(4,\mathbb{R}) \mid {}^tgJ_2g=J_2\right\}$ be the real symplectic group of real rank 2, where $J_2=\left(\begin{smallmatrix}0 & 1_2\\ -1_2 & 0\end{smallmatrix}\right)$ with the 2×2 unit matrix 1_2. Let P_s be the Siegel maximal parabolic subgroup of G. It has a Levi decomposition $P_s=L_s\ltimes N_s$; $L_s=\left\{\left(\begin{smallmatrix}m & 0\\ 0 & {}^tm^{-1}\end{smallmatrix}\right) \mid m\in \mathrm{GL}(2,\mathbb{R})\right\}$, $N_s=\left\{n_T=\left(\begin{smallmatrix}1_2 & T\\ 0 & 1_2\end{smallmatrix}\right) \mid T={}^tT=\left(\begin{smallmatrix}t_1 & t_3\\ t_3 & t_2\end{smallmatrix}\right)\right\}$. The nilpotent radical N_s is abelian. Fix a non-degenerate unitary character η of N_s; for $n_T\in N_s$, set

$$\eta(n_T)=\exp\bigl(2\pi\sqrt{-1}\ \mathrm{tr}(H_\eta T)\bigr)$$

with $H_\eta={}^tH_\eta=\left(\begin{smallmatrix}h_1 & h_3/2\\ h_3/2 & h_2\end{smallmatrix}\right)\in M_2(\mathbb{R})$, $\det H_\eta\neq 0$.

The Levi subgroup naturally acts on the set of unitary characters of N_s. Define $SO(\eta)$ to be the identity component of the subgroup of L_s which stabilizes η by the action. It is isomorphic to the group $SO(2)$, or $SO(1,1)$, according to the sign of H_η. Denote by R the semi direct product group $SO(\eta) \ltimes N_s$. For a character χ of $SO(\eta)$, the character $\chi \cdot \eta$ of R is well-defined by $\chi\cdot\eta(r) := \chi(m)\eta(n)$, $r = (m,n) \in R$. We use the same notations η, χ and $\chi\cdot\eta$ for the differentials of them on the Lie algebras $\mathfrak{n}_s$, $\mathfrak{so}(\eta)$, and $\mathfrak{r}$, of the corresponding groups, respectively. We fix a generator Y_η of $\mathfrak{so}(\eta)$ by

$$Y_\eta = \left(\begin{smallmatrix} B_\eta & 0 \\ 0 & -{}^tB_\eta \end{smallmatrix}\right), \quad B_\eta = H_\eta^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

We now consider the representation of G induced from $(\chi\cdot\eta,\ \mathbb{C}_{\chi\cdot\eta})$ in C^∞-context:

$$C^\infty\text{-}\mathrm{Ind}_R^G(\chi\cdot\eta) = \big\{f : G \to \mathbb{C}_{\chi\cdot\eta} \mid \text{smooth},\ f(rg) = \chi\cdot\eta(r)f(g),\\ (r,\ g) \in R\times G\big\}.$$

Here G acts by the right translations. This is the reduced generalized Gelfand-Graev representation [Y1] II, Sections 1, 2. Through differentiation it has a $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module structure for the Lie algebra $\mathfrak{g}$ and a maximal compact subgroup K of G.

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible admissible Hilbert representation of G. Then the set of $K_\mathbb{C}$-finite vectors $\mathcal{H}_{\pi,K}$ gives a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module $(\pi, \mathcal{H}_{\pi,K})$ through the differentiation.

1.2 Definition. We define a space of intertwining maps

$$\mathrm{Wh}_{\chi\cdot\eta}(\pi) = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K_\mathbb{C})}\big(\mathcal{H}_{\pi,K}, C^\infty\text{-}\mathrm{Ind}_R^G(\chi\cdot\eta)_K\big)$$

that is called the space of the algebraic generalized Whittaker realizations of $(\pi, \mathcal{H}_\pi)$.

1.3. For a finite dimensional K-representation (τ, V), denote by $C^\infty_{\chi\cdot\eta,\tau^*}(R\backslash G/K)$ the space of $\mathbb{C}_{\chi\cdot\eta}\otimes V^*$-valued functions f on G satisfying

$$f(rgk) = \chi\cdot\eta(r)\tau^*(k)^{-1}f(g), \quad (r,g,k) \in R\times G\times K.$$

If we restrict $(\pi, \mathcal{H}_{\pi,K})$ to the subgroup K, it decomposes into a direct Hilbert sum of irreducible finite dimensional K-modules, each of which occurs with finite multiplicity. We call a K-module (τ, V) with non-zero multiplicity a K-type of $(\pi, \mathcal{H}_{\pi,K})$. Fix a K-type (τ, V) of $(\pi, \mathcal{H}_{\pi,K})$ and a $K_{\mathbb{C}}$-embedding $\iota : V \hookrightarrow \mathcal{H}_{\pi,K}$. Considering the $K_{\mathbb{C}}$-map $\Phi \circ \iota$ for each functional $\Phi \in \mathrm{Wh}_{\chi\cdot\eta}(\pi)$, we define a function φ on G with values in $\mathbb{C}_{\chi\cdot\eta} \otimes V^*$ by

$$\Phi \circ \iota(v)(g) = \langle v, \varphi(g)\rangle \text{ for all } v \in V,$$

with the canonical dual $\mathbb{C}_{\chi\cdot\eta}$-valued pairing $\langle\ ,\ \rangle$ on $V \times (\mathbb{C}_{\chi\cdot\eta} \otimes V^*)$. Here (τ^*, V^*) denotes the contragradient of (τ, V). Then φ belongs to $C^\infty_{\chi\cdot\eta,\tau^*}(R\backslash G/K)$. We call it *a generalized Whittaker function for* π *with values in* $\mathbb{C}_{\chi\cdot\eta} \otimes V^*$.

2. Basic Notation, and the Structure of Lie Groups and Algebras

2.1. Let us introduce some notation. Take a maximal compact subgroup $K = \left\{\left(\begin{smallmatrix} A & B \\ -B & A\end{smallmatrix}\right) \in G \mid A, B \in M_2(\mathbb{R})\right\}$ of G, which is isomorphic to the unitary group $U(2)$ of degree 2. Denote by $\mathfrak{g}$ the Lie algebra of G; $\mathfrak{g} = \left\{X \in M_4(\mathbb{R}) \mid JX + {}^tXJ = 0\right\}$, on which we define the Cartan involution θ by $\theta(X) = -{}^tX$, $X \in \mathfrak{g}$. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. The space $\mathfrak{k} = \left\{X_{A,B} = \left(\begin{smallmatrix} A & B \\ -B & A\end{smallmatrix}\right) \in \mathfrak{g} \mid {}^tA = -A,\ {}^tB = B,\ A, B \in M_2(\mathbb{R})\right\}$ is the Lie algebra of K. An isomorphism between $\mathfrak{k}$ and $\mathfrak{u}(2)$ is given by $X_{A,B} \mapsto A + \sqrt{-1}B$. We give a basis of $\mathfrak{u}(2)$ by

$$\sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \quad Y' = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

The space $\mathfrak{p}$ is given by $\mathfrak{p} = \left\{\left(\begin{smallmatrix} A & B \\ B & -A\end{smallmatrix}\right) \in \mathfrak{g} \mid {}^tA = A, {}^tB = B; A, B \in M_2(\mathbb{R})\right\}$. It determines the adjoint representation of K. We express the complexification of a Lie algebra by putting the subscript $\mathbb{C}$; $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, etc.

2.2. *A compact Cartan subalgebra* $\mathfrak{h}$ *and its roots.* We take a basis of $\mathfrak{u}(2)_{\mathbb{C}}$ as

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \quad \overline{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

Here note that $2X = Y - \sqrt{-1}Y'$, $-2\overline{X} = Y + \sqrt{-1}Y'$ and $\left(H', X, \overline{X}\right)$ determines an $\mathfrak{sl}_2$-triple. Via the isomorphism $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{u}(2)_{\mathbb{C}}$, the preimage of the above basis is given by

$$Z = -\sqrt{-1}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \\ \hline -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -1 & & \end{array}\right),$$
$$H' = -\sqrt{-1}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & -1 \\ \hline -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 1 & & \end{array}\right),$$
$$Y = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & -1 & 0 \end{array}\right), \qquad Y' = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \end{array}\right).$$

A compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by $\mathfrak{h} = \mathbb{R}\cdot\sqrt{-1}Z + \mathbb{R}\cdot\sqrt{-1}H'$. Let us define $H_1' = \frac{1}{2}(Z + H')$ and $H_2' = \frac{1}{2}(Z - H')$. Then $\sqrt{-1}H_i'$ belong to $\mathfrak{h}$ for $i = 1, 2$.

We write the value $\beta_i = \beta(\sqrt{-1}H_i)$ of a linear form $\beta : \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$. Then β may be realized by the values (β_1, β_2) and the set of roots $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of $\mathfrak{h}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ is $\sqrt{-1}\{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\}$. This determines the C_2 root system. For each root $\beta \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, put $\mathfrak{g}_\beta = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \beta(H)X,\ \forall H \in \mathfrak{h}_{\mathbb{C}}\}$. We fix a root vector $X_\beta \in \mathfrak{g}_\beta$ as in Table 1. Then $X_{(1,-1)}$ and $X_{(-1,1)}$ are the compact roots and $\mathfrak{k}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathbb{C}\cdot X_{(1,-1)} + \mathbb{C}\cdot X_{(-1,1)}$.

We have a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where $\mathfrak{p}_+ = \mathbb{C}\cdot X_{(2,0)} + \mathbb{C}\cdot X_{(1,1)} + \mathbb{C}\cdot X_{(0,2)}$, and $\mathfrak{p}_- = \mathbb{C}\cdot X_{-(2,0)} + \mathbb{C}\cdot X_{-(1,1)} + \mathbb{C}\cdot X_{-(0,2)}$. This corresponds to the irreducible decomposition the adjoint representation of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$ and a complex structure on the Siegel upper half plane G/K. We put $\Sigma_c^+ = \{(1,-1)\}$ the compact positive roots, $\Sigma_{nc}^+ = \{(2,0), (1,1), (0,2)\}$ the set of non-compact positive roots, $\Sigma_c = \Sigma_c^+ \cup (-\Sigma_c^+)$ all the compact roots, and $\Sigma_{nc} = \Sigma_{nc}^+ \cup (-\Sigma_{nc}^+)$ all the non-compact roots, respectively.

For each root $\beta = (\beta_1, \beta_2)$, put $\|\beta\| = \sqrt{|\beta_1|^2 + |\beta_2|^2}$; here we note $\|\beta\|^2 = 4$ or 2. Then the set $\left\{c\cdot|\beta|(X_\beta + X_{-\beta}),\ c\cdot\sqrt{-1}|\beta|(X_\beta - X_{-\beta}),\ \beta \in \Sigma_{nc}^+\right\}$ forms an orthonormal basis of $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}}$ with respect to the Killing form under adjustment by a constant multiple.

Table 1.

$$
\begin{array}{c|cccc}
-\sqrt{-1}\beta & (2,0) & (1,1) & (0,2) & (1,-1) \\
\hline
X_{\beta} &
\left(\begin{array}{cc|cc} 1 & & i & \\ & 0 & & 0 \\ \hline i & & -1 & \\ & 0 & & 0 \end{array}\right) &
\left(\begin{array}{cc|cc} & 1 & & i \\ 1 & & i & \\ \hline & i & & -1 \\ i & & -1 & \end{array}\right) &
\left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & i \\ \hline 0 & & 0 & \\ & i & & -1 \end{array}\right) &
\left(\begin{array}{cc|cc} & 1 & & -i \\ -1 & & -i & \\ \hline & i & & 1 \\ i & & -1 & \end{array}\right) \\
X_{-\beta} &
\left(\begin{array}{cc|cc} 1 & & -i & \\ & 0 & & 0 \\ \hline -i & & -1 & \\ & 0 & & 0 \end{array}\right) &
\left(\begin{array}{cc|cc} & 1 & & -i \\ 1 & & -i & \\ \hline & -i & & -1 \\ -i & & -1 & \end{array}\right) &
\left(\begin{array}{cc|cc} 0 & & 0 & \\ & 1 & & -i \\ \hline 0 & & 0 & \\ & -i & & -1 \end{array}\right) &
\left(\begin{array}{cc|cc} & 1 & & i \\ -1 & & i & \\ \hline & -i & & 1 \\ -i & & -1 & \end{array}\right).
\end{array}
$$

2.3. *The restricted root system and the Iwasawa decomposition.* Let $\mathfrak{a} = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}\right) \in \mathfrak{g} \mid A = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right);\ t_1, t_2 \in \mathbb{R} \right\}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Fix a basis $\{H_1, H_2\}$ of $\mathfrak{a}$;

$$
H_1 = \left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -1 & 0 \\ & & 0 & 0 \end{array}\right) \qquad H_2 = \left(\begin{array}{cc|cc} 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & -1 \end{array}\right).
$$

Let e_i, $i = 1, 2$ be linear forms on $\mathfrak{a}$ defined by $e_i(H_j) = \delta_{i,j}$, $i, j = 1, 2$. Then the set of restricted roots $\Psi(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{a}$ on $\mathfrak{g}$ is given by $\{\pm 2e_1,\ \pm 2e_2,\ \pm e_1 \pm e_2\}$. This determines the C_2 root system. Fix the positive roots $\Psi_+ = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}$. Then $\mathfrak{n} = \sum_{\alpha \in \Psi_+} \mathfrak{g}_\alpha$ determines the nilradical of a minimal parabolic subalgebra, and we have Iwasawa decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We fix the root vectors E_α of $\alpha \in \Psi_+$ as follows:

$$
E_{2e_1} = \left(\begin{array}{c|cc} \multirow{2}{*}{0} & 1 & 0 \\ & 0 & 0 \\ \hline & \multicolumn{2}{c}{0} \end{array}\right); \qquad E_{e_1+e_2} = \left(\begin{array}{c|cc} \multirow{2}{*}{0} & 0 & 1 \\ & 1 & 0 \\ \hline & \multicolumn{2}{c}{0} \end{array}\right);
$$

$$
E_{2e_2} = \left(\begin{array}{c|cc} \multirow{2}{*}{0} & 0 & 0 \\ & 0 & 1 \\ \hline & \multicolumn{2}{c}{0} \end{array}\right); \qquad E_{e_1-e_2} = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & -1 & 0 \end{array}\right).
$$

3. Representations of the Maximal Compact Subgroup

We recall some basic facts about the representations of the maximal compact subgroup K. Because $K_{\mathbb{C}}$ is isomorphic to $GL(2,\mathbb{C})$, the irreducible finite-dimensional $U(\mathfrak{k}_{\mathbb{C}})$-modules are parameterized by the set $\{\lambda = (\lambda_1,\lambda_2) \in \mathbb{Z}^{\oplus 2} \mid \lambda_1 \geq \lambda_2\}$, which corresponds to dominant highest weights. For each dominant weight λ, define an integer $d = \lambda_1 - \lambda_2$. Then the module τ_λ with the highest weight λ is of $d+1$ dimension. Now we fix a realization of V_λ with a basis $(v_j^\lambda \mid 0 \leq j \leq d)$, which is used throughout this paper.

3.1 LEMMA. *We have a basis* $\langle\, v_j^\lambda \mid 0 \leqq j \leqq d \,\rangle$ *of* V_λ *such that the* $U(\mathfrak{k}_{\mathbb{C}})$*-module is realized as*

$$\tau_\lambda(Z)v_j^\lambda = (\lambda_1+\lambda_2)v_j^\lambda, \quad \tau_\lambda(H')v_j^\lambda = (2j-d)v_j^\lambda,$$
$$\tau_\lambda(X)v_j^\lambda = (j+1)v_{j+1}^\lambda, \quad \tau_\lambda(\overline{X})v_j^\lambda = (d+1-j)v_{j-1}^\lambda.$$

For the elements $H_1' = (Z+H')/2$ and $H_2' = (Z-H')/2$, we have

$$\tau_\lambda(H_1')v_j^\lambda = (j+\lambda_2)v_j^\lambda, \quad \tau_\lambda(H_2')v_j^\lambda = (-j+\lambda_1)v_j^\lambda.$$

3.2. For the adjoint representations of $K_{\mathbb{C}}$ on $\mathfrak{p}_\pm$, we have the isomorphisms $\mathfrak{p}_+ \cong V_{(2,0)}$; $X_{(0,2)} \mapsto v_0^{(2,0)}$, $X_{(1,1)} \mapsto v_1^{(2,0)}$, $X_{(2,0)} \mapsto v_2^{(2,0)}$, and $\mathfrak{p}_- \simeq V_{(0,-2)}$; $X_{(-2,0)} \mapsto v_0^{(0,-2)}$, $X_{(-1,-1)} \mapsto -v_1^{(0,-2)}$, $X_{(0,-2)} \mapsto v_2^{(0,-2)}$.

The tensor representations $V_\lambda \otimes \mathfrak{p}_\pm$ has the irreducible decompositions:

$$V_\lambda \otimes \mathfrak{p}_+ \cong V_\lambda \otimes V_{(2,0)} = V_{(\lambda_1+2,\lambda_2)} \oplus V_{(\lambda_1+1,\lambda_2+1)} \oplus V_{(\lambda_1,\lambda_2+2)},$$
$$V_\lambda \otimes \mathfrak{p}_+ \cong V_\lambda \otimes V_{(0,-2)} = V_{(\lambda_1-2,\lambda_2)} \oplus V_{(\lambda_1-1,\lambda_2-1)} \oplus V_{(\lambda_1,\lambda_2-2)},$$

where some summands may vanish. Let P^{up}, P^{even}, and P^{down} be the projections from $V_\lambda \otimes \mathfrak{p}_+$ (resp. $V_\lambda \otimes \mathfrak{p}_-$) to the components $V_{(\lambda_1+2,\lambda_2)}$ (resp. $V_{(\lambda_1,\lambda_2-2)}$); $V_{(\lambda_1+1,\lambda_2+1)}$ (resp. $V_{(\lambda_1-1,\lambda_2-1)}$); and $V_{(\lambda_1,\lambda_2+2)}$ (resp. $V_{(\lambda_1-2,\lambda_2)}$).

We denote by w_j the basis $v_j^{(2,0)}$ or $v_j^{(0,-2)}$, $j=0,1,2$. Then the following lemmas give formulas of these projections.

3.3 Lemma. *Set* $\mu = (\lambda_1+2, \lambda_2)$ (or (λ_1, λ_2-2)). *Then the projector* P^{up} *is;*

(i) $$P^{up}(v_j^\lambda \otimes w_2) = \frac{(j+1)(j+2)}{2} v_{j+2}^\mu;$$

(ii) $$P^{up}(v_j^\lambda \otimes w_1) = (j+1)(d+1-j) v_{j+1}^\mu;$$

(iii) $$P^{up}(v_j^\lambda \otimes w_0) = \frac{(d+1-j)(d+2-j)}{2} v_j^\mu.$$

3.4 Lemma. *Set* $\nu = (\lambda_1+1, \lambda_2+1)$ (or $(\lambda_1-1, \lambda_2-1)$). *Then the projector* P^{even} *is;*

(0) $$P^{even}(v_d^\lambda \otimes w_2) = 0$$

(i) $$P^{even}(v_j^\lambda \otimes w_2) = (j+1) v_{j+1}^\nu \qquad 0 \le j \le d-1;$$

(ii) $$P^{even}(v_j^\lambda \otimes w_1) = (d-2j) v_j^\nu \qquad 0 \le j \le d;$$

(iii) $$P^{even}(v_j^\lambda \otimes w_0) = -(d+1-j) v_{j-1}^\nu \qquad 1 \le j \le d.$$

3.5 Lemma. *Set* $\pi = (\lambda_1, \lambda_2+2)$ (or (λ_1-2, λ_2)). *Then the projector* P^{down} *is;*

(i) $$P^{down}(v_j^\lambda \otimes w_2) = v_j^\pi \qquad 0 \le j \le d-2;$$

(ii) $$P^{down}(v_j^\lambda \otimes w_1) = -2 v_{j-1}^\pi \qquad 1 \le j \le d-1;$$

(iii) $$P^{down}(v_j^\lambda \otimes w_0) = v_{j-2}^\pi \qquad 2 \le j \le d;$$

(iv) $$P^{down}(v_d^\lambda \otimes w_2) = P^{down}(v_d^\lambda \otimes w_1) = P^{down}(v_{d-1}^\lambda \otimes w_2) = 0.$$

Proof. To show these lemmas it is enough to find the highest weight vectors in $V_\lambda \otimes \mathfrak{p}_+$ corresponding to the factors V_μ, V_ν, and V_π, respectively. The other steps of the proof are completed by induction. □

4. The Schmid Operator

We introduce the Schmid operator and the shift operators, which we use to obtain the generalized Whittaker functions for an admissible Hilbert space representation.

4.1. *Definition of the Schmid operator and the shift operators.* Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Then the maximal compact subgroup K

acts on $\mathfrak{p}_{\mathbb{C}}$ by the adjoint action. We denote this K-module by $(\mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}, \mathfrak{p}_{\mathbb{C}})$. Let $C^{\infty}_{\tau}(G/K)$ be that space of V_{τ}-valued C^{∞}-functions ϕ on G satisfying $\phi(gk) = \tau(k)^{-1}\phi(g)$ for all $g \in G$ and $k \in K$. Then we define the left G-equivariant differential operator $\nabla : C^{\infty}_{\tau}(G/K) \to C^{\infty}_{\tau\otimes Ad_{\mathfrak{p}_{\mathbb{C}}}}(G/K)$ by

$$\nabla\phi = \sum_{i\in I} R_{X_i}\phi(\cdot) \otimes X_i,$$

for an orthonormal basis $(X_i)_{i\in I}$ of $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}}$ with respect to the Killing form on $\mathfrak{g}$. Here $R_X\phi$ means the right differentiation of the function ϕ by $X \in \mathfrak{g}$: $R_X\phi(g) = \frac{d}{dt}\phi(g \exp tX)|_{t=0}$. The definition does not depend on the choice of an orthonormal basis. We call this operator the Schmid operator. We composite it with the projections P^{up}, P^{even}, and P^{down} onto the irreducible components of $\tau_{\lambda} \otimes Ad_{\mathfrak{p}_{\mathbb{C}}}$ as K-modules. We call these composite operators $P^{\bullet} \circ \nabla$ *the shift operators.*

5. Radial Part of the Schmid Operator and the Casimir Operator

Define a subgroup $A = \{a = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1, a_2 > 0\}$ of G and denote its Lie algebra by $\mathfrak{a}$. There is a decomposition $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a^{-1})(\mathfrak{so}(\eta)_{\mathbb{C}} + \mathfrak{n}_{s\mathbb{C}}) + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ with $a \in A$, then $G = RAK$. Now we study the restriction $\phi|_A$ to A of a function $\phi \in C^{\infty}_{\chi\cdot\eta,\tau_{\lambda}}(R\backslash G/K)$ through an inclusion of $C^{\infty}_{\chi\cdot\eta,\tau_{\lambda}}(R\backslash G/K)$ into $C^{\infty}(A; V_{\tau_{\lambda}})$, where $C^{\infty}(A; V_{\tau_{\lambda}})$ is the space of $V_{\tau_{\lambda}}$-valued C^{∞}-functions on A. It is given by the left R- and right K-equivariance of the function ϕ. Then we need to describe how the each differential operator acts on the restriction when we fix a character $\chi \cdot \eta$ of R and a K-module τ_{λ}. We call the action the A-radial part of an operator, and denote by $R(\nabla^{\pm}_{\chi\cdot\eta,\tau_{\lambda}})$ the A-radial part of $\nabla^{\pm}$, for example. In this section we give formulas of the A-radial part of the Schmid operator and the Casimir operator acting on the generalized Whittaker functions with a fixed K-type. We use the notation given in the previous sections.

5.1. *Radial part of the Schmid operator.* Let us take an orthogonal basis of $\mathfrak{p}$ by $\left(C\|\beta\|(X_{\beta} + X_{-\beta}),\ \frac{C\|\beta\|}{\sqrt{-1}}(X_{\beta} - X_{-\beta}) \mid \beta \in \Sigma_n^+\right)$ where $X_{\pm\beta}$, $\beta \in \Sigma_n^+$ are the non-compact root vectors $\{X_{\pm\beta}; \beta \in \Sigma_n^+\}$ of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$, and C is a normalizing constant depending on the Killing form. Then the Schmid operator ∇ is given by

$$\nabla F = 2C^2 \sum_{\beta\in\sum_n^+} \|\beta\|^2 R_{X_{-\beta}} F \otimes X_\beta + 2C^2 \sum_{\beta\in\sum_n^+} \|\beta\|^2 R_{X_\beta} F \otimes X_{-\beta}.$$

We can write it as a sum of two operators corresponding to the irreducible decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as K-modules. These operators $\nabla^\pm : C^\infty_\tau(G/K) \to C^\infty_{\tau\otimes Ad_{\mathfrak{p}_\pm}}(G/K)$ are given by

$$\begin{aligned}(5.1)\qquad \nabla^+ F &= \frac{1}{4}\Sigma\|\beta\|^2 \cdot R_{X_{-\beta}} F \otimes X_\beta\\ &= R_{X_{(-2,0)}} F \otimes X_{(2,0)} + \frac{1}{2} R_{X_{(-1,-1)}} F \otimes X_{(1,1)}\\ &\quad + R_{X_{(0,-2)}} F \otimes X_{(0,2)},\end{aligned}$$

$$\begin{aligned}(5.2)\qquad \nabla^- F &= \frac{1}{4}\Sigma\|\beta\|^2 \cdot R_{X_\beta} F \otimes X_{-\beta}\\ &= R_{X_{(2,0)}} F \otimes X_{(-2,0)} + \frac{1}{2} R_{X_{(1,1)}} F \otimes X_{(-1,-1)}\\ &\quad + R_{X_{(0,2)}} F \otimes X_{(0,-2)}.\end{aligned}$$

We prepare the following lemma to describe the A-radial parts of the actions of $\nabla_\pm$ on the generalized Whittaker functions.

5.2 Lemma. *For the character η of N_s we suppose that H_η is invertible and both of h_1 and h_2 are not equal to zero. Then*

(i) $X_{(\pm2,0)} = H_1 \pm H'_1 \pm 2\sqrt{-1}a_1^2 \mathrm{Ad}(a^{-1}) E_{2e_1}$;

(ii) $X_{(0,\pm2)} = H_2 \pm H'_2 \pm 2\sqrt{-1}a_2^2 \mathrm{Ad}(a^{-1}) E_{2e_2}$;

(iii)
$$\begin{aligned}X_{(1,1)} &= 2E_{e_1-e_2} + 2\,\overline{X} + 2\sqrt{-1}a_1a_2 \mathrm{Ad}(a^{-1}) E_{e_1+e_2}\\ &= -2\big(a_1a_2/D\big)\big\{\det H_\eta \cdot \mathrm{Ad}(a^{-1})Y_\eta - \big(h_3/2\big)(H_1 - H_2)\\ &\quad - h_1\big(a_1/a_2\big)(X - \overline{X})\big\}\\ &\quad + 2\,\overline{X} + 2\sqrt{-1}a_1a_2 \mathrm{Ad}(a^{-1}) E_{e_1+e_2};\end{aligned}$$

(iv)
$$\begin{aligned}X_{(-1,-1)} &= 2E_{e_1-e_2} - 2\,X - 2\sqrt{-1}a_1a_2 \mathrm{Ad}(a^{-1}) E_{e_1+e_2}\\ &= -2\big(a_1a_2/D\big)\big\{\det H_\eta \cdot \mathrm{Ad}(a^{-1})Y_\eta - \big(h_3/2\big)(H_1 - H_2)\\ &\quad - h_1\big(a_1/a_2\big)(X - \overline{X})\big\}\\ &\quad - 2\,X - 2\sqrt{-1}a_1a_2 \mathrm{Ad}(a^{-1}) E_{e_1+e_2}.\end{aligned}$$

Here we use the symbols: $D = h_1 a_1^2 - h_2 a_2^2$ *and* Y_η *is a generator of* $\mathfrak{so}(\eta)$*. The second equalities in* (iii) *and* (iv) *are valid for generic elements* $a \in A$*, satisfying* $a_1 a_2 (h_1 a_1^2 - h_2 a_2^2) \neq 0$*.*

Proof. These formulas can be given along with the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(a^{-1})(\mathfrak{so}(\eta)_{\mathbb{C}} + \mathfrak{n}_{s\mathbb{C}}) + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}, \quad a \in A.$$

The equalities (i), (ii) and the first ones of (iii), (iv) can be checked by definition.

To get the second equality in (iii), or (iv), we note that a generator Y_η of $\mathfrak{so}(\eta)$ is given by

$$H_\eta^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto Y_\eta = (\det H_\eta)^{-1} \Big\{ \frac{h_3}{2}(H_1 - H_2) + h_2 E_{e_1 - e_2} - h_1 E_{-e_1 + e_2} \Big\}.$$

For $a \in A$ satisfying the generic condition, computing $\det H_\eta \ \mathrm{Ad}(a^{-1}) Y_\eta$, and solving it for $E_{e_1 - e_2}$, then we have

$$E_{e_1 - e_2} = -\frac{a_1 a_2}{D} \Big\{ \det H_\eta \cdot \mathrm{Ad}(a^{-1}) Y_\eta - \frac{h_3}{2}(H_1 - H_2) - h_1 \frac{a_1}{a_2}(X - \overline{X}) \Big\}, \tag{5.3}$$

in the decomposition of $\mathfrak{g}_{\mathbb{C}}$. This completes the proof of lemma. □

We give formulas of the A-radial parts of Schmid operators.

5.3 Proposition. *We assume the same condition on the character* η *of* N_s *as in Lemma 5.2. Then the* A*-radial parts of the Schmid operators:*

$R(\nabla^{\pm}_{\chi\cdot\eta,\tau_\lambda}) : C^\infty(A; V_{\tau_\lambda}) \to C^\infty(A; V_{\tau_\lambda}\otimes\mathfrak{p}_\pm)$, *are given by*

$$(5.4)\quad R(\nabla^{+}_{\chi\cdot\eta,\lambda})f(a)$$
$$= \Big(\partial_1 + 4\pi h_1 a_1^2 + (\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+})(H_1') + 2\frac{h_2a_2^2}{D} - 2\Big)(f(a)\otimes X_{(2,0)})$$
$$+ \Big(\mathcal{I}^- - \frac{h_2a_2^2}{D}(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+})(X) + \frac{h_1a_1^2}{D}(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+})(\overline{X})\Big)(f(a)\otimes X_{(1,1)})$$
$$+ \Big(\partial_2 + 4\pi h_2 a_2^2 + (\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+})(H_2') - 2\frac{h_1a_1^2}{D} - 2\Big)(f(a)\otimes X_{(0,2)});$$

$$(5.5)\quad R(\nabla^{-}_{\chi\cdot\eta,\lambda})f(a)$$
$$= \Big(\partial_1 - 4\pi h_1 a_1^2 - (\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_-})(H_1') + 2\frac{h_2a_2^2}{D} - 2\Big)(f(a)\otimes X_{(-2,0)})$$
$$+ \Big(\mathcal{I}^+ - \frac{h_1a_1^2}{D}(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_-})(X) + \frac{h_2a_2^2}{D}(\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_-})(\overline{X})\Big)(f(a)\otimes X_{(-1,-1)})$$
$$+ \Big(\partial_2 - 4\pi h_2 a_2^2 - (\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_-})(H_2') - 2\frac{h_1a_1^2}{D} - 2\Big)(f(a)\otimes X_{(0,-2)}).$$

Here we use the symbols

$$\partial_i = a_i\frac{\partial}{\partial a_i},\ i=1,2;\quad D = h_1a_1^2 - h_2a_2^2,\ \textit{and}$$

$$\mathcal{I}^\pm = \frac{h_3}{2}\frac{a_1a_2}{D}(\partial_1-\partial_2) \mp 2\pi h_3 a_1 a_2 - \frac{\chi(Y_\eta)\det H_\eta a_1a_2}{D}.$$

Proof. These are obtained by applying the equalities in Lemma 5.2 into the expressions (5.1) and (5.2) of $\nabla^\pm$. Here we show a computation of $R(\nabla^{+}_{\chi\cdot\eta,\tau_\lambda})$ from (5.1). Because $R_{H_1}F_{|A}(a) = \partial_1 F_{|A}(a)$, $R_{H_1'}F_{|A}(a) = -\tau_\lambda(H_1')F_{|A}(a)$, and $R_{Ad(a^{-1})E_{2e_1}}F_{|A}(a) = 2\pi\sqrt{-1}h_1F_{|A}(a)$, we have

$$R_{X_{(-2,0)}}F_{|A}(a)\otimes X_{(2,0)}$$
$$= \{(H_1 - H_1' - 2\sqrt{-1}a_1^2\mathrm{Ad}(a^{-1})E_{2e_1})F_{|A}(a)\}\otimes X_{(2,0)}$$
$$= \big\{\big(\partial_1 + \tau_\lambda(H_1') + 4\pi h_1 a_1^2\big)F_{|A}(a)\big\}\otimes X_{(2,0)}.$$

Note $\big(\tau_\lambda(H_1')F_{|A}\big)\otimes X_{(2,0)} = \tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+}(H_1')\big(F_{|A}(a)\otimes X_{(2,0)}\big) - F_{|A}(a)\otimes [X_{(2,0)}, -H_1'] = \tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_+}(H_1')\big(F_{|A}(a)\otimes X_{(2,0)}\big) - 2\big(F_{|A}(a)\otimes X_{(2,0)}\big)$. Computations of the other terms are done similarly, where we only remark that $\mathrm{Ad}(a^{-1})Y_\eta F_{|A}(a) = (d/dt)F(\exp(tY_\eta)a)|_{t=0} = \chi(Y_\eta)F_{|A}(a)$ in the term

$R_{X_{(-1,-1)}} F_{|A}(a) \otimes X_{(1,1)}$. The formula (5.5) for ∇^- is obtained in a parallel way. □

5.4. *The Radial part of the Casimir operator.* The Casimir element L in the center $Z(\mathfrak{g}_{\mathbb{C}})$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ is given by

$$\begin{aligned} L &= H_1^2 + H_2^2 - 4H_1 - 2H_2 \\ &\quad + 2E_{e_1-e_2} \cdot E_{-e_1+e_2} + 4E_{2e_1} \cdot E_{-2e_1} + 2E_{e_1+e_2} \cdot E_{-e_1-e_2} \\ &\quad + 4E_{2e_2} \cdot E_{-2e_2} \\ &= H_1^2 + H_2^2 - 4H_1 - 2H_2 + 2E_{e_1-e_2} \cdot E_{-e_1+e_2} + 4E_{2e_1}^2 + 2E_{e_1+e_2}^2 + 4E_{2e_2}^2 \\ &\quad - 4E_{2e_1}(E_{2e_1} - E_{-2e_1}) - 2E_{e_1+e_2}(E_{e_1+e_2} - E_{-e_1-e_2}) \\ &\quad - 4E_{2e_2}(E_{2e_2} - E_{-2e_2}) \end{aligned}$$

up to a scalar multiple, see [M-O1], §7. In this expression we note that the elements $E_{2e_1} - E_{-2e_1} = \sqrt{-1}H_1'$, $E_{2e_2} - E_{-2e_2} = \sqrt{-1}H_2'$, and $E_{e_1+e_2} - E_{-e_1-e_2} = \sqrt{-1}(X + \overline{X})$, are all contained in $\mathfrak{k}_{\mathbb{C}}$. Therefore, we can describe the A-radial part of L by fixing data $\chi \cdot \eta$ and τ_λ, for the terms other than $2E_{e_1-e_2} \cdot E_{-e_1+e_2}$. We have to calculate also the action of this term $2E_{e_1-e_2} \cdot E_{-e_1+e_2}$. To make our expression simple, we shall only treat the case $h_3 = 0$ for the character η of N_s. This assumption is not essentially restrictive for our purpose.

5.5 LEMMA. *Assume that the matrix H_η is non-degenerate and $h_3 = 0$ for the character η of N_s. Put $W = X - \overline{X}$ in $\mathfrak{k}_{\mathbb{C}}$. Then we get*

$$\begin{aligned} 2E_{e_1-e_2} \cdot E_{-e_1+e_2} &= 2\frac{h_1 a_1^2}{D}(H_1 - H_2) + 2\left(\frac{h_2 a_2^2}{D}\right)\left(\frac{h_1 a_1^2}{D}\right) W^2 \\ &\quad + 2\left(\frac{h_1 a_1 h_2 a_2}{D}\right)^2 \left(\mathrm{Ad}(a^{-1})(Y_\eta)\right)^2 \\ &\quad - 2\frac{h_1 a_1 h_2 a_2 \left(h_1 a_1^2 + h_2 a_2^2\right)}{D^2} \mathrm{Ad}(a^{-1})(Y_\eta) W \end{aligned}$$

for generic elements $a \in A$.

PROOF. The formula follows from the expression (5.3) of $E_{e_1-e_2}$ in the proof of Lemma 5.2 by a similar computation given in [Kn], Chap. 8, Proposition 8.16. □

Now we give a formula of the A-radial part of the Casimir operator.

5.6 Proposition. *Assume the same condition on* H_η *as in Lemma 5.5. Then the* A*-radial part* $R(L) = R(L_{\chi\cdot\eta,\tau_\lambda})$ *of the Casimir operator* L *acting on the space* $C^\infty(A; V_{\tau_\lambda})$ *is given by*

$$\begin{aligned} R(L) = {} & \partial_1^2 + \partial_2^2 - 2(\partial_1 + \partial_2) + 2\frac{h_2 a_2^2}{D}\partial_1 - 2\frac{h_1 a_1^2}{D}\partial_2 \\ & - 16\pi^2 h_1^2 a_1^4 - 16\pi^2 h_2^2 a_2^4 \\ & - 8\pi h_1 a_1^2\ \tau_\lambda(H_1') - 8\pi h_2 a_2^2\ \tau_\lambda(H_2') \\ & + 2\chi(Y_\eta)\left(\frac{h_1 a_1^2 + h_2 a_2^2}{D}\right)\left(\frac{h_1 a_1 h_2 a_2}{D}\right)\tau_\lambda(W) \\ & + 2\left(\frac{h_1 a_1^2}{D}\right)\left(\frac{h_2 a_2^2}{D}\right)\{\tau_\lambda(W)\}^2 + 2\mathcal{S}^2. \end{aligned}$$

Here we use the symbols: $\mathcal{S} := \frac{\chi(Y_\eta)h_1a_1h_2a_2}{D}$ *and* $W := X - \overline{X} \in \mathfrak{k}_{\mathbb{C}}$.

6. Generalized Principal Series Representations Induced from a Maximal Parabolic Subgroup and Their K-types

For a maximal parabolic subgroup P_1 of G we consider a generalized principal series representation $\pi = \mathrm{Ind}_{P_1}^{G}(\sigma \otimes (\nu_1 + \rho_1))$ of G.

6.1. Let P_1 be the maximal parabolic subgroup of G with Langlands decomposition $P_1 = M_1 A_1 N_1$, where the unipotent radical N_1 is the two step 3 dimensional nilpotent group, $A_1 = \{\mathrm{diag}(a, 1, a^{-1}, 1) \mid \ a > 0\}$, and $M_1 \simeq \{\pm 1\} \times \mathrm{SL}(2, \mathbb{R})$. We call P_1 the *Jacobi maximal parabolic subgroup of* G.

Fix a representation of M_1 by a pair $\sigma = (\varepsilon, D)$, where $\varepsilon : \{\pm 1\} \to \mathbb{C}^*$ is a character and D is a discrete series representation of $\mathrm{SL}(2, \mathbb{R})$. Take also $\nu_1 \in \mathfrak{a}_{1\mathbb{C}}^*$ and define the character $\exp(\nu_1) : A_1 \to \mathbb{C}^\times$. Then we define the generalized principal series representation $\pi = I(P_1; \sigma, \nu_1)$ of G by the smooth induced representation $\mathrm{Ind}_{P_1}^{G}(\sigma \otimes (\nu_1 + \rho_1))$ from P_1 to G ([Kn], Chap.7, §1). Here $2\rho_1 = (e_1 - e_2) + 2e_1 + (e_1 + e_2) = 4e_1$.

6.2 *The K-types of the generalized principal series representations.* We recall the K-type decomposition of a generalized principal series representation $I(P_1; \sigma, \nu_1)$. Each discrete series representation of $\mathrm{SL}(2, \mathbb{R})$ has the Harish-Chandra parameter parameterized by $\mathbb{Z} \setminus \{0\}$. For the Harish-Chandra parameter $\ell \in \mathbb{Z} \setminus \{0\}$ we know that the Blattner parameter k of the discrete series is written as $k = \ell + \mathrm{sgn}(\ell) \cdot 1$. We denote by D_k^+ the discrete series representation of $\mathrm{SL}(2, \mathbb{R})$ with Blattner parameter $k > 0$ (in fact $k \geq 2$) and also by D_k^- the discrete series representation with Blattner parameter $k < 0$ ($k \leq -2$). Then the K-types of $D_k^+|_{\mathrm{SO}(2)}$ are parameterized by the highest weights $k+2j$ where j runs through the all non-negative integers.

Irreducible finite dimensional representations of K are parameterized by the associated dominant highest weights; we write by $\tau_{\lambda_1,\lambda_2}$ the module with the highest weight $(\lambda_1, \lambda_2) \in \mathbb{Z} \oplus \mathbb{Z}$, $\lambda_1 \geq \lambda_2$. Define $\gamma_{2e_1} = \mathrm{diag}(-1, 1, -1, 1) \in M_1$.

6.3 PROPOSITION. *Let* $\pi = I(P_1; \sigma, \nu_1)$, $\sigma = (\varepsilon, D_k^{\pm})$, $\nu_1 \in \mathfrak{a}_{1\mathbb{C}}^*$, *be a generalized principal series representation of* G. *Then the multiplicity of* $\tau_{\lambda_1,\lambda_2}$ *in the restriction of* π *to* K *is given by*

$$[\pi : \tau_{\lambda_1,\lambda_2}] = \# \left\{ m \in \mathbb{Z} \mid \begin{array}{l} m \equiv k (\mathrm{mod}\ 2), \quad \mathrm{sgn}(D_k^{\pm}) \cdot (m-k) \geq 0, \\ (-1)^{\lambda_1+\lambda_2-m} = \varepsilon(\gamma_{2e_1}), \quad \lambda_2 \leq m \leq \lambda_1 \end{array} \right\},$$

which may be zero. Here we set that $\mathrm{sgn}(D_k^+) = +1$, $\mathrm{sgn}(D_k^-) = -1$.

PROOF. The multiplicity formula is derived from the Frobenius reciprocity for induced representations. It says that the multiplicity of $\tau_{\lambda_1,\lambda_2}$ in $I(P_1; \sigma, \nu_1)|_K$ is given by

$$[I(P_1; \sigma, \nu_1)|_K : \tau_{\lambda_1,\lambda_2}] = \sum_{\omega \in (K \cap M_1)^{\widehat{}}} [\sigma|_{K\cap M_1} : \omega] \cdot [\tau_{\lambda_1,\lambda_2}|_{K\cap M_1} : \omega],$$

([Kn], Chap.1, Theorem 1.14). Here we consider the restriction of the representation σ to $K \cap M_1$, $\sigma|_{K\cap M_1} = \sum_{\omega\in(K\cap M_1)^{\widehat{}}} [\sigma|_{K\cap M_1} : \omega]$. Since $K \cap M_1 \simeq \{\pm 1\} \times \mathrm{SO}(2)$, any character $\omega \in (K \cap M_1)^{\widehat{}}$ is specified by the value $\omega(\gamma_{2e_1})$ and the restriction $\omega|_{\mathrm{SO}(2)}$. The characters χ_m of $\mathrm{SO}(2)$ are parameterized by $m \in \mathbb{Z}$ as $\chi_m(r_\theta) = \exp(\sqrt{-1} m\theta)$, where $r_\theta \in \mathrm{SO}(2)$ is the rotation with angle θ. Then this fact for the $\mathrm{SO}(2)$-types for $D_k^{\pm}$ implies that the multiplicity $[\sigma|_{K\cap M_1} : \omega]$ is given in terms of the pair

$\{\omega(\gamma_{2e_1}),\ \chi_m := \omega|_{\mathrm{SO}(2)}\}$ as: 1, if $m \equiv k \pmod 2$, $\mathrm{sgn}(D_k^{\pm}) \cdot (m-k) \geq 0$, $\omega(\gamma_{2e_1}) = \varepsilon(\gamma_{2e_1})$, or 0, otherwise.

On the other hand, we have that $\tau_{\lambda_1,\lambda_2}|_{K\cap M_1} = \sum_{\lambda_2 \leq m \leq \lambda_1} \{(-1)^{\lambda_1+\lambda_2-m},\ \chi_m\}$, where $\{(-1)^{\lambda_1+\lambda_2-m},\ \chi_m\}$ denotes the value assigned to this pair in the previous paragraph, holds. Applying these equalities into the multiplicity formula, we obtain our assertion. □

6.4 Corollary. *(0)* $\tau_{\lambda_1,\lambda_2}$ *with* $\lambda_1 < k$ *(resp.* $\lambda_2 > k$*) does not occur in the* K*-type of* $I(P_1;\sigma,\nu_1)$*, if* $k > 0$*,* $\sigma = (\varepsilon, D_k^+)$ *(resp.* $k < 0$*,* $\sigma = (\varepsilon, D_k^-)$*).*
(i) *When* $\sigma = (\varepsilon, D_k^+)$*,* $\varepsilon(\gamma_{2e_1}) = (-1)^k$*,* $(k > 0)$ *then each of* $\tau_{\lambda,\lambda}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \geq k)$*, or* $\tau_{k,\lambda}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \leq k)$*, occurs in* $I(P_1;\sigma,\nu_1)$ *with multiplicity one.*
(ii) *When* $\sigma = (\varepsilon, D_k^+)$*,* $\varepsilon(\gamma_{2e_1}) = -(-1)^k$*,* $(k > 0)$ *then each of* $\tau_{\lambda,\lambda-1}$*,* $(\lambda \in \mathbb{Z},\ \lambda \geq k)$*, or* $\tau_{k,\lambda-1}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \leq k)$ *occurs in* $I(P_1;\sigma,\nu_1)$ *with multiplicity one.*
(iii) *When* $\sigma = (\varepsilon, D_k^-)$*,* $\varepsilon(\gamma_{2e_1}) = (-1)^k$*,* $(k < 0)$ *then each of* $\tau_{\lambda,\lambda}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \leq k)$*, or* $\tau_{\lambda,k}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \geq k)$ *occurs in* $I(P_1;\sigma,\nu_1)$ *with multiplicity one.*
(iv) *When* $\sigma = (\varepsilon, D_k^-)$*,* $\varepsilon(\gamma_{2e_1}) = -(-1)^k$*,* $(k < 0)$ *then each of* $\tau_{\lambda+1,\lambda}$*,* $(\lambda \in \mathbb{Z},\ \lambda \leq k)$*, or* $\tau_{\lambda+1,k}$*,* $(\lambda \in \mathbb{Z},\ \lambda \equiv k \pmod 2,\ \lambda \geq k)$ *occurs in* $I(P_1;\sigma,\nu_1)$ *with multiplicity one.*

We call the K-type $\tau_{k,k}$ in the case (i), $\tau_{k,k-1}$ in (ii), $\tau_{k,k}$ in (iii), or $\tau_{k+1,k}$ in (iv), *the corner* K*-type* of the principal series representation, respectively.

7. The Generalized Whittaker Functions for the Generalized Principal Series Representations from a Maximal Parabolic Subgroup P_1

We study the generalized Whittaker functions with the corner K-type for the generalized principal series representation $I(P_1;\sigma,\nu_1)$ of G.

Definition. We say that a generalized principal series $I(P_1;(\varepsilon, D_k^{\pm}), \nu_1)$ is of

(i) *even type*, if $\varepsilon(\gamma_{2e_1}) = (-1)^k$, or (ii) *odd type*, if $\varepsilon(\gamma_{2e_1}) = -(-1)^k$.

By Corollary 6.4, we have that the corner K-type of $I(P_1;\sigma,\nu_1)$ is one dimensional, if the principal series is of even type, or two dimensional, if it is of odd type.

To begin with, we have a remark. Set $M = \{\mathrm{diag}(\varepsilon_1,\varepsilon_2,\varepsilon_1,\varepsilon_2) \mid \varepsilon_i = \pm 1\}$ the centralizer of A in K. Then a generalized Whittaker function $\phi(a)$, $a \in A$, must satisfy $\phi(a) = \phi(\gamma a \gamma^{-1}) = \chi(\gamma)\tau(\gamma)\phi(a)$ for $\gamma \in M \cap SO(\eta)$. Assume that $H_\eta = 1_2$, hence $SO(\eta) = SO(2)$. Then $M \cap SO(\eta) = \{\mathrm{diag}(\varepsilon,\varepsilon,\varepsilon,\varepsilon) \mid \varepsilon = \pm 1\}$. Denote by γ_0 the nontrivial element in this group. In the even (resp. the odd) case with the corner K-type $\tau = \tau_{k,k}$ (resp. $\tau = \tau_{k+1,k}$), we have $\tau_{k,k}(\gamma_0) = \det(-1_2)^k = 1$ (resp. $\tau_{k+1,k}(\gamma_0) = \det(-1_2)^k \otimes \mathrm{Sym}^1(-1_2) = (-1)\cdot id$). Hence it should be satisfied that $\chi(\gamma_0) = 1$ (resp. $\chi(\gamma_0) = -1$) in the even case (resp. the odd case) for $\phi(a) \neq 0$. This parity condition appears again as a result of direct calculation, Lemmas 7.4 and 7.9.

7.1. *The generalized principal series of even type: a system of differential equations.* Since our treatment proceeds in parallel, we study only the case $k > 0$, $\sigma = (\varepsilon, D_k^+)$, for the generalized principal series $I(P_1;\sigma,\nu_1)$ of even type; $\varepsilon(\gamma_{2e_1}) = (-1)^k$. This module has the corner K-type $\tau_{k,k}$ with multiplicity one by Corollary 6.4. The A-radial part of a generalized Whittaker function with K-type $\tau_{-k,-k} = \tau_{k,k}^*$ for the principal series satisfies a set of differential equations, which we give by using the shift operators and the Casimir operators. We fix a base $\{v_0^{-k,-k}\}$ of $\tau_{-k,-k}$ given in Lemma 3.1.

7.2 PROPOSITION. *Suppose that $h_3 = 0$ and $H_\eta = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}$ is positive definite for the character η of N_s. Let $\phi(a_1,a_2) = b(a_1,a_2)\cdot v_0^{-k,-k} \in C^\infty(A;V_{\tau_{-k,-k}})$ be the restriction to A of a generalized Whittaker function with the K-type $\tau_{-k,-k}$ for $I(P_1;\sigma,\nu_1)$ of even type. Set $b(a_1,a_2) = (\sqrt{h_1}a_1)^{k+1}(\sqrt{h_2}a_2)^{k+1}e^{-2\pi(h_1a_1^2+h_2a_2^2)}c(a_1,a_2)$. Then the function $c(a_1,a_2)$ has to satisfy the following set of differential equations:*

$$\left(\partial_1\partial_2 - \frac{h_2a_2^2}{D}\partial_1 + \frac{h_1a_1^2}{D}\partial_2 - \mathcal{S}^2\right)c(a_1,a_2) = 0; \tag{7.1}$$

$$\begin{aligned}\{(\partial_1+\partial_2)^2 + 2k(\partial_1+\partial_2) - 8\pi h_1a_1^2\partial_1 - 8\pi h_2a_2^2\partial_2 \\ - 8\pi(h_1a_1^2+h_2a_2^2) + k^2 - \nu_1^2\}c(a_1,a_2) = 0,\end{aligned} \tag{7.2}$$

where we use the symbols: $D = h_1 a_1^2 - h_2 a_2^2$ *and* $\mathcal{S} = \frac{\chi(Y_\eta) h_1 a_1 h_2 a_2}{D}$.

Proof. The differential equation (7.1) is derived from the action of the shift operator on $\phi(a_1, a_2)$. (7.2) is essentially from the action of the Casimir operator. We explain first (7.1). We know, by Corollary 6.4, the K-module $\tau_{k-2,k-2}$ does not occur in the principal series. Take two shift operators: $P^{up} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k,-k}})$ and $P^{down} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k+2,-k}})$ which moves the K-types from $\tau_{-k,-k}$ to $\tau_{-k+2,-k}$, and from $\tau_{-k+2,-k}$ to $\tau_{-k+2,-k+2}$. Then the composition of these operators must annihilate a generalized Whittaker function ϕ with the K-type $\tau_{-k,-k}$ for $I(P_1; \sigma, \nu_1)$.

By Proposition 5.3, Lemma 3.1, and the projection formula in Lemma 3.3, we can write

$$\begin{aligned} P^{up} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k,-k}})\phi(a) = {} & \big(\partial_1 + 4\pi h_1 a_1^2 - k\big) b(a) v_2^{-k+2,-k} \\ & - \mathcal{S} b(a) v_1^{-k+2,-k} \\ & + \big(\partial_2 + 4\pi h_2 a_2^2 - k\big) b(a) v_0^{-k+2,-k} \end{aligned}$$

with the basis $(v_j^{-k+2,-k} \mid j = 0, 1, 2)$ of $\tau_{-k+2,-k}$. Also for a function $\widetilde{\phi}(a) = \sum_{j=0}^{2} \widetilde{b}_j(a) v_j^{-k+2,-k}$ in $C^\infty(A; V_{\tau_{-k+2,-k}})$, we have

$$\begin{aligned} & P^{down} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k+2,-k}})\widetilde{\phi}(a) \\ & \quad = \big\{ \big(\partial_1 + 4\pi h_1 a_1^2 + 2\tfrac{h_1 a_1^2}{D} - (k+2)\big)\widetilde{b}_0(a) + 2\mathcal{S}\widetilde{b}_1(a) \\ & \qquad + \big(\partial_2 + 4\pi h_2 a_2^2 - 2\tfrac{h_2 a_2^2}{D} - (k+2)\big)\widetilde{b}_2(a) \big\} v_0^{-k+2,-k+2}. \end{aligned}$$

Then the composition of these operators, which annihilates ϕ, reads

$$\begin{aligned} & \bigg\{ \Big(\partial_1 + 4\pi h_1 a_1^2 + 2\frac{h_1 a_1^2}{D} - (k+2)\Big)\Big(\partial_2 + 4\pi h_2 a_2^2 - k\Big) - 2\mathcal{S}^2 \\ & \quad + \Big(\partial_2 + 4\pi h_2 a_2^2 - 2\frac{h_2 a_2^2}{D} - (k+2)\Big)\Big(\partial_1 + 4\pi h_1 a_1^2 - k\Big) \bigg\} b(a_1, a_2) = 0. \end{aligned}$$

Rewrite this equation for $c(a_1, a_2)$, then it becomes the equation (7.1).

We produce the equation (7.2). We remark that the Casimir operator L acts by a scalar multiplication on $I(P_1; \sigma, \nu_1)$; it is written by $\nu_1^2 + (k-1)^2 - 5$

with the parameter of the module, [M-O1], §7. Therefore we have the following equation:

$$\Big(\partial_1^2+\partial_2^2-2(\partial_1+\partial_2)+2\frac{h_2a_2^2}{D}\partial_1-2\frac{h_1a_1^2}{D}\partial_2-16\pi^2h_1^2a_1^4-16\pi^2h_2^2a_2^4 \\ +8k\pi h_1a_1^2+8k\pi h_2a_2^2+2\mathcal{S}^2\Big)b(a)=\{\nu_1^2+(k-1)^2-5\}b(a)$$

by Proposition 5.6. Substituting (7.1) for the part $2\frac{h_2a_2^2}{D}\partial_1-2\frac{h_1a_1^2}{D}\partial_2$ in the above, and rewriting it for $c(a_1,a_2)$, we obtain the equation (7.2). □

7.3. *A solution with an integral expression in the even case.* We search a formal power series solution of (7.1) and (7.2) satisfying certain asymptotic behavior.

We introduce new variables x and y by

$$x=2\pi(h_1a_1^2-h_2a_2^2)\ \text{ and }\ y=2\pi(h_1a_1^2+h_2a_2^2),$$

[Ni]. Then the equations (7.1), (7.2) in Proposition 7.2 are rewritten into

$$\frac{x^2-y^2}{x^2}\Big\{x^2\Big(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Big)+x\frac{\partial}{\partial x}+\chi(Y_\eta)^2\frac{h_1h_2}{4}\Big\}c(x,y)=0, \tag{7.3}$$

$$\Big\{x^2\frac{\partial^2}{\partial x^2}+y^2\frac{\partial^2}{\partial y^2}+2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}+(k+1)\Big(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\Big) \\ -(x^2+y^2)\frac{\partial}{\partial y}-2xy\frac{\partial}{\partial x}-y+\frac{k^2-\nu_1^2}{4}\Big\}c(x,y)=0. \tag{7.4}$$

Consider a formal power series solution at $x=0$: $c(x,y)=\sum_{m=m_0}^{\infty}p_m(y)x^m$. We assume that it is holomorphic at $x=0$ in the variable x, so $m_0\ge 0$, and increase at most in polynomial order in y when $y\to+\infty$. The equations (7.3) and (7.4) yields the following differential recurrence equations satisfied by $p_m(y)$, $m\ge m_0$:

$$\Big(m^2+\frac{\chi(Y_\eta)^2h_1h_2}{4}\Big)p_m(y)=\frac{d^2}{dy^2}p_{m-2}(y), \tag{7.5}$$

$$\left\{\left(y\frac{d}{dy}\right)^2-(y-2m-k)y\frac{d}{dy}-(2m+1)y \right. \\ \left. -\frac{\nu_1^2-(2m+k)^2}{4}\right\}p_m(y)=\frac{d}{dy}p_{m-2}(y). \tag{7.6}$$

Then we obtain the following lemma:

7.4 Lemma. *Let m_0 be the degree of the first non-vanishing term $p_{m_0}(y)\neq 0$ of the series expansion. Then we have*

$$\left(4m_0^2+\chi(Y_\eta)^2h_1h_2\right)\ p_{m_0}(y)=0, \tag{7.7}$$

$$\left\{\left(y\frac{d}{dy}\right)^2-\big(y-(2m_0+k)\big)y\frac{d}{dy}-(2m_0+1)y \right. \\ \left. -\frac{\nu_1^2-(2m_0+k)^2}{4}\right\}p_{m_0}(y)=0. \tag{7.8}$$

So it must be satisfied that $\chi(Y_\eta)^2h_1h_2=-4m_0^2$, and m_0 is an integer. Moreover if $p_m(y)\neq 0$ for $m\geq m_0$, then $m=m_0+2\ell$ with a non-negative integer ℓ.

Proof. The remark in the top of this section implies that $-\sqrt{-1}\chi(Y_\eta)\sqrt{h_1h_2}=2m_0$ should be an even integer. Hence we conclude that m_0 is an integer. □

By this lemma we can write $c(x,y)=x^{m_0}\cdot\sum_{\ell=0}^{\infty}p_{m_0+2\ell}(y)x^{2\ell}$. Assume that each $p_{m_0+2\ell}(y)$ is written by a Laplace integral $p_{m_0+2\ell}(y)=\int_0^\infty q_{m_0+2\ell}(t)e^{-yt}dt$. Then (7.5) becomes $t^2q_{m_0+2\ell-2}(t)=4\ell(m_0+\ell)q_{m_0+2\ell}(t)$, $\ell\geq 0$. A general term is given recurrently by

$$q_{m_0+2\ell}(t)=\frac{m_0!\left(\frac{t}{2}\right)^{2\ell}}{\ell!\ \Gamma(m_0+\ell+1)}q_{m_0}(t).$$

One can also check these expressions and the equation (7.6) give the equation (7.8).

Recall that the ν-th Bessel function of the first kind has a series expansion

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}$$

with the gamma function $\Gamma(z)$, [M-O-S], 3.1, p.65. Hence we have

$$c(x,y) = m_0!\, e^{-\frac{m_0\pi\sqrt{-1}}{2}} x^{m_0} \int_0^\infty \left(\frac{tx}{2}\right)^{-m_0} J_{m_0}(\sqrt{-1}\, tx) q_{m_0}(t) e^{-yt} dt.$$

We have to determine the first non-vanishing term $p_{m_0}(y)$, or equivalently $q_{m_0}(t)$, satisfying (7.8). Put $p_{m_0}(y) = e^{\frac{y}{2}} y^{-\frac{2m_0+k+1}{2}} W(y)$. Then (7.8) gives the following equation for $W(y)$:

$$\text{(7.9)} \qquad \left\{ y^2 \frac{d^2}{dy^2} - \frac{y^2}{4} - \frac{2m_0-k+1}{2} y - \left(\frac{\nu_1^2}{4} - \frac{1}{4}\right) \right\} W(y) = 0.$$

It has a solution $W(y) = W_{\frac{k-2m_0-1}{2}, \frac{\nu_1}{2}}(y)$ with

$$W_{\kappa,m}(z) = \frac{e^{-\frac{z}{2}} z^\kappa}{\Gamma\left(\frac{1}{2}-\kappa+m\right)} \int_0^\infty t^{-\kappa-\frac{1}{2}+m} \left(1+\frac{t}{z}\right)^{\kappa-\frac{1}{2}+m} e^{-t} dt.$$

It gives a solution of (7.9) which decays rapidly as $\mathrm{Re}(z) \to \infty$, and such a solution is unique up to constant multiple, [M-O-S], §7.1, or [W-W], 16.12, p.340. We have $p_{m_0}(y) \sim y^{-(2m_0+1)}$ as $y \to +\infty$, since $W_{\frac{k-2m_0-1}{2}, \frac{\nu_1}{2}}(y) \sim e^{-\frac{y}{2}} y^{-\frac{2m_0+1-k}{2}}$. We can also express $p_{m_0}(y)$ by the Laplace transform, [M-O-S] §7.5.2, p.316, or [ET] I, §5.20, p.294, (9):

$$p_{m_0}(y) = \Gamma(2m_0+1)^{-1} \int_0^\infty t^{2m_0} F\left(\alpha + \tfrac{\nu_1}{2}, \alpha - \tfrac{\nu_1}{2}; 2m_0+1; -t\right) e^{-yt} dt,$$

where $F(a,b;c;z) = {}_2F_1(a,b;c;z)$ is the Gauss's hypergeometric function, and $\alpha = \frac{2-k+2m_0}{2}$. Note that m_0 is non-negative.

7.5 THEOREM. *Assume that h_1 and h_2 are both positive, and $h_3 = 0$ for the character η of N_s; in particular, H_η is positive definite. Consider the system of differential equations in Proposition 7.2. Then there is a unique* *solution* $\phi^{sol}(a_1, a_2) = b^{sol}(a_1, a_2) v_0^{-k,-k}$, *up to a constant multiple, that is characterized by the following conditions* (a) *and* (b): (a) *it is holomorphic at* $x = 0$, *and* (b) *it decays rapidly, when* a_1 *and* a_2 *tend to* $+\infty$. *The solution is expressed by the following integral:*

$$
\begin{aligned}
b^{sol}(a_1, a_2) = \frac{e^{-\frac{m_0\pi\sqrt{-1}}{2}}\big(a_1 a_2\big)^{k+1}}{\Gamma(2m_0+1)} & \int_0^\infty t^{m_0} J_{m_0}\big(2\pi\sqrt{-1}(h_1 a_1^2 - h_2 a_2^2)\ t\big) \\
& \times F\big(\tfrac{2-k+2m_0+\nu_1}{2}, \tfrac{2-k+2m_0-\nu_1}{2}; 2m_0+1; -t\big) \\
& \times e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)(t+1)} dt.
\end{aligned}
$$

Here $2m_0 = |\chi(Y_\eta)|\sqrt{h_1 h_2}$ *is a non-negative integer,* $J_\nu(z)$ *is the* ν*-th Bessel function of the first kind, and* $F(a, b; c; z)$ *is the Gauss's hypergeometric function.*

Proof. All of $p_m(y)$ with $m = m_0 + 2\ell$ are recursively determined, once the first non-zero term $p_{m_0}(y)$ is given. The equation (7.9) for $W(y)$ has 2 dimensional solution space. By Lemma 7.4 the integer m_0 should satisfy $4m_0^2 = -\chi(Y_\eta)^2 h_1 h_2$. Hence (7.1) and (7.2) determine a holonomic system of rank 4.

We have already seen that the integral in the statement of our theorem gives a solution of the set of differential equations in Proposition 7.2. We remark the convergence of the integral. To estimate the integrand as t tend to $+\infty$, we recall the asymptotic behavior of the functions appearing in the formula; they are $F(a, b; c; t) \sim C_1 t^{-a} + C_2 t^{-b} + O(t^{-a-1}) + O(t^{-b-1})$ as $t \to +\infty$, and $J_{m_0}(\sqrt{-1}xt) = e^{\frac{m_0\pi\sqrt{-1}}{2}} I_{m_0}(xt) \sim C_3 e^{xt}(2\pi xt)^{-\frac{1}{2}}$ as $t \to +\infty$, where $-\frac{\pi}{2} < \arg(xt) < \frac{3\pi}{2}$, $x = 2\pi(h_1 a_1^2 - h_2 a_2^2)$, [M-O-S] p.139; this is the asymptotic formula for the modified Bessel function $I_\nu(z)$ of second kind. Therefore the integrand grows at most $\sim C \times t^\kappa \times e^{-4\pi h_2 a_2^2 t} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)}$ as $t \to +\infty$, where we note h_1, $h_2 > 0$. This implies that the integral converges. Then it determines a solution satisfying the condition (a) and (b).

For the uniqueness, first remark that there are 2 choices of $c(x, y)$ with holomorphic expansion, hence $m_0 \leq 0$, at $x = 0$, which correspond to the solutions of (7.9). We use one of them in the above. Another solution corresponds to the solution $M_{\frac{k-2m_0-1}{2}, \frac{\nu_1}{2}}(y)$ of (7.9), [M-O-S], 7.1, p.296. Then the first non-zero term $p_{m_0}(y) = e^{\frac{y}{2}} y^{-\frac{2m_0+k+1}{2}} M_{\frac{k-2m_0-1}{2},\frac{\nu_1}{2}}(y)$ increases as $\sim C_1 e^y y^{-k} + C_2 y^{-(2m_0+1)}$ when $y \to +\infty$ along the real line, [M-O-S], 7.6.1, p.317. Multiplying $(x^2-y^2)^{\frac{k+1}{2}} e^{-y} = (\sqrt{h_1}a_1\sqrt{h_2}a_2)^{k+1} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)}$, we get an asymptotic formula of $b(a_1,a_2)$ on $x=0$ by $b(a_1,a_2) \sim C_1 y + C_2 e^{-y} y^{k-2m_0}$ when $y \to +\infty$. Hence it does not satisfy the condition (b). □

Let $\Phi(v)(g) = \langle v, \phi^{sol}(g)\rangle_K$ be the function in $C^\infty\text{-Ind}_R^G(\chi\cdot\eta)$, where v is a vector in the K-type $\tau_{k,k}$ of $I(P_1;\sigma,\nu_1)$. Consider the right $U(\mathfrak{g}_{\mathbb{C}})$ module $R_{U(\mathfrak{g}_{\mathbb{C}})}\Phi(v)$ in $C^\infty\text{-Ind}_R^G(\chi\cdot\eta)$ generated by $\Phi(v)$. Then we have the following theorem.

7.6 THEOREM. *The module $R_{U(\mathfrak{g}_{\mathbb{C}})}\Phi(v)$ determines the generalized principal series $I(P_1;\sigma,\nu_1)$ of even type. Hence the principal series has a unique (up to a constant multiple) non-trivial generalized Whittaker realization with the η and χ in Theorem 7.5, whose restriction to a K-type corresponds to K-finite functions satisfying the properties* (a), (b) *given above.*

PROOF. We prove this theorem by the following lemmas.

7.7 LEMMA. *The equations (7.1) and (7.2) determine the action of the A-radial parts of $Z(\mathfrak{g}_{\mathbb{C}})$, the center of the universal enveloping algebra, on the function ϕ with the one dimensional K-type $\tau_{-k,-k}$.*

PROOF. A set of generators of $Z(\mathfrak{g}_{\mathbb{C}})$ is given by the Casimir element in degree 2 and another element C_4 in degree 4. Proposition 7.2 explains the action of the Casimir operator. For C_4, the same argument as given in [M-O1] Lemma 10.2, tells us that the composition of the following four operators: $P^{up}\circ R(\nabla^+_{\tau_{-k,-k}})$, $P^{down}\circ R(\nabla^+_{\tau_{-k+2,-k}})$, $P^{up}\circ R(\nabla^-_{\tau_{-k+2,-k+2}})$, and $P^{down}\circ R(\nabla^-_{\tau_{-k+2,-k}})$ is written as a $\mathbb{C}$-linear sum of the C_4, the Casimir operator, and a scalar multiplication. Then we can determine the action of the C_4 because (7.2) implies that this composition, indeed, the composition of the first two, annihilates $\phi(a_1,a_2)$. □

By this lemma we know that the module $R_{U(\mathfrak{g}_{\mathbb{C}})}\Phi(v)$ has the same infinitesimal character ν as that of $I(P_1;\sigma,\nu_1)$. Since $R_{U(\mathfrak{g}_{\mathbb{C}})}\Phi(v)$ is generated by a K-finite vector, and it has an infinitesimal character, it is admissible of finite length, [W] Theorem 4.2.1.

7.8 Lemma. *The module $R_{U(\mathfrak{g}_\mathbb{C})}\Phi(v)$ has the K-type $\tau_{k,k}$ with multiplicity one.*

Proof. We want to use Lemma 3.5.3 of Wallach [W]. To apply it we have to know the right action of $U(\mathfrak{g})^K$ on $R_{U(\mathfrak{k}_\mathbb{C})}\Phi(v)$. Hereupon, since $R_{U(\mathfrak{k}_\mathbb{C})}\Phi(v) = V_{\tau_{k,k}}$ is one dimensional, Proposition 2.1 and Theorem 2.4 of Shimura [Sh] state that this action is realized by that of $Z(\mathfrak{g}_\mathbb{C})$ which has been specified by (7.1) and (7.2), Lemma 7.6. □

Considering the family of irreducible admissible modules of $Sp(2,\mathbb{R})$ with the common infinitesimal character ν, we conclude that $I(P_1;\sigma,\nu_1)$ is one of the constituents of $R_{U(\mathfrak{g}_\mathbb{C})}\Phi(v)$, occurs with multiplicity one. Moreover, it is contained as a submodule, since the equations (7.1) hold. If ν_1 is generic, then $I(P_1;\sigma,\nu_1)$ is irreducible, and the K-type $\tau_{k,k} \simeq R_{U(\mathfrak{k}_\mathbb{C})}\Phi(v)$ occurring in $R_{U(\mathfrak{g}_\mathbb{C})}\Phi(v)$ must coincide with the corner K-type of $I(P_1;\sigma,\nu_1)$ by Lemma 7.7. Hence $R_{U(\mathfrak{g}_\mathbb{C})}\Phi(v)$ is isomorphic to the generalized principal series.

7.9. *The generalized principal series of odd type: a system of differential equations.* In the next place, we study the generalized principal series $I(P_1;\sigma,\nu_1) = I(P_1;(\varepsilon,D_k^+),\nu_1)$, $k>0$, of odd type, that is, $\varepsilon(\gamma_{2e_1}) = -(-1)^k$. Then the corner K-type $\tau_{k,k-1}$ of the module is two dimensional and occurs with multiplicity one. Also we know that the K-type $\tau_{k-1,k-2}$ does not occur in $I(P_1;\sigma,\nu_1)$ by Corollary 6.4. Hence the map $P^{even}\circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k+1,-k}})$, which moves the K-types from $\tau_{-k+1,-k}$ to $\tau_{-k+2,-k+1}$, annihilates the generalized Whittaker functions with the K-type $\tau^*_{k,k-1} = \tau_{-k+1,-k}$ for $I(P_1;\sigma,\nu_1)$. Pairing it with an equation given by the Casimir operator, we set a system of differential equations.

7.10 Proposition. *Suppose that $h_3=0$ and h_1 and h_2 are both positive for the character η of N_s. Let $\phi(a_1,a_2) = \sum_{j=0,1} b_j(a_1,a_2)v_j^{-k+1,-k}$ a generalized Whittaker function with the K-type $\tau_{-k+1,-k}$ for $I(P_1;\sigma,\nu_1)$ of odd type. Set*

$$b_j(a_1,a_2) = (\sqrt{h_1}a_1)^{k+1+j}(\sqrt{h_2}a_2)^{k+2-j}e^{-2\pi(h_1a_1^2+h_2a_2^2)}c_j(a_1,a_2)$$

for $j=0$ and 1. Then $c_j(a_1,a_2)$, $j=0$ and 1, must satisfy the following set of differential equations:

$$\begin{aligned}(7.10)\qquad &\chi(Y_\eta)\sqrt{h_1h_2}\frac{h_1a_1^2h_2a_2^2}{D}c_0(a_1,a_2)\\ &+h_1a_1^2\Big(\partial_2-\frac{h_2a_2^2}{D}\Big)c_1(a_1,a_2)=0,\end{aligned}$$

$$\begin{aligned}(7.11)\qquad &h_2a_2^2\Big(\partial_1+\frac{h_1a_1^2}{D}\Big)c_0(a_1,a_2)\\ &+\chi(Y_\eta)\sqrt{h_1h_2}\frac{h_1a_1^2h_2a_2^2}{D}c_1(a_1,a_2)=0,\end{aligned}$$

$$\begin{aligned}(7.12)\qquad &\{(\partial_1+\partial_2)^2+2(k+1)(\partial_1+\partial_2)-8\pi h_1a_1^2\partial_1\\ &-8\pi h_2a_2^2\partial_2+8\pi(h_1a_1^2-h_2a_2^2)\\ &-16\pi(h_1a_1^2+h_2a_2^2)+(k+1)^2-\nu_1^2\}c_0(a_1,a_2)=0,\end{aligned}$$

$$\begin{aligned}(7.13)\qquad &\{(\partial_1+\partial_2)^2+2(k+1)(\partial_1+\partial_2)-8\pi h_1a_1^2\partial_1\\ &-8\pi h_2a_2^2\partial_2-8\pi(h_1a_1^2-h_2a_2^2)\\ &-16\pi(h_1a_1^2+h_2a_2^2)+(k+1)^2-\nu_1^2\}c_1(a_1,a_2)=0.\end{aligned}$$

Here we use the symbol: $D=h_1a_1^2-h_2a_2^2$.

PROOF. The equation $P^{even}\circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k+1,-k}})\phi=0$ yields (7.10) and (7.11). By the projection formula in Lemma 3.4 and Proposition 5.3, it says

$$\begin{aligned}&\big\{(\partial_1+4\pi h_1a_1^2+\tfrac{h_2a_2^2}{D}-k)b_0(a)+\mathcal{S}b_1(a)\big\}v_1^{-k+2,-k+1}\\ &\quad-\big\{\mathcal{S}b_0(a)+(\partial_2+4\pi h_2a_2^2-\tfrac{h_1a_1^2}{D}-k)b_1(a)\big\}v_0^{-k+2,-k+1}=0,\end{aligned}$$

where $\mathcal{S}=\frac{\chi(Y_\eta)h_1a_1h_2a_2}{D}$. Then each coefficient should vanish. Rewriting them for $c_j(a_1,a_2)$, we get the equations (7.10) and (7.11).

By Proposition 5.6, the action of the Casimir operator on ϕ is given by

$$\begin{aligned}&\begin{pmatrix}P+8k\pi h_1a_1^2+8(k-1)\pi h_2a_2^2 & -2\frac{h_1a_1^2+h_2a_2^2}{D}\mathcal{S}\\ 2\frac{h_1a_1^2+h_2a_2^2}{D}\mathcal{S} & P+8(k-1)\pi h_1a_1^2+8k\pi h_2a_2^2\end{pmatrix}\\ &\quad\times\begin{pmatrix}b_0(a)\\ b_1(a)\end{pmatrix}=0,\end{aligned}$$

where

$$P = \partial_1^2 + \partial_2^2 - 2(\partial_1 + \partial_2) + 2\frac{h_2 a_2^2}{D}\partial_1 - 2\frac{h_1 a_1^2}{D}\partial_2 - 16\pi^2 h_1^2 a_1^4 - 16\pi^2 h_2^2 a_2^4 - 2\frac{h_1 a_1^2 h_2 a_2^2}{D^2} + 2\mathcal{S}^2 - \nu_1^2 - (k+1)^2 + 5.$$

Here we note a relation between the Casimir operator L and the shift operator. Let us put

$$S_{+-} = \big(P^{even} \circ R(\nabla^-_{\chi\cdot\eta,\tau_{-k+2,-k+1}})\big) \circ \big(P^{even} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-k+1,-k}})\big)$$
$$S_{-+} = \big(P^{even} \circ R(\nabla^+_{\chi\cdot\eta,\ \tau_{-k,-k-1}})\big) \circ \big(P^{even} \circ R(\nabla^-_{\chi\cdot\eta,\tau_{-k+1,-k}})\big).$$

Then they satisfy the relation

$$R(L_{\chi\cdot\eta,\tau_{-k+1,-k}}) - 2(k+1)(k-2) = S_{+-} + S_{-+}.$$

Now the equation obtained above from the Casimir operator can be written as

$$\begin{pmatrix} Q_1 & -2\frac{h_1 a_1^2 + h_2 a_2^2}{D}\mathcal{S} \\ 2\frac{h_1 a_1^2 + h_2 a_2^2}{D}\mathcal{S} & Q_2 \end{pmatrix} \begin{pmatrix} b_0(a) \\ b_1(a) \end{pmatrix} = (\nu_1^2 - k^2) \begin{pmatrix} b_0(a) \\ b_1(a) \end{pmatrix},$$

with

$$Q_1 = \big(\partial_1 - 4\pi h_1 a_1^2 + \tfrac{h_2 a_2^2}{D} + k - 2\big)\big(\partial_1 + 4\pi h_1 a_1^2 + \tfrac{h_2 a_2^2}{D} - k\big) + \big(\partial_2 + 4\pi h_2 a_2^2 - \tfrac{h_1 a_1^2}{D} - k - 1\big)\big(\partial_2 - 4\pi h_2 a_2^2 - \tfrac{h_1 a_1^2}{D} + k - 1\big) + 2\mathcal{S}^2,$$

$$Q_2 = \big(\partial_2 - 4\pi h_2 a_2^2 - \tfrac{h_1 a_1^2}{D} + k - 2\big)\big(\partial_2 + 4\pi h_2 a_2^2 - \tfrac{h_1 a_1^2}{D} - k\big) + \big(\partial_1 + 4\pi h_1 a_1^2 + \tfrac{h_2 a_2^2}{D} - k - 1\big)\big(\partial_1 - 4\pi h_1 a_1^2 + \tfrac{h_2 a_2^2}{D} + k - 1\big) + 2\mathcal{S}^2.$$

If we take (7.10) and (7.11) into account, it provides us with equations of the single $c_j(a_1, a_2)$ for $j = 0$ and 1. For example, we obtain that

$$\Big\{\Big(\partial_1 - 8\pi h_1 a_1^2 + \frac{h_1 a_1^2}{D} + 2(k-1)\Big)\Big(\partial_1 + \frac{h_1 a_1^2}{D}\Big)$$

$$+ 2\frac{h_1a_1^2 + h_2a_2^2}{D}\Big(\partial_1 + \frac{h_1a_1^2}{D}\Big)$$
$$+ \Big(\partial_2 - \frac{h_2a_2^2}{D}\Big)\Big(\partial_2 - 8\pi h_2a_2^2 - \frac{h_2a_2^2}{D} + 2k\Big) + 2\mathcal{S}^2 + k^2 - \nu_1^2\Big\}c_0(a) = 0$$

for $c_0(a_1, a_2)$. On the other hand, (7.10) and (7.11) yield the equations

$$\{(\partial_2 - 3\frac{h_2a_2^2}{D})(\partial_1 + \frac{h_1a_1^2}{D}) - \mathcal{S}^2\}c_0(a_1, a_2) = 0,$$

$$\{(\partial_1 + 3\frac{h_1a_1^2}{D})(\partial_2 - \frac{h_2a_2^2}{D}) - \mathcal{S}^2\}c_1(a_1, a_2) = 0.$$

By canceling the term $2\mathcal{S}^2c_0(a_1, a_2)$ in the 2 equations for $c_0(a)$, we obtain (7.12). The equation (7.13) for $c_1(a)$ is obtained in a similar way. □

7.11. *A solution with an integral expression in the odd case.* We investigate a formal power series solution of the system of equations in Proposition 7.10. We change the variables a_1, a_2 by x and y, the same as in the even case, 7.3. Then the equations are written as

$$\frac{x^2 - y^2}{4x}\Big\{-\rho\ c_0(x, y) + \Big(2x\frac{d}{dx} - 2x\frac{d}{dy} + 1\Big)c_1(x, y)\Big\} = 0, \tag{7.14}$$

$$\frac{x^2 - y^2}{4x}\Big\{\Big(2x\frac{d}{dx} + 2x\frac{d}{dy} + 1\Big)c_0(x, y) + \rho\ c_1(x, y)\Big\} = 0, \tag{7.15}$$

$$\begin{aligned}\Big\{x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + 2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}& \\ + (k+2)\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big) - (x^2 + y^2)\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x}& \\ + x - 2y + \frac{(k+1)^2 - \nu_1^2}{4}\Big\}c_0(x, y) = 0,&\end{aligned} \tag{7.16}$$

$$\begin{aligned}\Big\{x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + 2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}& \\ + (k+2)\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big) - (x^2 + y^2)\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x}& \\ - x - 2y + \frac{(k+1)^2 - \nu_1^2}{4}\Big\}c_1(x, y) = 0.&\end{aligned} \tag{7.17}$$

Consider formal series solutions $c_j(x,y) = \sum_{m=m_0^j}^{\infty} p_m^j(y)x^m$ for $j=0$ and 1. We suppose that $m_0^j \geq 0$, hence they are holomorphic at the singular locus $x=0$. Then (7.14) and (7.15) provide us with the following recurrence differential equations,

$$2\frac{d}{dy}\begin{pmatrix} p_{m-1}^0(y) \\ p_{m-1}^1(y) \end{pmatrix} = M_m \cdot \begin{pmatrix} p_m^0(y) \\ p_m^1(y) \end{pmatrix} \tag{7.18}$$

with

$$M_m = \begin{pmatrix} -(2m+1) & -\chi(Y_\eta)\sqrt{h_1h_2} \\ -\chi(Y_\eta)\sqrt{h_1h_2} & 2m+1 \end{pmatrix}.$$

Now we obtain the following lemma concerning with the terms $p_{m_0}^j(y)$ in the first degree for $j=0$, 1.

7.12 Lemma. *The numbers m_0^0, m_0^1 must coincide with each other for the first non-vanishing terms. Denote the common one by $m_0 = m_0^0 = m_0^1$. Then it should be satisfied that $\chi(Y_\eta)^2h_1h_2 = -(2m_0+1)^2$ and m_0 is an integer.*

Proof. Consider the matrix M_{m_0} for $m_0 := \min(m_0^0, m_0^1)$. Then it is easy to obtain the first assertion. Also it should be satisfied that $\det M_{m_0} = 0$, which produces the second condition. The remark at the beginning of this section implies that $-\sqrt{-1}\chi(Y_\eta)\sqrt{h_1h_2} = 2m_0+1$ should be an odd integer. □

Express $p_m^j(y)$ by a Laplace integral: $p_m^j(y) = \int_0^\infty q_m^j(t)e^{-yt}dt$. Then (7.18) gives us that

$$-2t\begin{pmatrix} q_{m-1}^0(t) \\ q_{m-1}^1(t) \end{pmatrix} = M_m \begin{pmatrix} q_m^0(t) \\ q_m^1(t) \end{pmatrix}. \tag{7.19}$$

If we look at the first pair $(q_{m_0}^0(t), q_{m_0}^1(t))$, Lemma 7.12 tells us that it is written as

$$\begin{pmatrix} q_{m_0}^0(t) \\ q_{m_0}^1(t) \end{pmatrix} = Q_{m_0}(t)\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}. \tag{7.20}$$

with a non-zero function $Q_{m_0}(t)$ that does not depend on j. Here we note that the vector ${}^t\big(1, \sqrt{-1}\big)$ generates the kernel of M_{m_0}. By an inductive calculation we conclude that

7.13 LEMMA.

$$\begin{pmatrix} q^0_{m_0+r}(t) \\ q^1_{m_0+r}(t) \end{pmatrix} = \begin{cases} \frac{(t/2)^{2\ell} m_0!}{\ell!\ \Gamma(m_0+\ell+1)} Q_{m_0}(t) \binom{1}{\sqrt{-1}}, & \textit{if } r = 2\ell \geq 0 \\ \frac{(t/2)^{2\ell+1} m_0!}{\ell!\ \Gamma(m_0+1+\ell+1)} Q_{m_0}(t) \binom{1}{-\sqrt{-1}}, & \textit{if } r = 2\ell+1 > 0. \end{cases}$$

Hence we obtain that

$$c_j(x,y) = m_0!(\sqrt{-1})^j x^{m_0} \times \int_0^\infty \left(\frac{tx}{2}\right)^{-m_0} \big(I_{m_0}(xt) + (-1)^j I_{m_0+1}(xt)\big) Q_{m_0}(t) e^{-yt} dt$$

for $j = 0$ and 1. Here the ν-th modified Bessel function $I_\nu(z)$ of second kind is defined by

$$I_\nu(z) = \sum_{m=0}^\infty \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\ \Gamma(\nu+m+1)},$$

[M-O-S], 3.1, p.66. □

It remains to determine the $Q_{m_0}(t)$. The equations (7.16) and (7.17) read

$$\begin{aligned} &\frac{d}{dy} p^j_{m-2}(y) - (-1)^j p^j_{m-1}(y) \\ &\quad = \Big\{ y^2 \frac{d^2}{dy^2} - \{y - (2m+k+2)\} y \frac{d}{dy} - 2(m+1)y \\ &\qquad - \frac{\nu_1^2 - (2m+k+1)^2}{4} \Big\} p^j_m(y) \end{aligned}$$

for $j = 0$ and 1. If $m = m_0$, so $p^0_{m_0-1}(y) = p^0_{m_0-2}(y) = 0$, then each equation above has a solution

$$p^j_{m_0}(y) = (\sqrt{-1})^j e^{\frac{y}{2}} y^{-\frac{2m_0+k+2}{2}} W_{\frac{k-2m_0-2}{2}, \frac{\nu_1}{2}}(y)$$

that satisfies $p^j_{m_0}(y) \sim y^{-(2m_0+2)}$ when $y \to +\infty$. They can be expressed by Laplace integrals, then $Q_{m_0}(t)$ is determined. We state now the main result for $I(P_1;\sigma,\nu_1)$ of odd type.

7.14 THEOREM. *Suppose that* $h_3 = 0$ *and* h_1 *and* h_2 *are both positive for the character* η *of* N_s. *Consider the set of differential equations given in Proposition 7.10. Then there is a solution* $\phi^{sol}(a_1,a_2) = \sum_{j=0,1} b^{sol}_j(a_1,a_2)v_j^{-k+1,-k}$ *that is uniquely determined up to a constant multiple by the conditions* (a) *and* (b): (a) *it is holomorphic at* $x = 2\pi(h_1a_1^2 - h_2a_2^2) = 0$, *and* (b) *it decays rapidly, when* $a_1, a_2 \to +\infty$. *The functions* $b^{sol}_j(a_1,a_2)$, $j = 0$ *and* 1 *are given as follows:*

$$\begin{aligned} b^{sol}_j(a_1,a_2) = & \frac{(\sqrt{-1})^j e^{-\frac{m_0\pi\sqrt{-1}}{2}}(\sqrt{h_1}a_1)^{k+1+j}(\sqrt{h_2}a_2)^{k+2-j}}{\Gamma(2m_0+2)} \\ & \times \int_0^\infty t^{m_0+1}e^{-2\pi(h_1a_1^2+h_2a_2^2)(t+1)} \\ & \times F\big(\frac{2m_0+3+\nu_1-k}{2},\frac{2m_0+3-\nu_1-k}{2};2m_0+2;-t\big) \\ & \times \big(J_{m_0}(2\pi\sqrt{-1}t(h_1a_1^2-h_2a_2^2)) \\ & \quad + \big(\sqrt{-1}\big)^{2j+3}J_{m_0+1}(2\pi\sqrt{-1}t(h_1a_1^2-h_2a_2^2))\big)dt, \end{aligned}$$

for $j = 0$ *and* 1. *Here* m_0 *is a non-negative integer that satisfies* $\chi(Y_\eta)^2h_1h_2 = -(2m_0+1)^2$, $J_\nu(z)$ *is the* ν*-th Bessel function of the first kind, and* $F(a,b;c;z)$ *is the Gauss's hypergeometric function.*

PROOF. This is proved by a similar argument as in the even case. We mention that the differential equations in Proposition 7.10 determines a holonomic system of rank 4. □

7.15 REMARKS. We conjecture that the solution given above presents a non-trivial generalized Whittaker realization of the principal series $I(P_1;\sigma,\nu_1)$ of odd type, and the dimension of the space of the realizations whose images satisfy the conditions (a) and (b) above, is equal to one. But the author can not verify this conjecture. We know that the set of equations in Proposition 7.10 is necessary to characterize such realizations, but the author does not know whether all the solution generate the module. The problem is to determine the action of $U(\mathfrak{g}_{\mathbb{C}})^K$ on the space of functions whose values are in a two dimensional K-type. This was known in the even case, Lemma 7.7.

8. The Mellin Transform of a Generalized Whittaker Function for the Generalized Principal Series Representation

Andrianov [An] studied L-functions for a Hecke eigen holomorphic Siegel cusp forms of degree 2. In order to generalize his results to non-holomorphic forms, it becomes crucial to investigate the generalized Whittaker functions belonging to the other standard representations than holomorphic discrete series, and their Mellin transforms. Using an explicit formula of a class 1 generalized Whittaker function obtained by Niwa [Ni], Hori [H] carries out the steps for the Siegel wave forms. They treat a spherical vector in the principal series induced from a minimal parabolic subgroup of G. Now we study the same steps for the generalized principal series $I(P_1;\sigma,\nu_1)$.

To obtain the L-function we consider an integral transform of a Siegel modular form over a real three dimensional hyperbolic manifold [An]. We recall the paper [H], which studied the Siegel wave forms of degree two. Let $F(Z)$ be a Siegel wave form on the Siegel upper half space $\mathbb{H}_2$ of degree two, [H] Definition (1.1). It is, by definition, a class 1 (with the trivial K-type) function. The integral is given by

$$\widetilde{R}_F(s)=\int_0^\infty\int_{X_{1_2}(\mathbb{R})/X_{1_2}(\mathbb{Z})}F(X+\sqrt{-1}\ v1_2)v^{s-1}dX\,dv,$$

where $Z=X+\sqrt{-1}Y\in\mathbb{H}_2$, $X_{1_2}(\mathbb{R})=\{X\in M_2(\mathbb{R})\mid {}^tX=X,\ \mathrm{tr}(X)=0\}$, and $X_{1_2}(\mathbb{Z})=X_{1_2}(\mathbb{R})\cap M_2(\mathbb{Z})$. Consider the Fourier expansion of F, [H] Section 1: $F(Z)=\sum_{N\in\mathfrak{N}}a_F(N,Y)e^{2\pi\sqrt{-1}\mathrm{tr}(NX)}$, where $\mathfrak{N}=\{n\in M_2(\mathbb{Q})\mid {}^tN=N,$ semi integral$\}$. Also for a definite N, we consider the expansion: $a_F(N,Y)=\sum_{n\in\mathbb{Z}}a_{N,n}(F)W_{N,n}(Y)$ by the class 1 generalized Whittaker functions $W_{N,n}(Y)$, where $a_{N,n}(F)\in\mathbb{C}$. The the above integral is also written as

$$\widetilde{R}_F(s)=\left(\sum_{m\in\mathbb{N}}\frac{a_{m1_2,0}(F)+a_{-m1_2,0}(F)}{m^s}\right)\int_0^\infty W_{1_2,0}(u1_2)u^{s-1}du,$$

[H], Section 4. Here the function $\widetilde{R}_F(s)$ can be separated into the local parts corresponding to the nonarchimedean, or the archimedean, places. The above integral of the generalized Whittaker function $W_{1_2,0}(u1_2)$ is related to the gamma factor of the L-function associated with $F(Z)$, [H], Theorem (2.1), Sections 5 and 6.

Now we study the same integral transform of the solution obtained for $I(P_1;\sigma,\nu_1)$ in the last section.

8.1 THEOREM. *Take the special character η_0 of N_s by $h_1 = h_2 = 1$ and $h_3 = 0$. Let $\phi^{sol}(a_1,a_2) = b^{sol}(a_1,a_2)v_0^{-k,-k}$ be the unique solution given in Theorem 7.5, which represents the generalized Whittaker function with the corner K-type for $I(P_1;\sigma,\nu_1)$ of even type. Then we obtain the following formula of the generalized Mellin transform of ϕ^{sol}: if $m_0 = 0$, then for* $\mathrm{Re}(s + \frac{k-1}{2} \pm \frac{\nu_1}{2}) > 0$

$$\int_0^\infty b^{sol}(\sqrt{a},\sqrt{a})a^{s-\frac{3}{2}}\frac{da}{a} = \frac{\Gamma(s+\frac{k-1}{2}+\frac{\nu_1}{2})\Gamma(s+\frac{k-1}{2}-\frac{\nu_1}{2})}{(4\pi)^{s+k-\frac{1}{2}}\Gamma(s+\frac{1}{2})}.$$

This integral vanishes, if $m_0 > 0$.

PROOF. This is obtained by direct calculation. If $m_0 = 0$, then the integral equals

$$\int_0^\infty \Big(\int_0^\infty a^{s+k-\frac{1}{2}}e^{-4\pi a(t+1)}\frac{da}{a}\Big)F\big(\tfrac{2-k+\nu_1}{2},\tfrac{2-k+\nu_1}{2};1;-t\big)dt.$$

We note that $J_{m_0}(0) = 0$, if $m_0 \neq 0$. It is calculated as

$$= \frac{\Gamma(s+k-1/2)}{(4\pi)^{s+k-1/2}}\int_0^\infty (t+1)^{-s-k+\frac{1}{2}}F\big(\tfrac{2-k+\nu_1}{2},\tfrac{2-k+\nu_1}{2};1;-t\big)dt.$$

Then a formula [ET] II, §20.2, p. 400 (9), gives the result. □

We also give a formula of the Mellin transform in the case of $I(P_1;\sigma,\nu_1)$ of odd type.

8.2 THEOREM. *Take the same character η_0 as in Theorem 8.1. Let $\phi^{sol}(a_1,a_2) = \sum_{j=0,1} b_j^{sol}(a_1,a_2)v_j^{-k+1,-k}$ be the unique solution given in*

Theorem 7.11 in the case of $I(P_1;\sigma,\nu_1)$ of odd type. Then the generalized Mellin transform of each $b_j^{sol}(a_1,a_2)$ is given as follows: if $m_0=0$, then for $\mathrm{Re}(s+\frac{k-1}{2}\pm\frac{\nu_1}{2})>0$

$$\int_0^\infty b_j^{sol}(\sqrt{a},\sqrt{a})a^{s-\frac{3}{2}}\frac{da}{a}=\frac{\Gamma(s+\frac{k-1}{2}+\frac{\nu_1}{2})\Gamma(s+\frac{k-1}{2}-\frac{\nu_1}{2})}{(4\pi)^{s+k}\Gamma(s+1)}.$$

If $m_0>0$, then the integral vanishes.

9. Parameterization of the Discrete Series Representations

Now we study the generalized Whittaker functions for the large discrete series representations of $G=\mathrm{Sp}(2,\mathbb{R})$. We recall the parameterization of the discrete series representations of G, and its K-type decompositions in this section.

9.1. *The Harish-Chandra parameterization of the discrete series representations and their K-types.* Consider a compact Cartan subgroup of $G=\mathrm{Sp}(2,\mathbb{R})$

$$\exp(\mathfrak{h})=\left\{k(\theta_1,\theta_2)=\begin{pmatrix}\cos\theta_1 & 0 & \sin\theta_1 & 0\\ 0 & \cos\theta_2 & 0 & \sin\theta_2\\ -\sin\theta_1 & 0 & \cos\theta_1 & 0\\ 0 & -\sin\theta_2 & 0 & \cos\theta_2\end{pmatrix}\mid\theta_1,\theta_2\in\mathbb{R}\right\}$$

corresponding to the compact Cartan subalgebra $\mathfrak{h}$, Section 2. Then the characters of this group are given by $k(\theta_1,\theta_2)\mapsto\exp\big(\sqrt{-1}(m_1\theta_1+m_2\theta_2)\big)\in\mathbb{C}^\times$ with some integers m_1 and m_2. The derivations of these characters determine the weight lattice in $\mathfrak{h}_{\mathbb{C}}^*=\mathrm{Hom}(\mathfrak{h}_{\mathbb{C}},\mathbb{C})$.

In Section 2.2 we fixed the set of compact positive roots as $\Sigma_c^+=\{(1,-1)\}$. Then the set of dominant integral weights is given by $\{(\Lambda_1,\Lambda_2)\in\mathbb{Z}^{\oplus 2}\mid\Lambda_1\geq\Lambda_2\}$. Here we pick up all sets of the positive roots with respect to $\Sigma_c^+=\{(1,-1)\}$:

$$\begin{aligned}\Sigma_I^+&=\{(1,-1),(2,0),(1,1),(0,2)\},\\ \Sigma_{II}^+&=\{(1,-1),(2,0),(1,1),(0,-2)\},\\ \Sigma_{III}^+&=\{(1,-1),(2,0),(0,-2),(-1,-1)\},\\ \Sigma_{IV}^+&=\{(1,-1),(-2,0),(0,-2),(-1,-1)\}.\end{aligned}$$

For the index $J \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ we define $\Sigma^+_{J,nc} := \Sigma^+_J \setminus \Sigma^+_c$, the set of non-compact positive roots for J. Also we define a subset Ξ_J of the dominant weights for each J by $\Xi_J = \{\Lambda = (\Lambda_1, \Lambda_2) \mid \langle \Lambda, \beta \rangle > 0, \text{ for all } \beta \in \Sigma^+_J\}$. Then it is known that the union of Ξ_J, $J = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, gives a parameterization of the discrete series of G, which is called the Harish-Chandra parameterization.

Let us write π_Λ for the discrete series representation of G with the Harish-Chandra parameter $\Lambda \in \Xi_J$ for one J. Then its K-type decomposition $\pi_\Lambda|_K$ is given by the Blattner formula [H-S]. If a K-module τ occurs in the restriction, then its highest weight is of the form $\lambda + \sum_{\beta \in \Sigma^+_{J,nc}} m_\beta \beta$ with $m_\beta \in \mathbb{Z}_{\geq 0}$, where $\lambda = \Lambda - \rho_c + \rho_{nc}$, and ρ_c (resp. ρ_{nc}) is the half of the sum of compact positive roots (resp. non-compact positive roots) in Σ^+_J. We call λ the Blattner parameter of π_Λ. We also use the symbol π_λ for the discrete series representation with the Blattner parameter λ. Its minimal K-type τ_λ occurs with multiplicity one. The Blattner parameter λ associated with a Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2)$ is given by $\lambda = (\Lambda_1 + 1, \Lambda_2 + 2)$, if Λ is of type I; $(\Lambda_1 + 1, \Lambda_2)$, type II; $(\Lambda_1, \Lambda_2 - 1)$, type III; $(\Lambda_1 - 2, \Lambda_2 - 1)$, type IV. A discrete series representation π_Λ with the Harish-Chandra parameter $\Lambda \in \Xi_{II}$ or Ξ_{III} is called *a large discrete series representation*. The Gelfand-Kirillov dimension of a large discrete series is equal to 4, which is the dimension of the maximal unipotent subgroup of G. Hence the large discrete series representation has a non-degenerate Whittaker model for the maximal unipotent subgroup [V1].

10. The Generalized Whittaker Functions for the Large Discrete Series Representations

This section is devoted to a study on a generalized Whittaker function with the minimal K-type of a large discrete series. We give a system of differential equations satisfied by the Whittaker function. Then we check the holonomicity of the system.

10.1. *A system of differential equations.* Let (π, H_π) be a large discrete series representation of G with the Harish-Chandra parameter $(\Lambda_1, \Lambda_2) \in \Xi_{II}$ defined in Section 9. Its Blattner parameter, that is the highest weight of the minimal K-type of π, is given by $(\lambda_1, \lambda_2) = (\Lambda_1 + 1, \Lambda_2)$. The minimal K-type $\tau_{\lambda_1,\lambda_2}$ occurs with multiplicity one in the large discrete series

$\pi = \pi_{\lambda_1,\lambda_2}$. The K-type decomposition of $\pi_{\lambda_1,\lambda_2}$ tells us that each of the following 3 shift operators: $P^{down} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-\lambda_2,-\lambda_1}})$, $P^{even} \circ R(\nabla^+_{\chi\cdot\eta,\tau_{-\lambda_2,-\lambda_1}})$, and $P^{down} \circ R(\nabla^-_{\chi\cdot\eta,\tau_{-\lambda_2,-\lambda_1}})$ annihilates the generalized Whittaker functions ϕ with the K-type $\tau_{-\lambda_2,-\lambda_1} = \tau^*_{\lambda_1,\lambda_2}$ for $\pi_{\lambda_1,\lambda_2}$. On the other hand, Yamashita proved

Theorem. (Yamashita [Y1]) *The system of differential equations in the above determines the generalized Whittaker functions for the large discrete series representation.*

Write $d = \lambda_1 - \lambda_2$, then the minimal K-type $\tau_{\lambda_1,\lambda_2}$ is of $d+1$ dimension. We take the basis $\{v_j^{-\lambda_2,-\lambda_1}\}_{0\le j\le d}$ of $\tau_{-\lambda_2,-\lambda_1}$ defined in Lemma 3.1.

10.2 Proposition. *Suppose that $h_3 = 0$ and both h_1 and h_2 are positive for the character η of N_s. Let $\phi(a_1,a_2) = \sum_{j=0}^d b_j(a_1,a_2)v_j^{-\lambda_2,-\lambda_1}$ be the restriction to A of a generalized Whittaker function with the minimal K-type $\tau^*_{\lambda_1,\lambda_2} = \tau_{-\lambda_2,-\lambda_1}$ for a large discrete series $\pi_{\lambda_1,\lambda_2}$ of G. Then we have the following system of differential equations for $b_j(a) = b_j(a_1,a_2)$, $0 \le j \le d$:*

$$\begin{aligned}(10.1;\text{ j})\quad &\Big(\partial_1 + 4\pi h_1 a_1^2 + 2j\frac{h_2a_2^2}{D} + j - 1 - \lambda_1\Big)b_{j-1}(a) + 2\mathcal{S}b_j(a)\\ &+ \Big(\partial_2 + 4\pi h_2 a_2^2 - 2(d-j)\frac{h_1a_1^2}{D} - j - 1 - \lambda_2\Big)b_{j+1}(a) = 0\end{aligned}$$

for $1 \le j \le d-1$,

$$\begin{aligned}(10.2;\text{ j})\quad & j\Big(\partial_1 + 4\pi h_1 a_1^2 - (d-2j)\frac{h_2a_2^2}{D} + j - 1 - \lambda_1\Big)b_{j-1}(a)\\ &- (d-2j)\mathcal{S}b_j(a)\\ &- (d-j)\Big(\partial_2 + 4\pi h_2 a_2^2 - (d-2j)\frac{h_1a_1^2}{D} - j - 1 - \lambda_2\Big)b_{j+1}(a)\\ &= 0\end{aligned}$$

for $0 \le j \le d$, and

$$\begin{aligned}(10.3;\text{ j})\quad &\Big(\partial_2 - 4\pi h_2 a_2^2 - 2j\frac{h_1a_1^2}{D} + j - 1 + \lambda_2\Big)b_{j-1}(a) - 2\mathcal{S}b_j(a)\\ &+ \Big(\partial_1 - 4\pi h_1 a_1^2 + 2(d-j)\frac{h_2a_2^2}{D} - j - 1 + \lambda_1\Big)b_{j+1}(a) = 0\end{aligned}$$

for $1 \leq j \leq d-1$.

Here we use the symbols: $D = h_1 a_1^2 - h_2 a_2^2$, *and* $\mathcal{S} = \frac{\chi(Y_\eta) h_1 a_1 h_2 a_2}{D}$.

Proof. The equations (10.1; j), $1 \leq j \leq d-1$ are obtained from the equation $P^{down} \circ R(\nabla^{+}_{\chi\cdot\eta, \tau_{-\lambda_2, -\lambda_1}})\phi(a_1, a_2) = 0$ for ϕ, where we use Proposition 5.3, Lemma 3.5, and Lemma 3.1. The others are from $P^{even} \circ R(\nabla^{+}_{\chi\cdot\eta, \tau_{-\lambda_2, -\lambda_1}})\phi(a_1, a_2) = 0$, and $P^{down} \circ R(\nabla^{-}_{\chi\cdot\eta, \tau_{-\lambda_2, -\lambda_1}})\phi(a_1, a_2) = 0$. □

In the next place, we study about the holonomicity of the system of differential equations given above. We need to prepare some lemmas to check the holonomicity. To simplify our calculations, it is convenient to put

$$b_j(a_1, a_2) = (\sqrt{h_1} a_1)^{\lambda_1 - j} (\sqrt{h_2} a_2)^{\lambda_2 + j} e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)} c_j(a_1, a_2),$$

for $0 \leq j \leq d$ and consider the functions $c_j(a_1, a_2)$.

Making (10.1; j) × ($d - j$) + (10.2; j) and (10.1; j) × j − (10.2; j) for $1 \leq j \leq d-1$, we have the following system of equations for $c_j(a_1, a_2)$, $0 \leq j \leq d$, which is equivalent to the original system in Proposition 10.2:

$$\text{(10.4; j)} \quad \Big(\partial_1 + j\frac{h_2 a_2^2}{D}\Big) c_{j-1}(a) + \rho \frac{h_2 a_2^2}{D} c_j(a) - (d-j)\frac{h_2 a_2^2}{D} c_{j+1}(a) = 0$$

for $1 \leq j \leq d$,

$$\text{(10.5; j)} \quad j\frac{h_1 a_1^2}{D} c_{j-1}(a) + \rho \frac{h_1 a_1^2}{D} c_j(a) + \Big(\partial_2 - (d-j)\frac{h_1 a_1^2}{D}\Big) c_{j+1}(a) = 0$$

for $0 \leq j \leq d-1$, and

$$\begin{aligned} \text{(10.6; j)} \quad & h_1 a_1^2 \Big(\partial_2 - 8\pi h_2 a_2^2 - 2j\frac{h_2 a_2^2}{D} + 2\lambda_2 - 2\Big) c_{j-1}(a) \\ & - 2\rho \frac{h_1 a_1^2 h_2 a_2^2}{D} c_j(a) + h_2 a_2^2 \Big(\partial_1 - 8\pi h_1 a_1^2 \\ & + 2(d-j)\frac{h_1 a_1^2}{D} + 2\lambda_2 - 2\Big) c_{j+1}(a) = 0 \end{aligned}$$

for $1 \leq j \leq d-1$, where we put $\rho = \chi(Y_\eta)\sqrt{h_1 h_2}$.

From these equations we can obtain $\mathbb{C}(a_1, a_2)$-linear relations among the $c_j(a)$.

10.3 Lemma. *When $d = \lambda_1 - \lambda_2 \geq 4$, we obtain the following $d-3$ linear relations over $\mathbb{C}(a_1, a_2)$ among $c_j(a_1, a_2)$ with $0 \leq j \leq d$. These are given by*

$$
\begin{aligned}
\text{(10.7; j)}\quad & (j-2)\frac{(h_1a_1^2)^2}{D}c_{j-3}(a) + \rho\frac{(h_1a_1^2)^2}{D}c_{j-2}(a) \\
& - h_1a_1^2\Big((d-j+2)\frac{h_1a_1^2}{D} - 2j\frac{h_2a_2^2}{D} + 2\lambda_2 - 2 - 8\pi h_2a_2^2\Big)c_{j-1}(a) \\
& + 2\rho\frac{h_1a_1^2h_2a_2^2}{D}c_j(a) \\
& - h_2a_2^2\Big(2(d-j)\frac{h_1a_1^2}{D} - (j+2)\frac{h_2a_2^2}{D} + 2\lambda_2 - 2 - 8\pi h_1a_1^2\Big)c_{j+1}(a) \\
& + \rho\frac{(h_2a_2^2)^2}{D}c_{j+2}(a) - (d-j-2)\frac{(h_2a_2^2)^2}{D}c_{j+3}(a) = 0.
\end{aligned}
$$

for $2 \leq j \leq d-2$.

Proof. We use the equations (10.4; j), (10.5; j) to cancel the terms of differentials $\partial_1 c_{j+1}(a)$ and $\partial_2 c_{j-1}(a)$ in (10.6; j) for $2 \leq j \leq d-2$. □

10.4 Corollary. *Let $d \geq 4$ and $\chi(Y_\eta) \neq 0$. Then the above $d-3$ linear relations are mutually independent of each other. In particular, if we pick up arbitrary 4 functions among $c_j(a_1, a_2)$, $0 \leq j \leq d$, then the others can be written by the $\mathbb{C}(a_1, a_2)$-linear sums of them.*

Proof. For example, take $c_0(a)$, $c_1(a)$, $c_{d-1}(a)$ and $c_d(a)$ and write down the linear combinations of the other $d-3$ functions by these four functions in the natural order for $c_j(a)$. Considering the coefficient matrix of the other $d-3$ functions in the above, then it is a 7-gonal matrix (for $d \geq 10$ and we can check directly for smaller d cases) and in each coefficient appear only the terms of the degree, either 2 or 4, with respect to the variables a_1, a_2. Here "degree 2" terms mean the terms containing $\frac{(h_1a_1^2)^2}{D}$, $\frac{(h_2a_2^2)^2}{D}$, $\frac{h_1a_1^2h_2a_2^2}{D}$, or $h_ia_i^2$ as a factor. Also "degree 4" terms mean the terms containing $8\pi h_1a_1^2h_2a_2^2$. We find that the terms of "degree 4" appears only in the $(i, i+1)$-th, $1 \leq i \leq d-4$, or $(i, i-1)$-th, $2 \leq i \leq d-3$ coefficients of the matrix. From this observation on the degree we can easily show that the matrix has non zero determinant. □

10.5 Lemma. *Let $d \geq 4$ and $\chi(Y_\eta) \neq 0$. Then each of $\partial_1 c_j(a)$, $\partial_2 c_j(a)$, $0 \leq j \leq d$, can be written as a $\mathbb{C}(a_1, a_2)$-linear combination of 4 functions which are arbitrary chosen among $\{c_j(a_1, a_2),\ 0 \leq j \leq d\}$.*

Proof. From (10.4; j) with $1 \leq j \leq d$, and (10.5; j) with $0 \leq j \leq d-1$ we can describe $\partial_1 c_j(a)$ by a $\mathbb{C}(a_1, a_2)$-linear combination of $c_j(a)$, $c_{j+1}(a)$, and $c_{j+2}(a)$ for $0 \leq j \leq d-1$:

$$\text{(10.8; j)} \qquad \begin{aligned}\partial_1 c_j(a) = &-(j+1)\frac{h_2 a_2^2}{D} c_j(a) - \rho \frac{h_2 a_2^2}{D} c_{j+1}(a) \\ &+ (d-j-1)\frac{h_2 a_2^2}{D} c_{j+2}(a)\end{aligned}$$

Similarly we have

$$\text{(10.9; j)} \qquad \begin{aligned}\partial_2 c_j(a) = &-(j-1)\frac{h_1 a_1^2}{D} c_{j-2}(a) - \rho \frac{h_1 a_1^2}{D} c_{j-1}(a) \\ &+ (d-j+1)\frac{h_1 a_1^2}{D} c_j(a)\end{aligned}$$

for $1 \leq j \leq d$. Therefore, by Corollary 10.4, the assertion in the lemma is verified for $\partial_1 c_j(a)$, $\partial_2 c_j(a)$ with $1 \leq j \leq d-1$ and for $\partial_1 c_0(a)$, $\partial_2 c_d(a)$.

For the remaining ones: $\partial_2 c_0(a)$ and $\partial_1 c_d(a)$, we use the equations (10.6; j), $j = 1$ or $d-1$. We see $\partial_2 c_0(a)$, (resp. $\partial_1 c_d(a)$) can be expressed as $\mathbb{C}(a_1, a_2)$-linear combination of $c_0(a), c_1(a), c_2(a)$, and $\partial_1 c_2(a)$ (resp. $c_{d-2}(a), c_{d-1}(a), c_d(a)$, and $\partial_2 c_{d-2}(a)$). Then, combining the result above for $\partial_1 c_2(a)$ or $\partial_2 c_{d-2}(a)$, and Corollary 10.6, we conclude the lemma. □

By the lemmas above, we can conclude that, if the integrability condition is also satisfied, then the equations in 10.2 determine a holonomic system of rank 4. We now check the integrability condition.

10.6 Lemma. *The system of equations* (10.4; j) *with* $1 \leq j \leq d-1$, (10.5; j) *with* $0 \leq j \leq d$, *and* (10.6; j) *with* $1 \leq j \leq d-1$, *satisfies the integrability condition.*

Proof. The system of equations is equivalent to the set of (10.7; j) with $2 \leq j \leq d-2$, (10.8; j) with $0 \leq j \leq d-1$, (10.9; j) with $1 \leq j \leq d$, and (10.6; j) with $j = 1$, $d-1$. One has to check that these equations yield the integrability conditions: $\partial_1\partial_2 c_j(a) = \partial_2\partial_1 c_j(a)$ for $0 \le j \le d$, and also that these conditions add no more equation that is independent of the system.

The integrability conditions for $1 \le j \le d-1$ are obtained from (10.8; j) for $0 \le j \le d-1$ and (10.9; j) for $1 \le j \le d$. Applying ∂_2 to (10.8; j), $0 \le j \le d-1$ from the left, we have

$$\partial_2\partial_1 c_j(a) = -\frac{h_2 a_2^2}{D}\Big(\partial_2 + 2\frac{h_1 a_1^2}{D}\Big)\big((j+1)c_j(a) + \rho c_{j+1}(a) - (d-j-1)c_{j+2}(a)\big)$$

for $0 \le j \le d-1$. Then using (10.9; j) for $1 \le j \le d$, we see that the above formulas are equal to

$$\begin{aligned} =\big\{&(j+1)(j-1)c_{j-2}(a) + \rho(2j+1)c_{j-1}(a) \\ &- \big(2(j+1)(d-j+1) - \rho^2\big)c_j(a) \\ &- \rho(2d-2j+1)c_{j+1}(a) + (d-j+1)(d-j-1)c_{j+2}(a)\big\}\frac{h_1 a_1^2 h_2 a_2^2}{D^2} \end{aligned}$$

for $1 \le j \le d-1$. On the other hand, (10.9; j) with $1 \le j \le d$, and (10.8; j) with $0 \le j \le d-1$, give the same results for $\partial_1\partial_2 c_j(a)$ for $1 \le j \le d-1$. Therefore the conditions are shown to be satisfied with $c_j(a)$ for $1 \le j \le d-1$. For $c_0(a)$ and $c_d(a)$ we also use (10.6; j) with $j = 1$ and $d-1$. I omit a detailed computation. We mention finally that the integrability conditions are compatible with the equations (10.7; j) for $2 \le j \le d-2$. □

Combining Corollary 10.4, Lemmas 10.5, and 10.6, we conclude the following:

10.7 PROPOSITION. *If $\chi(Y_\eta) \neq 0$, then the system of differential equations in Proposition 10.2 determines a holonomic system of rank* 4.

11. Multiplicity Free Theorem and the Mellin Transforms of the Generalized Whittaker Functions of a Large Discrete Series

11.1. In the previous section we have shown that the system of differential equations in Proposition 10.2 is holonomic of rank 4. Using the integrable conditions, we rewrite the system into an equivalent one to obtain its solutions. Remind that we have set

$$b_j(a_1,a_2) = (\sqrt{h_1}a_1)^{\lambda_1-j}(\sqrt{h_2}a_2)^{\lambda_2+j}e^{-2\pi(h_1a_1^2+h_2a_2^2)}c_j(a_1,a_2)$$

for $0 \le j \le d$, and $\rho = \chi(Y_\eta)\sqrt{h_1h_2}$. Then we have the following equations for the set of $c_j(a)$:

$$\text{(11.1;j)} \qquad \Big(\frac{D}{h_2a_2^2}\partial_1 + j\Big)c_{j-1}(a) + \rho c_j(a) - (d-j)c_{j+1}(a) = 0$$

for $0 \le j \le d-1$,

$$\text{(11.2;j)} \qquad j\ c_{j-1}(a) + \rho c_j(a) + \Big(\frac{D}{h_1a_1^2}\partial_2 - (d-j)\Big)c_{j+1}(a) = 0$$

for $1 \le j \le d-1$, and

$$\text{(11.3; j)} \qquad \begin{aligned} &h_1a_1^2\big(\partial_1+\partial_2-8\pi h_2a_2^2+2\lambda_2-2\big)c_{j-1}(a)\\ &\quad + h_2a_2^2\big(\partial_1+\partial_2-8\pi h_1a_1^2+2\lambda_2-2\big)c_{j+1}(a) = 0\end{aligned}$$

for $1 \le j \le d-1$.

The equations (11.3; j) are obtained from (10.4; j), (10.5; j), and (10.6; j). The equations (11.1; j) and (11.2; j) rewrite (10.4; j) and (10.6; j), and they also yield

$$h_1a_1^2\partial_1c_{j-1}(a) - h_2a_2^2\partial_2c_{j+1}(a) = 0$$

for $1 \le j \le d-1$. Combining these and (11.3; j), we get

$$\text{(11.4;j)} \qquad \begin{aligned} &\big\{(\partial_1+\partial_2)^2 + 2(\lambda_2-2)(\partial_1+\partial_2) - 8\pi h_1a_1^2\partial_1\\ &\qquad - 8\pi h_2a_2^2\partial_2 - 4(\lambda_2-1)\big\}c_j(a) = 0\end{aligned}$$

for $0 \le j \le d$.

11.2. *A formal power series solution.* We introduce the following pair of variables

$$x = \frac{2\pi(h_1a_1^2 - h_2a_2^2)}{4\pi^2h_1a_1^2h_2a_2^2}, \qquad y = \frac{2\pi(h_1a_1^2 + h_2a_2^2)}{4\pi^2h_1a_1^2h_2a_2^2}.$$

We consider a holomorphic formal power series solution at $\{x=0\}$, which is a singular locus of the equations. We denote it by $c_j(a_1,a_2)=c_j(x,y)=\sum_{m=m_0^j}^{\infty} p_m^j(y)x^m$ with $m_0^j \geq 0$ for $0 \leq j \leq d$. The equations (11.1; j) and (11.2; j) lead to the following recurrence differential equations for $p_m^j(y)$'s:

$$(11.5;\ \mathrm{j})\qquad \frac{d}{dy}\big(p_{m-1}^{j-1}(y)+p_{m-1}^{j+1}(y)\big)=(m+j)p_m^{j-1}(y)+\rho\, p_m^j(y) - (m+d-j)p_m^{j+1}(y)$$

for $1 \leq j \leq d-1$,

$$(11.6;\ \mathrm{j})\qquad \frac{d}{dy}\big(p_{m-1}^{j-1}(y)-p_{m-1}^{j+1}(y)\big)=m\big(p_m^{j-1}(y)+p_m^{j+1}(y)\big)$$

for $1 \leq j \leq d-1$, and

$$(11.7)\qquad \begin{aligned} 2\frac{d}{dy}p_{m-1}^1(y) &= \rho\, p_m^0(y)-(2m+d)p_m^1(y),\\ 2\frac{d}{dy}p_{m-1}^{d-1}(y) &= (2m+d)p_m^{d-1}(y)+\rho\, p_m^d(y). \end{aligned}$$

Let m_0^j be the degree in x of the first non-vanishing term of $c_j(x,y)$; $p_m^j(y)=0$, if $m<m_0^j$, and $p_{m_0^j}(y)\neq 0$. Then we obtain a lemma.

11.3 LEMMA. *The degrees m_0^j coincide with each other for all j, $0 \leq j \leq d$. Denote the common value by m_0. If $m_0>0$, then there should be a relation that $\rho^2=-(d+2m_0)^2$. If $m_0=0$, then it should be that $\rho^2=-(d-2k)^2$ with $0 \leq k \leq \left[\frac{d}{2}\right]$. Moreover, if $m_0 \geq 0$ and $\rho=(d+2m_0)\sqrt{-1}$ (resp. $-(d+2m_0)\sqrt{-1}$), then $p_{m_0}^j(y)$ are given by*

$$p_{m_0}^j(y)=(\sqrt{-1})^j P_{m_0}(y)\quad (\mathrm{resp.}\ (\sqrt{-1})^{d-j}P_{m_0}(y))$$

with a nonzero function $P_{m_0}(y)$ that is independent of j.

Proof. Denote by m_0 the smallest number in the set of integers $\{m_0^j\}_{0\le j\le d}$. Let M_m be the matrix

$$M_m = \begin{pmatrix} \rho & -(2m+d) & 0 & & \mathbf{0} \\ m+1 & \rho & -(m+d-1) & & \\ \ddots & \ddots & \ddots & \ddots & \\ & & m+d-1 & \rho & -(m+1) \\ \mathbf{0} & & 0 & 2m+d & \rho \end{pmatrix}.$$

Then the equations (11.5; j) and (11.7) read

$$M_{m_0}\cdot {}^t\big(p^0_{m_0}(y),\cdots\cdots,p^d_{m_0}(y)\big) = {}^t\big(0,\cdots\cdots,0\big). \tag{11.8}$$

Further if $m_0\neq 0$, then (11.6; j) for $1\le j\le d-1$ tell us

$$p^{j-1}_{m_0}(y)+p^{j+1}_{m_0}(y)=0, \qquad 1\le j\le d-1. \tag{11.9; j}$$

From (11.8) we can conclude that neither $p^0_{m_0}(y)$ nor $p^d_{m_0}(y)$ vanishes, otherwise all $p^j_{m_0}(y)$ become zero, which contradicts the definition of m_0. So we obtain that $m_0^0=m_0^d=m_0$. Then we can also see easily that all the other m_0^j must be equal to m_0.

Using the first row of (11.8): $\rho p^0_{m_0}(y)=(2m_0+d)p^1_{m_0}$, and (11.9; j) for $j=1$, we can write that $p^0_{m_0}(y)=(d+2m_0)f(y)$, $p^1_{m_0}(y)=\rho f(y)$, and $p^2_{m_0}(y)=-(d+2m_0)f(y)$ with a function $f(y)\neq 0$. Then the second row of (11.8) gives $(\rho^2+(d+2m_0)^2)f(y)=0$. Hence $\rho^2=-(d+2m_0)^2$. On the other hand, if $\rho=(d+2m_0)\sqrt{-1}$ (resp. $-(d+2m_0)\sqrt{-1}$), then the vector ${}^t\big(1,\sqrt{-1},\ldots,(\sqrt{-1})^j,\ldots,(\sqrt{-1})^d\big)$ (resp. ${}^t\big((\sqrt{-1})^d,\ldots,(\sqrt{-1})^{d-j},\ldots,1)\big)$ generates the one dimensional kernel of M_{m_0}. Hence we get $p^j_{m_0}(y)=(\sqrt{-1})^jP_{m_0}(y)$ or $(\sqrt{-1})^{d-j}P_{m_0}(y)$, where $P_{m_0}(y)$ is independent of j.

We consider the case $m_0=0$. Then the kernel of the matrix $M_0=M_{m_0}$ becomes non-trivial if and only if $\rho^2=-(d-2k)^2$, $0\le k\le\left[\frac{d}{2}\right]$. If one of these is satisfied, then (11.8) has a non-trivial solution. □

Remark. If $m_0>0$, then $\rho=\chi(Y_\eta)\sqrt{h_1h_2}$ can not be equal to zero, and Proposition 10.7 is applicable to this case.

11.4. In the next place, we determine the function $P_{m_0}(y)$ appearing in the above lemma. The equations (11.4; j) give the following equations of second degree for $p^j_m(y)$ with general degree m:

$$\begin{aligned}(11.10)\qquad &\Big\{\Big(y\frac{d}{dy}\Big)^2 + (2m-\lambda_2+2)y\frac{d}{dy} \\ &\quad + 4\frac{d}{dy} + (m+1)(m+1-\lambda_2)\Big\}p^j_m(y) = 0\end{aligned}$$

for $0 \le j \le d$. Each of them has a solution

$$p^j_m(y) = C^j_m \times \frac{(-1)^{m-m_0} m!(m-\lambda_2)!}{m_0!(m_0-\lambda_2)!} e^{\frac{2}{y}} y^{-\frac{2m-\lambda_2+1}{2}} W_{-\frac{2m-\lambda_2+1}{2}, -\frac{\lambda_2}{2}}\big(\tfrac{4}{y}\big),$$

with a constant C^j_m. We note that these solutions take finite limits, when a_1 and a_2 tend infinity, and that any solution with this property is given by a constant multiple of the above. In particular we have

$$P_{m_0}(y) = e^{\frac{2}{y}} y^{-\frac{2m_0-\lambda_2+1}{2}} W_{-\frac{2m_0-\lambda_2+1}{2}, -\frac{\lambda_2}{2}}\big(\tfrac{4}{y}\big),$$

and, when $\chi(Y_\eta) \neq 0$, $C^j_{m_0} = \big(\sqrt{-1}\big)^j$ or $\big(\sqrt{-1}\big)^{d-j}$.

We should determine C^j_m for $0 \le j \le d$ and all integers $m \ge m_0$. By the formula in the last line of [M-O-S] p301, the recurrence relations among the set (C^j_m) are obtained from (11.5; j), (11.6; j), and (11.7) as

$$\begin{aligned} &2C^{j-1}_{m-1} = (2m+j)C^{j-1}_m + \rho\, C^j_m - (d-j)C^{j+1}_m \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \le j \le d, \\ (11.11)\qquad &2C^{j+1}_{m-1} = j\, C^{j-1}_m + \rho\, C^j_m - (2m+d-j)C^{j+1}_m \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \le j \le d-1. \end{aligned}$$

We note $C^j_{m_0-1} = 0$ for all $0 \le j \le d$ by the definition of m_0. Put $\mathbf{c}_m = {}^t\big(C^j_m\big)_{0\le j\le d}$. Starting from the vector $\mathbf{c}_{m_0}$ satisfying (11.11) for $m = m_0$, we can determine recurrently the $\mathbf{c}_m$ for $m \ge m_0$ by (11.11). Hence it yields a formal power series solution $\big(c_j(x,y)\big)$ of (11.1; j), (11.2; j), and (11.3; j), then also a formal solution $b_j(x,y) = (\sqrt{h_1}a_1)^{\lambda_1-j}(\sqrt{h_2a_2})^{\lambda_2+j}$.

$e^{-2\pi(h_1a_1^2+h_2a_2^2)}c_j(x,y)$ of the equations in Proposition 10.2. At this moment we obtain the following:

11.5 THEOREM. *Consider the space of solutions for the system of differential equations in Proposition 10.2. Then the dimension of the solutions satisfying the following conditions* (a), (b), *is less than or equal to one;* (a) *they decay rapidly, when* a_1^2 *and* a_2^2 *tend to* $+\infty$, *and* (b) *they are holomorphic at* $x=0$ *in the variable* x.

We make the following conjecture:

11.6 CONJECTURE. *There exists a unique (up to a constant multiple) non-zero generalized Whittaker realization of a large discrete series representation with an* $\eta \in \widehat{N}_s$ *definite, whose restriction to the minimal* K*-type corresponds to* K*-finite functions satisfying the conditions* (a), (b) *above. Equivalently, the formal solution defined by the recurrence relations in* (11.11) *converges actually on* A *and determine the solution with properties* (a) *and* (b).

Convergences of the formal solution are established when (i) $\rho = (4 + 2m_0)\sqrt{-1}$ with $m_0 \geq 0$ for $d=4$, (ii) $\rho = (5+2m_0)\sqrt{-1}$ with $m_0 \geq 0$ for $d=5$, and (iii) $\rho = 0$ with $m_0 = 0$ for any even d, hence we have

11.7 PROPOSITION. *The conjecture 11.6 is true in the cases* (i), (ii), (iii) *above.*

We will show these cases more precisely bellow.

(i) Consider the large discrete series with $d=4$. It exists uniquely, and its Blattner parameter is given by $(\lambda_1,\lambda_2)=(3,-1)$. Set $\rho=(4+2m_0)\sqrt{-1}$ with $m_0 \geq 0$. Then $C^j_{m_0} = \left(\sqrt{-1}\right)^j$ and the general $\mathbf{c}_m$ are explicitly determined by (11.11) as

$$\mathbf{c}_{m_0+2n} = \begin{pmatrix} \frac{m_0!(m_0^2+(8n+3)m_0+8n^2+14n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{\sqrt{-1}m_0!(m_0+2)(m_0+2n+1)}{2^{2n}(m_0+2+n)!n!} \\ \frac{-m_0!(m_0^2+3m_0-2n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{-\sqrt{-1}m_0!(m_0+2)(m_0+2n+1)}{2^{2n}(m_0+2+n)!n!} \\ \frac{m_0!(m_0^2+(8n+3)m_0+8n^2+14n+2)}{2^{2n}(m_0+2+n)!n!} \end{pmatrix},$$

$$\mathbf{c}_{m_0+2n+1} = \begin{pmatrix} \frac{m_0!(m_0+2n+2)}{2^{2n-1}(m_0+2+n)!n!} \\ \frac{\sqrt{-1}m_0!(m_0+2)}{2^{2n}(m_0+2+n)!n!} \\ 0 \\ \frac{\sqrt{-1}m_0!(m_0+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{-m_0!(m_0+2n+2)}{2^{2n-1}(m_0+2+n)!n!} \end{pmatrix},$$

where n runs over the all non-negative integers. The proof is given by a direct inductive calculation, which we omit. We have also that the following integral formulas express the formal power series solution,

$$b_j(a) = \frac{4(\sqrt{h_1}a_1)^{\lambda_1-j}(\sqrt{h_2a_2})^{\lambda_2+j}e^{-2\pi(h_1a_1^2+h_2a_2^2)}}{(m_0+1)!} \times \int_0^\infty \sqrt{t}\ K_1\big(4\sqrt{t}\big)R_j(xt)e^{-yt}dt$$

for $0 \leq j \leq 4$, where

$$R_0(z) = 8I_{m_0}(z) - 2\big(4 + \frac{2(4m_0+5)}{z}\big)I_{m_0+1}(z) + (m_0+2)\big(\frac{8}{z} + \frac{4(m_0+3)}{z^2}\big)I_{m_0+2}(z),$$

$$R_1(z) = \sqrt{-1}\big\{\big(\frac{4(m_0+2)}{z}\big)I_{m_0+1}(z) - (m_0+2)\big(\frac{4}{z} + \frac{4(m_0+3)}{z^2}\big)I_{m_0+2}(z)\big\},$$

$$R_2(z) = \big(\frac{4}{z}\big)I_{m_0+1}(z) + (m_0+2)\big(\frac{4}{z} - \frac{4(m_0+3)}{z^2}\big)I_{m_0+2}(z),$$

$$R_3(z) = -\sqrt{-1}\big\{\big(\frac{4(m_0+2)}{z}\big)I_{m_0+1}(z) + (m_0+2)\big(\frac{4}{z} - \frac{4(m_0+3)}{z^2}\big)I_{m_0+2}(z)\big\},$$

$$R_4(z) = 8I_{m_0}(z) + 2\big(4 - \frac{2(4m_0+5)}{z}\big)I_{m_0+1}(z) - (m_0+2)\big(\frac{8}{z} - \frac{4(m_0+3)}{z^2}\big)I_{m_0+2}(z),$$

and $K_\nu(z)$ and $I_\nu(z)$ are the modified Bessel functions [M-O-S] p.66. To obtain these we used the formula (37) in [ET], p.199.
(ii) If $d = 5$ and $\rho = (5+2m_0)\sqrt{-1}$ with $m_0 \geq 0$, we obtain that

$$\mathbf{c}_{m_0+2n} = \begin{pmatrix} \frac{m_0!(m_0^2+(12n+3)m_0+16n^2+18n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{\sqrt{-1}m_0!(m_0^2+(4n+3)m_0+10n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{-m_0!(m_0^2+3m_0-2n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{-\sqrt{-1}m_0!(m_0^2+3m_0-2n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{m_0!(m_0^2+(4n+3)m_0+10n+2)}{2^{2n}(m_0+2+n)!n!} \\ \frac{\sqrt{-1}m_0!(m_0^2+(12n+3)m_0+16n^2+18n+2)}{2^{2n}(m_0+2+n)!n!} \end{pmatrix},$$

$$\mathbf{c}_{m_0+2n+1} = \begin{pmatrix} \frac{m_0!(5m_0^2+5(4n+5)m_0+16n^2+54n+30)}{2^{2n+1}(m_0+3+n)!n!} \\ \frac{\sqrt{-1}m_0!(3m_0^2+(4n+15)m_0+10n+18)}{2^{2n+1}(m_0+3+n)!n!} \\ \frac{-m_0!(m_0^2+5m_0-2n+6)}{2^{2n+1}(m_0+3+n)!n!} \\ \frac{\sqrt{-1}m_0!(m_0^2+5m_0-2n+6)}{2^{2n+1}(m_0+3+n)!n!} \\ \frac{-m_0!(3m_0^2+(4n+15)m_0+10n+18)}{2^{2n+1}(m_0+3+n)!n!} \\ \frac{-\sqrt{-1}m_0!(5m_0^2+5(4n+5)m_0+16n^2+54n+30)}{2^{2n+1}(m_0+3+n)!n!} \end{pmatrix},$$

with non-negative integers n. This yields the similar solution with an integral expression as in the case of $d = 4$. For greater d, the author has not obtained the explicit formula for $\mathbf{c}_m$. Each coefficient of $\mathbf{c}_m$ may have the form

$$\text{a constant} \times \frac{\text{a polynomial in } n \text{ of degree } \left[\frac{d+1}{2}\right]\ (\pm 1)}{2^{2n\ (+1)}\left(m_0 + \left[\frac{d+1}{2}\right] + n\ (\pm 1)\right)!\ n!},$$

for $m = m_0 + 2n$, or $m_0 + 2n + 1$.
(iii) In the case $\rho = 0$ with $m_0 = 0$ for any even d, we obtain $p_m^1(y) = p_m^{d-1}(y) = 0$ for all $m \geq 0$ from the equations (11.7) by an induction on m. Also we can obtain that $p_m^{2k+1}(y) = 0$ for $0 \leq k \leq \frac{d}{2} - 1$ and $m \geq 0$, by (11.5; j) and (11.6; j), hence $c_{2k+1}(y) = 0$ for $0 \leq k \leq \frac{d}{2} - 1$. Although this case, indeed $\chi(Y_\eta) = 0$, was not treated in Proposition 10.7 (Corollary 10.4 in particular), we can conclude again the holonomicity of the system of equations in this case under this observation. Then we obtain the following

formula:

$$C_{2n}^{\ell-1} = \frac{(\frac{\ell-1}{2})!}{2^{2n}(\frac{\ell-1}{2}+n)!\ n!}, \quad C_{2n+1}^{\ell-1} = \frac{(\frac{\ell-1}{2})!}{2^{2n+1}(\frac{\ell+1}{2}+n)!\ n!};$$

$$C_{2n}^{\ell+1} = \frac{(\frac{\ell-1}{2})!}{2^{2n}(\frac{\ell-1}{2}+n)!\ n!}, \quad C_{2n+1}^{\ell+1} = -\frac{(\frac{\ell-1}{2})!}{2^{2n+1}(\frac{\ell+1}{2}+n)!\ n!},$$

if $\ell = \frac{d}{2}$ is odd, or

$$C_{2n}^{\ell-2} = \frac{(\frac{\ell}{2})!\ (\ell+1+4n)}{2^{2n}(\frac{\ell}{2}+n)!\ n!}, \quad C_{2n+1}^{\ell-2} = \frac{4(\frac{\ell}{2})!}{2^{2n+1}(\frac{\ell}{2}+n)!\ n!};$$

$$C_{2n}^{\ell}(t) = \frac{(\frac{\ell}{2})!\ (\ell-1)}{2^{2n}(\frac{\ell}{2}+n)!\ n!}, \quad C_{2n+1}^{\ell}(t) = 0;$$

$$C_{2n}^{\ell+2} = \frac{(\frac{\ell}{2})!\ (\ell+1+4n)}{2^{2n}(\frac{\ell}{2}+n)!\ n!}, \quad C_{2n+1}^{\ell+2} = -\frac{4(\frac{\ell}{2})!}{2^{2n+1}(\frac{\ell}{2}+n)!\ n!},$$

if $\ell = \frac{d}{2}$ is even. In the both cases, n runs over all the non-negative integers. The other C_m^{2k} for $0 \le k \le \frac{d}{2}$ are given by the equations (11.11). They are written in an integral formula like in the case (i), thus we obtain that the conjecture holds in these cases.

At the end of this section, we give a formula of the Mellin transform of the solution obtained above, which will be used in the next section.

11.8 THEOREM. (The Mellin transform) *Take the character η of N_s with $h_3 = 0$, and $h_1 = h_2 = h$ positive. Take $\rho = \chi(Y_\eta)h = (d+2m_0)\sqrt{-1}$ with a positive integer m_0. Let $\phi(a_1, a_2) = \sum_{j=0}^{d} b_j(a) v_j^{-\lambda_2,-\lambda_1}$ be the formal power series solution given in 11.4 for the large discrete series representation $\pi_{\lambda_1,\lambda_2}$. We assume that it converges globally on A, Conjecture 11.6, for the general cases. Then*

$$b_j(a^{\frac{1}{2}}, a^{\frac{1}{2}}) = \begin{cases} (\sqrt{-1})^j \pi^{\frac{-\lambda_2+1}{2}} (ha)^{\frac{\lambda_1+1}{2}} e^{-2\pi ha} W_{\frac{\lambda_2-1}{2}, -\frac{\lambda_2}{2}}(4\pi ha), & \text{if } m_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then its Mellin transform is given by

$$\int_0^\infty b_j(a^{\frac{1}{2}}, a^{\frac{1}{2}}) a^{s-\frac{3}{2}} \frac{da}{a} = \frac{\beta(\sqrt{-1})^j \Gamma(s+\frac{\lambda_1-1}{2}+\frac{\lambda_2}{2})\Gamma(s+\frac{\lambda_1-1}{2}-\frac{\lambda_2}{2})}{(4\pi h)^s \Gamma(s+\frac{\lambda_1-\lambda_2+1}{2})},$$

for $\mathrm{Re}(s+\frac{\lambda_1-1}{2}\pm\frac{\lambda_2}{2})>0$*, where* $\beta = 4^{1-\frac{\lambda_1}{2}}\pi^{\frac{3-\lambda_1-\lambda_2}{2}}h^{\frac{3}{2}}$*.*

Proof. This can be obtained using the formula in [M-O-S] p.316, line 3. □

Here we remark that $(\lambda_1-1, \lambda_2) = (\Lambda_1, \Lambda_2)$ is the Harish-Chandra parameter of the large discrete series representation with the Blattner parameter (λ_1, λ_2).

12. The Rankin-Selberg Integral and the Andrianov's *L*-function

12.1. The Andrianov's L-function was studied by Novodvorsky, Piatetski-Shapiro, Soudry by representation theoretical methods. We refer to [PS]. Here, according to an investigation by Sugano [Su], we present some applications of our archimedean results. Styles of necessary discussions, or proofs, are essentially given in [Su]. The only difference between [Su] and our treatments is concentrated on the objects at the real archimedean prime. We treat the large discrete series representations and their generalized Whittaker functions, whereas the (anti) holomorphic discrete series were considered in [Su].

Let $G_0 = GSp(2)$ be the symplectic algebraic group of degree 2 over the rational numbers. So we take $B = M_2(\mathbb{Q})$ in the paper [Su]. Let $K_0 = \prod_{v\le\infty} K_v$, $K_p = G_0(\mathbb{Q}_p)\cap GL_4(\mathbb{Z}_p)$ for p finite prime, and $K_\infty \simeq U(2)$. Set $K_f = \prod_{p<\infty} K_p$.

Consider the Siegel maximal parabolic subgroup of $G_0(\mathbb{Q}_A)$ and the unipotent radical $N_s(\mathbb{Q}_A)$. We define a character $\eta = \eta_H$ of $N_s(\mathbb{Q}_A)$ by $\eta(n(T)) = \widetilde{\eta}(\mathrm{tr}(HT))$ for $n(T) = \begin{pmatrix} 1_2 & T \\ 0 & 1_2 \end{pmatrix} \in N_s(\mathbb{Q}_A)$. Here $\widetilde{\eta}$ is a character on $\mathbb{Q}_A$ that is trivial on $\mathbb{Q}$ such that $\widetilde{\eta}_\infty(x) = e^{2\pi\sqrt{-1}x}$, and $H = \begin{pmatrix} 1 & h_3/2 \\ h_3/2 & h \end{pmatrix}$ is a primitive positive definite matrix with h, $h_3 \in \mathbb{Z}$.

Given H, we define the imaginary quadratic field $E = \mathbb{Q}(\sqrt{D})$ where $D = h_3^2 - 4h = d_E f^2$. Let $O(f)$ be the order of E with conductor f. Let

$\{1, \omega\}$ be a $\mathbb{Z}$-basis of $O(f)$. We can embed E into $G_0(\mathbb{Q})$; for $\beta = u + \omega v \in E$, define its image by

$$\begin{pmatrix} \iota_H(\beta) & 0 \\ 0 & N(\beta)^t\iota_H(\beta)^{-1} \end{pmatrix} \in G_0(\mathbb{Q})$$

with $\iota_H(\beta) = \begin{pmatrix} u & -hv \\ v & u+h_3v \end{pmatrix}$.

Denote by $G_0(\mathbb{R})^+$ the identity component of $G_0(\mathbb{R})$. Let $F(g)$ be a cusp form on $G_0(\mathbb{Q}_A)^+ = G_0(\mathbb{R})^+G_0(\mathbb{Q}_{A_f})$ of full level. We suppose that the central character of $F(g)$ is given by an unramified character $\lambda = \otimes_v \lambda_v$ on $\mathbb{Q}_A^\times$ such that $\lambda_\infty \equiv 1$. A large discrete series representation of $Sp(2, \mathbb{R})$ with the minimal K-type of odd dimension can be extended to a representation of $G_0(\mathbb{R})^+$ with trivial central character. We suppose that the real archimedean part of F belongs to the minimal K-type $\tau_{\lambda_1,\lambda_2}$ of a large discrete series extended as above, if $d = \lambda_1 - \lambda_2$ is even. Also extend F to $G_0(\mathbb{Q}_A)$, by a decomposition $G_0(\mathbb{Q}_A) = G_0(\mathbb{Q})G_0(\mathbb{R})^+K_f$, trivially on $G_0(\mathbb{Q})$ and K_f. Then we define a function on $G_0(\mathbb{Q}_A)$

$$F_\eta(g) = \int_{Sym_2(\mathbb{Q})\backslash Sym_2(\mathbb{Q}_A)} \eta(n(T))^{-1}F(n(T)g)dT.$$

We also take a non-trivial primitive character χ on $E_A^\times$ of conductor f, which is trivial on $E^\times \cdot \prod_{p<\infty} O(f)_p^\times$, $O(f)_p = O(f) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\chi|_{\mathbb{Q}_A^\times} \equiv \lambda$. Then we define the generalized Whittaker function associated with $F(g)$:

$$W_{\chi\cdot\eta}^F(g) = \int_{E^\times\mathbb{Q}_A^\times\backslash E_A^\times} \chi(\beta)^{-1}F_\eta(\beta g)d^\times\beta.$$

Further let $h_3 = 0$ for simplicity and $F(g)$ be a normalized Hecke common eigenfunction, [Su], (2-24) and p.546. Here we make an indispensable assumption.

ASSUMPTION. *We suppose that there exist a suitable pair of η and χ such that the global integral transform $W_{\chi\cdot\eta}^F(g)$ defined above does not vanish.*

We fix a suitable pair χ and η for the assumption in the following discussion. For $s \in \mathbb{C}$, define a function

$$A_{\chi\cdot\eta}(F,s) = \int_{\mathbb{Q}_A^\times} W^F_{\chi\cdot\eta}\left(\begin{pmatrix} t1_2 & 0 \\ 0 & 1_2 \end{pmatrix} R_\eta\right)|t|_A^{s-3/2} d^\times t,$$

with $R_{\eta,\infty} = \mathrm{diag}(\sqrt{|D|}, 2, 2, \sqrt{|D|})$, $R_{\eta,p} = 1_4$ for the finite primes. A version of [Su], Theorem 3-1 is described in our case.

12.2 Theorem. *If η and χ are suitable ones under the assumptions, we have*

$$A_{\chi\cdot\eta}(F,s) = C_0 \times \frac{\Gamma\big(s+\frac{\lambda_1-1}{2}+\frac{\lambda_2}{2}\big)\Gamma\big(s+\frac{\lambda_1-1}{2}-\frac{\lambda_2}{2}\big)}{\big(2\pi\sqrt{|D|}\big)^s\,\Gamma\big(s+\frac{\lambda_1-\lambda_2+1}{2}\big)} \times \frac{L(F,s)}{L(\overline{\chi}, s+\frac{1}{2})} W^F_{\chi\cdot\eta}(R_\eta), \tag{12.1}$$

with $C_0 = e^{\pi\sqrt{|D|}}\big(2\pi\sqrt{|D|}\big)^{-\frac{\lambda_1+1}{2}} W_{\frac{\lambda_2-1}{2},-\frac{\lambda_2}{2}}(2\pi\sqrt{|D|})^{-1}$. *Here $L(F,s)$ is the Andrianov's L-function,* [Su], p.547 (3-4), *$L(\overline{\chi},s)$ is the Hecke L-function for the grössencharacter $\overline{\chi}(z) := \chi(\overline{z})$ for $z \in E_A^\times$ with the canonical involution.*

Proof. Calculation at the non-archimedean places is exactly the same as in [Su]. We only have to replace the function, [Su], p.549, line 1, at the real place with our formula for the generalized Whittaker function, which was studied in Section 11. Then the calculation is similar. □

Define

$$\zeta(F,s) = (2\pi)^{-2s}\Gamma(s+\tfrac{\lambda_1-1}{2}+\tfrac{\lambda_2}{2})\Gamma(s+\tfrac{\lambda_1-1}{2}-\tfrac{\lambda_2}{2})L(F,s),$$

and

$$\begin{aligned} B_{\chi\cdot\eta}(F,s) &= (2\pi)^{s-\frac{\lambda_1-\lambda_2-1}{2}}\Gamma(s+\tfrac{\lambda_1-\lambda_2+1}{2})L(\overline{\chi},s+\tfrac{1}{2})A_{\chi\cdot\eta}(F,s) \\ &= C_1 \times \big(\sqrt{|D|}\big)^{-s}\zeta(F,s) \times W^F_{\chi\cdot\eta}(R_\eta), \end{aligned}$$

where $C_1 = (2\pi)^{-\frac{\lambda_2}{2}-1}\big(\sqrt{|D|}\big)^{-\frac{\lambda_1+1}{2}} e^{\pi\sqrt{|D|}} W_{\frac{\lambda_2-1}{2},-\frac{\lambda_2}{2}}(2\pi\sqrt{|D|})^{-1}$ is a non-zero constant which does not depend on s.

12.3. *Eisenstein series and Rankin-Selberg convolution.* Define G_1 an algebraic group over the rational numbers and its rational subgroup B_1 by

$$G_1(\mathbb{Q}) = \{g \in GL_2(E) \mid \det g \in \mathbb{Q}^\times\},$$
$$B_1(\mathbb{Q}) = \left\{\begin{pmatrix} t\beta & * \\ 0 & \overline{\beta}\end{pmatrix} \mid t \in \mathbb{Q}^\times,\ \beta \in E^\times\right\}.$$

Also fix compact subgroups at each place as $M_p = G_1(\mathbb{Q}_p) \cap GL_2(O(f)_p)$, $p < \infty$, and $M_\infty \simeq SU(2)$. We define a representation $(\widetilde{\tau}, V_{\widetilde{\tau}})$ of M_∞ by the restriction $(\tau_{\lambda_1,\lambda_2}|_{R_{\eta,\infty}^{-1}\psi_H(M_\infty)R_{\eta,\infty}}, V_{\tau_{\lambda_1,\lambda_2}})$. Here we fix an embedding ψ_H of G_1 into G_0

$$\psi_H\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e_0^{-1} \end{pmatrix}\begin{pmatrix} \iota_H(\alpha) & \iota_H(\beta) \\ \iota_H(\gamma) & \iota_H(\delta) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e_0 \end{pmatrix},$$
$$e_0 = \begin{pmatrix} h_3 & -1 \\ -1 & 0 \end{pmatrix}.$$

We shall define an Eisenstein series [Su], §3-3, on $G_1(\mathbb{Q}_A)$. Let $W = E \oplus E$ be a $\mathbb{Q}$-vector space. Define $V_{\widetilde{\tau}}$-valued Schwartz-Bruhat function φ on W_A, $\varphi = \prod_{v\leq\infty}\varphi_v$, by $\varphi_\infty(t(0,1)m_\infty) = t^d e^{-2\pi t^2}\widetilde{\tau}(m_\infty)$, $t \geq 0$, $m_\infty \in M_\infty$, and φ_p is the characteristic function of $O(f)_p \oplus O(f)_p$. For the above φ and $g_1 \in G_1(\mathbb{Q}_A)$, put

$$L_\varphi^{\overline{\chi}}(g_1, s) = |\det g_1|^{s+1/2}\int_{E_A^\times}\overline{\chi}(t)|t\overline{t}|^{s+1/2}\varphi(t(0,1)g_1)d^\times t.$$

Put

$$E_\varphi^{\overline{\chi}}(g_1, s) = \sum_{\gamma\in B_1(\mathbb{Q})\backslash G_1(\mathbb{Q})} L_\varphi^{\overline{\chi}}(\gamma g_1, s).$$

Then the proof of [Su], Lemma 3-2 also gives us

$$B_{\chi\cdot\eta}(F, s) = \int_{G_1(\mathbb{Q})\mathbb{Q}_A^\times\backslash G_1(\mathbb{Q}_A)} E_\varphi^{\overline{\chi}}(g_1, s)F(\psi_H(g_1)R_\eta)dg_1. \tag{12.2}$$

The Eisenstein series is meromorphically continued to the s-plane. Moreover it is entire in this case, since $d = \lambda_1 - \lambda_2$ is strictly positive, [Su], p.559, and Theorem 3-2. Hence $B_{\chi\cdot\eta}(F,s)$, then also $\zeta(F,s)$ are meromorphically continued to the whole plane, and, moreover, determine entire functions in s.

Sugano also studies the Fourier transform $\widehat{\varphi}$ of the Schwartz-Bruhat function φ, [Su], Lemma 3-3, then he obtains the functional equation for $\zeta(F,s)$, [Su], Theorem 3-2. His calculation can be applied to our case in the same way. Then the result is

12.4 Theorem. *Let $\zeta(F,s)$ be the one defined above. Then it is an entire function in the variable s. Moreover it has the following functional equation:*

$$\zeta(F,s) = (-1)^{\lambda_2}\zeta(F',1-s).$$

Here $F'(g) = \lambda^{-1}(m(g))F(g)$, and $m(g)$ is the similitude of $g \in G_0$.

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(Received July 30, 1998)
(Revised August 6, 1999)

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