Analysis of error constants for linear conforming and nonconforming finite elements

適合および非適合１次有限要素の誤差定数の解析

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Chapter 1

Introduction

With mathematical modeling and computer simulation, we are now able to solve problems in many fields such as weather and pollution forecasts, underwater measuring, semiconductor device simulation and so on. When dealing with these problems, we are often required to solve partial differential equations(PDEs). Because it is usually very difficult to solve such problems mathematically, numerical algorithms or schemes become quite indispensable to obtain acceptable approximate solutions. Nowadays, many methods have been developed, such as finite element method(FEM), the finite difference method(FDM), the finite volume method(FVM) and so on (cf.P.Knabner [30] for an overview and basic introduction). Among these methods, the finite element method is a popular one, and has been integrated into many computer aided engineering(CAE) systems. The popularity of FEM is proberbly due to its sound mathematical foundations such as a priori and a posteriori error estimations as well as its practicality. The focus of this work is on error analysis of the finite element methods.

There is a large number of literatures on FEM, for example, both textbooks of C. Johnson [24] and P.G. Ciarlet [15] have lists of numerous references in this area. Simply speaking, the finite element method is a kind of Galerkin's method, in which a variational form of the given PDE is solved in a finite dimensional space. The obtained solution u_h, as an approximation of the exact one u, is usually different from u. To assure the reliability and efficiency of the computations, it is important to estimate the error $u - u_h$ in some suitable norms. For this purpose, a priori and a posteriori error estimation theories for FEM have been developed to estimate and further to control the approximation errors. A priori error estimate is based on the exact but unknown solution together with given data to predict the final computation error, while a posteriori one also utilizes the knowledge of the obtained computational result u_h. In either case, there appear a number of positive constants besides the standard discretization parameter h and norms, but it has been proved very difficult to evaluate such constants explicitly. For quantitative purposes, however, it is essential to evaluate or bound these constants as accurately as possible, because sharper estimates enable more efficient finite element computations. Therefore such evaluations have become progressively and increasingly important and have specifically been attempted for adaptive finite element calculations relying on a posteriori error estimates. At the beginning of Chapter 2, we explain in detail how to derive these constants and demonstrate their role in the error estimation.

The need of explicit evaluation of the constants mentioned above also comes from mathematical proofs based on numerical verifictions. As is well known now, we can monitor the round-off error of floating point computation in the computer by the interval analysis [41, 42, 19]. Utilizing theories of verified computations, such as known as Nakao's theory [38], Nakao gave mathematical proofs of existence of the solutions for various elliptic problems (cf. [37, 52] and the references therein). However, there are also various error constants to be evaluated for quantitative error estimates. The accurate estimation of these constants has great effects on the success of the interval computation.

Evaluation of the error constants has been proved to be very difficult. Some people tried to give rough bounds by the path integration method [48, 40], or by the interpolation remainder theory [21, 10, 9, 44]. In [4, 47], the finite element method was used to provide approximate evaluations without estimation for the approximation error. The interval computing was also employed to provide quite satisfactory enclosing a certain constant [36, 39], where the computation was done with quite complex procedures.

As we will see, interpolation error on narrow element is related to the dependency of constant on the geometric shape of the element. Babuška and Aziz considered the case of triangular element and proposed the "maximum angle condition" [6], which states that, if the maximum interior angle is fixed, the H^1 norm of Lagrange interpolation error is bounded even the smallest interior angle approaches zero. In the 3D case, the maximum angle condition for nodal interpolation on tetrahedron element was also discussed in [1, 31], where it was shown that the error in H^1 norm cannot be bounded under the "maximum angle condition", so that some other kinds of interpolations rather than Lagrange one may be recommended.

Outline of our research

In the following chapters, we will study various error constants appearing in the error estimates of conforming and nonconforming linear triangular FEMs. The obtained constant values or upper bounds will be used to give quantitative error estimates for the finite element solutions.

In Chapter 2, we will derive some fundamental estimates for the interpolation error constants appearing in the conforming linear triangluar finite elements. For each constant, we characterize them by appropriate Rayleigh quotient over a specified linear space, and then study the properties of the constant, such as the continuity, monotonicity, asymptotic behaviours when one edge tends to zero, and so on. In this work, we again verify the "maximum angle condition" by analyzing the dependency of constants on geometric parameters of the element.

We will also try to determine the concrete values of constants. In the case of isosceles right triangle, we successfully determine several constants, including the Babuška-Aziz constant [28]. By showing these constants to be related with the root of some transcendental functions, we can evaluate the constants with arbitrary precision. For some other constants, we also find reasonable upper bounds. Thus it becomes possible to perform quantitative interpolation error estimation and consequently computable error evaluation of finite element solution.

In Chapter 3, we present quantitative error estimates for the linear nonconforming finite element. More specifically, we introduce the Fortin interpolation and another edge-wise interpolation, and then study the error constants appearing there. Although we cannot determine the concrete values for these constants even in the case of special triangles, we are able to give upper bounds for them by utilizing the methods established in Chapter 2. The research implies that the maximum angle condition is also important in the linear nonconforming FEM. Some results in triangular element are also extended to the 3D case.

In Chapter 4, we consider eigenvalue problems of the Laplace operator and propose a posteriori estimation method to evaluate the constants which are associated to second order ODE's. As we will see, search for concrete values of the error constants usually results in solving some eigenvalue problems for operator $-\Delta$ or Δ^2, where various constraint conditions are imposed on the associated function spaces, for example, vanishing of integration over the domain. Due to such constraints, the eigenfunctions will be subject to nonhomogeneous Dirichlet or Neumann bounding conditons and hence the eigenvalue is very difficult to obtain. We solve one of these problems by constructing an auxiliary function to make the boundary condition homogeneous, adopting the ideas of Nakao in [36, 39].

As an application of our proposed method, we also consider the eigenvalue problem of Laplacian over disk with the homogeneous Dirichlet condition, and give quantitative upper and lower bounds for the minimum eigenvalue.

In Chapter 5, we give a hypercircle-based a posteriori error estimates method for the FEM solutions of Poisson's equation, where the linear conforming FEM and nonconforming one are used together. Once the verified computation becomes truly reliable, the method is expected to give mathematically correct error estimate. It should be pointed out that our proposed method can even be applied to singularity problem, for example, Poisson's equation with homogeneous Dirichelt boundary condition on the L-shaped domain. The computational results demonstrate the validity of this method in such a case.

Chapter 2

Conforming P_1 triangular finite element

2.1 Motivation of research on error constants

Where do the error constants come from and why do we consider them?

As an answer to this question, we would like to explain the motivation of our research on the error constants and also demonstrate the important role of error constants especially in error analysis for finite element method (FEM).

2.1.1 Conforming P_1 finite element for model problem

We start with Poisson's equation as a model problem, and will apply the conforming P_1 finite element to find approximation of the solution.

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with the boundary Γ. Given $f \in L_2(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ that, in the sense of distribution, satisfies the following Poisson's equation with homogeneous Dirichlet boundary condition

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma\,. \tag{2.1.1}$$

Thus, the function $u \in H_0^1(\Omega)$ is the unique solution of the variational problem,

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega)\,. \tag{2.1.2}$$

For this well-posed problem, we can define an operator G by $G : f \in L_2(\Omega) \to u \in H_0^1(\Omega)$. Here we also assume that the problem is a regular one (cf. Chapter 3.2 of

[15]), that is, the solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and there exists a positive constant C' such that $\|u\|_{H^2(\Omega)} \leq C'\|f\|_{L^2}$. As is known, if the domain Ω is a convex polygonal one, the problem in (2.1.1) is a regular one, where the constant C' can be taken as the unity.

In most cases, due to the complexity of the domain Ω and the given data f, we cannot obtain the explicit solution for the given problem. However, we can approximate the solution in finite dimensional spaces by utilizing the corresponding variational forms, where the theories of finite element methods ensure the validity and reliability of the computation. In this chapter, we will focus on the conforming P_1 FEM and then in the next chapter, the case of nonconforming P_1 FEM.

To apply the triangular P_1 FEM to the problem above, let us consider a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω, (cf.[15] for the terminology *regular*) and then construct the finite element space $V_{conf}^h \subset H_0^1(\Omega)$ for each $\mathcal{T}^h$:

$$V_{conf}^h := \{v \in C(\overline{\Omega}) |\ v \text{ is linear on each } K \in \mathcal{T}^h; v = 0 \text{ on } \partial\Omega.\ \}, \tag{2.1.3}$$

where $C(\overline{\Omega})$ denotes all the continuous function over $\overline{\Omega}$(:= closure of Ω). Thus the finite element approximation $u_h \in V_{conf}^h$ of the above $u \in H_0^1(\Omega)$ is now uniquely determined by imitating (2.1.2) in V_{conf}^h:

$$(\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_{conf}^h\,. \tag{2.1.4}$$

Within this section, we will also abbreviate V_{conf}^h as V^h if there is no fear of confusion. Note that V^h may present other kind of spaces under various situations.

2.1.2 A priori error estimates

Letting u and u_h be theose defined above, an important fact in the error analysis of the Ritz-Galerkin FEM is the following best approximation property:

$$|u - u_h|_{1,\Omega} = \min_{v_h \in V^h} |u - v_h|_{1,\Omega}, \tag{2.1.5}$$

where $|\cdot|_{1,\Omega}$ is the standard H^1 semi-norm for functions over domain Ω. Another important one is the L_2-error estimate based on the Aubin-Nitsche trick: (See Theorem 3.2.4 of [15])

$$\|u - u_h\|_\Omega \leq |u - u_h|_{1,\Omega} \sup_{g \in L_2(\Omega)\setminus\{0\}} \inf_{v_h \in V^h} \frac{|Gg - v_h|_{1,\Omega}}{\|g\|_\Omega}\,. \tag{2.1.6}$$

Let Π_h^1 be a nodal value interpolation operator that maps a function $u \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C(\overline{\Omega})$ to V^h, that is

$$(\Pi_h^1 u)(p_i) = u(p_i) \text{ for each vertex } p_i \text{ of } \mathcal{T}_h\,. \tag{2.1.7}$$

From (2.1.5), an error estimate based on the interpolation function $\Pi_h^1 u$ is given by

$$|u - u_h|_{1,\Omega} \le |u - \Pi_h^1 u|_{1,\Omega} \le Ch|u|_{2,\Omega} \le CC'\|f\|_\Omega\,, \tag{2.1.8}$$

where C is a constant independent of u and h. Also, taking $v_h = \Pi_h^1(Gg)$ in (2.1.6), we have

$$\sup_{g\in L_2(\Omega)\backslash\{0\}} \frac{|Gg - \Pi_h^1 Gg|_{1,\Omega}}{\|g\|_\Omega} \le \sup_{g\in L_2(\Omega)\backslash\{0\}} Ch\frac{|Gg|_{2,\Omega}}{\|g\|_\Omega} \le CC'h\,.$$

Hence, by adopting (2.1.6) and (2.1.8), we have

$$\|u - u_h\|_\Omega \le CC'h|u - u_h|_{1,\Omega} \le (CC'h)^2\|f\|_\Omega\,. \tag{2.1.9}$$

From the analysis above, we can see that the boundedness of the constants C and C' ensures a priori error estimates for the FEM solution. However, the values of these constants are usually very difficult to obtain. The main objective of this dissertation is to give concrete values or upper bounds for various constants appearing in FEM error analysis and further to make quantitative error estimation for FEM solutions. As these constants are closely related to error estimates, we call them "**error constants**".

Before further discussing the error constants, we also recall one kind of a posteriori estimate for FEM to show the role of the error constants.

2.1.3 A posteriori error estimates

A posteriori error estimation is also feasible and effective in various situations such as adaptive FEM computation. Here, as a demonstration, we explain a special and rather classical a posteriori error estimate method briefly to show the indispensability of the interpolation function together with the error constants. Detailed analysis can be found in the subsequent sections.

Let q be an arbitrary vector function taken from

$$H(\mathrm{div};\Omega) := \{q \in L_2(\Omega)^2 |\ \mathrm{div}\, q \in L_2(\Omega)\}\,. \tag{2.1.10}$$

Using the Green theorem, we have,

$$\begin{aligned}
|u-u_h|_{1,\Omega}^2 &= (\nabla(u-u_h),\nabla(u-u_h))_\Omega = (u-u_h,-\Delta u)_\Omega - (\nabla(u-u_h),\nabla u_h)_\Omega \\
&= (u-u_h,f)_\Omega + (\nabla(u-u_h), q-\nabla u_h - q)_\Omega \\
&= (u-u_h, f+\mathrm{div}\, q)_\Omega + (\nabla(u-u_h), q-\nabla u_h)_\Omega \\
&\le \|u-u_h\|_\Omega \cdot \|f+\mathrm{div}\, q\|_\Omega + |u-u_h|_{1,\Omega}\cdot \|q-\nabla u_h\|_\Omega\,.
\end{aligned}$$

Applying the former part of (2.1.9), we have

$$|u-u_h|_{1,\Omega} \le CC'h\|f+\mathrm{div}\, q\|_\Omega + \|q-\nabla u_h\|_\Omega\,. \tag{2.1.11}$$

The estimate above becomes an a posteriori one if q is specified appropriately. The most elegant but quite a restrictive choice is based on the hyper-circle method [29], where q is chosen so that $f + \mathrm{div}\, q = 0$ and hence the the use of CC' becomes unnecessary. More common and practical approach is to obtain q by post-processing of u_h, for example, by averaging or smoothing ∇u_h so as to belong to $H(\mathrm{div};\Omega)$. To make this approach effective, it is necessary that $\|q-\nabla u_h\|_\Omega = O(h)$ and preferably $\|f + \mathrm{div}\ q\|_\Omega = o(h)$. A kind of a posteriori L_2-error estimate is also obtainable by using (2.1.9),

$$\|u-u_h\|_\Omega \le (CC'h)^2\|f+\mathrm{div}\, q\|_\Omega + CC'h\|q-\nabla u_h\|_\Omega\,. \tag{2.1.12}$$

Once again, we observe the importance of the concrete values of the error constants. In the following section, we will introduce necessary constants and develop methodology to give sharp estimates.

2.2 Interpolation functions and error constants

We have demonstrated the importance of the interpolation error constants in the error estimation for the finite element methods. From this section, we will investigate several error constants related to triangular finite elements.

First of all, we give the necessary notations and define the error constants. Let h, α and θ be positive constants such that

$$h>0,\ 0<\alpha\le 1,\ (\frac{\pi}{3}\le)\cos^{-1}\frac{\alpha}{2}\le\theta<\pi\,. \tag{2.2.1}$$

We denote by $T_{\alpha,\theta,h}$ the triangle $\triangle OAB$ with $O(0,0)$, $A(h,0)$, $B(\alpha h\cos\theta, \alpha h\sin\theta)$ as three vertexes. The conditions in (2.2.1) imply that AB is the edge of maximum length, while OA is the medium edge and OB the shortest one. Notice that the notation h is mostly used as the largest edge length in standard textbooks such as [15], but our usage of h as the medium one may be convenient for the present purposes. A point in $T_{\alpha,\theta,h}$ or over its closure is designated by $x = (x_1, x_2)$, and three edges e_1, e_2 and e_3 of $T_{\alpha,\theta,h}$ are defined as

$$e_1 = OA, e_2 = OB, e_3 = OC\,.$$

Thus each triangle can be configured with three parameter α, θ and h by an appropriate congruent transformation. Like the usage in [6], we will use abbreviated notations $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$ and $T = T_1$ (Fig 2.2).

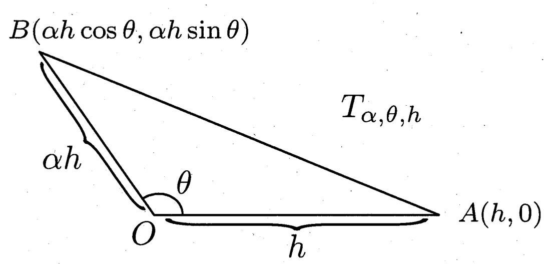


Figure 2.1: Triangular element $T_{\alpha,\theta,h}$

Before further considering the constants, let us introduce several function spaces. On domain $T_{\alpha,\theta,h}$, we use the popular Hilbert space $L_2(T_{\alpha,\theta,h})$, where the norm is denoted by $\|\cdot\|_{L_2(T_{\alpha,\theta,h})}$, or $\|\cdot\|_{T_{\alpha,\theta,h}}$ if there is no fear of confusion. When we need to use the L_2 space and its norm for other domains such as Ω, we will use notations such as $L_2(\Omega)$ and $\|\cdot\|_\Omega$. The spaces $H^1(T_{\alpha,\theta,h})$ and $H^2(T_{\alpha,\theta,h})$ are respectively the first and the second-order Sobolev spaces for real square integrable functions over $T_{\alpha,\theta,h}$ [2]. The symbols $\partial u/\partial x_i$, $\partial_i u$ and u_{x_i} will all denote the partial derivative of function u with respect the variable x_i. The standard semi-norms for $H^1(T_{\alpha,\theta,h})$ and $H^2(T_{\alpha,\theta,h})$ are represented by $|\cdot|_1 = (\sum_{i=1}^2 \|\partial v/\partial x_i\|^2)^{1/2}$ and $|v|_2 = (\sum_{i,j=1}^2 \|\partial^2 v/\partial x_i \partial x_j\|^2)^{1/2}$ respectively. Similarly we also use $|\cdot|_{1,\Omega}$ and $|\cdot|_{2,\Omega}$.

Let us define the following closed linear subspaces of $H^1(T_{\alpha,\theta,h})$ or $H^2(T_{\alpha,\theta,h})$ for functions over $T_{\alpha,\theta,h}$:

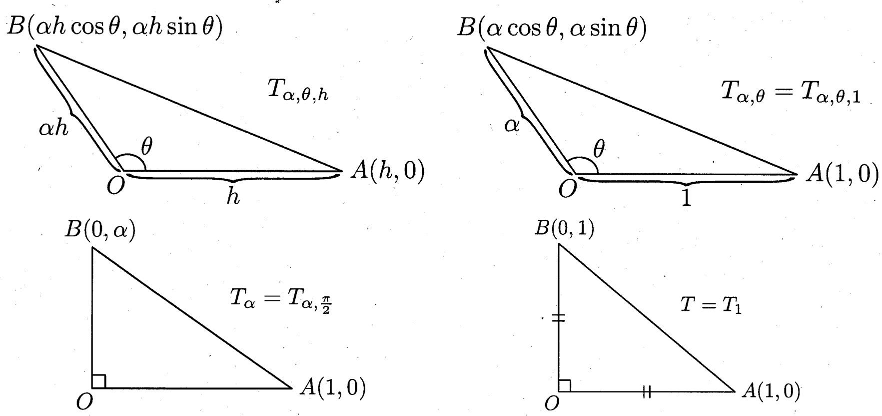


Figure 2.2: Notations for triangles

$$V^0_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h})| \int_{T_{\alpha,\theta,h}} v(x)dx = 0\}, \tag{2.2.2}$$

$$V^i_{\alpha,\theta,h} = \{v \in H^1(T_{\alpha,\theta,h})| \int_{e_i} v(s)ds = 0\} \quad (i = 1,2,3), \tag{2.2.3}$$

$$V^4_{\alpha,\theta,h} = \{v \in H^2(T_{\alpha,\theta,h})| \ v(O) = v(A) = v(B) = 0\}, \tag{2.2.4}$$

where ds is the line element. For other domains like Ω, we will also use spaces such as $H^1(\Omega)$ and $H^2(\Omega)$ later. For the above spaces, we will again use abbreviated notations $V^i_{\alpha,\theta} = V^i_{\alpha,\theta,1}$, $V^i_\alpha = V^i_{\alpha,\pi/2}$ and $V^i = V^i_1 (0 \le i \le 4)$.

The spaces above are introduced for the purpose of giving error estimate of conforming P_1 FEM. There will appear several other spaces introduced in the next chapter.

In the following, let us consider the usual P_0 interpolation operator $\Pi^0_{\alpha,\theta,h}$ and P_1 one $\Pi^1_{\alpha,\theta,h}$ for functions on $T_{\alpha,\theta,h}$ [12, 15].

Interpolation operators

Averaged interpolation function: For each $v \in H^1(T_{\alpha,\theta,h})$ (or even $v \in L_2(T_{\alpha,\theta,h})$), $\Pi^0_{\alpha,\theta,h}v$ is a constant function well-defined by

$$(\Pi^0_{\alpha,\theta,h}v)(x) = \int_{T_{\alpha,\theta,h}} v(y)dy \Big/ \int_{T_{\alpha,\theta,h}} dy \quad (\forall x \in T_{\alpha,\theta,h}). \tag{2.2.5}$$

Nodal Lagrange interpolation function: For each $v \in H^2(T_{\alpha,\theta,h})$, $\Pi^1_{\alpha,\theta,h}v$ is a linear polynomial function such that

$$(\Pi^1_{\alpha,\theta,h}v)(x) = v(x) \text{ for } x = O, A, B. \tag{2.2.6}$$

To give error estimates for these interpolation operators, it is natural to evaluate positive constants defined by

$$C_i(\alpha,\theta,h) = \sup_{v\in V^i_{\alpha,\theta,h}\backslash\{0\}} \frac{\|v\|_{T_{\alpha,\theta,h}}}{|v|_{1,T_{\alpha,\theta,h}}} \quad (i = 0,1,2,3), \tag{2.2.7}$$

$$C_4(\alpha,\theta,h) = \sup_{v\in V^4_{\alpha,\theta,h}\backslash\{0\}} \frac{|v|_{1,T_{\alpha,\theta,h}}}{|v|_{2,T_{\alpha,\theta,h}}}, \tag{2.2.8}$$

$$C_5(\alpha,\theta,h) = \sup_{v\in V^4_{\alpha,\theta,h}\backslash\{0\}} \frac{\|v\|_{T_{\alpha,\theta,h}}}{|v|_{2,T_{\alpha,\theta,h}}}. \tag{2.2.9}$$

The existence of these positive constants follows from the Rellich compactness theorem and the "sup" here can be actually replaced by "max". Due to the properties to become clear soon, such constants, together with some related ones, are often called *interpolation error constants*. We will again use abbreviated notations $C_i(\alpha,\theta) = C_i(\alpha,\theta,1)$, $C_i(\alpha) = C_i(\alpha,\pi/2)$ and $C_i = C_i(1)$ for $0 \le i \le 5$.

By a simple scale change, we find that $C_i(\alpha,\theta,h) = hC_i(\alpha,\theta)$ $(i = 0,1,2,3,4)$ and $C_5(\alpha,\theta,h) = h^2C_5(\alpha,\theta)$. These relations and constants are used to derive popular interpolation error estimates for $\Pi^i_{\alpha,\theta,h}$ $(i = 0,1)$ applied to functions on $T_{\alpha,\theta,h}$ [15, 30, 12]:

$$\|v - \Pi^0_{\alpha,\theta,h}v\| \le C_0(\alpha,\theta)h|v|_1, \quad \forall v \in H^1(T_{\alpha,\theta,h}), \tag{2.2.10}$$

$$|v - \Pi^1_{\alpha,\theta,h}v|_1 \le C_4(\alpha,\theta)h|v|_2, \quad \forall v \in H^2(T_{\alpha,\theta,h}), \tag{2.2.11}$$

$$\|v - \Pi^1_{\alpha,\theta,h}v\|_1 \le C_5(\alpha,\theta)h|v|_2, \quad \forall v \in H^2(T_{\alpha,\theta,h}), \tag{2.2.12}$$

where we have used the facts that $v - \Pi^0_{\alpha,\theta,h}v \in V^0_{\alpha,\theta,h}$ for $v \in H^1(T_{\alpha,\theta,h})$ and $v - \Pi^1_{\alpha,\theta,h}v \in V^4_{\alpha,\theta,h}$ for $v \in H^2(T_{\alpha,\theta,h})$.

Moreover, in the present coordinate system (Figure 2.1), we have, for the partial derivative $\partial_1 v (= \partial v/\partial x_1)$ of $v \in H^2(T_{\alpha,\theta,h})$,

$$\|\partial_1(v - \Pi^1_{\alpha,\theta,h}v)\| \le C_1(\alpha,\theta)h\,|\partial_1 v|_1, \tag{2.2.13}$$

since $\partial_1(v - \Pi^1_{\alpha,\theta,h}\, v) \in V^1_{\alpha,\theta,h}$. On the other hand, we can give an interpolation estimate in terms of $C_2(\alpha,\theta)$:

$$\|\partial(v - \Pi^1_{\alpha,\theta,h}v)/\partial\beta\| \le C_2(\alpha,\theta)h\,|\partial v/\partial\beta|_1, \tag{2.2.14}$$

where $\partial(v - \Pi^1_{\alpha,\theta,h}\, v)/\partial\beta$ denotes the directional derivative of $v - \Pi^1_{\alpha,\theta,h} v$ in the direction $\beta := (\cos\theta, \sin\theta)$, that is, $\nabla(v - \Pi^1_{\alpha,\theta,h} v) \cdot (\cos\beta, \sin\beta)$.

The above two estimates (2.2.13) and (2.2.14) are in a sense sharper than (2.2.11) as noted in [12].

Remark 2.2.1. *We can also consider anisotropic error estimates such as*

$$|v - \Pi^1_{\alpha,\theta,h} v|_{1,T_{\alpha,\theta,h}} \leq h \left(\sum_{i,j=1}^{2} c_{ij} \|\partial_{ij} v\|^2_{T_{\alpha,\theta,h}} \right)^{1/2}, \tag{2.2.15}$$

where the constants c_{ij}'s $(1 \leq i, j \leq 2)$ can be different from each other to give better error estimates. Notice that we are here considering the special cases where $c_{ij} = C_4(\alpha, \theta)$ for all i and j. Such kind of error estimates can be used to control anisotropic elements in adaptive FEM [20, 18]. However, we will not include such topics here.

Remark 2.2.2. *The constants $C_1(1, \pi/2, 1) = C_2(1, \pi/2, 1)$ are first introduced by I.Babuška and A.K.Aziz [6] to give an upper bound for $C_4(1, \pi/2, 1)$, that is $C_4(1, \pi/2, 1) \leq C_1(1, \pi/2, 1) = C_2(1, \pi/2, 1)$. In the following sections, we will also show that $C_4(\alpha) \leq \max\{C_1(\alpha), C_2(\alpha)\}$ for $\alpha > 0$. Thus the estimates for $C_1(\alpha)$ and $C_2(\alpha)$ can be used to give upper bound for $C_4(\alpha)$. Relations between $C_4(\alpha, \theta)$ and $C_i(\alpha, \theta)$ (i=1,2,3) will be discussed in Section 2.4.2. One of the merits of considering $C_i(\alpha, \theta)$ $(i = 1, 2, 3)$ is that the estimates for these constants are much easier than the one for $C_4(\alpha, \theta)$.*

Thus we can give quantitative interpolation estimates, provided that we succeed in evaluating or bounding the constant $C_i(\alpha, \theta)$'s explicitly. So we will try to bound these constants by fairly simple functions of α and θ. Notice here that each of such constants can be characterized by minimization of a kind of Rayleigh quotient. Then it is equivalent to finding the minimum eigenvalue of a certain eigenvalue problem expressed by a weak formulation, which is further expressed by a partial differential equation with some auxiliary conditions.

More specifically, we can characterize the constants $C_i(\alpha, \theta)$'s by minimization of Rayleigh's quotients $R^{(i)}_{\alpha,\theta}$'s:

$$C_i^{-2}(\alpha,\theta) = \inf_{v\in V^i_{\alpha,\theta}\backslash\{0\}} R^{(i)}_{\alpha,\theta}(v); \quad R^{(i)}_{\alpha,\theta}(v) = \frac{|v|^2_{1,T_{\alpha,\theta}}}{\|v\|^2_{T_{\alpha,\theta}}} \ (i=0,1,2,3)\,, \tag{2.2.16}$$

$$C_4^{-2}(\alpha,\theta) = \inf_{v\in V^4_{\alpha,\theta}\backslash\{0\}} R^{(4)}_{\alpha,\theta}(v); \quad R^{(4)}_{\alpha,\theta}(v) = \frac{|v|^2_{2,T_{\alpha,\theta}}}{|v|^2_{1,T_{\alpha,\theta}}}, \tag{2.2.17}$$

$$C_5^{-2}(\alpha,\theta) = \inf_{v\in V^4_{\alpha,\theta}\backslash\{0\}} R^{(5)}_{\alpha,\theta}(v); \quad R^{(5)}_{\alpha,\theta}(v) = \frac{|v|^2_{2,T_{\alpha,\theta}}}{\|v\|^2_{T_{\alpha,\theta}}}, \tag{2.2.18}$$

where all the notations and functions are for $T_{\alpha,\theta}$. Here we also introduce several quantities $\lambda_i(\alpha,\theta)$'s by

$$\lambda_i(\alpha,\theta) := C_i^{-2}(\alpha,\theta) \quad (0\le i\le 5)\,, \tag{2.2.19}$$

which will often appear in the forms of eigenvalue problems (see below).

By the standard compactness arguments, each infimum above is actually a minimum and is the smallest eigenvalue of a certain eigenvalue problem. For example, the eigenvalue problem associated with $C_0(\alpha,\theta)$ is to *find* $\lambda\in\mathbb{R}$ *and* $u\in V^0_{\alpha,\theta}\setminus\{0\}$ *that satisfy*

$$(\nabla u,\nabla v)_{T_{\alpha,\theta}} = \lambda(u,v)_{T_{\alpha,\theta}}, \quad \forall v\in V^0_{\alpha,\theta}. \tag{2.2.20}$$

Here $(\cdot,\cdot)_{T_{\alpha,\theta}}$ denotes the inner products of both $L_2(T_{\alpha,\theta})$ and $L_2(T_{\alpha,\theta})^2$. The present eigenvalue problem is also expressed by a partial differential equation with a linear constraint for $V^0_{\alpha,\theta}$ and a boundary condition [36, 39].

$$-\Delta u = \lambda u \text{ in } T_{\alpha,\theta}, \quad \int_{T_{\alpha,\theta}} u(x)dx = 0, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial T_{\alpha,\theta}\,, \tag{2.2.21}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative to the boundary $\partial T_{\alpha,\theta}$. The above boundary condition is the homogeneous Neumann one, and the desired value $C_0(\alpha,\theta)^{-2}$ is just the second eigenvalue for the same PDE problem without the linear constraint.

For $C_1(\alpha,\theta)$, it is characterized in essentially the same fashion as (2.2.20), if the associated space $V^0_{\alpha,\theta}$ is replaced with $V^1_{\alpha,\theta}$: *find* $\lambda\in\mathbb{R}$ *and* $u\in V^1_{\alpha,\theta}\setminus\{0\}$ *that satisfy*

$$(\nabla u,\nabla v)_{T_{\alpha,\theta}} = \lambda(u,v)_{T_{\alpha,\theta}}, \quad \forall v\in V^1_{\alpha,\theta}. \tag{2.2.22}$$

On the other hand, the equations corresponding to (2.2.21) become [36, 39]:

$$-\Delta u = \lambda u \text{ in } T_{\alpha,\theta}, \quad \int_0^1 (x_1,0)dx_1 = 0, \quad \frac{\partial u}{\partial n} = \begin{cases} 0 & \text{on edges } OB \text{ and } AB, \\ c & \text{on edge } OA, \end{cases} \tag{2.2.23}$$

where c denotes an unknown constant to be decided simultaneously with u and λ.

The other constants are characterized similarly. For example, the eigenvalue problem associated to $C_4(\alpha,\theta)$ is to *find* $\lambda \in \mathbb{R}$ *and* $u \in V^4_{\alpha,\theta} \setminus \{0\}$ *that satisfy*

$$\sum_{i,j=1}^{2} (\partial_{ij} u, \partial_{ij} v)_{T_{\alpha,\theta}} = \lambda (\nabla u, \nabla v)_{T_{\alpha,\theta}}, \quad \forall v \in V^4_{\alpha,\theta}\,. \tag{2.2.24}$$

But the partial differential equation related to the above and also that to $C_5(\alpha,\theta)$ are the ones of fourth order with special linear constraints and boundary conditions, and are more difficult to deal with than the second order equations such as in (2.2.21) and (2.2.23), cf.[4, 44]. Since $T_{\alpha,\theta}$ is a triangle, it is difficult to solve such eigenvalue problems explicitly even in the case of second order equations, except in some rare cases to be shown later.

2.3 Dependence of constants on geometric parameters

The classical method to estimate the interpolation error is to consider the interpolation on a reference element, e.g., the isosceles right triangle, and then introduce appropriate affine coordinate transformations between the given elements and the reference one (cf. Chapter 3 of [15]), where only the convergence orders have been usually assured with many unknown constants. Here we will follow essentially the same technique as above to consider the dependence of the constants on geometric parameters, and then give concrete estimates for $C_i(\alpha,\theta)$'s by using the ones on the reference triangular element.

2.3.1 Relation between $C_i(\alpha)$'s $(i = 1, 2)$ and $C_4(\alpha)$

In this section, we simply extend the result of [6] to show the role of $C_i(\alpha)$ $(i = 1, 2, 3)$ in estimating $C_4(\alpha)$.

Lemma 2.3.1. *For $\alpha > 0$, it holds that*

$$C_4(\alpha) \le \max\{C_1(\alpha), C_2(\alpha)\}\,. \tag{2.3.1}$$

Proof. From the definition,

$$C_4(\alpha)^{-2} = \inf_{v \in V^4_\alpha \setminus \{0\}} \frac{|v|_2^2}{|v|_1^2} = \inf_{v \in V^4_\alpha \setminus \{0\}} \frac{|\partial v/\partial x_1|_1^2 + |\partial v/\partial x_2|_1^2}{\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2}\,.$$

We can see that $\partial v/\partial x_i \in V_\alpha^i$ $(i = 1, 2)$ for $v \in V_\alpha^4$, so that

$$|\partial v/\partial x_i|_{1,T_\alpha} \geq C_i(\alpha)^{-1} \|\partial v/\partial x_i\|_{T_\alpha} (i = 1, 2) \,.$$

Then,

$$\begin{aligned} C_4(\alpha)^{-2} &\geq \inf_{v \in V_\alpha^4 \setminus \{0\}} \frac{C_1(\alpha)^{-2} \|\partial v/\partial x_1\|^2 + C_2(\alpha)^{-2} \|\partial v/\partial x_2\|^2}{\|\partial v/\partial x_1\|^2 + \|\partial v/\partial x_2\|^2} \\ &\geq \min\{C_1(\alpha)^{-2}, C_2(\alpha)^{-2}\} \,. \end{aligned}$$

Now we obtain the desired result. □

As shown in the above proof, neglecting the curl-free condition for $v \in V_\alpha^4$, that is, $\partial_1(\partial_2 v) - \partial_2(\partial_1 v) = 0$ required for $v \in V_\alpha^4$, leads to an upper bound for $C_4(\alpha)$. As we will see in the later computational results in Figure 2.3, the constants $C_1(\alpha)$ and $C_2(\alpha)$ give reasonable upper bound for $C_4(\alpha)$. Moreover, the orders of derivative in the PDEs corresponding to $C_1(\alpha)$ and $C_2(\alpha)$ are lower than that for $C_4(\alpha)$, which fact makes $C_1(\alpha)$ and $C_2(\alpha)$ easier to deal with. Therefore, we will pay more efforts on these two constants instead of the primary one $C_4(\alpha)$ [36, 39].

The method used in Lemma 2.3.1 can also be extended to general cases to give estimate for $C_4(\alpha, \theta)$ by utilizing $C_i(\alpha, \theta)$ $(i = 1, 2, 3)$, cf. Section 2.4.2.

2.3.2 Dependence of constants on α

Monotonicity of constants $C_i(\alpha)$ in α

In the case of $\alpha = \pi/2$, we can easily prove the monotonicity of $C_i(\alpha)$, $(0 \leq i \leq 5; i \neq 4)$, as will be shown below. However, it appears to be difficult to show the monotonicity of $C_4(\alpha)$, although our numerical results suggest that it holds even in this case. In general case where $\theta \neq \frac{\pi}{2}$, it would be much more difficult or even impossible to show the monotonicity of $C_i(\alpha, \theta)$ even when one of α and θ is fixed.

Before going into further discussion, let us introduce new Rayleigh quotients $\hat{R}_\alpha^{(i)}$'s for $u \in H^1(T)$ or $u \in H^2(T)$, where $T = T_{1,\pi/2,1}$:

$$\hat{R}_\alpha^{(i)}(u) = \frac{\|\partial_1 u\|_T^2 + \alpha^{-2}\|\partial_2 u\|_T^2}{\|u\|_T^2} \qquad \text{for } i = 0,1,2,3, \tag{2.3.2}$$

$$\hat{R}_\alpha^{(4)}(u) = \frac{\|\partial_{11} u\|_T^2 + 2\alpha^{-2}\|\partial_{12} u\|_T^2 + \alpha^{-4}\|\partial_{22} u\|_T^2}{\|\partial_1 u\|_T^2 + \alpha^{-2}\|\partial_2 u\|_T^2} \qquad \text{for } i = 4\,, \tag{2.3.3}$$

$$\hat{R}_\alpha^{(5)}(u) = \frac{\|\partial_{11} u\|_T^2 + 2\alpha^{-2}\|\partial_{12} u\| + \alpha^{-4}\|\partial_{22} u\|_T^2}{\|u\|_T^2} \qquad \text{for } i = 5\,. \tag{2.3.4}$$

Lemma 2.3.2. *For $\alpha > 0$, $C_i(\alpha)$'s $(i = 0,1,2,3,5; i \neq 4)$ are strictly monotonically increasing with respect to α.*

Proof. We only show the proof for $C_1(\alpha)$, while the other ones can be done in analogous ways. Let us consider the transformation between $x = (x_1, x_2) \in T_\alpha$ and $\xi = (\xi_1, \xi_2) \in T$ by $\xi_1 = x_1, \xi_2 = x_2/\alpha$ and let $\hat{u}(\xi_1, \xi_2) := u(x_1, x_2)$ for the corresponding ξ and x. Using the Rayleigh quotient in equation (2.3.2), we have $\hat{R}_\alpha^{(1)}(\hat{u}) = R_\alpha^{(1)}(u)$. Also, notice that $\hat{R}_\alpha^{(1)}(\hat{u})$ is strictly monotonically decreasing in α for fixed $\hat{u}$ if $\partial_{\xi_2}\hat{u} \neq 0$.

As $R_\alpha^{(1)}(u) = \hat{R}_\alpha^{(1)}(\hat{u})$ and from the definition of $\lambda_1(\alpha)$ in (2.2.19), we can see that

$$\lambda_1(\alpha) = \inf_{\hat{v} \in V^1 \setminus \{0\}} \hat{R}_\alpha^{(1)}(\hat{v})\,, \tag{2.3.5}$$

where "inf" is actually "min". For each α, let $\hat{u}_\alpha^{(1)} \in V^1$ be the minimizing function corresponding to $\lambda_1(\alpha)$. We can see that $\partial_{\xi_2}\hat{u}_\alpha^{(1)} \neq 0$ although we omit the details (cf. Sec.2.5). Hence, for given $0 < \alpha_1 < \alpha_2$, we have

$$\lambda_1(\alpha_1) = \hat{R}_{\alpha_1}(\hat{u}_{\alpha_1}) > \hat{R}_{\alpha_2}(\hat{u}_{\alpha_1}) \geq \hat{R}_{\alpha_2}(\hat{u}_{\alpha_2}) = \lambda_1(\alpha_2)\,, \tag{2.3.6}$$

where the second inequality follows from the definition of minimizing function $\hat{u}_{\alpha_2}$. Now, we have proved that $C_1(\alpha) = \lambda_1(\alpha)^{-1/2}$ is strictly monotonically increasing as α increases, and the proof is completed. □

Remark 2.3.1. *Summarizing the results in Lemma 2.3.1 and 2.3.2, we have*

$$C_4(\alpha) \leq \max\{C_1(\alpha), C_2(\alpha)\} \leq C_1 = C_2 \quad \textit{for } \alpha \leq 1,$$

which fact makes it possible to give an upper bound for $C_4(\alpha)$, provided that the values of $C_1(\alpha)$ and $C_2(\alpha)$, or even the single value of $C_1 = C_2$, are available.

Continuity of constants in α

For all $\alpha \in (0, \infty)$, we will show that $C_i(\alpha)$'s $(0 \le i \le 5)$ are continuous with respect to α. The proof for each constant adopts essentially the same technique.

Lemma 2.3.3. *For $\alpha > 0$, $C_i(\alpha)$'s $(0 \le i \le 5)$ are continuous with respect to α.*

Proof. We describe the proof only for $C_4(\alpha)$, while it is easier to prove in other cases since the associated $C_i(\alpha)$'s are monotone. Let us recall the Rayleigh quotient in equation(2.3.3), and the constant $\lambda_4(\alpha)$ introduced by (2.2.19):

$$\lambda_4(\alpha) := \frac{1}{C_4(\alpha)^2} = \inf_{v \in V^4(T)\setminus\{0\}} \hat{R}_\alpha^{(4)}(v) \,. \tag{2.3.7}$$

Within the present proof, we will denote the denominator of $\hat{R}_\alpha^{(4)}(v)$ by $b_\alpha(v)$ and the numerator by $a_\alpha(v)$, that is, $\hat{R}_\alpha^{(4)}(v) = a_\alpha(v)/b_\alpha(v)$. Let $v_\alpha \in V^4 \setminus \{0\}$ be one of the minimization function corresponding to $\lambda_4(\alpha)$, for which we assume that $b_\alpha(v_\alpha) = 1$.

For a fixed $\alpha > 0$, let $I_\alpha := [\alpha - \epsilon, \alpha + \epsilon] \subset (0, \infty)$ for sufficiently small $\epsilon > 0$. As we can see that $\lambda_4(\alpha)$ is uniformly bounded for $\beta \in I_\alpha$, both $\overline{\lambda}_4 := \limsup_{\beta \to \alpha} \lambda_4(\beta)$ and $\underline{\lambda}_4 := \liminf_{\beta\to\alpha} \lambda_4(\beta)$ exist.

To show the continuity of $\lambda_4(\beta)$ at $\beta = \alpha$, we need to prove that

$$\lambda_4(\alpha) = \liminf_{\beta\to\alpha} \lambda_4(\beta) = \limsup_{\beta\to\alpha} \lambda_4(\beta) \,. \tag{2.3.8}$$

In fact, as $\underline{\lambda}_4 \le \overline{\lambda}_4$, it is sufficient to show

$$(\limsup_{\beta\to\alpha} \lambda_4(\beta) =)\, \overline{\lambda}_4 \le \lambda_4(\alpha) \le \underline{\lambda}_4 \,(= \liminf_{\beta\to\alpha} \lambda_4(\beta)) \,. \tag{2.3.9}$$

From the definitions of lim inf and lim sup, there exist a sequence $\{\beta_i\}_{i=1}^\infty$ such that $\beta_i \to \alpha$ and $\lambda_4(\beta_i) \to \underline{\lambda}_4$, and also another one $\{\hat{\beta}_i\}_{i=1}^\infty$ such that $\hat{\beta}_i \to \alpha$ and $\lambda_4(\hat{\beta}_i) \to \overline{\lambda}_4$ as $i \to \infty$.

Firstly, we will show that

$$\overline{\lambda}_4 \le \lambda_4(\alpha) \,. \tag{2.3.10}$$

which is true by noticing the relation $\lambda_4(\hat{\beta}_i) \le R_{\hat{\beta}_i}^{(4)}(v_\alpha)$, and the fact that $R_{\hat{\beta}_i}(v_\alpha) \to R_\alpha(v_\alpha)$ and $\lambda_4(\hat{\beta}_i) \to \overline{\lambda}_4$ as $i \to \infty$.

Secondly, we will show

$$\lambda_4(\alpha) \le \underline{\lambda}_4 \,, \tag{2.3.11}$$

for which we give the proof as below.

1) $\|v_\beta\|_{H^2(T)}$ are uniformly bounded for all $\beta \in I_\alpha$:

Firstly, there exist positive constants $k_i(I_\alpha)$ $(i = 1, 2, 3, 4)$, such that

$$k_1(I_\alpha)|v|^2_{2,T} \le a_\alpha(v) \le k_2(I_\alpha)|v|^2_{2,T} \,,$$

$$k_3(I_\alpha)|v|^2_{1,T} \le b_\alpha(v) \le k_4(I_\alpha)|v|^2_{1,T} \,.$$

Considering the boundedness of $\{\lambda_4(\beta)\}$ on I_α and the assumption $b_\alpha(v_\beta) = 1$, we find that $\{a_\beta(v_\beta)\}$ and $\{b_\beta(v_\beta)\}$ are uniformly bounded on I_α, so that $\{|v_\beta|_T\}$ are uniformly bounded. Secondly, noting that $\|u\|_T \le C_5|u|_{2,T}$ for $u \in V^4$ (c.f. 2.2.7), we have that $\{\|v_\beta\|_T\}$ is uniformly bounded. Therefore, $\{\|v_\beta\|_{2,T}\}$ is uniformly bounded for all v_β with $\beta \in I_\alpha$.

2) Since $\{v_{\beta_i}\}$ are uniformly bounded in $H^2(T)$, we can apply the compactness theorem in the Sobolev space (Rellich's theorem) to show that there exist $v_0(\neq 0) \in H^2(T)$ and a sub-sequence of $\{v_{\beta_i}\}$, still using the same notation, such that $v_{\beta_i} \rightharpoonup v_0$ in $H^2(T)$ and $v_{\beta_i} \to v_0$ in $H^1(T)$ as $i \to \infty$. Moreover, we have $\frac{\partial^2 v_{\beta_i}}{\partial x_k \partial x_j} \rightharpoonup \frac{\partial^2 v_0}{\partial x_k \partial x_j}$ $(1 \le k, j \le 2)$ in $L_2(T)$ and $\frac{\partial v_{\beta_i}}{\partial x_j} \to \frac{\partial v_0}{\partial x_j}$ $(j = 1, 2)$ in $L_2(T)$. Here, "$\to$" and "$\rightharpoonup$" respectively denote the strong and weak convergence in normed spaces.

3) $\lim_{i\to\infty} a_{\beta_i}(v_{\beta_i}) \ge a_\alpha(v_{\beta_i})$ and $\lim_{i\to\infty} b_{\beta_i}(v_{\beta_i}) = b_\alpha(v_0) = 1$:

The latter equality is easier to show. For the former inequality, we use the weakly lower semi-continuity of Hilbertian norms: for $\{w_i\}_{i=1}^\infty$ such that $w_i \rightharpoonup w_0$ in $L_2(T)$, we have $\|w_0\|_{L_2(T)} \le \liminf_{i\to\infty} \|w_i\|_T$. Then

$$\begin{aligned}
\lim_{i\to\infty} a_{\beta_i}(v_{\beta_i}) &= \lim_{i\to\infty} \left\{ \|\frac{\partial^2 v_{\beta_i}}{\partial x_1^2}\|_T^2 + \frac{2}{{\beta_i}^2}\|\frac{\partial^2 v_{\beta_i}}{\partial x_1 \partial x_2}\|_T^2 + \frac{1}{{\beta_i}^4}\|\frac{\partial^2 v_{\beta_i}}{\partial x_2^2}\|_T^2 \right\} \\
&\ge \liminf_{i\to\infty} \|\frac{\partial^2 v_{\beta_i}}{\partial x_1^2}\|_T^2 + \liminf_{i\to\infty} \frac{2}{{\beta_i}^2}\|\frac{\partial^2 v_{\beta_i}}{\partial x_1 \partial x_2}\|_T^2 \\
&\quad + \liminf_{i\to\infty} \frac{1}{{\beta_i}^4}\|\frac{\partial^2 v_{\beta_i}}{\partial x_2^2}\|_T^2 \\
&\ge \|\frac{\partial^2 v_0}{\partial x_1^2}\|_T^2 + \frac{2}{\alpha^2}\|\frac{\partial^2 v_0}{\partial x_1 \partial x_2}\|_T^2 + \frac{1}{\alpha^4}\|\frac{\partial^2 v_0}{\partial x_2^2}\|_T^2 \\
&= a_\alpha(v_0) \,.
\end{aligned}$$

Thus we have $\lambda_4(\alpha) = \hat{R}^{(4)}_\alpha(v_\alpha) \le \hat{R}^{(4)}_\alpha(v_0) \le \lim_{i\to\infty} R^{(4)}_{\beta_i}(v_{\beta_i}) = \underline{\lambda}_4$.

Now, both (2.3.10) and (2.3.11) are proved, so that (2.3.8) holds. Therefore the continuity of $C_4(\alpha)$ is assured. $\square$

Remark: Here we only consider the continuity of constants on parameter α in the case where $\theta = \pi/2$. Actually, by extending the technique used here, we can prove that all these constants are continuous in two parameters α and θ for $\alpha \in (0, 1]$ and $\theta \in [\frac{\pi}{3}, \pi)$.

We summarize the results above as follows.

Theorem 2.3.1. *In the case of $h = 1$ and $\theta = \pi/2$, $C_i(\alpha)$'s $(0 \le i \le 5)$ are continuous and positive-valued functions of $\alpha \in (0, +\infty)$ ($\alpha > 1$ is also considered here). Except for $i = 4$, they are strictly monotonically increasing with respect to α. In particular,*

$$C_i(\alpha) \le C_i, \quad \forall\, \alpha \in (0, 1] \quad (0 \le i \le 5; i \ne 4)\,. \tag{2.3.12}$$

Furthermore, $C_4(\alpha)$ has the property

$$C_4(\alpha) \le \max\{C_1(\alpha), C_2(\alpha)\} \le C_1 = C_2 \text{ for } \alpha \in (0, 1]\,. \tag{2.3.13}$$

Here we see that each $C_i(\alpha)$ $(0 \le i \le 5; i \ne 4)$ is bounded from above by C_i, and $C_4(\alpha)$ is so by $C_1 = C_2$. Fortunately, since the value of $C_0(= 1/\pi)$ and $C_1 = C_2$ will be available (to be shown in the next chapter), we can give rough but correct upper bounds for $C_i(\alpha)$'s $(i = 0, 1, 2, 3)$.

In Figure 2.3, we show the numerical results for $C_1(\alpha)$, $C_2(\alpha)$ and $C_4(\alpha)$ to check the validity of the present theorem. As may be seen from the figure, $C_4(\alpha)$ is actually bounded from above by $\max\{C_1(\alpha), C_2(\alpha)\}$ for every $\alpha \le 1$. Moreover their monotonicity is seen to hold, although such a property is not yet proved for $C_4(\alpha)$. It is also interesting that C_4 is numerically close to $C_1 = C_2$ at $\alpha = 1$.

2.3.3 Dependence of constants on θ

Since various properties of error constants in the case of $\alpha = \pi/2$ become clearer now, we now try to estimate $C_i(\alpha, \theta)$ by $C_i(\alpha)$ for each fixed α. There are also some other ways to estimate $C_i(\alpha, \theta)$'s by considering the coordinate transformation between, for example, $T_{\alpha,\theta}$ and $T_{1,\pi/2}$, which will be studied in the next section.

For fixed value of parameter α, we can estimate $C_i(\alpha, \theta)$ by $C_i(\alpha)$ as follows.

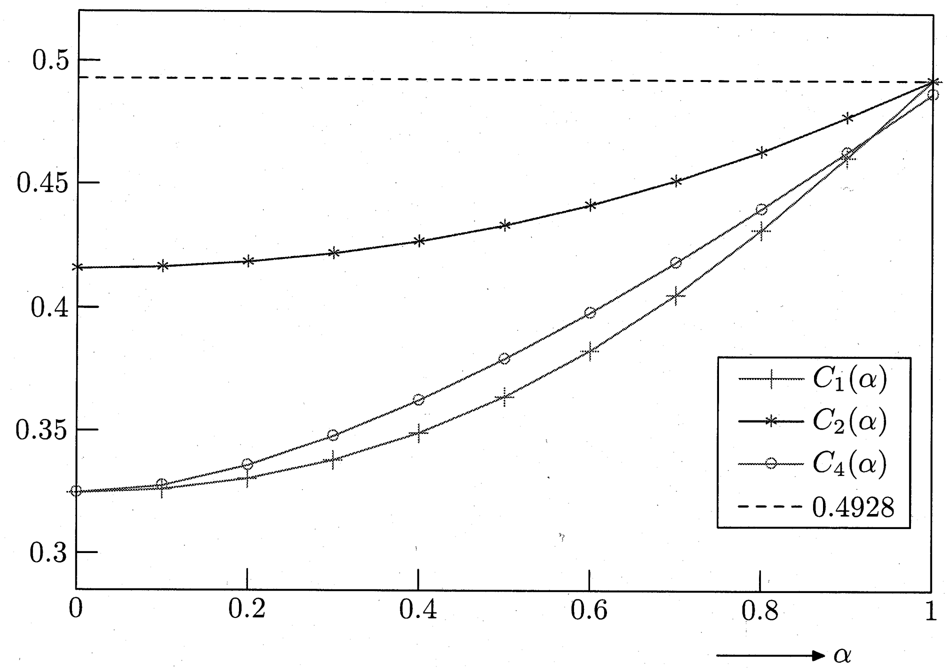


Figure 2.3: Numerical results for $C_1(\alpha), C_2(\alpha)$ and $C_4(\alpha)$

Theorem 2.3.2. *For each $\alpha \in (0, 1]$ and $\theta \in [\pi/3, \pi)$, the following relations hold:*

$$\psi_i(\theta)C_i(\alpha) \leq C_i(\alpha, \theta) \leq \phi_i(\theta)C_i(\alpha) \quad (0 \leq i \leq 5)\,. \tag{2.3.14}$$

Here,

$$\phi_i(\theta) = \sqrt{1 + |\cos\theta|}\ (i = 0, 1, 2, 3),\ \phi_4(\theta) = \frac{1 + |\cos\theta|}{\sqrt{1 - |\cos\theta|}},\ \phi_5(\theta) = 1 + |\cos\theta|\,; \tag{2.3.15}$$

$$\psi_i(\theta) = \sqrt{1 - |\cos\theta|}\ (i = 0, 1, 2, 3),\ \psi_4(\theta) = \frac{1 - |\cos\theta|}{\sqrt{1 + |\cos\theta|}},\ \psi_5(\theta) = 1 - |\cos\theta|\,. \tag{2.3.16}$$

Proof. Given the triangle $T_{\alpha,\theta}$, we define the affine transformation between $x = (x_1, x_2) \in T_{\alpha,\theta}$ and $\xi = (\xi_1, \xi_2) \in T_\alpha$ by (cf. Figure 2.4):

$$\xi_1 = x_1 - \cot\theta x_2, \quad \xi_2 = \frac{x_2}{\sin\theta}\,. \tag{2.3.17}$$

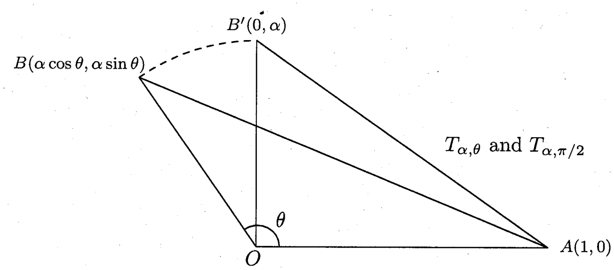


Figure 2.4: Transformation between $T_{\alpha,\theta}$ and $T_{\alpha,\pi/2}$

Further, define new function $\hat{u}(\xi_1, \xi_2) := u(x_1, x_2)$ over T_α. The transformation above in the matrix form is given by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ with } T = (t_{i,j})_{i,j=1,2} := \begin{pmatrix} 1 & -\cot\theta \\ 0 & 1/\sin\theta \end{pmatrix}.$$

To investigate the relation between the derivatives of the two functions $\hat{u}(\xi_1, \xi_2)$ and $u(x_1, x_2)$, we are required to consider the eigenvalue problem of the matrices related to the transformation (2.3.17). Firstly we give the Jacobian of this transformation,

$$\left| \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} \right| = \sin\theta\,. \tag{2.3.18}$$

The derivatives of u and $\hat{u}$ are related with each other as

$$\begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} = T^t \begin{pmatrix} \hat{u}_{\xi_1} \\ \hat{u}_{\xi_2} \end{pmatrix}, \quad \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} \\ u_{x_2x_1} & u_{x_2x_2} \end{pmatrix} = T^t \begin{pmatrix} \hat{u}_{\xi_1\xi_1} & \hat{u}_{\xi_1\xi_2} \\ \hat{u}_{\xi_2\xi_1} & \hat{u}_{\xi_2\xi_2} \end{pmatrix} T\,, \tag{2.3.19}$$

where we use the notations such as u_{x_i} to denote the partial derivative $\partial u/\partial x_i$ and denote by T^t the transpose of the matrix T.

Noticing that semi-norm $|u|_{T_{\alpha,\theta}}$ can be presented by $|u|^2_{2,T_{\alpha,\theta}} = \|\beta^t\beta\|_{T_{\alpha,\theta}}$ where β is a vector function defined by $\beta = (u_{x_1x_1}, u_{x_2,x_2}, \sqrt{2}u_{x_1x_2})$. we consider the following equations

$$\begin{pmatrix} u_{x_1x_1} \\ u_{x_2x_2} \\ \sqrt{2}\,u_{x_1x_2} \end{pmatrix} = \begin{pmatrix} t_{11}^2 & t_{21}^2 & \sqrt{2}\,t_{11}t_{21} \\ t_{12}^2 & t_{22}^2 & \sqrt{2}\,t_{12}t_{22} \\ \sqrt{2}\,t_{11}t_{12} & \sqrt{2}\,t_{21}t_{22} & t_{11}t_{22} + t_{12}t_{21} \end{pmatrix} \begin{pmatrix} \hat{u}_{\xi_1\xi_1} \\ \hat{u}_{\xi_2\xi_2} \\ \sqrt{2}\,\hat{u}_{\xi_1\xi_2} \end{pmatrix} := L^t \begin{pmatrix} \hat{u}_{\xi_1\xi_1} \\ \hat{u}_{\xi_2\xi_2} \\ \sqrt{2}\,\hat{u}_{\xi_1\xi_2} \end{pmatrix}.$$

Hence,

$$\begin{cases} \lambda_{\min}(TT^t)(\hat{u}^2_{\xi_1} + \hat{u}^2_{\xi_2}) \le (u^2_{x_1} + u^2_{x_2}) \le \lambda_{\max}(TT^t)(\hat{u}^2_{\xi_1} + \hat{u}^2_{\xi_2})\,, \\ \lambda_{\min}(LL^t) \sum_{1\le i,j\le 2} \hat{u}^2_{\xi_i\xi_j} \le \sum_{1\le i,j\le 2} u^2_{x_ix_j} \le \lambda_{\max}(LL^t) \sum_{1\le i,j\le 2} \hat{u}^2_{\xi_i\xi_j}\,, \end{cases}$$

where λ_{min} and λ_{max} denote respectively the minimum and maximum eigenvalues for the corresponding matrices:

$$TT^t = \frac{1}{\sin^2\theta}\begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix},$$

$$LL^t = \frac{1}{\sin^4\theta}\begin{pmatrix} 1 & \cos^2\theta & -\sqrt{2}\cos^2\theta \\ \cos^2\theta & 1 & -\sqrt{2}\cos\theta \\ -\sqrt{2}\cos^2\theta & -\sqrt{2}\cos\theta & 1+\cos^2\theta \end{pmatrix}.$$

As we can find that the eigenvalues of TT^t are $(1\pm|\cos\theta|)/\sin^2\theta$, and those of L^TL are $(1-\cos^2\theta)^{-1}$ and $(1\pm|\cos\theta|)^{-2}$, we have

$$\begin{cases} (1-|\cos\theta|)(\hat{u}_{\xi_1}^2+\hat{u}_{\xi_2}^2)/\sin^2\theta \ \leq (u_{x_1}^2+u_{x_2}^2) \leq (1+|\cos\theta|)(\hat{u}_{\xi_1}^2+\hat{u}_{\xi_2}^2)/\sin^2\theta\,, \\ \sum_{1\leq i,j\leq 2}(\hat{u}_{\xi_i\xi_j})^2/(1+|\cos\theta|)^2 \leq \sum_{1\leq i,j\leq 2}(u_{x_ix_j})^2 \leq \sum_{1\leq i,j\leq 2}(\hat{u}_{\xi_i\xi_j})^2/(1-|\cos\theta|)^2\,. \end{cases}$$

Adopting the Jacobian in (2.3.18) and the inequalities above, we have

$$\frac{(1-|\cos\theta|)/\sin\theta}{\sin\theta/(1-|\cos\theta|)^2}\cdot\frac{|\hat{u}|_{1,T_\alpha}^2}{|\hat{u}|_{2,T_\alpha}^2} \leq \frac{|u|_{1,T_{\alpha,\theta}}^2}{|u|_{2,T_{\alpha,\theta}}^2} \leq \frac{(1+|\cos\theta|)/\sin\theta}{\sin\theta/(1+|\cos\theta|)^2}\cdot\frac{|\hat{u}|_{1,T_\alpha}^2}{|\hat{u}|_{2,T_\alpha}^2}\,,$$

which finally leads to

$$\frac{1-|\cos\theta|}{\sqrt{1+|\cos\theta|}}C_4(\alpha) \leq C_4(\alpha,\theta) \leq \frac{1+|\cos\theta|}{\sqrt{1-|\cos\theta|}}C_4(\alpha)\,.$$

Similarly, we can obtain the estimates for the other constants.

□

Remark 2.3.2. *The results for the dependence of constants on θ are consistent with the well known maximum interior angle condition. That is, given a triangular element with bounded diameter, the smallest interior angle can tend to 0 while the $\Pi^1_{\alpha,\theta,h}$ interpolation error in H^1 norm is bounded if the maximum interior angle is bounded above from π. Babuška and Aziz proposed this condition in [6] by considering the transformation between $T_{\alpha,\theta}$ and $T_{\alpha\sin\theta,\pi/2}$ (See Figure 2.5).*

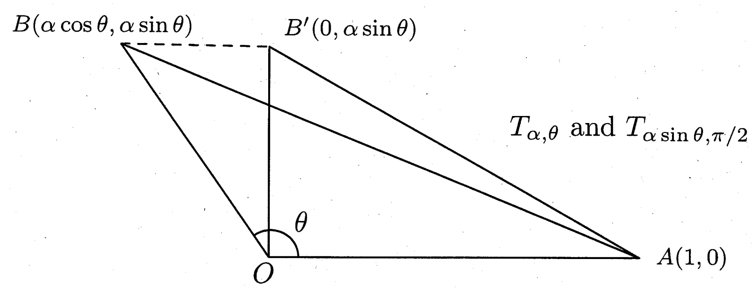


Figure 2.5: Transformation between $T_{\alpha,\theta}$ and $T_{\alpha\sin\theta,\pi/2}$

Remark 2.3.3. *If the values of $C_0, C_1 = C_2$ and C_5 are available, we can then give quantitative error estimates for the interpolation operators $\Pi^0_{\alpha,\theta,h}$ and $\Pi^1_{\alpha,\theta,h}$:*

$$\begin{aligned}
\Pi^0_{\alpha,\theta,h}: \quad & \|v - \Pi^0_{\alpha,\theta,h}v\|_{T_{\alpha,\theta,h}} \leq C_0\phi_0(\theta)h\,|v|_{1,T_{\alpha,\theta,h}}; && \forall v \in H^1(T_{\alpha,\theta,h}),\\
\Pi^1_{\alpha,\theta,h}: \quad & |v - \Pi^1_{\alpha,\theta,h}v|_{1,T_{\alpha,\theta,h}} \leq C_1\phi_4(\theta)h\,|v|_{2,T_{\alpha,\theta,h}}; && \forall v \in H^2(T_{\alpha,\theta,h}),\\
& \|v - \Pi^1_{\alpha,\theta,h}v\|_{T_{\alpha,\theta,h}} \leq C_5\phi_5(\theta)h^2\,|v|_{2,T_{\alpha,\theta,h}}; && \forall v \in H^2(T_{\alpha,\theta,h}).
\end{aligned} \tag{2.3.20}$$

In the following sections, we will determine the concrete values of C_0 and $C_1 = C_2$, while for C_5, we had a known rough upper bound as $C_5 \leq 0.361$ [21].

2.3.4 Natterer's estimate for $C_4(\alpha, \theta)$

To consider the dependence of the constants on geometric parameters, an intuitive idea is to consider the affine transformation between $T_{\alpha,\theta}$ and $T_{1,\pi/2}$. Such a method was in fact applied to give estimate for $C_4(\alpha,\theta)$ by F. Natterer [40]. Here we will apply this method to all the constants mentioned above, where the result of Natterer, expressed in our notations, is also included as a special case.

To this end, let us introduce the following simple affine transformation $\xi = \Phi_{\alpha,\theta}(x)$ between $x = (x_1, x_2) \in T_{\alpha,\theta}$ and $\eta = \{\xi_1, \xi_2\} \in T = T_{1,\pi/2}$ (See Fig 2.6):

$$\left\{\begin{array}{l} \xi_1 = x_1 - x_2\cot\theta \\ \xi_2 = x_2/(\alpha\sin\theta) \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} x_1 = \xi_1 + \xi_2\ \alpha\cos\theta \\ x_2 = \xi_2\ \alpha\sin\theta \end{array}\right. \tag{2.3.21}$$

In an analogous way as in the proof of Theorem 2.3.2, we can deduce the estimates as follows. For detailed proof, refer to Theorem 1 of [34].

Theorem 2.3.3. *For $\alpha \in (0, +\infty)$ and $\theta \in (0, \pi)$, $C_i(\alpha,\theta)$'s are bounded as*

$$\psi_i(\alpha,\theta)C_i \leq C_i(\alpha,\theta) \leq \phi_i(\alpha,\theta)C_i \quad (0 \leq i \leq 5), \tag{2.3.22}$$

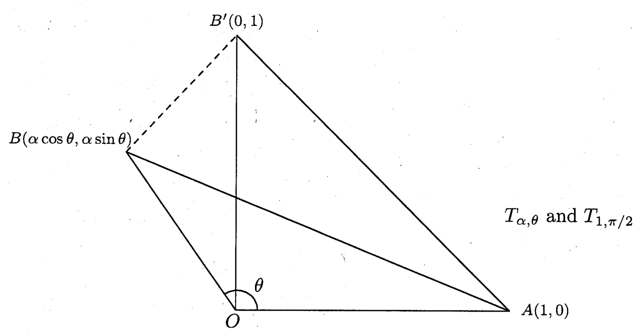


Figure 2.6: Transformation between $T_{\alpha,\theta}$ and $T_{1,\pi/2}$

where $C_i = C_i(1, \frac{\pi}{2})$ $(0 \le i \le 5)$,

$$\psi_i(\alpha,\theta) = \sqrt{\frac{\nu_-(\alpha,\theta)}{2}} (0 \le i \le 3), \psi_4(\alpha,\theta) = \frac{\nu_-(\alpha,\theta)}{\sqrt{2\nu_+(\alpha,\theta)}}, \psi_5(\alpha,\theta) = \frac{\nu_-(\alpha,\theta)}{2}, \tag{2.3.23}$$

$$\phi_i(\alpha,\theta) = \sqrt{\frac{\nu_+(\alpha,\theta)}{2}} (0 \le i \le 3), \phi_4(\alpha,\theta) = \frac{\nu_+(\alpha,\theta)}{\sqrt{2\nu_-(\alpha,\theta)}}, \phi_5(\alpha,\theta) = \frac{\nu_+(\alpha,\theta)}{2}, \tag{2.3.24}$$

with

$$\nu_- = 1 + \alpha^2 - \sqrt{1 + 2\alpha^2\cos 2\theta + \alpha^4}, \ \nu_+ = 1 + \alpha^2 + \sqrt{1 + 2\alpha^2\cos 2\theta + \alpha^4}. \tag{2.3.25}$$

It should be noticed that the upper bound for $C_4(\alpha,\theta)$ above is just the same one as Natterer's result [40], although the notation here is different from his.

Remark 2.3.4. *We can see that, except for* $i = 4$, *the upper bounds given for the constants are uniformly bounded as may be seen in Theorem 2.3.3. On the other hand, the upper bound for* $C_4(\alpha,\theta)$ *is not so, which will lead to the minimum angle condition [15]: the minimum angle of* $T_{\alpha,\theta}$ *is bounded above from below by a certain positive constant. This may be seen by using the identity* $\nu_-(\alpha,\theta)\nu_+(\alpha,\theta) = 4\alpha^2\sin^2\theta$ *and rewriting the upper bound inequality as*

$$C_4(\alpha,\theta) \le \frac{C_4}{\alpha\sin\theta}\left(\frac{\nu_+(\alpha,\theta)}{2}\right)^{\frac{3}{2}}. \tag{2.3.26}$$

Namely, we can see, for each fixed $\theta \in (0,\pi)$, the right hand side diverges to $+\infty$ as $\alpha \to +0$ or the minimum angle of the triangle tends to $+0$, which does not reflect the essential maximum angle condition. Hence, the above estimate for $C_4(\alpha,\theta)$ is weaker than the one in (3.2.15).

2.4 Estimation of the error constants

2.4.1 Exact value determination of particular constants

Up to now, we have analyzed the dependence of the constants on the geometric parameters. Here we will further consider determination of the exact values of the constants, which provides usually very difficult problems to solve. However, for several constants, we can use the symmetry method to give the concrete values of the constants in the case $T_{1,\pi/2}$.

Firstly, we summarize the results to be proved in this section.

Theorem 2.4.1. *As for the constants $C_i = C_i(1, \pi/2)$'s $(0 \le i \le 3)$, we have that*
1) $C_0 = 1/\pi$.
2) C_1 and C_2 satisfy $C_1 = C_2$ and are given as the maximum positive solution of the transcendental equation for μ:

$$\frac{1}{\mu} + \tan\frac{1}{\mu} = 0\,. \tag{2.4.1}$$

The concrete value of C_1 can be obtained numerically with verification. For example, we have the estimation as

$$0.49282 < C_1 < 0.49293\,. \tag{2.4.2}$$

3) $C_3 = C_1/\sqrt{2}$ and $0.34847 < C_3 < 0.34856$.

Remark 2.4.1. *Simple numerical algorithm without verification, such as the Newton method, gives $C_1 = 0.49291245\cdots$ and $C_3 = 0.34854173\cdots$. The present transcendental equation can be commonly seen in vibration analysis of strings with special boundary conditions [43]. The constant C_1 plays an important role in various situations and is called the Babuška-Aziz constant in [27, 28].*

Remark 2.4.2. *At present, $C_1(= C_2)$ is a nice upper bound of C_4 as we will see in Sections 2.4.2. Numerically $C_4 \approx 0.489$ as was reported in [4, 45, 47]. As for C_5, estimate $C_5 < 0.361$ is a correct but probably rough one given in [21], while an exact lower bound estimation is $C_5 \ge [(15 + \sqrt{193})/1440]^{1/2} = 0.1416\cdots$, which is derived by the Ritz-Galerkin method using $x_1 + x_2 - x_1^2 - x_2^2$ and x_1x_2 as the basis of the trial space employed in [36]. Our own numerical computations suggest that $C_5 \approx 0.168$.*

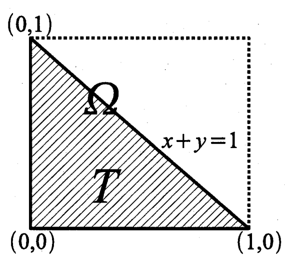
In the following, we will first demonstrate the method of symmetry by determining the value of C_0, which is actually already known, cf.[37]. Then, we show the proof for other constants.

Determination of C_0

As explained in the preview section, $\lambda_0 = C_0^{-2}$ is the minimum eigenvalue of the following eigenvalue problem: *Find* $\lambda > 0$ *and* $u \in V^0 \setminus \{0\}$ *such that*

$$(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in V^0 . \tag{2.4.3}$$

Within the following proof, instead of the notation (x_1, x_2), we will denote a point in 2-dimensional domains by (x, y). Let us modify the problem above to be the one over extended domain $\Omega = (0,1)^2$, a unit square. For each u in V^0, we can define an extended function $\hat{u}$ over Ω by reflection along $x + y = 1$, that is,



$$\hat{u}(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in T, \\ u(1-y, 1-x) & \text{if } (x,y) \in \Omega \setminus T , \end{cases} \tag{2.4.4}$$

where T is the original triangle domain already defined. We should be aware that $\hat{u}$ also belongs to $H^1(\Omega)$.

Define also a space $\hat{V}^0$ by

$$\hat{V}^0 = \{\hat{v} \in H^1(\Omega) | \int_\Omega \hat{v}(x,y)dxdy = 0\} , \tag{2.4.5}$$

then $\hat{V}^0$ can be expressed as a direct sum:

$$\hat{V}^0 = \hat{V}_s^0 \oplus \hat{V}_a^0 ,$$

where

$$\begin{cases} \hat{V}_s^0 = \text{ the set of functions in } \hat{V}^0 \text{ that are symmetric with respect to } x + y = 1 , \\ \hat{V}_a^0 = \text{ the set of functions in } \hat{V}^0 \text{ that are antisymmetric with respect to } x + y = 1 . \end{cases}$$

Moreover, $\hat{V}_s^0$ and $\hat{V}_a^0$ are orthogonal to each other in both $L_2(\Omega)$ and $H^1(\Omega)$. As a result, $\hat{V}_s$ and $\hat{V}_a$ are orthogonal with respect to the bi-linear forms $(\nabla\cdot, \nabla\cdot)_\Omega$. We can see the extension of eigenfunction of eigenvalue problem in (2.4.3) is also the one of the eigenvalue problem: *Find* $\lambda > 0$ *and* $\hat{u} \in \hat{V}^0 \setminus \{0\}$ *such that*

$$(\nabla \hat{u}, \nabla \hat{v}) = \lambda(\hat{u}, \hat{v}), \quad \forall \hat{v} \in \hat{V}^0. \tag{2.4.6}$$

On the other hand, the restriction of a symmetric eigenfunction $\hat{u}$ is also the one of (2.4.3). Therefore, it is sufficient to consider only the eigenvalue problem of (2.4.6).

As is well known, a complete system of functions for $H^1(\Omega)$ is given by the totality of eigenfunctions of (2.4.6) with $\hat{V}^0$ replaced with the whole $H^1(\Omega)$:

$$\varphi_{m,n}(x,y) = \cos m\pi x \cos n\pi y \quad (m, n \geq 0)\,.$$

Since we are interested in symmetric eigenfunctions only, we should make a complete system of symmetric functions in $H^1(\Omega)$ from the above: for $m \geq n; m, n = 0, 1, 2, 3, \cdots$,

$$\varphi_{m,n}(x,y) = \cos m\pi x \cos n\pi y + \cos m\pi(1-y) \cos n\pi(1-x)\,.$$

The functions above are orthogonal in $L_2(\Omega)$, and also orthogonal with respect to the bi-linear form $(\nabla\cdot, \nabla\cdot)_\Omega$ (and in $H^1(\Omega)$). A fact to be pointed out is that, except for $\varphi_{0,0} \equiv 2$, all $\varphi_{m,n}$'s for $m \geq n$ belong to $\hat{V}_s^0$ and are eigenfunctions of (2.4.6). Thus the desired eigenvalue λ_0 is π^2, which is just the one associated to $\varphi_{1,0}$. Hence, we obtain $C_0 = 1/\sqrt{\lambda_0} = 1/\pi$.

Determination of $C_1 = C_2$

Recall the corresponding eigenvalue problem for $\lambda_1 = C_1^{-2}$ in the variational form: *find* $u \in V^1 \setminus \{0\}$ *and* $\lambda > 0$ *such that*

$$(\nabla u, \nabla v)_T = \lambda(u, v)_T \quad \forall v \in V^1\,. \tag{2.4.7}$$

By adopting similar techniques used for C_0, we prove the second part of Theorem 2.4.1 in 5 steps:

Proof. **1)** In an analogous way, we consider the extended domain $\Omega = (0,1)^2$ and introduce a new space $\hat{V}^1$ on $\Omega = (0,1)^2$ by

$$\hat{V}^1(\Omega) = \{v \in H^1(\Omega) | \int_0^1 v(x,0)dx = \int_0^1 v(1,y)dy = 0\}\,. \tag{2.4.8}$$

In the same way as in (2.4.5), we decompose $\hat{V}^1$ into $\hat{V}^1 = \hat{V}^1_a \oplus \hat{V}^1_s$, where $\hat{V}^1_s$ is the subspace of symmetric functions and $\hat{V}^1_a$ the one of antisymmetric functions. As before, $\hat{V}^1_s$ and $\hat{V}^1_a$ are orthogonal to each other with respect to the inner products of both $L_2(\Omega)$ and $H^1(\Omega)$.

Let $\{\lambda, u\} \in R\times V^1\backslash\{0\}$ be one of eigenpairs of (2.4.7), and define the symmetric extension $\hat{u}$ over Ω by reflection with respect to $x+y=1$, c.f. (2.4.4). Then $\{\lambda, \hat{u}\}$ is an eigenpair of the eigenvalue problem over Ω: *Find* $\lambda > 0$ *and* $\hat{u} \in \hat{V}^1(\Omega) \setminus \{0\}$ *such that*

$$(\nabla \hat{u}, \nabla \hat{v}) = \lambda(\hat{u}, \hat{v}), \quad \forall \hat{v} \in \hat{V}^1(\Omega)\,. \tag{2.4.9}$$

Conversely, suppose $\hat{u}$ is one of symmetric eigenfunctions for problem (2.4.9), then the restriction of $\hat{u}$ to T is the also the one for (2.4.7). Consequently, for the present purposes, it suffices to deal with the eigenvalue problem in $V^1_s(\Omega)$: *Find* $\lambda > 0$ *and* $\hat{u} \in \hat{V}^1_s(\Omega) \setminus \{0\}$ *such that*

$$(\nabla \hat{u}, \nabla \hat{v}) = \lambda(\hat{u}, \hat{v}), \quad \forall \hat{v} \in \hat{V}^1_s(\Omega)\,. \tag{2.4.10}$$

2) We use the complete system of functions $\{\psi_{m,n}\}$ $(m \ge n; m, n = 0, 1, 2, ...)$ defined by

$$\psi_{m,n}(x,y) := \cos m\pi x \cos n\pi y + (-1)^{m+n} \cos n\pi x \cos m\pi y, m \ge n \ge 0\,.$$

A function $\hat{v} \in \hat{V}^1_s(\Omega)$ expressed by

$$\hat{v} = \sum_{m\ge n\ge 0}^{\infty} a_{m,n}\psi_{m,n} \quad (\, a_{m,n} \in \mathbb{R}\,)$$

must satisfy

$$\int_0^1 \hat{v}(x,0)dx = \int_0^1 \sum_{m,n\ge 0} a_{m,n}\psi_{m,n}(x,0)dx = 0 \text{ and } \int_\Omega (|\hat{v}|^2 + |D\hat{v}|^2)dxdy < \infty\,.$$

Hence,

$$2a_{0,0} + \sum_{m=1}^{\infty}(-1)^m a_{m,0} = 0 \;\text{ and }\; \sum_{m\ge n\ge 0}^{\infty} (1+m^2+n^2)a_{m,n}^2 < \infty\,.$$

We can show the sum of the series $\sum_{m=1}^{\infty}(-1)^m a_{m,0}$ is absolutely convergent under the condition imposed above on the coefficients. Eliminating $a_{0,0}$ by the above equation, every $\hat{v} \in \hat{V}^1_s$ is expressed by $\hat{v} = \sum_{m=1}^{\infty} a_{m,0}[\psi_{m,0} - (-1)^m] + \sum_{m\ge n\ge 1}^{\infty} a_{m,n}\psi_{m,n}$.

Clearly, $\psi_{m,n}$'s for $m \geq n \geq 1$ are eigenfunctions of (2.4.9) with completely homogeneous Neumann's boundary condition, and the minimum of the associated eigenvalues is $2\pi^2$.

3) Let W_1 be the closure of linear combinations of $\psi_{m,0} - (-1)^m (m \geq 1)$ and W_2 the closure of linear combinations of $\psi_{m,n}(m \geq n \geq 1)$. We have $\hat{V}_s^1 = W_1 \oplus W_2$. Here, W_1 and W_2 are orthogonal to each other in both $L_2(\Omega)$ and $H^1(\Omega)$. Since all the eigenfunctions and associated eigenvalues of W_2 are known and the smallest one to be $2\pi^2$, we just need to consider the eigenvalues in W_1: if its minimum is smaller than $2\pi^2$, it is just the one we need.

4) Let us now solve the eigenvalue problem restricted to W_1 by expressing $\hat{u} \in W_1 \setminus \{0\}$ by

$$\hat{u} = \sum_{m=1}^{\infty} a_m \phi_m \text{ with } \sum_{m=1}^{\infty} a_m^2 < \infty, \text{ where } \phi_m = \psi_{m,0} - (-1)^m . \tag{2.4.11}$$

Noting that $\hat{u}$ has the form $\hat{u} = \sum_{m=1}^{\infty} a_m \left(\cos m\pi x + (-1)^m \cos m\pi(y) + (-1)^m\right)$, it must be of the form, for an unknown single variable function $g = g(t)$,

$$\hat{u}(x, y) = g(x) + g(1 - y) .$$

Substituting the expression above into (2.4.9), we have

$$-g''(t) = \lambda g(t)(0 < t < 1),\ g'(0) = 0,\ g(1) + \int_0^1 g(t)dt = 0 .$$

Solving the eigenvalue problem above, we have that the eigenfunction associated with the smallest eigenvalue is $g(t) = \cos(\sqrt{\lambda_1} t)$, where λ_1 is the first positive root of

$$\sqrt{\lambda} + \tan\sqrt{\lambda} = 0 .$$

Clearly, λ_1 lies in the interval $(\pi^2/4, \pi^2)$ and is the unique solution there. Since $\lambda_1 < 2\pi^2$, it is exactly the desired eigenvalue of eigenvalue problem in (2.4.10). Moreover, an eigenfunction associated to λ_1 is $\hat{u}(x, y) = \cos\sqrt{\lambda_1}x + \cos\sqrt{\lambda_2}(1 - y)$.

5) To obtain the concrete value of $\sqrt{\lambda_1}$, we are just required to find the first positive root of

$$f(t) := \cos t + t^{-1} \sin t = 2 \sum_{m=0}^{\infty} \frac{(-1)^m (m + 1) t^{2m}}{(2m + 1)!} \ (t > 0) .$$

Moreover, the series above is an alternating one, and for a fixed t and sufficiently large m, the absolute values of its terms converge monotonically to 0 as $m \to \infty$. Thus we can use its truncated finite series to give both lower and upper bounds for $f(t)$. Let us define f_n by

$$f_n(t) = 2\sum_{m=0}^{n} \frac{(-1)^m (m+1) t^{2m}}{(2m+1)!} .$$

It is to be noted here that, as least in principle, all the computations can be performed in the finite-digit binary arithmetic without rounding errors, provided t is a rational number. For example, by taking $n = 4, 5$, we can bound $t_0 = \sqrt{\lambda_1}$ as $2.0287 < t_0 < 2.0291$, since $f(2.0291) < f_4(2.2091) < 0$ and $f(2.2087) > f_5(2.2087) > 0$. $\square$

Remark: Here we show another way to derive the determination equation for C_1. Substituting (2.4.11) into (2.4.9) and letting $\hat{v}_s$ be each ψ_m, we have the equations for coefficients a_m's:

$$(m^2\pi^2 - \lambda)a_m = \lambda(-1)^m \sum_{n=1}^{\infty} (-1)^n a_n \quad (m \in N) ,$$

where we can show $\sum_{n=1}^{\infty}(-1)^n a_n \neq 0$, $\lambda \neq m^2\pi^2$ and $a_m \neq 0$ $(\forall m \in N)$. So

$$(-1)^m a_m = (m^2\pi^2 - \lambda)^{-1}\lambda \sum_{n=1}^{\infty} (-1)^n a_n \quad (m \in N)$$

and

$$\sum_{m=1}^{\infty} (-1)^m a_m = \sum_{m=1}^{\infty} (m^2\pi^2 - \lambda)^{-1}\lambda \sum_{n=1}^{\infty} (-1)^n a_n .$$

Hence

$$1 = \sum_{m=1}^{\infty} \frac{\lambda}{m^2\pi^2 - \lambda} = \sum_{m=1}^{\infty} \frac{1}{m^2(\pi/\sqrt{\lambda})^2 - 1} .$$

Notice here the Fourier expansion of $\cos ax$ on $[-\pi, \pi]$:

$$\cos ax = \frac{\sin a\pi}{\pi} \left(1/2 + \sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 - n^2} \cos nx \right) ,$$

where a is a non-integer real number. Letting $x = \pi$ above, we have

$$\sum_{m=1}^{\infty} \frac{1}{(m/a)^2 - 1} = \frac{1}{2} - \frac{\pi a}{2\tan \pi a} .$$

Further, substituting $a = \sqrt{\lambda}/\pi$ into equation above, we obtain (2.4.1).

Determination of C_3

The method for determining C_3 determining is essentially the same as used for C_0 and $C_1 = C_2$. Here we show the outline of the proof in three steps.

1) The eigenvalue problem associated to C_3 is given by: *Find* $\{\lambda, u\} \in \mathbb{R} \times V^3 \setminus \{0\}$ *such that*

$$(\nabla u, \nabla v)_T = \lambda(u, v)_T \quad (\forall v \in V^3) . \tag{2.4.12}$$

Here, T is the unit right isosceles triangle $T_{1,\pi/2,1}$, $V^3 = V^3_{1,\pi/2,1}$ is defined in (2.2.3), and the inner products are those for T. Notice that we are interested only in the minimum eigenvalue and the associated eigenfunctions.

Let us divide T into two congruent parts by the line $x_1 = x_2$, which is also the line of symmetry for T. Moreover, one of the congruent parts is denoted by $\hat{T}$:

$$\hat{T} = \{x = \{x_1, x_2\} \in T; x_1 > x_2\} .$$

The eigenfunction $u \neq 0$ can be uniquely decomposed into the symmetric part u_s and the antisymmetric one u_a:

$$u = u_s + u_a,$$

where the symmetry and antisymmetry are those with respect to $x_1 = x_2$. Due to the orthogonality of u_s and u_a for the bi-linear forms $(\cdot, \cdot)_T$ and $(\nabla \cdot, \nabla \cdot)_T$, the functions u_s and u_a can be dealt with separately: u_s and u_a both belong to V^3 and satisfy (2.4.12) for the minimum eigenvalue λ.

2) We first consider the case where $u_s \neq 0$. In this case, the restriction $\hat{u}$ of u_s to $\hat{T}$ is not zero and satisfies the following eigenvalue problem related to $\hat{T}$:

$$\hat{u} \in \hat{V}^3 \setminus \{0\}; (\nabla \hat{u}, \nabla \hat{v})_{\hat{T}} = \lambda(\hat{u}, \hat{v})_{\hat{T}} \quad (\forall \hat{v} \in \hat{V}^3) , \tag{2.4.13}$$

where λ is identical to the former one, the inner products are the L_2 ones for $\hat{T}$, and $\hat{V}^3$ is defined by

$$\hat{V}^3 = \{\hat{v} \in H^1(\hat{T}); \int_0^{\frac{1}{2}} \hat{v}(1-s, s) ds = 0 \ \} . \tag{2.4.14}$$

Now we can see that this is essentially the same problem as the eigenvalue problem for $C_1(1, \pi/2, 1/\sqrt{2})$, since $\hat{T}$ is congruent to $T_{1,\pi/2,1/\sqrt{2}}$. It is also fairly easy to see that the eigenpair for the minimum eigenvalue of (2.4.13) satisfies (2.4.12), if the eigenfunction is extended to whole T symmetrically with respect to $x_1 = x_2$. Thus $\hat{u}$ is an eigenfunction for the minimum eigenvalue of (2.4.12) in the present case. Then we find that $C_3 = C_1/\sqrt{2}$, since $C_1(\alpha, \theta, 1/\sqrt{2}) = C_1(\alpha, \theta)/\sqrt{2}$. Of course, this conclusion is derived under the assumption $u_s \neq 0$.

3) Secondly, we consider the case where $u_a \neq 0$. Due to the antisymmetry, the trace of u_a to the line of symmetry $x_1 = x_2$ inside T is just 0. Moreover, any antisymmetric function in $H^1(T)$ automatically satisfies the line integration condition imposed on V^3. Thus the restriction $u^\dagger$ of u_a to $\hat{T}$ is not zero and is an eigenfunction of the eigenvalue problem:

$$u^\dagger \in V^\dagger \setminus \{0\}; (\nabla u^\dagger, \nabla v^\dagger = \lambda(u^\dagger, v^\dagger)_{T^\dagger} \quad (\forall v^\dagger \in V^\dagger)\,, \tag{2.4.15}$$

where λ is identical to the former one, and $V^\dagger$ is defined by

$$V^\dagger = \{v^\dagger \in H^1(T^\dagger); v^\dagger(s, s) = 0 \quad (0 < s < \frac{1}{2})\ \}\,.$$

If we consider the reflection with respect to the line $x_1 = 1/2$, (2.4.15) becomes the problem of the same form if we replace $V^\dagger$ by

$$V^* = \{v^* \in H^1(\hat{T});\ v^*(1-s, s) = 0(0 < s < \frac{1}{2})\}\,.$$

Clearly, the eigenvalues remain the same under such a transformation. Since $V^* \subset \hat{V}^3$, the minimum eigenvalue of (2.4.15) cannot be smaller than that of (2.4.13), as can be seen by considering the characterization of the minimum eigenvalue by the Rayleigh quotient. Thus it is sufficient to consider only the case where $u_s \neq 0$, and the proof is complete.

2.4.2 Estimating $C_4(\alpha, \theta)$ by $C_i(\alpha, \theta)$'s ($i = 1, 2, 3$)

In section 2.3.1, we extend the method of Babuška-Aziz to deduce an upper bound for $C_4(\alpha)$. Here we will further consider the problem of estimating $C_4(\alpha, \theta)$ by using $C_i(\alpha, \theta)$'s ($i = 1, 2, 3$).

Firstly, let us observe the characterization of $C_4(\alpha, \theta)$ again:

$$C_4(\alpha, \theta)^2 = \sup_{u \in V^4_{\alpha,\theta} \setminus \{0\}} \frac{|u|^2_{1,T_{\alpha,\theta}}}{|u|^2_{2,T_{\alpha,\theta}}} = \sup_{u \in V^4_{\alpha,\theta} \setminus \{0\}} \frac{\|\partial_1 u\|^2_{T_{\alpha,\theta}} + \|\partial_2 u\|^2_{T_{\alpha,\theta}}}{|\partial_1 u|^2_{1,T_{\alpha,\theta}} + |\partial_2 u|^2_{1,T_{\alpha,\theta}}}\,.$$

The key idea for estimating $C_4(\alpha, \theta)$ is to relax the curl-free condition $\partial_{12}u = \partial_{21}u$ by weaker ones, e.g. $\int_{e_i} \nabla u \cdot t_i ds = 0$ for $i = 1, 2, 3$, where t_i denotes the unit vector along the direction of the edges e_i in clockwise, that is

$$t_1 = (-1, 0), \quad t_2 = (\cos\theta, \sin\theta), \quad t_3 = \frac{(1-\cos\theta, -\sin\theta)}{\sqrt{2(1-\cos\theta)}} .$$

Let us introduce two constants $C_{\{4,e12\}}(\alpha, \theta, h)$ and $C_{\{4,e123\}}(\alpha, \theta, h)$ by

$$C_{\{4,e123\}}(\alpha, \theta, h)^2 := \sup_{\substack{u, v \in H^1(T_{\alpha,\theta,h}) \setminus \{0\} \\ (u,v)\cdot t_i \in V^i_{\alpha,\theta,h} (i=1,2,3)}} \frac{\|u\|^2 + \|v\|^2}{\|\nabla u\|^2 + \|\nabla v\|^2} \tag{2.4.16}$$

and

$$C_{\{4,e12\}}(\alpha, \theta, h)^2 := \sup_{\substack{u, v \in H^1(T_{\alpha,\theta,h}) \setminus \{0\} \\ (u,v)\cdot t_i \in V^i_{\alpha,\theta,h} (i=1,2)}} \frac{\|u\|^2 + \|v\|^2}{\|\nabla u\|^2 + \|\nabla v\|^2} . \tag{2.4.17}$$

Denote $C_i(\alpha, \theta, 1)$ by $C_i(\alpha, \theta)$ for $i = \{4, e12\}, \{4, e123\}$. Then we find

$$C_4(\alpha, \theta) \le C_{\{4,e123\}}(\alpha, \theta) \le C_{\{4,e12\}}(\alpha, \theta) . \tag{2.4.18}$$

Firstly, we will utilize the second inequality in (2.4.18) to give an explicit upper bounds for $C_4(\alpha, \theta)$ by using $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$. One thing to be pointed out is that the values of $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$ can be well evaluated with a posteriori estimates, as we will discuss in Chapter 3.

Theorem 2.4.2. *Given a triangle $T_{\alpha,\theta}$ for $\alpha \in (0, 1]$ and $\theta \in (0, \pi)$, we can give an upper bound for $C_4(\alpha, \theta)$ in terms of $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$ as below: (We write $C_1(\alpha, \theta)$, $C_2(\alpha, \theta)$ as c_1, c_2 for purpose of abbreviation.)*

$$C_4(\alpha, \theta) \le \frac{1}{\sqrt{2}\sin\theta} \left(c_1^2 + c_2^2 + 2c_1c_2 \cos^2\theta + (c_1 + c_2)\sqrt{(c_1 - c_2)^2 + 4c_1c_2\cos^2\theta} \right)^{1/2} . \tag{2.4.19}$$

Proof. For any $w \in H^2(T_{\alpha,\theta})$, let $u := \partial_1 w$ and $v := \partial_2 w$ and introduce a new quantity $\hat{v} := u\cos\theta + v\sin\theta \in H^1(T_{\alpha,\theta})$. Clearly, $u \in V^1_{\alpha,\theta}$. By noticing that

$$\int_{e2} (u, v) \cdot t_2 ds = \int_{e2} u\cos\theta + v\sin\theta ds = 0 ,$$

we have $\int_{e_2} \hat{v} = 0$, which means $\hat{v} \in V^2_{\alpha,\theta}$.

Considering the definitions of $C_1(\alpha, \theta)$ and $C_2(\alpha, \theta)$, such results are clear:

$$\|u\| \le C_1(\alpha,\theta)\|\nabla u\|, \quad \|\hat{v}\| \le C_2(\alpha,\theta)\|\nabla \hat{v}\| . \tag{2.4.20}$$

As $v = (\hat{v} - u\cos\theta)/\sin\theta$, we have

$$\begin{aligned} \|v\|^2 &= \sin^{-2}\theta \|\hat{v} - u\cos\theta\|^2 \\ &= \sin^{-2}\theta \left(\|\hat{v}\|^2 + \cos^2\theta\|u\|^2 - 2\cos\theta(\hat{v}, u)\right) \\ &\le \sin^{-2}\theta \left(\|\hat{v}\|^2 + \cos^2\theta\|u\|^2 + 2|\cos\theta|\|\hat{v}\|\|u\|\right) . \end{aligned}$$

So we obtain:

$$\|u\|^2 + \|v\|^2 \le \sin^{-2}\theta \left(\|u\|^2 + \|\hat{v}\|^2 + 2|\cos\theta|\, \|\hat{v}\|\, \|u\|\right) . \tag{2.4.21}$$

Similarly, we have

$$\|\nabla u\|^2 + \|\nabla v\|^2 \ge \sin^{-2}\theta \left(\|\nabla u\|^2 + \|\nabla \hat{v}\|^2 - 2|\cos\theta|\, \|\nabla\hat{v}\|\, \|\nabla u\|\right) . \tag{2.4.22}$$

Considering (2.4.18), we find

$$C_4(\alpha,\theta)^2 \le C_{\{4,e12\}}(\alpha,\theta)^2 \le \frac{\|u\|^2 + \|v\|^2}{\|\nabla u\|^2 + \|\nabla v\|^2} .$$

Now, considering the inequalities (2.4.20) and those of (2.4.21) and (2.4.22), we have

$$\begin{aligned} C_4(\alpha,\theta)^2 &\le \frac{\sin^{-2}\theta\,(\|u\|^2 + \|\hat{v}\|^2 + 2|\cos\theta|\|\hat{v}\|\,\|u\|)}{\sin^{-2}\theta\,(\|\nabla u\|^2 + \|\nabla\hat{v}\|^2 - 2|\cos\theta|\|\nabla\hat{v}\|\,\|\nabla u\|)} \\ &\le \frac{c_1^2\|\nabla u\|^2 + c_2^2\|\nabla\hat{v}\|^2 + 2c_1c_2|\cos\theta|\,\|\nabla\hat{v}\|\,\|\nabla u\|}{\|\nabla u\|^2 + \|\nabla\hat{v}\|^2 - 2|\cos\theta|\|\nabla\hat{v}\|\,\|\nabla u\|} \\ &= \frac{e^tAe}{e^tBe} , \end{aligned}$$

where e is the vector $(\|\nabla u\|, \|\nabla\hat{v}\|)^t$, and A and B are the matrices defined by

$$A = \begin{pmatrix} c_1^2 & c_1c_2\,|\cos\theta| \\ c_1c_2\,|\cos\theta| & c_2^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -|\cos\theta| \\ -|\cos\theta| & 1 \end{pmatrix} .$$

The generalized eigenvalue problem $Ax = \lambda Bx$ has the maximum eigenvalue as

$$\lambda_{max} = \frac{1}{2\sin^2\theta}\left(c_1^2 + c_2^2 + 2\cos^2\theta c_1c_2 + (c_1 + c_2)\sqrt{(c_1 - c_2)^2 + 4\cos^2\theta c_1c_2}\right) .$$

So, for arbitrary vector $e \ne 0$,

$$\frac{e^tAe}{e^tBe} \le \lambda_{max} .$$

Thus, we obtain one upper bound for $C_4(\alpha,\theta)$, which is just the one in (2.4.19). $\square$

Remark 2.4.3. *We can get the same estimation of $C_4(\alpha,\theta)$ as the one in (2.4.19) in another way. From equation (2.4.21), we have*

$$\begin{aligned}
\sin^2\theta(\|u\|^2+\|v\|^2) &\le c_1^2\|\nabla u\|^2+2c_1c_2|\cos\theta|\|\nabla u\|\|\nabla\hat{v}\|+c_2^2\|\hat{v}\|^2\\
&= c_1^2\|\nabla u\|^2+2c_1c_2|\cos\theta|\,\|\nabla u\|\,\|\cos\theta\nabla u+\sin\theta\nabla v\|\\
&\quad +c_2^2\|\cos\theta\nabla u+\sin\theta\nabla v\|\\
&\le (c_1^2+2c_1c_2\cos^2\theta+c_2^2\cos^2\theta)\|\nabla u\|^2+c_2^2\sin^2\theta\|\nabla v\|^2\\
&\quad +2(c_1c_2+c_2^2)\sin\theta\,|\cos\theta|\|\nabla u\|\|\nabla v\|\\
&=: \tilde{e}^t\tilde{A}\tilde{e}\,,
\end{aligned}$$

where $\tilde{e}=(\|\nabla u\|,\|\nabla v\|)^t$, and $\tilde{A}$ is defined by

$$\tilde{A}:=\begin{pmatrix} c_1^2+2c_1c_2\cos^2\theta+\cos^2\theta c_2^2 & (c_1c_2+c_2^2)\sin\theta\,|\cos\theta| \\ (c_1c_2+c_2^2)\sin\theta\,|\cos\theta| & c_2^2\sin^2\theta \end{pmatrix}$$

which has the maximum eigenvalue $\tilde{\lambda}_{max}$ as

$$\tilde{\lambda}_{max}=\frac{1}{2}\left(c_1^2+c_2^2+2\cos^2\theta c_1c_2+(c_1+c_2)\sqrt{(c_1-c_2)^2+4\cos^2\theta c_1c_2}\right)\,.$$

Hence

$$\|u\|^2+\|v\|^2\le\sin^{-2}\theta\,\tilde{\lambda}_{max}\,(\|\nabla u\|^2+\|\nabla v\|^2)\,,$$

which finally leads to the estimate in (2.4.19).

Remark 2.4.4. *In the proof above, the intermediate problem of $C_{\{4,e12\}}(\alpha,\theta)$ gives an estimate for $C_4(\alpha,\theta)$ as in (2.4.19). Another possibility is to apply the constant $C_{\{4,e123\}}(\alpha,\theta)$ to deduce a new estimate, which is very interesting but not done yet.*

An important thing to be pointed out is that, through the deduction of (2.4.19), there may be over and under estimates in the inequalities (2.4.21) and (2.4.22). Therefore, to have better estimates, we may evaluate the constants $C_{\{4,e12\}}(\alpha,\theta)$ and $C_{\{4,e123\}}(\alpha,\theta)$ directly. This is not difficult since the derivatives of functions associated to the constants are only of second order. The piecewise linear finite element space can be used to construct conforming subspaces of $H^1(\Omega)^2$ with the corresponding constraint conditions satisfied. Also, it may be possible to give a posteriori error estimates for the finite element solutions, which will be left for our future research. For the moment, we have only executed numerical computations.

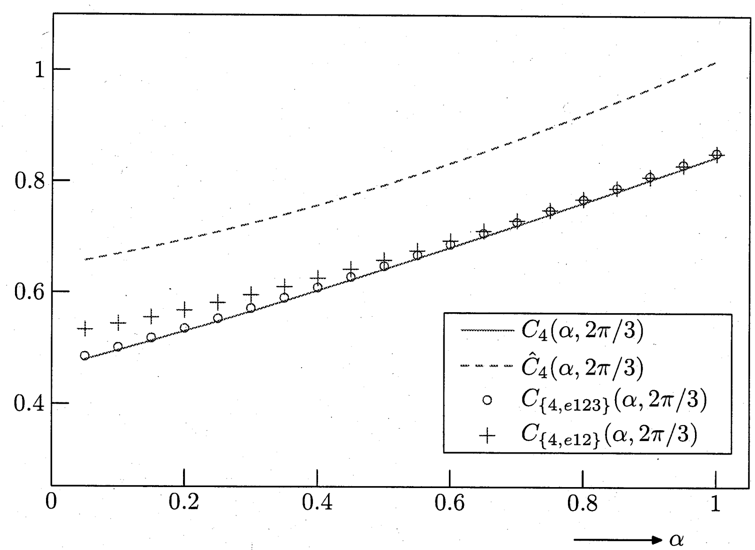


Figure 2.7: Upper bounds for $C_4(\alpha, 2\pi/3)$

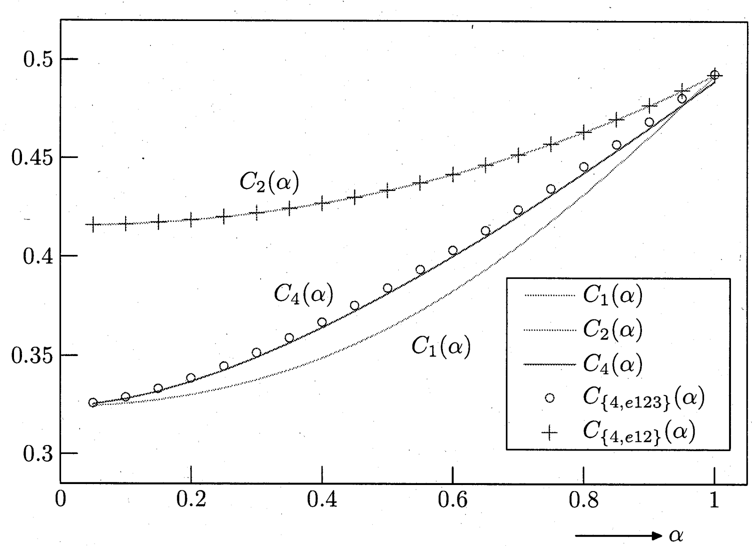


Figure 2.8: Upper bounds for $C_4(\alpha, \pi/2)$

In Figure 2.7, we show the estimate of (2.4.19), which we denote as $\hat{C}_4(\alpha,\theta)$*, and the numerical evaluation of* $C_{\{4,e12\}}(\alpha,\theta)$ *in the case of* $\theta = 2\pi/3$*. We can see that* $C_{\{4,e123\}}(\alpha, 2\pi/3)$ *gives quite good upper bound for* $C_4(\alpha, 2\pi/3)$*. Although the gap between* $C_{\{4,e123\}}(\alpha, 2\pi/3)$ *and* $C_4(\alpha, 2\pi/3)$ *is very small, we cannot expect it to be zero.*

As a complement, we also show the computational results for $\theta = \pi/2$ *in Figure 2.8. Again both* $C_{\{4,e12\}}$ *and* $C_{\{4,e123\}}$ *give upper bound for* $C_4(\alpha)$*. Now it is clear to see the difference between* $C_4(1, \pi/2)$ *and* $C_{\{4,e123\}}(1, \pi/2)$*, although here only the approximate values are available. Also, we can find that the numerical values of* $C_2(\alpha)$ *agrees with* $C_{\{4,e12\}}(\alpha)$*. We can easily show that* $C_{\{4,e12\}}(\alpha) = \max\{C_1(\alpha), C_2(\alpha)\}$*, but are not yet able to prove that* $\max\{C_1(\alpha), C_2(\alpha)\} = C_2(\alpha)$ *or* $C_2(\alpha) \geq C_1(\alpha)$ *for* $\alpha \leq 1$*.*

2.5 Asymptotic behaviour of error constants on slender triangular domain

2.5.1 Preliminary and main results

We will now analyze the asymptotic behaviors of the constants $C_i(\alpha)$'s ($0 \leq i \leq 5$) as $\alpha \to +0$ by adopting various techniques developed e.g. in [33]. In particular, the right limit values $C_i(+0)$'s are given by zeros of certain transcendental equations (derived from eigenvalue problems of ordinary differential equations, ODE's) in terms of the hyper-geometric functions [51]. For example, $C_2(+0)^{-1}$ is equal to the first positive zero of the Bessel function $J_0(x)$. Moreover, these right limits give lower bounds for respective $C_i(\alpha)$'s, including the non-trivial case $i = 4$. Such results can be of use for understanding and analyzing the so called "anisotropic triangulations" discussed e.g. in [1, 8, 20].

We first introduce several function spaces, which play important role in the following discussion.

$$H^{k,Z}(T) = \{v \in H^k(T); \partial v/\partial x_2 = 0\} \quad (k = 1, 2), \tag{2.5.1}$$

$$V^{i,Z} = \{v \in V^i; \partial v/\partial x_2 = 0\} \quad (0 \leq i \leq 4), \tag{2.5.2}$$

which are actually identified with the spaces of functions dependent only on the variable x_1 as we will see later. Let us introduce bilinear forms $a_Z^{(i)}(\cdot,\cdot)$'s for $i = 1, 2$:

$$a_Z^{(1)}(u, v) := \left(\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1}\right)_T, \quad \forall u, v \in H^1(T), \tag{2.5.3}$$

$$a_Z^{(2)}(u,v) := \left(\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_1^2}\right)_T , \quad \forall u, v \in H^2(T) . \tag{2.5.4}$$

Although these are defined over the whole H^1 and H^2 spaces for convenience, the partial derivatives above can be actually replaced with the ordinary ones when they are considered over the respective $H^{1,Z}$ and $H^{2,Z}$ spaces.

As a characterization of the above $H^{1,Z}(T)$, let us state a fundamental lemma to be used for our analysis. Its proof is omitted here since it can be performed by slightly modifying that for Theorem 3.1.4 of [23]. Of course, we can draw the same conclusions for other spaces mentioned in (2.5.1) and (2.5.2).

Lemma 2.5.1. *Any $v \in H^{1,Z}(T)$ can be identified with a function v^* of single variable x_1:*

$$v(x_1, x_2) = v^*(x_1) \text{ for a.e. } x = \{x_1, x_2\} \in T . \tag{2.5.5}$$

Remark 2.5.1. *The present lemma does not necessarily hold for general domains. In the 2-dimensional case where we are considering here, it holds for a domain $\Omega \subset \mathbb{R}^2$ which is "connected in x_2 direction" in the sense that for any two points x and x' in Ω with a common x_1 component, the segment connecting these points is contained in Ω.*

We first quoted the main results below, while the proof is given in the following sub-sections.

Theorem 2.5.1. *For each i $(0 \le i \le 5)$, $C_i(+0) = \lim_{\alpha\to+0} C_i(\alpha)$ exists and is positive. Moreover, they are the lower limits of the respective constants, i.e., $C_i(+0) = \inf_{\alpha>0} C_i(\alpha)$ for $0 \le i \le 5$. They are characterized by $C_i(+0) = 1/\sqrt{\lambda^{(i)}}$ for $0 \le i \le 5$, where $\lambda^{(i)}$'s are the minimum eigenvalues of the following eigenvalue problems.*

$0 \le i \le 3$*: Find $\lambda = \lambda^{(i)} \in \mathbb{R}$ and $u \in V^{i,Z} \setminus \{0\}$ such that*

$$a_Z^{(1)}(u,v) = \lambda(u,v)_T; \forall v \in V^{i,Z}, \tag{2.5.6}$$

$i = 4$*: Find $\lambda = \lambda^{(4)} \in \mathbb{R}$ and $u \in V^{4,Z} \setminus \{0\}$ such that*

$$a_Z^{(2)}(u,v) = \lambda a_Z^{(1)}(u,v)_T; \forall v \in V^{4,Z}, \tag{2.5.7}$$

$i = 5$*: Find $\lambda = \lambda^{(5)} \in \mathbb{R}$ and $u \in V^{4,Z} \setminus \{0\}$ such that*

$$a_Z^{(2)}(u,v) = \lambda(u,v)_T; \forall v \in V^{4,Z} . \tag{2.5.8}$$

These eigenvalue problems are also expressed by those for the following 2rd- or 4th-order ordinary differential equations for $u = u(s)$ over the interval $[0, 1]$.

$i = 0$:

$$-[(1-s)u'(s)]' = \lambda^{(0)}(1-s)u(s) \quad (0 < s < 1), \int_0^1 (1-s)u(s)ds = u'(0) = 0, \quad (2.5.9)$$

$i = 1$:

$$-[(1-s)u'(s)]' = \lambda^{(1)}(1-s)u(s)+C \quad (0 < s < 1), \int_0^1 u(s)ds = u'(0) = 0, \quad (2.5.10)$$

$i = 2$:

$$-[(1-s)u'(s)]' = \lambda^{(2)}(1-s)u(s) \quad (0 < s < 1),\ u(0) = 0, \quad (2.5.11)$$

$i = 3$: *essentially the same as for* $i = 1$;

$$-[(1-s)u'(s)]' = \lambda^{(3)}(1-s)u(s)+C \quad (0 < s < 1), \int_0^1 u(s)ds = u'(0) = 0, \quad (2.5.12)$$

$i = 4$: *actually reduces to the case* $i = 1$;

$$[(1-s)u''(s)]'' = -\lambda^{(4)}[(1-s)u'(s)]' \quad (0 < s < 1),\ u(0) = u(1) = u''(0) = 0\,, \quad (2.5.13)$$

$i = 5$:

$$[(1-s)u''(s)]'' = \lambda^{(5)}(1-s)u(s) \quad (0 < s < 1),\ u(0) = u(1) = u''(0) = 0\,. \quad (2.5.14)$$

Here, C *is an unknown constant to be determined simultaneously with* u *and* $\lambda^i (i = 1, 3)$.

Recall that the triangle T here is still referred as a unit isosceles right triangle. Let us also recall the definition of Rayleigh quotients $\hat{R}_\alpha^{(i)}$'s defined in equations (2.3.2), (2.3.3), (2.3.4), and introduce new quantities $\lambda_i(\alpha)$'s by

$$\lambda_i(\alpha) := C_i(\alpha)^{-2} = \inf_{v \in V^i \setminus \{0\}} \hat{R}_\alpha^{(i)}(v) \quad (0 \le i \le 5)\,. \quad (2.5.15)$$

Uniform boundedness of $\lambda_i(\alpha)$'s: One of the common important properties for these constants is that $\lambda_i(\alpha)$'s are uniformly bounded for $\alpha \in (0, \infty)$, because for a fixed $w \in V^{\min\{i,4\}}$ with $w \neq 0$ and $\partial_2 w \equiv 0$,

$$\lambda_i(\alpha) \le \hat{R}_\alpha^{(i)}(w) \equiv \tilde{C}^{(i)}\ (0 \le i \le 5)\,, \quad (2.5.16)$$

where the right-hand sides are constants independent of α for a fixed w.

For $i \neq 4$, the proofs for the determination of $\lambda_i(+0)$'s are similar, so we will only show the one for $\lambda_0(+0)$ as an example. For $i = 4$, the proof is more complex and will be given separately.

Before giving the details of the proof, we show in Table 2.1 the numerical results for $C_i(+0)$'s $(0 \le i \le 5)$.

Table 2.1: Numerical values of $C_i(+0)$'s $(0 \le i \le 5)$

i	0	1,3,4	2	5
$C_i(+0)$	0.26098	0.32454	0.41583	0.10790

2.5.2 Determination of $\lambda_i(+0)$'s $(0 \le i \le 5; i \neq 4)$

Here we only discuss $\lambda_0(+0)$. Let $u_\alpha \in V^0$ be the minimizing function in (2.5.15) corresponding to $\lambda_0(\alpha)$ and assume that $\|u_\alpha\| = 1$.

Define $\hat{\lambda}_0$ to be the infimum of the following infimum problem:

$$\widehat{\lambda}_0 = \inf_{u \in V^{0,Z} \setminus \{0\}} \frac{\|\partial_1 u\|^2}{\|u\|^2}, \tag{2.5.17}$$

where $V^{0,Z}$ is defined in (2.5.2).

Theorem 2.5.2. *Let $\lambda_0(\alpha)$ be defined as above. Then the limit $\lambda_0(+0) := \lim_{\alpha \to +0} \lambda_0(\alpha)$ exists and is given by $\lambda_0(+0) = \hat{\lambda}_0$.*

Proof. **1)** First, it is easy to see the existence of $\lambda_0(+0) = \lim_{\alpha \to +0} \lambda_0(\alpha)$ by considering two facts that $\lambda_0(\alpha)$ is monotonically increasing as α decreases to $+0$, and that $\lambda_0(\alpha)$ is uniformly bounded for all $\alpha \in (0, 1]$, as we have already shown. Actually we have $\lambda_0(\alpha) \le \hat{\lambda}_0$ for $\hat{\lambda}_0$ in (2.5.17).

Since $\|u_\alpha\|_{L^2(T)} = 1$, we have $\lambda_0(\alpha) = \|\partial_1 u_\alpha\|^2 + \frac{1}{\alpha^2}\|\partial_2 u_\alpha\|^2$, so that $\|\partial_1 u_\alpha\|$ and $\alpha^{-2}\|\partial_2 u_\alpha\|$ are uniformly bounded for $\alpha \in (0, 1]$. Thus, $\|u_\alpha\|_{H^1(T)}$ is uniformly bounded. From Rellich's theorem, there exists a sequence $\{u_{\alpha_i}\}_{i=1}^{\infty}$ with $\alpha_i \to +0$ and $u_0 \in H^1(T)$ such that

$$\begin{cases} u_{\alpha_i} \rightharpoonup u_0 \text{ in } H^1(T)\,, \\ u_{\alpha_i} \to u_0 \text{ in } L^2(T)\,, \end{cases} \tag{2.5.18}$$

where '$\rightharpoonup$' (and '$\to$') denotes the weak (and respectively the strong) convergence of the sequence in the corresponding spaces. As $\{u_{\alpha_i}, \lambda(\alpha_i)\}$ satisfies $\|\partial_2 u_{\alpha_i}\|^2 \leq \alpha_i^2 \lambda_0(\alpha_i) \leq \alpha_i^2 \hat{\lambda}_0$, we have $\lim_{i\to\infty} \|\partial_2 u_{\alpha_i}\| = 0$. Since $\partial_2 u_{\alpha_i} \rightharpoonup \partial_2 u_0$ in $L_2(T)$, we have $\partial_2 u_0 = 0$, so that $u_0 \in V^{0,Z}$. Moreover, we can see $\lim_{i\to\infty} \|u_{\alpha_i}\| = \|u_0\| = 1$, so that $u_0 \neq 0$.

2) As $u_0 \in V^{0,Z}$ and $\|u_0\|$=1, we have from the definition of $\hat{\lambda}_0$ that

$$\hat{\lambda}_0 \leq \|\partial_1 u_0\|^2 \,.$$

Also, considering the weak convergence of $\{u_{\alpha_i}\}$ in $H^1(T)$, we get

$$\begin{aligned} \|\partial_1 u_0\|^2 &\leq \liminf_{i\to\infty} \|\partial_1 u_{\alpha_i}\|^2 \\ &\leq \lim_{i\to\infty} (\|\partial_1 u_{\alpha_i}\|^2 + \frac{1}{\alpha_i^2} \|\partial_2 u_{\alpha_i}\|^2) \\ &= \lim_{i\to\infty} \hat{R}^{(0)}_{\alpha_i}(u_{\alpha_i}) \\ &= \lambda_0(+0)\,. \end{aligned}$$

Hence,

$$\hat{\lambda}_0 \leq \lambda_0(+0)\,. \tag{2.5.19}$$

On the other hand, since

$$\hat{\lambda}_0 = \inf_{v\in V^{0,Z}\setminus\{0\}} \hat{R}^{(0)}_{\alpha}(v) \geq \inf_{v\in V^{0}\setminus\{0\}} \hat{R}^{(0)}_{\alpha}(v) = \lambda_0(\alpha)\,,$$

and considering the convergence of $\{\lambda_0(\alpha_i)\}$, we get

$$\hat{\lambda}_0 \geq \lim_{i\to\infty} \lambda_0(\alpha_i) = \lambda_0(+0)\,. \tag{2.5.20}$$

From the inequalities (2.5.19) and (2.5.20), we can now conclude that

$$\lambda_0(+0) = \hat{\lambda}_0\,.$$

□

2.5.3 Determination of $\lambda_4(+0)$

Recall that

$$\lambda_4(\alpha) := \inf_{v \in V^4 \setminus \{0\}} \hat{R}_\alpha^{(4)}(v) ,$$

where $\hat{R}_\alpha^{(4)}(v)$ can be expressed by

$$\hat{R}_\alpha^{(4)}(v) = \frac{\|\partial_{11}v\|_T^2 + 2\alpha^{-2}\|\partial_{12}v\|_T^2 + \alpha^{-4}\|\partial_{22}v\|_T^2}{\|\partial_1 v\|_T^2 + \alpha^{-2}\|\partial_2 v\|_T^2} =: \frac{a_\alpha(v,v)}{b_\alpha(v,v)} .$$

In the following proof, we will omit the subscript of $\lambda_4(\alpha)$ as $\lambda(\alpha)$. Also, assume one of the minimizing functions for $\lambda(\alpha)$ to be denoted by u_α and $b_\alpha(u_\alpha, u_\alpha) = 1$.

Theorem 2.5.3. *The limit $\lambda(+0) := \lim_{\alpha \to +0} \lambda(\alpha)$ exists. Moreover, $\lambda(+0)$ is the smallest eigenvalue of the eigenvalue problem for $\lambda > 0$ and $u \in V^{4,Z} \setminus \{0\}$:*

$$(\partial_{11}u, \partial_{11}w) = \lambda(\partial_1 u, \partial_1 w), \quad \forall w \in V^{4,Z} , \tag{2.5.21}$$

where $V^{4,Z}$ is defined in(2.5.2).

Proof. As we have shown, $\lambda(\alpha)$ is continuous and uniformly bounded in $\alpha > 0$. Thus both $\liminf_{\alpha \to +0} \lambda(\alpha)$ and $\limsup_{\alpha \to +0} \lambda(\alpha)$ exist, and what we must prove is:

$$\liminf_{\alpha \to +0} \lambda(\alpha) = \limsup_{\alpha \to +0} \lambda(\alpha) .$$

That is, we will show that $\{\lambda(\alpha)\}_{\alpha > 0}$ has a unique accumulation point as $\alpha \to +0$.

From the definition of $\lambda(\alpha)$ and u_α, the eigenpair $(u_\alpha, \lambda(\alpha))$ satisfies:

$$\begin{aligned}(\partial_{11}u_\alpha, \partial_{11}w)_T &+ \frac{2}{\alpha^2}(\partial_{12}u_\alpha, \partial_{12}w)_T + \frac{1}{\alpha^4}(\partial_{22}u_\alpha, \partial_{22}w)_T \\ &= \lambda(\alpha)\left((\partial_1 u_\alpha, \partial_1 w)_T + \frac{1}{\alpha^2}(\partial_2 u_\alpha, \partial_2 w)_T\right) \text{ for } \forall w \in V^3 .\end{aligned} \tag{2.5.22}$$

The proof is performed by the following several steps.

1. As $\lambda(\alpha)$ is uniformly bounded for all $\alpha \in (0,1]$, we can find a sequence $\{\alpha_i\}_{i=1}^\infty$ and λ^* such that $\alpha_i \to +0$, $\lambda(\alpha_i) \to \lambda^*$ as $i \to \infty$. We will show that the value of λ^* is independent of the choice of $\{\alpha_i\}$.

2. Since $b_\alpha(u_\alpha, u_\alpha) = 1$ and $\lambda(\alpha)$ is uniformly bounded, both $|u_{\alpha_i}|_{2,T}$ and $|u_{\alpha_i}|_{1,T}$ are uniformly bounded. Considering the inequality that $\|u\|_{L^2(T)} \leq C_5|u|_{2,T}$ for $u \in V^4$, cf.(2.2.9), we have $\|u_\alpha\|_{L_2(T)}$ are uniformly bounded. Hence, $\{u_\alpha\}$ are uniformly bounded in $H^2(T)$ for $\alpha > 0$. By the compact theorem in Hilbert space, we can find a sub-sequence of $\{\alpha_i\}_{i=1}^{\infty}$, still using the same notation, and $u_0 \in H^2(T)$ such that

$$\begin{cases} u_{\alpha i} \rightharpoonup u_0 \text{ weakly in } H^2(T)\,, \\ u_{\alpha i} \rightarrow u_0 \text{ strongly in } H^1(T)\,. \end{cases}$$

Considering the limit for $b_{\alpha_i}(u_{\alpha_i}, u_{\alpha_i})$=1, we find $u_0 \in V^4$ satisfies:

$$\partial_2 u_0 = 0, \ i.e., \ u_0 \in V^{4,Z}\,.$$

As there are two possible cases: $u_0 = 0$ and $u_0 \neq 0$, we will discuss each case as follows.

3. (Case: $u_0 \neq 0$) In (2.5.22), let the test function w be chosen from $V^{4,Z}(T)$ and α be $\{\alpha_i\}$, then it holds that

$$(\partial_{11}u_{\alpha_i}, \partial_{11}w)_T = \lambda(\alpha_i)(\partial_1 u_{\alpha_i}, \partial_1 v)_T\,; \quad \forall v \in V^{4,Z}(T)\,.$$

Taking the limit for $i \rightarrow \infty$, we have

$$(\partial_{11}u_0, \partial_{11}w)_T = \lambda^*(\partial_1 u_0, \partial_1 w)_T\,; \quad \forall w \in V^{4,Z}(T)\,. \tag{2.5.23}$$

Thus, $\{u_0, \lambda^*\} \in \{V^{4,Z}\backslash\{0\}\} \times \mathbb{R}$ is an eigenpair of eigenvalue problem defined by (2.5.23). It is easy to see that λ^* is actually the smallest eigenvalue by using the arguments similar to those in the preceding subsection.

4. (Case: $u_0 = 0$)

Define $v_\alpha = \partial_2 u_\alpha/\alpha$, then we can see that $v_\alpha \in V^2(\subset H^1(T))$. As $u_0 = 0$ and $b_{\alpha_i}(u_{\alpha_i}, u_{\alpha_i}) = 1$, we have $\|v_{\alpha_i}\| \rightarrow 1$ as $i \rightarrow \infty$. Further, considering the boundedness of $a_{\alpha_i}(u_{\alpha_i}, u_{\alpha_i}) = \lambda(\alpha_i)$, we find $\{v_{\alpha_i}\}$ are also uniformly bounded in $H^1(T)$. In the same way as before, we find that there exists a sub-sequence of $\{v_{\alpha_i}\}$, still using the same notation, and $v_0 \in H^1(T)$ such that

$$\begin{cases} v_{\alpha i} \rightharpoonup v_0 \text{ in } H^1(T)\,, \\ v_{\alpha i} \rightarrow v_0 \text{ in } L^2(T)\,. \end{cases} \tag{2.5.24}$$

Since $\|\partial_2 v_{\alpha_i}\|^2 \le \alpha_i^2 a_{\alpha_i}(u_{\alpha_i}, u_{\alpha_i})$ and the $\alpha_i^2 a_{\alpha_i}(u_{\alpha_i}, u_{\alpha_i})$ tends to 0 as $i \to \infty$, we can deduce that $\partial_2 v_0 = 0$. Further by Lemma (2.5.1), v_0 can be identified with a function v_0^* of single varible x_1.

Multiply each side of (2.5.22) by α, and choose the test function $w \in V^4$ such that $\partial_{22} w \equiv 0$, then we get:

$$\alpha(\partial_{11}u_\alpha, \partial_{11}w)_T + 2(\partial_{12}u_\alpha/\alpha, \partial_{12}w)_T \\ = \lambda(\alpha)\left(\alpha(\partial_1 u_\alpha, \partial_1 w)_T + (\partial_2 u_\alpha/\alpha, \partial_2 w)_T\right) . \quad (2.5.25)$$

Substituting $v_{\alpha_i} = \partial_2 u_{\alpha_i}/\alpha_i$ in the equation above and letting $i \to \infty$, we find λ^* and v_0^* satisfy

$$2(\partial_1 v_0^*, \partial_{12} w)_T = \lambda^*(v_0^*, \partial_2 w)_T . \quad (2.5.26)$$

For each $v \in C_0^\infty(0,1)$, take $w(x_1, x_2) := v(x_1)x_2$. Then

$$2(\partial_1 v_0^*, \partial_1 v)_T = \lambda^*(v_0^*, v)_T , \quad (2.5.27)$$

that is,

$$2\int_0^1 (1-x_1)\frac{dv_0^*(x_1)}{dx_1}\frac{dv}{dx_1}dx_1 = \lambda^* \int_0^1 (1-x_1)v_0^*(x_1)v(x_1)dx_1, \quad \forall v \in C_0^\infty(0,1) . \quad (2.5.28)$$

Finally, we can conclude that v_0^* together with λ^* satisfies

$$\begin{cases} -((1-s)u'(s))' = \frac{\lambda}{2}u(s)(1-s) \text{ for } s \in (0,1) , \\ u(0) = 0 . \end{cases} \quad (2.5.29)$$

As $\lambda^* > 0$, the solution of the above is of the form, with arbitrary constants c_1 and c_2,

$$v_0^*(s) = c_1 J_0\left(\sqrt{\frac{\lambda^*}{2}}(1-s)\right) + c_2 Y_0\left(\sqrt{\frac{\lambda^*}{2}}(1-s)\right) , \quad (2.5.30)$$

where $J_0(s)$ and $Y_0(s)$ are the 0-th order Bessel functions of the frist and second kinds, respectively. As is well known, $J_0(s)$ is sufficiently smooth, while $Y_0(s)$ is of the form $Y_0(s) = c_3 \log s + r(s)$ for $s > 0$, where $c_3 \neq 0$ is a constant and $r(s)$ a sufficiently smooth remainder term [51]. To make v_0^* has the extension over T belong to $V^{2,Z} \subset H^1(T)$, the constant c_2 must be zero. Also to satisfy the boundary condition, $\sqrt{\lambda^*/2}$ needs to be the positive zero of $J_0(s)$. In fact, $J_0(s)$ has countably many positive zeros without any accumulation points except $+\infty$. Denoting the smallest positive zero by $\gamma_0 > 0$, we have

$$\lambda^* \geq 2\gamma_0^2. \tag{2.5.31}$$

We can show that $\gamma_0 > 2.25$, so that $\lambda^* > 10$. Also, considering the function $\tilde{u}(x_1, x_2) = \sin \pi x_1$, we have $\hat{R}_\alpha^{(4)}(\tilde{u}) = \pi^2$, hence

$$\lambda^* = \lim_{i\to\infty} \hat{R}_{\alpha_i}^{(4)}(u_{\alpha_i}) \leq \lim_{i\to\infty} \hat{R}_{\alpha_i}^{(4)}(\tilde{u}) = \pi^2 < 10\,. \tag{2.5.32}$$

The two equations (2.5.31) and (2.5.32) lead to a contradiction. Hence the case that $u_0 = 0$ does not occur.

Now, we can conclude that λ^* is the minimum eigenvalue of (2.5.23) (or (2.5.21)) and is independent of the selected sequence $\{\alpha_i\}$.

□

2.6 Numerical results

We performed floating-point number computations to see the actual dependence of various error constants on α and θ.

2.6.1 Computational methods

To obtain approximate values of error constants, we can utilize the FEM quite effectively. In particular we used the most popular P_1 triangular finite element for numerical computations of $C_i(\alpha, \theta)$'s for $0 \leq i \leq 3$ by preparing appropriate triangulations of $T_{\alpha,\theta}$. For $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$, it is natural to use various triangular finite elements for Kirchhoff plate bending problems, since the associated partial differential equations are of 4th order as is noted in Section 2.2. In our actual computations, we used the discrete Kirchhoff triangular element presented in [26]. On the other hand, we can also use the Siganevich approach for computation of $C_4(\alpha, \theta)$, which also adopts the P_1 element and a kind of penalty method for a system of 2-nd order partial differential equations similar to the incompressible Stoke system [47]. This method works well if the penalty parameter is carefully chosen.

In every case, we have a matrix eigenvalue problem as the discretization of the original eigenvalue problem described by a weak form. More specifically, it is a generalized matrix eigenvalue problem with respect to unknown eigenvectors of nodal values of approximate eigenfunctions, and it can be solved for example by the inverse iteration method and the subspace iteration method [13]. A difficulty in deriving such matrix eigenvalue problems come from linear constraint conditions imposed on the spaces $V^i_{\alpha,\theta}$ for $i = 0, 1, 2, 3$. Similar constraint conditions are also necessary to deal with, if we compute $C_4(\alpha, \theta)$ by the method of Siganevich [47]. On the other hand, we do not have such a difficulty in computing $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$ by Kirchhoff elements, where the linear constraints $v(O) = v(A) = v(B) = 0$ for $V^4_{\alpha,\theta}$ can be handled as homogeneous "nodal" conditions.

One possible method for removing the constraints is to construct new function bases that satisfy the constraint conditions, but then we have the final matrix that is not sparse. Another method is to use the Lagrange multiplier method, which does not essentially destroy the global sparseness of the matrices. We tested both approaches and obtained reasonable results. Various iteration methods may be also available for the same purposes.

The numerical results below are obtained by the double or quadruple precision arithmetic, and we do not employ the interval analysis. But their accuracy appears to be reasonable at least in graphical level, since finer mesh computations give essentially the same graphs. We hope that the effective verification methods will be established in near future, so that the numerical results can be of strictly mathematical significance.

2.6.2 Numerical results for error constants

Here, we first show some results for $C_i(\alpha)$'s $(0 \leq i \leq 5)$ by the P_1 conforming finite element and the Kirchhoff triangular element in [26] with the uniform triangulation of the domain T_α. In such calculations, T_α is subdivided into a number of small triangles congruent to $T_{\alpha,\pi/2,h}$ with e.g. $h = 1/20$. The penalty method in [47] is also tested to calculate $C_4(\alpha)$ approximately.

Figure 2.9 consists of two parts and illustrates the graphs of approximate $C_i(\alpha)$'s $(0 \leq i \leq 5)$ versus $\alpha \in (0, 1]$. Exact values of C_0 and $C_1 = C_2$ together with an approximate value of C_5 are also included as horizontal lines in graphs. At $\alpha = 1$, the approximate values coincide well with the available exact ones in Theorem 2.4.1, and we can numerically see that $C_1 = (C_2)$ is a nice upper bound of C_4. For general α, the monotonically increasing behaviors theoretically predicted for $C_i(\alpha)$'s $(i = 0, 1, 2, 3, 5\ )$ as well as the relation $C_4(\alpha) \leq \min\{C_1(\alpha), C_2(\alpha)\}$ are also well observable in the graphs. The present numerical results suggest that $C_4(\alpha)$ is also monotonically increasing, but we have not succeeded in proving such a conjecture. Moreover, when $\alpha \approx 0$, the numerical results agree well with the exact right limits given in Table 2.1 based on the asymptotic analysis.

For $C_4(\alpha)$, we tested two methods, that is, the P_1 conforming triangular finite element with the penalty method and the Kirchhoff triangluar finite elemnt . These two methods turned out to give almost the same results if the meshes are relatively fine and the penalty parameter is appropriately chosen. The graph for $C_4(\alpha)$ in Figure 2.9 is actually obtained by the Kirchhoff element, but is indistinguishable in graphical level from the one by the penalty method.

Figure 2.10 and 2.11 illustrate numerically obtained contour lines for $C_i(\alpha, \theta)$'s in the $\alpha - \theta$ polar coordinates, where the abscissa denotes $\alpha \cos\theta$, and the ordinate does $\alpha \sin\theta$. The unit circle $\alpha = 1$ is also shown by a dotted curve. The minimum required range for α and θ is specified by equation (2.2.1), but the contour lines are shown for wider ranges, so that we can easily see global behaviors of error constants. These results can be also useful for practical adaptive computations to specify constants in error indicators approximately. Of course, for strict mathematical analysis like numerical verification, we need correct upper bounds to error constants. The contour lines are sometimes cut off in the portions where the expected accuracy may be insufficient. For example, when $\alpha \approx 0$ or $|\theta - \pi/2| \approx \pi/2$, it requires extraordinarily fine meshes to retain sufficient accuracy. The behavior of $C_4(\alpha, \theta)$ appears to be the most complicated among all the constants, and the necessity of the maximum angle condition can be visually recognized. The other constants seem to be uniformly bounded over the unit disk $\alpha \leq 1$.

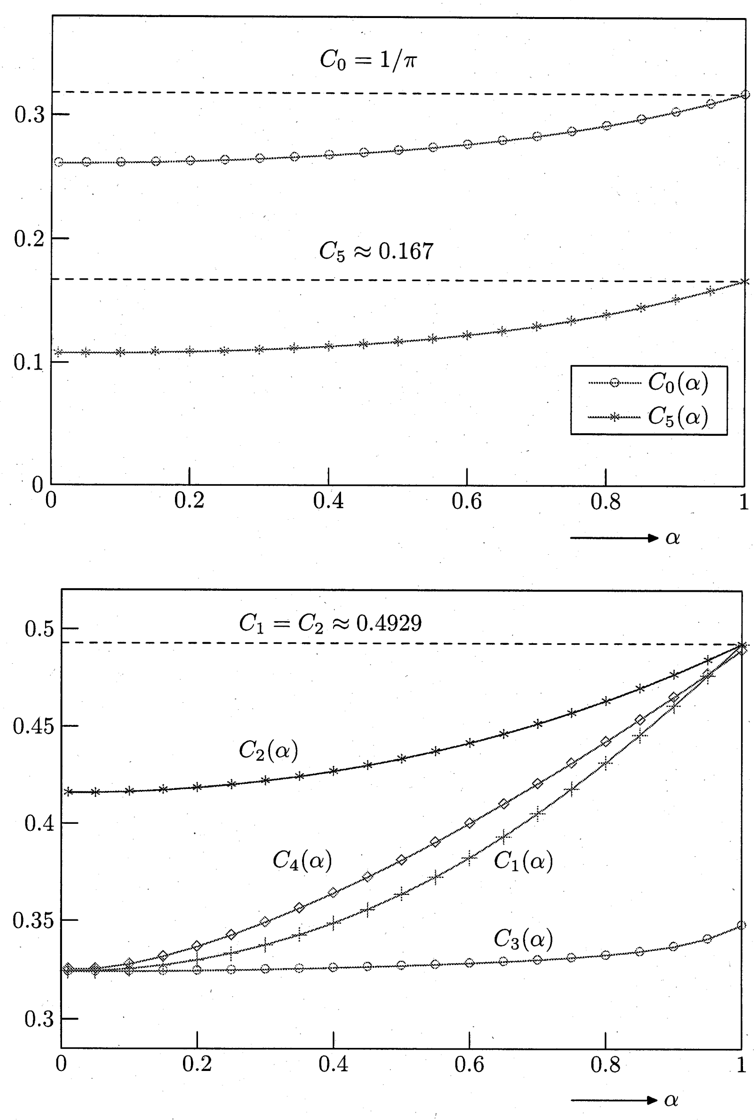


Figure 2.9: Numerically obtained graphs for $C_i(\alpha)$ $(0 \leq i \leq 5; 0 < \alpha \leq 1)$

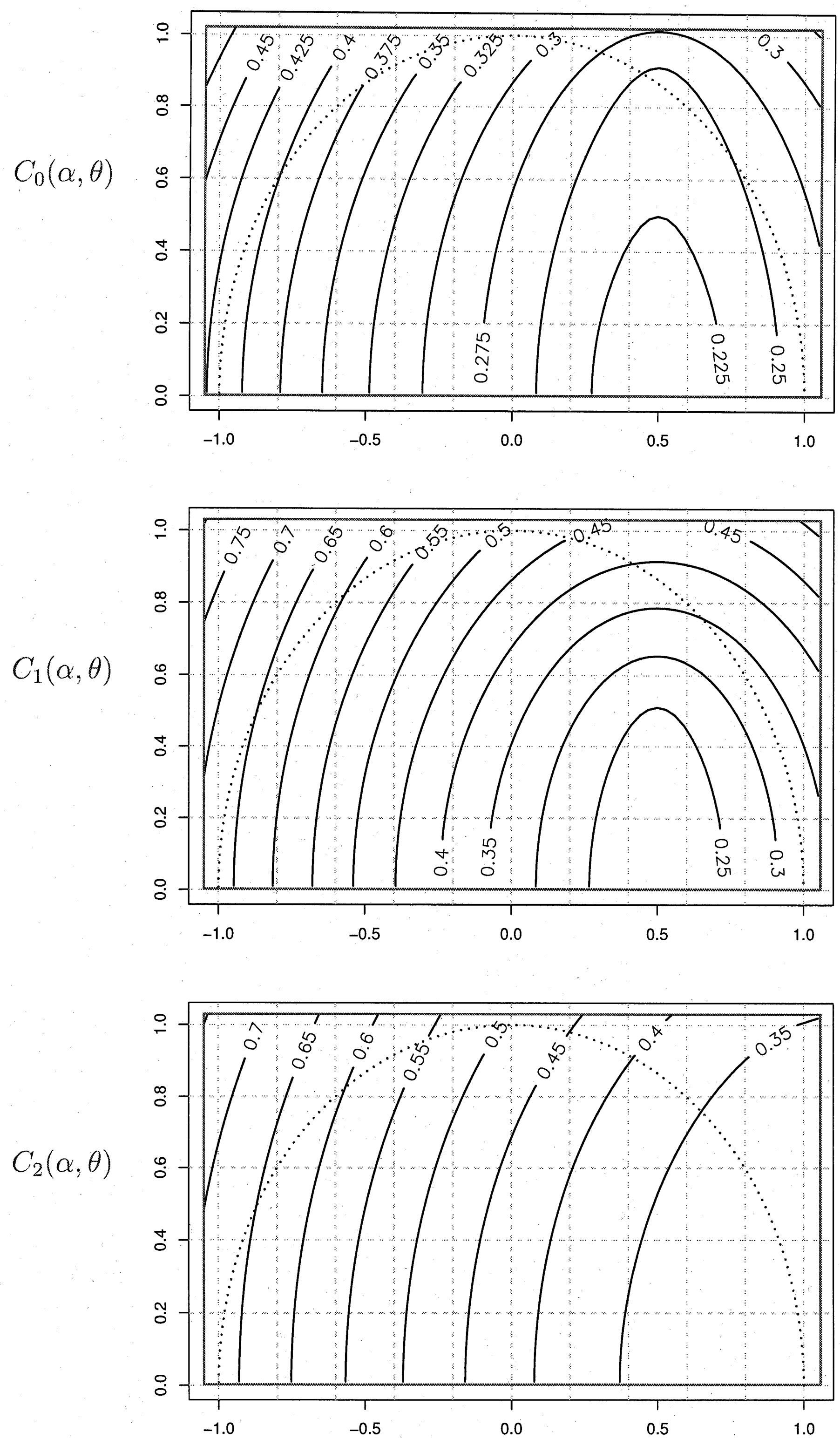


Figure 2.10: Contour lines for constants (I)

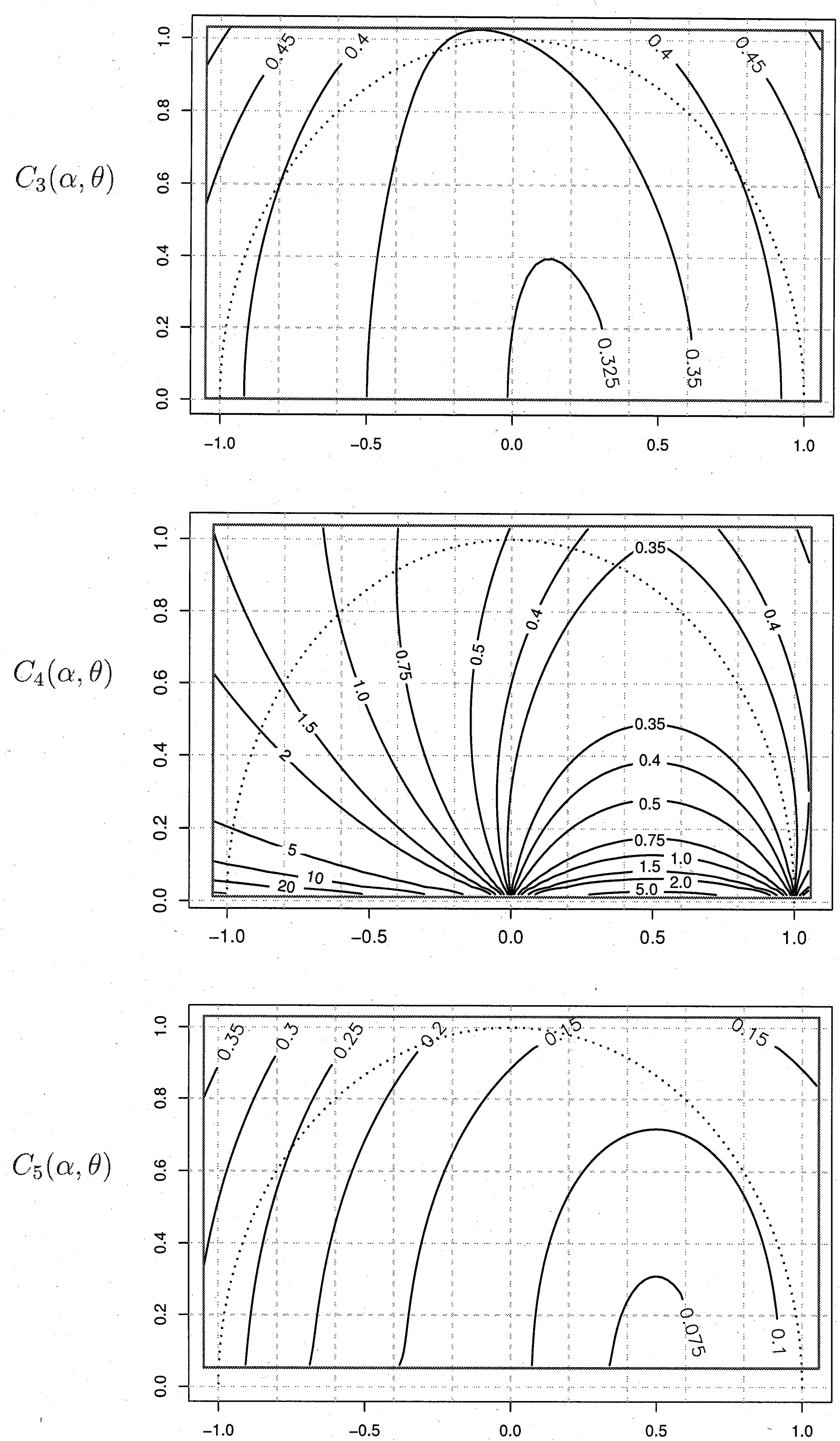


Figure 2.11: Contour lines for constants (II)

Chapter 3

Non-conforming P_1 triangular finite element

As a well-known alternative to the conforming linear (P_1) triangular finite element for approximation of the first-order Sobolev space (H^1), the nonconforming P_1 element is considered a classical discontinuous Galerkin finite element [16] and has various interesting properties from both theoretical and practical standpoints [15, 49]. In particular, its a priori error analysis was performed in fairly early stage of mathematical analysis of FEM, and recently a posteriori error analysis is rapidly developing as well. There are also various error constants to be evaluated quantitatively [3, 7, 13, 15] in order to give accurate error estimation of such nonconforming FEM.

Based on the research for the ones related to conforming P_1 FEM, we investigate several error constants required in the error analysis for nonconforming P_1 FEM, Thus quantitative a priori error estimates for the nonconforming P_1 FEM solutions become available. A kind of a posteriori error estimate is introduced in Chapter 5, which adopts the conforming FEM solution as well as the nonconforming one. At the end of this chapter, we illustrate the validity of error estimation by numerical results.

3.1 A priori error estimation

We here summarize a priori error estimation for the nonconforming P_1 triangular FEM. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and recall the Poisson equation in a weak form with the homogeneous Dirichlet boundary condition:

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{3.1.1}$$

As we mentioned in Chapter 2, the notations $L_2(\Omega)$ and $H_0^1(\Omega)$ are the usual Hilbertian Sobolev spaces associated to Ω, ∇ is the gradient operator, and $(\cdot,\cdot)$ stands for the inner products for both $L_2(\Omega)$ and $L_2(\Omega)^2$. It is well known that the solution exists uniquely in $H_0^1(\Omega)$ and also belongs to $H^2(\Omega)$.

Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of triangulations of Ω, to which we associate a family of nonconforming P_1 finite element spaces $\{V_{nc}^h\}_{h>0}$. Each V_{nc}^h is constructed as below [15, 49]:

V_{nc}^h :={ piecewise linear functions over $\mathcal{T}^h$ with continuity at midpoints of interior edges and zero values at midpoints on boundary edges } . (3.1.2)

Notice that the homogeneous Dirichlet condition is not exactly satisfied. If there is no ambiguity, within the current chapter, we will often omit the subscript of V_{nc}^h as V^h.

Then the finite element solution $u_h \in V^h$ is determined by, for a given $f \in L_2(\Omega)$,

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h), \quad \forall v_h \in V^h\,, \tag{3.1.3}$$

where ∇_h is the "nonconforming" or discrete gradient operator defined by the element-wise relations $(\nabla_h v)|_K := \nabla(v|_K)$ for any $v \in V^h + H^1(\Omega)$ and any $K \in \mathcal{T}^h$.

Eq.(3.1.3) is formally of the same form as in the conforming case, so that, for error analysis, it is natural to consider an appropriate interpolation operator Π_h^1 from $H_0^1(\Omega)$ (or its intersection with some other spaces) to V^h. However, the situation is not so simple. That is, using the Green formula, we have

$$(\nabla_h u_h, \nabla_h v_h) = (\nabla u, \nabla_h v_h) - \sum_{K \in \mathcal{T}^h} \int_{\partial K} v_h \frac{\partial u}{\partial n}|_{\partial K} d\gamma, \quad \forall v_h \in V^h, \tag{3.1.4}$$

where $\frac{\partial u}{\partial n}|_K$ denotes the trace of the derivative of u in the outward normal direction of ∂K, and $d\gamma$ does the infinitesimal element of ∂K. Because of the line integral term above, we cannot appreciate the best approximation property that holds in the conforming case, e.g., equation (2.1.5). The conventional efforts of error analysis have been focused on the estimation of such a term.

Before going into the details of analysis, let us quote Lemma 6 of [25], which is a refined and specialized form of Strang's second lemma for general nonconforming FEM [15].

Lemma 3.1.1. *Let $u \in H_0^1(\Omega)$ and u_h the solutions of (3.1.1) and (3.1.3), respectively. Then it holds that*

$$\|\nabla u - \nabla_h u_h\|^2 = \inf_{v_h \in V^h} \|\nabla u - \nabla_h v_h\|^2 + \left[\sup_{w_h \in V^h \setminus \{0\}} \frac{(\nabla u, \nabla_h w_h) - (f, w_h)}{\|\nabla_h w_h\|} \right]^2 . \quad (3.1.5)$$

Remark 3.1.1. *The present estimate is essentially the same as the original one by Strang, which is based on the triangle inequality. However, the above is better for quantitative purposes because of the equality form and the smallness of the coefficients.*

Proof. We sketch the proof since the Strang lemma of this equality form is not necessarily widely known. Define $\tilde{u}_h \in V^h$ by

$$(\nabla_h \tilde{u}_h, \nabla_h v_h) = (\nabla u, \nabla_h v_h), \quad \forall v_h \in V^h . \quad (3.1.6)$$

The present $\tilde{u}_h$ exists uniquely in V^h, and satisfies the best approximation property

$$\|\nabla u - \nabla_h \tilde{u}_h\| = \inf_{v_h \in V^h} \|\nabla u - \nabla_h v_h\|, \quad (3.1.7)$$

as well as a kind of Pythagorean equality

$$\|\nabla u - \nabla_h u_h\|^2 = \|\nabla u - \nabla_h \tilde{u}_h\|^2 + \|\nabla_h (\tilde{u}_h - u_h)\|^2 . \quad (3.1.8)$$

Here the last term above can be rewritten by

$$\|\nabla_h (\tilde{u}_h - u_h)\| = \sup_{w_h \in V^h \setminus \{0\}} \frac{(\nabla_h (\tilde{u}_h - u_h), \nabla_h w_h)}{\|\nabla_h w_h\|} = \sup_{w_h \in V^h \setminus \{0\}} \frac{(\nabla u, \nabla_h w_h) - (f, w_h)}{\|\nabla_h w_h\|} . \quad (3.1.9)$$

From the last three equalities, we obtain (3.1.5). □

We introduce the lowest-order Raviart-Thomas triangular $H(\mathrm{div})$ finite element space W^h associated to each $\mathcal{T}^h$ [14, 29]:

$W^h(\mathcal{T}^h) := \{$ Each $q_h \in W^h$ is piecewise vector function such that on each $K \in \mathcal{T}^h$, $q_h = (a_K + c_K x_1, b_K + c_K x_2)$. Moreover, the normal component of q_h is constant and continuous along each inter-element edge of $\mathcal{T}^h$ $\}$. (3.1.10)

For $q_h \in W^h$ and $v_h \in V^h$, because the integral of v_h over each edge on $\partial\Omega$ vanishes, we can derive by Green formula that

$$(q_h, \nabla_h v_h) + (\mathrm{div}\, q_h, v_h) = 0 .$$

Hence

$$(\nabla_h u_h - \nabla u, \nabla_h v_h) = (q_h - \nabla u, \nabla_h v_h) + (\text{div}\, q_h + f, v_h); \forall q_h \in W^h, \forall v_h \in V^h . \tag{3.1.11}$$

Thus, from (3.1.5) we have

$$\|\nabla u - \nabla_h u_h\|^2 = \inf_{v_h \in V^h} \|\nabla u - \nabla_h v_h\|^2 + \left[\sup_{w_h \in V^h \setminus \{0\}} \frac{(q_h - \nabla u, \nabla_h w_h) + (\text{div}\, q_h + f, w_h)}{\|\nabla_h w_h\|} \right]^2 . \tag{3.1.12}$$

Using the Fortin operator $\Pi_h^F : H(\text{div}; \Omega) \cap H^{\frac{1}{2}+\delta}(\Omega)^2 \to W^h (\delta > 0)$ (to be given later or cf. [14]) and the orthogonal projection one $Q_h : L_2(\Omega) \to X^h :=$ space of step functions over $\mathcal{T}^h$, we obtain a priori error estimate:

$$\|\nabla u - \nabla_h u_h\|^2 \le \|\nabla u - \nabla_h \Pi_h u\|^2 + \left[\|\nabla u - \Pi_h^F \nabla u\| + \sup_{w_h \in V^h \setminus \{0\}} \frac{(f - Q_h f, w_h - Q_h w_h)}{\|\nabla_h w_h\|} \right]^2 . \tag{3.1.13}$$

Here v_h in (3.1.12) is replace by $\Pi_h u$, where Π_h is a kind of interpolation to be specified later which maps $u \in H^1(\Omega)$ into V^h. Also q_h is taken as $\Pi_h^F \nabla u$, for which will show that div $q_h =$ div $\Pi_h^F \nabla u = -Q_h f$ [14].

We can obtain a more concrete error estimate in terms of the mesh parameter $h_* > 0$ (h will be used in a different meaning later) by deriving estimates such as, for $v \in H_0^1(\Omega) \cap H^2(\Omega)$ and $g \in H^1(\Omega) + V^h$,

$$\begin{aligned} &\|v - \Pi_h v\| \le \gamma_0 h_*^2 |v|_2, \qquad &&\|\nabla v - \nabla \Pi_h v\| \le \gamma_1 h_* |v|_2 , \\ &\|\nabla v - \Pi_h^F \nabla_h v\| \le \gamma_2 h_* |v|_2, &&\|g - Q_h g\| \le \gamma_3 h_* \|\nabla_h g\| . \end{aligned} \tag{3.1.14}$$

Then we obtain, for the solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\|\nabla u - \nabla_h u_h\| \le \begin{cases} h_* \{\gamma_1^2 |u|_2^2 + (\gamma_2 |u|_2 + \gamma_3 \|f\|)^2\}^{1/2} & \text{for } f \in L_2(\Omega), \\ h_* \{\gamma_1^2 |u|_2^2 + (\gamma_2 |u|_2 + \gamma_3^2 h_* |f|_1)^2\}^{1/2} & \text{for } f \in H^1(\Omega), \end{cases} \tag{3.1.15}$$

where the term $|u|_2$ can be bounded as $|u|_2 \le \|f\|$ for the present Ω.

We can also use Nitsche's trick to evaluate a priori L_2 error of u_h [15, 30]. That is, let us define $\psi \in H_0^1(\Omega)(\cap H^2(\Omega))$ for $e^h := u - u_h$ by

$$(\nabla \psi, \nabla v) = (e^h, v), \quad \forall v \in H_0^1(\Omega) .$$

Then we have the following lemma.

Lemma 3.1.2. *It holds for the above $e^h = u - u_h$ that*

$$\begin{aligned}\|e^h\|^2 &= (\tilde{q}_h - \nabla_h v_h, \nabla_h e^h) + (\nabla_h v_h - \nabla\psi, \nabla u - q_h) + (\psi - v_h, \operatorname{div} q_h + f)\\ &+ (\operatorname{div}\tilde{q}_h + e^h, e^h); \quad \forall v_h \in V^h, \forall q_h, \tilde{q}_h \in W^h .\end{aligned}$$

Proof. As in the derivation of (3.1.11), we have for the above ψ, e^h, v_h and $\tilde{q}_h$ that

$$(\tilde{q}_h, \nabla_h e^h) + (\operatorname{div}\tilde{q}_h, e^h) = 0, (\nabla_h v_h, q_h) + (v_h, \operatorname{div} q_h) = 0, (\nabla\psi, q_h) + (\psi, \operatorname{div} q_h) = 0 .$$

On the other hand, since u and u_h are the solutions of (3.1.1) and (3.1.3), respectively, we find that

$$(\nabla\psi, \nabla u) = (\psi, f), \quad (\nabla_h v_h, \nabla_h e^h) = (\nabla_h v_h, \nabla u - \nabla_h u_h) = (\nabla_h v_h, \nabla u) - (v_h, f) .$$

From the above equalities and $\|e^h\|^2 = (e^h, e^h)$, we can obtain the desired identity. □

Substituting $v_h = \Pi_h\psi, q_h = \Pi_h^F \nabla u$ and $\tilde{q}_h = \Pi_h^F \nabla\psi$ into the equation (3.1.16), we obtain

$$\begin{aligned}\|e^h\| = (\Pi_h^F\nabla\psi - \nabla\psi + \nabla\psi - \nabla_h\Pi_h\psi, \nabla_h e^h) + (\nabla_h\Pi_h\psi - \nabla\psi, \nabla u - \Pi_h^F\nabla u) +\\ (\psi - \Pi_h\psi, f - Q_h f) + (e^h - Q_h e^h, e^h - Q_h e^h)\,, \quad (3.1.16)\end{aligned}$$

where we utilize the relations such that $\operatorname{div} q_h = \operatorname{div}\Pi_h^F\nabla u = -Q_h f$ and $\operatorname{div}\tilde{q}_h = \operatorname{div}\Pi_h^F\nabla\psi = -Q_h e^h$. Then we have, by (3.1.14) as well as the relations $|u|_2 \le \|f\|$ and $|\psi|_2 \le \|e^h\|$,

$$\|e^h\|^2 \le \left[(\gamma_1 + \gamma_2)h_*\|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2)h_*^2\|f\|\right]\|e^h\| + \gamma_3^2 h_*^2\|\nabla_h e^h\|^2\,, \quad (3.1.17)$$

where the term $\gamma_0 h_*^2\|f\|$ can be replaced with $\gamma_0\gamma_3 h_*^3|f|_1$ if $f \in H^1(\Omega)$. This may be considered a quadratic inequality for $\|e^h\|$, and solving it gives an expected order estimate $\|u - u_h\| = \|e^h\| = O(h_*^2)$:

$$\|e^h\| \le \frac{h_*}{2}(A_1 + \sqrt{A_1^2 + 4A_2}); \qquad (3.1.18)$$

where $A_1 := (\gamma_1 + \gamma_2)\|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2)h_*\|f\|$, $A_2 := \gamma_3^2\|\nabla_h e^h\|^2$.

3.2 Error constants for nonconforming FEM

To analyze the error constants in (3.1.14), let us consider their element-wise counterparts. First we configure the triangular element in the same way of section 2.2. Here we recall the definition of the geometric parameters: h, α and θ are positive constants such that

$$h > 0, \quad 0 < \alpha \leq 1, \quad (\frac{\pi}{3} \leq) \cos^{-1}\frac{\alpha}{2} \leq \theta < \pi \,. \tag{3.2.1}$$

Each triangle $T_{\alpha,\theta,h}$ has three vertices $O(0,0), A(h,0)$ and $B(\alpha h\cos\theta, \alpha h\sin\theta)$ and three edges e_i's $(i = 1,2,3)$ defined by $\{e_1, e_2, e_3\} = \{OA, OB, AB\}$ (Figure 3.2). Hence $h = \overline{OA}$ still denotes the medium edge length. The abbreviation of notations will be the same as the one in last chapter, e.g., $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\frac{\pi}{2}}$ and $T = T_1$. Also we will use the notations $\|\cdot\|$ and $|\cdot|_k$ as the norm and standard semi-norms for functions over $T_{\alpha,\theta,h}$, where the subscript of domain is often omitted.

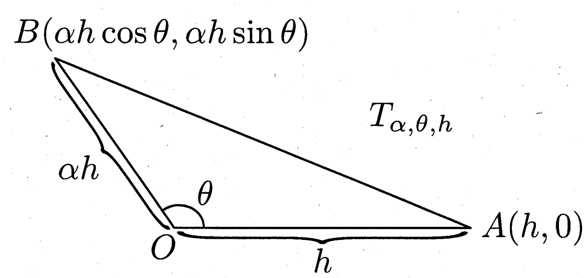


Figure 3.1: Triangular element $T_{\alpha,\theta,h}$

In addition to the linear spaces $V^i_{\alpha,\theta,h}$, $i \in \{0,1,2,3,4\}$ defined in Section 2.2, we introduce several new closed linear spaces for functions over $T_{\alpha,\theta,h}$:

$$V^{\{1,2\}}_{\alpha,\theta,h} = \{ \; v \in H^1(T_{\alpha,\theta,h}) \mid \int_{e_1} v(s)ds = \int_{e_2} v(s)ds = 0 \; \}, \tag{3.2.2}$$

$$V^{\{1,2,3\}}_{\alpha,\theta,h} = \{ \; v \in H^1(T_{\alpha,\theta,h}) \mid \int_{e_i} v(s)ds = 0, \quad (i = 1,2,3) \; \}, \tag{3.2.3}$$

$$V^{4,n}_{\alpha,\theta,h} = \{ \; v \in H^2(T_{\alpha,\theta,h}) \mid \int_{e_i} v(s)ds = 0, \quad (i = 1,2,3) \; \}. \tag{3.2.4}$$

The abbreviations for notations $V^i_{\alpha,\theta,h}$ are also used here, e.g., $V^{\{1,2\}}_{\alpha,\theta} = V^{\{1,2\}}_{\alpha,\theta,1}$, $V^{\{1,2\}}_{\alpha} = V^{\{1,2\}}_{\alpha,\frac{\pi}{2}}$, $V^{\{1,2\}} = V^{\{1,2\}}_1$ etc. For the purpose of error analysis for nonconforming FEM, we define nonconforming P_1 interpolation operator $\Pi^{1,n}_{\alpha,\theta,h}$ for functions on

$T_{\alpha,\theta,h}$ [13, 15]: For $v \in H^1(T_{\alpha,\theta,h})$, $\Pi^{1,n}_{\alpha,\theta,h}v$ is a linear function such that

$$\int_{e_i} (\Pi^{1,n}_{\alpha,\theta,h}v)(s)ds = \int_{e_i} v(s)ds \quad (i = 1, 2, 3). \tag{3.2.5}$$

For simplicity, we will often use $\Pi^{1,n}$ instead of $\Pi^{1,n}_{\alpha,\theta,h}$, where the subscript is omitted.

In the same way as we define $C_i(\alpha, \theta, h)$ $(0 \le i \le 5)$, let us consider several other positive constants for the purpose of estimating the interpolation operator mentioned above,

$$C_J(\alpha, \theta, h) = \sup_{v \in V^J_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|_{T_{\alpha,\theta,h}}}{|v|_{1,T_{\alpha,\theta,h}}} \quad (J = \{1, 2\}, \{1, 2, 3\}), \tag{3.2.6}$$

$$C_{\{4,n\}}(\alpha, \theta, h) = \sup_{v \in V^{4,n}_{\alpha,\theta,h}\setminus\{0\}} \frac{|v|_{1,T_{\alpha,\theta,h}}}{|v|_{2,T_{\alpha,\theta,h}}}, \quad C_{\{5,n\}}(\alpha, \theta, h) = \sup_{v \in V^{4,n}_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|_{T_{\alpha,\theta}}}{|v|_{2,T_{\alpha,\theta}}} . \tag{3.2.7}$$

We will again use abbreviated notations $C_J(\alpha, \theta) = C_J(\alpha, \theta, 1)$, $C_J(\alpha) = C_J(\alpha, \pi/2)$, $C_J = C_J(1)$ and also $C_{J,\alpha,\theta} := C_J(\alpha, \theta)$ for every possible subscript J.

By a simple scale change, we can easily find that $C_J(\alpha, \theta, h) = hC_J(\alpha, \theta)$ $(J \neq \{5, n\})$ and $C_{\{5,n\}}(\alpha, \theta, h) = h^2 C_{\{5,n\}}(\alpha, \theta)$. Now, by noting $v - \Pi^{1,n}_{\alpha,\theta,h}v \in V^{4,n}_{\alpha,\theta,h}$ for $v \in H^2(T_{\alpha,\theta,h})$, we can easily have the popular interpolation error estimates on $T_{\alpha,\theta,h}$: [13, 15].

$$|v - \Pi^{1,n}_{\alpha,\theta,h}v|_1 \le C_{\{4,n\}}(\alpha, \theta)h|v|_2, \quad \forall v \in H^2(T_{\alpha,\theta,h}) , \tag{3.2.8}$$
$$\|v - \Pi^{1,n}_{\alpha,\theta,h}v\| \le C_{\{5,n\}}(\alpha, \theta)h^2|v|_2, \quad \forall v \in H^2(T_{\alpha,\theta,h}) . \tag{3.2.9}$$

Below, we show some fundamental properties of the constants.

Lemma 3.2.1. *For the constant $C_J(\alpha, \theta)$, we have*

$$C_{\{4,n\}}(\alpha, \theta) \le C_0(\alpha, \theta),\ C_{\{5,n\}}(\alpha, \theta) \le C_0(\alpha, \theta)C_{\{1,2,3\}}(\alpha, \theta) \le C_0(\alpha, \theta)C_{\{1,2\}}(\alpha, \theta) . \tag{3.2.10}$$

Proof. To show the former of (3.2.10), we notice that function in $V^{4,n}_{\alpha,\theta}$ has the zero integral on each edge, and then apply the Gauss formula to obtain

$$\int_{T_{\alpha,\theta}} \frac{\partial v}{\partial x_i} dx = 0 \text{ for } v \in V^{4,n}_{\alpha,\theta} \quad (i = 1, 2) . \tag{3.2.11}$$

Hence we can easily obtain $C_{\{4,n\}}(\alpha,\theta) \leq C_0(\alpha,\theta)$ by noting the definition of $C_0(\alpha,\theta)$. To derive the latter of (3.2.10), we notice that

$$C_{\{5,n\}}(\alpha,\theta) = \sup_{v\in V^{4,n}_{\alpha,\theta}\backslash\{0\}} \frac{\|v\|_{T_{\alpha,\theta}}}{|v|_{2,T_{\alpha,\theta}}} \tag{3.2.12}$$

$$\leq \sup_{v\in V^{4,n}_{\alpha,\theta}\backslash\{0\}} \frac{\|v\|_{T_{\alpha,\theta}}}{|v|_{1,T_{\alpha,\theta}}} \cdot \sup_{v\in V^{4,n}_{\alpha,\theta}\backslash\{0\}} \frac{|v|_{1,T_{\alpha,\theta}}}{|v|_{2,T_{\alpha,\theta}}} \tag{3.2.13}$$

$$= C_{\{4,n\}}(\alpha,\theta)\ C_{\{1,2,3\}}(\alpha,\theta)\,. \tag{3.2.14}$$

By further noticing that $C_{\{1,2,3\}}(\alpha,\theta) \leq C_{\{1,2\}}(\alpha,\theta)$, we prove the latter of (3.2.10). Also notice that an estimate possibly rougher than the latter of equation (3.2.10) is $C_{\{5,n\}}(\alpha,\theta) \leq C_0(\alpha,\theta)\min_{i=1,2,3} C_i(\alpha,\theta)$ by utilizing the relation $C_{\{1,2,3\}}(\alpha,\theta) \leq \min_{i=1,2,3} C_i(\alpha,\theta)$.

□

Thus we can give quantitative interpolation estimates (3.2.8) and (3.2.9), if we succeed in evaluating or bounding the constants $C_J(\alpha,\theta)$'s explicitly for all possible J. Among them, $C_0(\alpha,\theta)$ and $C_{\{1,2\}}(\alpha,\theta)$ are important as may be seen from (3.2.10). Just as we did in Chapter 2, we execute analogous analysis to show the following properties for the newly introduced constants:

Lemma 3.2.2. *The constants $C_i(\alpha)$'s ($J = \{1,2\},\{1,2,3\},\{4,n\},\{5,n\}$) are continuous with respect to variable α. Moreover, except for $C_{\{4,n\}}(\alpha)$, these constants are strictly monotonically increasing with respect to α. (Numerical computations suggest that the constant $C_{\{4,n\}}(\alpha)$ to be monotonically increasing on α, while it has not been proved yet.) The dependence of these constants on α and θ is given as follows:*

$$\psi_J(\theta)C_J(\alpha) \leq C_J(\alpha,\theta) \leq \phi_J(\theta)C_J(\alpha) \quad (J = \{1,2\},\{1,2,3\},\{4,n\},\{5,n\}), \tag{3.2.15}$$

where

$$\begin{cases} \phi_J(\theta) = \sqrt{1+|\cos\theta|}, \quad \psi_J(\theta) = \sqrt{1-|\cos\theta|} \quad (J = \{1,2\},\{1,2,3\}), \\ \phi_{\{4,n\}}(\theta) = (1+|\cos\theta|)/\sqrt{1-|\cos\theta|}, \\ \psi_{\{4,n\}}(\theta) = (1-|\cos\theta|)/\sqrt{1+|\cos\theta|}, \\ \phi_{\{5,n\}}(\theta) = 1+|\cos\theta|\,, \quad \psi_{\{5,n\}}(\theta) = 1-|\cos\theta|\,. \end{cases}$$

As a result, the interpolation by the nonconforming P_1 triangle is robust to the distortion of $T_{\alpha,\theta}$. This fact does not necessarily imply the robustness of the final error estimates for $u - u_h$, since analysis of the Fortin interpolation has not been performed yet.

Remark 3.2.1. *Instead of $\Pi^{1,n}_{\alpha,\theta,h}$, it is also possible to consider an interpolation operator using the function values at midpoints of edges. Such an operator is definable for continuous functions over $\overline{T}_{\alpha,\theta,h}$, but not so for general functions in $H^1(T_{\alpha,\theta,h})$. Moreover, its analysis would be different from the one for $\Pi^{1,n}_{\alpha,\theta,h}$.*

Determination of $C_{\{1,2\}}$

From the preceding observations, we can give explicit upper bounds of various interpolation constants associated to the nonconforming P_1 element, provided that the value of $C_{\{1,2\}}$ is determined. This becomes indeed possible by adopting essentially the same idea and techniques to determine C_0 and $C_1(= C_2)$:

Theorem 3.2.1. *$C_{\{1,2\}} = C_{\{1,2\}}(1, \pi/2, 1)$ is equal to the maximum positive solution of the transcendental equation for μ:*

$$\frac{1}{2\mu} + \tan\frac{1}{2\mu} = 0\,. \tag{3.2.16}$$

The above implies that $C_{\{1,2\}} = \frac{1}{2}C_1(= \frac{1}{2}C_2)$, and hence is bounded as, with numerical verification,

$$0.24641 < C_{\{1,2\}} < 0.24647\,. \tag{3.2.17}$$

Remark 3.2.1. *Thus $1/4$ is a simple but nice upper bound. Numerically, we have $C_{\{1,2\}} = 0.2464562258\cdots$.*

Proof. By the use of the technique for determination of C_0 and $C_1 = C_2$ in [27, 29], we obtain the following equation for μ:

$$1 + \frac{1}{2\mu}\sin\frac{1}{\mu} - \cos\frac{1}{\mu} = 0\,, \tag{3.2.18}$$

whose maximum positive solution is the desired $C_{\{1,2\}}$. By the double-angle formulas, the above is transformed into

$$(2\sin\frac{1}{2\mu} + \frac{1}{\mu}\cos\frac{1}{2\mu})\sin\frac{1}{2\mu} = 0\,. \tag{3.2.19}$$

It is now easy to derive (3.2.16), and also to draw other conclusions by using the results in [27, 29]. $\square$

3.3 Analysis of Fortin's interpolation

This section is devoted to analysis of the Fortin interpolation operator $\Pi^F_{\alpha,\theta}$ for each $T_{\alpha,\theta}$ [14]. First, let us introduce the following transformation between $x = \{x_1, x_2\} \in T_{\alpha,\theta}$ and $\hat{x} = \{\hat{x}_1, \hat{x}_2\}$:

$$\hat{x}_1 = x_1 \sin\theta - x_2 \cos\theta, \quad \hat{x}_2 = x_1 \cos\theta + x_2 \sin\theta\,. \tag{3.3.1}$$

For each $q = \{q_1, q_2\} \in H(\mathrm{div}; T_{\alpha,\theta})$, we also consider the (contravariant) expression $\hat{q} = \{\hat{q}_1, \hat{q}_2\}$:

$$\hat{q}_1 = q_1 \sin\theta - q_2 \cos\theta, \quad \hat{q}_2 = q_1 \cos\theta + q_2 \sin\theta\,, \tag{3.3.2}$$

for which we loosely use both x and $\hat{x}$ as variables. The Raviart-Thomas type approximate function $q_h = \{q_{h1}, q_{h2}\}$ are given, together with the expression for $\hat{q}_h = \{\hat{q}_{h1}, \hat{q}_{h2}\}$, by

$$\left\{\begin{array}{l} q_{h1} = \alpha_1 + \alpha_3 x_1 \\ q_{h2} = \alpha_2 + \alpha_3 x_2 \end{array}\right., \quad \left\{\begin{array}{l} \hat{q}_{h1} = \alpha_1 \sin\theta - \alpha_2 \cos\theta + \alpha_3 \hat{x}_1 \\ \hat{q}_{h2} = \alpha_1 \cos\theta + \alpha_2 \sin\theta + \alpha_3 \hat{x}_2 \end{array}\right.. \tag{3.3.3}$$

The Fortin interpolation $q^*_h = \{q^*_{h1}, q^*_{h2}\} = \Pi^F_{\alpha,\theta} q$ for $q \in H(\mathrm{div}; T_{\alpha,\theta}) \cap H^{\frac{1}{2}+\delta}(T_{\alpha,\theta})^2 (\delta > 0)$ is of the form in (3.3.3) and characterized by the conditions:

$$\int_{e_1} (q^*_{h2} - q_2)\,ds = \int_{e_2} (\hat{q}^*_{h1} - \hat{q}_1)\,ds = 0, \quad \int_{T_{\alpha,\theta}} \mathrm{div}(q^*_h - q)\,dx = 0, \tag{3.3.4}$$

where $\hat{q}$ for q and $\hat{q}^*_h$ for q^*_h are defined in (3.3.2),(3.3.3), respectively.

Let us now introduce another interpolation $\Pi^{\{1,2\}}_{\alpha,\theta} q = q^\dagger_h = \{q^\dagger_{h1}, q^\dagger_{h2}\}$ for the same q, which is a constant vector function that satisfies only the former two conditions of (3.3.4). Then we can have the L_2 estimate:

$$\begin{aligned} \|q - \Pi^F_{\alpha,\theta} q\| &\leq \|q - \Pi^{\{1,2\}}_{\alpha,\theta} q\| + \frac{\|\mathrm{div}\, q\|}{2\sqrt{|T_{\alpha,\theta}|}} \left(\int_{T_{\alpha,\theta}} |x|^2 dx \right)^{1/2} && (3.3.5) \\ &= \|q - \Pi^{\{1,2\}}_{\alpha,\theta} q\| + \sqrt{\frac{1 + \alpha\cos\theta + \alpha^2}{24}} \|\mathrm{div}\, q\|\,. && (3.3.6) \end{aligned}$$

Here we introduce another quantity $C_{F,1}$ for later purpose,

$$C_{F,1}(\alpha, \theta) := \sqrt{\frac{1 + \alpha\cos\theta + \alpha^2}{24}}\,. \tag{3.3.7}$$

To bound $\|q - \Pi^{\{1,2\}}_{\alpha,\theta} q\|$, let us evaluate $\|\hat{q}_1 - \hat{q}^\dagger_{h1}\|$ and $\|q_2 - q^\dagger_{h2}\|$ by using $C_1(\alpha,\theta)$ and $C_2(\alpha,\theta)$ and there is no difficulty to get the following theorem.

Theorem 3.3.1. *It holds for* $q = \{q_1, q_2\} \in H^1(T_{\alpha,\theta})^2$ *that*

$$\|q - \Pi_{\alpha,\theta}^{\{1,2\}} q\|_{T_{\alpha,\theta}} \le C_{F,2}(\alpha,\theta)|q|_{1,T_{\alpha,\theta}}, \tag{3.3.8}$$

$$C_{F,2}(\alpha,\theta) := \frac{1}{\sqrt{2}\sin\theta}\left\{c_1^2 + c_2^2 + 2c_1c_2\cos^2\theta + (c_1+c_2)\sqrt{c_1^2+c_2^2+2c_1c_2\cos 2\theta}\right\}^{1/2}, \tag{3.3.9}$$

where c_i *presents* $C_i(\alpha,\theta)$ $(i = 1, 2)$ *for the purpose of abbreviation.*

Remark 3.3.1. *From (3.3.6) and (3.3.8), it is easy to derive the following estimate for the Fortin interpolation operator* $\Pi_{\alpha,\theta,h}^F$*:*

$$\|q - \Pi_{\alpha,\theta,h}^F q\|_{T_{\alpha,\theta,h}} \le C_{F,1}(\alpha;\theta)\, h\|\operatorname{div} q\|_{T_{\alpha,\theta,h}} + C_{F,2}(\alpha,\theta)h|q|_{1,T_{\alpha,\theta,h}}, \forall q \in H^1(T_{\alpha,\theta,h})^2. \tag{3.3.10}$$

Because of the factor $\sin\theta$ *in (3.3.9), the maximum angle condition [1, 6, 29] works for estimate (3.3.8), and consequently for (3.3.10). On the other hand, the estimates for* $\Pi_{\alpha,\theta,h}^0$ *and* $\Pi_{\alpha,\theta,h}^{1,n}$ *are free from such conditions as may be seen from (3.2.10) and the comments there.*

3.4 Summary of a priori error estimate

So far, we have introduced and analyzed local interpolation operators $\Pi_{\alpha,\theta,h}^0$, $\Pi_{\alpha,\theta,h}^{1,n}$ and $\Pi_{\alpha,\theta,h}^F$. For each $K \in \mathcal{T}^h$, we can find an appropriate $T_{\alpha,\theta,h}$ congruent to K under an appropriate mapping $\Psi_K : K \Rightarrow T_{\alpha,\theta,h}$. Then it is natural to define the P_1 nonconforming interpolation operator $\Pi_h^{nc} : H_0^1(\Omega) \Rightarrow V^h$ by

$$(\Pi_h^{nc} v)|_K = \left(\Pi_{\alpha,\theta,h}^{1,n}(v|_K \circ \Psi_K^{-1})\right) \circ \Psi_K \tag{3.4.1}$$

for $v \in H_0^1(\Omega)$ and $K \in \mathcal{T}^h$. Similarly, the orthogonal projection operator $Q_h : L_2(\Omega) \Rightarrow X^h$ is related to $\Pi_{\alpha,\theta,h}^0$, while the global Fortin operator Π_h^F is defined through $\Pi_{\alpha,\theta,h}^F$. Concretely, function Ψ_K is the Piola transformation for 2D convariant vector fields [5].

We define $\{\alpha_K, \theta_K, h_K\}$ by the parameters $\{\alpha, \theta, h\}$ for each $K \in \mathcal{T}^h$. Also, we introduce the global parameters

$$h_* = \max_{K \in \mathcal{T}^h} h_K, \quad C_J^h := \max_{K \in \mathcal{T}^h} C_J(\alpha_K, \theta_K) \quad \text{for all index J}.$$

Let us recall the interpolation operators mentioned in (3.1.13) and (3.1.14). By taking Π_h to be Π_h^{nc}, we have the interpolation estimates in (3.1.14) as, for $u \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\left\{ \begin{array}{rcl} \|u - \Pi_h^{nc} u\| & \leq & C_5^h h_*^2 |u|_2 \leq C_0^h C_{\{1,2\}}^h h_*^2 |u|_2 \, , \\ \|\nabla u - \nabla_h \Pi_h^{nc} u\| & \leq & C_4^h h_* |u|_2 \leq C_0^h h_* |u|_2 \, , \\ \|\nabla u - \Pi_h^F \nabla u\| & \leq & C_{F,1}^h h_* \|\Delta u\| + C_{F,2}^h h_* |u|_2 \, , \end{array} \right.$$

and for $g \in H^1(\Omega) + V^h$,

$$\|g - Q_h g\| \leq C_0^h h_* \|\nabla_h g\| \, .$$

Substituting the constants above into (3.1.15), we now obtain the computable a priori error estimate as follows: given data $f \in L_2(\Omega)$, we have

$$\|\nabla u - \nabla_h u_h\| \leq h_* \left\{ C_0^{h^2} |u|_2^2 + \left(C_{F,2}^h |u|_2 + (C_0^h + C_{F,1}^h) \|f\| \right)^2 \right\}^{1/2} . \qquad (3.4.2)$$

If f belongs to $H^1(\Omega)$ as well, then

$$\|\nabla u - \nabla_h u_h\| \leq h_* \left\{ C_0^{h^2} |u|_2^2 + \left(C_{F,2}^h |u|_2 + C_0^{h^2} h_* |f|_1 + C_{F,1}^h \|f\| \right)^2 \right\}^{1/2} . \qquad (3.4.3)$$

Similarly, we can derive a computable estimate for $\|u - u_h\|_\Omega$ in explicit form, which is omitted here.

Remark 3.4.1. *As we will see in the following chapters, relations such as (5.2.7), (5.2.10) and (5.2.12) may suggest the possibility of finding interpolations for ∇u in W^h other than the one by the Fortin operator, which are free from the maximum angle condition [6]. However, $\nabla_h(\Pi_h u + \alpha_h)$, for example, is not shown to belong to W^h, because we cannot prove the inter-element continuity of normal components unlike $\nabla_h \hat{u}_h$. Our numerical results show that the maximum angle condition is probably essential for the nonconforming P_1 triangle. See also [1] for related topics.*

3.5 Asymptotic analysis of constants on narrow element

In this section, we will investigate the behaviours of the constants $C_{\{i,n\}}(\alpha)$'s, ($i = 4, 5$), when the shortest edge of triangle tends to be zero. The method to be used is almost the same as those for the constants appearing in the case of the conforming FEM.

As is well known, the constants $C_{\{4,n\}}(\alpha,\theta)$ and $C_{\{5,n\}}(\alpha,\theta)$ can be characterized by the following variational problems:

$\lambda_{\{4,n\}}(\alpha,\theta) = \left(1/C_{\{4,n\}}(\alpha,\theta)\right)^2$: *Find $u(\neq 0) \in V_{\alpha,\theta}^{4,n}$, and minimal $\lambda > 0$ such that*

$$\sum_{i,j=1}^{2}(\partial_{ij}u, \partial_{ij}v)_T = \lambda(\nabla u, \nabla v)_T, \quad \forall v \in V_{\alpha,\theta}^{4,n} . \tag{3.5.1}$$

$\lambda_{\{5,n\}}(\alpha,\theta) = \left(1/C_{\{5,n\}}(\alpha,\theta)\right)^2$: *Find $u(\neq 0) \in V_{\alpha,\theta}^{4,n}$ and minimal $\lambda > 0$ such that*

$$\sum_{i,j=1}^{2}(\partial_{ij}u, \partial_{ij}v)_T = \lambda(u, v)_T, \quad \forall v \in V_{\alpha,\theta}^{4,n} . \tag{3.5.2}$$

The existence of these $C_J(+0) := \lim_{\alpha\to+0} C_J(\alpha)$ $(J = \{4,n\}, \{5,n\})$ is easy to see by considering the boundedness of the constants over $(0,1]$ and the compactness theories such as Rellich's theorem. Let us introduce two new quantities $\lambda_{\{4,n\}}(+0) := C_{\{4,n\}}^{-2}(+0)$ and $\lambda_{\{5,n\}}(+0) := C_{\{5,n\}}^{-2}(+0)$ and also a subspace of $V^{\{1,2,3\}}$,

$$W^{nc}(T) := \{v \in V^{4,n} | \partial_2 v = 0\} . \tag{3.5.3}$$

From the Lemma 2.5.1, we see the function $u(x_1,x_2) \in W^{nc}(T)$ can be identified with a single variable one $\hat{u}(x_1)$. In the following, the symbol $u(x) \in W^{nc}(T)$ is just $u(x) := u(x_1,x_2) = \hat{u}(x_1)$, $u^{(1)}(x) := d\hat{u}(x_1)/dx_1$ and $u^{(2)}(x) := d^2\hat{u}(x_1)/dx_1^2$.

We can show that these two constants are characterized by the following eigenvalue problems:

Problem for $C_{\{4,n\}}(+0)$: *Find minimum $\lambda > 0$ and $u \in W^{nc}(T)\setminus\{0\}$ such that*

$$(\partial_{11}u, \partial_{11}v)_T = \lambda(\partial_1 u, \partial_1 v)_T, \quad \forall v \in W^{nc}(T) , \tag{3.5.4}$$

or

$$\begin{cases} (u^{(2)}(1-x))^{(2)} = -\lambda(u^{(1)}(1-x))^{(1)} + C, \\ u(0) = \int_0^1 u(t)dt = u^{(2)}(0) = u^{(2)}(1) = 0 , \end{cases} \tag{3.5.5}$$

where C is an unknown constant to be determined. By using the hyper-geometric functions, the general solution of the ordinary differential equation here can be presented by

$$
\begin{aligned}
u(x) &= c_1 + c_2(1-x)^2 \ {}_2F_3(1,1;\frac{3}{2},\frac{3}{2},2;-\frac{\lambda(1-x)^2}{4}) \\
&+c_3\left((1-x) - \frac{1}{12}(1-x)^3 \ {}_2F_3(1,\frac{3}{2};2,2,\frac{5}{2};-\frac{\lambda(1-x)^2}{4})\right) \\
&+C\,\frac{1}{12\lambda^{1/2}}(1-x)^3 \ {}_2F_3(1,\frac{3}{2};2,2,\frac{5}{2};-\frac{\lambda(1-x)^2}{4})
\end{aligned} \tag{3.5.6}
$$

with proper selection of c_i's , C and λ to make u satisfy the conditions in (3.5.5). Numerical computations show that

$$
\lambda_{\{4,n\}}(+0) \approx 14.682, \quad C_{\{4,n\}}(+0) \approx 0.26098\,. \tag{3.5.7}
$$

By taking $u := x(x-2/3)$, we can easily obtain an upper bound for $\lambda_{\{4,n\}}(+0)$ as 18.

Problem for $C_{\{5,n\}}(+0)$: *Find minimum* $\lambda > 0$ *and* $u \in W^{nc}(T)\setminus\{0\}$ *such that*

$$
(\partial_{11}u, \partial_{11}v)_T = \lambda(u,v)_T, \quad \forall v \in W^{nc}(T)\,, \tag{3.5.8}
$$

or

$$
\left\{
\begin{array}{l}
(u^{(2)}(1-x))^{(2)} = \lambda(1-x)u + C \\
u(0) = \int_0^1 u(t)dt = u^{(2)}(0) = u^{(2)}(1) = 0\,.
\end{array}
\right. \tag{3.5.9}
$$

The general solution in the form of hyper-geometric form is

$$
\begin{aligned}
u(x) &= c_1\,{}_0F_3(;\frac{1}{2},\frac{3}{4},\frac{3}{4};\frac{\lambda(1-x)^4}{256}) + c_2\,(1-x)\,{}_0F_3(;\frac{3}{4},1,\frac{5}{4};\frac{\lambda(1-x)^4}{256}) \\
&+ c_3\,(1-x)^2\,{}_0F_3(;\frac{5}{4},\frac{5}{4},\frac{6}{4};\frac{\lambda(1-x)^4}{256}) \\
&+ C\,\frac{\lambda}{12}\,(1-x)^3\,{}_1F_3(1;\frac{5}{4},\frac{3}{2},\frac{3}{2},\frac{7}{4};\frac{\lambda(1-x)^4}{256})
\end{aligned} \tag{3.5.10}
$$

with proper selection of c_i's and λ to make u satisfy the conditions of Eq.(3.5.9). Also numerical computations show that

$$
\lambda_{\{5,n\}}(+0) \approx 428.31, \quad C_{\{5,n\}}(+0) \approx 0.048319\,. \tag{3.5.11}
$$

Sketch of determining $\lambda_{\{4,n\}}(+0)$

The process of determining $\lambda_{\{4,n\}}(+0)$ is analogous to the one in Theorem 2.5.3. Here we only show the sketch. Before going into further discussion, let us recall the Rayleigh quotient defined by equation (2.3.3) for function $v \in V^{4,n}$ over $T(=T_{1,\pi/2,1})$:

$$\hat{R}_\alpha^{(4)}(v) := \frac{a_\alpha(v)}{b_\alpha(v)} = \frac{\|\partial_{11}v\|_T^2 + 2\alpha^{-2}\|\partial_{12}v\|_T^2 + \alpha^{-4}\|\partial_{22}v\|_T^2}{\|\partial_1 v\|_T^2 + \alpha^{-2}\|\partial_2 v\|_T^2}, \tag{3.5.12}$$

and $\lambda_{\{4,n\}}$ can be presented in the following form:

$$\lambda_{\{4,n\}}(\alpha) := \inf_{v \in V^{4,n}\setminus\{0\}} \hat{R}_\alpha^{(4)}(v). \tag{3.5.13}$$

On considering a special function $\tilde{u}(x_1, x_2) = \sin(2\pi x_1) \in V^{4,n}$, we can easily show that

$$\lambda_{\{4,n\}}(\alpha) = \hat{R}_{\alpha_n}^{(4)}(u_n) \le \frac{|\tilde{u}|_{2,T}}{|\tilde{u}|_{1,T}} < \infty. \tag{3.5.14}$$

What we aim to show is that $\lambda_{\{4,n\}}(\alpha)$ has a limit when $\alpha \to +0$:

$$\lambda_{\{4,n\}}(+0) = \liminf_{\alpha_n \to 0} \lambda_{\{4,n\}}(\alpha_n) = \limsup_{\alpha_n \to 0} \lambda_{\{4,n\}}(\alpha_n). \tag{3.5.15}$$

For any convergent sequence $\lambda_{\{4,n\}}(\alpha_n) \to \lambda^*$ as $\alpha_n \to +0$, $(0 < \alpha_n < 1)$, we will prove that the limit λ^* here is independent of the choice of $\{\alpha_n\}$.

For any $\lambda_{\{4,n\}}(\alpha)$, let $u_{\alpha_n} \in V^{4,n}$ be one of the corresponding eigenfunctions, that is,

$$\lambda_{\{4,n\}}(\alpha_n) = \hat{R}_{\alpha_n}^{(4)}(u_{\alpha_n}).$$

Here we also assume that $b_{\alpha_n}(u_{\alpha_n}) = 1$. The uniform boundedness $\|u_{\alpha_n}\|_{2,T}$ is clear since $\lambda_{\{4,n\}}$ is uniformly bounded. By the compactness theories in Sobolev spaces and the same technique adopted in Theorem 2.5.3, we find there exists a sub-sequence of u_{α_n}, which we still denote by u_{α_n}, that satisfies, when $\alpha_n \to +0$,

$$\begin{cases} u_{\alpha_n} \rightharpoonup u_0 \text{ weakly in } H^2(T_\alpha), \\ u_{\alpha_n} \to u_0 \text{ strongly in } H^1(T_\alpha). \end{cases}$$

Since the limit u_0 may be zero, we should discuss the following two cases separately.

$$\lim_{\alpha_n \to +0} \|u_{\alpha_n}\|_{2,T} = 0, \text{ or } \lim_{\alpha_n \to +0} \|u_{\alpha_n}\|_{2,T} \neq 0. \tag{3.5.16}$$

The former finally leads to the eigenvalue problem in equation (3.5.5), for which we omit the proof here. For the second case, we define a new sequence $\{v_n\} = \{\alpha_n^{-1}\partial_2 u_{\alpha_n}\}$, and will show that $\{v_n\}$ weakly convergences to $v \in W^{nc}(T)$, which is the eigenfunction corresponding to one of the eigenvalues of:

$$\begin{cases} -(v^{(1)}(x)(1-x))^{(1)} = \frac{\lambda}{2}\, v(x)\,(1-x) \text{ for } x \in (0,1)\,, \\ \int_0^1 (1-x)v(x)dx = 0, \quad v^{(1)}(0) = 0\,. \end{cases} \tag{3.5.17}$$

By numerical computations, we can show the problem above has the minimal eigenvalue $\lambda \approx 2 \times 3.8317^2 > 18$. Hence we see the solution of this problem is not the required eigenfunction since $\lambda_{\{4,n\}}(\alpha_n) < 18$. One thing to be pointed out is that here the computations are executed by floating-point arithmetic. To give strict conclusion, we still need the verified computation technique to guarantee the computational results.

In the following, we show how to deduce the eigenvalue problem (3.5.17) from the assumption $\|u_0\|_{2,T} = 0$, which is analogous to the 4th part of Theorem 2.5.3, or (4.3.2) in our paper [28].

Let $w_n := \alpha_n^{-1}\partial_2 u_{\alpha_n}$ $(n = 1, 2, ...)$. Then $w_n \in H^1(T)$ satisfies

$$\int_T w_n \; dx_1 dx_2 = 0, \quad \text{for } n = 1, 2, ...$$

Moreover,

$$\|\partial_{11} u_{\alpha_n}\|_T^2 + 2\|\partial_1 w_n\|_T^2 + \alpha_n^{-2}\|\partial_2 w_n\|^2 = \lambda_{\{4,n\}}(\alpha_n)\ (n = 1, 2, ...)\,.$$

As $\lambda_{\{4,n\}}(\alpha_n)$ is uniformly bounded, $\{w_n\}$ is bounded in $H^1(T)$ and $\partial w_n / \partial x_2 \to 0$ $(n \to \infty)$. Thus, by choosing a sub-sequence of $\{w_n\}$ and denoting it by the same notation for simplicity, we can show the existence of $w_0 \in H^1(T)$ such that, for $n \longrightarrow \infty$,

$$w_n \to w_0 \text{ weakly in } H^1(T) \text{ and strongly in } L^2(T).$$

It is obvious that $\partial w_0 / \partial x_2 = 0$ a.e. on T, then we can identify w_0 by a function depending on only variable x_1, which we still denote by $w_0(x_1)$. Also, w_0 still satisfies

$$\int_T w_0 \; dx_1 dx_2 = 0\,.$$

Let v^* be an arbitrary function of variable x_1 such that $v^* \in C^\infty([0, 1])$. Notice that such v^* can be extended to the one over domain T, which is only depending on the variable x_1. For simplicity, we denote the extended one by the same symbol v^*.

For the aforementioned $v^* \in C^\infty([0,1])$, define another function $P_1 v^*$ of x_1 by

$$(P_1 v^*)(x_1) = v^*(x_1) - \frac{\int_0^1 (1-s)v^*(s)ds}{\int_0^1 (1-s)ds} \cdot 1\,. \tag{3.5.18}$$

We take $v(x_1,x_2) := (P_1 v^*)(x_1) \cdot x_2 + g(x_1)$, where $g(x_1)$ is selected to satisfy $\int_0^1 g(x_1)dx_1 = 0$ and $\int_0^1 [(P_1 v^*)(0) \cdot x_2 + g(0)]\, dx_2 = 0$. Hence, the function v belongs to $V^{\{1,2,3\}}$, that is,

$$\int_{e_i} v ds = 0 \quad (i = 1,2,3)\,.$$

Considering the variational equation for w_n together with the test function v given above, we have

$$\alpha_n\,(\frac{\partial^2 u_{\alpha_n}}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_1^2})_T, +2\left(\frac{\partial w_n}{\partial x_1}, \frac{\partial (P_1 v^*)}{\partial x_1}\right)_T$$

$$= \lambda_{\{4,n\}}(\alpha_n)[\alpha_n(\frac{\partial u_n}{\partial x_1}, \frac{\partial v}{\partial x_1})_T + (w_n, P_1 v^*)_T]\,. \tag{3.5.19}$$

Taking the limit of the equation (3.5.19) and noticing that $\partial_1 (P_1 v^*) = \partial_1 v^*$, we find that w_0 in $H^1(T)$ satisfies

$$2(\frac{\partial w_0}{\partial x_1}, \frac{\partial v^*}{\partial x_1})_T = \lambda^*(w_0, P_1 v^*)_T\,.$$

Now we obtain the following eigenvalue problem:

$$2\int_0^1 (1-x_1)\frac{dw_0}{dx_1}\,\frac{dv^*}{dx_1}dx_1 = \lambda^* \int_0^1 (1-x_1)w_0\,(P_1 v^*)\,dx_1;\ \forall v^* \in C^\infty([0,1]).$$

From the arbitrariness of v^* and the relation $\int_0^1 (P_1 v^*)(x_1)(1-x_1)dx_1 = 0$, we can deduce that

$$2\frac{d}{dx_1}[(1-x_1)\frac{dw_0}{dx_1}] + \lambda^*(1-x_1)w_0(x_1) = C(1-x_1) \text{ and } \frac{dw_0}{dx_1}(0) = 0\,. \tag{3.5.20}$$

Considering the integration of the former ODE in (3.5.20) over $(0,1)$,we deduce that $C = 0$. So the eigenvalue problem is just the one in equation (3.5.17).

3.6 Estimate of interpolation constants in 3D case

As an extension of the results which we obtained in 2D case, we here consider the the nonconforming finite element in 3-dimensional space, for which only partial results are given. For further investigation, we still need much more efforts.

Let us consider the P_1 nonconforming tetrahedral element space V^h that is defined over the subdivision of domain with tetrahedra. The function in V^h is piecewise linear function whose integrations on inter-element faces are continuous. To approximate the homogeneous Dirichlet boundary conditions, the function in V^h is forced to have vanishing integration on boundary faces. In the following, we will consider and analyze an important interpolation analogous to the 2D case.

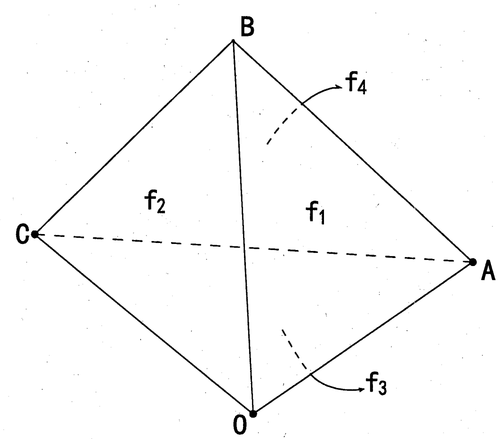


Figure 3.2: Tetrahedron element K

Firstly, let us consider the tetrahedral element in 3D space. With $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$, the vectors in $\mathbb{R}^3$, we define a tetrahedron K (See Figure 3.2):

$$K = \text{ convex hull of } \{\mathbf{0}, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\} \text{ with the boundary omitted.} \tag{3.6.1}$$

To orient the vectors $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$, we define the matrix $M = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ and require that $\det(M) > 0$.

Let us denote the the nodes of K by $O, A(\mathbf{t}_1), B(\mathbf{t}_2)$ and $C(\mathbf{t}_3)$, and the faces $f_1(OAB)$, $f_2(OBC)$, $f_3(OAB)$ and $f_4(ABC)$. The Cartesian coordinates of point in K denoted by $x = (x_1, x_2, x_3)$.

Introduction of interpolation operator Π_K^{nc}

On tetrahedron K, which is supposed to be an open set, let $H^m(K)$ denote the Sobolev spaces of functions of $L^2(K)$ with distributional derivatives up to the order m. The norm of $u \in H^m(K)$ is written as $\|u\|_{H^m(K)}$ or $\|u\|_{m,K}$ and the standard semi-norm to be $|u|_{H^m(K)}$ or $|u|_{m,K}$. The L^2 norm of u, $\|u\|_{L^2(K)}$, will be abbreviated as $\|u\|_K$ or $\|u\|$.

Given $u \in H^1(K)$, which may not be continuous on K, we consider the interpolation operator Π_K^{nc}, which maps u to a linear function $\Pi_K^{nc}u$ such that

$$\int_{f_i} (\Pi_K^{nc}\, u - u)\, dS \;= 0 \text{ for } 1 \le i \le 4\,, \tag{3.6.2}$$

where dS is the surface element on f_i.

In the application of FEM, the following two kinds of interpolation error estimates are widely used: Given $u \in H^2(K)\,(\subset H^1(K))$, there exist constants $C_0^{nc}(K)$ and $C_1^{nc}(K)$, which depend only on the geometry of K, such that,

$$\|u - \Pi_K^{nc}u\|_K \le C_0^{nc}(K)\, |u|_{2,K}\,, \tag{3.6.3}$$

$$|u - \Pi_K^{nc}u|_{1,K} \le C_1^{nc}(K)\, |u|_{2,K}\,. \tag{3.6.4}$$

The existence of these two constants are easy to prove. For simplicity, we will usually write $C_i^{nc}(K)$ as C_i^{nc}.

Let us introduce a subspace of Sobolev space $H^2(K)$:

$$V_0^{nc}(K) = \{v \in H^2(K) |\, v \text{ has the zero integration on each face.}\}\,. \tag{3.6.5}$$

It is easy to check that $u - \Pi_K^{nc}u \in V_0^{nc}(K)$ and $|u - \Pi_K^{nc}u|_{2,K} = |u|_{2,K}$. We then characterize the optimal constants above by the Rayleigh quotients:

$$(1/C_0^{nc})^2 := \lambda_0^{nc} = \inf_{u \in V_0^{nc}(K)\setminus\{0\}} \frac{|u|_{2,K}^2}{\|u\|_K^2}\,, \tag{3.6.6}$$

$$(1/C_1^{nc})^2 := \lambda_1^{nc} = \inf_{u \in V_0^{nc}(K)\setminus\{0\}} \frac{|u|_{2,K}^2}{|u|_{1,K}^2}\,. \tag{3.6.7}$$

In addition, we consider the average interpolation Π_K^A: for $u \in H^1(K)$, $\Pi_K^A u$ is constant function over K such that

$$\int_K (\Pi_K^A u - u) dx = 0\,. \tag{3.6.8}$$

Let us introduce the constant $C^A(K)$ by

$$C^A(K) := \sup_{u \in H^1(K)\backslash\{0\}} \frac{\|u - \Pi_K^A u\|_K}{|u|_{1,K}}, \tag{3.6.9}$$

then we have the estimate for interpolation Π_K^A:

$$\|u - \Pi_K^A u\|_K \leq C^A(K)|u|_{1,K}, \tag{3.6.10}$$

where the optimal constant $C^A(K)$ is a kind of the Poincare constant. Such a constant plays an important role in bounding the constants C_0^{nc} and C_1^{nc}, as will be shown below. From the results in [32] and [11], where the latter one [11] corrected a mistake in the former [32], we have

$$C^A(K) \leq \frac{\text{diam}(K)}{\pi}, \ \text{diam(K)} = \text{the diameter of } K.$$

To give estimates to the constants C_0^{nc} and C_1^{nc}, we still need another several constants. Let $\mathcal{P}$ be the power set of $\{1, 2, 3, 4\}$. Then define, for each index set $I \in \mathcal{P} \setminus \{\emptyset\}$,

$$C_I^{-2}(K) := \lambda_I = \inf_{\substack{u \in H^1(K) \setminus \{0\} \\ \int_{f_i} u ds = 0, \ \forall i \in I.}} \frac{|u|_{1,K}^2}{\|u\|_K^2}. \tag{3.6.11}$$

Thus, we have constants such as $C_{\{1\}}(K)$, $C_{\{2,3\}}(K)$, $C_{\{1,2,3,4\}}(K)$ and so on.

Upper bound for constants $C_0^{nc}(K)$ and $C_1^{nc}(K)$

Theorem 3.6.1. *The following etimates hold:*

$$C_1^{nc}(K) \leq C^A(K), \tag{3.6.12}$$

$$C_0^{nc}(K) \leq C^A(K) \cdot \min_{I \in \mathcal{P}\backslash\{\emptyset\}} C_I(K). \tag{3.6.13}$$

Proof. We first consider the inequality (3.6.12). For $u \in V_0(K)$, since the integration on each face is zero and by the Green formula, each partial derivative $\partial u/\partial x_i$ ($i = 1, 2, 3$) satisfies

$$\int_K \frac{\partial u}{\partial x_i} dx = \sum_{k=1,2,3,4} \int_{f_k} u \ n_i ds = 0; \quad i = 1, 2, 3, \tag{3.6.14}$$

where n_i is the i-th component of the unit normal vector on the face f_k. Hence,

$$\|\frac{\partial u}{\partial x_i}\|_K \leq C^A(K)|\frac{\partial u}{\partial x_i}|_{1,K}\ (i = 1, 2, 3)\,. \tag{3.6.15}$$

which lead to

$$|u|_{1,K} \leq C^A(K)|u|_{2,K}\,. \tag{3.6.16}$$

For the second inequality, we should consider the following fact:

$$C_0^{nc}(K) = \sup_{u \in V_0^{nc} \backslash \{0\}} \frac{\|u\|_K}{|u|_{2,K}} \leq \sup_{u \in V_0^{nc} \backslash \{0\}} \frac{\|u\|_K}{|u|_{1,K}} \sup_{v \in V_0^{nc} \backslash \{0\}} \frac{|v|_{1,K}}{|v|_{2,K}}\,.$$

As we can easily see

$$\sup_{u \in V_0^{nc} \backslash \{0\}} \frac{\|u\|_K}{|u|_{1,K}} \leq \min_{I \in \mathcal{P} \backslash \{\emptyset\}} C_I(K)\,,$$

together with the inequality in (3.6.12), we can deduce the inequality (3.6.13). □

Remark 3.6.1. *In the two dimensional case, we can give concrete values to some constants by using the so-called "symmetry techniques", i.e., the isosceles right triangle can be extended to the unit square by reflection. However, in three dimensional case, such technique fails completely. So, we are planning to develop numerical method with a posteriori estimates to obtain the upper and lower bounds for these constants.*

Numerical results

In the case where the tetrahedron K is constructed by the convex hull of the canonical unit vector $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$, we evaluate $C^A(K)$ by the finite element method with linear tetrahedral elements, and obtain that

$$C^A(K) \approx 0.262\,,$$

which is compatible with the above mentioned theoretical one, that is, $C^A(K) \leq \sqrt{2}/\pi (\approx 0.451)$.

3.7 Numerical results

3.7.1 Evaluation of constants $C_{\{4,n\}}(\alpha, \theta)$ and $C_{\{5,n\}}(\alpha, \theta)$

Firstly, we perform numerical computations to see the actual dependence of various constants on α and θ by adopting the conforming P_1 element and a kind of discrete Kirchhoff plate bending element [26], the latter of which is used to deal with directly the 4-th order partial differential equations corresponding to $C_4(\alpha, \theta)$ and $C_5(\alpha, \theta)$. We obtain numerical results for $C_4(\alpha)$ and $C_5(\alpha)$ ($\theta = \pi/2$) together with their upper bounds. The uniform triangulation of the entire domain T_α is adopted, that is, T_α is subdivided into small triangles, all being congruent to $T_{\alpha,\pi/2,h}$ with, e.g., $h = 1/20$.

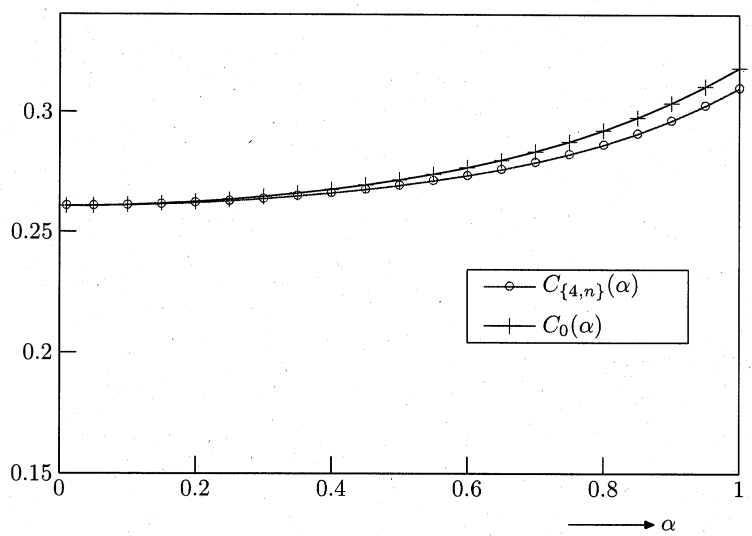


Figure 3.3: Numerically obtained graphs for $C_{\{4,n\}}(\alpha)$ and its upper bound

Figure 3.3 illustrates the graphs of approximate values of $C_{\{4,n\}}(\alpha)$ and $C_0(\alpha)$ versus $\alpha \in]0, 1]$, while Figure 3.4 shows similar graphs for $C_{\{5,n\}}(\alpha)$ together with its upper bounds $C_0(\alpha)C_{\{1,2\}}(\alpha)$ and $C_0(\alpha)C_{\{1,2,3\}}(\alpha)$. In both cases, the theoretical upper bounds give fairly good approximations to the considered constants $C_{\{4,n\}}(\alpha)$ and $C_{\{5,n\}}(\alpha)$. The asymptotic analysis result that $C_{\{4,n\}}(+0) = C_0(+0)$ can also be oberserved in the Figure 3.3. Meanwhile, the limit $C_{\{5,n\}}(+0)$ is different from $C_0(+0)C_{\{1,2,3\}}(+0) = C_0(+0)C_{\{1,2\}}(+0)$, although the numerical values are close to each other.

3.7.2 Computation for a priori error estimates

We test numerically the validity of our a priori error estimate for $\|\nabla u - \nabla_h u_h\|$. That is, we choose Ω as the unit square $\{x = \{x_1, x_2\}; 0 < x_1, x_2 < 1\}$ and f as $f(x_1, x_2) = \sin \pi x_1 \sin \pi x_2$. So the solution is $u(x_1, x_2) = \frac{1}{2\pi^2} \sin \pi x_1 \sin \pi x_2$. The

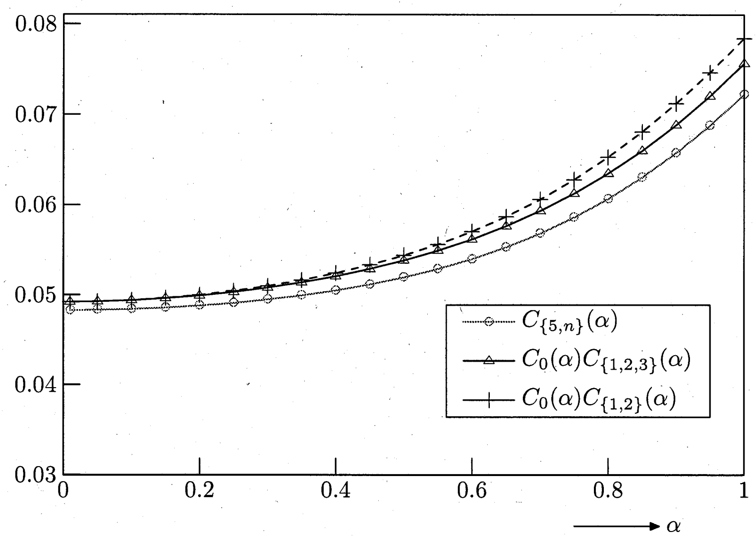


Figure 3.4: Numerically obtained graphs for $C_{\{5,n\}}(\alpha)$ and its upper bound

$N \times N$ Friedrichs-Keller type uniform triangulations ($N \in \mathbb{N}$) is used for computations. In such situation, all the triangles are congruent to a right isosceles triangle $T_{1,\pi/2,1/N}$, i.e., $h_* = h = 1/N$. Moreover, we can use the following values or their upper bounds for the necessary constants:

$$C_0^h = C_0 = 1/\pi, \quad C_{\{1,2\}}^h = 1/4, \quad C_{\{4,n\}}^h = 1/\pi, \quad C_{\{5,n\}}^h = 1/12.$$

Figure 3.5 illustrates the comparison of the actual $\|\nabla u - \nabla_h u_h\|$ and its a priori estimate based on our analysis. The difference is still large, but anyway our analysis appears to give correct upper bounds and order of errors, i.e., $O(h_*)$. In Chapter 5, we will consider a kind of hypercircle-based a posteriori estimation, which gives relatively better error estimate than the current one.

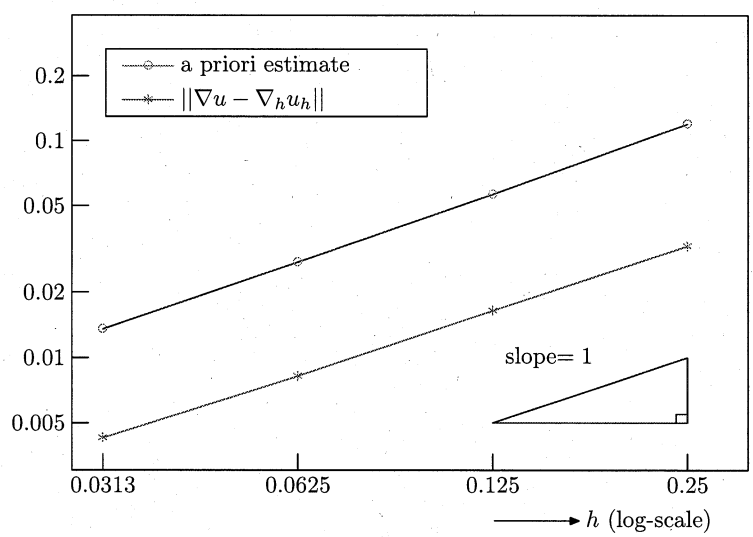


Figure 3.5: Numerical results for $\|\nabla u - \nabla_h u_h\|$ and their order plots for h

Chapter 4

Enclosing eigenvalue of Laplacian and its application to evaluation of error constants

In the preceding two chapters, we have introduced various constants related to error estimation of both conforming and nonconforming FEMs. These constants are characterized by Rayleigh quotients and hence related to eigenvalue problems with various kinds of constraints. For example, the constant $C_0(\alpha, \theta)$ is related to the first positive eigenvalue of $-\Delta$ in the space over $T_{\alpha,\theta}$, where the function has zero integration over $T_{\alpha,\theta}$.

As we have already seen, we can give exact values or proper estimates for the constants only in very rare cases, e.g., C_0, $C_1 = C_2$. It is in fact very difficult to determine the exact values of constants related to $T_{\alpha,\theta}$ of general shape. On the other hand, we can adopt the FEM to obtain approximate values for such constants as may be found in, e.g., [4, 44, 29, 47], but their quantitative error estimates for the approximation are often unavailable.

In this chapter, we will give quantitative a posteriori estimation for the evaluation of $C_i(\alpha, \theta)$'s $(0 \le i \le 3)$ by utilizing the piecewise linear FEM and the obtained estimates for the constants. The basic idea adopted here can be found in many texbooks such as that of Schultz[46]. To see the validity of the method in section 4.5, we will consider the evaluation of the minimum eigenvalue of the Laplacian eigenvalue problems on disk under the homogeneous Dirichlet condition.

At present, our approach gives only approximate or numerical boundings of the constants, but they can be turned into mathematically correct ones provided that appropriate numerical verification methods become available. Refer to [41, 38] for the interval computation method and the theories required by efficient verified computations.

4.1 Preliminaries

Let Ω be a bounded convex polygonal domain, which is in many cases the triangular one $T_{\alpha,\theta}$. Let us also consider a closed linear subspace $H^1_s(\Omega)$ of $H^1(\Omega)$, which can be finite-dimensional and satisfies

$$H^1_s(\Omega) \neq \{0\}, \quad 1 \notin H^1_s(\Omega), \tag{4.1.1}$$

where 1 is the constant function of unit value in Ω. A typical example of such $H^1_s(\Omega)$ is $H^1_0(\Omega)$.

As a generalization of variational form (2.1.2), we consider the problem of finding $u \in H^1_s(\Omega)$, for a given $f \in L_2(\Omega)$, such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \quad \forall v \in H^1_s(\Omega) . \tag{4.1.2}$$

The uniqueness and existence of u in $H^1_s(\Omega)$ are also trivial, so that we can define an operator G_s by

$$G_s : f \in L_2(\Omega) \to u \in H^1_s(\Omega) \text{ determined by (4.1.2)} . \tag{4.1.3}$$

As a generalization of the problem related to (2.2.18), let us also consider a minimization problem for the Rayleigh quotient:

$$R^s(v) := \frac{|v|^2_{1,\Omega}}{\|v\|^2_\Omega} ; \quad \forall v \in H^1_s(\Omega) \setminus \{0\} . \tag{4.1.4}$$

The minimum actually exists and is positive under (4.1.1) as may be shown by the compactness arguments. Moreover, denoting the minimum and the associated minimizer by $\lambda > 0$ and $u \in H^1_s(\Omega) \setminus \{0\}$, respectively, they satisfy the following variational equation:

$$(\nabla u, \nabla v)_\Omega = \lambda (u, v)_\Omega, \quad \forall v \in H^1_s(\Omega) . \tag{4.1.5}$$

By using G_s in (4.1.3), the present $u \in H^1_s(\Omega)$ is shown to satisfy $u = \lambda G_s u$.

To apply the P_1 FEM to the above two problems, we first introduce a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω as we mentioned in Section 2.1, and then construct the piecewise linear finite element space $S^h \subset H^1(\Omega)$ for each $\mathcal{T}^h$ as

$$S^h := \{v_h \in C(\overline{\Omega}) |\, v_h|_K \text{ is a linear function for each } K \in \mathcal{T}^h\}\,. \tag{4.1.6}$$

For $u \in H^2(\Omega) \left(\subset C(\overline{\Omega})\right)$, recall the definition the piecewise linear interpolation $\Pi_h^1 u \in S^h$ in (2.1.7):

$$(\Pi_h^1 u)(p_i) = u(p_i) \text{ for each vertex } p_i \text{ of } \mathcal{T}^h\,. \tag{4.1.7}$$

We will also use the parameters $h = \max_{K\in\mathcal{T}^h} h_K$, $C_4^h = \max_{K\in\mathcal{T}^h} C_4(\alpha_K, \theta_K)$ and $C_5^h = \max_{K\in\mathcal{T}^h} C_5(\alpha_K, \theta_K)$ defined in Section 2.2. Then we have the following interpolation estimates for the above u as was discussed in Section 2.2:

$$|u - \Pi_h^1 u|_{1,\Omega} \leq C_4^h\, h|u|_{2,\Omega}, \quad \|u - \Pi_h^1 u\|_\Omega \leq C_5^h\, h^2|u|_{2,\Omega}\,. \tag{4.1.8}$$

To construct approximate problems to (4.1.2) and the minimization of (4.1.5), let us consider the subspace $S^{h,s}$ of S^h defined by

$$S^{h,s} := S^h \cap H_s^1(\Omega)\,, \tag{4.1.9}$$

which we assume to be different from $\{0\}$. Of course, various other finite-dimensional subspaces of $H_s^1(\Omega)$ are available in place of $S^{h,s}$, but the above one is theoretically simple and also practically favorable in many cases.

Then an approximation to (4.1.2) is to find $u_h \in S^{h,s}$, for a given $f \in L_2(\Omega)$, such that

$$(\nabla u_h, \nabla v_h)_\Omega = (f, v_h)_\Omega, \quad \forall v_h \in S^{h,s}\,. \tag{4.1.10}$$

The uniqueness and existence of u_h in $S^{h,s}$ are trivial, so that we can define an operator G_s^h by

$$G_s^h : f \in L_2(\Omega) \to u_h \in S^{h,s} \text{ determined by (4.1.10)}\,. \tag{4.1.11}$$

Noticing that $u = G_s f$ and $u_h = G_s^h f$, we generalize the estimations in (2.1.5) and (2.1.6) as below:

$$|G_s f - G_s^h f|_{1,\Omega} = \min_{v_h \in S^{h,s}} |G_s f - v_h|_{1,\Omega}\,, \tag{4.1.12}$$

$$\|G_s f - G_s^h f\|_\Omega \leq |G_s f - G_s^h f|_{1,\Omega} \sup_{g\in L_2(\Omega)\setminus\{0\}} \inf_{v_h\in S^{h,s}} \frac{|G_s g - v_h|_{1,\Omega}}{\|g\|_\Omega}\,. \tag{4.1.13}$$

On the other hand, an approximation problem related to $R^s(\cdot)$ is to find the minimizer in $S^{h,s} \setminus \{0\}$. In this case, the existence of the minimum is again trivial, and the minimum λ^h and an associated minimizer $u_h \in S^{h,s} \setminus \{0\}$ satisfy the relation analogous to (4.1.5):

$$(\nabla u_h, \nabla v_h)_\Omega = \lambda^h (u_h, v_h)_\Omega, \quad \forall v_h \in S^{h,s} . \tag{4.1.14}$$

The following results are easy to derive but will play an essential role in our approach, cf. e.g. Theorem 8.3 of [46].

Lemma 4.1.1. *Let λ and λ^h be respectively defined by $\lambda = min_{v \in H^1_s(\Omega)\setminus\{0\}} R^s(v)$ and $\lambda^h = min_{v \in S^{h,s}\setminus\{0\}} R^s(v)$, and $u \in H^1_s(\Omega)$ be an minimizer associated to λ such that $\|u\|_\Omega = 1$. Then it holds that, for $\forall v_h \in S^{h,s} \setminus \{0\}$ with $\|u - v_h\| < 1$,*

$$\lambda \leq \lambda^h \leq \lambda + \frac{|u - v_h|^2_{1,\Omega}}{(1 - \|u - v_h\|_\Omega)^2} . \tag{4.1.15}$$

The following results are also well known and will be used later, cf.[22]

Lemma 4.1.2. *For the present Ω and a given $f \in L_2(\Omega)$, consider the problem of finding $u \in H^1(\Omega)$ such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega , \quad \forall v \in H^1(\Omega) . \tag{4.1.16}$$

Such u exists if and only if

$$(f, 1)_\Omega = 0 , \tag{4.1.17}$$

and is unique up to an additive arbitrary constant function. Moreover, $u \in H^2(\Omega)$ with

$$|u|_{2,\Omega} \leq \|\Delta u\|_\Omega = \|f\|_\Omega . \tag{4.1.18}$$

Remark 4.1.1. *To assure the uniqueness to u, we can for example impose the condition $(u, 1)_\Omega = 0$ on u. The present problem corresponds to the one for the Poisson equation with the homogeneous Neumann boundary condition:*

$$-\Delta u = f \ in \ \Omega, \quad \frac{\partial u}{\partial n} = 0 \ on \ \partial\Omega . \tag{4.1.19}$$

Lemma 4.1.3. *Given data $f \in L_2(\Omega)$, the problem of finding $u \in H^1_0(\Omega)$ such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega , \quad \forall v \in H^1_0(\Omega), \tag{4.1.20}$$

has a unique solution. Moreover, $u \in H^2(\Omega) \cap H^1_0(\Omega)$ with

$$|u|_{2,\Omega} \leq \|\Delta u\|_\Omega = \|f\|_\Omega . \tag{4.1.21}$$

4.2 A posteriori estimation of $C_0(\alpha, \theta)$

We first give a posteriori estimates to $C_0(\alpha, \theta)$. In this case, $\Omega = T_{\alpha,\theta}$ and $H^1_s(\Omega) = V^0_{\alpha,\theta}$. Let us define an orthogonal projection operator $P^0 : L_2(T_{\alpha,\theta}) \to L^0_2(T_{\alpha,\theta}) := \{g \in L_2(T_{\alpha,\theta}) | (g,1)_{T_{\alpha,\theta}} = 0\}$ by

$$P^0 g := g - \frac{\int_{T_{\alpha,\theta}} g(x)dx}{\int_{T_{\alpha,\theta}} dx} = g - \frac{(g,1)_{T_{\alpha,\theta}}}{|T_{\alpha,\theta}|}, \quad \forall g \in L_2(T_{\alpha,\theta}), \tag{4.2.1}$$

where $|T_{\alpha,\theta}|$ denotes the measure of $T_{\alpha,\theta}$. We can easily show that P^0 is also an orthogonal projection operator from $H^1(T_{\alpha,\theta})$ to $V^0_{\alpha,\theta}$, defined in (2.2.3), with respect to the standard inner product of $H^1(T_{\alpha,\theta})$. Notice that the present G_s, G^h_s and S^h are now those corresponding to domain $\Omega = T_{\alpha,\theta}$. Denote by $S^h_{T_{\alpha,\theta}}$ the finite element space S^h over domain $\Omega = T_{\alpha,\theta}$. Then we find that $S^h_{T_{\alpha,\theta}}$ contains the constant functions and the $S^{h,s}$ for the present $H^1_s(\Omega)$ is

$$S^{h,0}_{T_{\alpha,\theta}} = P^0 S^h_{T_{\alpha,\theta}} . \tag{4.2.2}$$

From now on, we will omit the subscript $T_{\alpha,\theta}$ for the norms, semi-norms and inner products related to the domain $T_{\alpha,\theta}$.

Noting that $\nabla P^0 v = \nabla v$ and $(f, P^0 v) = (P^0 f, v)$ for $v \in H^1(T_{\alpha,\theta})$, equation (4.1.2) for the present $u \in V^0_{\alpha,\theta}$ becomes

$$(\nabla u, \nabla v) = (P^0 f, v), \quad \forall v \in H^1_0(T_{\alpha,\theta}) , \tag{4.2.3}$$

which reduces to (4.1.16) under (4.1.17). Likewise, Eq.(4.1.5) for the present $\{\lambda, u\} \in \mathbb{R} \times \left(V^0_{\alpha,\theta} \setminus \{0\}\right)$ becomes

$$(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in H^1_0(T_{\alpha,\theta}) , \tag{4.2.4}$$

since $P^0 u = u$. By Lemma 4.1.2, the above u belongs to $H^2(T_{\alpha,\theta}) \cap V^0_{\alpha,\theta}$ with

$$|u|_2 \leq \lambda \|u\| . \tag{4.2.5}$$

The same way, we denote by $\lambda^{h,0}$ the minimum eigenvalue λ^h in equation (4.1.14), where $S^{h,s}$ is relaced by $S^{h,0}$.

Under the preceding preparations, let us apply Lemma 4.1.1 to estimate $\lambda^{h,0}$ in terms of the one λ^0 of (4.1.5) or (4.2.4). The minimizer associated to λ^0 is denoted by u^0 with the normalization condition $\|u^0\| = 1$. As v_h in (4.1.15) can be taken arbitrarily, we can choose various candidates from $S^{h,0}$. One possibility is to utilize the interpolation $\Pi_h^1 u^0 \left(\in S_{\alpha,\theta}^h\right)$ of u^0. Unfortunately, it may be outside of $S^{h,0}$, but its projection $P^0(\Pi_h^1 u^0)$ can be used thanks to (4.2.2). By taking advantage of properties of the orthogonal projection (4.2.1), we find that

$$|u^0 - P^0\left(\Pi_h^1 u^0\right)|_1 = |u^0 - \Pi_h^1 u^0|_1, \tag{4.2.6}$$

$$\|u^0 - P^0\left(\Pi_h^1 u^0\right)\| = \|P^0\left(u^0 - \Pi_h^1 u^0\right)\| \le \|u^0 - \Pi_h^1 u^0\|. \tag{4.2.7}$$

Using (4.1.8) and (4.2.5), we can evaluate the above in terms of h, λ^0, C_4^h, and C_5^h. Unfortunately, we have not obtained sufficiently accurate theoretical upper bounds for C_5^h as was noted in Section 2.4.1. So we should avoid the use of such a constant from theoretical standpoint.

Another possibility is to use $\tilde{u}_h^0 := \lambda^0 G^{h,0} u^0$, which is surely in $S^{h,0}$ and is suggested by the identity $u^0 = \lambda^0 G^0 u^0$. For this choice, we have

$$|u^0 - \tilde{u}_h^0|_1 \le |u^0 - P^0\left(\Pi_h^1 u^0\right)|_1 = |u^0 - \Pi_h^1 u^0|_1\,, \tag{4.2.8}$$

$$\|u^0 - \tilde{u}_h^0\| \le |u^0 - \tilde{u}_h^0|_1 \sup_{g\in L_2(T_{\alpha,\theta})\setminus\{0\}} \inf_{v_h\in S^{h,0}} \frac{|G^0 g - v_h|_1}{\|g\|}\,. \tag{4.2.9}$$

In this case, we only need the estimate in H^1 semi-norm (4.1.8), that is , the values of h, λ^0 and C_4^h. Hence we avoid the use of C_5^h.

Based on the above considerations, we have now the following two a priori error estimates.

Lemma 4.2.1. *(A priori estimates for $\lambda^{h,0}$) Let λ^0 and $\lambda^{h,0}$ be defined as above. Then if $C_5^h h^2 \lambda^0 < 1$,*

$$\lambda^0 \le \lambda^{h,0} \le \lambda^0 + \frac{(C_4^h \lambda^0)^2}{(1 - C_5^h h^2 \lambda^0)^2}\,. \tag{4.2.10}$$

Similarly, if ${C_4^h}^2 h^2 \lambda^0 < 1$, then

$$\lambda^0 \le \lambda^{h,0} \le \lambda^0 + \frac{(C_4^h \lambda^0)^2}{(1 - {C_4^h}^2 h^2 \lambda^0)^2}\,. \tag{4.2.11}$$

Remark 4.2.1. *In actual application of the above estimates, where the exact value of C_4^h (C_5^h resp.) may not be available, we can use an appropriate upper bound $\tilde{C}_4^h$ ($\tilde{C}_5^h$ resp.). From the considerations in Section 2.4.1 for concrete values of these constants, (4.2.10) would give a better bounding than (4.2.11), if an accurate upper bound $\tilde{C}_5^h$ of C_5^h becomes available.*

Let us define two functions related to (4.2.11) and (4.2.10):

$$\psi_{0,1}(t) := t + \frac{(C_4^h t)^2}{(1 - C_5^h h^2 t)^2} \quad (0 < t < \frac{1}{C_5^h h^2}), \tag{4.2.12}$$

$$\psi_{0,2}(t) := t + \frac{(C_4^h t)^2}{(1 - C_4^{h^2} h^2 t)^2} \quad (0 < t < \frac{1}{C_4^{h^2} h^2}), \tag{4.2.13}$$

where t is the variable, while other quantities are considered just parameters. Since these two functions are continuous and monotonically increasing on their domains of definition, they have their inverse functions, $\psi_{0,1}^{-1}$ and $\psi_{0,2}^{-1}$, to be monotonically continuous over in $(0, \infty)$. Then we can easily obtain the following a posteriori estimates for bounding λ^0 by numerically obtained $\lambda^{h,0}$.

Theorem 4.2.1. *(A posteriori estimates for λ^0) Let λ^0, $\lambda^{h,0}$, $\psi_{0,1}^{-1}$ and $\psi_{0,2}^{-1}$ be defined as above. Then it holds that*

$$\psi_{0,1}^{-1}(\lambda^{h,0}) \leq \lambda^0 \leq \lambda^{h,0} \quad \textit{if } \lambda^{h,0} < \frac{1}{C_5^h h^2}, \tag{4.2.14}$$

$$\psi_{0,2}^{-1}(\lambda^{h,0}) \leq \lambda^0 \leq \lambda^{h,0} \cdot \quad \textit{if } \lambda^{h,0} < \frac{1}{C_4^{h^2} h^2}. \tag{4.2.15}$$

Proof. From the preceding theorem, we have for example, $(0 <)\lambda^{h,0} \leq \psi_{0,1}(\lambda^0) \leq \psi_{0,1}(\lambda^{h,0})$ if $\lambda^{h,0} < 1/(C_5^h h^2)$. Then (4.2.14) follows immediately by operating $\psi_{0,1}^{-1}$ to this inequality, while (4.2.15) can be obtained similarly.

□

It is now straightforward to obtain boundings to the constant $C_0(\alpha, \theta)$. For example, we have from (4.2.14) that

$$1/\sqrt{\lambda^{h,0}} \leq C_0(\alpha, \theta) \leq 1/\sqrt{\psi_{0,1}^{-1}(\lambda^{h,0})} \quad \text{if } \lambda^{h,0} < \frac{1}{C_5^h h^2}. \tag{4.2.16}$$

Remark 4.2.2. *The method above to give a posteriori estimate for λ^0 can be also used to give estimates for the classical Dirichlet type eigenvalue problem over the bounded convex domain Ω: Find the smallest $\lambda \in \mathbb{R}$ and associated $u \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in H_0^1(\Omega). \tag{4.2.17}$$

For this purpose, we need to define the finite dimensional space $S^h \cap H_0^1(\Omega)$ and adopt the result for the regularity of solution as in Lemma 4.1.3, while the projection operator similar to P^0 is not necessary since $\Pi_h^1 u \in S^h \cap H_0^1(\Omega)$.

4.3 A posteriori estimation of $C_i(\alpha,\theta)$'s $(i = 1, 2, 3)$

Secondly, we give a posteriori estimates to $C_i(\alpha,\theta)$'s ($i = 1, 2, 3$). In current cases, the notations defined in preliminary have concrete forms: $\Omega = T_{\alpha,\theta}$, $H_s^1(\Omega) = V_{\alpha,\theta}^i$ and $S^{h,s} = S^{h,i} := S^h \cap V_{\alpha,\theta}^i$ ($i = 1, 2, 3$). Also, let us define an operator $P^i : H^1(T_{\alpha,\theta}) \to V_{\alpha,\theta}^i (i \in \{1, 2, 3\})$ by

$$P^i v := v - \frac{1}{|e_i|} \int_{e_i} v\, ds, \quad \forall v \in H^1(T_{\alpha,\theta}), \tag{4.3.1}$$

where $|e_i|$ denotes the length of edge e_i. Unlike P^0, the above operators are not well defined over $L_2(T_{\alpha,\theta})$, but the following relations similar to (4.2.2) still hold:

$$S^{h,i} = P^i\, S^h \quad (1 \le i \le 3)\,. \tag{4.3.2}$$

Let $p^{(i)}$'s ($i = 1, 2, 3$) be the three vertexes of triangle $T_{\alpha,\theta}$, that is,

$$p^{(1)} = O(0,0), \quad p^{(2)} = A(1,0), \quad p^{(3)} = B(\alpha\cos\theta, \alpha\sin\theta)\,.$$

Suggested by [36], we introduce quadratic functions f_i's ($1 \le i \le 3$) of $x = (x_1, x_2)$ by

$$f_i(x_1, x_2) := \frac{|e_i|}{4|T_{\alpha,\theta}|}|x - p^{(i)}|^2\,, \tag{4.3.3}$$

where $|x - p^{(i)}|$ denotes the Euclidean distance between x and $p^{(i)}$.

These functions are sufficiently smooth and satisfy

$$\frac{\partial f_i}{\partial n} = \delta_{ij} \text{ on } e_j\,, \quad \forall i, j \in \{1, 2, 3\}\,.$$

Then, for each $v \in H^1(T_{\alpha,\theta})$, we find that

$$\int_{e_i} v ds = (\nabla f_i, \nabla v) + (\Delta f_i, v)\,,$$

so that (4.3.1) can be rewritten by

$$P^i\, v := v - \frac{1}{|e_i|}[(\nabla f_i, \nabla v) + (\Delta f_i, v)]\,, \quad \forall v \in H^1(T_{\alpha,\theta})\,.$$

Similarly to (4.2.3), (4.1.5) for the present $u \in V_{\alpha,\theta}^i$ becomes

$$(\nabla u, \nabla v) = (f, P^i\, v), \quad \forall v \in H^1(T_{\alpha,\theta})\,, \tag{4.3.4}$$

which can be rewritten by

$$(\nabla(u + \frac{(f,1)}{|e_i|} f_i), \nabla v) = (f - \frac{(f,1)}{|e_i|} \Delta f_i, v)\,, \quad \forall v \in H^1(T_{\alpha,\theta})\,. \tag{4.3.5}$$

By Lemma 4.1.2, we find that $u + \frac{(f,1)}{|e_i|} f_i \in H^2(T_{\alpha,\theta})$ with

$$\left| u + \frac{(f,1)}{|e_i|} f_i \right|_2 \leq \| f - \frac{(f,1)}{|e_i|} \Delta f_i \|\,. \tag{4.3.6}$$

Hence, by using the triangle and Schwarz inequalities, we have

$$|u|_2 \leq \|f\| + \frac{(f,1)}{|e_i|}(|f_i|_2 + \|\Delta f_i\|) \leq \|f\| \left\{ 1 + \frac{\sqrt{|T_{\alpha,\theta}|}}{|e_i|}(|f_i|_2 + \|\Delta f_i\|) \right\}\,. \tag{4.3.7}$$

Clearly, it holds that

$$|T_{\alpha,\theta}| = \frac{\alpha}{2}\sin\theta, |e_1| = 1, |e_2| = \alpha, |e_3| = \sqrt{1+\alpha^2 - 2\alpha\cos\theta},$$

$$|f_i|_2 = \frac{\sqrt{2}}{2}\|\Delta f_i\|, \quad \Delta f_i = \frac{|e_i|}{|T_{\alpha,\theta}|},$$

so that we have, for $i \in \{1,2,3\}$,

$$|u|_2 \leq (2 + \sqrt{2}/2)\|f\|\,. \tag{4.3.8}$$

Also, the eigenvalue problem for the present $\{\lambda, u\} \in \mathbb{R} \times \left(V^i_{\alpha,\theta} \setminus \{0\}\right)$ $(1 \leq i \leq 3)$ becomes

$$(\nabla u, \nabla v) = \lambda(u, P^i\, v), \quad \forall v \in H^1(T_{\alpha,\theta})\,. \tag{4.3.9}$$

Thus, we can utilize the results for (4.3.4) by taking f in (4.3.4) as λu in (4.3.9).

The approximation problems corresponding to 4.1.10 and 4.1.14 are also given by using $S^{h,i}$'s $(1 \leq i \leq 3)$. Then, just like Lemma 4.1.1 and Theorem 4.2.1 for $C_0(\alpha,\theta)$, we have the following results for $C_i(\alpha,\theta)$'s $(1 \leq i \leq 3)$.

Theorem 4.3.1. *[A priori and a posteriori estimates for $\lambda^{h,i}$, $(i = 1,2,3)$] For each $i \in \{1,2,3\}$, let λ^i and $\lambda^{h,i}$ be the smallest eigenvalues of (4.1.5) and (4.1.14) in the case where $H^1_s(\Omega) = V^i_{\alpha,\theta}$ and $S^{h,s} = S^{h,i}$ cf.(4.3.2). Then, if $(MC^h_4 h)^2 < 1$ with $M := 2 + \sqrt{2}/2$, it holds that*

$$\lambda^i \leq \lambda^{h,i} \leq \lambda^i + \frac{(MC^h_4\lambda^i)^2}{(1 - M^2(C^h_4)^2 h^2 \lambda^i)^2}\,, \tag{4.3.10}$$

and, if $\lambda^{h,i} < 1/(MC_4^h h)^2 < 1$,

$$\psi_i^{-1}(\lambda^{h,i}) \le \lambda^i \le \lambda^{h,i}\,, \tag{4.3.11}$$

where

$$\psi_i(t) := t + \frac{(MC_4^h t)^2}{(1 - M^2(C_4^h)^2 h^2 t)^2} \quad \left(0 < t < \frac{1}{(MC_4^h h)^2}; 1 \le i \le 3\right), \tag{4.3.12}$$

which is continuous and monotonically increasing.

Remark 4.3.1. *Because of the factor* $M \approx 2.7071\cdots$, *efficiency of (4.3.10) is worse than that of (4.2.10). In the present case, estimates corresponding to (4.2.11) and using* C_5^h *do not appear to be fully effective unlike those in the preceding subsection. This is attributed to the fact that we cannot at present obtain desirable estimates for* $\|u - P^i(\Pi_h^1 u)\|$ $(\forall u \in V_{\alpha,\theta}^i \cap H^2(T_{\alpha,\theta});\ 1 \le i \le 3)$, *since* P^i *is not definable over* $L_2(T_{\alpha,\theta})$ *and hence we cannot take advantage of the best approximation property with respect to the* L_2 *norm.*

Remark 4.3.2. *In the procedure of obtaining (4.3.8), we find that the coefficient* $M := (2 + \sqrt{2}/2)$ *depends on the selection of* f_i*'s, which reminds us of finding improved functions for smaller* M. *We leave this work to future research.*

Remark 4.3.3. *By using the similar techniques, we may further give a posteriori estimate for constants such as* $C^{\{1,2\}}$ *and* $C^{\{1,2,3\}}$, *where we need a priori estimate for the eigenvalue problem: Find* $\{\lambda, u\} \in \mathbb{R} \times \left(V_{\alpha,\theta}^{\{1,2,3\}} \setminus \{0\}\right)$ *such that*

$$(\nabla u, \nabla v) = \lambda(u, v)\,, \quad \forall v \in V^{\{1,2,3\}}(T_{\alpha,\theta})\,.$$

To deal with the constraint conditions associated to $V^{\{1,2,3\}}$, *we need to specify the functions like* f_i*'s in (4.3.3), which is not so obvious. However, such associated functions may be constructed in the finite element spaces, although we do not discuss such topics here.*

4.4 Numerical results for a posteriori estimates for constants

To show the validity of the a posteriori estimates developed in the previous sections, we will take the constants $C_0 = C_0(1, \pi/2)$ and $C_1 = C_1(1, \pi/2)$ as examples and perform numerical evaluations. With no further efforts, the quantitative estimates for other constants $C_i(\alpha, \theta)$'s $(i = 0, 1, 2, 3)$ can also be done similarly.

We denote the associated eigenvalues by $\lambda_0 = C_0^{-2}$ and $\lambda_1 = C_1^{-2}$ and show the results as below.

N	bounds for λ_0 by $\psi_{0,1}^{-1}$	bounds for λ_0 by $\psi_{0,2}^{-1}$	bounds for λ_1 by ψ_1^{-1}
2	$5.9890 < \lambda_0 < 11.7155$	$6.5550 < \lambda_0 < 11.7155$	$\lambda_1 < 4.3071^\dagger$
3	$7.8874 < \lambda_0 < 10.7213$	$8.1463 < \lambda_0 < 10.7213$	$1.9780 < \lambda_1 < 4.2102$
4	$8.7512 < \lambda_0 < 10.3570$	$8.8616 < \lambda_0 < 10.3570$	$2.6006 < \lambda_1 < 4.1713$
8	$9.6055 < \lambda_0 < 9.9946$	$9.6143 < \lambda_0 < 9.9946$	$3.6537 < \lambda_1 < 4.1304$
16	$9.8054 < \lambda_0 < 9.9012$	$9.8060 < \lambda_0 < 9.9012$	$3.9982 < \lambda_1 < 4.1196$
32	$9.8537 < \lambda_0 < 9.8776$	$9.8537 < \lambda_0 < 9.8776$	$4.0864 < \lambda_1 < 4.1168$
64	$9.8656 < \lambda_0 < 9.8716$	$9.8656 < \lambda_0 < 9.8716$	$4.1085 < \lambda_1 < 4.1161$
(∞)	$\lambda = \pi^2 = 9.869604\ldots$		$\lambda_1 \approx 4.115858$

† In this case, the obtained $\lambda^{h,0}$ is outside the domain of definition for ψ_1^{-1}.

Table 4.1: A posteriori estimate for λ_0 and λ_1

Table 4.1 gives the boundings for λ_0 based on (4.2.15) and (4.2.14) of Theorem 4.2.1 and those for λ_1 based on (4.3.11) of Theorem 4.3.1, where the conforming P_1 finite element method is adopted. We tested several meshes, which are uniform ones composed of small triangles similar to the entire domain T (See Figure 4.1). In practical computations, the values for the important parameters C_4^h and C_5^h and h are specified below as

$$C_4^h < \hat{C}_4^h = 0.5, \quad C_5^h < \hat{C}_5^h = 0.17, \quad h = 1/N,$$

where N is the number of subdivision for the mesh, for example, $N = 4$ in the Figure 4.1. Notice that $\hat{C}_4^h = 0.5$ is a theoretical upper bound of C_4^h, but the one

$\hat{C}_5^h$ is a numerically obtained approximate upper bound of C_5^h at present. We tested (4.2.14) only to see its effectiveness experimentally.

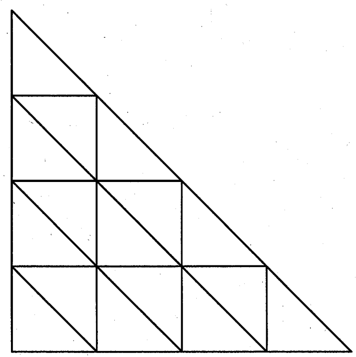


Figure 4.1: Triangulation of T $(N = 4)$

We can observe that the present simple methods can actually bound C_0 and C_1 from both above and below. As is expected, (4.2.14) gives better lower bounds than (4.2.15) for coarser meshes. Table 4.1 also shows that the lower bounds obtained for C_1 are in general rougher that those for C_0. This is probably attributed to the existence of the factor $M = 2 + \sqrt{2}/2$. Even in this case, we can obtain reasonable results by mesh refinement.

4.5 A posteriori estimates for eigenvalue of Laplacian operator over disk

As an application of the approach we constructed in the previous sections, here we will try to evaluate of the first eigenvalue of Laplacian over the unit disk.

Let $\lambda(> 0)$ be the one characterized by the eigenvalue problem over unit disk Ω: *Find minimal* $\lambda > 0$ *and* $u \in H^2(\Omega) \setminus \{0\}$ *such that*

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega\,. \tag{4.5.1}$$

As we can show, the eigenvalue λ is the square of the first zero of the Bessel function J_0, which will help us test the precision of the estimation.

As the circular boundary cannot be presented by polygon, we use the regular n-polygonal domain Ω^n (Figure 4.2) to approximate the disk, and then consider the determination of λ^n of the eigenvalue problem over there: *Find minimal* $\lambda^n > 0$ *and*

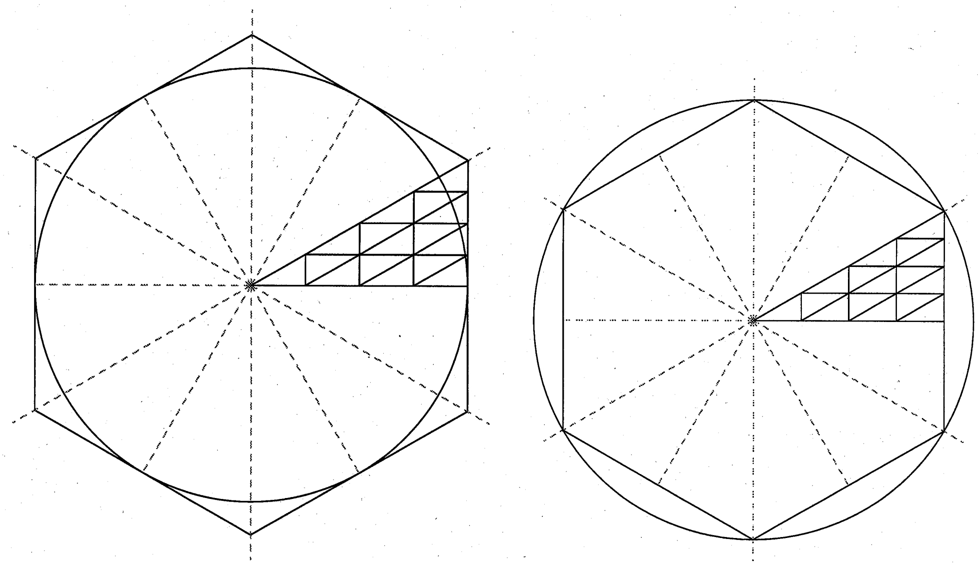


Figure 4.2: Circumscribed regular hexagonal polygon Ω_c^6 (left) and inscribed one Ω_i^6 (right) associated to unit circle

$u \in H^2(\Omega) \setminus \{0\}$ *such that,*

$$-\Delta u = \lambda^n u \text{ in } \Omega^n, \quad u = 0 \text{ on } \partial\Omega^n . \tag{4.5.2}$$

Here, Ω^n can be either Ω_{i}^n or Ω_{c}^n, which are the inscribed n-polygon and the circumscribed one of the unit disk, respectively.

By considering the Rayleigh quotient and the extension theory of Sobolev spaces, we can show that the exact solution of eigenvalue problem (4.5.2) over an inscribed regular n-polygon Ω_{i}^n of the unit disk can give an upper bound for λ in (4.5.1), while the one on circumscribed polygon Ω_{c}^n will supply a lower bound. For each polygonal domain, we can apply the piecewise linear FEM to evaluate the eigenvalues λ^n.

As for meshes, we first triangulate the right triangle ΔOAB with $OA = 1$ and $AB = \tan \pi/n$ and $\angle OAB = \pi/2$ just as we did for T and T_α in the preceding problems by dividing each edges uniformly into N segments. Notice that by a reflection and rotations, we can immediately obtain whole meshes for Ω_c^n, see Figure 4.3. The constants in Theorem 4.2.1 can be taken as

$$\tilde{C}_4^h = 0.5, \quad h = \begin{cases} \sqrt{3}/N & \text{if } n = 3 \\ 1/N & \text{if } n \geq 4 \end{cases},$$

where $\alpha \leq 1$ in all cases.

We solve the problem of (4.5.2) with $\Omega^n = \Omega_c^n$, and summarize the results in Table 4.2, from which we can experimentally see the effectiveness of our bounding method.

n	N	bounds for λ	N	bounds for λ	N	bounds for λ
3	5	$3.9082 < \lambda < 4.4963$	10	$4.2688 < \lambda < 4.4147$	100	$4.3853 < \lambda < 4.3868$
4	5	$4.7700 < \lambda < 5.0211$	10	$4.8954 < \lambda < 4.9569$	100	$4.9344 < \lambda < 4.9351$
5	5	$5.0049 < \lambda < 5.2826$	10	$5.1590 < \lambda < 5.2273$	100	$5.2075 < \lambda < 5.2082$
6	5	$5.1387 < \lambda < 5.4323$	10	$5.3114 < \lambda < 5.3839$	100	$5.3659 < \lambda < 5.3667$
7	5	$5.2220 < \lambda < 5.5257$	10	$5.4078 < \lambda < 5.4831$	100	$5.4666 < \lambda < 5.4674$
8	5	$5.2774 < \lambda < 5.5879$	10	$5.4727 < \lambda < 5.5498$	100	$5.5346 < \lambda < 5.5354$
9	5	$5.3160 < \lambda < 5.6313$	10	$5.5185 < \lambda < 5.5969$	100	$5.5827 < \lambda < 5.5836$
10	5	$5.3440 < \lambda < 5.6628$	10	$5.5520 < \lambda < 5.6313$	100	$5.6181 < \lambda < 5.6190$

Table 4.2: A posteriori estimates for the first eigenvalue λ associated to Ω_c^n

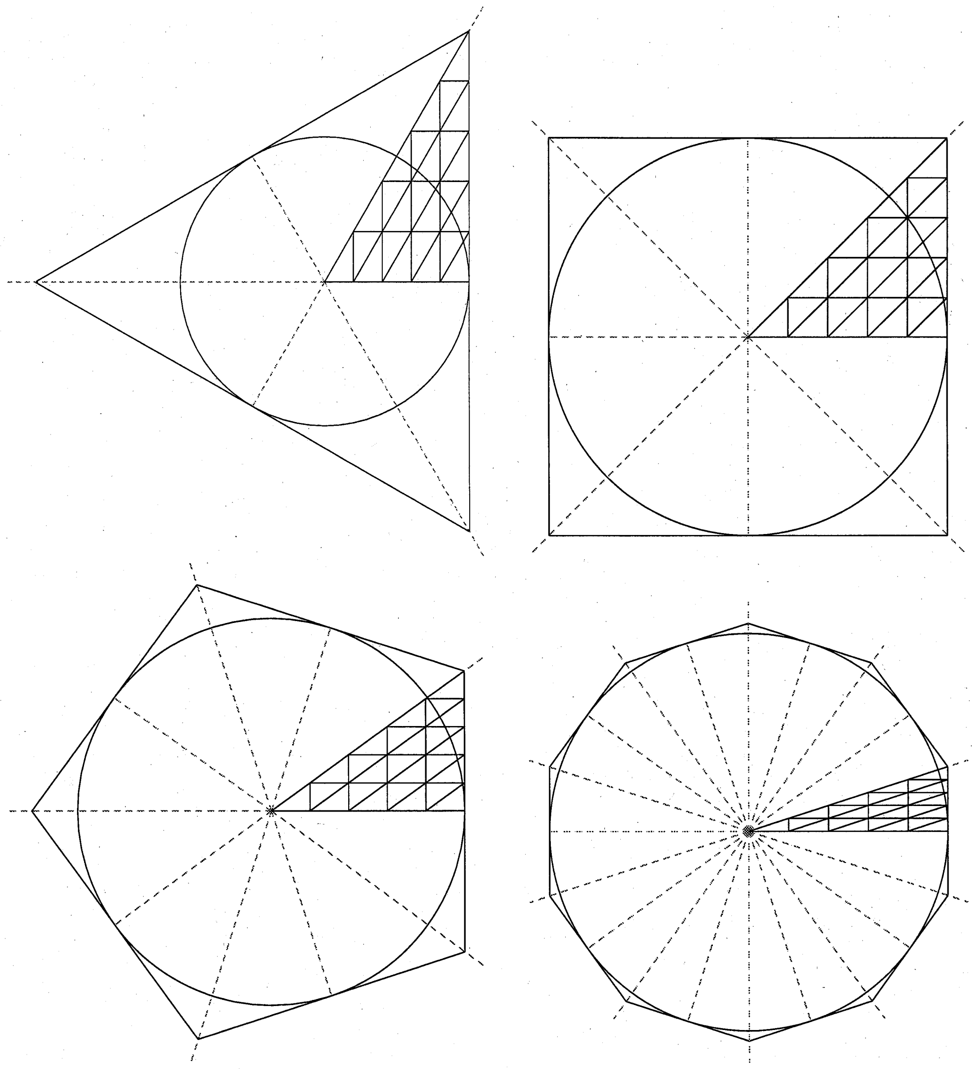


Figure 4.3: Meshes for n-polygonal domains Ω^n with $N = 5$, $n = 3, 4, 5, 10$

Chapter 5

Quantitative a posteriori error estimates for FEM solutions of Poisson's equation

In the past four chapters, we have paid efforts to give quantitative estimates to various error constants. As we noted at the beginning of this dissertation, the concrete values or upper bounds will enable quantitative error estimation for the FEM solutions of PDE problems. In this chapter, by applying the obtained results for the error constants, we consider a hypercircle-based a posteriori error estimation method for Poisson's equation, which gives computable estimates for the FEM solutions.

Also, to demonstrate the feasibility of this method, we will examine Poisson's equation over L-shaped domain and propose quantitative error estimation for its FEM solutions.

5.1 Hypercircle-based a posteriori error estimates

We reconsider the problem of Poisson's equation over a polygonal domain Ω, which may be nonconvex one, with regular familiy of triangulation $\{\mathcal{T}^h\}$ $(h > 0)$. For each $\mathcal{T}^h$, we consider the lowest-order Raviart-Thomas triangular finite element space $W^h \subset H(\text{div}, \Omega)$, cf. Eq.(3.1.10) or [14, 29]. In the following, we introduce the hypercircle-based a posteriori error estimation [17, 29].

As in Chapter 2, assume $u \in H_0^1(\Omega)$ to be the solution of the variational problem with $f \in L_2(\Omega)$:

$$(\nabla u, \nabla v) = (f, v)\,, \forall v \in H_0^1(\Omega)\,. \tag{5.1.1}$$

Let $u^h \in H_0^1(\Omega)$ be the solution of (4.1.16) with f replaced by $Q_h f$, that is,

$$(\nabla u^h, \nabla v) = (Q_h f, v)\,, \quad \forall v \in H_0^1(\Omega)\,. \tag{5.1.2}$$

Noting that $(f - Q_h f, v) = (f - Q_h f, v - Q_h v)$ for each $v \in H_0^1(\Omega)\,(\subset L_2(\Omega))$, we have that

$$\|\nabla(u - u^h)\| \le C_0^h h \|f - Q_h f\| \quad (\le C_0^{h2} h^2 |f|_1 \text{ if } f \in H^1(\Omega)\ )\,, \tag{5.1.3}$$

where

$$C_0^h := \max_{K \in \mathcal{T}^h} C_0(\alpha_K, \theta_K), \quad h := \max_{K \in \mathcal{T}^h} h_K\,. \tag{5.1.4}$$

Here, $\{\alpha_K, \theta_K, h_K\}$ are the ones related to the element K as in Section 2.2.

For $p_h \in W^h\,(\subset H(\mathrm{div};\Omega))$ with $\operatorname{div} p_h = Q^h f$, we find that, for each $v \in H_0^1(\Omega)$,

$$\|\nabla v - p_h\|^2 = \|\nabla(v - u^h)\|^2 + \|\nabla u^h - p_h\|^2, \quad \|\nabla u^h - \frac{1}{2}(\nabla v + p_h)\| = \frac{1}{2}\|\nabla v - p_h\|\,. \tag{5.1.5}$$

The equations above imply that the three points ∇u^h, ∇v and p_h in $L_2(\Omega)^2$ make a hypercircle, the first having a right inscribed angle. Here, the vector function p_h is available as a FEM approximation of u, e.g., the Raviart-Thomas mixed FEM solution. By a proper choice of concrete function $v \in H_0^1(\Omega)$ and applying the hypercircle equalities (5.1.5), we can obtain a posteriori error estimates for $(\nabla u - p_h)$:

$$\|\nabla u - p_h\|_\Omega \le \|\nabla(u - u^h)\|_\Omega + \|p_h - \nabla u^h\|_\Omega \le \|\nabla(u - u^h)\|_\Omega + \|\nabla v - p_h\|_\Omega. \tag{5.1.6}$$

Another approximation of u is given by $(\nabla v + p_h)/2$ with the error estimate:

$$\|\nabla u - \frac{1}{2}(\nabla v + p_h)\|_\Omega \le \|\nabla(u - u^h)\|_\Omega + \frac{1}{2}\|\nabla v - p_h\|_\Omega\,. \tag{5.1.7}$$

A typical example of v is the conforming P_1 finite element solution, for example, the continuous piecewise linear function $u_h \in V_0^h$ defined in (2.1.4). Another example is obtained by appropriate post-processing, such as nodal averaging or smoothing, of nonconforming FEM solution such as the $u_h \in V_{nc}^h$ characterized in (3.1.3). A cheap method of constructing a nice v may be also an interesting subject. If we use $\nabla_h u_h$, the one in V_{nc}^h in (3.1.3), instead of the modified one $\tilde{u}_h \in V^h$, we must evaluate some additional terms. Fortunately, such evaluation can be done explicitly by using C_0^h and some positive constants.

5.2 Nonconforming FEM and Raviart-Thomas mixed FEM

We have already introduced the Raviart-Thomas space W^h for auxiliary purposes. But it is well known that the present nonconforming FEM is closely related to the mixed Raviart-Thomas FEM [5, 35]. Here we will summarize the implementation of such a mixed FEM by slightly modifying the original nonconforming P_1 scheme described by Eq.(3.1.3). The original idea in [5, 35] is based on the enrichment by the conforming cubic bubble functions with the L_2 projection into W^h, but we here adopt nonconforming quadratic bubble ones to make the modification procedure a little simpler.

Here, we write V_{nc}^h defined in Eq. (3.1.2) as V^h for simplicity. Firstly, we replace f in Eq.(3.1.3) by $Q_h f$. Then u_h is modified to $u_h^* \in V^h$ defined by

$$(\nabla_h u_h^*, \nabla_h v_h) = (Q_h f, v_h)\,, \quad \forall v_h \in V^h. \tag{5.2.1}$$

Secondly, we introduce the space V_B^h of nonconforming quadratic bubble functions by defining its basis function ϕ_K associated to each $K \in \mathcal{T}^h$ as follows: ϕ_K vanished outside K and its value at $x \in K$ is given by

$$\phi_K(x) = \frac{1}{2}|x^G|^2 - \frac{1}{12}\sum_{i=1}^{3}|x^{(i)} - x^G|, \tag{5.2.2}$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^2$, x^G is the barycenter of K, and $x^{(i)}$'s for $i = 1, 2, 3$ is the i-th vertex of K. It is easy to see that the line integration of ϕ_K for each e of K vanishes:

$$\int_e \phi_K d\gamma = 0\,. \tag{5.2.3}$$

Now the enriched nonconforming finite element space $\tilde{V}^h$ is defined by the following direct sum:

$$\tilde{V}^h = V^h \oplus V_B^h\,. \tag{5.2.4}$$

By Eq.(5.2.3) and the Green formula, we find the following orthogonality relation for $(\nabla_h \cdot, \nabla_h \cdot)$:

$$(\nabla_h v_h, \nabla_h \beta_h) = 0\,, \quad \forall v_h \in V^h, \forall \beta_h \in V_B^h\,. \tag{5.2.5}$$

Then the modified finite element solution $\tilde{u}_h \in \tilde{V}^h$ is defined by

$$(\nabla_h \tilde{u}_h, \nabla_h \tilde{v}_h) = (Q_h f, \tilde{v}_h), \quad \forall \tilde{v}_h \in \tilde{V}^h\,. \tag{5.2.6}$$

Thanks to Eq.(5.2.4), the present $\tilde{u}_h$ can be obtained as the sum:

$$\tilde{u}_h = u_h^* + \alpha_h \,, \tag{5.2.7}$$

where $u_h^* \in V^h$ is the solution of (5.2.1), and $\alpha_h \in V_B^h$ is determined by

$$(\nabla_h \alpha_h, \nabla_h \beta_h) = (Q_h f, \beta_h), \quad \forall \beta_h \in V_B^h, \tag{5.2.8}$$

i.e., completely independent of u_h^*. Moreover, α_h can be decided by element-by-element computations. More specifically, denoting $\alpha_h|_K$ as $\alpha_K \phi_K|_K$, Eq.(5.2.8) leads to

$$\alpha_K (\nabla \phi_K, \nabla \phi_K)_K = (Q_h f, \phi_K)_K, \quad \forall K \in \mathcal{T}^h, \tag{5.2.9}$$

where $(\cdot, \cdot)$ denotes the inner products of both $L_2(K)$ and $L_2(K)^2$.

Let X^h be the piecewise constant function space over triangulation $\mathcal{T}^h$. Define $\{p_h, \overline{u}_h\} \in L_2(\Omega)^2 \times X^h$ by

$$p_h = \nabla_h \tilde{u}_h, \; \overline{u}_h = Q_h \tilde{u}_h \,. \tag{5.2.10}$$

By applying the Green formula to Eq.(5.2.6), we can show that $p_h \in W^h$, and also that the present pair $\{p_h, \overline{u}_h\}$ satisfies the determination equations of the lowest-order Raviart-Thomas mixed FEM:

$$\begin{cases} (p_h, q_h) + (\overline{u}_h, \operatorname{div} q_h) = 0; & \forall q_h \in W^h, \\ (\operatorname{div} p_h, \overline{v}_h) = -(Q_h f, \overline{v}_h); & \forall \overline{v}_h \in X^h. \end{cases} \tag{5.2.11}$$

By the uniqueness of the solutions, $\{p_h, \overline{u}_h\}$ is nothing but the unique solution of Eq.(5.2.11).

In conclusion, denoting the constant value of $Q_h f|_K$ by $\overline{f}_K \left(= \int_K f dx / \text{meas}(K)\right)$, we have for $K \in \mathcal{T}^h$ and $x \in K$ that

$$\alpha_K = -\frac{1}{2}\overline{f}_K, \; \tilde{u}_h(x) = u_h^*(x) + \alpha_K \phi_K(x) = u_h^*(x) - \frac{1}{4}\overline{f}_K(|x - x^G|^2 - \frac{1}{6}\sum_{i=1}^{3} |x^{(i)} - x^G|^2),$$

$$p_h(x) = \nabla_h u_h^*(x) - \frac{1}{2}\overline{f}_K(x - x^G), \overline{u}_h(x) = u_h^*(x^G) - \frac{1}{16}\overline{f}_K(|x^G|^2 - \frac{1}{3}\sum_{i=1}^{3} |x^{(i)}|^2), \tag{5.2.12}$$

which coincide with those in [35] and are easy to compute by post-processing.

Now we state the following theorem.

Theorem 5.2.1. *Given data function $f \in L_2(\Omega)$, suppose $u \in H_0^1(\Omega)$ to be the solution of (2.1.2) or (4.1.16) and $u_h^* \in V_{nc}^h$ the nonconforming FEM solution of (5.2.1). We post-process u_h^* to construct $p_h \in W^h\ (\subset H(\mathrm{div};\Omega))$ as we do in (5.2.12). Then for any $v \in H_0^1(\Omega)$, we have*

$$\|\nabla u - \nabla v\|_\Omega + \|\nabla u - p_h\|_\Omega \le \|\nabla v - p_h\|_\Omega + C_0^h\, h\|Q_h f - f\|_\Omega\,. \tag{5.2.13}$$

If f belongs to $H^1(\Omega)$ as well, the estimate can be further improved as

$$\|\nabla u - \nabla v\|_\Omega + \|\nabla u - p_h\|_\Omega \le \|\nabla v - p_h\|_\Omega + (C_0^h)^2 h^2 |f|_{1,\Omega}\,. \tag{5.2.14}$$

Remark 5.2.1. *If we prefer to give an error estimate for nonconforming solution u_h, we can have a rough one based on equation (5.2.12),*

$$\|\nabla u - \nabla_h u_h\| \le \|\nabla u - p_h\| + \frac{1}{2}\|\sum_{K\in\mathcal{T}^h} \overline{f}_K \phi_K\| \tag{5.2.15}$$

$$\le \|\nabla u - p_h\| + \frac{\sqrt{2}}{2}\, h\, \|f\| \tag{5.2.16}$$

Remark 5.2.2. *As we can see, the error estimations in Theorem 5.2.1 is based on the auxiliary problem of modified Poisson's equation with $Q_h f$. In practical computation, we can omit such pre-processing of f and solve the variational equation (3.1.3) directly to obtain u_h. Then after post-processing as we do in (5.2.12) , that is, on each $K \in \mathcal{T}^h$,*

$$\hat{p}_h(x) = \nabla_h u_h(x) - \frac{1}{2}\overline{f}_K(x - x^G),$$

we can obtain $\hat{p}_h$, which may not belong to W^h any more. To give an error estimate for $\|\nabla u - \hat{p}_h\|$, we estimate the term $p_h - \hat{p}_h = \nabla_h u_h^* - \nabla_h u_h$ as

$$\|\nabla_h u_h^* - \nabla u_h\| \le C_0^h h \|Q_h f - f\|\,.$$

Thus, taking p_h in equation (5.2.13) as $p_h = \hat{p}_h + (p_h - \hat{p}_h)$, we can deduce an error estimate as

$$\|\nabla u - \hat{p}_h\|_\Omega \le \|\nabla v - \hat{p}_h\|_\Omega + 3C_0^h h\|f - Q_h f\|_\Omega\,. \tag{5.2.17}$$

Notice that the part $\|f - Q_h f\|$ converges to zero faster if f belongs to $H^1(\Omega)$.

5.3 Numerical results

To confirm the validity of the error estimates in Theorem 5.2.1, we will consider two computational examples for Poisson's equation with the homogeneous Dirichlet boundary condition: one is over the unit square domain and the other the L-shaped domain.

5.3.1 Poisson's equation over the unit square

As in Section 3.7, we take $f = \sin \pi x_1 \sin \pi x_2 \in H^1(\Omega)$. We here show only the estimate (5.2.13) to see its efficiency.

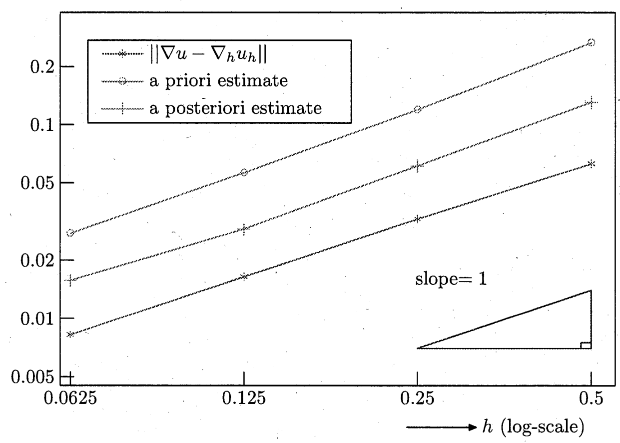


Figure 5.1: Numerical results for $\|\nabla u - \nabla_h u_h\|$ and its estimates versus h

As the domain is triangulated in the same way as in Section 3.7, we can take the value of the constant C_0^h as $C_0^h = 1/\pi$. The function v in (5.2.13) is taken as the P_1 conforming FEM solution defined in (2.1.4). The computational results are shown in Figure 5.1, where we can see that the a posteriori one gives a better result than the a priori one.

5.3.2 Poisson's equation over L-shaped domain

Secondly, we consider the Poisson's problem on L-shaped domain (see Fig 5.2) with homogeneous Dirichlet condition:

$$-\Delta u = f \text{ in } \Omega; u = 0 \text{ on } \partial\Omega\,,$$

where $f \in L^2(\Omega)$ is given by

$$f = \sin\frac{2}{3}\theta\left(2r^{2/3} + \frac{14}{3}(r-1)r^{-1/3}\right), r \leq 1;\ \ f = 0, r > 1\,.$$

Here (r, θ) are the variables in the polar coordinates. Also, the fact that $f \notin H^1(\Omega)$ is easy to verify.

Since the domain has an interior angle to be obtuse, we know that the solution belongs to $H^1(\Omega)$ but may not belong to $H^2(\Omega)$, which make both the a priori and a posteriori error estimates difficult. In current case, we know the exact solution for the problem is

$$u = (r-1)^2 r^{2/3}\sin(\frac{2}{3}\,\theta), r \leq 1;\ \ u = 0, r > 1\,.$$

It is easy to see that $u \in H^1(\Omega)$ but $u \notin H^2(\Omega)$.

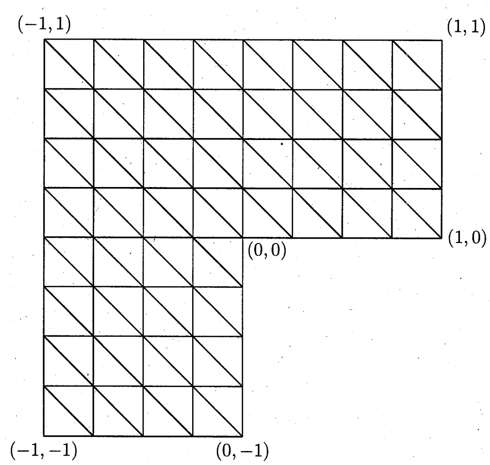


Figure 5.2: L-shaped domain

Subdivide the domain by right triangles as in Figure 5.2 and then solve the given problem by utilizing the conforming finite element space V^h_{conf} and the nonconforming one V^h_{nc}. Let $u^N_h \in V^h_{nc}$ be the one that

$$(\nabla_h u^N_h, \nabla_h v_h) = (Q_h f, v_h), \quad \forall v_h \in V^h_{nc}(\Omega)\,, \tag{5.3.1}$$

and $u_h^C \in V_{conf}^h$ be the one for

$$(\nabla u_h^C, \nabla v_h) = (f, v_h); \quad \forall v_h \in V_{conf}^h(\Omega)\,. \tag{5.3.2}$$

By post-processing u_h^N as we deal with u_h^* in (5.2.12), we can also obtain $\hat{u}_h^N$ and $p_h = \nabla_h \hat{u}_h^N$. Then we apply the estimate in (5.2.13) to give an estimate for both $\|\nabla u - \nabla u_h^C\|$ and $\|\nabla u - \nabla_h \hat{u}_h^N\|$, where the constant C_0^h can be taken as $C_0^h = C_0 = 1/\pi$. We summarize the computational results in Table 5.1.

h	$\|\nabla u - \nabla u_h^C\|$	$\|\nabla u_h^C - \nabla_h \hat{u}_h^N\|$	+	$C_0 h\|f - Q_h f\|$	=	total estimate
1/2	0.4315	0.4476	+	0.2576	=	0.7052
1/4	0.2778	0.3719	+	0.0865	=	0.4584
1/8	0.1661	0.2064	+	0.0291	=	0.2355
1/16	0.0985	0.1280	+	0.0098	=	0.1378
1/32	0.0587	0.0786	+	0.0033	=	0.0819

Table 5.1: A posteriori estimates in the case of L-shaped domain

Chapter 6

Overview and future work

6.1 Summary of present research

For the well known linear conforming and nonconforming triangular FEMs, we have studied the corresponding interpolation errors to give quantitative a priori and a posteriori error estimates for the FEM solutions.

In this process, we have given systematic analysis for the error constants that appear in the interpolation error estimation. For each constant, we have studied the dependency of the constants on geometric parameters of the element and tried to determine the concrete values or give suitable upper bounds in special cases. Thus the quantitative but rough interpolation error estimation becomes available for arbitrary element. Here we summarize the results below.

On triangle $T_{\alpha,\theta,h}$, for $u \in H^2(T_{\alpha,\theta,h})$, $v \in H^1(T_{\alpha,\theta,h})$ and $q \in H(\mathrm{div};T_{\alpha,\theta,h})$,

$$
\begin{array}{lrl}
\Pi^0_{\alpha,\theta,h}: & \|v-\Pi^0_{\alpha,\theta,h}v\| & \le 1/\pi\ \phi_0(\theta)h\ |u|_1, \\
\Pi^1_{\alpha,\theta,h}: & |u-\Pi^1_{\alpha,\theta,h}u|_1 & \le 1/2\ \phi_4(\theta)h\ |u|_2, \\
 & \|u-\Pi^1_{\alpha,\theta,h}u\| & \le 0.36\ \phi_5(\theta)h^2\ |u|_2\ , \\
\Pi^{1,n}_{\alpha,\theta,h}: & \|u-\Pi^{1,n}_{\alpha,\theta,h}u\| & \le\ 1/(4\pi)\phi_0(\theta)\ h^2\ |u|_2, \\
 & \|\nabla u-\nabla_h\Pi^{1,n}_{\alpha,\theta,h}u\| & \le 1/\pi\phi_0(\theta)\ h\ |u|_2, \\
\Pi^F_{\alpha,\theta,h}: & \|q-\Pi^F_{\alpha,\theta}q\| & \le C_{F,1}(\alpha,\theta)h\|\mathrm{div}\, q\|+C_{F,2}(\alpha,\theta)h|q|_1\ ,
\end{array}
$$

where $\phi_i(\theta)$'s are defined in (3.2.15) and $C_{F,i}$'s defined in (3.3.7) and (3.3.9).

Also, the analysis of the dependency of the constants on geometric parameters ensures the uniform boundedness of $C_i(\alpha, \theta)$'s ($i = 0, 1, 2, 3, 5, \{4, n\}, \{5, n\}$) on arbitrary element with fixed medium edge length. On the contrary, the constants $C_4(\alpha, \theta)$ and $C_{F,2}(\alpha, \theta)$ will tend to ∞ when the maximum angle tends to π. Therefore, when we use either conforming FEM or nonconforming one, we should follow the "maximum angle condition" in the triangulations of the domain, that is, the maximum interior angle of the triangle should be bounded above from π, while the smallest angle can be close to zero.

To evaluate the constants on arbitrary triangular element, we also developed an a posteriori estimation method to give computable lower and upper bounds for the constants. Such method is based on the theories of the eigenvalue problem for Laplacian. Not limited to triangular domains, the method we developed can also be used to estimate the minimum positive eigenvalue of Laplacian on more general domains. One example of convex polygonal domain has been executed to show the validity of the method.

Combining the explicit error estimates for interpolation and the analysis for conforming FEM and nonconforming one in Section 2.1.2 and 3.1, we can obtain computable a priori error estimates, which are summarized in (2.1.8) and (2.1.9) in the conforming case, and (3.1.15) and (3.1.17) for the nonconforming one. Compared with the earlier results of [4, 45] for the conforming FEM, our error bounds is much shaper since better estimates of the constants are adopted. The a posteriori error estimate based on the hypercircle method is also developed for the Poisson equation, which uses both conforming FEM solution and the nonconforming one.

6.2 Future research

The research on evaluating error constants and enclosing eigenvalues is very interesting and challenging work. Compared with the results obtained in this dissertation, there are much more left to do in the future. Here, let us list up some possible and meaningful researches in the future.

In the following subsections, we will give a sketch of three topics, that is,

1. evaluation of the second and also n-th eigenvalues of Laplacian,

2. use of conforming space of vector functions for evaluating eigenvalues of bi-harmonic operator,

3. study of error constants in anisotropic element.

6.2.1 Enclosing the second eigenvalue of Laplacian

In the previous chapters, we have developed methods to give quantitative estimates for error constants, which correspond to the first positive eigenvalue of linear operators. Here, we will consider the problem of estimating the second eigenvalue of Laplace operator. This work is still under progress and is not completed yet.

Preliminary

Unlike the notation in previous chapters, we here define by T a general triangle. The edges of T are denoted by e_1, e_2 and e_3. Let us introduce a linear space $V^1(T)$ or V^1:

$$V^1(T) := \{v \in H^1(T) | \int_{e_1} v \, ds = 0\} .$$

and the constant C be defined in term of the Rayleigh quotient:

$$C^{-2} := \lambda = \inf_{u \in V^1 \setminus \{0\}} R(u), \text{ where } R(u) := \frac{|u|^2_{H^1(T)}}{\|u\|^2_{L^2(T)}} \text{ for } u \in H^1(T) \setminus \{0\} .$$

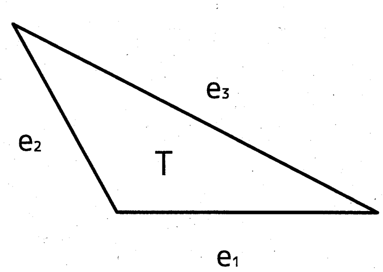


Figure 6.1: Triangle T

Recall that when T is a unit isosceles right triangle and e_1 is one of the edges of right interior angle, the constant above reduces to the Babuška-Aziz constant.

We can also characterize λ in the variational form: *Find the eigenpair* $(\lambda, u) \in \mathbb{R} \times (V^1(T) \setminus \{0\})$ *with* λ *being the smallest positive eigenvalue such that*

$$(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in V^1(T). \tag{6.2.1}$$

In this section, we will try to consider the second eigenpair (λ_2, u_2) of the eigenvalue problem (6.2.1). To distinguish the pairs from each other, we write the first eigenpair by (λ_1, u_1). Moreover, the eigenfunction u_i's $(i = 1, 2)$ are assumed to be normalized and orthogonal to each other in $L_2(T)$, that is,

$$\|u_1\|_{L_2(T)} = \|u_2\|_{L_2(T)} = 1, \quad (u_1, u_2) = 0.$$

Approximation in conforming finite element space

Let S^h be the conforming piecewise linear finite element space over a triangulation $\mathcal{T}^h$ of T, and $V^{1,h} := S^h \cap V^1$. We consider the eigenvalue problem in $V^{1,h}$: *Find* $(\lambda_h, u_h) \in \mathbb{R} \times \left(V^{1,h} \setminus \{0\}\right)$ *such that*

$$(\nabla u_h, \nabla v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V^{1,h}. \tag{6.2.2}$$

The i-th eigenpair is denoted by $(\lambda_{i,h}, u_{i,h})$. By the minimum-maximum principle to be mentioned below, it is easy to see $\lambda_i \leq \lambda_{i,h}$ $(i = 1, 2)$. The estimate for $|\lambda_1 - \lambda_{1,h}|$ was given in Section 4.3 (or cf. [34]).

To approximate the functions u_1 and u_2, we introduce two associated functions $\overline{u}_{1,h}$ and $\overline{u}_{2,h}$, which are characterized by the following variational equations: for $\overline{u}_{1,h} \in V^{1,h}$,

$$(\nabla \overline{u}_{1,h}, \nabla v_h) = \lambda_1(u_1, v_h), \quad \forall v_h \in V^{1,h}. \tag{6.2.3}$$

and for $\overline{u}_{2,h} \in V^{1,h}$,

$$(\nabla \overline{u}_{2,h}, \nabla v_h) = \lambda_2(u_2, v_h), \quad \forall v_h \in V^{1,h}. \tag{6.2.4}$$

By arguments analogous to those in Section 4.3, we find $|u_i|_{2,T} \leq M\|\Delta u_i\|_{L_2(T)}$ $(i = 1, 2)$. Further noticing the fact that $-\Delta u_i = \lambda_i u$, we have

$$|u_i|_{H^2(T)} \leq M\|\Delta u_i\|_{L_2(T)} \leq M\lambda_i \quad (i = 1, 2). \tag{6.2.5}$$

Considering the finite element error estimates for the approximation $\overline{u}_{1,h}$ and $\overline{u}_{2,h}$ as in (2.1.8) and (2.1.9), we have for each $i = 1, 2$ that

$$\|\nabla(u_i - \overline{u}_{i,h})\| \le \kappa h |u_i|_{2,T} \le \kappa h M \lambda_i, \quad \|u_i - \overline{u}_{i,h}\| \le \kappa^2 h^2 |u_i|_{2,T} \le \kappa^2 h^2 M \lambda_i, \quad (6.2.6)$$

where $M = 2 + \sqrt{2}/2$ is the same one as in Theorem 4.3.1, and κ and h are defined by

$$\kappa := \max_{K \in T^h} C_4(\alpha_K, \theta_K), \quad h := \max_{K \in T^h} h_K .$$

Here α_K, θ_K and h_K are parameters related to element K (cf. Section 2.2). Notice that κ can be given concrete upper bounds due to the estimates in Chapter 2.

Minimum-maximum principle

By the minimum-maximum principle [50], the n-th eigenvalue of the problem (6.2.1) and (6.2.2) are characterized respectively by

$$\lambda_n = \min_{\mathcal{B}_n} \max_{u \in \mathcal{B}_n} R(u), \quad \lambda_{n,h} = \min_{\mathcal{B}_{n,h}} \max_{u_h \in \mathcal{B}_{n,h}} R(u_h) , \qquad (6.2.7)$$

where $\mathcal{B}_n$ and $\mathcal{B}_{n,h}$ present any n-dimensional subspaces of V^1 and $V^{1,h}$ respectively, i.e., $\dim|\mathcal{B}_n| = \dim|\mathcal{B}_{n,h}| = n \in \mathbb{N}$.

From the assumption that $(u_1, u_2)_T = 0$ and the error estimates in (6.2.6), we can show that $\overline{u}_1$ and $\overline{u}_2$ are linearly independent if

$$\kappa^2 h^2 M \lambda_i < 1/2 \quad (i = 1, 2) . \qquad (6.2.8)$$

Notice that the above condition can be numerically verified by considering the relation that $\lambda_i \le \lambda_{i,h}$ for $i = 1, 2$, where $\lambda_{i,h}$'s are computable. Therefore, with the condition (6.2.8) satisfied, we can construct one 2-dimensional space $\overline{\mathcal{B}}_{2,h} \subset V^{1,h}$ by

$$\overline{\mathcal{B}}_{2,h} = \text{span}\{\overline{u}_{1,h}, \overline{u}_{2,h}\} . \qquad (6.2.9)$$

Let us introduce a new quantity $\overline{\lambda}_h := \max_{u_h \in \overline{\mathcal{B}}_{2,h}} R(u_h)$. Then we have

$$\lambda_1 \le \lambda_{1,h} \le R(\overline{u}_{1,h}), \quad \lambda_2 \le \lambda_{2,h} \le \overline{\lambda}_h . \qquad (6.2.10)$$

Estimate of second eigenvalue

As is well known, the value $\overline{\lambda}_h$ can be characterized by the maximum eigenvalue of the following eigenvalue problem:

$$\begin{pmatrix} (\nabla\overline{u}_{1,h},\nabla\overline{u}_{1,h}) & (\nabla\overline{u}_{1,h},\nabla\overline{u}_{2,h}) \\ (\nabla\overline{u}_{2,h},\nabla\overline{u}_{1,h}) & (\nabla\overline{u}_{2,h},\nabla\overline{u}_{2,h}) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \overline{\lambda}_h \begin{pmatrix} (\overline{u}_{1,h},\overline{u}_{1,h}) & (\overline{u}_{1,h},\overline{u}_{2,h}) \\ (\overline{u}_{2,h},\overline{u}_{1,h}) & (\overline{u}_{2,h},\overline{u}_{2,h}) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{6.2.11}$$

where $(c_1, c_2)^T$ is the eigenvector corresponding to $\overline{\lambda}_h$. The matrix equation above is in fact an approximation of the one below:

$$\begin{pmatrix} (\nabla u_1,\nabla u_1) & 0 \\ 0 & (\nabla u_2,\nabla u_2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} (u_1,u_1) & 0 \\ 0 & (u_2,u_2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{6.2.12}$$

As each component of matrices in (6.2.12) can be well approximated by the corresponding one in (6.2.11), the eigenvalue $\overline{\lambda}_h$ is expected to be close to λ_2.

By considering Eq.(6.2.3) and Eq.(6.2.4), we can transform (6.2.11) into

$$\begin{pmatrix} (\lambda_1 u_1 - \overline{\lambda}_h\overline{u}_{1,h},\overline{u}_{1,h}) & (\lambda_1 u_1 - \overline{\lambda}_h\overline{u}_{1,h},\overline{u}_{2,h}) \\ (\lambda_2 u_2 - \overline{\lambda}_h\overline{u}_{2,h},\overline{u}_{1,h}) & (\lambda_2 u_2 - \overline{\lambda}_h\overline{u}_{2,h},\overline{u}_{2,h}) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0\,. \tag{6.2.13}$$

Hence, we obtain a determination equation for $\overline{\lambda}_h$ as follows:

$$a\ \overline{\lambda}_h^2 + b\ \overline{\lambda}_h + c = 0\,, \tag{6.2.14}$$

where the coefficients $\{a, b, c\}$ are

$$\left\{\begin{array}{l} a = \|\overline{u}_{1,h}\|^2\cdot\|\overline{u}_{2,h}\|^2 - (\overline{u}_{1,h},\overline{u}_{2,h})^2\,, \\ b = -\lambda_2(u_2,\overline{u}_{2,h})\cdot\|\overline{u}_{1,h}\|^2 - \lambda_1(u_1,\overline{u}_{1,h})\cdot\|\overline{u}_{2,h}\|^2 + (\lambda_1(u_1,\overline{u}_{2,h}) + \lambda_2(u_2,\overline{u}_{1,h}))\cdot(\overline{u}_{1,h},\overline{u}_{2,h})\,, \\ c = \lambda_1\lambda_2(u_1,\overline{u}_{1,h})\cdot(u_2,\overline{u}_{2,h}) - \lambda_1\lambda_2(u_1,\overline{u}_{2,h})\cdot(u_2,\overline{u}_{1,h})\,. \end{array}\right.$$

Before giving bounds for $\{a, b, c\}$, we summarize the estimates for the terms appearing above: for $i, j = 1, 2, (i \neq j)$

$$\left\{\begin{array}{l} 1-\kappa^2h^2M\lambda_i \le \|\overline{u}_{i,h}\| \le 1+\kappa^2h^2M\lambda_i\,, \\ 1-\kappa^2h^2M\lambda_i \le (u_i,\overline{u}_{i,h}) \le 1+\kappa^2h^2M\lambda_i\,, \\ |(u_i,\overline{u}_{j,h})| \le \kappa^2h^2M\lambda_j\,, \\ |(\overline{u}_{i,h},\overline{u}_{j,h})| \le |(u_{i,h}-u_i,u_{j,h})| + |(u_i,u_{j,h})| \le (\kappa^2h^2M\lambda_i)(1+\kappa^2h^2M\lambda_j) + \kappa^2h^2M\lambda_j\,. \end{array}\right. \tag{6.2.15}$$

Hence, we can give estimates for a, b and c as

$$\left\{\begin{array}{l} a \in [1-\epsilon_1, 1+\epsilon_1]\,, \\ b \in [-(\lambda_1+\lambda_2)-\epsilon_2, -(\lambda_1+\lambda_2)+\epsilon_2]\,, \\ c \in [\lambda_1\lambda_2-\epsilon_3\,, \lambda_1\lambda_2+\epsilon_3]\,, \end{array}\right.$$

where each ϵ_i ($i = 1, 2, 3$), depending on variables λ_1 and λ_2, is constructed to be monotonically increasing with respect to each variable if we fix the other one. Clearly, $\overline{\lambda}_h$ is the maximum solution of Eq.(6.2.14), that is,

$$\overline{\lambda}_h = \frac{-b + \sqrt{b^2 - 4ac}}{2a} . \tag{6.2.16}$$

By using the fact that $\lambda_1 \leq \lambda_2$ and the known estimate for λ_1, we can obtain an upper bound for $\overline{\lambda}_h$ as

$$(\lambda_{2,h} \leq)\, \overline{\lambda}_h \leq \phi(M, \kappa, \lambda_1, h; \lambda_2) , \tag{6.2.17}$$

where $\phi(M, \kappa, \lambda_1, h; \lambda_2)$, to be denoted by notation $\phi(\lambda_2)$ for simplicity, is selected to be a monotonically increasing function with respect to λ_2 and $\phi(\lambda_2) \to \lambda_2$ as $h \to 0$. The inequality above is in fact an a priori estimate of $\overline{\lambda}_h$.

Combining the estimate in (6.2.10) and (6.2.17), we have one computable a posteriori estimate for λ_2:

$$\phi^{-1}(\lambda_{2,h}) \leq \lambda_2 \leq \lambda_{2,h} , \tag{6.2.18}$$

provided that the estimate for λ_1 is known. At present, we have not fully succeeded in evaluating the error of λ_1 and its influence to ϕ.

Remark 6.2.1. *The procedure above aims to give an upper bound for $\overline{\lambda}_h$ by λ_1 and λ_2, where we rely on the quadratic formula (6.2.16) to give an explicit form of $\overline{\lambda}_h$. However, to evaluate the n-th eigenvalue, the eigenvalues of (6.2.11) are the zeros of polynomial of degree n, which does not have any explicit formula for $n \geq 5$. To solve this problem, we are considering other methods by adopting the interval computations.*

Remark 6.2.2. *If the computation shows that $\lambda_{1,h} < \lambda_{2,h}$ and $|\lambda_{2,h} - \lambda_2| < |\lambda_{2,h} - \lambda_{1,h}|$, then we have $\lambda_1 \leq \lambda_{1,h} < \lambda_2$. Thus λ_2 is separated from λ_1, which means the multiplicity of the first eigenvalue λ_1 to be 1. Also, by adopting the minimum-maximum principle, we may hope to develop a posteriori estimation method to evaluate the n-th eigenvalue.*

6.2.2 Space of conforming vector functions for estimating eigenvalues of biharmonic operator

For the constants such as $C_4(\alpha, \theta)$ and $C_{\{4,n\}}(\alpha, \theta)$, the corresponding weak forms

Once the theoretical analysis for the conforming vector function spaces is done, it may be possible to design a posteriori estimation method to deal with the eigenvalue problems of biharmonic operator, e.g., those for $C_4(\alpha, \theta)$.

6.2.3 Error constants for anisotropic element

An anisotropic element is a finite element that can be very slender in one direction, and the maximum angle of the triangular element may be close to π. Such an element is often required in the analysis of the convection-dominated equations, which appears in the problems of heat transport in water flow, carrier transport in semiconductors and so on. In view of singular perturbations, these problems accompany special boundary layers where the solution varies much faster in the normal direction than in the tangential direction. Therefore, the anisotropic mesh optimization with the adaptive finite element method is indispensable. We hope to give sharper a posteriori error estimation and improve mesh optimization, by which computation time can be greatly saved.

As we mentioned in Remark 2.2.1, we may give error estimation of the form

$$|v - \Pi^1_{\alpha,\theta,h} v|_{1,T_{\alpha,\theta,h}} \leq h \left(\sum_{i,j=1}^{2} c_{ij} \|\partial_{ij} v\|^2_{T_{\alpha,\theta,h}} \right)^{1/2} \quad \text{for } v \in H^2(T_{\alpha,\theta}) \,, \tag{6.2.22}$$

where c_{ij}'s are to be suitably chosen to give sharp estimates for given function v.

A conventional way in error analysis is first to consider the interpolation error constants on a reference element and then consider an appropriate transformation between an arbitrary element and the reference one. Therefore, we can find proper choice of c_{ij}'s and decide the element direction according to the a priori estimate for the given function v [20]. But the optimal estimate cannot be obtained in such a way. To sharpen the estimate, we need to directly consider the interpolation error estimate on slender triangle. Moreover, the interpolation function considered there usually has a priori information, which will lead to the nonlinear constraints for the corresponding minimization problems. The processing of these nonlinear constraints may still remain as a challenge.

have 2nd order derivatives, which make the problems very difficult to solve. As for this, in section 2.4.2, we have introduced a new constant $C_{\{4,e123\}}(\alpha,\theta)$, which has the Rayleigh quotient defined over space of vector fileds. By using the finite dimensional space for conforming vector functions, we can get approximate values of $C_{\{4,e123\}}(\alpha,\theta)$ (Figure 2.7 and Figure 2.8), which seem to give nice upper bounds for $C_4(\alpha,\theta)$. As an alternative to direct estimation of $C_4(\alpha,\theta)$, it may be meaningfurl to evaluate the constant $C_{\{4,e12\}}$ by applying the conforming finite element methods. Since there are only derivatives up to the first order in the corresponding PDE, it is desirable to give a posteriori estimates like those for $C_i(\alpha,\theta)$ $(i=0,1,2,3)$.

The conforming vector function space suggested above is defined on triangle $T_{\alpha,\theta}$ by

$$M^h := \{w=(u,v)\in S^h(T_{\alpha,\theta})^2 \mid \int_{e_i} w\cdot t_i ds = 0, i=1,2,3\}, \tag{6.2.19}$$

where each t_i is the normalized vector in the direction of edge e_i and $S^h(T_{\alpha,\theta})$ is the conforming finite element space composed of the piecewise linear functions. Noticing that $S^h(T_{\alpha,\theta})\subset H^1(T_{\alpha,\theta})$, it is easy to see that M^h is a subspace of the following one:

$$M := \{w=(u,v)\in H^1(T_{\alpha,\theta})^2 \mid \int_{e_i} w\cdot t_i ds = 0, i=1,2,3\}. \tag{6.2.20}$$

Such finite element space is effective to compute approximate values of $C_{\{4,e123\}}$.

However, the space M^h is not included to

$$M_{\text{grad}} := \{(\partial_1 u, \partial_2 u) \mid u \in V^4(T_{\alpha,\theta})\},$$

where the curl-free condition $\partial_{12}u - \partial_{21}u = 0$ is required. So we hope to design a new conforming FE space such that

$$M^h_{\text{grad}} := \{w=(u_h,v_h)\in S^h(T_{\alpha,\theta})^2 \mid \int_{e_i} w\cdot t_i ds = 0 \ \ (i=1,2,3) \text{ and } \partial_2 u_h = \partial_1 v_h \text{ in } T_{\alpha,\theta}.\}. \tag{6.2.21}$$

The approximation capability of the functions in M^h_{grad} may be doubtful, since the curl-free constraint requires the same number of algebraic relations as that of finite elements. But our analysis shows that the vector $(\partial_1 u, \partial_2 u)$ seems to be well approximated by functions in M^h_{grad} if u is smooth enough. Numerical computations in several cases also demonstrate the validity of such conforming vector function spaces. However, there are still much efforts needed for systematic analysis.

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