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Zeta Functions of Finite Graphs

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Abstract. Poles of the *Ihara zeta function* associated with a finite graph are described by graph-theoretic quantities. Elementary proofs based on the notions of *oriented line graphs*, *Perron-Frobenius operators*, and *discrete Laplacians* are provided for Bass's theorem on the determinant expression of the zeta function and Hashimoto's theorems on the pole at $u = 1$.

1. Introduction

We shall start with fixing the terminology required to state our results. Let $X = (V, E)$ be a finite connected graph with a set V of vertices and a set E of oriented edges. We allow X to have loop edges and multiple edges. We denote by $o(e)$ (resp. $t(e)$) the *origin* (resp. *terminus*) of an edge $e \in E$, and by $\overline{e}$ the *inverse edge*. The *adjacency operator* $\mathcal{A}$ is an operator acting on the space $C(V)$ of functions on V defined by

$$(\mathcal{A}f)(x) = \sum_{e \in E_x} f(t(e)),$$

where $E_x = \{e \in E | \ o(e) = x\}$. We write deg $x = \#E_x$, the *degree* of x. Throughout we assume that deg $x \geq 2$ for every $x \in V$.

A *closed path* in X is a sequence $c = (e_1, \dots, e_k)$ of edges with $t(e_i) = o(e_{i+1})$ $(i \in \mathbb{Z}/k\mathbb{Z})$. If $e_i \neq \overline{e}_{i+1}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$, c is called a *closed geodesic*. We may form the *m-multiple* c^m of a closed geodesic c by repeating c m-times. If c is not a m-multiple of a closed geodesic with $m \geq 2$, c is said to be *prime*. Two prime closed geodesics are said to be *equivalent* if one is obtained from another by a cyclic permutation of edges. An equivalence class of a prime closed geodesic is called a *prime cycle*. The *length* of a prime cycle $\mathfrak{p}$ is defined as the number of edges in a representative of $\mathfrak{p}$, and is denoted by $|\mathfrak{p}|$.

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The (*Ihara*) *zeta function* $Z(u) = Z(u, X)$ of X, the main object in this note, is defined by

$$Z(u) = \prod_{\mathfrak{p} \in P} (1 - u^{|\mathfrak{p}|})^{-1},$$

where P denotes the set of all prime cycles in X. We denote by α^{-1} the radius of convergence of $Z(u)$.

The primary purpose of this note is to give a short and conceptually simple proof for the following beautiful theorem.

THEOREM 1.1. (H. Bass [3])

$$Z(u) = (1 - u^2)^{\chi(X)} \det\Big(I - u\mathcal{A} + u^2(\mathcal{D} - I)\Big)^{-1},$$

where $\chi(X)$ *denotes the Euler number of* X *and* $\mathcal{D}$ *is the operator on* $C(V)$ *defined by* $(\mathcal{D}f)(x) = (\deg x)f(x)$.

It should be noted that $\chi(X) \leq 0$, and the equality holds if and only if X is a circuit graph, i.e. a graph with $\deg x = 2$ for every vertex x, or equivalently a graph homeomorphic to the circle. From the theorem above, we conclude that $Z(u)^{-1}$ is a polynomial of degree $\#E(= -2\chi(X) + 2\#V)$ whose leading coefficient is

$$(-1)^{\chi(X)} \prod_{x \in V} (\deg x - 1).$$

COROLLARY 1.2. *If* X *is a finite regular connected graph of degree* $q+1$ *with* $N = \#V$, *then* $\alpha = q$ *and*

$$Z(u) = (1 - u^2)^{(1-q)N/2} \det\Big(I - u\mathcal{A} + qu^2 I\Big)^{-1}. \tag{1}$$

The above corollary was originally established by Y. Ihara [12] in his study of a p-adic analogue of the Selberg zeta functions. His result is interpreted as above in terms of the regular graphs (of degree $p + 1$) associated with a co-compact discrete subgroup in the p-adic linear group $SL_2(\mathbb{Q}_p)$. In his proof, Ihara employed a combinatorial nature of the underlying Hecke algebra on X which we can not apply to the general case (see also [20], [21] and J. P. Serre [19]).

Our idea of the proof of Theorem 1.1 is to relate $Z(u)$ to the zeta function associated with an oriented graph (the *oriented line graph*) to which we easily give a determinant form by using the *Perron-Frobenius operator* (this is actually a well-known fact; see R. Bowen and O. Lanford [7]), and to use only elementary linear algebra to transform it to the desired form. Thus the proof given here is much simpler than that in H. Bass [3] where the notion of "non-commutative" determinant was used (it should be pointed out that his method applies also to "ramified" cases). In the course of our discussion, we observe that α coincides with the *maximal* positive eigenvalue of the Perron-Frobenius operator (the *Perron-Frobenius root*).

In the case of regular graphs, the determinant expression (1) for the zeta function allows us to locate the poles in terms of the eigenvalues of the adjacency operator $\mathcal{A}$; especially it is concluded that the real poles u satisfy $q^{-1} \leq |u| \leq 1$, and the imaginary poles are on the circle $\{u \in \mathbb{C};\ |u| = q^{-1/2}\}$. We also observe that $u = q^{-1}$ is a (simple) pole, and that $u = -q^{-1}$ is a pole if and only if X is *bipartite*. Incidentally, a regular graph is called a *Ramanujan graph* if the zeta function satisfies an analogue of "Riemann Hypothesis"; that is, its real poles are only those u with $|u| = 1$ or q^{-1}. The notion of Ramanujan graphs is related to a model of efficient communication networks (see F. Bien [5], P. Sarnak [18] and A. Lubotzky [16]).

In the general case, the eigenvalues of the adjacency operator are not enough to describe the poles of $Z(u)$ in an exact way because of the presence of the non-scalar operator $\mathcal{D}$. However we can establish the following weak result.

THEOREM 1.3. *Let X be a non-circuit graph and write*

$$d_m = \min_{x \in V} \deg x, \quad d_M = \max_{x \in V} \deg x.$$

(1) *$d_m - 1 \leq \alpha \leq d_M - 1$ and α^{-1} is a simple pole of $Z(u)$. Every pole u satisfies $\alpha^{-1} \leq |u| \leq 1$.*

(2) *Every imaginary pole u satisfies $(d_M - 1)^{-1/2} \leq |u| \leq (d_m - 1)^{-1/2}$.*

For a regular graph of degree $q + 1$, the poles u with $|u| = q^{-1}$ are $u = q^{-1}$ or $u = -q^{-1}$ (in bipartite case). The following theorem, which is proved by means of the Perron-Frobenius theorem, concerns the poles on the circle $\{u \in \mathbb{C};\ |u| = \alpha^{-1}\}$ in the general case.

THEOREM 1.4. *Let $\nu \geq 1$ be the greatest common divisor of $\{|\mathfrak{p}|\ ; \mathfrak{p} \in P\}$. Then the poles on the circle $\{u \in \mathbb{C};\ |u| = \alpha^{-1}\}$ are just $\alpha^{-1}e^{2\pi\sqrt{-1}k/\nu}$, $0 \leq k < \nu$.*

A graph X with $\nu \geq 3$ has a very special feature as is seen in the following theorem.

THEOREM 1.5. *If $\nu \geq 3$ and is odd, then X is the $(\nu - 1)$-subdivision of a non-bipartite graph. If $\nu \geq 3$ and is even, then X is the $(\nu/2 - 1)$-subdivision of a bipartite graph. In particular, if $d_m \geq 3$, then $\nu = 1$ (X being non-bipartite) or $\nu = 2$ (X being bipartite).*

In the above, the *k-subdivision* $Y^{(k)}$ of a graph Y is the graph obtained by adding k vertices on each edge of Y. It is easy to see that $Z(u, Y^{(k)}) = Z(u^k, Y)$ since there is a one-to-one correspondence $\mathfrak{p} \leftrightarrow \mathfrak{p}'$ between prime cycles $\mathfrak{p}$ in Y and $\mathfrak{p}'$ in $Y^{(k)}$ with $|\mathfrak{p}'| = k|\mathfrak{p}|$. The proof of Theorem 1.5 relies heavily on the notion of oriented line graphs.

In the case of regular graphs of degree $q+1$, the value $\alpha = q$ corresponds to the maximal eigenvalue $q+1$ of the adjacency operator $\mathcal{A}$. The following theorem characterizes α in the general case.

THEOREM 1.6. *Let $0 < u < 1$. There exists a positive-valued function $f \in C(V)$ such that $(I - u\mathcal{A} + u^2(\mathcal{D} - I))f = 0$ and $f(t(e)) - uf(o(e)) > 0$ for every $e \in E$ if and only if $u = \alpha^{-1}$.*

We shall also give an elementary proof for the following theorem due to K. Hashimoto [11] (compare our proof with the one by H. Bass [3]).

THEOREM 1.7. *If X is a non-circuit graph, then $u = 1$ is a pole of $Z(u)$ of order $n =$ rank $H_1(X)$(= the first betti number). Furthermore*

$$\lim_{u \to 1}(1-u)^{-n}Z(u)^{-1} = 2^n\chi(X)K(X),$$

where $K(X)$ is the complexity of X, the number of spannning trees in X.

This paper is a byproduct of our reserach project on *discrete spectral geometry* which is concerned with the spectra of *discrete Laplacians* on locally finite graphs.

2. Zeta Functions of Oriented Graphs

We first treat the zeta functions of finite *oriented* graphs which are much easier to handle than the zeta functions of unoriented graphs. As a matter of fact, what we shall explain here is more or less a reproduction of the result on symbolic dynamics by R. Bowen and O. Lanford [7] in terms of oriented graphs.

Let $X^o = (V, E^o)$ be an oriented finite graph which are supposed to be *strongly connected* in the sense that, for any $x, y \in V$, there exists an admissible path c with $o(c) = x$ and $t(c) = y$. Here a path $c = (e_1, \ldots, e_k)$ is said to be *admissible* if $e_i \in E^o$ for every i. We further put $o(c) = o(e_1)$ and $t(c) = t(e_k)$.

For $x \in V$, we let $\Lambda(x)$ be the set of admissible loops with the base point x. We denote by $\nu(x)$ the greatest common divisor of the set of integers $\{|c|; c \in \Lambda(x)\}$. The number $\nu(x)$ does not depend on the choice of x. We call the number $\nu(x)$ the *period* of X^o, which we denote by $\nu = \nu(X^o)$. When $\nu = 1$, the oriented graph X^o is called *primitive*.

In case $\nu > 1$, we may decompose V into disjoint subsets $\{V_i\}$ parametrized by $i \in \mathbb{Z}/\nu\mathbb{Z}$ such that, if $o(e) \in V_i$, then $t(e) \in V_{i+1}$. Moreover, if $o(c) \in V_i$ and $t(c) \in V_j$ for an admissible path c, then $|c| \equiv i - j$ (mod. ν).

For $m \geq 1$, we let N_m be the number of admissible closed paths in X^o with length m, and put

$$Z^o(u) = Z^o(u, X^o) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m\right).$$

We call $Z^o(u)$ the *zeta function* of X^o (the motivation of this definition comes obviously from the shape of the Weil zeta functions of projective algebraic varieties defined over finite fields; see [13] and [14]).

Rationality of the zeta function $Z^o(u)$ is easily deduced by using the *Perron-Frobenius operator* $\mathcal{L} : C(V) \longrightarrow C(V)$ defined by

$$(\mathcal{L}f)(x) = \sum_{e \in E^o_x} f(t(e)),$$

where $E^o_x = \{e \in E^o;\ o(e) = x\}$. We readily check that

$$(\mathcal{L}^n f)(x) = \sum_{c; |c|=n, o(c)=x} f(t(c)).$$

By the Perron-Frobenius theorem (see [8]), we have

LEMMA 2.1. *Let ν be the period of X^o.*

(1) *$\mathcal{L}$ has at least one positive eigenvalue. The maximal positive eigenvalue is simple (as a characteristic root) and has a positive-valued eigenfunction.*

(2) *$|\lambda| \leq \alpha$ for any eigenvalue λ of $\mathcal{L}$.*

(3) *$\min_{x\in V}(\mathcal{L}1)(x) \leq \alpha \leq \max_{x\in V}(\mathcal{L}1)(x)$.*

(4) *The eigenvalues λ with $|\lambda| = \alpha$ are just $\alpha e^{2\pi\sqrt{-1}k/\nu}$, $0 \leq k < \nu$.*

(5) *If $\mathcal{L}f = \lambda f$, $f \geq 0$, $f \not\equiv 0$, then $\lambda = \alpha$ and $f > 0$.*

It is easily checked that if X^o is a non-circuit graph, then $\alpha > 1$. The following lemma is essentially due to R. Bowen and O. Lanford [7].

LEMMA 2.2. (1) *The power series $\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m$ converges absolutely in $|u| < \alpha^{-1}$.*

(2) *$Z^o(u) = \det(I - u\mathcal{L})^{-1}$. In particular, $Z^o(u)$ is a rational function of u and has a simple pole at $u = \alpha^{-1}$.*

PROOF. (1) Note that

$$(\mathcal{L}^m\delta_y)(x) = \#\{c|\ \text{admissible paths with } o(c) = x, t(c) = y, |c| = m\}.$$

Here δ_x denotes the defining function of the set $\{x\}$. Hence

$$\mathrm{tr}\mathcal{L}^m = \sum_{x\in V} (\mathcal{L}^m\delta_x)(x) = N_m,$$

from which it follows that the series $\sum_{m=1}^{\infty} \frac{1}{m} N_m u^m$ converges absolutely in $|u| < \alpha^{-1}$.

(2) Let $\lambda_1, \ldots, \lambda_N$ be the characteristic roots of $\mathcal{L}$ ($N = \#V$). Using the equality $-\log(1 - x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$, we have

$$\begin{aligned} Z^o(u) &= \exp\left(\sum_{m=1}^{\infty}\sum_{i=1}^{N} \frac{1}{m}\lambda_i^m u^m\right) = \prod_{i=1}^{N} \exp(-\log(1 - \lambda_i u)) \\ &= \prod_{i=1}^{N} \frac{1}{1 - \lambda_i u} = \det(I - u\mathcal{L})^{-1}. \end{aligned}$$

This completes the proof. □

We define the notion of *admissible prime cycles* in X^o just in the same manner as the definition of prime cycles, and denote by P^o the set of admissible prime cycles. If we denote by M_k the number of admissible prime cycles of length k, then we have

$$\sum_{k|m} kM_k = N_m,$$

where k runs over all divisiors of m. The following theorem tells that $Z^o(u)$ has an *Euler product* expression.

TJHEOREM 2.3. $Z^o(u) = \prod_{\mathfrak{p}\in P^o} (1 - u^{|\mathfrak{p}|})^{-1}.$

PROOF. The claim is shown by the following computation.

$$\begin{aligned}\log Z^o(u) &= \sum_{\mathfrak{p}\in P^o}\sum_{k=1}^{\infty}\frac{1}{k}u^{k|\mathfrak{p}|} = \sum_{k=1}^{\infty}\frac{1}{k}\sum_{l=1}^{\infty}\sum_{|\mathfrak{p}|=l}u^{kl} = \sum_{k=1}^{\infty}\frac{1}{k}\sum_{l=1}^{\infty}M_l u^{kl}\\ &= \sum_{l,k=1}^{\infty}\frac{1}{k}M_l u^{kl} = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{l|m}kM_l u^m = \sum_{m=1}^{\infty}\frac{1}{m}N_m u^m. \ \square\end{aligned}$$

3. Oriented Line Graphs

Let $X = (V, E)$ be a non-circuit graph. To relate the zeta function $Z(u)$ of $X = (V, E)$ to the zeta function $Z^o(u)$ of an oriented graph defined in the previous section, we introduce the *oriented line graph* $X_L^o = (V_L, E_L^o)$ associated with X by setting

$$\begin{aligned} V_L &= E,\\ E_L^o &= \{(e_1, e_2) \in E \times E;\ \overline{e}_1 \neq e_2,\ t(e_1) = o(e_2)\},\end{aligned}$$

(namely, E_L^o is the set of geodesics of length 2). The incidence map (o, e) : $E_L^o \longrightarrow V_L \times V_L$ is induced from the identity map of $E \times E$.

It might be worthwhile to recall the definition of the *line graph* X_L, a standard notion in graph theory. Vertices of X_L are unoriented edges of X and edges of X_L are pairs of edges in X which have exactly one vertex in common. As is seen below, the notion of oriented line graphs is much convenient for counting closed geodesics.

An admissible path in X_L^o is expressed as $((e_1, e_2), (e_2, e_3), \ldots, (e_{k-1}, e_k))$ $((e_i, e_{i+1}) \in E_L^o)$ and corresponds to the geodesic $(e_1, \ldots, e_k)$ in X. Conversely a geodesic $(e_1, \ldots, e_k)$ in X corresponds to the admissible path $((e_1, e_2), (e_2, e_3), \ldots, (e_{k-1}, e_k))$. It is straightforward to check that X_L^o is strongly connected (actually, strong connectivity for X_L^o is equivalent to that, for given two edges e, e', there exists a geodesic $c = (e, \ldots, e')$). The period of X_L^o turns out to coincide with the greatest common divisor of length of closed geodesics in X. We also deduce that N_m coincides with the number of closed geodesics in X of length m. We also readily observe that there is a one-to-one correspondence between admissible prime cycles in X_L^o and prime cycles in X, and hence $Z(u, X) = Z^o(u, X_L^o)$. In particular, $Z(u)$ is a rational function in u (Lemma 2.2).

At this stage, we have the following characterization of bipartiteness of graphs in terms of zeta functions.

PROPOSITION 3.1. *X is bipartite if and only if $Z(u)$ is an even function, i.e. $Z(-u) = Z(u)$.*

PROOF. Recall that X is bipartite if and only if every closed path has even length, or equivalently every closed geodesic has even length. Therefore if X is bipartite, $|\mathfrak{p}|$ is even for every $\mathfrak{p} \in P$. In view of the Euler product expression, we conclude that $Z(u)$ is even. Conversely suppose that $Z(-u) = Z(u)$. Then

$$\sum_{m=1}^{\infty} \frac{1}{m}(-1)^m N_m u^m = \sum_{m=1}^{\infty} \frac{1}{m} N_m u^m.$$

This implies that $N_m = 0$ for odd m. Thus every closed geodesic has even length. □

The following proposition is related to Theorem 1.5.

PROPOSITION 3.2. *Let $\nu = \nu(X_L^o)$ be the period of X_L^o.*

(1) *If $\nu \geq 2$ and is odd, then X is the $(\nu - 1)$-subdivision of a non-bipartite garph Y with $\nu(Y_L^o) = 1$.*

(2) *If $\nu \geq 2$ and is even, then X is the $(\nu/2-1)$-subdivision of a bipartite garph Y with $\nu(Y_L^o) = 2$.*

In particular, if $d_m \geq 3$, then $\nu = 1$ (X being non-bipartite) or $\nu = 2$ (X being bipartite).

Proof. We note that X is bipartite if and only if ν is even.

From the definition of oriented line graphs together with the definition of the period, we obtain the decomposition:

$$E = E_0 \coprod E_1 \coprod \cdots \coprod E_{\nu-1}$$

such that, if $(e_1, e_2) \in E_L^o$ and $e_1 \in E_i$, then $e_2 \in E_{i+1}$ $(i \in \mathbb{Z}/\nu\mathbb{Z})$.

Since X is a non-circuit graph, there exists an edge e with deg $o(e) \geq 3$. Without of generality, we may assume that $e \in E_0$.

Let e' be another edge with deg $o(e') \geq 3$. Take a geodesic $c = (e_0, e_1, \ldots, e_k)$ such that $e_0 = e$ and $e_k = e'$. Note $e_i \in E_i$. From the assumption that deg $o(c) \geq 3$, it follows that there exist e_{-1} and e_0' such that $(e_{-1}, e_0), (e_{-1}, e_0') \in E_L^o$ and $e_0' \neq e_0$. Since $e_{-1} \in E_{-1} (= E_{\nu-1})$, we find that $e_0' \in E_0$, and hence $\overline{e_0} \in E_{-1}$. We then observe by induction that $\overline{e_i} \in E_{-i-1}$. Since deg $o(e_k) \geq 3$, there exist $e_k' \neq e_k$ with $(e_{k-1}, e_k') \in E_L^o$. Note that $\overline{e_k'} \in E_{-k-1}$, and hence $e_k \in E_{-k}$. This implies that $2k = k - (-k) \equiv 0$ (mod. ν). Therefore if ν is odd, then $k \equiv 0$ (mod. ν), and if $\nu = 2\mu$, then $k \equiv 0$ (mod. μ).

We now put, in the case ν is odd,

$$\begin{aligned} V_Y &= \{o(e) |\ e \in E_0\}, \\ E_Y &= \{c = (e_0, \ldots, e_{\nu-1}) |\ \text{geodesics with } e_0 \in E_0\}. \end{aligned}$$

We see that, if $c = (e_0, \ldots, e_{\nu-1}) \in E_Y$, then $\overline{e_{\nu-1}} \in E_0$. Indeed, by using the discussion above, this turns out to be true if deg $t(e_{\nu-1}) \geq 3$. When deg $t(e_{\nu-1}) = 2$, we add edges $e_\nu, e_{\nu+1}, \ldots, e_{k\nu}$ to c untill we obtain a geodesic $c' = (e_0, \ldots, e_{\nu-1}, e_\nu, e_{\nu+1}, \ldots, e_{k\nu})$ with deg $t(e_{k\nu-1}) \geq 3$ (such k exists since X is not a circuit graph). We then have $\overline{e_{k\nu-1}} \in E_0$, and hence $\overline{e_{\nu-1}} \in E_0$. We now consider the graph $Y = (V_Y, E_Y)$ where the incidence map (o, e) is defined in a natural manner. The inversion of $c = (e_0, \ldots, e_{\nu-1}) \in E_Y$ is given by $\overline{c} = (\overline{e_{\nu-1}}, \ldots, \overline{e_0})$. From the way of construction, we observe that X is the $(\nu-1)$-subdivision of Y. It is easy to check that $\nu(Y_L^0) = 1$ (in fact, if $\nu(Y_L^o) = k$, then $\nu(X_L^o) = k\nu$).

Next consider the case $\nu = 2\mu$. We put

$$\begin{aligned} A_Y &= \{o(e)|\ e \in E_0\}, \\ B_Y &= \{o(e)|\ e \in E_\mu\}, \\ V_Y &= A_Y \cup B_Y, \\ E_Y &= \{c = (e_0, \ldots, e_{\mu-1})|\ \text{geodesics with } e_0 \in E_0 \text{ or } e_0 \in E_\mu\}. \end{aligned}$$

In the same way as above, we find that $Y = (V_Y, E_Y)$ is a bipartite graph with the bipartition $V_Y = A_Y \cup B_Y$, and X is the $(\mu - 1)$-subdivision of Y. It is clear that $\nu(Y_L^0) = 2$. □

Theorem 1.4 is a consequence of Lemma 2.1 and Proposition 3.2

4. Perron-Frobenius Operators on Oriented Line Graphs

To obtain more information on $Z(u)$, we shall take a closer look at the Perron-Frobenius operator $\mathcal{L}$ on X_L^o.

Define inner products on $C(V)$ and $C(E)$ by

$$\begin{aligned} \langle f_1, f_2 \rangle &= \sum_{x \in V} f_1(x)\overline{f_2(x)}, \\ \langle \omega_1, \omega_2 \rangle &= \frac{1}{2} \sum_{e \in E} \omega_1(e)\overline{\omega_2(e)}. \end{aligned}$$

We put

$$\begin{aligned} C_-(E) &= \{\omega \in C(E)|\ \omega(\overline{e}) = -\omega(e)\}, \\ C_+(E) &= \{\omega \in C(E)|\ \omega(\overline{e}) = \omega(e)\}. \end{aligned}$$

We easily observe that $C(V_L) = C(E) = C_-(E) \oplus C_+(E)$ (an orthogonal direct sum), and

$$\mathcal{L}\omega(e) = \sum_{e'; (e,e') \in E_L^o} \omega(e') = \sum_{e'; o(e') = t(e)} \omega(e') - \omega(\overline{e})$$

We also observe, from (3) in Lemma 2.1, that the maximal positive eigenvalue α of $\mathcal{L}$ satisfies

$$d_m - 1 \le \alpha \le d_M - 1$$

In particular, $\alpha = q$ for a regular graph of degree $q+1$.

We denote by $P_{\pm} : C(E) \longrightarrow C_{\pm}(E)$ the orthogonal projection; say

$$\begin{aligned} P_{-}\omega(e) &= \frac{1}{2}(\omega(e) - \omega(\overline{e})), \\ P_{+}\omega(e) &= \frac{1}{2}(\omega(e) + \omega(\overline{e})). \end{aligned}$$

According to the direct sum decomposition of $C(V_L)$, we express the Perron-Frobenius operator $\mathcal{L}$ as the following matrix form

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{--} & \mathcal{L}_{-+} \\ \mathcal{L}_{+-} & \mathcal{L}_{++} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{L}_{--} &= P_{-}\mathcal{L}P_{-} : C_{-}(E) \longrightarrow C_{-}(E), \\ \mathcal{L}_{-+} &= P_{-}\mathcal{L}P_{+} : C_{+}(E) \longrightarrow C_{-}(E), \\ \mathcal{L}_{+-} &= P_{+}\mathcal{L}P_{-} : C_{-}(E) \longrightarrow C_{+}(E), \\ \mathcal{L}_{++} &= P_{+}\mathcal{L}P_{+} : C_{+}(E) \longrightarrow C_{+}(E). \end{aligned}$$

Define the operators $d_{-} : C(V) \longrightarrow C_{-}(E)$ and $d_{+} : C(V) \longrightarrow C_{+}(E)$ by

$$\begin{aligned} (d_{-}f)(e) &= f(t(e)) - f(o(e)), \\ (d_{+}f)(e) &= f(t(e)) + f(o(e)), \end{aligned}$$

and let $\delta_{-} : C_{-}(E) \longrightarrow C(V)$ and $\delta_{+} : C_{+}(E) \longrightarrow C(V)$ be the adjoint operator of d_{-} and d_{+} respectively. More explicitly,

$$(\delta_{\pm}\omega)(x) = \pm \sum_{e \in E_x} \omega(e) \quad (\omega \in C_{\pm}(E))$$

It should be noted that d_{-} is nothing but the *coboundary operator* of the chain complex associated with X (we regard X as a 1-dimensional cell complex). Thus $\chi(X) = \dim \operatorname{Ker} d_{-} - \dim \operatorname{Ker} \delta_{-}$. We also observe that $\dim \operatorname{Ker} d_{+} \leq 1$, and $\dim \operatorname{Ker} d_{+} = 1$ if and only if X is bipartite.

LEMMA 4.1.

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{--} & \mathcal{L}_{-+} \\ \\ \mathcal{L}_{+-} & \mathcal{L}_{++} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}d_-\delta_- + I & \frac{1}{2}d_-\delta_+ \\ \\ -\frac{1}{2}d_+\delta_- & \frac{1}{2}d_+\delta_+ - I \end{pmatrix}.$$

PROOF. We shall only show that $\mathcal{L}_{--} = -\frac{1}{2}d_-\delta_- + I$ since the proof for the others is similarly done. Let $\omega \in C_-(E)$. Then

$$\begin{aligned}
\mathcal{L}_{--}\omega(e) &= P_-\mathcal{L}\omega(e) = \frac{1}{2}(\mathcal{L}\omega(e) - \mathcal{L}\omega(\overline{e})) \\
&= \frac{1}{2}\left(\sum_{e';o(e')=t(e)} \omega(e') - \omega(\overline{e}) - \sum_{e';o(e')=t(\overline{e})} \omega(e') + \omega(e)\right) \\
&= \frac{1}{2}\left(-\delta_-\omega(t(e)) + \delta_-\omega(o(e)) + 2\omega(e)\right) \\
&= -\frac{1}{2}d_-\delta_-\omega(e) + \omega(e),
\end{aligned}$$

so that $\mathcal{L}_{--} = -\frac{1}{2}d_-\delta_- + I$. □

The operator $\Delta_- = \delta_- d_- = \mathcal{D} - \mathcal{A}$ is what we call the *discrete Laplacian* on X (actually, this is a special case of Laplacians defined on *weighted graphs*).

5. Proof of Theorem 1.1

To study the operator $\mathcal{L}$ more closely, we shall introduce several auxiliary operators. Define

$$\begin{aligned}
\tau &: C(E) \longrightarrow C(E), \\
\mathcal{S} &: C(V) \longrightarrow C(E), \\
\mathcal{T} &: C(E) \longrightarrow C(V)
\end{aligned}$$

by setting, for $\omega = (\omega_-, \omega_+) \in C(E) = C_-(E) \oplus C_+(E)$ and $f \in C(V)$,

$$\begin{aligned}
\tau(\omega_-, \omega_+) &= (-\omega_-, \omega_+), \\
\mathcal{S}f &= (d_-f, d_+f), \\
\mathcal{T}(\omega_-, \omega_+) &= \delta_-\omega_- - \delta_+\omega_+.
\end{aligned}$$

It should be noted that $\mathcal{S}$ is injective. By easy calculations, we obtain

$$\begin{aligned}
\mathcal{L} &= -\tau - \frac{1}{2}\mathcal{S}\mathcal{T}, \\
I - u\mathcal{L} &= I + u\tau + \frac{1}{2}u\mathcal{S}\mathcal{T}, \\
\mathcal{T}\mathcal{S} &= -2\mathcal{A}, \\
\mathcal{T}\tau\mathcal{S} &= -2\mathcal{D}, \\
(I - u\mathcal{L})(I - u\tau) &= (1-u^2)I + \frac{1}{2}u\mathcal{S}\mathcal{T}(1-u\tau), \qquad (2)\\
(I - u\tau)(I - u\mathcal{L}) &= (1-u^2)I + \frac{1}{2}u(1-u\tau)\mathcal{S}\mathcal{T},
\end{aligned}$$

and hence

$$(I - u\mathcal{L})(I - u\tau)\mathcal{S} = \mathcal{S}(I - u\mathcal{A} + u^2(\mathcal{D} - I)), \qquad (3)$$

$$\mathcal{T}(I - u\tau)(I - u\mathcal{L}) = (I - u\mathcal{A} + u^2(\mathcal{D} - I))\mathcal{T}. \qquad (4)$$

From the last two formulae (3), (4), we conclude that the operator $(I - u\mathcal{L})(I - u\tau)$ preserves the subspaces Image $\mathcal{S}$ and Ker $\mathcal{T}(I - u\tau)$.

We now write $N = \#V$ and $M = \#E^u$, the number of unotiented edges, so that $\chi(X) = N - M$.

Lemma 5.1. *Let $u \neq \pm 1$. The linear space $C(E)$ is the direct sum of* Image $\mathcal{S}$ *and* Ker $\mathcal{T}(I - u\tau)$ *if and only if* $\det(-\mathcal{A} + u\mathcal{D}) \neq 0$. *In particular* Image $\mathcal{S}$ *and* Ker $\mathcal{T}(I - u\tau)$ *are invariant subspaces of the operator $(I - u\mathcal{L})(I - u\tau)$ which are complementary to each other in $C(E)$ for a generic u.*

Proof. Note $\det(I - u\tau) = (1-u^2)^M$ so that $I - u\tau$ is bijective for $u \neq \pm 1$. Since $\mathcal{S}^* = -\mathcal{T}\tau$ and $(\text{Image } \mathcal{S})^\perp = \text{Ker } \mathcal{S}^*$, we have

$$\begin{aligned}
\dim \text{Image } \mathcal{S} + \dim \text{Ker } \mathcal{T}(I - \tau) &= \dim \text{Image } \mathcal{S} + \dim \text{Ker } \mathcal{T} \\
&= \dim \text{Image } \mathcal{S} + \dim \text{Ker } \mathcal{T}\tau \\
&= \dim \text{Image } \mathcal{S} + \dim \text{Ker } \mathcal{S}^* \\
&= \dim C(E).
\end{aligned}$$

What remains to check is that Image $\mathcal{S} \cap$ Ker $\mathcal{T}(I - u\tau) = \{0\}$ if and only if Ker $(-\mathcal{A} + u\mathcal{D}) = \{0\}$. For this, let $\omega = \mathcal{S}f \in$ Image $\mathcal{S} \cap$ Ker $\mathcal{T}(I - u\tau)$.

Then $0 = \mathcal{T}\mathcal{S}f - u\mathcal{T}\tau\mathcal{S}f = -2\mathcal{A}f + 2u\mathcal{D}f$ so that, if Ker $(-\mathcal{A}+u\mathcal{D}) = \{0\}$, then $f = 0$ and $\omega = \mathcal{S}f = 0$. The converse is obvious. □

From (2), it follows that $(I - u\mathcal{L})(I - u\tau) = (1 - u^2)I$ on Ker $\mathcal{T}(I - u\tau)$. Therefore in view of (3), we have the following expression of $(I - u\mathcal{L})(I - u\tau)$ for a generic u, according to the direct sum decomposition $C(E) =$ Image $\mathcal{S} \oplus$ Ker $\mathcal{T}(I - u\tau)$:

$$(I - u\mathcal{L})(I - u\tau) = \begin{pmatrix} \mathcal{S}(I - u\mathcal{A} + u^2(\mathcal{D} - I))\mathcal{S}^{-1} & O \\ O & (1 - u^2)I \end{pmatrix},$$

from which we find

$$\begin{aligned} (1 - u^2)^M \det(I - u\mathcal{L}) &= \det(I - u\mathcal{L})(I - u\tau) \\ &= (1 - u^2)^{2M-N} \det(I - u\mathcal{A} + u^2(\mathcal{D} - I)), \end{aligned}$$

and

$$\begin{aligned} \det(I - u\mathcal{L}) &= (1 - u^2)^{M-N} \det(I - u\mathcal{A} + u^2(\mathcal{D} - I)) \\ &= (1 - u^2)^{-\chi(X)} \det(I - u\mathcal{A} + u^2(\mathcal{D} - I)), \end{aligned}$$

where we should note that

$$\dim \text{Ker } \mathcal{T}(I - u\tau) = \dim C(E) - \dim \text{Image } \mathcal{S} = 2M - N.$$

Since the above equality holds for generic u, so does for all u. This completes the proof of Theorem 1.1.

6. Poles of the Zeta Functions

In view of Lemma 2.1 (2), we first see that $|u| \geq \alpha^{-1}$ for every pole u of $Z(u)$.

We next observe that $|\langle \mathcal{A}f, f\rangle| \leq \langle \mathcal{D}f, f\rangle$. Indeed,

$$\langle \mathcal{D}f, f\rangle - \langle \mathcal{A}f, f\rangle = \langle (\mathcal{D} - \mathcal{A})f, f\rangle = \langle \delta_- d_- f, f\rangle = \langle d_- f, d_- f\rangle \geq 0,$$

$$\langle \mathcal{A}f, f\rangle + \langle \mathcal{D}f, f\rangle = \langle (\mathcal{D} + \mathcal{A})f, f\rangle = \langle \delta_+ d_+ f, f\rangle = \langle d_+ f, d_+ f\rangle \geq 0.$$

Let $u \neq \pm 1$ be a pole of $Z(u)$. Then there exists a non-zero $f \in C(V)$ such that $(I - u\mathcal{A} + u^2(\mathcal{D} - I))f = 0$. We then have

$$\|f\|^2 - u\langle \mathcal{A}f, f\rangle + u^2\langle (\mathcal{D} - I)f, f\rangle = 0$$

Put

$$\lambda = \frac{\langle \mathcal{A}f, f\rangle}{\|f\|^2}, \qquad \mu = \frac{\langle \mathcal{D}f, f\rangle}{\|f\|^2},$$

so that $1 - \lambda u + (\mu - 1)u^2 = 0$, and

$$u = \frac{\lambda \pm \sqrt{\lambda^2 - 4(\mu - 1)}}{2(\mu - 1)}$$

From what we have said above, we obtain $|\lambda| \le \mu$. It is also straightforward to check that $d_m \le \mu \le d_M$.

(1) If u is real, then

$$\frac{\lambda + \sqrt{\lambda^2 - 4(\mu - 1)}}{2(\mu - 1)} \le \frac{\mu + \sqrt{\mu^2 - 4(\mu - 1)}}{2(\mu - 1)} = 1,$$

and

$$\frac{\lambda - \sqrt{\lambda^2 - 4(\mu - 1)}}{2(\mu - 1)} \ge \frac{-\mu - \sqrt{\mu^2 - 4(\mu - 1)}}{2(\mu - 1)} = -1,$$

and hence $|u| \le 1$.

(2) If u is imaginary, then

$$|u|^2 = \frac{\lambda^2 + (4(\mu - 1) - \lambda^2)}{4(\mu - 1)^2} = (\mu - 1)^{-1},$$

from which we obtain

$$(d_M - 1)^{-1/2} \le |u| \le (d_m - 1)^{-1/2}.$$

This completes the proof of Theorem 1.3.

We now proceed to the proof of Theorem 1.6. Let $u = \alpha^{-1}$. By Lemma 2.1 (1), there exists a positive-valued $\omega \in C(E)$ with $(I - u\mathcal{L})\omega = 0$. Put

$$f = \frac{-u}{2(1 - u^2)} \mathcal{T}\omega.$$

Since

$$(\mathcal{T}\omega)(x) = -\sum_{e \in E_x} \omega(e),$$

and $0 < u = \alpha^{-1} < 1$, we find that $f > 0$. By (4), $(I - u\mathcal{A} + u^2(\mathcal{D} - I))f = 0$. Furhtermore

$$\begin{aligned}(I-u\tau)\mathcal{S}f &= \frac{-u}{2(1-u^2)}(I-u\tau)\mathcal{S}\mathcal{T}\omega \\ &= \frac{-u}{2(1-u^2)}(I-u\tau)(-\tau-\mathcal{L})\omega \\ &= \frac{u}{2(1-u^2)}(I-u\tau)(\tau+u^{-1})\omega \\ &= \omega\end{aligned}$$

Since $((I-u\tau)\mathcal{S}f)(e) = ((1+u)d_- f + (1-u)d_+ f)(e) = 2f(t(e)) - 2uf(o(e))$, we conclude that $f(t(e)) > uf(o(e))$. Conversely, suppose that there exists f such that $(I - u\mathcal{A} - u^2(\mathcal{D} - I))f = 0$ and $f(t(e)) - uf(o(e)) > 0$ for every $e \in E$. Then $\omega = (I - u\tau)\mathcal{S}f$ is positive-valued, and $(I - u\mathcal{L})\omega = 0$ in view of (3). By Lemma 2.1 (5), we have $u = \alpha^{-1}$.

7. Proof of Theorem 1.7

The proof which we are going to give relies on the following classical result:

LEMMA 7.1.

$$\det(\Delta_-|_{(\mathrm{Ker}\ d)^\perp}) = NK(X).$$

See N. Biggs [4] and B. Bollobás [6] for the proof. The argument in these references is actually based on homological nature of the complexity. It is easy to translate it to a cohomological context.

What we have to prove is the following lemma.

LEMMA 7.2. $\det(I - u\mathcal{A} + u^2(\mathcal{D} - I)) = (1 - u)F(u)$, *where* $F(u)$ *is a polynomial with* $F(0) = 1, F(1) = 2\chi(X)K(X)$.

PROOF. Let H be the orthogonal projection of $C(V)$ onto the space of constant functions;

$$Hf = \frac{1}{N}\sum_{x\in V} f(x).$$

The orthogonal projection P onto $(\text{Ker } d)^{\perp}$ is given by $I - H$. Note

$$I - u\mathcal{A} + u^2(\mathcal{D} - I) = 1 - u^2 + u\Delta_- + u(u-1)\mathcal{D}$$

and

$$H\Delta_- = \Delta_- H = O, \quad H\mathcal{D}H = \frac{2M}{N}H.$$

We thus have

$$\begin{aligned}
H(I - u\mathcal{A} + u^2(\mathcal{D} - I))H &= (1 - u^2 + u(u-1)\frac{2M}{N})H, \\
H(I - u\mathcal{A} + u^2(\mathcal{D} - I))P &= u(u-1)H\mathcal{D}P, \\
P(I - u\mathcal{A} + u^2(\mathcal{D} - I))H &= u(u-1)P\mathcal{D}H, \\
P(I - u\mathcal{A} + u^2(\mathcal{D} - I))P &= (1-u^2)P + uP\Delta_- P + u(u-1)P\mathcal{D}P,
\end{aligned}$$

and hence $I - u\mathcal{A} + u^2(\mathcal{D} - I)$ is expressed as

$$\begin{pmatrix} (1-u)(1+u-u\frac{2M}{N})H & u(u-1)H\mathcal{D}P \\ u(u-1)P\mathcal{D}H & (1-u^2)P + u\Delta_- P + u(u-1)P\mathcal{D}P \end{pmatrix}.$$

Therefore $\det(I - u\mathcal{A} + u^2(\mathcal{D} - I))$ is equal to

$$(1-u)\begin{vmatrix} 1+u-u\frac{2M}{N} & -uH\mathcal{D}P \\ u(u-1)P\mathcal{D}H & (1-u^2)P + uP\Delta_- P + u(u-1)P\mathcal{D}P \end{vmatrix},$$

and

$$\begin{aligned}
&\{(1-u)^{-1}\det(I - u\mathcal{A} + u^2(\mathcal{D} - I))\}|_{u=1} \\
= &\begin{vmatrix} \frac{2}{N}(N-M) & -H\mathcal{D}P \\ 0 & P\Delta_- P \end{vmatrix} \\
= &\frac{2}{N}\chi(X)\det P\Delta_- P.
\end{aligned}$$

By the lemma above, we have $\det P\Delta_- P = NK(X)$ from which the claim follows immediately. □

8. Concluding Remarks

(1) What we have had in mind during the discussion is an analogy with the geodesic flows over a negatively curve manifold. Namely if a finite graph X is regarded as a discrete analogue of a closed Riemannian manifold Y, the associated oriented line graph X^o_L is regarded as a counterpart of the tangent unit sphere bundle UY. Remember that the geodesic flow on UY is identified with the one parameter transformation group $\{\varphi_t\}_{t\in\mathbb{R}}$ acting as $(\varphi_t c)(s) = c(s+t)$ on the set of geodesic curves $c : \mathbb{R} \longrightarrow Y$ of constant speed 1. Thus it is natural to consider the shift on the set of infinite admissible paths in X^o_L as a discrete analogue of the geodesic flow. Actually this discrete geodesic flow is nothing but the symbolic dynamical system associated with the oriented line graph, and periodic orbits in this dynamical system are identified with prime cycles in X. Therefore the dynamical zeta function defined by E. Artin and B. Mazur [2] coincides with $Z(u)$. It is worthwhile to point out that primitiveness of X^o_L is equivalent to that the discrete geodesic flow is *topologically mixing* as a symbolic dynamical system (see W. Parry and M. Pollicott [17]).

(2) Our idea in this paper works also for the *L-function*, a generalization of zeta functions, defined by

$$L(u,\rho) = \prod_{\mathfrak{p}\in P} \det(1-\rho([\mathfrak{p}])u^{|\mathfrak{p}|})^{-1},$$

where ρ is a finite dimensional unitary representation of the fundmental group $\pi_1(X)$, and $[\mathfrak{p}]$ denotes the (free) homotopy class of $\mathfrak{p}$. What we need are "twisted" objects for the operators which we introduced in our arguments. These twisted operators are defined, in a natural manner, by considering the "flat vector bundles" over X and X^o_L associated with the representation ρ (see [1]).

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