Generalized Whittaker functions for degenerate principal series of $GL(4,\mathbb{R})$.

($GL(4,\mathbb{R})$ の退化主系列表現の一般 Whittaker 関数.)

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Generalized Whittaker functions for degenerate principal series of $GL(4, \mathbb{R})$

Kazuki Hiroe*

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.

Abstract

We give a characterization of a generalized Whittaker model of a degenerate principal series representation of $GL(n, \mathbb{R})$ as the kernel of some differential operators. By this characterization, we investigate some examples on $GL(4, \mathbb{R})$. We obtain the dimensions of the generalized Whittaker models and give their basis in terms of hypergeometric functions of one and two variables. We show the multiplicity one of the generalized Whittaker models by using the theory of hypergeometric functions.

1 Introduction

Our interest in this paper is generalized Whittaker models of degenerate principal representations. There are many studies about them for admissible (non-degenerate) characters of unipotent radicals of parabolic subgroups (for example [6],[16],[30],[31],[32]). In the case of degenerate principal series representations, Yamashita gives existence theorem and multiplicity formula for wide classes of generalized Whittaker models, i.e., generalized Whittaker models for generalized Gelfand-Graev representations in [32]. However, their techniques strongly depend on the admissibility of the characters of the unipotent subgroups. On the other hand, if we regard the Whittaker models as an analogue of Fourier coefficients of an automoprhic form at a cusp, we often meet the necessity to consider non-admissible characters. For example, Terras gives an expansion of the Epstein zeta function in terms of modified Bessel functions [27]. Non-admissible characters play important roles there. The Epstein zeta function corresponds to the degenerate principal series representation of $GL(n, \mathbb{R})$ induced from the character of the maximal parabolic subgroup $P_{1,n}$ which fixes the unit vector $e_n = (0, \ldots, 0, 1)$ (cf. [17]). Hence the Fourier coefficients given by Terras can be seen as the generalized Whittaker functions for this representation. It seems, however, widely open about the problem of the existence and the multiplicity formula of the generalized Whittaker models for non-admissible characters of unipotent subgroups. In this paper, we give some examples about this problems in the case of $GL(4, \mathbb{R})$. As related studies, we should mention about recent

*E-mail:kazuki@ms.u-tokyo.ac.jp

works of Abe and Oshima obtained independently [1],[21] which give the solutions of such problems in the case of degenerate characters of maximal unipotent subgroup.

The other purpose of this paper is to give an expression of the generalized Whittaker function as a hypergeometric function of several variables. According to the recent work of Oshima and Shimeno [23], Whittaker functions can be seen as the confluent hypergeometric functions obtained from Heckman-Opdam hypergeometric functions. The similarities of Whittaker functions with spherical functions were already pointed out by Hashizume in [7]. Also there are various explicit pictures of Whittaker functions as hypergeometric functions of several variables given by Hirano, Ishii ,Oda and many other researchers (see [10] for the reference). We show that the generalized Whittaker functions of the degenerate principal series representations of $GL(4,\mathbb{R})$ are written by modified Bessel functions and Horn's hypergeometric function H_{10} in this paper. There is a similar work on $SL(3,\mathbb{R})$ in [12].

Let us explain the contents of this paper. In Section 2 and Section 3, we give a characterization of a Whittaker model of a degenerate principal series representation of $GL(n,\mathbb{R})$ as the kernel of a family of differential operators. More precisely, let $G=GL(n,\mathbb{R})$ and consider an Iwasawa decomposition $G=KAN$ where $K=O(n)$, A is the group of diagonal matrices with positive real entries, and N is the group of lower triangular matrices with 1s on diagonal entries. We take an increasing sequence of positive integers stopped at n, i.e., $\Theta=\{n_1,\dots,n_L\}$ with $0<n_1<n_2<\cdots<n_L=n$. Then let P_Θ be the parabolic subgroup corresponding to the sequence Θ and take the Langlands decomposition $P_\Theta=M_\Theta A_\Theta N_\Theta$. For a linear mapping $\lambda\in\mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(A_\Theta),\mathbb{C})$, we can consider an induced representation $C^\infty\text{-}\mathrm{Ind}_{P_\Theta}^{G}(1_{M_\Theta}\otimes e^\lambda\otimes 1_{N_\Theta})$. We call this representation a degenerate principal series representation. The underlying representation space of this is

$$C^\infty(G/P_\Theta,\lambda)=\\ \{f\in C^\infty(G)\mid f(gp)=(1_{M_\Theta}\otimes e^\lambda\otimes 1_{N_\Theta})(p^{-1})f(g),\ g\in G,p\in P_\Theta\}$$

and the action of G is defined by the left translation. Then we consider an ideal of $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{gl}(n,\mathbb{C})$ such that $I_\Theta(\lambda)=\{X\in U(\mathfrak{g})\mid R_Xf=0,\ f\in C^\infty(G/P_\Theta,\lambda)\}$. Here R_X is the right derivation by $X\in U(\mathfrak{g})$. We consider λ as an element of $\mathfrak{a}_{\mathbb{C}}^*=\mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(A),\mathbb{C})$ and we assume it is regular and dominant. Under this assumption, the generators of the ideal $I_\Theta(\lambda)$ is known by Oshima (cf. Theorem 2.7). Let U be a closed subgroup N and (η,V_η) an irreducible unitary representation of U. We consider the space $C_\eta^\infty(U\backslash G)=\{f\colon G\to V_\eta^\infty \text{ smooth}\mid f(ug)=\eta(u)f(g),u\in U,g\in G\}$ where V_η^∞ is the space of smooth vectors in V_η. Let $X_{\Theta,\lambda}$ be the Harish-Chandra module of $C^\infty(G/P_\Theta,\lambda)$ and $X_{\Theta,\lambda}^*$ its dual Harish-Chandra module, i.e., the space of K-finite vectors of $\mathrm{Hom}_{\mathbb{C}}(X_{\Theta,\lambda},\mathbb{C})$. The generalized Whittaker model is the image of $X_{\Theta,\lambda}$ by the element of $\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda},C_\eta^\infty(U\backslash G))$. Then we can show the following characterization theorem of the generalized Whittaker model.

Theorem 1.1 (see Theorem 3.7). *Suppose that $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ is regular and dominant.*

We take a nonzero K-fixed vector f_0 in $X^*_{\Theta,\lambda}$. Then the following mapping

$$\begin{array}{cccc} \tilde{\Phi}\colon & \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X^*_{\Theta,\lambda}, C^\infty_\eta(U\backslash G)) & \xrightarrow{\sim} & C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda)) \\ & W & \longmapsto & W(f_0)(g) \end{array}$$

is a linear isomorphism. Here

$$\begin{aligned} &C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda)) \\ &\quad = \{f\colon G \to V^\infty_\eta \text{ smooth} \mid f(ngk) = \eta(n)f(g), g \in G, n \in U,\ k \in K \\ &\qquad\qquad \text{and } R_X f(g) = 0,\ X \in I_\Theta(\lambda)\}. \end{aligned}$$

This theorem is an analogue of the theorem for the generalized Whittaker models of unitary highest weight modules obtained by Yamashita [33], [34].

From Section 4, we consider examples on degenerate principal series representations of $GL(4,\mathbb{R})$ induced from characters of maximal parabolic subgroups $P_{1,4}$ and $P_{2,4}$ by using above theorem. We determine the dimension of $\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X^*_{\Theta,\lambda}, C^\infty_\eta(U\backslash G))$ and the basis of $C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda))$. Let us explain more detailed settings. As the space $C^\infty_\eta(U\backslash G)$, we consider the space defined as follows.

1. the group U is a closed subgroup of N and η is its unitary character,
2. the unitary induced representation $L^2\text{-}\mathrm{Ind}_U^N \eta$ is an irreducible unitary representation of N.

We classify the G-equivalent classes of these $C^\infty_\eta(U\backslash G)$ in Section 4.1 (see Proposition 4.10).

There is a linear isomorphism from the space $C^\infty_\eta(U\backslash G/K)$ onto $C^\infty(U\backslash N \times A)$ (cf. Lemma 4.11). In Section 4.2, we see how the the action of the Lie algebra $\mathfrak{g}$ is written as differential operators on $C^\infty(U\backslash N \times A)$.

Our main results are in Section 4.3. In this section, we give the dimensions of $C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda))$ and the basis of them as the functions on $C^\infty(U\backslash N \times A)$. These basis can be written in terms of modified Bessel functions and Horn's hypergeometric functions H_{10} (see Theorem 4.22, Theorem 4.25, Theorem 4.27, Theorem 4.28, Theorem 4.29, Theorem 4.31). According to these theorems, we can conclude the following. For the degenerate principal series representation induced from a character of $P_{1,4}$, the multiplicity one theorem is true for the generalized Whittaker models for characters of the closed proper subgroups of N. On the other hand, for the degenerate principal series representation induced from a character of $P_{2,4}$, the multiplicity one theorem is no longer true. This fact seems to correspond to the result of Terras in [28]. In that paper, she could determine only the nonsingular terms in the Fourier expansion of Eisenstein series corresponding to this degenerate principal series representation (Theorem 1 in [28]). And she could not say anything about degenerate terms in Fourier expansion. The multiplicities of the generalized Whittaker models corresponding to these Fourier coefficients seems to be one of the cause of this phenomenon.

Finally, we give some facts about Horn's hypergeometric functions in Appendix.

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2 Spherical degenerate principal series representations of $GL(n,\mathbb{R})$

In this section, we study degenerate principal series representations of $GL(n,\mathbb{R})$ and their annihilators in the enveloping algebra $U(\mathfrak{gl}(n,\mathbb{C}))$. T.Oshima shows that the image of a degenerate principal series representation by the Poisson transform is characterized by the kernel of the annihilator of the degenerate principal series representation [19]. He also give the explicit generators for its annihilator [20], [22]. We give a brief review of these results here.

2.1 Spherical degenerate principal series representations of $GL(n,\mathbb{R})$.

Let $G = GL(n,\mathbb{R})$, its Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$. We take the Iwasawa decomposition of G as $G = KAN$, where $K = O(n)$, A is the group of $n\times n$ diagonal matrices with positive real entries and N is the group of lower triangular matrices with 1s on the diagonal entries. Let E_{ij} be the matrix with 1 in the (i,j)-entry and 0 elsewhere. We introduce a non-degenerate bilinear form on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$ by

$$\langle X, Y\rangle = \mathrm{tr}(XY) \text{ for } X, Y \in \mathfrak{g}_{\mathbb{C}}.$$

By this, we identify $\mathfrak{g}_{\mathbb{C}}$ with its dual space $\mathfrak{g}_{\mathbb{C}}^*$. The dual basis $\{E_{ij}^*\}$ of $\{E_{ij}\}$ is given by $E_{ij}^* = E_{ji}$. For simplicity, we write $e_i = E_{ii}^*$.

We consider the Lie algebra

$$\mathfrak{a} = \{\sum_{i=1}^{n} a_i E_{ii} \mid a_i \in \mathbb{R},\ i = 1,\dots,n\},$$

of A, and the root system of $(\mathfrak{g},\mathfrak{a})$ is

$$\triangle(\mathfrak{g},\mathfrak{a}) = \{e_i - e_j \mid 1 \le i \ne j \le n\}.$$

We put $\alpha_i = e_{i+1} - e_i$ for $i = 1,\dots,n-1$ and fix a simple system of $\triangle(\mathfrak{g},\mathfrak{a})$ as

$$\Pi(\mathfrak{g},\mathfrak{a}) = \{\alpha_1,\dots,\alpha_{n-1}\}.$$

Then the positive system of $\triangle(\mathfrak{g},\mathfrak{a})$ associated to $\Pi(\mathfrak{g},\mathfrak{a})$ is $\triangle^+(\mathfrak{g},\mathfrak{a}) = \{e_i - e_j \mid 1 \le j < i \le n\}$. The Lie algebra $\mathfrak{n}$ of N is written by

$$\mathfrak{n} = \sum_{\alpha\in\triangle^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha = \sum_{i>j} \mathbb{R}E_{ij}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \alpha(H)X \text{ for } H \in \mathfrak{a}\}$. On the other hand, let $\overline{N}$ be the group of upper triangular matrices with 1s on the diagonal entries. Then the Lie algebra $\overline{\mathfrak{n}}$ of $\overline{N}$ is also written by

$$\begin{aligned}\overline{\mathfrak{n}} &= \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_{-\alpha} \\ &= \sum_{i<j} \mathbb{R}E_{ij}.\end{aligned}$$

Let $\Theta = \{n_1, \dots, n_L\}$ be a sequence of strictly increasing positive integers stopped at n, i.e., $(0 = n_0 <) n_1 < n_2 < \cdots < n_L (= n)$. For this Θ, the associated standard parabolic subgroup P_Θ can be defined as follows. Let

$$\mathfrak{a}_\Theta = \{\sum_{k=1}^{L} a_k \sum_{i=n_{k-1}+1}^{n_k} E_{ii} \mid a_k \in \mathbb{R}, k = 1, \dots, L\}.$$

Let L_Θ be the centralizer of $\mathfrak{a}_\Theta$ in G, i.e.,

$$L_\Theta = \left\{ l = \begin{pmatrix} l_1 & & & \\ & l_2 & & \\ & & \ddots & \\ & & & l_L \end{pmatrix} \mid l_i \in GL(n_i - n_{i-1}, \mathbb{R}) \right\}$$

and $\mathfrak{l}_\Theta$ its Lie algebra which is the centralizer of $\mathfrak{a}_\Theta$ in $\mathfrak{g}$. We put

$$\mathfrak{n}_\Theta = \sum_{\iota_\Theta(i) > \iota_\Theta(j)} \mathbb{R}E_{ij}$$

where

$$\iota_\Theta(\nu) = i \text{ if } n_{i-1} < \nu \leq n_i \text{ for } i = 1, \dots, L. \tag{2.1}$$

The corresponding analytic subgroup of G is $N_\Theta = \exp \mathfrak{n}_\Theta$, i.e.,

$$N_\Theta = \left\{ n = \begin{pmatrix} I_{n'_1} & & & & \\ N_{21} & I_{n'_2} & & & \\ N_{31} & N_{32} & I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ N_{L1} & N_{L2} & N_{L3} & \cdots & I_{n'_L} \end{pmatrix} \mid N_{ij} \in M(n'_i, n'_j; \mathbb{R}), n'_i = n_i - n_{i-1} \right\}.$$

Here I_m denotes the identity matrix of size m and $M(k, l; \mathbb{R})$ denotes the space of matrices of size $k \times l$ with components in $\mathbb{R}$. We also define $\overline{\mathfrak{n}}_\Theta = \sum_{\iota_\Theta(i) < \iota_\Theta(j)} \mathbb{R}E_{ij}$ and $\overline{N}_\Theta = \exp \overline{\mathfrak{n}}_\Theta$ as well.

Then we define the parabolic subgroup $P_\Theta = L_\Theta N_\Theta$, i.e.,

$$P_\Theta = \left\{ p = \begin{pmatrix} g_1 & & & \\ * & g_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & g_L \end{pmatrix} \in GL(n, \mathbb{R}) \mid g_i \in GL(n_i - n_{i-1}, \mathbb{R}) \right\}.$$

Its Lie algebra is written as $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$.

For $(\lambda_1, \lambda_2, \cdots, \lambda_L) \in \mathbb{C}^L$, we define a 1-dimensional representation of P_Θ, $\lambda\colon P_\Theta \to \mathbb{C}^\times$ as follows,

$$\lambda(p) = |\det(g_1)|^{\lambda_1}|\det(g_2)|^{\lambda_2}\cdots|\det(g_L)|^{\lambda_L}, \text{for } p \in P_\Theta.$$

We define a spherical degenerate principal series representation of G, denote by $\pi_{\Theta,\lambda} = C^\infty\text{-ind}_{P_\Theta}^G(\lambda)$. The underlying representation space is

$$C^\infty(G/P_\Theta;\lambda) = \{\phi \in C^\infty(G) \mid \phi(gp) = \lambda(p)\phi(g),\ g \in G, p \in P_\Theta\}$$

where $C^\infty(G)$ is the space of C^∞-functions on G. The action of G on this space is defined by the left translation, $\pi_{\Theta,\lambda}(g)\phi(x) = \phi(g^{-1}x)$ for $g \in G$ and $\phi \in C^\infty(G/P_\Theta;\lambda)$.

We consider the annihilator of $C^\infty(G/P_\Theta;\lambda)$ in the universal enveloping algebra. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. We can see $U(\mathfrak{g})$ as the ring of left G-invariant differential operators on $C^\infty(G)$ by the natural extent ion of the differentiation of the right translation,

$$R_X(f)(g) = \frac{d}{dt}f(g\exp(tX))|_{t=0}.$$

for $X \in \mathfrak{g}, f \in C^\infty(G)$. The representation of $U(\mathfrak{g})$ on $C^\infty(G/P_\Theta;\lambda)$ is defined by the differentiation of $\pi_{\Theta,\lambda}$, i.e., for $X \in \mathfrak{g}$, $\phi \in C^\infty(G/P_\Theta;\lambda)$, $\pi_{\Theta,\lambda}(X)\phi(x) = \frac{d}{dt}\phi(\exp(-tX)x)|_{t=0}$.

Let L_g and R_g be the left and right translations by $g \in G$ respectively, i.e., $L_g f(x) = f(g^{-1}x)$ and $R_g f(x) = f(xg)$ for $f \in C^\infty(G)$.

Definition 2.1. *We define the annihilator of $C^\infty(G/P_\Theta;\lambda)$ in $U(\mathfrak{g})$ by*

$$\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}) = \{X \in U(\mathfrak{g})\ ;\ \pi_{\Theta,\lambda}(X)\phi(x) = 0, \textit{ for all } \phi \in C^\infty(G/P_\Theta;\lambda)\}.$$

This is a two-sided ideal of $U(\mathfrak{g})$.

We consider an antiautomorphism ι of $U(\mathfrak{g})$ defined by $\iota(XY) = (-Y)(-X)$ for $X, Y \in \mathfrak{g}_\mathbb{C}$. We denote the differentiation of λ by $d\lambda\colon \mathfrak{p}_\Theta \to \mathbb{C}$. Although the proposition below is a well-known fact, we give a proof for the completeness of the paper.

Proposition 2.2. *The annihilator of $\pi_{\Theta,\lambda}$ is written as follows,*

$$\iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})) = \bigcap_{g \in G} \mathrm{Ad}(g)J_\Theta(d\lambda).$$

Here

$$J_\Theta(d\lambda) = \sum_{X \in \mathfrak{p}_\Theta} U(\mathfrak{g})(X - d\lambda(X))$$

is a left ideal of $U(\mathfrak{g})$.

Proof. For $X \in \mathfrak{p}_\Theta$ and $f \in C^\infty(G/P_\Theta;\lambda)$, we have

$$\begin{aligned} R_X f(g) &= \frac{d}{dt}f(g\cdot\exp tX)|_{t=0} \\ &= \frac{d}{dt}\lambda(\exp tX)|_{t=0}f(g). \end{aligned} \tag{2.2}$$

This implies $R_X f = 0$ for $X \in J_\Theta(d\lambda)$. We recall the equation $L_X f(g) = R_{\mathrm{Ad}(g^{-1})\iota(X)} f(g)$, $X \in U(\mathfrak{g})$. Since $X \in \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda)$ implies $\mathrm{Ad}(g)X \in J_\Theta(d\lambda)$, we have

$$L_{\iota(X)} f(g) = R_{\mathrm{Ad}(g^{-1})X} f(g) = 0,$$

for $X \in \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda)$. Hence we have the inclusion $\bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda) \subset \iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}))$.

On the other hand, we take $X \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})$, and put $X_{g_0} = \mathrm{Ad}(g_0^{-1})\iota(X)$ for $g_0 \in G$. Then we have

$$\begin{aligned} R_{X_{g_0}} f(g) &= L_{g_0 g^{-1}}(R_{X_{g_0}} f)(g_0) = R_{X_{g_0}}(L_{g_0 g^{-1}} f)(g_0) \\ &= L_X(L_{g_0 g^{-1}} f)(g_0) = L_X(\pi_{\Theta,\lambda}(g_0 g^{-1}) f)(g_0) = 0 \end{aligned} \tag{2.3}$$

for $f \in C^\infty(G/P_\Theta;\lambda)$. By the decomposition $\mathfrak{g} = \overline{\mathfrak{n}}_\Theta \oplus \mathfrak{p}_\Theta$ and the Poincaré-Birkoff-Witt theorem, we have

$$U(\mathfrak{g}) = U(\overline{\mathfrak{n}}_\Theta) \oplus J_\Theta(d\lambda)$$

where $U(\overline{\mathfrak{n}}_\Theta)$ is the universal enveloping algebra of $\overline{\mathfrak{n}}_\Theta \otimes_{\mathbb{R}} \mathbb{C}$. Hence there exist $Y \in U(\overline{\mathfrak{n}}_\Theta)$ and $Z \in J_\Theta(d\lambda)$ such that $X_{g_0} = Y + Z$. By the equation (2.2), we have $R_Z f(g) = 0$ for $g \in G$ and $f \in C^\infty(G/P_\Theta;\lambda)$. Therefore the equation (2.3) lead us that $0 = R_{X_{g_0}} f(g) = R_Y f(g)$. We show $Y = 0$. Then this means $X_{g_0} \in J_\Theta(d\lambda)$. Therefore we can show the inclusion $\bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda) \subset \iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}))$.

We consider the space of compactly supported C^∞-functions on $\overline{N}_\Theta$, and denote it by $C_o^\infty(\overline{N}_\Theta)$. For $g \in \overline{N}_\Theta P_\Theta$, we take $\bar{n}(g) \in \overline{N}_\Theta$ and $p(g) \in P_\Theta$ such that $g = \bar{n}(g)p(g)$. Then we have an injection

$$\begin{array}{ccc} C_o^\infty(\overline{N}_\Theta) & \longrightarrow & C^\infty(G/P_\Theta;\lambda) \\ f & \longmapsto & \begin{cases} \lambda(p(g))f(\bar{n}(g)) & \text{if } g \in \overline{N}_\Theta P_\Theta \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

By this injection, we can consider $C_o^\infty(\overline{N}_\Theta) \subset C^\infty(G/P_\Theta;\lambda)$. Therefore if we recall that $R_Y f(g) = 0$ for $g \in G, f \in C^\infty(G/P_\Theta;\lambda)$, we have

$$R_Y f(\bar{n}) = 0 \text{ for } \bar{n} \in \overline{N}_\Theta, f \in C_o^\infty(\overline{N}_\Theta).$$

For any $\psi \in C^\infty(\overline{N}_\Theta)$ and $\bar{n} \in \overline{N}_\Theta$, there exists $f \in C_o^\infty(\overline{N}_\Theta)$ such that $\psi = f$ on some neighbourhood of $\bar{n}$ in $\overline{N}_\Theta$. Hence this implies

$$R_Y \psi(\bar{n}) = 0 \text{ for } \bar{n} \in \overline{N}_\Theta, \psi \in C^\infty(\overline{N}_\Theta).$$

Therefore $Y \in U(\overline{\mathfrak{n}}_\Theta)$ must be 0, because of the fact that $U(\overline{\mathfrak{n}}_\Theta)$ is identified with the ring of all left invariant differential operators in $\overline{N}_\Theta$. Hence $X_{g_0} \in J_\Theta(\lambda)$ for any $g_0 \in G$. This complete the proof. □

2.2 The Poisson transform for the degenerate principal series representation.

For simplicity we write $I_\Theta(\lambda) = \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda)$. Then we ee that this ideal $I_\Theta(\lambda)$ characterizes the image of the Poisson transform from the degenerate principal series. To explain this fact, we should extend the representation space of the degenerate principal series to the space of hyperfunctions on G.

The space $\mathcal{B}(G)$ of hyperfunctions on G is a left G-module by the left translation $G \times \mathcal{B}(G) \ni (g, f(x)) \mapsto f(g^{-1}x)$. We take a parabolic subgroup P_Θ of G. Also we take a character $\lambda\colon P_\Theta \to \mathbb{C}^\times$ for $(\lambda_1, \cdots, \lambda_L) \in \mathbb{C}^L$. Then we can define a G-submodule

$$\mathcal{B}(G/P_\Theta; \lambda) = \{f \in \mathcal{B}(G) \mid f(xp) = \lambda(p)f(x) \text{ for } p \in P_\Theta\},$$

as in Section 2.1. Let $M = \{k \in K \mid kak^{-1} = a,\ a \in A\}$, then we can define the minimal parabolic subgroup $P_o = P_{\{1,2,\cdots,n\}} = MAN$. We define a character of P_o by

$$\begin{array}{rccc} \lambda_\Theta\colon & P_o & \longrightarrow & \mathbb{C}^\times \\ & man & \longmapsto & \displaystyle\prod_{i=1}^{L} \prod_{j=n_i+1}^{n_{i+1}} a_j^{\lambda_i}, \end{array}$$

for $m \in M, a \in A, n \in N$. Now we introduce the Poisson transform of $\mathcal{B}(G/P_o; \lambda_\Theta)$.

Definition 2.3. *The Poisson transform is a G-homomorphism*

$$\begin{array}{rccc} \mathcal{P}^\lambda\colon & \mathcal{B}(G/P_o; \lambda_\Theta) & \longrightarrow & \mathcal{B}(G/K) \\ & f & \longmapsto & F(x) = \displaystyle\int_K f(xk)\, dk, \quad x \in G. \end{array}$$

Here dk is the normalized Haar measure on K so that $\int_K dk = 1$.

We define a character of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Let $d\lambda_\Theta\colon \mathrm{Lie}(P_o) \to \mathbb{C}$ be the differentiation of λ_Θ. By the restriction to $\mathfrak{a} \subset \mathrm{Lie}(P_o)$, we can regard $d\lambda_\Theta \in \mathfrak{a}_\mathbb{C}^*$. Let ω be a projection map from $U(\mathfrak{g})$ to the symmetric algebra $S(\mathfrak{a})$ of $\mathfrak{a}_\mathbb{C} = \mathfrak{a} \otimes_\mathbb{R} \mathbb{C}$ along the decomposition

$$U(\mathfrak{g}) = S(\mathfrak{a}) \oplus (\bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}).$$

It is known that ω is an algebra homomorphism from $Z(\mathfrak{g})$ into $S(\mathfrak{a})$. We can identify the symmetric algebra $S(\mathfrak{a})$ with the algebra of polynomials on $\mathfrak{a}_\mathbb{C}^*$. Hence if we consider the evaluation of $\omega(\cdot) \in S(\mathfrak{a})$ at $d\lambda_\Theta$, we obtain a character of $Z(\mathfrak{g})$ as follows

$$\chi_\lambda\colon Z(\mathfrak{g}) \ni X \longmapsto \omega(X)(d\lambda_\Theta) \in \mathbb{C}.$$

We define a subspace of $C^\infty(G/K)$ by

$$C^\infty(G/K; \mathcal{M}_\lambda) = \{f \in C^\infty(G/K) \mid R_X f = \chi_\lambda(X) f \text{ for } X \in Z(\mathfrak{g})\}.$$

We put

$$e(\lambda_\Theta) = \prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \Gamma\left(\frac{1}{4}(3 + \frac{2\langle \lambda_\Theta, \alpha\rangle}{\langle \alpha, \alpha\rangle})\right)^{-1} \Gamma\left(\frac{1}{4}(1 + \frac{2\langle \lambda_\Theta, \alpha\rangle}{\langle \alpha, \alpha\rangle})\right)^{-1}.$$

The following theorem is known as Helgason's conjecture [8].

Theorem 2.4 ([14]). *The Poisson transform $\mathcal{P}^\lambda$ gives G-isomorphism*

$$\mathcal{B}(G/P_o; \lambda_\Theta) \cong C^\infty(G/K; \mathcal{M}_\lambda)$$

if and only if $e(\lambda_\Theta) \neq 0$.

We can also define the Poisson transform for the subspace $\mathcal{B}(G/P_\Theta; \lambda)$ of $\mathcal{B}(G/P_o; \lambda_\Theta)$. We discuss the characterization of the image of $\mathcal{B}(G/P_\Theta; \lambda)$. We consider the subspace

$$C^\infty(G/K; I_\Theta(\lambda)) = \{f \in C^\infty(G/K) \mid R_X f = 0 \text{ for } X \in I_\Theta(\lambda)\}$$

of $C^\infty(G/K; \mathcal{M}_\lambda)$.

Remark 2.5. *We can easily show that*

$$I_\Theta(\lambda) \supset \sum_{D \in Z(\mathfrak{g})} U(\mathfrak{g})(D - \omega(D)(\lambda_\Theta))$$

(cf. Remark 4.3 in [22]). Hence actually $C^\infty(G/K; I_\Theta(\lambda))$ is a subspace of $C^\infty(G/K; \mathcal{M}_\lambda)$.

We assume

$$\lambda_\Theta + \rho \in \mathfrak{a}^*_{\mathbb{C}} \text{ is regular and dominant.}$$

Here $\rho = \frac{1}{2}\mathrm{tr}(\mathrm{ad}|_{\mathfrak{n}}) \in \mathfrak{a}^*_{\mathbb{C}}$, i.e.,

$$\rho = \frac{1}{2} \sum_{1 \le i < j \le n} (e_j - e_i) = \sum_{i=1}^{n} (i - \frac{n+1}{2}) e_i.$$

This assumption is equivalent to

$$\frac{2\langle \lambda_\Theta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \{0, -1, -2, \cdots\} \text{ for } \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}),$$

i.e.,

$$(\lambda_j + \nu_j) - (\lambda_i + \nu_i) \notin \{0, -1, -2, \cdots\}$$

for $i < j$ and ν_k are integers which satisfy $n_{k-1} + 1 \le \nu_k \le n_k$ ($k = i$ or j). We keep this assumption all through the remaining of this paper.

Theorem 2.6 (Oshima. Theorem 5.1 in [22]). *Under the above assumption, the Poisson tranform*

$$\begin{array}{rcl} \mathcal{P}^\lambda_\Theta \colon \mathcal{B}(G/P_\Theta; \lambda) & \longrightarrow & C^\infty(G/K, I_\Theta(\lambda)) \\ f & \longmapsto & F(x) = \int_K f(xk)\, dk, \quad x \in G. \end{array}$$

is a G-isomorphism.

2.3 The explicit generators of $I_\Theta(\lambda)$.

In the previous section, we see that a degenerate principal series representation has a realization on the subspace of $C^\infty(G/K)$ which is the kernel of the annihilator ideal $I_\Theta(\lambda)$. In [19],[20] and [22], T.Oshima obtained several good generator systems of $I_\Theta(\lambda)$. We introduce one of his generators here.

We denote the space of $n \times n$ matrices with entries in $U(\mathfrak{g})$ by $M(n; U(\mathfrak{g}))$. For $\mathbb{E} = (E_{ij})_{ij} \in M(n; U(\mathfrak{g}))$, we define elements in $Z(\mathfrak{g})$ by

$$\Delta_k = \mathrm{tr}(\mathbb{E}^k), \quad \text{for } k = 1, \ldots, n.$$

Then it is known that $Z(\mathfrak{g}) \cong \mathbb{C}[\Delta_1, \ldots, \Delta_n]$ as $\mathbb{C}$-algebras.

Theorem 2.7 (Oshima. Corollary 4.6 in [22]). *Asuume $\lambda_\Theta + \rho \in \mathfrak{a}_\mathbb{C}^*$ is regular and dominant. Then we have*

$$I_\Theta(\lambda) = \sum_{i=1}^{n}\sum_{j=1}^{n} U(\mathfrak{g}) \prod_{k=1}^{L} (\mathbb{E} - \lambda_k - n_{k-1})_{ij} + \sum_{k=1}^{L-1} U(\mathfrak{g})(\Delta_k - \chi_\lambda(\Delta_k)).$$

3 Generalized Whittaker models

The generalized Whittaker model is the main theme of this paper. We give a characterization of the space of the generalized Whittaker models of a degenerate principal series $\pi_{\Theta,\lambda}$ as the kernel of $I_\Theta(\lambda)$. This is an analogy of Yamashita's method in the case of irreducible highest weight modules [34]. The substantial part of his method is that the maximal globalization (in the sense of W.Schmid [26]) of highest weight modules is given by the kernel of a certain differential operator. The corresponding theorem for the degenerate principal series is obtained in Theorem 2.6 in Section 2.2. Moreover thanks to Theorem 2.7, we know explicit structures of these differential operators. Hence we can carry out the explicit calculations about the space of the generalized Whittaker models.

Let V_K be the space of K-finite vectors for a continuous representation of G on a complete Hausdorff locally convex space V. Let $X_{\Theta,\lambda}$ be $C^\infty(G/P_\Theta; \lambda)_K$. This becomes a $(\mathfrak{g}_\mathbb{C}, K)$-module, i.e., the $\mathfrak{g}_\mathbb{C}$-action is the differentiation of $\pi_{\Theta,\lambda}$ and the K-action is the restriction of $\pi_{\Theta,\lambda}$, furthermore the actions of $\mathfrak{g}_\mathbb{C}$ and K are compatible. Also $X_{\Theta,\lambda}$ is a Harish-Chandra module, i.e., finitely generated as a $U(\mathfrak{g})$-module and with finite K-multiplicities.

3.1 Maximal globalization

For the Harish-Chandara module $X_{\Theta,\lambda}$, let us consider its dual Harish-Chandra module X_{Θ,λ^*}. Here the character λ^* of P_Θ is defined by

$$\lambda^* = -\bar{\lambda} - 2\rho_\Theta = (n - n_0 - n_1 - \bar{\lambda}, \ldots, n - n_{L-1} - n_L - \bar{\lambda}),$$

where $\rho_\Theta = \frac{1}{2}\mathrm{tr}(\mathrm{ad}|_{\mathfrak{n}_\Theta}) \in \mathfrak{a}_\mathbb{C}^*$, i.e.,

$$\rho_\Theta = \sum_{i=1}^{L} \frac{n_{i-1} + n_i - n}{2} \sum_{j=n_{i-1}}^{n_i} e_j.$$

Actually, if we consider the pairing $\langle\ ,\ \rangle\colon C^\infty(G/P_\Theta;\lambda)\times C^\infty(G/P_\Theta;\lambda^*)\to\mathbb{C}$ defined by

$$\langle f,g\rangle_{\lambda,\lambda^*}=\int_K f(k)\overline{g(k)}\,dk$$

for $(f,g)\in C^\infty(G/P_\Theta;\lambda)\times C^\infty(G/P_\Theta;\lambda^*)$, this is a G-equivariant non-degenerate sesquilinear pairing. By this pairing, the Harish-Chandra module $X_{\Theta,\lambda^*}=C^\infty(G/P_{\Theta,\lambda^*})_K$ can be identified with the dual Harish-Chandra module $(X_{\Theta,\lambda})^*$,i.e., all K-finite vectors in $\mathrm{Hom}_{\mathbb{C}}(X_{\Theta,\lambda},\mathbb{C})_K$. Here K acts on $\mathrm{Hom}_{\mathbb{C}}(X_{\Theta,\lambda},\mathbb{C})$ by $k\cdot I(v)=I(\pi_{\Theta,\lambda}(k^{-1})v)$ for $I\in\mathrm{Hom}_{\mathbb{C}}(X_{\Theta,\lambda},\mathbb{C})_K$ and $v\in X_{\Theta,\lambda}$.

We can consider the natural $(\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}},K\times K)$-bimodule structures on $X_{\Theta,\lambda}\otimes X_{\Theta,\lambda^*}$ and $C^\infty(G)$. For $X_1,X_2\in\mathfrak{g}_{\mathbb{C}}$ and $k_1,k_2\in K$, we put

$$\begin{aligned}(X_1,X_2)(f\otimes f^*)=&\pi_{\Theta,\lambda^*}(X_1)f\otimes f^*+f\otimes\pi_{\Theta,\lambda^*}(X_2)f^*,\\ (k_1,k_2)(f\otimes f^*)=&\pi_{\Theta,\lambda}(k_1)f\otimes\pi_{\Theta,\lambda^*}(k_2)f^*\end{aligned}$$

for $f\in X_{\Theta,\lambda}$ and $f^*\in X_{\Theta,\lambda^*}$. Also we define

$$\begin{aligned}(X_1,X_2)g=&L_{X_1}g+R_{X_2}g,\\ (k_1,k_2)g=&L_{k_1}R_{k_2}g\end{aligned}$$

for $g\in C^\infty(G)$. Then we introduce the matrix coefficient map (cf. [4]) from $(\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}},K\times K)$-bimodule $X_{\Theta,\lambda}\otimes X_{\Theta,\lambda^*}$ to $C^\infty(G)$ so that,

1. the map $c\colon X_{\Theta,\lambda}\otimes X_{\Theta,\lambda^*}\to C^\infty(G)$ is a $(\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}},K\times K)$-bimodule homomorphism,
2. for any $f\in X_{\Theta,\lambda}$ and $f^*\in X_{\Theta,\lambda^*}$, the evaluation at the origin $e\in G$ becomes
$$c(f\otimes f^*)(e)=\langle f,f^*\rangle_{\lambda,\lambda^*}.$$

It is known that this matrix coefficient map is uniquely determined (cf. Theorem 8.7 in [4]).

If we consider the restriction of Poisson transform $\mathcal{P}^\lambda_\Theta$ on $X_{\Theta,\lambda}$, Theorem 2.6 gives us the $(\mathfrak{g}_{\mathbb{C}},K)$-isomorphism

$$\mathcal{P}^\lambda_\Theta\colon X_{\Theta,\lambda}\xrightarrow{\sim}C^\infty(G/K;I_\Theta(\lambda))_K.$$

Lemma 3.1. *Take the K-fixed vector $f_0\in X_{\Theta,\lambda^*}$ such that $f_0|_K\equiv 1$. Then the restriction of the Poisson transform on $X_{\Theta,\lambda}$ is a matrix coefficient of an element of $X_{\Theta,\lambda}$ with $f_0\in X_{\Theta,\lambda^*}$, i.e.,*

$$\mathcal{P}^\lambda_\Theta(f)=c(f\otimes f_0).$$

Proof. By the pairing of $C^\infty(G/P_\Theta;\lambda)\times C^\infty(G/P_\Theta;\lambda^*)$ defined above, we can define a map $X_{\Theta,\lambda}\otimes X_{\Theta,\lambda^*}\to C^\infty(G)$ as follows,

$$f\otimes f^*\longmapsto\langle\pi_{\Theta,\lambda}(g^{-1})f,f^*\rangle=\int_K f(gk)\overline{f^*(k)}\,dk,$$

for $f \in X_{\Theta,\lambda}$ and $f^* \in X_{\Theta,\lambda^*}$. This map satisfies the conditions of the matrix coefficient map. Hence for $f \in X_{\Theta,\lambda}$, we have

$$\begin{aligned}
\mathcal{P}_\Theta^\lambda(f)(g) &= \int_K f(gk)\,dk \\
&= \int_K f(gk)\overline{f_0(k)}\,dk \\
&= \int_K (\pi_{\Theta,\lambda}(g^{-1})f)(k)\overline{f_0(k)}\,dk \\
&= \langle \pi_{\Theta,\lambda}(g^{-1})f, f_0\rangle_{\lambda,\lambda^*} \\
&= c(f \otimes f_0)(g),
\end{aligned}$$

by the uniqueness of the matrix coefficient map. □

Lemma 3.2. *The dual Harish-Chandra module X_{Θ,λ^*} is a cyclic $U(\mathfrak{g})$-module with the cyclic vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$.*

Proof. We put $W = \{\pi_{\Theta,\lambda^*}(X)f_0 \mid X \in U(\mathfrak{g})\}$. This is a $(\mathfrak{g}_\mathbb{C}, K)$-module. We restrict the pairing $\langle\ ,\ \rangle_{\lambda,\lambda^*}$ to $X_{\Theta,\lambda} \times W$. Take an element $f \in X_{\Theta,\lambda}$ and assume $\langle f, w\rangle_{\lambda,\lambda^*} = 0$ for any $w \in W$. Since $\mathcal{P}_\lambda(f)(g)$ is K-finite and $Z(\mathfrak{g})$-finite, it is a real analytic function on G. Let C be an sufficiently small open neighbourhood of 0 in $\mathfrak{g}$. Then we have the Taylor expansion at the origin $e \in G$,

$$\begin{aligned}
\mathcal{P}_\lambda(f)(\exp X) &= \sum_{n=0}^{\infty} \frac{1}{n!} R_{X^n}(\mathcal{P}_\lambda(f))(e) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} R_{X^n}(c(f \otimes f_0))(e) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \langle f, \pi_{\Theta,\lambda^*}(X^n)f_0\rangle_{\lambda,\lambda^*} \\
&= 0
\end{aligned}$$

for $X \in C$. Here we used Lemma 3.1. We can extend this equality to the identity component G^o of G because both functions are real analytic. Also we can extend to G by the equation $G = KG^o$. The injectivity of the Poisson transfrom $\mathcal{P}_\lambda$ tells us $f = 0$. Hence the bilinear form of $X_{\Theta,\lambda} \times W$ is non-degenerate. Then $W = X_{\Theta,\lambda^*}$ by Lemma 2 in Section 5.2 of [29]. □

Let us consider the space of $(\mathfrak{g}_\mathbb{C}, K)$-homomorphisms of X_{Θ,λ^*} into $C^\infty(G)$,

$$\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G)).$$

Here we regard $C^\infty(G)$ as a $(\mathfrak{g}_\mathbb{C}, K)$-module by the right translation. Moreover, this space of $(\mathfrak{g}_\mathbb{C}, K)$-homomorphisms inherits a Fréchet topology and a continuous G-action from $C^\infty(G)$. More precisely, we define a semi-norm on this space as follows. The space $C^\infty(G)$ is a Fréchet space of uniformly convergence on compact sets for functions on G and their derivatives. Let $\{|\cdot|_\alpha\}_{\alpha\in\Lambda}$ be a family of countable many semi-norms on $C^\infty(G)$ which defines the Fréchet topology on $C^\infty(G)$ where Λ is the index set. Take a semi-norm

$|\cdot|_\alpha \in \{|\cdot|_\alpha\}_{\alpha\in\Lambda}$ and $v \in X_{\Theta,\lambda^*}$. Then we define a real-valued function $|\cdot|_{\alpha,v}\colon \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G)) \to \mathbb{R}_{\geq 0}$ by

$$|I|_{\alpha,v} = |I(v)|_\alpha$$

for $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$. We can see that the function $|\cdot|_{\alpha,v}$ defines a semi-norm on $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$ for $\alpha \in \Lambda$ and $v \in X_{\Theta,\lambda^*}$.

Lemma 3.3. *Let $\{v_n\}$ be a countable vector space basis of the Harish-Chandra module X_{Θ,λ^*}. Then the family of semi-norms $\{|\cdot|_{\alpha,v_m}\}_{\alpha\in\Lambda, v_m\in\{v_n\}}$ defines a Fréchet topology on* $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$.

Proof. Take $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$, and we assume $|I|_{\alpha,v_m} = 0$ for any $|\cdot|_{\alpha,v_m} \in \{|\cdot|_{\alpha,v_m}\}_{\alpha\in\Lambda,v_m\in\{v_n\}}$. Since $\{v_n\}$ is a basis of X_{Θ,λ^*}, it follows that $|I(v)|_\alpha = 0$ for any $v \in X_{\Theta,\lambda^*}$ and $\alpha \in \Lambda$. This means $I(v) = 0$ for any $v \in X_{\Theta,\lambda^*}$ because $C^\infty(G)$ is the Hausdorff space. Thus we have $I = 0$. This implies that $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$ is a Hausdorff space as well. Since $\{|\cdot|_{\alpha,v_m}\}_{\alpha\in\Lambda,v_m\in\{v_n\}}$ consists of countable many semi-norms, the space $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$ is metrizable by this family of semi-norms. Finally we need to check the completeness. Suppose that there exists a Cauchy sequence $\{I_k\}$ of the elements of $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$, i.e.,

$$|I_k - I_l|_{\alpha,v_m} \to 0 \quad \text{for } k, l \to \infty,$$

for any $|\cdot|_{\alpha,v_m} \in \{|\cdot|_{\alpha,v_m}\}_{\alpha\in\Lambda,v_m\in\{v_n\}}$. Then this implies that for any $v \in X_{\Theta,\lambda^*}$, the sequence $\{I_k(v)\} \subset C^\infty(G)$ is a Cauchy sequence. Hence there exists $\lim_{k\to\infty} I_k(v) \in C^\infty(G)$. We define the map $\tilde{I}\colon X_{\Theta,\lambda^*} \to C^\infty(G)$ by $\tilde{I}(v) = \lim_{k\to\infty} I_k(v)$ for $v \in X_{\Theta,\lambda^*}$. Then we show that $\tilde{I}$ is the element in $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$. For any $Z \in \mathfrak{g}_\mathbb{C}$, $v \in X_{\Theta,\lambda^*}$ and $\alpha \in \Lambda$, we have

$$\begin{aligned}
&|\tilde{I}(\pi_{\Theta,\lambda^*}(Z)v) - R_Z\tilde{I}(v)|_\alpha \\
&\quad = |\tilde{I}(\pi_{\Theta,\lambda^*}(Z)v) - I_k(\pi_{\Theta,\lambda^*}(Z)v) + R_Z I_k(v) - R_Z\tilde{I}(v)|_\alpha \\
&\quad \leq |\tilde{I}(\pi_{\Theta,\lambda^*}(Z)v) - I_k(\pi_{\Theta,\lambda^*}(Z)v)|_\alpha + |R_Z I_k(v) - R_Z\tilde{I}(v)|_\alpha \\
&\quad \to 0 \quad \text{for } k \to \infty.
\end{aligned}$$

Thus $\tilde{I}(\pi_{\Theta,\lambda^*}(Z)v) = R_Z\tilde{I}(v)$ for any $Z \in \mathfrak{g}_\mathbb{C}$ and $v \in X_{\Theta,\lambda^*}$. Hence $\tilde{I}$ is a $\mathfrak{g}_\mathbb{C}$-homomorphism. Similarly we can show that $\tilde{I}$ is a K-homomorphism and a linear map. Hence we could show that $\tilde{I} \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$. By the construction of $\tilde{I}$, we can see that $I_k \to \tilde{I}$ $(k \to \infty)$ in $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$. This proves the lemma. □

We could define a Fréchet topology on $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$. Then a continuous G-action on this space is defined by left translation on $C^\infty(G)$. Hence this space defines a continuous Fréchet representation of G. This is called the maximal globalization of the Harish-Chandra module $X_{\Theta,\lambda}$ (cf. [26] and [15]).

Lemma 3.4. *Take the K-fixed vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$. We consider a mapping*

$$\begin{array}{rccc}
\Phi\colon & \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G)) & \longrightarrow & C^\infty(G) \\
& I & \longmapsto & I(f_0)(g) \quad (g \in G).
\end{array}$$

Then Φ is a continuous mapping. Moreover for any semi-norm $|\cdot|_{\alpha,v_m} \in \{|\cdot|_{\alpha,v_m}\}_{\alpha\in\Lambda,v_m\in\{v_n\}}$ on $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$, there exists a continuous semi-norm μ_{α,v_m} on $C^\infty(G)$ such that

$$\mu_{\alpha,v_m}(\Phi(I)) = |I|_{\alpha,v_m},$$

for $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$. Thus Φ is injective.

Proof. For any semi-norm $|\cdot|_\alpha$ on $C^\infty(G)$, there exists a continuous semi-norm $|\cdot|_{\alpha,f_0}$ on $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$ such that

$$|I|_{\alpha,f_0} = |I(f_0)|_\alpha = |\Phi(I)|_\alpha$$

for $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$. Hence Φ is the continuous. Conversely, we take a semi-norm $|\cdot|_{\alpha,v_m}$ on $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$ for $\alpha \in \Lambda$ and $v_m \in \{v_n\}$. By Lemma 3.2, X_{Θ,λ^*} is a cyclic $U(\mathfrak{g})$-module with cyclic vector f_0. Thus there exists an element $X \in U(\mathfrak{g})$ such that $\pi_{\Theta,\lambda^*}(X)f_0 = v_m$. Then we have

$$|I|_{\alpha,v_m} = |I(v_m)|_\alpha = |I(\pi_{\Theta,\lambda^*}(X)f_0)|_\alpha = |R_X I(f_0)|_\alpha.$$

If we recall that $U(\mathfrak{g})$ can be identified with the ring of left invariant differential operators on $C^\infty(G)$, then $\mu_{\alpha,v_m}(f) = |R_X f|_\alpha$ defines a continuous semi-norm on $C^\infty(G)$. This proves the lemma. □

The maximal globalization of $X_{\Theta,\lambda}$ is isomrphic to the subspace of $C^\infty(G/K)$ as follows.

Proposition 3.5. *Take the K-fixed vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$. Then we have a following topological G-isomorphism,*

$$\begin{array}{rccc}\Phi\colon & \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G)) & \xrightarrow{\sim} & C^\infty(G/K; I_\Theta(\lambda)) \\ & I & \longmapsto & I(f_0)(g) \quad (g \in G).\end{array}$$

Here $C^\infty(G/K; I_\Theta(\lambda))$ has the Fréchet topology as the closed subspace of $C^\infty(G)$.

Proof. We can immediately see that Φ preserves the action of G by definition. First we show that Φ is well-defined. Take a K-finite element $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K$. Then by the evaluation at the origin $e \in G$, we can regard $I(\cdot)(e)$ as the element of $X_{\Theta,\lambda} \cong (X_{\Theta,\lambda^*})^*$. Since $I(f_0)(g) \in C^\infty(G)$ is K-finite and $Z(\mathfrak{g})$-finite, it is a real analytic function on G. Let C be an sufficiently small open neighbourhood of 0 in $\mathfrak{g}$. Then we have the Taylor expansion at the origin $e \in G$,

$$\begin{aligned}I(f_0)(\exp X) &= \sum_{n=0}^{\infty} \frac{1}{n!} R_{X^n}(I(f_0))(e) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle I(\cdot)(e), \pi_{\Theta,\lambda^*}(X^n) f_0 \rangle_{\lambda,\lambda_*} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} c(I(\cdot)(e) \otimes \pi_{\Theta,\lambda^*}(X^n) f_0)(e) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} R_{X^n} c(I(\cdot)(e) \otimes f_0)(e) \\ &= c(I(\cdot)(e) \otimes f_0)(\exp X)\end{aligned}$$

for $X \in C$. We can extend this equality for the identity component G^o of G because both functions are real analytic. And the fact $G = G^o \cdot K$ implies $I(f_0)(g) = c(I(\cdot)(e) \otimes f_0)(g)$ for all $g \in G$. Hence by Theorem 2.6 and Lemma 3.1, we have the inclusion

$$\Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K) \subset C^\infty(G/K; I_\Theta(\lambda)).$$

We recall that for a continuous representation of G on a locally convex complete space V, the space of K-finite vectors V_K is dense in V (for example, Lemma 1.9, Ch.IV in [9]). Since Φ is a continuous mapping by Lemma 3.4, we have

$$\begin{aligned}
&\Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))) = \Phi(\mathrm{Cl}(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K) \\
&\subset \mathrm{Cl}(\Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K)) \subset \mathrm{Cl}(C^\infty(G/K; I_\Theta(\lambda))) \\
&= C^\infty(G/K; I_\Theta(\lambda)).
\end{aligned}$$

Here $\mathrm{Cl}(\cdot)$ is the closure. Hence Φ is well-defined. Next we prove Φ is the bijective map. By Lemma 3.4, Φ is injective. We prove that Φ is surjective.

For any $F(g) \in C^\infty(G/K; I_\Theta(\lambda))_K$, there exist $h \in X_{\Theta,\lambda}$ and $F(g) = c(h \otimes f_0)(g)$ $(g \in G)$ by Theorem 2.6 and Lemma 3.1. We define an element of $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ so that

$$I_h(v)(g) = c(h \otimes v)(g),$$

for $v \in X_{\Theta,\lambda^*}$. Then we can see that $\Phi(I_h)(g) = I_h(f_0)(g) = c(h \otimes f_0)(g) = F(g)$. Hence we have an inclusion $C^\infty(G/K; I_\Theta(\lambda))_K \subset \Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G)))$. Because $C^\infty(G/K; I_\Theta(\lambda))_K$ is a dense subspace of $C^\infty(G/K; I_\Theta(\lambda))$, for any $f \in C^\infty(G/K; I_\Theta(\lambda))$ we can choose a convergent sequence $f_n \to f$ $(n \to \infty)$ where $f_n \in C^\infty(G/K; I_\Theta(\lambda))_K$ for $n \in \mathbb{N}$. The above inclusion shows that there exist $I_n \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ such that $\Phi(I_n) = f_n$. From the second assertion in Lemma 3.4, the sequence $\{I_n\}$ is a Cauchy sequence in $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$. Since $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ is a Fréchet space, i.e., complete space, there exist $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ such that $I_n \to I$ $(n \to \infty)$. Thus we have $\Phi(I) = (f)$ by the continuity of Φ. This shows that Φ is a surjective map. The open mapping theorem leads that Φ is a homeomorphism. □

3.2 Generalized Whittaker models

We define a generalized Whittaker model for $X_{\Theta,\lambda}$. Let us fix a closed subgroup U of N. We take an irreducible unitary representation η of U on a Hilbert space V_η. Let V_η^∞ be the space of C^∞-vectors in V_η. Let us consider the space $C_\eta^\infty(U\backslash G) = \{f : G \to V_\eta^\infty \text{ smooth} \mid f(ng) = \eta(n)f(g), g \in G, n \in U\}$. This becomes a G-module by the right translation.

Definition 3.6. *We consider the following intertwining space*

$$\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G)).$$

We call images of X_{Θ,λ^*} by these $(\mathfrak{g}_\mathbb{C}, K)$-homomorphisms generalized Whittaker models of X_{Θ,λ^*}.

Theorem 3.7. *We take the K-fixed vector in X_{Θ,λ^*} such that $f_0|_K \equiv 1$. Then the following mapping*

$$\begin{array}{rccc} \tilde{\Phi}\colon \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty_\eta(U\backslash G)) & \xrightarrow{\sim} & C^\infty_\eta(U\backslash G/K; I_\Theta(\lambda)) \\ W & \longmapsto & W(f_0)(g) \end{array}$$

is a linear isomorphism. Here

$$\begin{aligned} &C^\infty_\eta(U\backslash G/K; I_\Theta(\lambda)) \\ &\quad = \{f\colon G \to V^\infty_\eta \ \textit{smooth} \,|\, f(ngk) = \eta(n)f(g), g \in G, n \in U,\ k \in K \\ &\qquad\qquad\qquad\qquad\qquad \textit{and}\ R_X f(g) = 0,\ X \in I_\Theta(\lambda)\}. \end{aligned}$$

Proof. Fix a nonzero element $\xi \in V_\eta$. Then we consider the linear mapping

$$T\colon C^\infty_\eta(U\backslash G) \in f \longmapsto \langle \xi, f(g)\rangle_\eta \in C^\infty(G),$$

which commutes with G and $\mathfrak{g}_{\mathbb{C}}$ actions from the right where $\langle\ ,\ \rangle_\eta$ is an inner product on V_η. Since (η, V_η) is an irreducible unitary representation of U, this mapping T is injective. In fact, if $T(f) \equiv 0$ for $f \in C^\infty_\eta(U\backslash G)$, then we have

$$0 = T(f)(ng) = \langle \xi, f(ng)\rangle = \langle \xi, \eta(n)f(g)\rangle = \langle \eta(n^{-1})\xi, f(g)\rangle,$$

for any $n \in U$ and $g \in G$. Since V_η is irreducible, this implies $f \equiv 0$. By this map T, we also have an injective map

$$\begin{array}{rccc} \tilde{T}\ \colon \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty_\eta(U\backslash G)) & \longrightarrow & \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty(G)) \\ W & \longmapsto & T \circ W. \end{array}$$

For any $W \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty(U\backslash G,\eta))$, we have $T(\tilde{\Phi}(W)) = T(W(f_0)) = T \circ W(f_0) = \tilde{T}(W)(f_0) = \Phi(\tilde{T}(W))$. Hence we have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty_\eta(U\backslash G)) & \xrightarrow{\tilde{\Phi}} & C^\infty_\eta(U\backslash G/K) \\ \tilde{T}\big\downarrow & & \big\downarrow T \\ \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty(G)) & \xrightarrow[\Phi]{} & C^\infty(G/K) \end{array}.$$

Since Φ, T and $\tilde{T}$ are injective, we see $\tilde{\Phi}$ is injective. Next we show that $\operatorname{Im}\tilde{\Phi} \subset C^\infty_\eta(U\backslash G/K; I_\Theta(\lambda))$. Take $W \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty_\eta(U\backslash G))$, then $T(\tilde{\Phi}(W))(g) = \langle \xi, W(f_0)(g)\rangle \in C^\infty(G/K; I_\Theta(\lambda))$ where $g \in G$. Hence we have $0 = R_X T(\tilde{\Phi}(W))(g) = T(R_X\tilde{\Phi}(W))(g)$ for $X \in I_\Theta(\lambda)$ and $g \in G$. Since T is injective, we have $R_X W(f_0)(g) = 0$ for $X \in I_\Theta(\lambda)$ and $g \in G$, i.e, $\operatorname{Im}\tilde{\Phi} \subset C^\infty_\eta(U\backslash G/K; I_\Theta(\lambda))$.

Finally, we show that $\tilde{\Phi}$ is surjective. Let $f \in C^\infty_\eta(U\backslash G/K; I_\Theta(\lambda))$. For $v \in X_{\Theta,\lambda^*}$ there exist $X_v \in U(\mathfrak{g})$ such that $v = \pi_{\Theta,\lambda^*}(X_v)f_0$ since X_{Θ,λ^*} is irreducible. Then we define a mapping $W_f\colon X_{\Theta,\lambda^*} \ni v = \pi_{\Theta,\lambda^*}(X_v)f_0 \mapsto R_{X_v}f(g) \in C^\infty_\eta(U\backslash G)$. We need to check that it is a well-defined mapping. If for $X_v, X'_v \in \mathfrak{g}$ we have $v = \pi_{\Theta,\lambda^*}(X_v)f_0 = \pi_{\Theta,\lambda^*}(X'_v)f_0$, then we have $\pi_{\Theta,\lambda^*}(X_v - X'_v)f_0 = 0$. On the other hand we have $T(f) \in C^\infty(G/K; I_\Theta(\lambda))$.

Thus there exists $I_f \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$ such that $T(f) = \Phi(I_f)$ by Proposition 3.5. We put $Z = X_v - X'_v$. Then we have $T(R_Z f) = R_Z T(f) = \Phi(R_Z I_f) = R_Z I_f(f_0)(g) = I_f(\pi_{\Theta,\lambda^*}(Z)f_0)(g) = 0$. Hence by the injectivity of T, we have $R_Z f(g) = 0$, i.e., $R_{X_v} f = R_{X'_v} f$. This implies that W_f is well-defined. Also we can check that W_f is compatible with $\mathfrak{g}_\mathbb{C}$ and K actions. Hence $W_f \in \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(X_{\Theta,\lambda^*}, C^\infty_\eta(U\backslash G))$ and $\tilde{\Phi}(W_f) = W_f(f_0) = f$. Hence $\tilde{\Phi}$ is surjective. $\square$

Remark 3.8. *This theorem is an analogue of Yamashita's result for the generalized Whittaker models of discrete series representations (Theroem 2.4. in [33]) and more general settings (Corollary 1.8. in [34]).*

4 Calculus on the case of $GL(4,\mathbb{R})$

In previous sections, we gave a characterization of the space of generalized Whittaker model as the kernel of some explicit differential operators. We calculate some examples on $GL(4,\mathbb{R})$ by using these theories. In particular we take the spherical degenerate principal series representations induced from the maximal parabolic subgroups $P_{1,4}$, $P_{2,4}$ and compute dimensions of the spaces of generalized Whittaker models and find the basis for them.

Let us explain the detailed settings. We consider the case $n = 4$. Hence $G = GL(4,\mathbb{R})$, $K = O(4)$, A is the group of the 4×4 diagonal matrices with positive real entries and N is the group of 4×4 strict lower triangular matrices with 1s in the diagonal entries. We put $P_k = P_{k,4}, k = 1,2$. For $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, we define the character $\lambda\colon P_k \to \mathbb{C}^\times$ and define degenerate principal series representation induced from P_k and λ as before. Let $X_{k,\lambda}$ be their Harish-Chandra modules which consist of K-finite vectors of these degenerate principal series representations. Then by Theorem 2.7, the annihilator ideal in $U(\mathfrak{g})$ of the degenerate principal series representation are written by

$$I_k(\lambda) = I_{\{k,4\}}(\lambda)$$
$$\sum_{i=1}^{4}\sum_{j=1}^{4} U(\mathfrak{g})(\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k) + U(\mathfrak{g})(\sum_{i=1}^{4} E_{ii} - \lambda_1 - (4-k)\lambda_2). \quad (4.1)$$

for $k = 1, 2$. We put a stronger condition for λ,

$$\lambda_1 - \lambda_2 \notin \mathbb{Z}.$$

4.1 Equivalent classes of $C^\infty_\eta(U\backslash G)$.

A generalized Whittaker model is an image of an embeddings of X_{Θ,λ^*} into $C^\infty_\eta(U\backslash G)$ where U is a closed subgroup of N and η is its irreducible unitary representation. In this paper, we consider the space $C^\infty_\eta(U\backslash G)$ defined as follows.

1. the group U is a closed subgroup of N and η is its unitary character,
2. the unitary induced representation $L^2\text{-}\mathrm{Ind}_U^N \eta$ is an irreducible unitary representation of N.

We classify the G-equivalent classes of these $C^\infty_\eta(U\backslash G)$.

4.1.1 The classification of $\hat{N}$

Firstly we give the classification of the unitary dual of maximal unipotent subgroup N of G by using the Killirov's method for coadjoint orbits. Since the contents of this subsection is standard, the details of this method can be found in [3] for example.

Let $\mathfrak{n} = \mathrm{Lie}(N)$, i.e.,

$$\mathfrak{n} = \left\{ n(z, y_1, y_2, x_1, x_2, x_3) = \begin{pmatrix} 0 & & & \\ x_1 & 0 & & \\ y_1 & x_2 & 0 & \\ z & y_2 & x_3 & 0 \end{pmatrix} \; ; \; x_1, \ldots, z \in \mathbb{R} \right\}.$$

We denote its dual $\mathbb{R}$-vector space by $\mathfrak{n}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{n}, \mathbb{R})$. We identify this space with a subspace of $M(4, \mathbb{R})$,

$$\left\{ l(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} 0 & \gamma_1 & \beta_1 & \alpha \\ & 0 & \gamma_2 & \beta_2 \\ & & 0 & \gamma_3 \\ & & & 0 \end{pmatrix} \; ; \; \alpha, \ldots, \gamma_3 \in \mathbb{R} \right\},$$

so that

$$\begin{aligned} & l(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3) \cdot n(z, y_1, y_2, x_1, x_2, x_3) \\ & \quad = \mathrm{tr}(l(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3) n(z, y_1, y_2, x_1, x_2, x_3)) \\ & \quad\quad = \alpha z + \beta_1 y_1 + \beta_2 y_2 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3. \end{aligned}$$

We define the coadjoint action of N on $\mathfrak{n}$ by $(\mathrm{Ad}^*(n)l)(X) = l(\mathrm{Ad}(n^{-1})X)$ for $n \in N, l \in \mathfrak{n}^*, X \in \mathfrak{n}$. Take a basis $X_1, X_2, X_3, Y_1, Y_2, Z$ of $\mathfrak{n}$ and its dual basis $X_1^*, X_2^*, X_3^*, Y_1^*, Y_2^*, Z^*$ of $\mathfrak{n}^*$ so that

$$n(z, y_1, y_2, x_1, x_2, x_3) = zZ + \sum_{i=1}^{2} y_i Y_i + \sum_{j=1}^{3} x_j X_j$$

and

$$l(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3) = \alpha Z^* + \sum_{i=1}^{2} \beta_i Y_i^* + \sum_{j=1}^{3} \gamma_j X_j^*.$$

Under our coordinate system, the coadjoint action is written as follows,

$$\begin{aligned} & (\mathrm{Ad}^* \exp(n(z, \cdots, x_3)))(l(\alpha, \cdots, \gamma_3)) \\ & = \alpha Z^* + (\beta_1 + \alpha x_3) Y_1^* + (\beta_2 - \alpha x_1) Y_2^* \\ & + (\gamma_1 + \beta_1 x_2 + \alpha(y_2 + \frac{x_2 x_3}{2})) X_1^* + (\gamma_2 + x_3 \beta_2 - x_1 \beta_1 - x_1 x_3 \alpha) X_2^* \\ & + (\gamma_3 - x_2 \beta_2 + \alpha(\frac{x_1 x_2}{2} - y_1)) X_3^*. \end{aligned} \tag{4.2}$$

We consider the classification of coadjoint orbits of $\mathfrak{n}^*$. First, we assume that $\alpha \neq 0$. Then by the equation (4.2), if we choose appropriate x_3, x_1, y_2, y_1, we can find in the Ad^*N-orbit a point with $\beta_1 = \beta_2 = \gamma_1 = \gamma_3 = 0$. Hence if we write $l_{\alpha,\gamma_2} = l(\alpha, 0, 0, 0, \gamma_2, 0)$, the coadjoint orbit is written as follows,

$$(\mathrm{Ad}^*N)l_{\alpha,\gamma_2} = \{\alpha Z^* + t_1 Y_1^* + t_2 Y_2^* + s_1 X_1^* \\ + (\gamma_2 + \frac{t_1 t_2}{\alpha})X_2^* + s_2 X_3^* \ ; \ t_1, t_2, s_1, s_2 \in \mathbb{R}\}.$$

Next, we consider the case $\alpha = 0$, i.e., $l(0, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$. We assume $\beta_1 \neq 0$ or $\beta_2 \neq 0$. Then from the equation (4.2), we can see that there is an element $l(0, \beta_1', \beta_2', \gamma_1', 0, \gamma_3')$ in $(\mathrm{Ad}^*N)l(0, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$. Hence in this case, it is enough to consider the orbit

$$(\mathrm{Ad}^*N)l(0, \beta_1, \beta_2, \gamma_1, 0, \gamma_3) \\ = \{\beta_1 Y_1^* + \beta_2 Y_2^* + (\beta_1 t_1 + \gamma_1)X_1^* \\ + t_2 X_2^* + (\gamma_3 - \beta_2 t_1)X_3^* \ ; \ t_1, t_2 \in \mathbb{R}\}.$$

Finally we consider the case $\beta_1 = \beta_2 = 0$. Then we have

$$(\mathrm{Ad}^* n^*(0, 0, 0, \gamma_1, \gamma_2, \gamma_3)) = \{\gamma_1 X_1^* + \gamma_2 X_2^* + \gamma_3 X_3^*\}.$$

We summarize these as a proposition.

Proposition 4.1. *We can classify coadjoint orbits of* $\mathfrak{n}^*$ *in following cases.*

(I) For $\alpha \in \mathbb{R}\backslash\{0\}$ *and* $\gamma_2 \in \mathbb{R}$,

$$\mathcal{O}_{\alpha,\gamma_2} = (Ad^*N)l(\alpha, 0, 0, 0, \gamma_2, 0) \\ = \{\alpha Z^* + t_1 Y_1^* + t_2 Y_2^* + s_1 X_1^* + (\gamma_2 + \frac{t_1 t_2}{\alpha})X_2^* \\ + s_2 X_3^* \ ; \ t_1, t_2, s_1, s_2 \in \mathbb{R}\}.$$

We have $\dim \mathcal{O}_{\alpha,\gamma_2} = 4$.

(II) For $\beta_1, \beta_2, \gamma_1, \gamma_3 \in \mathbb{R}$ *such that* $\beta_1 \neq 0$, *or* $\beta_2 \neq 0$,

$$\mathcal{O}_{\beta_1,\beta_2,\gamma_1,\gamma_3} = (Ad^*N)l(0, \beta_1, \beta_2, \gamma_1, 0, \gamma_3) \\ = \{\beta_1 Y_1^* + \beta_2 Y_2^* + (\beta_1 t_1 + \gamma_1)X_1^* + t_2 X_2 + (\gamma_3 - \beta_2 t_1)X_3^* \ ; \\ t_1, t_2 \in \mathbb{R}\}.$$

We have $\dim \mathcal{O}_{\beta_1,\beta_2,\gamma_1,\gamma_3} = 2$.

(III) For $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$,

$$\mathcal{O}_{\gamma_1,\gamma_2,\gamma_3} = (Ad^*N)l(0, 0, 0, \gamma_1, \gamma_2, \gamma_3) \\ = \{\gamma_1 X_1^* + \gamma_2 X_2^* + \gamma_3 X_3^*\}.$$

We have $\dim \mathcal{O}_{\gamma_1,\gamma_2,\gamma_3} = 0$.

To construct the irreducible unitary representation of N from the coadjoint orbit of $l \in \mathfrak{n}^*$, we should determine its radical $\mathfrak{r}_l$ and maximal subordinate subalgebra $\mathfrak{s}_l$. We define the coadjoint action of the Lie algebra $\mathfrak{n}$ on $l \in \mathfrak{n}^*$ by $((\mathrm{ad}^*X)l)(Y) = l([Y, X])$ for $X, Y \in \mathfrak{n}$.

Definition 4.2. *For $l \in \mathfrak{n}^*$, we define the subalgebra of $\mathfrak{n}$ such that*

$$\mathfrak{r}_l = \{X \in \mathfrak{n} \ ; \ (ad^* X)l = 0\}.$$

We call this subalgebra the radical of $l \in \mathfrak{n}^*$.

If V is a $\mathbb{R}$-vector space with an alternating bilinear form B, its isotropic subspace W is the subspace such that $B(w, w') = 0$ for all $w, w' \in W$. We define the radical of B by $\mathrm{rad}B = \{v \in V \ ; \ B(v, w) = 0, \text{all } w \in V\}$. It is known that any maximal isotropic subspaces of V have codimention $\frac{1}{2}\dim_{\mathbb{R}}(V/\mathrm{rad}B)$.

Definition 4.3. *For $l \in \mathfrak{n}^*$, we can regard $l([X, Y])$ as a bilinear form for $(X, Y) \in \mathfrak{n} \times \mathfrak{n}$. By the antisymmetry of Lie bracket $[X, Y] = -[Y, X]$ $(X, Y \in \mathfrak{n})$, this is an alternating form on $\mathfrak{n} \times \mathfrak{n}$. The subalgebra $\mathfrak{s}_l \subset \mathfrak{n}$ which is isotropic for l and has codimension $\frac{1}{2} dim_{\mathbb{R}}(\mathfrak{n}/\mathfrak{r}_l)$ is called maximal subordinate subalgebra of $\mathfrak{n}$ for l.*

Remark 4.4. *For a nilpotent Lie algebra $\mathfrak{n}$, there exists at least one maximal subordinate subalgebra for any $l \in \mathfrak{n}^*$. Although the radical for l is uniquely determined, maximal subordinate subalgebras are not unique for l.*

Let us construct radicals and maximal subordinate subalgebras for coadjoint orbits (I), (II), (III) which are classified in Proposition 4.1.

The case (I). By the equation (4.2), the coadjoint action of N on $l_{\alpha,\gamma_2} = l(\alpha, 0, 0, 0, \gamma_2, 0)$ is written as follows,

$$\begin{aligned}(\mathrm{Ad}^* \exp(n(z, \cdots, x_3)))&l(\alpha, 0, 0, 0, \gamma_2, 0) \\ &= \{\alpha Z^* + y_1 Y_1^* + y_2 Y_2^* + x_1 X_1^* + (\gamma_2 + \frac{y_1 y_2}{\alpha}) X_2^* + x_2 X_3^*\}.\end{aligned}$$

Hence we can see that

$$\mathfrak{r}_{l_{\alpha,\gamma_2}} = \{\mathbb{R}Z + \mathbb{R}X_2\}.$$

As we noted before, there are some choices of maximal subordinate subalgebra even if it contains the radical $\mathfrak{r}_{l_{\alpha,\gamma_2}}$. Among these choices, we pick up a maximal subordinate subalgebra

$$\mathfrak{s}_{l_{\alpha,\gamma_2}} = \{\mathbb{R}X_2 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}.$$

It is easy to check that this subspace is isotropic and its codimension is equal to $\frac{1}{2}\dim_{\mathbb{R}}(\mathfrak{n}/\mathfrak{r}_{l_{\alpha,\gamma_2}}) = 2$. Also this becomes a subalgebra of $\mathfrak{n}$. We can see that $\mathfrak{s}_{l_{\alpha,\gamma_2}}$ does not depend on the choice of $\alpha \in \mathbb{R}\backslash\{0\}$ and $\gamma_2 \in \mathbb{R}$. Hence we simply write $\mathfrak{s}_{(I)} = \mathfrak{s}_{l_{\alpha,\gamma_2}}$.

The case (II). As well as the case (I), we can see that the radical for $l_{\beta_1,\beta_2,\gamma_1,\gamma_3} = l(0, \beta_1, \beta_2, \gamma_1, 0, \gamma_3)$ is given by

$$\mathfrak{r}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}} = \{\mathbb{R}(\beta_1 X_3 + \beta_2 X_1) + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}.$$

The codimension of maximal subordinate subalgebras are $\frac{1}{2}\dim_{\mathbb{R}}(\mathfrak{n}/\mathfrak{r}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}) = 1$. We recall a fact for the codimension 1 subalgebra of $\mathfrak{n}$.

Proposition 4.5 (cf. Proposition 1.3.4 in [3]). *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{g}_0$ a codimension 1 subalgebra of $\mathfrak{g}$. For $l \in \mathfrak{g}^*$, let $l_0 = l|_{\mathfrak{g}_0}$ be the restriction to $\mathfrak{g}_0$. If the radical of l in $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, any maximal subordinate subalgebra of $\mathfrak{g}_0$ for l_0 is also maximal subordinate subalgebra of $\mathfrak{g}$ for l.*

For any codimension 1 subalgebra $\mathfrak{n}_0$ of $\mathfrak{n}$ containing $\mathfrak{r}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}$, there exist a maximal subordinate subalgebra of $\mathfrak{n}_0$ for $l_{\beta_1,\beta_2,\gamma_1,\gamma_3}|_{\mathfrak{n}_0}$ from Remark 4.4. By this proposition, this is also a maximal subordinate subalgebra of $\mathfrak{n}$ for $l_{(\mathrm{II})}$. By the calculation done above, this maximal subordinate subalgebra should have codimension 1. Hence this is nothing but $\mathfrak{n}_0$. This implies that any codimension 1 subalgebra $\mathfrak{n}_0$ of $\mathfrak{n}$ containing $\mathfrak{r}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}$ is maximal subordinate subalgebra for $l_{\beta_1,\beta_2,\gamma_1,\gamma_3}$. Among these, we pick up a maximal subordinate subalgebra

$$\mathfrak{s}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}} = \{\mathbb{R}X_1 + \mathbb{R}X_3 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}.$$

As in the case (I), the subalgebra $\mathfrak{s}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}$ does not depends on the choice of $\beta_1, \beta_2, \gamma_1, \gamma_3 \in \mathbb{R}$. Hence we simply write $\mathfrak{s}_{(\mathrm{II})} = \mathfrak{s}_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}$.

The case (III). The coadjoint action on $l_{\gamma_1,\gamma_2,\gamma_3} = l(0,0,0,\gamma_1,\gamma_2,\gamma_3)$ is given by

$$(\mathrm{Ad}^* \exp(n(z,\cdots,x_3)))l(0,0,0,\gamma_1,\gamma_2,\gamma_3) = \{\gamma_1 X_1^* + \gamma_2 X_2^* + \gamma_3 X_3^*\}.$$

It is obvious that the radical of $l_{\gamma_1,\gamma_2,\gamma_3}$ is

$$\mathfrak{r}_{l_{\gamma_1,\gamma_2,\gamma_3}} = \mathfrak{n}.$$

Also it is obvious that a maximal subordinate subalgebra of $\gamma_1, \gamma_2, \gamma_3$ is

$$\mathfrak{s}_{\gamma_1,\gamma_2,\gamma_3} = \mathfrak{n}.$$

Let us recall Kirillov's theory for irreducible unitary representations of nilpotent Lie group N. For $l \in \mathfrak{n}^*$, let $\mathfrak{s}_l$ be a maximal subordinate subalgebra for l and let $S_l = \exp \mathfrak{s}_l$. We can extend $l|_{\mathfrak{s}_l} : \mathfrak{s}_l \to \mathbb{R}$ to the map $\chi_l : S_l \to \mathbb{C}^1$ by

$$\chi_l(\exp X) = e^{2\pi i l(X)}, \quad X \in \mathfrak{s}_l.$$

This is a group homomorphism, i.e., an unitary character of S_l because $\mathfrak{s}_l$ is an isotropic subspace for l, i.e., $l([X,Y]) = 0$ for $X, Y \in \mathfrak{s}_l$. We consider a Hilbert space induced from χ_l,

$$\begin{aligned}\mathcal{H}_{\chi_l} =& \{f : N \to \mathbb{C} \text{ measurable } ; \ f(sx) = \chi_l(s)f(x) \text{ for } s \in S_l, x \in N, \\ & \text{and } \int_{S_l \backslash N} |f(x)|^2 \, d\dot{x} < +\infty\},\end{aligned}$$

where $d\dot{x}$ is the right-invariant measure on $S_l\backslash N$. The inner product is defined by

$$\langle f, f' \rangle = \int_{S_l \backslash N} f(x)\overline{f'(x)} \, d\dot{x}.$$

It can be shown that $\mathcal{H}_{\chi_l}$ is complete by this inner product. The action of N on $\mathcal{H}_{\chi_l}$ is given by the right translation. This is the unitary representation by the right-invariance of $d\dot{x}$. This representation is called the representation of N induced from χ_l, and denoted by $L^2\text{-}\mathrm{Ind}_{S_l}^N(\chi_l)$.

Theorem 4.6 (Kirillov [13]). *Take $l \in \mathfrak{n}^*$ and let $\mathfrak{s}_l$ be a maximal subordinate subalgebra of $\mathfrak{n}$ for l^*.*

1. *The induced representation $L^2\text{-}\mathrm{Ind}_{S_l}^N(\chi_l)$ is an irreducible representation of N.*

2. *Let $\mathfrak{s}'_l$ be a maximal subordinate subalgebra of $\mathfrak{n}$ for l and $S'_l = \exp \mathfrak{s}'_l$. Then $L^2\text{-Ind}_{S'_l}^N(\chi_l)$ is unitarily equivalent to $L^2\text{-Ind}_{S_l}^N(\chi_l)$. Hence we may write π_l for $L^2\text{-Ind}_{S_l}^N(\chi_l)$.*

3. *Let $l' \in \mathfrak{n}^*$. Then $\pi_{l'}$ is unitarily equivalent to π_l if and only if $l' \in (\mathrm{Ad}^* N)l$.*

4. *Let π be an irreducible unitary representation of N. Then there exists an $l \in \mathfrak{n}^*$ such that π is unitarily equivalent to π_l.*

This Kirillov's theorem implies that irreducible unitary representations of nilpotent subgroup N of G are equivalent to induced representations $\mathrm{Ind}_{S_l}^N(\chi_l)$ and their equivalent classes only depend on coadjoint orbits of $l \in \mathfrak{n}^*$. We have already classified coadjoint orbits of $l \in \mathfrak{n}^*$ and determined their maximal subordinate subalgebras $\mathfrak{s}_l$. Hence we can obtain equivalent classes of irreducible unitary representations of N.

Proposition 4.7. *We retain the notation as above. The every irreducible unitary representation of N is unitarily equivalent to one of the following representations.*

(I) For $l_{\alpha,\gamma_2} = l(\alpha, 0, 0, 0, \gamma_2, 0) \in \mathfrak{n}^*$ and its maximal subordinate subalgebra $\mathfrak{s}_{(\mathrm{I})} = \{\mathbb{R}X_2 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}$, we define the representation

$$L^2\text{-Ind}_{S_{(I)}}^N \chi_{l_{\alpha,\gamma_2}}.$$

Here $S_{(I)} = \exp \mathfrak{s}_{(I)}$ and $\alpha \in \mathbb{R}\backslash\{0\}, \gamma_2 \in \mathbb{R}$.

(II) For $l_{\beta_1,\beta_2,\gamma_1,\gamma_3} = l(0, \beta_1, \beta_2, \gamma_1, 0, \gamma_3) \in \mathfrak{n}^*$ and its maximal subordinate subalgebra $\mathfrak{s}_{(\mathrm{II})} = \{\mathbb{R}X_1 + \mathbb{R}X_3 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}$, we define the representation

$$L^2\text{-Ind}_{S_{(II)}}^N \chi_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}$$

Here $S_{l_{(II)}} = \exp \mathfrak{s}_{l_{(II)}}$ and $\beta_1, \beta_2, \gamma_1, \gamma_3 \in \mathbb{R}$, $(\beta_1, \beta_2) \neq (0, 0)$.

(III) For $l_{\gamma_1,\gamma_2,\gamma_3} = l(0, 0, 0, \gamma_1, \gamma_2, \gamma_3) \in \mathfrak{n}^*$, we define the unitary character of N,

$$\chi_{l_{\gamma_1,\gamma_2,\gamma_3}}.$$

4.1.2 Conjugacy classes of $C_\eta^\infty(U\backslash G)$

In previous section, we describe the unitary dual of N. The next step is the classification of G-equivalent classes of the following spaces:

$$C^\infty_{\chi_{l_{\alpha,\gamma_2}}}(S_{(I)}\backslash G), \quad \alpha \in \mathbb{R}\backslash\{0\},\ \gamma_2 \in \mathbb{R}, \tag{I}$$

$$C^\infty_{\chi_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}}(S_{(II)}\backslash G), \quad \beta_1, \beta_2 \in \mathbb{R},\ (\beta_1, \beta_2) \neq (0, 0), \tag{II}$$

$$C^\infty_{\chi_{l_{\gamma_1,\gamma_2,\gamma_3}}}(N\backslash G), \quad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}. \tag{III}$$

For $x \in G$, we write the conjugation of $g \in G$ by x as $g^x = xgx^{-1}$. Let H be a closed subgroup of G and π a continuous representation of H on a complete locally convex space E. Then for $x \in N_G(H) = \{g \in G \mid h^g \in H, \text{ for any } h \in H\}$, we can define conjugation of π as $\pi^x(h) = \pi(h^x)$. Then we have the following fact about the induced representation $C^\infty_\pi(H\backslash G) = \{f\colon G \to E \,\text{smooth} \mid f(hg) = \pi(h)f(g),\, g \in G, h \in H\}$ on which G acts by the right translation.

Lemma 4.8. *We retain the notations as above. The map*

$$\begin{array}{ccc} C^\infty_\pi(H\backslash G) & \xrightarrow{\sim} & C^\infty_{\pi^x}(H\backslash G) \\ f(g) & \longmapsto & F(g) = f(xg) \end{array}$$

gives isomorphism as G-modules.

Proof. We see that this map is well-defined. Take $f \in C^\infty_\pi(H\backslash G)$ and define $F(g) = f(xg)$. Then we have

$$\begin{aligned} F(hg) &= f(xhg) \\ &= f(xhx^{-1}xg) \\ &= \pi(xhx^{-1})f(xg) \\ &= \pi^x(h)F(g), \end{aligned}$$

for $h \in H$. Hence $F \in C^\infty_{\pi^x}(H\backslash G)$. Obviously this map is bijective and preserves the action of G. □

Lemma 4.9. *Fix a maximal subordinate subalgebra $\mathfrak{s}_l \subset \mathfrak{n}$ for $l \in \mathfrak{n}^*$. We put $S_l = \exp \mathfrak{s}_l$. Let us define a character $\chi_l\colon S_l \to \mathbb{C}^1$ so that $\chi_l(\exp X) = e^{2\pi\sqrt{-1}l(X)}$ for $X \in \mathfrak{s}_l$. Then the character χ_l is invariant by the conjugation by S_l, i.e.,*

$$\chi_l^x(s) = \chi_l(s)$$

for $s, x \in S_l$.

Proof. Take an element of $x \in S_l$. Then there exists $Z_x \in \mathfrak{s}_l$ such that $x = \exp Z_x$. By the Campbell-Hausdorff formula, for any $X \in \mathfrak{s}_l$ we have

$$\begin{aligned} x(\exp X)x^{-1} &= \exp Z_x \exp X \exp(-Z_x) \\ &\equiv \exp(Z_x + X - Z_x + Y) \\ &= \exp(X + Y) \end{aligned}$$

for some element $Y \in [\mathfrak{s}_l, \mathfrak{s}_l] = \{[V, W] \mid V, W \in \mathfrak{s}_l\}$. If we recall that $\mathfrak{s}_l$ is a maximal subordinate subalgebra for l, we have $l(Y) = 0$ for $Y \in [\mathfrak{s}_l, \mathfrak{s}_l]$. Thus we have

$$\begin{aligned} \chi_l^x(\exp X) &= \chi_l(x(\exp X)x^{-1}) \\ &= e^{2\pi\sqrt{-1}l(X)} \\ &= \chi_l(\exp X). \end{aligned}$$

for any $X \in \mathfrak{s}_l$. □

By these lemmas, we have the following classifications.

Proposition 4.10. *Case(I). For $\alpha \in \mathbb{R}\backslash\{0\}$ and $\gamma_2 \in \mathbb{R}$, we have*

$$C^\infty_{\chi_{l_{\alpha,\gamma_2}}}(S_{(I)}\backslash G) \cong \begin{cases} C^\infty_{\chi_{l(0,1,1,0,0,0)}}(S_{(I)}\backslash G) & \text{if } \gamma_2 \neq 0, \qquad (I_1) \\ C^\infty_{\chi_{l(0,0,1,0,0,0)}}(S_{(I)}\backslash G) & \text{if } \gamma_2 = 0 \qquad (I_2) \end{cases}$$

Case(II). We take $\beta_1, \beta_2, \gamma_1, \gamma_3 \in \mathbb{R}$ and assume $\beta_1 \neq 0$, or $\beta_2 \neq 0$. Then we have

$$C^{\infty}_{\chi_{l_{\beta_1,\beta_2,\gamma_1,\gamma_3}}}(S_{(II)}\backslash G) \cong \begin{cases} C^{\infty}_{\chi_{l(0,0,1,1,0,0)}}(S_{(II)}\backslash G) & \text{if } (\beta_1,\gamma_1)\cdot(\gamma_3,\beta_2) \neq 0, & (II_1) \\ C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash G) & \begin{array}{c}\text{if } (\beta_1,\gamma_1)\cdot(\gamma_3,\beta_2) = 0 \\ \text{and } \beta_1 \neq 0, \beta_2 \neq 0\end{array}, & (II_2) \\ C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash G) & \begin{array}{c}\text{if } (\beta_1,\gamma_1)\cdot(\gamma_3,\beta_2) = 0 \\ \text{and } \beta_1 \neq 0, \beta_2 = 0\end{array}, & (II_3) \\ C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(S_{(II)}\backslash G) & \begin{array}{c}\text{if } (\beta_1,\gamma_1)\cdot(\gamma_3,\beta_2) = 0 \\ \text{and } \beta_1 = 0, \beta_2 \neq 0\end{array}, & (II_4) \end{cases}$$

where $(a,b)\cdot(c,d) = ac + bd$ for $a,b,c,d \in \mathbb{R}$ is a natural inner product in $\mathbb{R}^2$.
Case(III). For $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, we have

$$C^{\infty}_{\chi_{l_{\gamma_1,\gamma_2,\gamma_3}}}(N\backslash G) \cong \begin{cases} C^{\infty}_{\chi_{l(0,0,0,1,1,1)}}(N\backslash G) & \text{if } \gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_3 \neq 0, & (III_1) \\ C^{\infty}_{\chi_{l(0,0,0,1,1,0)}}(N\backslash G) & \text{if } \gamma_1 \neq 0, \gamma_2 \neq 0, \gamma_3 = 0, & (III_2) \\ C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(N\backslash G) & \text{if } \gamma_1 \neq 0, \gamma_2 = 0, \gamma_3 \neq 0, & (III_3) \\ C^{\infty}_{\chi_{l(0,0,0,0,1,1)}}(N\backslash G) & \text{if } \gamma_1 = 0, \gamma_2 \neq 0, \gamma_3 \neq 0, & (III_4) \\ C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(N\backslash G) & \text{if } \gamma_1 \neq 0, \gamma_2 = 0, \gamma_3 = 0, & (III_5) \\ C^{\infty}_{\chi_{l(0,0,0,0,1,0)}}(N\backslash G) & \text{if } \gamma_1 = 0, \gamma_2 \neq 0, \gamma_3 = 0, & (III_6) \\ C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(N\backslash G) & \text{if } \gamma_1 = 0, \gamma_2 = 0, \gamma_3 \neq 0, & (III_7) \\ C^{\infty}_{\chi_{l(0,0,0,0,0,0)}}(N\backslash G) & \text{if } \gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0, & (III_8) \end{cases}$$

Proof. The case (I). The normalizer $N_G(S_{(\mathrm{I})})$ of $S_{(\mathrm{I})}$ in G is written as the semi-direct product $L_I \ltimes S_I$ where

$$L_I = \left\{ \left(\begin{array}{c|c} A & 0_2 \\ \hline 0_2 & B \end{array} \right) \mid A, B \in GL(2,\mathbb{R}) \right\}.$$

Here $0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M(2,\mathbb{R})$. We define the action of $N_G(S_{(\mathrm{I})})$ on $\chi_{(\alpha,\gamma_2)}$ as $(x\cdot\chi_{(\alpha,\gamma_2)})(s) = \chi_{(\alpha,\gamma_2)}(s^{x^{-1}})$ for $x \in N_G(S_{(\mathrm{I})})$ and $s \in S_{(\mathrm{I})}$. Then by lemma 4.8, if $\chi_{l_{\alpha,\gamma_2}}$ and $\chi_{l_{\alpha',\gamma_2'}}$ are in the same $N_G(S_{(\mathrm{I})})$-orbit, the spaces $C^{\infty}_{\chi_{l_{\alpha,\gamma_2}}}(S_I\backslash G)$ and $C^{\infty}_{\chi_{l_{\alpha',\gamma_2'}}}(S_I\backslash G)$ are G-equivalent. Also by Lemma 4.9, we only need to classify the $L_{(I)}$-orbits of $\chi_{l_{\alpha,\gamma_2}}$ for $\alpha \in \mathbb{R}\backslash\{0\}$ and $\gamma_2 \in \mathbb{R}$. Then it is easy to see that $l_{(\alpha,0,0,0,\gamma_2,0)} \in \mathrm{Ad}^*(N_G(S_{(\mathrm{I})}))(l_{(0,1,0,0,0,1)})$ if $\gamma_2 \neq 0$ and $l_{(\alpha,0,0,0,\gamma_2,0)} \in \mathrm{Ad}^*(N_G(S_{(\mathrm{I})}))(l_{(0,1,0,0,0,0)})$ if $\gamma_2 = 0$.

The case(II). The normalizer of $S_{(\mathrm{II})}$ in G is written as the semi-direct product $L_{(II)} \ltimes S_{(II)}$ where

$$L_{(II)} = \left\{ n_{(a,b,A)} = \begin{pmatrix} a & \mathbf{0}_2 & 0 \\ {}^t\mathbf{0}_2 & A & {}^t\mathbf{0}_2 \\ 0 & \mathbf{0}_2 & b \end{pmatrix} \in G \mid a, b \in \mathbb{R}^\times, A \in GL(2,\mathbb{R}) \right\}.$$

Here $\mathbf{0}_2 = (0,0)$ and ${}^t\mathbf{0}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then there are following $L_{(II)}$-orbits of $\chi_{l_{\beta_1,\beta_2,\gamma_1,\gamma_2}}$ for $\beta_1, \beta_2 \in \mathbb{R}$ ($\beta_1 \neq$, or $\beta_2 \neq 0$) and $\gamma_1, \gamma_2 \in \mathbb{R}$,

$$\mathcal{O}_1 = \{\chi_{l(0,0,v_1,v_2,w_1,0,w_2)} \mid v_1w_1 + v_2w_2 \neq 0\}$$

$$\mathcal{O}_2 = \{\chi_{l(0,0,v_1,v_2,w_1,0,w_2)} \mid (v_1,v_2) \neq (0,0),\ (w_1,w_2) \neq 0 \text{ and } v_1w_1 + v_2w_2 = 0\},$$
$$\mathcal{O}_3 = \{\chi_{l(0,0,v_1,v_2,w_1,0,w_2)} \mid (v_1,v_2) \neq (0,0),\ (w_1,w_2) = 0 \text{ and } v_1w_1 + v_2w_2 = 0\},$$
$$\mathcal{O}_4 = \{\chi_{l(0,0,v_1,v_2,w_1,0,w_2)} \mid (v_1,v_2) = (0,0),\ (w_1,w_2) \neq 0 \text{ and } v_1w_1 + v_2w_2 = 0\}.$$

The case(III). The normalizer of N in G is written as the semi-direct product $L_{(III)} \ltimes S_{(III)}$ where

$$L_{(III)} = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \mid a_1, \dots, a_4 \in \mathbb{R}^\times \right\}.$$

Then the lemma is easily follows. □

4.2 Differential operators on the generalized Whittaker models

Let U be a closed subgroup of N and χ a character of U. By Theorem 3.7, the space of the generalized Whittaker model is isomorphic to the subspace of

$$C^\infty_\chi(U\backslash G/K) = \{f \in C^\infty(G) \mid f(ugk) = \chi(u)f(g) \text{ for } (u,g,k) \in U \times G \times K\}.$$

Lemma 4.11. *We retain the above notations. There exists a linear bijection*

$$\Xi\colon C^\infty_\chi(U\backslash G/K) \xrightarrow{\sim} C^\infty(U\backslash N \times A).$$

Proof. Because N is a nilpotent group and U is its closed subgroup, there is a smooth cross section $\theta\colon U\backslash N \to N$ with the smooth splitting of $n \in N$ so that $n = u(n)s(n)$ for $u(n) \in U$ and $s(n) \in \theta(U\backslash N)$ (cf. Theorem 1.2.12 in [3]). This smooth cross section gives us a linear mapping

$$\begin{array}{rccc} \Xi\colon & C^\infty_\chi(U\backslash G/K) & \xrightarrow{\sim} & C^\infty(U\backslash N \times A) \\ & f & \longmapsto & \Xi(f)(x,a) = f(\theta(x)a), \end{array}$$

for $x \in U\backslash N$ and $a \in A$. Take an element $\phi \in C^\infty(U\backslash N \times A)$. If we define an element of $f_\phi \in C^\infty_\chi(U\backslash G/K)$ by

$$f_\phi(usak) = \chi(u)\phi(sa)$$

for $u \in U, s \in \theta(U\backslash N), a \in A$ and $k \in K$. Since $G \cong U \times U\backslash N \times A \times K$, this is well defined. We denote this map Π. Then it is easy to see that $\Pi \circ \Xi = \mathrm{id}_{C^\infty_\chi(U\backslash G/K)}$ and $\Xi \circ \Pi = \mathrm{id}_{C^\infty(U\backslash N \times A)}$. Hence Ξ is bijective. □

We define the action of $U(\mathfrak{g})$ on $C^\infty(U\backslash N \times A)$ by $X \cdot \Xi(f) = \Xi(R_X f)$ for $X \in U(\mathfrak{g})$ and $f \in C^\infty_\chi(U\backslash G/K)$. In this section, we give the explicit expression of the action of $U(\mathfrak{g})$ on $C^\infty(U\backslash N \times A)$.

According to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, it suffices to see the action of $\mathfrak{n}, \mathfrak{a}$ and $\mathfrak{k}$ respectively. We can see that $E_{ii} \in \mathfrak{a}, i = 1, \dots, 4$ acts on $C^\infty(U\backslash N \times A)$ as $\vartheta_{a_i} = a_i \frac{\partial}{\partial a_i}, i = 1, \dots, 4$ if we denote the elements of A by $a = \mathrm{diag}(a_1, \dots, a_4)$. By the right K-invariance of $C^\infty_\chi(U\backslash G/K)$, the elements in $\mathfrak{k}$ acts trivially. Hence we have the following symmetric relation among the generators of the annihilator ideal $I_k(\lambda)$.

Lemma 4.12. *For $F \in C^\infty(G/K)$, we have*

$$((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ij} F = ((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ji} F,$$

where $1 \le i, j \le 4$, and $k = 1, 2$.

Proof. Take elements $(E_{ij} - E_{ji})$, $1 \le i < j \le 4$ as the generators of $\mathfrak{k}$. Then we have $(E_{ij} - E_{ji})F = 0$, $1 \le i < j \le 4$ for $F \in C^\infty(G/K)$, i.e., $E_{ij}F = E_{ji}F$, $1 \le i < j \le 4$. This implies that

$$\begin{aligned}
&((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ij} F \\
&= (\sum_{l=1}^{4} E_{il}E_{lj} - (\lambda_1 + \lambda_2 + k)E_{ij} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= (\sum_{l=1}^{4} E_{il}E_{jl} - (\lambda_1 + \lambda_2 + k)E_{ji} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= (\sum_{l=1}^{4} (E_{jl}E_{il} - [E_{jl}, E_{il}]) - (\lambda_1 + \lambda_2 + k)E_{ij} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= (\sum_{l=1}^{4} (E_{jl}E_{li} - (\delta_{li}E_{jl} - \delta_{jl}E_{il})) - (\lambda_1 + \lambda_2 + k)E_{ji} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= (\sum_{l=1}^{4} E_{jl}E_{li} - (E_{ji} - E_{ij}) - (\lambda_1 + \lambda_2 + k)E_{ji} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= (\sum_{l=1}^{4} E_{jl}E_{li} - (\lambda_1 + \lambda_2 + k)E_{ji} + \lambda_1(\lambda_2 + k)\delta_{ij})F \\
&= ((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ji} F.
\end{aligned}$$

This is the required equation. □

We give more precise expressions of $((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ij}$ mod $U(\mathfrak{g})\mathfrak{k}$ for $k = 1, 2$ and $1 \le i < j \le 4$ below.

Lemma 4.13. *Representatives $((\mathbb{E} - \lambda_1)(\mathbb{E} - \lambda_2 - k))_{ij}$ modulo $U(\mathfrak{g})\mathfrak{k}$ ($k = 1, 2$ and $1 \le i < j \le 4$) are written as follows,*

$$E_{11}^2 + E_{21}^2 + E_{31}^2 + E_{41}^2 - (\lambda_1 + \lambda_2 + k - 3)E_{11} - (E_2 + E_3 + E_4) + \lambda_1(\lambda_2 + k), \quad ((i,j) = (1,1))$$

$$E_{21}(E_{11} + E_{22} - (\lambda_1 + \lambda_2 + k - 3)) + E_{32}E_{31} + E_{42}E_{41}, \quad ((i,j) = (1,2))$$

$$E_{31}(E_{11} + E_{33} - (\lambda_1 + \lambda_2 + k - 2)) + E_{32}E_{21} + E_{43}E_{41}, \quad ((i,j) = (1,3))$$

$$E_{41}(E_{11} + E_{44} - (\lambda_1 + \lambda_2 + k - 2)) + E_{42}E_{21} + E_{31}E_{43}, \quad ((i,j) = (1,4))$$

$$E_{22}^2 - (\lambda_1 + \lambda_2 + k - 2)E_{22} + E_{21}^2 + E_{32}^2 + E_{42}^2 - (E_{33} + E_{44}) + \lambda_1(\lambda_2 + k), \quad ((i,j) = (2,2))$$

$$E_{32}(E_{22} + E_{33} - (\lambda_1 + \lambda_2 + k - 2)) + E_{21}E_{31} + E_{43}E_{42}, \quad ((i,j) = (2,3))$$

$$E_{42}(E_{22} + E_{44} - (\lambda_1 + \lambda_2 + k - 2)) + E_{21}E_{41} + E_{32}E_{43}, \quad ((i,j) = (2,4))$$

$$E_{33}^2 - (\lambda_1 + \lambda_2 + k - 1)E_{33} + E_{31}^2 + E_{32}^2 + E_{43}^2 - E_{44} + \lambda_1(\lambda_2 + k), \quad ((i,j) = (3,3))$$

$$E_{43}(E_{33} + E_{44} - (\lambda_1 + \lambda_2 + k - 1)) + E_{31}E_{41} + E_{32}E_{42}, \quad ((i,j) = (3,4))$$

$$E_{44}^2 - (\lambda_1 + \lambda_2 + k)E_{44} + E_{41}^2 + E_{42}^2 + E_{43}^2. \quad ((i,j) = (4,4))$$

Proof. If we note that $(E_{ij} - E_{ji})$ $1 \le i < j \le 4$ are the generators of $\mathfrak{k}$, this lemma can be obtained by the direct computations. □

Along the classification obtained in Proposition 4.10, we express the action of $\mathfrak{n}$.

The case (I). We consider the space

$$C^\infty_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K)$$

Here $\varepsilon = 1$ (resp. $= 0$) corresponds to the case (I_1) (resp. (I_2)) classified in Proposition 4.10. If we notice that $\mathfrak{s}_{(\mathrm{I})} = \{\mathbb{R}X_2 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}$ is not only a subalgebra of $\mathfrak{n}$ but also an ideal of $\mathfrak{n}$, then $\mathfrak{s}_{(\mathrm{I})}\backslash\mathfrak{n} \cong \{\mathbb{R}X_1 + \mathbb{R}X_3\}$ can be seen as a subalgebra of $\mathfrak{n}$. Hence $S_{(\mathrm{I})}\backslash N$ is isomorphic to the subgroup $\{\exp(uX_1 + vX_3) \mid u, v \in \mathbb{R}\}$ of N. This isomorphism gives a smooth cross section $\theta_{(\mathrm{I})}\colon S_{(\mathrm{I})}\backslash N \to N$. Then we have the linear isomorphism

$$\Xi_{(\mathrm{I})}\colon C^\infty_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K) \xrightarrow{\sim} C^\infty(S_{(\mathrm{I})}\backslash N \times A)$$

by Lemma 4.11. We introduce a coordinate system on $S_{(\mathrm{I})}\backslash N \times A$ as follows,

$$\begin{array}{ccc} \mathbb{R}^2 \times (\mathbb{R}_{>0})^4 & \xrightarrow{\sim} & S_{(\mathrm{I})}\backslash N \times A \\ (u,v) \times (a_1,a_2,a_3,a_4) & \longmapsto & \exp(uX_1 + vX_3) \times \mathrm{diag}(a_1,a_2,a_3,a_4) \end{array}$$

Proposition 4.14. *We regard the space $C^\infty(S_{(I)}\backslash N \times A)$ as the image of the space $C^\infty_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(I)}\backslash G/K)$ by the mapping $\Xi_{(I)}$ for each $\varepsilon = 0, 1$. Then $\mathfrak{n}$ acts on $C^\infty(S_{(I)}\backslash N \times A)$ as follows,*

$$E_{21}F = \frac{a_2}{a_1}\frac{\partial}{\partial u}F, \qquad E_{31}F = \varepsilon 2\pi\sqrt{-1}\frac{a_3}{a_1}, F$$

$$E_{41}F = 0, \qquad E_{32}F = 2\pi\sqrt{-1}\frac{a_3}{a_2}(v - \varepsilon u)F,$$

$$E_{42}F = 2\pi\sqrt{-1}\frac{a_4}{a_2}F, \qquad E_{43}F = \frac{a_4}{a_3}\frac{\partial}{\partial v},$$

for $F \in C^\infty(S_{(I)}\backslash N \times A)$.

Proof. For $F \in C^\infty(S_{(\mathrm{I})}\backslash N \times A)$, there exists a $f \in C^\infty_{\chi_{l(0,\varepsilon,1,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K)$ such that $F(u,v;a) = \Xi_{(\mathrm{I})}(f) = f(\exp(uX_1 + vX_3)a)$ for $u, v \in \mathbb{R}$ and $a \in A$. Hence for $E_{ij}(1 \leq j < i \leq 4)$ we have

$$\begin{aligned}(E_{ij}F)(u,v;a) &= \Xi_{(\mathrm{I})}(R_{E_{ij}}f) \\ &= \frac{d}{dt}f(\exp(uX_1 + vX_3)a\exp(tE_{ij}))|_{t=0} \\ &= \frac{d}{dt}f(\exp(uX_1 + vX_3)\exp(t\mathrm{Ad}(a)E_{ij})a)|_{t=0} \\ &= \frac{a_i}{a_j}\frac{d}{dt}f(\exp(uX_1 + vX_3)\exp(tE_{ij})a)|_{t=0}.\end{aligned} \tag{4.3}$$

By the direct computation, we have

$$\begin{aligned}&\exp(uX_1 + vX_3)\cdot \exp n(z,\cdots,x_3) \\ &\qquad = \exp n(z', y_1', y_2', 0, x_2, 0)\cdot \exp((u+x_1)X_1 + (v+x_3)X_3),\end{aligned}$$

where

$$\begin{aligned}z' &= z + vy_1 - uy_2 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_1y_2 - uvx_2 - \frac{1}{2}vx_1x_2 - \frac{1}{2}ux_2x_3 - \frac{1}{3}x_1x_2x_3, \\ y_1' &= y_1 - ux_2 - \frac{x_1x_2}{2}, \\ y_2' &= y_2 + vx_2 + \frac{x_2x_3}{2}.\end{aligned}$$

Hence we have

$$\begin{aligned}&f(\exp(uX_1 + vX_3)\exp n(z,\cdots,x_3)) \\ &= f(\exp n(z', y_1', y_2', 0, x_2, 0)\exp((u+x_1)X_1 + (v+x_3)X_3)) \\ &= \chi_{l(0,\varepsilon,1,0,0,0)}(\exp n(z', y_1', y_2', 0, x_2, 0)) f(\exp((u+x_1)X_1 + (v+x_3)X_3)) \\ &= e^{2\pi\sqrt{-1}(\varepsilon y_1' + y_2')} f(\exp((u+x_1)X_1 + (v+x_3)X_3)).\end{aligned} \tag{4.4}$$

Combining formulas (4.3) and (4.4), we have the proposition. □

The case (II). We consider the space

$$C^\infty_{\chi_{l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}}(S_{(\mathrm{II})}\backslash G/K).$$

Here each $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ corresponds to the case (II_i) $(i = 1, \ldots, 4)$ in Proposition 4.10 as follows,

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \begin{cases}(1,1,0) & \text{if the case } (\mathrm{II}_1), \\ (0,1,1) & \text{if the case } (\mathrm{II}_2), \\ (0,1,0) & \text{if the case } (\mathrm{II}_3), \\ (0,0,1) & \text{if the case } (\mathrm{II}_4).\end{cases}$$

The subalgebra $\mathfrak{s}_{(\mathrm{II})} = \{\mathbb{R}X_1 + \mathbb{R}X_3 + \mathbb{R}Y_1 + \mathbb{R}Y_2 + \mathbb{R}Z\}$ is a codimension 1 subalgebra of $\mathfrak{n}$. Because any codimension 1 subalgebra of a nilpotent Lie algebra becomes an ideal (cf. Lemma 1.1.8 in [3]), the subalgebra $\mathfrak{s}_{(\mathrm{II})}$ is an ideal of $\mathfrak{n}$. By the similar argument as in the case (I), the homogeneous space $S_{(\mathrm{II})}\backslash N$ is isomorphic to the subgroup $\{\exp(uX_2) \mid u \in \mathbb{R}\}$ of N. This isomorphism gives a smooth section $\theta_{(\mathrm{II})}: S_{(\mathrm{II})}\backslash N \to N$. Then we have a linear bijection

$$\Xi_{(\mathrm{II})}: C^\infty_{\chi_l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}(S_{(\mathrm{II})}\backslash G/K) \xrightarrow{\sim} C^\infty(S_{(\mathrm{II})}\backslash N \times A).$$

We introduce a coordinate system on $S_{(\mathrm{II})}\backslash N \times A$ as follows,

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R}_{>0})^4 & \xrightarrow{\sim} & S_{(\mathrm{II})}\backslash N \times A \\ u \times (a_1, a_2, a_3, a_4) & \longmapsto & \exp(uX_2) \times \mathrm{diag}(a_1, a_2, a_3, a_4) \end{array}$$

Then we can write down the action of $\mathfrak{n}$ on $C^\infty(S_{(\mathrm{II})}\backslash N \times A)$.

Proposition 4.15. *We regard the space $C^\infty(S_{(II)}\backslash N \times A)$ as the image of the space $C^\infty_{\chi_l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}(S_{(II)}\backslash G/K)$ by the mapping $\Xi_{(II)}$. Then $\mathfrak{n}$ acts on $C^\infty(S_{(II)}\backslash N \times A)$ as follows,*

$$E_{21}F = (2\pi\sqrt{-1}(\frac{a_2}{a_1})\varepsilon_2)F, \qquad E_{31}F = 0,$$

$$E_{41}F = 0, \qquad E_{32}F = \frac{a_3}{a_2}\frac{\partial}{\partial u}F,$$

$$E_{42}F = (2\pi\sqrt{-1}(\frac{a_4}{a_2})\varepsilon_1)F, \qquad E_{43}F = (2\pi\sqrt{-1}(\frac{a_4}{a_3})(\varepsilon_3 - \varepsilon_1 u))F.$$

Here $F \in C^\infty(S_{(II)}\backslash N \times A)$ and

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} (1,1,0) & \text{if the case } (II_1), \\ (0,1,1) & \text{if the case } (II_2), \\ (0,1,0) & \text{if the case } (II_3), \\ (0,0,1) & \text{if the case } (II_4). \end{cases}$$

Proof. The proposition can be obtained in the same way as the case (I) by the formula,

$$\exp(uX_2) \cdot \exp n(z, \cdots, x_3) = \exp(n(z', y_1', y_2', x_1, 0, x_3)) \cdot \exp(u + x_2),$$

where

$$z' = z + \frac{1}{6}x_1x_2x_3$$
$$y_1' = y_1 + x_1u + \frac{1}{2}x_1x_2$$
$$y_2' = y_2 - x_3u - \frac{1}{2}x_2x_3.$$

□

The case (III). We consider the space

$$C^\infty_{\chi_l(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}(N\backslash G/K)$$

where

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{cases}(1,1,1) & \text{if the case }(\mathrm{III}_1),\\ (1,1,0) & \text{if the case }(\mathrm{III}_2),\\ (1,0,1) & \text{if the case }(\mathrm{III}_3),\\ (0,1,1) & \text{if the case }(\mathrm{III}_4),\\ (1,0,0) & \text{if the case }(\mathrm{III}_5),\\ (0,1,0) & \text{if the case }(\mathrm{III}_6),\\ (0,0,1) & \text{if the case }(\mathrm{III}_7),\\ (0,0,0) & \text{if the case }(\mathrm{III}_8).\end{cases}$$

By the Iwasawa decomposition, we have the linear bijection

$$\Xi_{(\mathrm{III})}: C^{\infty}_{\chi_{l(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(N\backslash G/K)\ni f\mapsto f|_A\in C^{\infty}(A).$$

by the restriction to A. Then we have the following proposition.

Proposition 4.16. *Let us consider the space $C^{\infty}(A)$ as image of the space $C^{\infty}_{\chi_{l(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(N\backslash G/K)$ by the mapping $\Xi_{(III)}$. Then $\mathfrak{n}$ acts on $C^{\infty}(A)$ as follows,*

$$E_{21}F=(2\pi\sqrt{-1}(\frac{a_2}{a_1})\varepsilon_1)F, \qquad E_{31}F=0,$$

$$E_{41}F=0, \qquad E_{32}F=(2\pi\sqrt{-1}(\frac{a_3}{a_2})\varepsilon_2)F,$$

$$E_{42}F=0, \qquad E_{43}F=(2\pi\sqrt{-1}(\frac{a_4}{a_3})\varepsilon_3)F.$$

Here $F\in C^{\infty}(A)$ and

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{cases}(1,1,1) & \textit{if the case }(III_1),\\ (1,1,0) & \textit{if the case }(III_2),\\ (1,0,1) & \textit{if the case }(III_3),\\ (0,1,1) & \textit{if the case }(III_4),\\ (1,0,0) & \textit{if the case }(III_5),\\ (0,1,0) & \textit{if the case }(III_6),\\ (0,0,1) & \textit{if the case }(III_7),\\ (0,0,0) & \textit{if the case }(III_8).\end{cases}$$

Proof. It is obvious from the formula,

$$(E_{ij}F)(a)=\frac{d}{dt}F(a\exp(tE_{ij}))|_{t=0}=\frac{d}{dt}F(\exp(t\mathrm{Ad}(a)E_{ij})a)|_{t=0}.$$

for $1\le i\ne j\le 4$. □

4.3 Generalized Whittaker models of $GL(4,\mathbb{R})$

After these preparations given in the preceeding sections, we can investigate explicit structures of the generalized Whittaker models. More precisely to say, we give the dimensions of these spaces and their basis in terms of the hypergeometric functions one and two variables.

4.3.1 The embeddings into the spaces (I)

We give the explicit structures of the embeddings of the Harish-Chandra modules $X_{k,\lambda}$ ($k = 1, 2$) into the spaces (I_1) and (I_2) which are classified in Proposition 4.10. These spaces are isomorphic to $C^{\infty}_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K\,;\, I_k(\lambda))$ by Theorem 3.7. We consider the image of $C^{\infty}_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K\,;\, I_k(\lambda))$ by the mapping $\Xi_{(\mathrm{I})}$ defined in Section 4.2. Here $\varepsilon = 1$ (resp. $= 0$) for (I_1) (resp. (I_2)). Hence our purpose is to investigate these spaces

$$C^{\infty}_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash N \times A;\, I_k(\lambda)) = \Xi_{(I)}(C^{\infty}_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(\mathrm{I})}\backslash G/K\,;\, I_k(\lambda))).$$

Proposition 4.17. *For $\varepsilon = 0, 1$, we consider the spaces $C^{\infty}_{\chi_{l(0,1,\varepsilon,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_k(\lambda))$. Then these are the solution spaces of the following systems of the differential equations on $C^{\infty}(S_{(I)}\backslash N \times A)$.*

$$\begin{aligned}[\vartheta_{a_1}^2 &-(\lambda_1+\lambda_2+k-3)\vartheta_{a_1} + \lambda_1(\lambda_2+k) + (\frac{a_2}{a_1})^2\frac{\partial^2}{\partial u^2} \\ &+ \varepsilon(\frac{a_3}{a_1})^2(2\pi\sqrt{-1})^2 - (\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4}) + \lambda_1(\lambda_2+k)]\phi = 0,\end{aligned} \tag{4.5}$$

$$[\frac{\partial}{\partial u}(\vartheta_{a_1}+\vartheta_{a_2}-(\lambda_1+\lambda_2+k-3)) + \varepsilon(\frac{a_3}{a_2})^2(2\pi\sqrt{-1})^2(v-\varepsilon u)]\phi = 0, \tag{4.6}$$

$$[\varepsilon(\vartheta_{a_1}+\vartheta_{a_3}-(\lambda_1+\lambda_2+k-2)) + (v-\varepsilon u)\frac{\partial}{\partial u}]\phi = 0, \tag{4.7}$$

$$[\frac{\partial}{\partial u} + \varepsilon\frac{\partial}{\partial v}]\phi = 0, \tag{4.8}$$

$$\begin{aligned}[\vartheta_{a_2}^2 &- (\lambda_1+\lambda_2+k-2)\vartheta_{a_2} + (\frac{a_2}{a_1})^2\frac{\partial^2}{\partial u^2} + (\frac{a_3}{a_2})^2(2\pi\sqrt{-1})^2(v-\varepsilon u)^2 \\ &+(\frac{a_4}{a_2})^2(2\pi\sqrt{-1})^2 - (\vartheta_{a_3}+\vartheta_{a_4}) + \lambda_1(\lambda_2+k)]\phi = 0,\end{aligned} \tag{4.9}$$

$$[(v-\varepsilon u)(\vartheta_{a_2}+\vartheta_{a_3}-(\lambda_1+\lambda_2+k-2)) + \varepsilon(\frac{a_2}{a_1})^2\frac{\partial}{\partial u} + (\frac{a_4}{a_3})^2\frac{\partial}{\partial v}]\phi = 0, \tag{4.10}$$

$$[(\vartheta_{a_2}+\vartheta_{a_4}-(\lambda_1+\lambda_2+k-2)) + (v-\varepsilon u)\frac{\partial}{\partial v}]\phi = 0, \tag{4.11}$$

$$\begin{aligned}[\vartheta_{a_3}^2 &-(\lambda_1+\lambda_2+k-1)\vartheta_{a_3} + \varepsilon(\frac{a_3}{a_1})^2(2\pi\sqrt{-1})^2 \\ &+ (\frac{a_3}{a_2})^2(2\pi\sqrt{-1})^2(v-\varepsilon u)^2 + (\frac{a_4}{a_3})^2\frac{\partial^2}{\partial v^2} - \vartheta_{a_4} + \lambda_1(\lambda_2+k)]\phi = 0,\end{aligned} \tag{4.12}$$

$$[\frac{\partial}{\partial v}(\vartheta_{a_3}+\vartheta_{a_4}-(\lambda_1+\lambda_2+k-1)) + (\frac{a_3}{a_2})^2(2\pi\sqrt{-1})^2(v-\varepsilon u)]\phi = 0, \tag{4.13}$$

$$[\vartheta_{a_4}^2 - (\lambda_1+\lambda_2+k)\vartheta_{a_4} + (\frac{a_4}{a_2})^2(2\pi\sqrt{-1})^2 + (\frac{a_4}{a_3})^2\frac{\partial^2}{\partial v^2} + \lambda_1(\lambda_2+k)]\phi = 0, \tag{4.14}$$

$$[(\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4} - k\lambda_1 - (4-k)\lambda_2)]\phi = 0. \tag{4.15}$$

Here $\phi \in C^{\infty}(S_{(I)}\backslash N \times A)$.

Proof. Recall that $I_k(\lambda)$ is written as (4.1). Then these differential equations immediately follows from Lemma 4.12, Proposition 4.13 and Proposition 4.14. □

We solve these systems of the differential equations.

(i) *The case* $\varepsilon = 1$.

We investigate the space $C^{\infty}_{\chi_{l(0,1,1,0,0,0)}}(S_{(\mathrm{I})}\backslash N \times A;\, I_k(\lambda))$, i.e, the case $\varepsilon = 1$.

We consider the case $k = 1$, i.e., the embedding of the Harish-Chandra module $X_{1,\lambda}$.

Proposition 4.18. *For the case $\varepsilon = 1$ and $k = 1$, the solution space of the system of the differential equations defined in Proposition 4.17 is $\{0\}$.*

Proof. The equations (4.7) and (4.11) give us the equation,

$$[\vartheta_{a_1} + \vartheta_{a_2} + \vartheta_{a_3} + \vartheta_{a_4} - 2\lambda_1 - 2\lambda_2 + 2 + 2(v-u)(\frac{\partial}{\partial u} + \frac{\partial}{\partial v})]\phi = 0.$$

By the equation (4.8), the term $\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ can be eliminated. Hence the only remaining term is

$$[\vartheta_{a_1} + \vartheta_{a_2} + \vartheta_{a_3} + \vartheta_{a_4} - 2\lambda_1 - 2\lambda_2 + 2]\phi = 0.$$

However if we compare this with the equation (4.15),

$$[\vartheta_{a_1} + \vartheta_{a_2} + \vartheta_{a_3} + \vartheta_{a_4} - \lambda_1 - 3\lambda_2]\phi = 0,$$

we can conclude the solution space of these equations must be $\{0\}$ because we assume $\lambda_1 - \lambda_2 \notin \mathbb{Z}$. □

Next, we consider the case $k = 2$, i.e., the embedding of the Harish-Chandra module $X_{2,\lambda}$. We introduce a new coordinate system below,

$$\begin{aligned} x_1 &= a_1 a_2 a_3 a_4, \\ x_2 &= (\pi\sqrt{-1})^2 \left((\frac{a_3}{a_2})^2 (v-u)^2 + (\frac{a_4}{a_2})^2 + (\frac{a_3}{a_1})^2 \right), \\ x_3 &= \left(\frac{a_1 a_3}{a_2 a_4}(v-u)^2 + \frac{a_2 a_3}{a_1 a_4} + \frac{a_1 a_4}{a_2 a_3} \right)^{-2}, \\ x_4 &= \frac{a_1 a_3}{a_2 a_4}, \\ x_5 &= \frac{a_1 a_4}{a_2 a_3}, \\ x_6 &= u. \end{aligned} \tag{4.16}$$

Proposition 4.19. *Let $\varepsilon = 1$ and $k = 2$. We consider the system of the differential equations in Proposition 4.17. By adding and substituting each differential equations and multiplying some rational functions, the system of the differential* *equations under the new coordinate system $x_1, \dots, x_6$ is written as followings ,*

$$(\vartheta_{x_1} - \frac{\lambda_1 + \lambda_2}{2})\phi = 0, \tag{4.17}$$

$$[x_2 - (\vartheta_{x_2} - \frac{1}{2})(2\vartheta_{x_3} - \vartheta_{x_2})]\phi = 0, \tag{4.18}$$

$$\begin{aligned}&[x_3(\vartheta_{x_2} - 2\vartheta_{x_3})(\vartheta_{x_2} - 2\vartheta_{x_3} - 1) \\ &\quad - (\vartheta_{x_3} - \frac{1}{4}(\lambda_2 - \lambda_1) - 1)(\vartheta_{x_3} + \frac{1}{4}(\lambda_2 - \lambda_1))]\phi = 0,\end{aligned} \tag{4.19}$$

$$\frac{\partial}{\partial x_4}\phi = 0, \tag{4.20}$$

$$\frac{\partial}{\partial x_5}\phi = 0, \tag{4.21}$$

$$\frac{\partial}{\partial x_6}\phi = 0. \tag{4.22}$$

Proof. First, we put

$$\alpha_1 = a_1 a_2, \qquad \alpha_2 = a_1 a_2^{-1}, \qquad \alpha_3 = a_3 a_4,$$
$$\alpha_4 = a_3 a_4^{-1}, \qquad u' = u, \qquad v' = (v - u).$$

Then the differential equation (4.8) becomes

$$\frac{\partial}{\partial u'}\phi = 0. \tag{4.23}$$

Furthermore we exchange the variables α_2, α_4, v' to

$$w = \alpha_2\alpha_4 v'^2 + \alpha_2\alpha_4^{-1} + \alpha_2^{-1}\alpha_4, \qquad \beta_2 = \alpha_2\alpha_4, \qquad \beta_4 = \alpha_2\alpha_4^{-1}.$$

Then equations (4.10) and (4.11) become

$$\vartheta_{\beta_4}\phi = 0, \tag{4.24}$$

$$\vartheta_{\beta_2}\phi = 0, \tag{4.25}$$

respectively. Setting

$$\beta_1 = \alpha_1\alpha_3, \qquad \beta_3 = \alpha_1\alpha_3^{-1},$$

the equation (4.15) becomes

$$(2\vartheta_{\beta_1} - (\lambda_1 + \lambda_2))\phi = 0. \tag{4.26}$$

Also we can see that the equation (4.6) is written as

$$[2w\frac{\partial}{\partial w}(2(\vartheta_{\beta_1} + \vartheta_{\beta_3}) - (\lambda_1 + \lambda_2 - 1)) - (2\pi\sqrt{-1})^2\beta_3^{-1}w]\phi = 0. \tag{4.27}$$

If we eliminate ϑ_{β_1} from (4.27) by using the equation (4.26), it can be written as

$$[2w\frac{\partial}{\partial w}(2\vartheta_{\beta_3} + 1) - (2\pi\sqrt{-1})^2\beta_3^{-1}w]\phi = 0. \tag{4.28}$$

We note that the equation (4.13) can be reduced to the same equation. Taking into account the equations (4.26) and (4.28), the equation (4.5) can be reduced to the equation

$$[(\vartheta_{\beta_3}+\vartheta_w-\frac{1}{2}(\lambda_1-\lambda_2-4))(\vartheta_{\beta_3}+\vartheta_w+\frac{1}{2}(\lambda_1-\lambda_2))-4\frac{\partial^2}{\partial w^2}]\phi=0. \quad (4.29)$$

We can also see that the equations (4.9), (4.12) and (4.14) can be written as the same equation (4.29). Finally, we put

$$\gamma_1=(\pi\sqrt{-1})^2\beta_3^{-1}w, \qquad \gamma_2=w^{-2}.$$

Then the equation (4.28) is equivalent to

$$[(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2})(\frac{1}{2}-\vartheta_{\gamma_1})-\gamma_1]\phi=0. \quad (4.30)$$

Also the equation (4.29) is written as

$$[(\vartheta_{\gamma_2}-\frac{1}{4}(\lambda_2-\lambda_1)-1)(\vartheta_{\gamma_2}+\frac{1}{4}(\lambda_2-\lambda_1))-\gamma_2(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2})(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2}-1)]\phi=0. \quad (4.31)$$

If we put

$$x_1=\beta_1, \qquad x_2=\gamma_1, \qquad x_3=\gamma_2,$$
$$x_4=\beta_2, \qquad x_5=\beta_4, \qquad x_6=u',$$

then the theorem follows from the equations (4.23), (4.24), (4.25), (4.26), (4.30) and (4.31).

□

Corollary 4.20. *The change of variables (4.16) gives a diffeomorphism from* $\{(a_1,\dots,a_4,u,v)\in\mathbb{R}^6 \mid a_i\in\mathbb{R}_{>0}\ (i=1,\dots,4)\}\cong S_{(I)}\backslash N\times A$ *to the domain* $D_1=\{(x_1,\dots,x_6)\mid x_i\in\mathbb{R}_{>0}\,(i=1,3,4,5), x_2\in\mathbb{R}_{<0}, x_6\in\mathbb{R}\}$.

Proof. We should show that this gives bijection and the Jacobi determinant is not zero, i.e.,

$$\left|\frac{\partial(x_1,\dots,x_6)}{\partial(a_1,\dots,a_4,u,v)}\right|(p)=\begin{vmatrix}\frac{\partial x_1}{\partial a_1}(p) & \cdots & \frac{\partial x_1}{\partial a_4}(p) & \frac{\partial x_1}{\partial u}(p) & \frac{\partial x_1}{\partial v}(p)\\ \vdots & & \vdots & \vdots & \vdots \\ \frac{\partial x_6}{\partial a_1}(p) & \cdots & \frac{\partial x_6}{\partial a_4}(p) & \frac{\partial x_6}{\partial u}(p) & \frac{\partial x_6}{\partial v}(p)\end{vmatrix}\neq 0,$$

for any $p\in\{(a_1,\dots,a_4,u,v)\in\mathbb{R}^6 \mid a_i\in\mathbb{R}_{>0}\ (i=1,\dots,4)\}$. Here $|X|$ means the determinant of $X\in M(6,\mathbb{R})$. As we see in the proof of the previous proposition, this change of variables is the composition of the following change of variables.

Step1.

$$\alpha_1=a_1a_2, \qquad \alpha_2=a_1a_2^{-1}, \qquad \alpha_3=a_3a_4,$$
$$\alpha_4=a_3a_4^{-1}, \qquad u'=u, \qquad v'=(v-u).$$

Here we can see this gives a bijection from $\{(a_1,\dots,a_4,u,v)\in\mathbb{R}^6 \mid a_i\in\mathbb{R}_{>0}\ (i=1,\dots,4)\}$ to $\{(\alpha_1,\dots,\alpha_4,u',v')\in\mathbb{R}^6 \mid \alpha_i\in\mathbb{R}_{>0}\ (i=1,\dots,4)\}$.

Step2.

$$\beta_1 = \alpha_1\alpha_3, \qquad \beta_2 = \alpha_2\alpha_4, \qquad \beta_3 = \alpha_1\alpha_3^{-1},$$
$$\beta_4 = \alpha_2\alpha_4^{-1}, \qquad w = \alpha_2\alpha_4 v'^2 + \alpha_2\alpha_4^{-1} + \alpha_2^{-1}\alpha_4, \qquad u' = u'.$$

Here we can see this gives a bijection from $\{(\alpha_1,\ldots,\alpha_4,u',v') \in \mathbb{R}^6 \mid \alpha_i \in \mathbb{R}_{>0}\ (i=1,\ldots,4)\}$ to $\{(\beta_1,\ldots,\beta_4,w,u') \in \mathbb{R}^6 \mid \beta_i, w \in \mathbb{R}_{>0}\ (i=1,\ldots,4)\}$.

Step3.

$$x_1 = \beta_1, \qquad x_2 = (\pi\sqrt{-1})^2\beta_3^{-1}w, \qquad x_3 = w^{-2},$$
$$x_4 = \beta_2, \qquad x_5 = \beta_4, \qquad x_6 = u'.$$

Here we can see this gives a bijection from $\{(\beta_1,\ldots,\beta_4,w,u') \in \mathbb{R}^6 \mid \beta_i, w \in \mathbb{R}_{>0}\ (i=1,\ldots,4)\}$ to $\{(x_1,\ldots,x_6') \in \mathbb{R}^6 \mid x_i \in \mathbb{R}_{>0}\,(i=1,3,4,5), x_2 \in \mathbb{R}_{<0}, x_6 \in \mathbb{R}\}$.

Also it is not hard to see that

$$\left|\frac{\partial(x_1,\ldots,x_6)}{\partial(a_1,\ldots,a_4,u,v)}\right|(p)$$
$$= \left|\frac{\partial(x_1,\ldots,x_6)}{\partial(\beta_1,\ldots,\beta_4,w,u')}\frac{\partial(\beta_1,\ldots,\beta_4,w,u')}{\partial(\alpha_1,\ldots,\alpha_4,u',v')}\frac{\partial(\alpha_1,\ldots,\alpha_4,u',v')}{\partial(a_1,\ldots,a_4,u,v)}\right|(p) \neq 0,$$

for any $p \in \{(a_1,\ldots,a_4,u,v) \in \mathbb{R}^6 \mid a_i \in \mathbb{R}_{>0}\ (i=1,\ldots,4)\}$. Thus we have the proposition. □

Pick up the differential equations (4.18) and (4.19). We take $f(x_2,x_3)$ as a solution of them. We take a function $F(x_2,x_3)$ such that

$$f = x_2^{\frac{1}{2}} x_3^{\frac{1}{4}(\lambda_1-\lambda_2)} F.$$

Then this $F(x_2,x_3)$ satisfies

$$[x_2 - \vartheta_{x_2}(2\vartheta_{x_3} - \vartheta_{x_2} + \frac{1}{2}(\lambda_1-\lambda_2-1))]F(x_2,x_3) = 0, \tag{4.32}$$

$$\begin{aligned}[x_3(2\vartheta_{x_3} - \vartheta_{x_2} + \frac{1}{2}(\lambda_1-\lambda_2-1))(2\vartheta_{x_3} - \vartheta_{x_2} + \frac{1}{2}(\lambda_1-\lambda_2-1)+1)\\ -\vartheta_{x_3}(\vartheta_{x_3} + \frac{1}{2}(\lambda_1-\lambda_2)-1)]F(x_2,x_3) = 0.\end{aligned} \tag{4.33}$$

These are the differential equations for Horn's hypergeometric function $\mathrm{H}_{10}(\frac{1}{2}(\lambda_1-\lambda_2-1), \frac{1}{2}(\lambda_1-\lambda_2); x_2, x_3)$ (cf. [11]). Let $\mathfrak{H}_{10}(a,d;x,y)$ be the solution space of the system of the partial differential equations for Horn's hypergeometric function $\mathrm{H}_{10}(a,d;x,y)$, i.e.,

$$[x(2\vartheta_x - \vartheta_y + a)(2\vartheta_x - \vartheta_y + a + 1) - \vartheta_x(\vartheta_x + d - 1)]f(x,y) = 0,$$
$$[y - \vartheta_y(2\vartheta_x - \vartheta_y + a)]f(x,y) = 0.$$

We can see more detailed properties of $\mathfrak{H}_{10}(a,d;x,y)$ in Appendix.

Definition 4.21. *Let $U \subset \mathbb{R}^n$ be a domain. A function $f(x)$ on U is called rapidly decreasing on U if it satisfies,*

$$\sup_{x\in U} |x^\alpha f(x)| < \infty$$

for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, *where* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Also a function $f(x)$ *on* U *is called slowly increasing on* U *if there exists* $N \in \mathbb{N}$ *such that*

$$\sup_{x \in U} (1 + |x|)^{-N} |f(x)| < \infty$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ *for* $x \in U$.

Theorem 4.22. *The space* $C^\infty_{\chi_{l(0,1,1,0,0,0)}}(S_{(I)}\backslash N \times A;\ I_k(\lambda))$ *consists of the elements,*

$$x_1^{\frac{\lambda_1+\lambda_2}{2}} x_2^{\frac{1}{2}} x_3^{\frac{1}{4}(\lambda_1-\lambda_2)} f(x_2, x_3),$$

for $f(x,y) \in \mathfrak{H}_{10}(\frac{1}{2}(\lambda_1 - \lambda_2 - 1), \frac{1}{2}(\lambda_1 - \lambda_2); x, y)$. *Here*

$$x_1 = a_1 a_2 a_3 a_4,$$

$$x_2 = (\pi\sqrt{-1})^2 \left((\frac{a_3}{a_2})^2 (v-u)^2 + (\frac{a_4}{a_2})^2 + (\frac{a_3}{a_1})^2 \right),$$

$$x_3 = \left(\frac{a_1 a_3}{a_2 a_4} (v-u)^2 + \frac{a_2 a_3}{a_1 a_4} + \frac{a_1 a_4}{a_2 a_3} \right)^{-2},$$

$$x_4 = \frac{a_1 a_3}{a_2 a_4},$$

$$x_5 = \frac{a_1 a_4}{a_2 a_3},$$

$$x_6 = u.$$

Thus we have

$$\dim_{\mathbb{C}} C^\infty_{\chi_{l(0,1,1,0,0,0)}}(S_{(I)}\backslash N \times A;\ I_k(\lambda)) = 4.$$

In $C^\infty_{\chi_{l(0,1,1,0,0,0)}}(S_{(I)}\backslash N \times A;\ I_k(\lambda))$, *there is a rapidly decreasing function on* $\{(x_2, \ldots, x_6) \mid x_2 \in \mathbb{R}_{<0}, x_3, x_4, x_5 \in \mathbb{R}_{>0}, x_6 \in \mathbb{R}\}$ *and it is unique up to constant multiple.*

Proof. This follows immediately from Proposition 4.19 and the argument above. The second assertion follows from Appendix B and Thorem C.1. □

(ii) *the case* $\varepsilon = 0$.

We investigate the space $C^\infty_{\chi_{l(0,1,0,0,0,0)}}(S_{(\mathrm{I})}\backslash N \times A;\ I_k(\lambda))$, i.e, the case $\varepsilon = 0$.

We introduce a new cooridinate system as below,

$$x_1 = a_1, \qquad x_2 = a_2,$$

$$x_3 = a_4^2 + a_3^2 v^2, \qquad x_4 = (a_3^{-2} + a_4^{-2} v^2)^{-1},$$

$$x_5 = a_3 a_4^{-1}, \qquad x_6 = u.$$

Lemma 4.23. *The change of variables given above is the diffeomorphism from* $\{(a_1, \ldots, a_4, u, v) \in \mathbb{R}^6 \mid a_i \in \mathbb{R}_{>0}\ (i = 1, \ldots, 4)\} \cong S_{(I)}\backslash N \times A$ *to the domain* $D_2 = \{(x_1, \ldots, x_6) \mid x_i \in \mathbb{R}_{>0}\ (i = 1, \ldots, 5),\ x_6 \in \mathbb{R}\}$.

Proof. We can see that this is a bijection. Also it is not hard to see that

$$\left| \frac{\partial(x_1, \ldots, x_6)}{\partial(a_1, \ldots, a_4, u, v)} \right| (p) \neq 0,$$

for $p \in \{(a_1, \ldots, a_4, u, v) \in \mathbb{R}^6 \mid a_i \in \mathbb{R}_{>0}\ (i = 1, \ldots, 4)\}$ by direct computation. □

Then the systems of the differential equations in Proposition 4.17 are reduced to the followings

Proposition 4.24. *For $k = 1, 2$, we consider the systems of the differential equations in Proposition 4.17. Then by adding and substituting each differential equations and multiplying some rational functions, these systems are written on* $\{(a_1, \dots, a_4, u, v) \in \mathbb{R}^6 \mid a_i \in \mathbb{R}_{>0}\ (i = 1, \dots, 4)\}$ *as follows.*

$$(\vartheta_{x_1} - (\lambda_1 - (4-k)))(\vartheta_{x_1} - \lambda_2)\phi = 0, \tag{4.34}$$

$$\begin{aligned}[\vartheta_{x_2}^2 - (\lambda_1 + \lambda_2 - 3 + k)\vartheta_{x_2} + \vartheta_{x_1} \\ + (2\pi\sqrt{-1})^2 x_2^{-2} x_3 + \lambda_2(\lambda_1 - (4-k))]\phi = 0,\end{aligned} \tag{4.35}$$

$$\begin{aligned}[4(\vartheta_{x_3}^2 + \vartheta_{x_4}^2) - 2(\lambda_1 + \lambda_2 - 1 + k)\vartheta_{x_3} + 2\vartheta_{x_4} \\ + (2\pi\sqrt{-1})^2 x_2^{-2} x_3 + 2\lambda_1(\lambda_2 + k)]\phi = 0,\end{aligned} \tag{4.36}$$

$$(2\vartheta_{x_4} - \lambda_1)(2\vartheta_{x_4} - (\lambda_2 + k))\phi = 0, \tag{4.37}$$

$$\vartheta_{x_5}\phi = 0, \tag{4.38}$$

$$\vartheta_{x_6}\phi = 0. \tag{4.39}$$

Proof. By the equations (4.5) and (4.15), we have the new equation

$$[\vartheta_{a_1}^2 - (\lambda_1 + \lambda_2 - 4 + k)\vartheta_{a_1} + \left(\frac{a_2}{a_1}\right)^2 \frac{\partial^2}{\partial u^2} + \lambda_2(\lambda_1 - (4-k))]\phi = 0.$$

By the equation (4.8), we can eliminate the term $\frac{\partial}{\partial u}$ from the above equation, then it can be written as

$$(\vartheta_{a_1} - (\lambda_1 - (4-k)))(\vartheta_{a_1} - \lambda_2)\phi = 0. \tag{4.40}$$

Next, from the equations (4.9) and (4.15), we have a new equation

$$[\vartheta_{a_2} - (\lambda_1 + \lambda_2 - 3 + k)\vartheta_{a_2} + \vartheta_{a_1} + (2\pi\sqrt{-1})^2(a_4^2 + a_3^2 v^2)a_2^{-2} + \lambda_2(\lambda_1 - (4-k))]\phi = 0, \tag{4.41}$$

as well. If we put $\alpha_3 = a_3 a_4$ and $\alpha_4 = a_3 a_4^{-1}$, we have a new equation

$$[2\vartheta_{\alpha_4} + (\alpha_4^{-2} v^{-1} - v)\frac{\partial}{\partial v}]\phi = 0$$

from the equation (4.10) and (4.11). Moreover if we put $w = \alpha_4^{-1} + \alpha_4 v^2$, $\beta_3 = \alpha_3$ and $\beta_4 = \alpha_4$, the above equation is reduced to

$$\vartheta_{\beta_4}\phi = 0. \tag{4.42}$$

And the equation (4.13) becomes

$$[\vartheta_w(2\vartheta_{\beta_3} - (\lambda_1 + \lambda_2 - 1 + k)) + \frac{1}{2}(2\pi\sqrt{-1})^2 \beta_3 w a_2^{-2}]\phi = 0. \tag{4.43}$$

Also if we consider the sum of the equations (4.12) and (4.14), we have a new equation

$$[2\vartheta_{\beta_3}^2 - 2(\lambda_1 + \lambda_2 + k)\vartheta_{\beta_3} + 2\vartheta_w^2 + 2\vartheta_w + (2\pi\sqrt{-1})^2 w\beta_3 a_2^{-2} + 2\lambda_1(\lambda_2 + k)]\phi = 0. \tag{4.44}$$

By the equation (4.43), we can eliminate the term $\beta_3 w a_2^{-2}$ from (4.44). Then we have

$$(\vartheta_{\beta_3} - \vartheta_w - \lambda_1)(\vartheta_{\beta_3} - \vartheta_w - (\lambda_2 + k))\phi = 0. \tag{4.45}$$

Hence if we put

$$x_1 = a_1, \qquad x_2 = a_2, \qquad x_3 = \beta_3 w, \tag{4.46}$$

$$x_4 = \beta_3 w^{-1}, \qquad x_5 = \beta_2, \qquad x_6 = u, \tag{4.47}$$

then we have the proposition from the equations (4.40), (4.41), (4.42), (4.44), (4.45) and (4.8). □

Let $\mathfrak{MB}(\nu\,;\,x)$ be the solution space of the differential equation

$$[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - (1 + \frac{\nu^2}{x^2})]f(x) = 0,$$

i.e., the solution space of the modified Bessel equation. We have $\dim_{\mathbb{C}} \mathfrak{MB}(\nu\,;\,x) = 2$. In $\mathfrak{MB}(\nu\,;\,x)$, there is a series solution

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^{\nu+m}}{m!\Gamma(\nu+m+1)}.$$

Also there is a solution as a slowly increasing function

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

and any slowly increasing function in $\mathfrak{MB}(\nu\,;\,x)$ are constant multiples of $K_\nu(x)$.

Theorem 4.25. *For the cases $k = 1, 2$, $C^\infty_{\chi_{l(0,1,0,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_k(\lambda))$ are written as follows under the coordinate system,*

$$x_1 = a_1, \qquad x_2 = a_2,$$

$$x_3 = a_4^2 + a_3^2 v^2, \qquad x_4 = (a_3^{-2} + a_4^{-2} v^2)^{-1},$$

$$x_5 = a_3 a_4^{-1}, \qquad x_6 = u.$$

(i) For $k = 1$, $C^\infty_{\chi_{l(0,1,0,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_1(\lambda))$ consists of

$$x_1^{\lambda_2} x_2^{\frac{\lambda_1+\lambda_2-2}{2}} x_3^{\frac{\lambda_1+\lambda_2}{4}} x_4^{\frac{\lambda_2+1}{2}} f(2\pi x_2^{-1}\sqrt{x_3})$$

for $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}\,;\,x)$. Thus we have

$$\dim_{\mathbb{C}} C^\infty_{\chi_{l(0,1,0,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_1(\lambda)) = 2,$$

and there is a slowly increasing function on $\{(x_1, \ldots, x_6) \mid x_i \in \mathbb{R}_{>0}\, (i = 1, \ldots, 5),\, x_6 \in \mathbb{R}\}$ and it is unique up to constant multiple.

(ii) For $k = 2$, $C^\infty_{\chi_{l(0,1,0,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_2(\lambda))$ consists of

$$C x_1^{\lambda_2} x_2^{\frac{\lambda_1+\lambda_2-1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{4}} x_4^{\frac{\lambda_1}{2}} f(2\pi x_2^{-1}\sqrt{x_3})$$

$$+ C' x_1^{\lambda_1-2} x_2^{\frac{\lambda_1+\lambda_2-1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{4}} x_4^{\frac{\lambda_2+2}{2}} g(2\pi x_2^{-1}\sqrt{x_3}),$$

where $C, C' \in \mathbb{C}$, $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-1}{2}\,;x)$ *and* $g(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-3}{2}\,;x)$. *Thus we have*

$$\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,1,0,0,0,0)}}(S_{(I)}\backslash N \times A;\, I_2(\lambda)) = 4,$$

and there is a 2-dimensional subspace which consists of slowly increasing functions on $\{(x_1,\ldots,x_6) \mid x_i \in \mathbb{R}_{>0}\,(i=1,\ldots,5),\, x_6 \in \mathbb{R}\}$.

Proof. The solution space of the equation (4.34) consists of the elements

$$C_1\phi_1(x_2,x_3,x_4)x_1^{\lambda_1-(4-k)} + C_2\phi_2(x_2,x_3,x_4)x_1^{\lambda_2},$$

where C_i are constants and ϕ_i are arbitrary functions for $i=1,2$. We determine functions ϕ_i by the equation (4.35). Then these functions satisfy following equations,

$$[\vartheta_{x_2}^2 - (\lambda_1+\lambda_2-3+k)\vartheta_{x_2} + (2\pi\sqrt{-1})^2 x_2^{-2}x_3 + (\lambda_1-4+k)(\lambda_2+1)]\phi_1 = 0,$$
$$[\vartheta_{x_2}^2 - (\lambda_1+\lambda_2-3+k)\vartheta_{x_2} + (2\pi\sqrt{-1})^2 x_2^{-2}x_3 + (\lambda_1-3+k)\lambda_2]\phi_2 = 0.$$

We define the functions ϕ'_i so that $\phi_i = x_2^{\frac{\lambda_1+\lambda_2-3+k}{2}}\phi'_i$ for $i=1,2$, then ϕ'_i satisfy following equations,

$$[\vartheta_{x_2}^2 - ((2\pi x_2^{-1}\sqrt{x_3})^2 + (\frac{\lambda_1-\lambda_2-5+k}{2})^2)]\phi'_1 = 0,$$
$$[\vartheta_{x_2}^2 - ((2\pi x_2^{-1}\sqrt{x_3})^2 + (\frac{\lambda_1-\lambda_2-3+k}{2})^2)]\phi'_2 = 0.$$

For some fixed x_3, if we put $z = (2\pi x_2^{-1}\sqrt{x_3})$, these equations are nothing but modified Bessel equations

$$[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - (1+\frac{\nu_i^2}{z^2})]\phi'_i = 0,$$

where $\nu_1 = \frac{\lambda_1-\lambda_2-5+k}{2}$ and $\nu_2 = \frac{\lambda_1-\lambda_2-3+k}{2}$. Hence the intersection of the solution space of the equations (4.34) and (4.35) consist of functions written as follows,

$$C_1\zeta_1(x_3,x_4)f_1(x_2,x_3)x_1^{\lambda_1-(4-k)}x_2^{\frac{\lambda_1+\lambda_2-3+k}{2}} + C_2\zeta_2(x_3,x_4)f_2(x_2,x_3)x_1^{\lambda_2}x_2^{\frac{\lambda_1+\lambda_2-3+k}{2}},$$

where C_i are constants, ζ_i are arbitrary functions and $f_i \in \mathfrak{MB}(\nu_i\,; 2\pi x_2^{-1}\sqrt{x_3})$ for $i=1,2$.

By the same argument, we can also see that the intersection of the solution spaces of the equations (4.36) and (4.37) are written as follows,

$$D_1\xi_1(x_1,x_2)g_1(x_2,x_3)x_3^{\frac{\lambda_1+\lambda_2-1+k}{4}}x_4^{\frac{\lambda_1}{2}} + D_2\xi_2(x_1,x_2)g_2(x_2,x_3)x_3^{\frac{\lambda_1+\lambda_2-1+k}{4}}x_4^{\frac{\lambda_2+k}{2}},$$

where D_i are constants, ξ_i are arbitrary functions and $g_i \in \mathfrak{MB}(\mu_i\,; 2\pi x_2^{-1}\sqrt{x_3})$ for $i=1,2$. Here we put $\mu_1 = \frac{\lambda_1-\lambda_2+1-k}{2}$ and $\mu_2 = \frac{\lambda_1-\lambda_2-1-k}{2}$.

Note that $\mathfrak{MB}(\nu\,;\,x) \cap \mathfrak{MB}(\mu\,;\,x) = \{0\}$ if $\nu \neq \mu$. Then intersections of the solution spaces of the equations (4.34), (4.35), (4.36) and (4.37) are written as follows. For $k = 1$,

$$x_1^{\lambda_2} x_2^{\frac{\lambda_1+\lambda_2-2}{2}} x_3^{\frac{\lambda_1+\lambda_2}{4}} x_4^{\frac{\lambda_2+1}{2}} f(2\pi x_2^{-1}\sqrt{x_3})$$

where $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}\,;\,x)$. And for $k = 2$,

$$C x_1^{\lambda_2} x_2^{\frac{\lambda_1+\lambda_2-1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{4}} x_4^{\frac{\lambda_1}{2}} f(2\pi x_2^{-1}\sqrt{x_3}) \\ + C' x_1^{\lambda_1-2} x_2^{\frac{\lambda_1+\lambda_2-1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{4}} x_4^{\lambda_2+2} g(2\pi x_2^{-1}\sqrt{x_3}),$$

where $C, C' \in \mathbb{C}$, $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-1}{2}\,;\,x)$ and $g(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-3}{2}\,;\,x)$. □

4.3.2 The embeddings into the space (II)

We consider the embeddings of $X_{k,\lambda}$ ($k = 1, 2$) into the spaces (II_i) ($i = 1, \ldots, 4$) which are classified in Proposition 4.10. These spaces are isomorphic to $C^\infty_{\chi((0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3))}(S_{(\mathrm{II})}\backslash G/K\,;\,I_k(\lambda))$. We consider the image of this space by the map $\Xi_{(\mathrm{II})}$ defined in Section 4.2. Here

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} (1,1,0) & \text{if the case } (\mathrm{II}_1), \\ (0,1,1) & \text{if the case } (\mathrm{II}_2), \\ (0,1,0) & \text{if the case } (\mathrm{II}_3), \\ (0,0,1) & \text{if the case } (\mathrm{II}_4). \end{cases}$$

We denote these space by

$$C^\infty_{\chi_{l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}}(S_{(\mathrm{II})}\backslash N \times A;\, I_k(\lambda)) = \Xi_{(II)}(C^\infty_{\chi_{l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}}(S_{(\mathrm{II})}\backslash G/K;\, I_k(\lambda)).$$

Proposition 4.26. *The function space $C^\infty_{\chi_{l(0,0,\varepsilon_1,\varepsilon_2,0,\varepsilon_3)}}(S_{(II)}\backslash N \times A;\, I_k(\lambda))$ for $k = 1, 2$ are equal to the solution spaces of the following systems of the differen-*

tial equations in $C^\infty(S_{(II)}\backslash N\times A)$.

$$[\vartheta_{a_1}^2-(\lambda_1+\lambda_2-3+k)\vartheta_{a_1}+(2\pi\sqrt{-1})^2(\frac{a_2}{a_1})^2\varepsilon_2 + \lambda_1(\lambda_2+k)-(\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4})]\phi=0, \tag{4.48}$$

$$\varepsilon_2(\vartheta_{a_1}+\vartheta_{a_2}-(\lambda_1+\lambda_2-3+k))\phi=0, \tag{4.49}$$

$$\frac{\partial}{\partial u}\phi=0, \tag{4.50}$$

$$\varepsilon_1\varepsilon_2\phi=0, \tag{4.51}$$

$$[\vartheta_{a_2}^2-(\lambda_1+\lambda_2-2+k)\vartheta_{a_2}+(2\pi\sqrt{-1})^2(\frac{a_2}{a_1})^2\varepsilon_2+(\frac{a_3}{a_2})^2\frac{\partial^2}{\partial u^2} +(2\pi\sqrt{-1})^2(\frac{a_4}{a_2})^2\varepsilon_1+\lambda_1(\lambda_2+k)-(\vartheta_{a_3}+\vartheta_{a_4})]\phi=0, \tag{4.52}$$

$$[\frac{a_3}{a_2}\frac{\partial}{\partial u}(\vartheta_{a_2}+\vartheta_{a_3}-(\lambda_1+\lambda_2-2+k))+(2\pi\sqrt{-1})^2\frac{a_4^2}{a_2a_3}\varepsilon_1(\varepsilon_3-\varepsilon_1 u)]\phi=0, \tag{4.53}$$

$$[\varepsilon_1(\vartheta_{a_2}+\vartheta_{a_4}-(\lambda_1+\lambda_2-1+k))+(\vartheta_3-\vartheta_1 u)\frac{\partial}{\partial u}]\phi=0, \tag{4.54}$$

$$[\vartheta_{a_3}^2-(\lambda_1+\lambda_2-1+k)\vartheta_{a_3}+(\frac{a_3}{a_2})^2\frac{\partial^2}{\partial u^2} + (2\pi\sqrt{-1})^2(\frac{a_4}{a_3})^2(\varepsilon_3-\varepsilon_1 u)^2-\vartheta_{a_4}+\lambda_1(\lambda_2+k)]\phi=0, \tag{4.55}$$

$$[(\varepsilon_3-\varepsilon_1 u)(\vartheta_{a_3}+\vartheta_{a_4}-(\lambda_1+\lambda_2-1+k))+\varepsilon_1\frac{\partial}{\partial u}]\phi=0, \tag{4.56}$$

$$[\vartheta_{a_4}^2-(\lambda_1+\lambda_2+k)\vartheta_{a_4}+(2\pi\sqrt{-1})^2(\frac{a_4}{a_2})^2\varepsilon_1 + (2\pi\sqrt{-1})^2(\frac{a_4}{a_3})^2(\varepsilon_3-\varepsilon_1 u)^2+\lambda_1(\lambda_2+k)]\phi=0, \tag{4.57}$$

$$[\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4}-k\lambda_1-(4-k)\lambda_2]\phi=0. \tag{4.58}$$

Here $\phi\in C^\infty(S_{(II)}\backslash N\times A)$.

Proof. As well as Proposition 4.17, these are obtained by the direct computation from Lemma 4.12, Proposition 4.13 and Proposition 4.15. □

(i) *The case* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,1,0)$.

Theorem 4.27. *When* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,1,0)$, *we have*

$$C^\infty_{\chi_{l(0,0,1,1,0,0)}}(S_{(II)}\backslash N\times A;\, I_k(\lambda))=\{0\},$$

for $k=1,2$.

Proof. It is immediate from the equation (4.51). □

(ii) *The case* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(0,1,1)$.

Theorem 4.28. *When* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(0,1,1)$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash N\times A;\, I_k(\lambda))$ $(k=1,2)$ *are written as follows.*

(i) If $k=1$, *we have*

$$C^\infty_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash N\times A;\, I_1(\lambda))=\{0\}.$$

(ii) If $k=2$, the space $C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash N\times A;\ I_2(\lambda))$ consists of

$$x_1^{\frac{\lambda_1+\lambda_1-1}{2}} x_2^{\frac{1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{2}} x_4^{\frac{1}{2}} f(2\pi x_2) g(2\pi x_3)$$

for $f(x), g(x)\in \mathcal{MB}(\frac{\lambda_1-\lambda_2-2}{2}; x)$. Here we put

$$x_1 = a_1a_2, x_2 = a_1^{-1}a_2, x_3 = a_3a_4, x_4 = a_3^{-1}a_4.$$

Thus we have $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash N\times A;\ I_2(\lambda)) = 4$. There exists a slowly increasing function on the domain $\{(x_1,\ldots,x_4,u)\mid x_i\in\mathbb{R}_{>0}, u\in\mathbb{R}\}$ in $C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(S_{(II)}\backslash N\times A;\ I_2(\lambda))$ and it is unique up to constant.

Proof. We show the case $k=1$ first. By the equations (4.49) and (4.56), we have the new one,

$$[\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4}-2\lambda_1-2\lambda_2+2]\phi = 0.$$

Comparing this equation with the equation (4.58), we can conclude that the space $C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(S_{(\mathrm{II})}\backslash N\times A;\ I_1(\lambda))$ is equal to $\{0\}$. Next, we consider the case $k=2$. By the equation (4.50), we can eliminate the terms $\frac{\partial}{\partial u}$ from the other differential equations. Then the equations in Proposition 4.26 are reduced to the followings,

$$\begin{aligned}
&[\vartheta_{a_1}+\vartheta_{a_2}-(\lambda_1+\lambda_2-1)]\phi = 0,\\
&[\vartheta_{a_1}^2-(\lambda_1+\lambda_2-1)\vartheta_{a_1}+\lambda_1(\lambda_2+2)\\
&\qquad +(2\pi\sqrt{-1})a_1^{-2}a_2^2-(\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4})]\phi = 0,\\
&[\vartheta_{a_2}^2-(\lambda_1+\lambda_2)\vartheta_{a_2}+(2\pi\sqrt{-1})^2+\lambda_1(\lambda_2+2)-(\vartheta_{a_3}+\vartheta_{a_4})]\phi = 0,\\
&[\vartheta_{a_3}+\vartheta_{a_4}-(\lambda_1+\lambda_2+1)]\phi = 0,\\
&[\vartheta_{a_3}^2-(\lambda_1+\lambda_2+1)\vartheta_{a_3}+(2\pi\sqrt{-1})^2a_3^{-2}a_4^2-\vartheta_{a_4}+\lambda_1(\lambda_2+2)]\phi = 0,\\
&[\vartheta_{a_4}^2-(\lambda_1+\lambda_2+2)\vartheta_{a_4}+(2\pi\sqrt{-1})^2a_3^{-2}a_4^2+\lambda_1(\lambda_2+2)]\phi = 0.
\end{aligned}$$

We put

$$\begin{aligned}
x_1 &= a_1a_2, & x_2 &= a_1^{-1}a_2,\\
x_3 &= a_3a_4, & x_4 &= a_3^{-1}a_4,
\end{aligned}$$

Then we can rewrite above differential equations as follows,

$$\begin{aligned}
&[2\vartheta_{x_1}-(\lambda_1+\lambda_2-1)]\phi = 0,\\
&[2\vartheta_{x_3}-(\lambda_1+\lambda_2+1)]\phi = 0,\\
&[\vartheta_{x_2}^2-\vartheta_{x_2}+(2\pi\sqrt{-1}x_2)^2-(\frac{\lambda_1-\lambda_2-2}{2})^2-\frac{1}{4}]\phi = 0,\\
&[\vartheta_{x_4}^2-\vartheta_{x_4}+(2\pi\sqrt{-1}x_4)^2-(\frac{\lambda_1-\lambda_2-2}{2})^2-\frac{1}{4}]\phi = 0.
\end{aligned}$$

We take ϕ' as $\phi = x_2^{\frac{1}{2}} x_4^{\frac{1}{2}} \phi'$. Then we can see that ϕ' satisfies following equations,

$$[2\vartheta_{x_1} - (\lambda_1 + \lambda_2 - 1)]\phi' = 0,$$
$$[2\vartheta_{x_3} - (\lambda_1 + \lambda_2 + 1)]\phi' = 0,$$
$$[\vartheta_{x_2}^2 - ((2\pi\sqrt{-1}x_2)^2 + (\frac{\lambda_1 - \lambda_2 - 2}{2})^2)]\phi' = 0,$$
$$[\vartheta_{x_4}^2 - ((2\pi\sqrt{-1}x_4)^2 + (\frac{\lambda_1 - \lambda_2 - 2}{2})^2)]\phi = 0.$$

Then we can conclude that

$$\phi(x_1, x_2, x_3, x_4) = x_1^{\frac{\lambda_1+\lambda_1-1}{2}} x_2^{\frac{1}{2}} x_3^{\frac{\lambda_1+\lambda_2+1}{2}} x_4^{\frac{1}{2}} f(2\pi x_2) g(2\pi x_3),$$

where $f, g \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}; x)$. Hence we have the proposition. □

(iii) *The case* $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 0)$.

Theorem 4.29. *When* $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 0)$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_k(\lambda))$ $(k = 1, 2)$ *are written as follows.*

(i) If $k = 1$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_1(\lambda))$ *consists of*

$$x_1^{\frac{\lambda_1+\lambda_2-2}{2}} x_2^{\frac{1}{2}} (x_3 x_4)^{\lambda_2+1} f(2\pi x_2)$$

for $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-3}{2}; x)$. *Here we put*

$$x_1 = a_1 a_2, x_2 = a_1^{-1} a_2, x_3 = a_3, x_4 = a_4.$$

Thus we have $\dim_{\mathbb{C}} C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_1(\lambda)) = 2$. *There exists a slowly increasing function on the domain* $\{(x_1, \ldots, x_4, u) \mid x_i \in \mathbb{R}_{>0}, u \in \mathbb{R}\}$ *in* $C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_2(\lambda))$ *and it is unique up to constant.*

(ii) If $k = 2$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_2(\lambda))$ *consists of*

$$(C_1 x_3^{\lambda_2} x_4^{\lambda_1} + C_2 x_3^{\lambda_1 - 1} x_4^{\lambda_2+2}) \times x_1^{\frac{\lambda_1+\lambda_2-1}{2}} x_2^{\frac{1}{2}} f(2\pi x_2)$$

for $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}; x)$ *and* $C_1, C_2 \in \mathbb{C}$. *Here* x_i $(i = 1 \ldots, 4)$ *are same as (i).*

Thus we have $\dim_{\mathbb{C}} C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_2(\lambda)) = 4$. *There exists a 2-dimentional supspace of* $C^\infty_{\chi_{l(0,0,0,1,0,0)}}(S_{(II)}\backslash N \times A;\ I_2(\lambda))$ *which consits of slowly increasing functions on* $\{(x_1, \ldots, x_4, u) \mid x_i \in \mathbb{R}_{>0}, u \in \mathbb{R}\}$.

Proof. By the equation (4.50), we can eliminate the term $\frac{\partial}{\partial u}$ from the other equations. Then the equations in Proposition 4.26 are reduced to the followings,

$$[\vartheta_{a_1}^2 - (\lambda_1 + \lambda_2 - 4 + k)\vartheta_{a_1} + (2\pi\sqrt{-1})^2 a_1^{-2} a_2^2 + \lambda_2(\lambda_1 - (4 - k))]\phi = 0,$$
$$[\vartheta_{a_1} + \vartheta_{a_2} - (\lambda_1 + \lambda_2 - 3 + k)]\phi = 0,$$
$$[\vartheta_{a_2}^2 - (\lambda_1 + \lambda_2 - 3 + k)\vartheta_{a_2} + (2\pi\sqrt{-1})^2 a_1^{-2} a_2^2 + \vartheta_{a_1} + \lambda_2(\lambda_1 - (4 - k))]\phi = 0,$$
$$[\vartheta_{a_3}^2 - (\lambda_1 + \lambda_2 - 1 + k)\vartheta_{a_3} - \vartheta_{a_4} + \lambda_1(\lambda_2 + k)]\phi = 0,$$
$$[(\vartheta_{a_4} - \lambda_1)(\vartheta_{a_4} - (\lambda_2 + k))]\phi = 0.$$

We put

$$x_1 = a_1 a_2, \qquad x_2 = a_1^{-1} a_2.$$

We take ϕ' as $\phi = x_2^{\frac{1}{2}} \phi'$. Then we have

$$[\vartheta_{x_1} - \frac{\lambda_1 + \lambda_2 - 3 + k}{2}]\phi' = 0, \tag{4.59}$$

$$[\vartheta_{x_2}^2 - ((2\pi x_2)^2 + (\frac{\lambda_1 - \lambda_2 - (4-k)}{2})^2)]\phi' = 0, \tag{4.60}$$

$$[\vartheta_{a_3}^2 - (\lambda_1 + \lambda_2 - 1 + k)\vartheta_{a_3} - \vartheta_{a_4} + \lambda_1(\lambda_2 + k)]\phi' = 0, \tag{4.61}$$

$$(\vartheta_{a_4} - \lambda_1)(\vartheta_{a_4} - (\lambda_2 + k))\phi' = 0. \tag{4.62}$$

The solution of (4.59) and (4.60) is

$$\phi'(x_1, x_2, a_3, a_4) = c(a_3, a_4) x_1^{\frac{\lambda_1 + \lambda_2 - 3 + k}{2}} f(2\pi x_2),$$

for an arbitrary function $c(a_3, a_4)$ and $f(x) \in \mathfrak{MB}(\frac{\lambda_1 - \lambda_2 - (4-k)}{2}; x)$. We solve the equations (4.61) and (4.62) to determine the function $c(a_3, c_4)$. Then we can see that

$$c(a_3, a_4) = \begin{cases} (a_3 a_4)^{\lambda_2 + 1} & \text{for k=1,} \\ C_1 a_3^{\lambda_2 + 1} a_4^{\lambda_1} + C_2 a_3^{\lambda_1 - 1} a_4^{\lambda_2 + 2} & \text{for } k = 2, \end{cases}$$

for some constants $C_1, C_2 \in \mathbb{C}$. This concludes the proposition. □

4.3.3 The embeddings into the space (III)

We consider the embeddings of $X_{k,\lambda}$ $(k = 1, 2)$ into the spaces type (III_i) $(i = 1, \ldots, 8)$ which are classified in Proposition 4.10. These spaces are isomorphic to $C^\infty_{\chi((0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3))}(N\backslash G/K;\, I_k(\lambda))$. We consider the image of this space by the map $\Xi_{(\mathrm{III})}$ defined in Section 4.2. Here

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} (1,1,1) & \text{if the case } (\mathrm{III}_1), \\ (1,1,0) & \text{if the case } (\mathrm{III}_2), \\ (1,0,1) & \text{if the case } (\mathrm{III}_3), \\ (0,1,1) & \text{if the case } (\mathrm{III}_4), \\ (1,0,0) & \text{if the case } (\mathrm{III}_5), \\ (0,1,0) & \text{if the case } (\mathrm{III}_6), \\ (0,0,1) & \text{if the case } (\mathrm{III}_7), \\ (0,0,0) & \text{if the case } (\mathrm{III}_8). \end{cases}$$

We denote the image of $\Xi_{(III)}$ by

$$C^\infty_{\chi_{\iota(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(A; I_k(\lambda)) = \Xi_{(III)}(C^\infty_{\chi((0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3))}(N\backslash G/K\,;\, I_k(\lambda))).$$

Proposition 4.30. *The space $C^\infty_{\chi_{\iota(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(A; I_k(\lambda))$ is equal to the solution space of the following system of the differential equations on $C^\infty(A)$.*

$$[\vartheta_{a_1} - (\lambda_1+\lambda_2+k-3)\vartheta_{a_1} + (2\pi\sqrt{-1}\frac{a_2}{a_1})^2\varepsilon_1 - (\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4}) + \lambda_1(\lambda_2+k)]\phi = 0, \tag{4.63}$$

$$\varepsilon_1 2\pi\sqrt{-1}\frac{a_2}{a_1}(\vartheta_{a_1}+\vartheta_{a_2} - (\lambda_1+\lambda_2+k-3))\phi = 0, \tag{4.64}$$

$$\varepsilon_1\varepsilon_2\phi = 0, \tag{4.65}$$

$$[\vartheta_{a_2} - (\lambda_1+\lambda_2+k-2)\vartheta_{a_2} + (2\pi\sqrt{-1}\frac{a_2}{a_1})^2\varepsilon_1 + (2\pi\sqrt{-1}\frac{a_3}{a_2})^2\varepsilon_2 - (\vartheta_{a_3}+\vartheta_{a_4}) + \lambda_1(\lambda_2+k)]\phi = 0, \tag{4.66}$$

$$\varepsilon_2 2\pi\sqrt{-1}\frac{a_3}{a_2}(\vartheta_{a_2}+\vartheta_{a_3} - (\lambda_1+\lambda_2+k-2))\phi = 0, \tag{4.67}$$

$$\varepsilon_2\varepsilon_3\phi = 0, \tag{4.68}$$

$$[\vartheta_{a_3}^2 - (\lambda_1+\lambda_2+k-1)\vartheta_{a_3} + (2\pi\sqrt{-1}\frac{a_3}{a_2})^2\varepsilon_2 + (2\pi\sqrt{-1}\frac{a_4}{a_3})^2\varepsilon_3 - \vartheta_{a_4} + \lambda_1(\lambda_2+k)]\phi = 0, \tag{4.69}$$

$$\varepsilon_3 2\pi\sqrt{-1}\frac{a_4}{a_3}(\vartheta_{a_3}+\vartheta_{a_4} - (\lambda_1+\lambda_2+k-1))\phi = 0, \tag{4.70}$$

$$[\vartheta_{a_4}^2 - (\lambda_1+\lambda_2+k)\vartheta_{a_4} + (2\pi\sqrt{-1}\frac{a_4}{a_3})^2\varepsilon_3 + \lambda_1(\lambda_2+k)]\phi = 0, \tag{4.71}$$

$$[\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0. \tag{4.72}$$

Here $\phi \in C^\infty(A)$.

Proof. As well as Proposition 4.17 and Proposition 4.26, this system of differential equations are obtained by the direct computation by using Lemma 4.12, Proposition 4.13 and Proposition 4.16. □

Theorem 4.31. *The space* $C^\infty_{\chi_{l(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(A; I_k(\lambda))$ *are written as follows.*

(i) When

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \begin{cases}(1,1,1)\\(1,1,0)\\(0,1,1)\end{cases},$$

then we have

$$C^\infty_{\chi_{l(0,0,0,\varepsilon_1,\varepsilon_2,\varepsilon_3)}}(A; I_k(\lambda)) = \{0\}.$$

(ii) When $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (1,0,1)$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,1)}}(A; I_k(\lambda))$ *consists of followings. If* $k=1$, *we have* $C^\infty_{\chi_{l(0,0,0,1,0,1)}}(A; I_1(\lambda)) = \{0\}$. *If* $k=2$, *the space* $C^\infty_{\chi_{l(0,0,0,1,0,1)}}(A; I_2(\lambda))$ *consists of*

$$x_1^{\frac{1}{2}} x_2^{\frac{\lambda_1+\lambda_2-1}{2}} x_3^{\frac{1}{2}} x_4^{\frac{\lambda_1+\lambda_2+1}{2}} f(2\pi x_1) g(2\pi x_3)$$

for $f(x), g(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}; x)$. *Here we put*

$$x_1 = \frac{a_2}{a_1}, x_2 = a_1 a_2, x_3 = \frac{a_4}{a_3}, x_4 = a_3 a_4.$$

Thus we have $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(A; I_2(\lambda)) = 4$. *There is a 1-dimensional subspace of* $C^{\infty}_{\chi_{l(0,0,0,1,0,1)}}(A; I_2(\lambda))$ *which consists slowly increasing functions on* $\{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}_{>0}, i = 1, \dots, 4\}$.

(iii) *When* $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 0)$, *the space* $C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_k(\lambda))$ *consists of*

$$x_1^{\frac{1}{2}} x_2^{\frac{\lambda_1+\lambda_2+k-3}{2}} f(2\pi x_1) \times \begin{cases} x_3^{\lambda_2+1} x_4^{\lambda_2+1} & \text{if } k = 1, \\ C_1 x_3^{\lambda_2+1} x_4^{\lambda_1} + C_2 x_3^{\lambda_1-1} x_4^{\lambda_2+2} & \text{if } k = 2, \end{cases}$$

for $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2+k-4}{2}; x)$ *and* $C_1, C_2 \in \mathbb{C}$. *Here we put*

$$x_1 = \frac{a_2}{a_1}, x_2 = a_1 a_2, x_3 = a_3, x_4 = a_4.$$

Thus if $k = 1$, *we have* $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_1(\lambda)) = 2$. *There is a 1-dimensional subspace of* $C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_1(\lambda))$ *which consists slowly increasing functions on* $\{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}_{>0}, i = 1, \dots, 4\}$.
Also if $k = 2$, *we have* $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_2(\lambda)) = 4$. *There is a 2-dimensional subspace of* $C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_2(\lambda))$ *which consists slowly increasing functions on* $\{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}_{>0}, i = 1, \dots, 4\}$.

(iv) *When* $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 0)$, *the space* $C^{\infty}_{\chi_{l(0,0,0,0,1,0)}}(A; I_k(\lambda))$ *consists of followings. We put*

$$x_1 = a_1, x_2 = \frac{a_3}{a_2}, x_3 = a_2 a_3, x_4 = a_4.$$

If $k = 1$,

$$x_1^{\lambda_2} x_2^{\frac{1}{2}} x_3^{\frac{\lambda_1+\lambda_2+k-2}{2}} x_4^{\lambda_2+1} f(2\pi x_2)$$

for $f(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-2}{2}; x)$. *Thus we have* $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,0,1,0)}}(A; I_1(\lambda)) = 2$. *There is a 1-dimensional subspace of* $C^{\infty}_{\chi_{l(0,0,0,0,1,0)}}(A; I_k(\lambda))$ *which consists slowly increasing functions on* $\{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}_{>0}, i = 1, \dots, 4\}$.
Also if $k = 2$,

$$C x_1^{\lambda_1-2} x_2^{\frac{1}{2}} x_3^{\frac{\lambda_1+\lambda_2+k-2}{2}} x_4^{\lambda_2+2} g_1(2\pi x_2) + C' x_1^{\lambda_2} x_2^{\frac{1}{2}} x_3^{\frac{\lambda_1+\lambda_2+k-2}{2}} x_4^{\lambda_1} g_2(2\pi x_2)$$

for $C, C' \in \mathbb{C}$, $g_1 \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-3}{2}; x)$ *and* $g_2 \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2}{2}; x)$. *Thus we have* $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,0,1,0)}}(A; I_2(\lambda)) = 4$. *There is a 1-dimensional subspace of* $C^{\infty}_{\chi_{l(0,0,0,1,0,0)}}(A; I_k(\lambda))$ *which consists slowly increasing functions on* $\{(x_1, \dots, x_4) \mid x_i \in \mathbb{R}_{>0}, i = 1, \dots, 4\}$.

(v) *When* $(\vartheta_1, \vartheta_2, \vartheta_3) = (0, 0, 1)$, *the space* $C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(A; I_k(\lambda))$ *consists of*

$$x_3^{\frac{1}{2}} x_4^{\frac{\lambda_1+\lambda_2+k-1}{2}} g(2\pi x_3) \times \begin{cases} x_1^{\lambda_2} x_2^{\lambda_2} & \text{if } k = 1, \\ C_1 x_1^{\lambda_2} x_2^{\lambda_1-1} + C_2 x_1^{\lambda_1^2} x_2^{\lambda_2+1} & \text{if } k = 2, \end{cases}$$

for $g(x) \in \mathfrak{MB}(\frac{\lambda_1-\lambda_2-k}{2}; x)$ *and* $C_1, C_2 \in \mathbb{C}$. *Here we put*

$$x_1 = a_1, x_2 = a_2, x_3 = \frac{a_4}{a_3}, x_4 = a_3 a_4.$$

Thus if $k=1$, we have $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(A;I_1(\lambda))=2$. There is a 1-dimensional subspace of $C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(A;I_1(\lambda))$ which consists slowly increasing functions on $\{(x_1,\dots,x_4)\mid x_i\in\mathbb{R}_{>0}, i=1,\dots,4\}$.

Also if $k=2$, we have $\dim_{\mathbb{C}} C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(A;I_2(\lambda))=4$. There is a 2-dimensional subspace of $C^{\infty}_{\chi_{l(0,0,0,0,0,1)}}(A;I_2(\lambda))$ which consists slowly increasing functions on $\{(x_1,\dots,x_4)\mid x_i\in\mathbb{R}_{>0}, i=1,\dots,4\}$.

(vi) *When $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(0,0,0)$, the space $C^{\infty}_{\chi_{l(0,0,0,0,0,0)}}(A;I_k(\lambda))$ consists of the followings. If $k=1$,*

$$C_1 a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_2}a_4^{\lambda_1}+C_2 a_1^{\lambda_2}a_2^{\lambda_1-2}a_3^{\lambda_2+1}a_4^{\lambda_2+1}$$
$$+C_3 a_1^{\lambda_1-3}a_2^{\lambda_2+1}a_3^{\lambda_2+1}a_4^{\lambda_2+1}+C_4 a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_1-1}a_4^{\lambda_2+1},$$

for $C_i\in\mathbb{C}$, $i=1,\dots,4$.

Also if $k=2$,

$$C_1 a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_1}a_4^{\lambda_1}+C_2 a_1^{\lambda_1-2}a_2^{\lambda_1-1}a_3^{\lambda_2+1}a_4^{\lambda_1}$$
$$+C_3 a_1^{\lambda_1-2}a_2^{\lambda_2+1}a_3^{\lambda_2+1}a_4^{\lambda_1}+C_4 a_1^{\lambda_2}a_2^{\lambda_1-1}a_3^{\lambda_1-1}a_4^{\lambda_2+2}$$
$$+C_5 a_1^{\lambda_1-2}a_2^{\lambda_2+1}a_3^{\lambda_1-1}a_4^{\lambda_2+2}+C_6 a_1^{\lambda_1-2}a_2^{\lambda_1-2}a_3^{\lambda_2+2}a_4^{\lambda_2+2}$$

for $C_i\in\mathbb{C}$, $i=1,\dots,6$.

Proof. (i) It is immediately follows from equations (4.65) and (4.68).

The remainig cases are show by solving the following differential equations.

(ii) If we put $x_1=\frac{a_2}{a_1}, x_2=a_1a_2, x_3=\frac{a_4}{a_3}, x_4=a_3a_4$, differential equations are written as

$$[\vartheta_{x_1}^2-\vartheta_{x_1}-\frac{1}{4}(\lambda_1+\lambda_2+k-3)(\lambda_1+\lambda_2+k-1)$$
$$+(2\pi\sqrt{-1}x_1)^2+(\lambda_1+k-3)(\lambda_2+1)]\phi=0,$$
$$[2\vartheta_{x_2}-(\lambda_1+\lambda_2+k-3)]\phi=0,$$
$$[\vartheta_{x_3}^2-\vartheta_{x_3}-\frac{1}{4}((\lambda_1+\lambda_2+k)^2+1)+(2\pi\sqrt{-1}x_3)^2+\lambda_1(\lambda_2+k)]\phi=0,$$
$$[2\vartheta_{x_4}-(\lambda_1+\lambda_2+k-1)]\phi=0,$$
$$[2\vartheta_{x_2}+2\vartheta_{x_4}-k\lambda_1-(4-k)\lambda_2]\phi=0.$$

(iii) If we put $x_1=\frac{a_2}{a_1}, x_2=a_1a_2, x_3=a_3, x_4=a_4$, differential equations are written as

$$[\vartheta_{x_1}^2-\vartheta_{x_1}-\frac{1}{4}(\lambda_1+\lambda_2+k-3)(\lambda_1+\lambda_2+k-1)$$
$$+(2\pi\sqrt{-1}x_1)^2+(\lambda_1+k-3)(\lambda_2+1)]\phi=0,$$
$$[2\vartheta_{x_2}-(\lambda_1+\lambda_2+k-3)]\phi=0,$$
$$(\vartheta_{x_3}-(\lambda_1+k-3))(\vartheta_{x_3}-(\lambda_2+1))\phi=0,$$
$$(\vartheta_{x_4}-\lambda_1)(\vartheta_{x_4}-(\lambda_2+k))\phi=0,$$
$$[\vartheta_{x_3}+\vartheta_{x_4}+(1-k)\lambda_1+(k-3)\lambda_2+(k-3)]\phi=0,$$
$$[2\vartheta_{x_2}+\vartheta_{x_3}+\vartheta_{x_4}-k\lambda_1-(4-k)\lambda_2]\phi=0.$$

(iv) If we put $x_1 = a_1, x_2 = \frac{a_3}{a_2}, x_3 = a_2a_3, x_4 = a_4$, differential equations are written as

$$\begin{aligned}
(\vartheta_{x_1} - (\lambda_1 + (k-4)))(\vartheta_{x_1} - \lambda_2)\phi = 0,\\
[\vartheta_{x_2}^2 - \vartheta_{x_2} - \frac{1}{4}(\lambda_1 + \lambda_2 + k - 2)(\lambda_1 + \lambda_2 + k)\\
+ (2\pi\sqrt{-1}x_2)^2 - \vartheta_{x_4} + \lambda_1(\lambda_2 + k)]\phi = 0,\\
(2\vartheta_{x_3} - (\lambda_1 + \lambda_2 + k - 2))\phi = 0,\\
(\vartheta_{x_4} - \lambda_1)(\vartheta_{x_4} - (\lambda_2 + k))\phi = 0,\\
[\vartheta_{x_1} + 2\vartheta_{x_3} + \vartheta_{x_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0.
\end{aligned}$$

(v) If we put $x_1 = a_1, x_2 = a_2, x_3 = \frac{a_4}{a_3}, x_4 = a_3a_4$, differential equations are written as

$$\begin{aligned}
(\vartheta_{x_1} - (\lambda_1 - 4 + k))(\vartheta_{x_1} - \lambda_2)\phi = 0,\\
(\vartheta_{x_2} - (\lambda_1 - 1))(\vartheta_{x_2} - (\lambda_2 - 1 + k))\phi = 0,\\
[\vartheta_{x_3}^2 - \vartheta_{x_3} - \frac{1}{4}((\lambda_1 + \lambda_2 + k)^2 + 1) + (2\pi\sqrt{-1}x_3)^2 + \lambda_1(\lambda_2 + k)]\phi = 0,\\
[2\vartheta_{x_4} - (\lambda_1 + \lambda_2 + k - 1)]\phi = 0,\\
[\vartheta_{x_1} + \vartheta_{x_2} + 2\vartheta_{x_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0.
\end{aligned}$$

(vi) Differential equations are written as

$$\begin{aligned}
(\vartheta_{a_1} - (\lambda_1 - (k-4)))(\vartheta_{a_1} - \lambda_2)\phi = 0,\\
[\vartheta_{a_2}^2 - (\lambda_1 + \lambda_2 + k - 2)\vartheta_{a_2} - (\vartheta_{a_3} + \vartheta_{a_4}) + \lambda_1(\lambda_2 + k)]\phi = 0,\\
[\vartheta_{a_3}^2 - (\lambda_1 + \lambda_2 + k - 1)\vartheta_{a_3} - \vartheta_{a_4} + \lambda_1(\lambda_2 + k)]\phi = 0,\\
(\vartheta_{a_4} - \lambda_1)(\vartheta_{a_4} - (\lambda_2 + k))\phi = 0,\\
[\vartheta_{a_1} + \vartheta_{a_2} + \vartheta_{a_3} + \vartheta_{a_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0.
\end{aligned}$$

□

Appendix

A The table of dimentions of generalized Whittaker functions of $GL(4,\mathbb{R})$

We summarize the dimensions of generalized Whittaker functions of $GL(4,\mathbb{R})$ as the table below. The notations are same as in Section 4. The first row of the table describes the basis of the space of generalized Whittaker functions. The second row describes the dimensions and the third row the dimensions of the spaces of functions which satisfy the growth conditions. For detailed conditions, see Section 4.
(i) Generalized Whittaker functions for $X_{1,\lambda}$.

	I_1	I_2	II_1	II_2	II_3	II_4
basis	0	$\mathfrak{MB}$	0	0	$\mathfrak{MB}$	$\mathfrak{MB}$
dim	0	2	0	0	2	2
$\dim^{\text{growth}}$	0	1	0	0	1	1

,

III_1	III_2	III_3	III_4	III_5	III_6	III_7	III_8
0	0	0	0	$\mathfrak{MB}$	$\mathfrak{MB}$	$\mathfrak{MB}$	x^α
0	0	0	0	2	2	2	4
0	0	0	0	1	1	1	4

.

(ii) Generalized Whittaker functions for $X_{2,\lambda}$.

	I_1	I_2	II_1	II_2	II_3	II_4
basis	$\mathfrak{H}_{10}$	$\mathfrak{MB}+\mathfrak{MB}$	0	$\mathfrak{MB}\times\mathfrak{MB}$	$(x^\alpha+x^\beta)\mathfrak{MB}$	$(x^\alpha+x^\beta)\mathfrak{MB}$
dim	4	4	0	4	4	4
$\dim^{\text{growth}}$	1	2	0	1	2	2

,

III_1	III_2	III_3	III_4	III_5	III_6
0	0	$\mathfrak{MB}\times\mathfrak{MB}$	0	$(x^\alpha+x^\beta)\mathfrak{MB}$	$\mathfrak{MB}+\mathfrak{MB}$
0	0	4	0	4	4
0	0	1	0	2	2

,

III_7	III_8
$(x^\alpha+x^\beta)\mathfrak{MB}$	x^α
4	6
2	6

.

B The multiplicity one theorem for Horn's hypergeometric functions

We consider asymptotic behaviors at the infinity of Horn's hypergeometric functions for the purpose of the application to the multiplicity theorem for generalized Whittaker models.

Let $P_i(x)$ and $Q_i(x)$ are nonzero polynomials of variables $x=(x_1,\dots,x_n)$ for $i=1,\dots,n$. Then the Horn's hypergeometric functions are defined as solutions of the system of linear partial differential equations

$$[x_iP_i(\vartheta)-Q_i(\vartheta)]f(x)=0,\quad i=1,\dots,n. \tag{B.1}$$

Here $\vartheta_i=x_i\frac{\partial}{\partial x_i}$ and $\vartheta=(\vartheta_1,\dots,\vartheta_n)$. In this note, we assume that P_i and Q_i can be decomposed by products of linear factors, i.e.,

$$P_i(s)=\prod_{k=1}^{p}(\langle A_k,s\rangle-c_i),\qquad Q_i(s)=\prod_{l=1}^{q}(\langle B_l,s\rangle-d_l)$$

for $s\in\mathbb{R}^n$, $A_k,B_l\in\mathbb{R}^n$, $c_k,d_l\in\mathbb{C}$ and $\langle\ ,\ \rangle$ denote the natural inner product in $\mathbb{R}^n$. We also assume $P_i(s)$, $Q_i(s+e_i)$ are relatively prime for $i=1,\dots,n$. Here $e_i=(0,\dots,0,1,0,\dots,0)$ (1 in the ith position).

We consider the following system of difference equations associated with the system of differential equations (B.1),

$$P_i(-(s+e_i))\phi(s+e_i)=Q_i(-s)\phi(s)\quad i=1,\dots,n. \tag{B.2}$$

Remark B.1. *Let ϕ be a solution of the system of difference equations (B.2). We consider the following integral,*

$$f(x) = \int_C \phi(s) x^{-s}\, ds.$$

Then under the following assumptions, we can see that $f(x)$ is a solution of the system of differential equations (B.1).

1. *For any $i = 1, \ldots, n$, the translation of the contour C with respect to the basis e_i is homologically equivalent to C in the complement of the set of the singularities of the integrand $\phi(s)$ in $\mathbb{C}^n$.*

2. *The integral converges absolutely and it can be differentiated with respect to x sufficiently many times.*

We put

$$R_i(s) = \frac{Q_i(-s)}{P_i(-(s+e_i))} \quad i = 1, \ldots, n.$$

Theorem B.2 (Ore [18], Sato [25], Sadykov [24]). *1. The system of difference equations (B.2) is solvable if and only if*

$$R_i(s+e_j)R_j(s) = R_j(s+e_i)R_i(s), \quad i,j = 1, \ldots, n. \qquad \text{(B.3)}$$

2. If the system (B.2) is solvable, then its solution is unique up to an arbitrary periodic function $\psi(s)$ with respect to e_i, i.e.,

$$\psi(s+e_i) = \psi(s),$$

for $i = 1, \ldots, n$. Furthermore, there exist $p', q' \in \mathbb{N}$, $A'_k, B'_l \in \mathbb{R}^n$ ($1 \le k \le p', 1 \le l \le q'$), $c'_k, d'_l \in \mathbb{C}$ ($1 \le k \le p', 1 \le l \le q'$) and $t_i \in \mathbb{R}$ ($i = 1, \ldots, n$) such that the general solution of (B.2) is written as follows,

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)} \psi(s),$$

where $t^{-s} = t_1^{-s_1} \cdots t_n^{-s_n}$ and $\psi(s)$ is an arbitrary periodic function satisfying $\psi(s+e_i) = \psi(s)$.

We put an assumption as follows for the multiplicity theorem.

(A). The system of difference equations (B.2) is solvable, i.e., the condition (B.3) is satisfied and we can choose a solution,

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)}$$

which satisfies following conditions;

(i) We have the inequality,

$$\sum_{l=1}^{q'} |\langle B'_l, s\rangle| - \sum_{k=1}^{p'} |\langle A'_k, s\rangle| \ge \sum_{i=1}^{n} |s_i|$$

for $s \in \mathbb{R}^n$.

(ii) The function $\phi(s)$ has no zero if each $\mathrm{Re}(s_i)$ are sufficiently large for $i = 1, \ldots, n$.

Remark B.3. *We consider the integral*

$$f(x) = \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty} \cdots \int_{\sigma_n-\sqrt{-1}\infty}^{\sigma_n+\sqrt{-1}\infty} \phi(s) x^{-s}\, ds,$$

for appropriate $\sigma_i \in \mathbb{R}$ $i = 1, \ldots, n$. *Under the assumption (A)-(i), it follows that the integral is absolutely convergent in the set* $\{x \in \mathbb{R}^n \mid (t_1x_1, \ldots, t_nx_n) \in (\mathbb{R}_{\geq 0})^n\}$.

The following theorem is a generalization of the theorem of Diaconu and Goldfeld (Theorem 6.1.6 in [5])

Theorem B.4 (Multiplicity one). *Suppose that the system of difference equations* (B.2) *associated with the one of differential equations* (B.1) *satisfies the assumption (A). Let* $f(x)$ *be a solution of the system* (B.1) *which satisfies the growth condition*

$$\sup_{x \in (\mathbb{R}_{\geq 0})^n} |x^\alpha f(tx)| < +\infty$$

for sufficiently large integers $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$. *Then it is unique up to constant multiple. Here* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *and* $tx = (t_1x_1, \ldots, t_nx_n)$.

Proof. We consider the Mellin transform of $f(tx)$ as the function of x,

$$\mathcal{M}[f, s] = \int_0^\infty \cdots \int_0^\infty f(tx) x^{s-1}\, dx.$$

This integral converges absolutely and $\mathcal{M}[f, s]$ is analytic function of s if each $\mathrm{Re}(s_i)$ is sufficiently large by the assumption of $f(x)$. Changing the variables x to $tx = (t_1x_1, \cdots, t_nx_n)$, then we have

$$\mathcal{M}[f, s] = t^{-s} \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) x^{s-1}\, dx.$$

By the growth condition of $f(x)$, we have

$$\int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} \frac{\partial^k}{\partial x_i^k} f(x) x^{s-1}\, dx$$
$$= (-1)^k \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) \frac{\partial^k}{\partial x_i^k} x^{s-1}\, dx,$$

by integration by parts for $i = 1, \ldots, n$. Recall that $f(x)$ satisfies the system of the partial differential equations (B.1), then we have the system of the difference equations for $\mathcal{M}[f, s]$,

$$P_i(-(s + e_i))\mathcal{M}[f, s + e_i] = Q_i(-s)\mathcal{M}[f, s] \quad i = 1, \ldots, n.$$

Hence by Theorem B.2, there is a periodic function $\psi(s)$ and we have

$$\frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)} \psi(s) = \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) x^{s-1}\, dx. \qquad \text{(B.4)}$$

By Stirling's formula and the assumption (A)-(i), we obtain the estimate for $\mathrm{Re}(s_i) > 0$ $(i = 1, \ldots, n)$,

$$\frac{\prod_{l=1}^{q} \Gamma(\langle B_l, s\rangle - d_l)}{\prod_{k=1}^{p} \Gamma(\langle A_k, s\rangle - c_i)} = O\left(\exp(-\frac{1}{2}\pi \sum_{i=1}^{n} |\mathrm{Im}(s_i)|)\right) \quad \text{as} \quad \sum_{i=1}^{n} |\mathrm{Im}(s_i)| \to +\infty.$$

Also by the Riemann-Lebesgue theorem, we have

$$\mathcal{M}[f, s] \to 0 \quad \text{as} \quad \sum_{i=1}^{n} |\mathrm{Im}(s_i)| \to +\infty.$$

Combining these estimates, we obtain the asymptotic behaviour of the periodic function

$$\psi(s) = O(\exp(\frac{1}{2}\pi |\mathrm{Im}(s_i)|)), \tag{B.5}$$

as $\mathrm{Im}(s_i) \to \infty$ and the other s_j $(i \neq j)$ are fixed. The right hand side of the equation (B.4) is the analytic function of s when $\mathrm{Re}(s_i)$ $(i = 1, \ldots, n)$ are sufficiently large. Thus if we recall that the assumption (A)-(ii) and the periodicy of $\psi(s)$, we can see that $\psi(s)$ is an entire function. We put $z_i = \exp 2\pi\sqrt{-1}s_i$ for $i = 1, \ldots, n$. And we consider the Laurant expansion of $\phi(s)$ with respect to z_1,

$$\psi(s) = \sum_{k=-\infty}^{\infty} c_k^{(1)}(s_2, \ldots, s_n) z_1^k.$$

Here $c_k^{(1)}(s_2, \ldots, s_n)$ are periodic and entire functions for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. We write $s_i = \sigma_i + \sqrt{-1}\tau_i$ for $\sigma_i, \tau_i \in \mathbb{R}$, $i = 1, \ldots, n$. We consider an integration

$$\begin{aligned} \int_0^1 |\psi(s)|^2 \, d\sigma_i &= \sum_{k=-\infty}^{\infty} |c_k^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi k \tau_i) \\ &\geq |c_t^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi t \tau_i) \end{aligned}$$

for every $t = 0, \pm 1, \pm 2, \ldots$. However the estimate (B.5) tells us that there exist constants $M_i \in \mathbb{R}_{>0}$ and we have

$$\exp(\pi|\tau_i|) > M_i \int_0^1 |\psi(s)|^2 \, d\sigma_i$$

for sufficiently large τ_i. Thus we have $c_t^{(1)}(s_2, \ldots, s_n) = 0$ for $t = \pm 1, \pm 2, \ldots$. The remaining coefficient $c_0^{(1)}(s_2, \ldots, s_n)$ is also the periodic and entire functions for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. Hence we apply the same argument for $c_0^{(1)}(s_2, \ldots, s_n)$ with respect to s_2. And also we can proceed inductively for $i = 3, \ldots, n$. Thus we can conclude $\psi(s)$ must be a constant. This completes the proof of the theorem. □

C Horn's hypergeometric function H_{10}

We give some facts about Horn's two variables hypergeometric function H_{10}. Horn's hypergeometric function H_{10} is the hypergeometric seires defined as follows,

$$\mathrm{H}_{10}(a, d; x, y) = \sum_{m=0, n=0}^{\infty} \frac{(a)_{2m-n}}{(d)_m m! n!} x^m y^n.$$

Here the symbol $(a)_m$ means the Pochhammer symbol, i.e., $(a)_m = a(a+1)\cdots(a+(m-1))$ for $a \in \mathbb{C}$ and $m \in \mathbb{N}$. It is not hard to see that this power series satisfies the system of hypergeometric partial differential equations,

$$\begin{aligned} \{x(2\vartheta_x - \vartheta_y + a)(2\vartheta_x - \vartheta_y + a + 1) - \vartheta_x(\vartheta_x + d - 1)\}\phi(x,y) = 0, \\ \{y - \vartheta_y(2\vartheta_x - \vartheta_y + a)\}\phi(x,y) = 0. \end{aligned} \tag{C.1}$$

It is known that the dimension of the solution space is 4 (cf. [2]). We define another convergent series

$$\tilde{\mathbb{H}}_{10}(a,d\,;\,x,y) = \sum_{m=0,n=0}^{\infty} \frac{(-1)^{m+2n}}{(a+1)_{m+2n}(d)_n m! n!} x^m y^n.$$

Then the basis of the solution space are written by the power series below,

$$\begin{gathered} \mathrm{H}_{10}(a,d\,;\,x,y) \\ y^{-d+1}\mathrm{H}_{10}(a-2d+2,-d+2\,;\,x,y), \\ x^a\tilde{\mathbb{H}}_{10}(a,d\,;\,x,x^2y), \\ x^a y^{-d+1}\tilde{\mathbb{H}}_{10}(a-2d+3,-d+2\,;\,x,x^2y). \end{gathered}$$

The system of hypergeometric differential equations $(C.1)$ has the solution which has the Mellin-Barnes integral representation. This is written as follows,

$$\phi(x,y) = \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} \Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)(-x)^{-s_1}y^{-s_2}\,ds_1\,ds_2.$$

Here $\sigma_1 \in \mathbb{R}$ and $\sigma_2 \in \mathbb{R}$ satisfy the conditions, $\sigma_1 > 0$, $\sigma_2 > \max\{0, \mathrm{Re}(d-1)\}$ and $\sigma_1 - 2\sigma_2 > \mathrm{Re}(a)$. This integral converges absolutely for $x \in \mathbb{R}_{\leq 0}$ and $y \in \mathbb{R}_{\geq 0}$.

Theorem C.1. *If $f(x,y)$ is a solution of the system (C.1) which satisfies that*

$$\sup_{x,y\in\mathbb{R}_{\geq 0}} |x^{\alpha_1}y^{\alpha_2}f(-x,y)| < +\infty$$

for sufficiently large $\alpha_1, \alpha_2 \in \mathbb{N}$, then

$$f(x,y) = C\int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} \Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)(-x)^{-s_1}y^{-s_2}\,ds_1\,ds_2,$$

for some constant C.

Proof. For

$$\phi(x,y) = \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} \Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)(-x)^{-s_1}y^{-s_2}\,ds_1\,ds_2,$$

it is easy to see that ϕ satisfies the assumptions of Theorem B.4. Hence we only need to check that ϕ satisfies the growth condition. If we write a complex number $s = \sigma + \sqrt{-1}\tau$, we have

$$|x|^{-s} = |x|^{-\sigma}.$$

Thus we have the inequality,

$$|\phi(x, y)| \leq M|x|^{-\sigma_1}|y|^{-\sigma_2},$$

for $x \in \mathbb{R}_{\leq 0}$ and $y \in \mathbb{R}_{\geq 0}$. Here the constant

$$M = \left| \int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty} \int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty} \Gamma(s_1)\Gamma(s_1 - 2s_2 - a)\Gamma(s_2)\Gamma(s_2 - d + 1)\, ds_1\, ds_2 \right|.$$

We can choose σ_1 and σ_2 as $\sigma_1 > 0$, $\sigma_2 > \max\{0, \mathrm{Re}(d-1)\}$ and $\sigma_1 - 2\sigma_2 > \mathrm{Re}(a)$. Thus $\phi(x, y)$ satisfies the growth condition. □

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