J. Math. Sci. Univ. Tokyo
6 (1999), 315–333.

Some Weighted Inequalities for the Kakeya Maximal Operator on Functions of Product Type

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Abstract. We shall prove some weighted inequalities for the Kakeya maximal operator restricting it to functions of product type. We shall also describe a detailed proof of a comparison theorem for two maximal operators of Kakeya type which is used in the proof.

1. Introduction and Theorems

In this paper we shall prove some weighted inequalities for the Kakeya maximal operator, restricting it to functions of product type. In the proof we shall use a comparison theorem for two maximal operators of Kakeya type a detailed proof of which will also be presented.

Fix $N >> 1$. For a real number $a > 0$ let $\mathcal{B}_{a,N}$ be the family of all rectangles in the d-dimensional Euclidean space $\mathbf{R}^d$, $d \geq 2$, which are congruent to the rectangle $(0,a)^{d-1} \times (0,Na)$ but with arbitrary direction and center. The so-called small Kakeya maximal operator $M_{a,N}$ is defined on locally integrable functions f on $\mathbf{R}^d$ by

$$(M_{a,N}f)(x) = \sup_{x \in R \in \mathcal{B}_{a,N}} \frac{1}{|R|} \int_R |f(y)|dy,$$

where $|A|$ represents the Lebesgue measure of a set A.

Let $\mathcal{B}_N$ be the family of rectangles defined by $\mathcal{B}_N = \bigcup_{a>0} \mathcal{B}_{a,N}$. The Kakeya maximal operator K_N is defined by

$$(K_N f)(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)|dy.$$

*Supported by Japan Society for the Promotion of Sciences and Fūjyukai Foundation.
1991 *Mathematics Subject Classification*. Primary 42B25.
This paper is a part of the thesis of the doctor of science [Ta4] Chapter 4 submitted to Gakushuin University.

A weight w is defined as a nonnegative locally integrable function on $\mathbf{R}^d$ and we will represent the norm of the function space $L^p(\mathbf{R}^d, w)$ as

$$\|f\|_{L^p(\mathbf{R}^d,w)} = (\int_{\mathbf{R}^d} |f(x)|^p w(x) dx)^{1/p}.$$

If $d = 2$, then for f in $L^d(\mathbf{R}^d)$ the inequalities

$$\|M_{a,N} f\|_d \leq C(\log N)^{\alpha_d} \|f\|_d, \quad \forall a > 0, \tag{1}$$

and

$$\|K_N f\|_d \leq C'(\log N)^{\alpha'_d} \|f\|_d \tag{2}$$

hold with $\alpha_d = 1/d$ and $\alpha'_d = 1 + 1/d$. (Córdoba [Co]). For $d \geq 3$ (1) is known to be true for functions of the form $f(x) = \prod_{l=1}^d f_l(x_l)$ (cf. Igari [Ig1] and also Tanaka [Ta1]) and for functions of square radial type (cf. Tanaka [Ta2]). For $d \geq 3$ (2) is known to be true for functions of radial type (cf. Carbery, Hernández and Soria [CHS] and Igari [Ig2]).

If $d = 2$, then the weighted inequality

$$\|K_N f\|_{L^p(\mathbf{R}^d,w)} \leq C_{N,p} \|f\|_{L^p(\mathbf{R}^d,K_N w)} \tag{3}$$

holds with

$$C_{N,p} = \begin{cases} O(N^{d/p-1}(\log N)^{\beta_{p,d}}), & 1 < p \leq d, \\ O((\log N)^{\beta_{p,d}}), & d < p < \infty, \end{cases}$$

for some constant $\beta_{p,d} > 0$. (Müller and Soria [MS]). For $d \geq 3$ (3) is known to be true for the range $1 < p \leq (d+1)/2$ (cf. Vargas [Va]) and for functions of radial type (cf. Tanaka [Ta3]).

In this parer we shall prove that a strong-type d estimate for $M_{a,N}$:

$$\|M_{a,N} f\|_{L^d(\mathbf{R}^d,w)} \leq C(\log N) \|f\|_{L^d(\mathbf{R}^d,K_N w)}, \quad \forall a > 0,$$

and a weak-type d estimate for K_N:

$$w(\{x \in \mathbf{R}^d \,|\, (K_N f)(x) > \lambda\})^{1/d} \leq C' \frac{\log N}{\lambda} \|f\|_{L^d(\mathbf{R}^d,K_N w)}, \quad \forall \lambda > 0,$$

hold for f in $L^d(\mathbf{R}^d, K_N w)$ of the form $f(x) = \prod_{l=1}^d f_l(x_l)$, where $w(A)$ denotes $w(x)dx$ measure of a set A. As yet we have not been able to prove weighted strong type d estimate for K_N. We also note that when $w \equiv 1$, the factor $\log N$ in strong type d estimate is weaker than the factor $(\log N)^{1-1/d}$, which was already obtained in [Ta1].

We shall restate our results in the form of thorems.

THEOREM 1. *Let $d \geq 2$. There exists a constant C depending only on the dimension d such that for every $a > 0$ and $N >> 1$ and for every nonnegative locally integrable weight w*

$$\|M_{a,N} f\|_{L^d(\mathbf{R}^d,w)} \leq C \log N \|f\|_{L^d(\mathbf{R}^d,K_N w)}$$

holds for all f in $L^d(\mathbf{R}^d, K_N w)$ of the form

$$f(x_1, \ldots, x_d) = \prod_{l=1}^{d} f_l(x_l).$$

THEOREM 2. *Let $d \geq 2$. There exists a constant C depending only on the dimension d such that for every $N >> 1$ and $\lambda > 0$ and for every nonnegative locally integrable weight w*

$$(w(\{x \in \mathbf{R}^d \,|\, (K_N f)(x) > \lambda\}))^{1/d} \leq C \frac{\log N}{\lambda} \|f\|_{L^d(\mathbf{R}^d,K_N w)}$$

holds for all f in $L^d(\mathbf{R}^d, K_N w)$ of the form

$$f(x_1, \ldots, x_d) = \prod_{l=1}^{d} f_l(x_l).$$

To prove these theorems we will need a comparison theorem for two maximal operators of Kakeya type.

Let $\mathcal{B}_{\leq N}$ be the class of all rectangles in $\mathbf{R}^d$ which satisfy

$$1 \leq (\text{the length of longest sides})/(\text{the length of shortest sides}) \leq N.$$

The corresponding maximal operator associated to this base $\mathcal{B}_{\leq N}$ will be denoted by $K_{\leq N}$.

THEOREM 3. *Let $d \geq 2$. There exists a constant C depending only on the dimension d such that*

$$(K_N f)(x) \leq (K_{\leq N} f)(x) \leq C(K_N f)(x) \tag{4}$$

holds for every locally integrable function f on $\mathbf{R}^d$ and for every point x in $\mathbf{R}^d$.

The maximal operator $K_{\leq N}$ was considered in [Mu]. But the above theorem seems not to have been noticed in the literature. This theorem will be proved in Section 4.

In the following C's will denote constants which may be different in each occasion but depend only on the dimension d.

2. Proof of Theorem 1

In this section we shall prove Theorem 1.

We may assume that $f_l \geq 0$ and N is a positive integer. By dilation invariance it suffices to consider only the case $a = 1$. We write $M_{1,N}$ as M_N. We will linearize the problem first. We divide $\mathbf{R}^d$ into open unit cubes Q_i (and their boundaries) which have center at lattice points $i \in \mathbf{Z}^d$ and whose sides are parallel to the axes. By the local integrablity of f we can find for every cube Q_i a rectangle R_i from $\mathcal{B}_{1,N}$ such that

$$Q_i \cap R_i \neq \emptyset,$$

and

$$(M_N f)(x) \leq \frac{C}{|R_i|} \int_{R_i} f(y)dy, \quad \forall x \in Q_i. \tag{5}$$

This shows that for proving the theorem it is sufficient to estimate

$$\sum_{i \in \mathbf{Z}^d} \frac{1}{N} \int_{R_i} f(y)dy \cdot \chi_{Q_i}(x). \tag{6}$$

In the proof we use the following notations.

$$\gamma_i = \{j \in \mathbf{Z}^d \,|\, Q_j \cap R_i \neq \emptyset\},$$

$$\begin{aligned} P_l(Q_j) &= \text{(the projection of } Q_j \text{ on the } l\text{-th axis)} \\ &= (j_l - \frac{1}{2}, j_l + \frac{1}{2}), \quad j = (j_1, \dots, j_d). \end{aligned}$$

We shall prove a weighted estimate of (6) by the method we used in [Ta1], but with some necessary modifications due to the presence of the weight. By the same manipulation as in [Ta1], which we shall repeat for the convenience of the reader, we see that

$$
\begin{aligned}
(7) \qquad & N^d \int_{\mathbf{R}^d} (\sum_{i \in \mathbf{Z}^d} \frac{1}{N} \int_{R_i} f(y) dy \cdot \chi_{Q_i}(x))^d w(x) dx \\
&= \sum_{i \in \mathbf{Z}^d} (\int_{R_i} f(y) dy)^d w(Q_i) \\
&\le \sum_{i \in \mathbf{Z}^d} (\sum_{j \in \gamma_i} \int_{Q_j} f(y) dy)^d w(Q_i) \\
&= \sum_{i \in \mathbf{Z}^d} (\sum_{j \in \gamma_i} \prod_{l=1}^{d} \int_{P_l(Q_j)} f_l(y_l) dy_l)^d w(Q_i) \\
&\le \sum_{i \in \mathbf{Z}^d} (\sum_{j \in \gamma_i} \prod_{l=1}^{d} (\int_{P_l(Q_j)} f_l(y_l)^d dy_l)^{1/d})^d w(Q_i) \\
&\le \sum_{i \in \mathbf{Z}^d} \prod_{l=1}^{d} (\sum_{j \in \gamma_i} \int_{P_l(Q_j)} f_l(y_l)^d dy_l) w(Q_i).
\end{aligned}
$$

On the right hand side of (7) we compute as follows.

$$
\begin{aligned}
& \prod_{l=1}^{d} \sum_{j \in \gamma_i} \int_{P_l(Q_j)} f_l(y_l)^d dy_l \\
&\quad = \sum_{j_1, \dots, j_d \in \gamma_i} \prod_{l=1}^{d} \int_{P_l(Q_{j_l})} f_l(y_l)^d dy_l \\
&\quad = \sum_{j_1, \dots, j_d \in \gamma_i} \int_{Q_{((j_1)_1, \dots, (j_d)_d)}} f(y)^d dy,
\end{aligned}
$$

where $(j_l)_l$ is the l-th component of $j_l \in \mathbf{Z}^d$ and $((j_1)_1, \dots, (j_d)_d) \in \mathbf{Z}^d$.

Now, fix $\iota = (\iota_1, \dots, \iota_d) \in \mathbf{Z}^d$ and put

$$
\Omega_l^\iota = \mathbf{R}^{l-1} \times (\iota_l - \frac{1}{2}, \iota_l + \frac{1}{2}) \times \mathbf{R}^{d-l}.
$$

Then by a simple counting we see easily that the number of d-tuples

$(j_1,\dots,j_d)\in\gamma_i\times\ldots\times\gamma_i$ such that $((j_1)_1,\dots,(j_d)_d)=\iota$ is

$$\prod_{l=1}^{d}\mathrm{card}(\{j\in\mathbf{Z}^d\,|\,Q_j\cap\Omega_l^\iota\cap R_i\neq\emptyset\}).$$

Thus, we see that

$$\text{RHS of (7)}=\sum_{\iota\in\mathbf{Z}^d}X_\iota\int_{Q_\iota}f(y)^d dy,$$

where

$$X_\iota=\sum_{i\in\mathbf{Z}^d}\left(\prod_{l=1}^{d}\mathrm{card}(\{j\in\mathbf{Z}^d\,|\,Q_j\cap\Omega_l^\iota\cap R_i\neq\emptyset\})\right)w(Q_i). \tag{8}$$

Now we shall show that

$$X_0\le CN^d(\log N)^d\inf_{y\in Q_0}(K_{\le N}w)(y). \tag{9}$$

Let I_1 be

$$I_1=\{i=(i_1,\dots,i_d)\in\mathbf{Z}^d\,|\,0\le i_l\le N+1,\ l=1,\dots,d\}.$$

Then we may restrict the sum of (8) (with $\iota=0$) to I_1 by the symmetry and the fact that $\Omega_l^0\cap R_i\neq\emptyset$. Indeed, $\Omega_l^0\cap R_i\neq\emptyset$ implies $0\le i_l\le N+\sqrt{2}$. By a simple geometric consideration we have

$$\mathrm{card}(\{j\in\mathbf{Z}^d\,|\,Q_j\cap\Omega_l^0\cap R_i\neq\emptyset\})\le C\frac{N}{i_l+1} \tag{10}$$

for every $i=(i_1,\dots,i_d)$ in I_1. From this inequality and (8) we have

$$X_0\le CN^d\sum_{i\in I_1}(\prod_{l=1}^{d}\frac{1}{i_l+1})w(Q_i). \tag{11}$$

Thus, (9) follows from (11) and the following proposition.

PROPOSITION 4. *Let w be a nonnegative locally integrable weight on $\mathbf{R}^d$. Then we have*

$$\sum_{i\in I_1}(\prod_{l=1}^{d}\frac{1}{i_l+1})w(Q_i)\le C(\log N)^d\inf_{y\in Q_0}(K_{\le N}w)(y).$$

PROOF. Let the sequence $\{a(j)\}$ be

$$a(j) = \begin{cases} \frac{1}{j+1}, & j = 0, 1, \dots, N+1, \\ 1, & j = N+2, \\ 0, & j > N+2. \end{cases}$$

Then we see that $\dfrac{1}{l+1} = \sum_{k \geq l} a(k)a(k+1)$, for $0 \leq l \leq N+1$. It follows by this equality and by reversing the order of summation that

$$\begin{aligned}
&\sum_{i \in I_1} \Big(\prod_{l=1}^{d} \frac{1}{i_l + 1}\Big) w(Q_i) \\
&\quad = \sum_{i \in I_1} w(Q_i) \sum_{i_1 \leq j_1, \dots, i_d \leq j_d} \prod_{l=1}^{d} a(j_l) a(j_l + 1) \\
&\quad = \sum_{j \in I_1} \Big(\prod_{l=1}^{d} a(j_l + 1)\Big) \Big\{\Big(\prod_{l=1}^{d} a(j_l)\Big)\Big(\sum_{0 \leq i_1 \leq j_1, \dots, 0 \leq i_d \leq j_d} w(Q_i)\Big)\Big\} \\
&\quad \leq C \inf_{y \in Q_0} (K_{\leq N} w)(y) \times \sum_{j \in I_1} \prod_{l=1}^{d} a(j_l + 1) \leq C (\log N)^d \inf_{y \in Q_0} (K_{\leq N} w)(y). \ \square
\end{aligned}$$

By the translation invariance we see that the same inequality as (9) holds for every X_ι, $\iota \in \mathbf{Z}^d$. Thus, from (5)–(9) and Theorem 3 we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d} ((M_N f)(x))^d w(x) dx \\
&\quad \leq C (\log N)^d \sum_{i \in \mathbf{Z}^d} \inf_{y \in Q_i} (K_{\leq N} w)(y) \int_{Q_i} f(y)^d dy \\
&\quad \leq C (\log N)^d \int_{\mathbf{R}^d} f(y)^d (K_{\leq N} w)(y) dy \\
&\quad \leq C (\log N)^d \int_{\mathbf{R}^d} f(x)^d (K_N w)(x) dx.
\end{aligned}$$

Therefore, we have proved the theorem.

3. Proof of Theorem 2

In this section we shall prove Theorem 2.

Let $\tilde{\mathcal{B}}_{\leq N}$ be the class of all rectangles in $\mathbf{R}^d$ whose sides are parallel to the axes and which satisfy

$$1 \leq (\text{the length of longest sides})/(\text{the length of shortest sides}) \leq N.$$

The corresponding maximal operator associated to this base $\tilde{\mathcal{B}}_{\leq N}$ will be denoted by $M_{\leq N}$. Obviously, we have that

$$(M_{\leq N} f)(x) \leq (K_{\leq N} f)(x), \quad \forall x \in \mathbf{R}^d.$$

The proof is based on a couple of lemmas.

LEMMA 5. *Let $d \geq 2$. The inequality*

$$(K_N f)(x) \leq C((M_{\leq N} f^d)(x))^{1/d}, \quad \forall x \in \mathbf{R}^d,$$

holds for every locally integrable function f on $\mathbf{R}^d$ of the form $\prod_{l=1}^d f_l(x_l)$.

PROOF. We may assume that $f_l \geq 0$. Fix x in $\mathbf{R}^d$. For all $\epsilon > 0$ we can select some R from $\mathcal{B}_N$ such that $x \in R$ and

$$(K_N f)(x) - \epsilon \leq \frac{1}{|R|} \int_R f(y) dy. \tag{12}$$

Let $\omega = (\omega_1, \ldots, \omega_d)$ be a unit vector which is parallel to the axis of R. If we allow an extra factor C on the right hand side of (12), then we can further assume that

$$|\omega_l| \geq \frac{1}{N}, \quad l = 1, \ldots, d. \tag{13}$$

By the definition of $\mathcal{B}_N$ there exists a $(d-1)$-dimensional cube Q with the side length a such that

$$R = \{q + t\omega \mid q \in Q,\, 0 \leq t \leq Na\}.$$

By Fubini's theorem we can select a point $q = (q_1, \ldots, q_d)$ from Q such that

$$\int_R f(y) dy \leq |Q| \int_0^{Na} f(q + t\omega) dt.$$

It follows by Hölder's inequality that

$$\int_0^{Na} f(q+t\omega)dt = \int_0^{Na} \prod_{l=1}^{d} f_l(q_l+t\omega_l)dt$$
$$\leq (\prod_{l=1}^{d} \int_0^{Na} f_l(q_l+t\omega_l)^d dt)^{1/d} = (\prod_{l=1}^{d} \frac{\mathrm{sign}\omega_l}{|\omega_l|} \int_{q_l}^{q_l+Na\omega_l} f_l(t)^d dt)^{1/d}.$$

From (13) the triple of

$$R' = \prod_{l=1}^{d} (\min(q_l, q_l+Na\omega_l), \max(q_l, q_l+Na\omega_l))$$

contains x. Since $R' \in \mathcal{B}_{\leq N}$ by (13) we therefore obtain

$$(K_N f)(x) - \epsilon \leq C\frac{1}{|R|}\int_R f(y)dy$$
$$\leq C\frac{1}{Na}\frac{1}{(\prod_{l=1}^{d}|\omega_l|)^{1/d}}(\int_{R'} f(y)^d dy)^{1/d}$$
$$\leq C(\frac{1}{|R'|}\int_{R'} f(y)^d dy)^{1/d} \leq C((M_{\leq N}f^d)(x))^{1/d}.$$

Thus we have proved the lemma. □

LEMMA 6. *For every nonnegative locally integrable weight w on $\mathbf{R}^d$ and for every function f in $L^1(\mathbf{R}^d, M_{\leq N}w))$ we have*

$$w(\{x \in \mathbf{R}^d \,|\, (M_{\leq N}f)(x) > \lambda\}) \leq C\frac{(\log N)^d}{\lambda}\|f\|_{L^1(\mathbf{R}^d, M_{\leq N}w)}, \quad \forall \lambda > 0.$$

To prove this lemma we use the following proposition.

Let ν be $\nu = [\log N/\log 2] + 1$. Here, $[a]$ denotes the largest integer not greater than a. Let I_2 be

$$I_2 = [1, \nu]^d \cap \mathbf{Z}^d.$$

We define B_i, $i = (i_1, \ldots, i_d) \in I_2$, as the class of all rectangles in $\mathbf{R}^d$ which are translations of some dilations of the rectangle

$$\prod_{l=1}^{d}(0, 2^{i_l}).$$

The corresponding maximal operators associated to these bases will be denoted by M_i.

PROPOSITION 7. *For every nonnegative locally integrable weight w on $\mathbf{R}^d$ and for every function f in $L^1(\mathbf{R}^d, M_i w)$ we have*

$$w(\{x \in \mathbf{R}^d \,|\, (M_i f)(x) > \lambda\}) \le C\frac{1}{\lambda}\|f\|_{L^1(\mathbf{R}^d, M_i w)}, \quad \forall \lambda > 0.$$

PROOF. This proposition can be proved in the same way as in the proof of well-known result for the Hardy-Littlewood maximal operator M. Namely,

$$w(\{x \in \mathbf{R}^d \,|\, (Mf)(x) > \lambda\}) \le C\frac{1}{\lambda}\|f\|_{L^1(\mathbf{R}^d, Mw)}, \quad \forall \lambda > 0$$

(see [GR]). □

PROOF OF LEMMA 6. We see that for every rectangle R in $\tilde{\mathcal{B}}_{\le N}$ we can select some i from I_2 and $\tilde{R}$ from B_i as

$$R \subset \tilde{R}, \quad |\tilde{R}| \le 2^d |R|.$$

From these facts we obtain

$$\{x \,|\, (M_{\le N} f)(x) > \lambda\} \subset \bigcup_{i \in I_2} \{x \,|\, (M_i f)(x) > \frac{\lambda}{2^d}\}.$$

On the other hand we see that for every $x \in \mathbf{R}^d$ and $i \in I_2$, we have

$$(M_i w)(x) \le (M_{\le N} w)(x).$$

From this inequality and Proposition 7 we obtain

$$\begin{aligned} &w(\{x \,|\, (M_{\le N} f)(x) > \lambda\}) \\ &\quad \le \sum_{i \in I_2} w(\{x \,|\, (M_i f)(x) > \frac{\lambda}{2^d}\}) \\ &\quad \le C \sum_{i \in I_2} \frac{1}{\lambda}\|f\|_{L^1(\mathbf{R}^d, M_i w)} \le C\frac{(\log N)^d}{\lambda}\|f\|_{L^1(\mathbf{R}^d, M_{\le N} w)}. \end{aligned}$$

Thus we have proved the lemma. □

Proof of Theorem 2.

Using Lemmas 5, 6, and Theorem 3 we have

$$w(\{x \mid (K_N f)(x) > \lambda\})$$
$$\leq w(\{x \mid (M_{\leq N} f^d)(x) > \frac{\lambda^d}{C}\}) \leq C(\frac{\log N}{\lambda})^d \int_{\mathbf{R}^d} f^d(x) \cdot (K_N w)(x) dx.$$

Thus we have proved Theorem 2. □

4. Proof of Theorem 3

We see that the first inequality of (4) follows by the definitions of K_N and $K_{\leq N}$. We shall prove the second inequality of (4) by proving a covering lemma.

Let S^{d-1} be the unit sphere in $\mathbf{R}^d$, i.e. $S^{d-1} = \{x \in \mathbf{R}^d \mid |x| = 1\}$. Let $B(x, r)$ be the closed ball of radius r centered at x. For $\rho > 1$, $H > 0$ and $\omega \in S^{d-1}$ let the icecream-cone like domain $C(d, \rho, H, \omega)$ be defined by

$$C(d, \rho, H, \omega) = \bigcup_{0 \leq t \leq H} B(t\omega, \frac{t}{2\rho}).$$

In what follows we call this domain a cone. For $0 < H_1 < H_2 < \infty$ and $\rho > 1$ we define the family of cones $\mathcal{C}(d, \rho, [H_1, H_2])$ by

$$\mathcal{C}(d, \rho, [H_1, H_2]) = \{C(d, \rho, H, \omega) \mid H \in [H_1, H_2], \omega \in S^{d-1}\}.$$

The following covering lemma is a major part of the proof.

Lemma 8. *Given $k \geq 1$ and the rectangle $R(d)$:*

$$R(d) = \prod_{l=1}^{d} [0, m_l], \quad 1 \leq m_1 \leq \ldots \leq m_d \leq kN, \tag{14}$$

we can select a finite number of cones $C_j = C(d, kN, H_j, \omega_j)$ such that

$$C_j \in \mathcal{C}(d, kN, [m_1, (\sum_1^d m_l^2)^{1/2}]), \tag{15}$$

$$R(d) \subset \bigcup_j C_j, \tag{16}$$

$$\sum_j |C_j| \leq C|R(d)|. \tag{17}$$

We shall divide the proof of this lemma into two cases.

CASE 1. $d=2$.

We write as $O(0,0)$, $A(m_1,0)$, $B(m_1,m_2)$, $C(0,m_2)$ to denote the origin, the point with the coordinate $(m_1,0)$ etc. Put $\angle AOB=\Theta_1$ and $\angle BOC=\Theta_2$. We start from the relation

$$1\le \frac{\theta}{\sin\theta}\le 2,\quad \theta\in[0,\frac{\pi}{2}].$$

Putting $\theta=\Theta_i$, $i=1,2$, in this inequality and dividing each term by $2\sqrt{(2kN)^2+1}$, we have

$$\frac{1}{2\sqrt{(2kN)^2+1}}\le\frac{\Theta_i}{2\sqrt{(2kN)^2+1}\sin\Theta_i}\le\frac{1}{\sqrt{(2kN)^2+1}}.$$

By $1\le m_1\le m_2\le kN$ we see that

$$\sin\Theta_i\sqrt{(2kN)^2+1}>1.$$

From these inequalities we obtain

$$\begin{aligned}\frac{2}{\sqrt{319(kN)^2+1}}&\le\frac{1}{4\sqrt{(2kN)^2+1}}\\ &\le\frac{\Theta_i}{4\sqrt{(2kN)^2+1}\sin\Theta_i}\le\frac{\Theta_i}{2\sqrt{(2kN)^2+1}\sin\Theta_i+1}\\ &\le\frac{\Theta_i}{[2\sqrt{(2kN)^2+1}\sin\Theta_i]+1}\le\frac{\Theta_i}{2\sqrt{(2kN)^2+1}\sin\Theta_i}\\ &\le\frac{1}{\sqrt{(2kN)^2+1}}.\end{aligned}$$

Let n_i, $i=1,2$, be

$$n_i=[2\sqrt{(2kN)^2+1}\sin\Theta_i]+1,$$

and let θ_i be

$$\theta_i=\frac{\Theta_i}{n_i}.$$

Let $0 < \psi < \psi' < \frac{\pi}{2}$ be $\sin\psi = 1/\sqrt{319(kN)^2+1}$ and $\sin\psi' = 1/\sqrt{(2kN)^2+1}$. Then, from above inequalities we have

$$\psi \le 2\sin\psi \le \theta_i \le \sin\psi' \le \psi',$$

$$\tan\psi \le \tan\theta_i \le \tan\psi'$$

and computing $\tan\psi$, $\tan\psi'$ we obtain

$$\frac{2}{\sqrt{319}} \cdot \frac{1}{2kN} \le \tan\theta_i \le \frac{1}{2kN}. \tag{18}$$

We divide $R(2)$ into two triangles $\triangle AOB$ and $\triangle BOC$. It suffices to prove Lemma 8 with $R(2)$ replaced by $\triangle AOB$ and $\triangle BOC$, respectively.

We shall consider $\triangle AOB$ first. On AB we define points P_j, $j = 0, 1, \ldots, n_1$ as $P_0 = A$, $\angle P_{j-1}OP_j = \theta_1$. We extend OP_j to OQ_j in such a way that

$$\angle P_{j+1}Q_jO = \frac{\pi}{2}.$$

Let the cones C_j be

$$C_j = C(d, kN, \overline{OQ_j}, \frac{\overrightarrow{OQ_j}}{\overline{OQ_j}}).$$

Then, we see that $m_1 \le \overline{OQ_j} \le (m_1^2 + m_2^2)^{1/2}$ and $\triangle Q_jOP_{j+1} \subset C_j$. Thus, we obtain

$$\triangle AOB \subset \bigcup_j \triangle Q_jOP_{j+1} \subset \bigcup_j C_j.$$

We next note that

$$\frac{|\triangle Q_jOP_{j+1}|}{|\triangle P_jOP_{j+1}|} = \frac{\overline{OQ_j}}{\overline{OP_j}} = \frac{\cos\theta_1\overline{OP_{j+1}}}{\overline{OP_j}} = \frac{1}{1-\tan\theta_1\tan j\theta_1} \le 2,$$

where in the last step we used $\tan\theta_1 \le 1/(2kN)$ and $\tan j\theta_1 \le m_2/m_1 \le kN$. From (18) and this inequality we finally obtain

$$\begin{aligned}
\sum_j |C_j| &\le C\sum_j \frac{1}{2kN}(\overline{OQ_j})^2 \\
&= C(\sqrt{319}/2)\sum_j (2/\sqrt{319}) \cdot \frac{1}{2kN}(\overline{OQ_j})^2 \le C\sum_j \tan\theta_1(\overline{OQ_j})^2 \\
&\le C\sum_j |\triangle Q_jOP_{j+1}| \le C\sum_j |\triangle P_jOP_{j+1}| = C|\triangle AOB|.
\end{aligned}$$

The other triangle $\triangle BOC$ can be dealt with by the same argument.

CASE 2. $d \geq 3$.

The proof is by induction on the dimension d.

We assume that the lemma is valid for the dimension $d-1$. To prove the lemma for the dimension d we fix $k \geq 1$ and fix $R(d)$ as in (14). For the purpose of the induction we write $k_1 = 3\sqrt{d-1}k$ and $R(d-1) = \prod_{l=2}^{d}[0, m_l]$. We apply the induction assumption to k_1 and $R(d-1)$. Since $k_1 > k$ the condition $1 \leq m_2 \leq \ldots \leq m_d \leq kN \leq k_1 N$ in (14) is satisfied. Therefore, we can select a finite number of cones C_j from $\mathcal{C}(d-1, k_1 N, [m_2, (\sum_2^d m_l^2)^{1/2}])$ such that

$$R(d-1) \subset \bigcup_j C_j, \quad \sum_j |C_j| \leq C|R(d-1)|. \tag{19}$$

Now we shall show that for every $[0, m_1] \times C_j$ we can select a finite number of cones $C_{j,k}$ such that

$$C_{j,k} \in \mathcal{C}(d, kN, [m_1, (\sum_1^d m_l^2)^{1/2}]) \tag{20}$$

$$[0, m_1] \times C_j \subset \bigcup_k C_{j,k}, \tag{21}$$

$$\sum_k |C_{j,k}| \leq C|[0, m_1] \times C_j|. \tag{22}$$

If this can be done, the proof of the lemma will be finished by (19).

Let ω_j be the axis of C_j and let H_j be the height of C_j. By the action of orthogonal transformation in $\mathbf{R}^{d-1}$ we may assume that $\omega_j = (0, 1, 0, \ldots, 0)$. We apply the case 1 to the two-dimensional rectangle $S_j = [0, m_1] \times [0, H_j]$ in the (x_1, x_2)-plain with k_1. (This is justified by the fact that $m_1 \leq m_2 \leq H_j \leq \sqrt{d-1}kN < k_1 N$). Then we have $C'_{j,k} \in \mathcal{C}(2, k_1 N, [m_1, (m_1^2 + H_j^2)^{1/2}])$ satisfying

$$C'_{j,k} = \bigcup_{0 \leq t \leq H_{j,k}} B(t\omega_{j,k}, t/(2k_1 N)),$$

$$S_j \subset \bigcup_k C'_{j,k}, \tag{23}$$

$$\sum_k |C'_{j,k}| \leq C|S_j|. \tag{24}$$

We introduce d-dimensional cones $C_{j,k}$ which have the same axis and the same height as $C'_{j,k}$ but their projections are a little fatter than $C'_{j,k}$:

$$C_{j,k} = \bigcup_{0 \le t \le H_{j,k}} B(t\omega_{j,k}, t/(2kN)).$$

Then these cones $C_{j,k}$ will satisfy our assertion.

Proof of (22). It follows from

$$H_{j,k} \le (m_1^2 + H_j^2)^{1/2} \le (\sum_1^d m_l^2)^{1/2}$$

that

$$C_{j,k} \in \mathcal{C}(d, kN, [m_1, (\sum_1^d m_l^2)^{1/2}]).$$

And it follows from $H_{j,k} \le \sqrt{2} H_j$ that

$$\begin{aligned}\sum_k |C_{j,k}| &\le C \sum_k H_{j,k} (\frac{1}{2kN} H_{j,k})^{d-1} \le C \sum_k H_{j,k} (\frac{1}{2k_1 N} H_{j,k})^{d-1} \\ &\le C(\frac{1}{2k_1 N} H_j)^{d-2} \sum_k H_{j,k} (\frac{1}{2k_1 N} H_{j,k}) \le C(\frac{1}{2k_1 N} H_j)^{d-2} \sum_k |C'_{j,k}| \\ &\le C(\frac{1}{2k_1 N} H_j)^{d-2} |S_j| \le C |[0, m_1] \times C_j|.\end{aligned}$$

Therefore, we obtain (22).

Proof of (21). Fix x in $[0, m_1] \times C_j$. Then x can be written as

$$x = (s, t + b_2, b_3, \dots, b_d), \quad (\sum_2^d b_l^2)^{1/2} \le t/(2k_1 N), \quad 0 \le t \le H_j.$$

Let ξ in S_j be $\xi = (s, t, 0, \dots, 0)$. Then by (23) we can find a cone C'_{j,k_0} such that $\xi \in C'_{j,k_0}$. Let ξ' be $\xi' = t'\omega_{j,k_0}$ such that $\angle \xi \xi' O = \frac{\pi}{2}$. Then, we shall show that

$$x \in B(t'\omega_{j,k_0}, t'/(2kN)). \tag{25}$$

Let $\theta \in [0, \frac{\pi}{2}]$ be $\theta = \tan^{-1}(\frac{1}{2k_1 N})$ and let θ' be the angle between ω_{j,k_0} and $\overrightarrow{O\xi}$. Then, by $\xi \in C'_{j,k_0}$ we have $0 \le \theta' \le \theta$ and hence

$$t' = \sqrt{s^2 + t^2} \cos\theta' \ge \sqrt{s^2 + t^2} \cos\theta \ge \frac{1}{\sqrt{2}} \sqrt{s^2 + t^2}.$$

We then see that

$$|\xi' - x| \le |\xi' - \xi| + |\xi - x| \le \sqrt{s^2+t^2}\sin\theta' + (\sum_l b_l^2)^{1/2}$$
$$\le \sqrt{s^2+t^2}\sin\theta + \frac{t}{2k_1N} \le \frac{\sqrt{s^2+t^2}}{2k_1N} + \frac{t}{2k_1N} \le \frac{\sqrt{s^2+t^2}}{k_1N}$$
$$\le \frac{2}{3}\cdot\frac{t'}{2kN} < \frac{t'}{2kN}.$$

This proves (25).

Now, if $t' \le H_{j,k_0}$, then (25) shows that $x \in C_{j,k_0}$ and (21) is proved.

If $t' > H_{j,k_0}$, we use $H_{j,k_0}\omega_{j,k_0}$ instead of ξ'. By the fact that $t' > H_{j,k_0}$ we see that $\xi \in B(H_{j,k_0}\omega_{j,k_0}, H_{j,k_0}/(2k_1N))$. Hence we have

$$|H_{j,k_0}\omega_{j,k_0} - \xi| \le \frac{H_{j,k_0}}{2k_1N},$$

and

$$t \le (s^2+t^2)^{1/2} = \overline{O\xi} \le H_{j,k_0} + \frac{H_{j,k_0}}{2k_1N} \le 2H_{j,k_0}.$$

It follows from these inequalies that

$$|H_{j,k_0}\omega_{j,k_0} - x| \le |H_{j,k_0}\omega_{j,k_0} - \xi| + |\xi - x|$$
$$\le \frac{H_{j,k_0}}{2k_1N} + \frac{t}{2k_1N} \le \frac{H_{j,k_0}}{2k_1N} + \frac{2H_{j,k_0}}{2k_1N} \le \frac{H_{j,k_0}}{\sqrt{d-1}2kN} \le \frac{H_{j,k_0}}{2kN}.$$

Hence we have $x \in C_{j,k_0}$ also in this case. Thus, we have proved the lemma.

COROLLARY 9. *For every rectangle R in $\mathcal{B}_{\le N}$ and for every point x in R we can select a finite number of rectangles R_j from $\mathcal{B}_N$ such that*

$$x \in R_j, \quad R \subset \bigcup_j R_j, \quad \sum_j |R_j| \le C|R|.$$

PROOF. By translation, rotation, inversion and dilation (and their inverses) we may assume that $x = 0$ and

$$R = \prod_{l=1}^{d}(-a_l, b_l), \quad a_l, b_l \ge 0, \quad 1 \le a_1 + b_1 \le a_2 + b_2 \le \ldots \le a_d + b_d \le N.$$

Let $\tilde{R}$ be

$$\tilde{R} = \prod_{l=1}^{d} (-(a_l + b_l), (a_l + b_l)).$$

Then if we can prove the corollary for the rectangle $\tilde{R}$ with $x = 0$, the corollary will follow by $R \subset \tilde{R}$ and $|\tilde{R}| = 2^d |R|$.

By symmetry it suffices to show that the corollary holds for

$$R' = \prod_{l=1}^{d} [0, (a_l + b_l))$$

with $x = 0$. By the above lemma this R' is covered by a finite number of cones C_j as described in that lemma. Now for each C_j we can find R_j from $\mathcal{B}_N$ such that

$$C_j \subset R_j, \quad |R_j| \leq C|C_j|.$$

The proof of the corollary is now complete. □

By this corollary we shall prove Theorem 3.

Let x be fixed. We may assume that $(K_{\leq N} f)(x) < \infty$. By the definition of $K_{\leq N}$ we can select for any $\epsilon > 0$ some R from $\mathcal{B}_{\leq N}$ such that $x \in R$ and

$$(K_{\leq N} f)(x) - \epsilon \leq \frac{1}{|R|} \int_R |f(y)| dy. \tag{26}$$

Applying Corollary 9 to R, we can find a finite number of rectangles R_j from $\mathcal{B}_N$ such that $x \in R_j$ and

$$R \subset \bigcup_j R_j, \quad \sum_j |R_j| \leq C|R|.$$

From these inequalities and (26) we have

$$\begin{aligned}(K_{\leq N} f)(x) - \epsilon &\leq \frac{1}{|R|} \int_R |f(y)| dy \leq \frac{1}{|R|} \sum_j \int_{R_j} |f(y)| dy \\ &\leq \frac{1}{|R|} \sum_j |R_j| (K_N f)(x) \leq C (K_N f)(x).\end{aligned}$$

Thus, we have proved the theorem. □

By Lemma 8, Corollary 9 and the above arguments we see easily the following remark.

REMARK 10. Fix $a > 0$. Let $\mathcal{B}_{a,\le N}$ denote the class of all rectangles in $\mathbf{R}^d$ which satisfy

$$a \le (\text{the length of shortest sides}) \le (\text{the length of longest sides}) \le Na.$$

The corresponding maximal operator associated to this base is denoted by $M_{a,\le N}$. Then for every locally integrable function f on $\mathbf{R}^d$ there exists a constant C independent of a and N such that

$$(M_{a,N}f)(x) \le (M_{a,\le N}f)(x) \le C \sup_{\alpha\in[a/N,\sqrt{d}a]} (M_{\alpha,N}f)(x)$$

holds for every x in $\mathbf{R}^d$.

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(Received June 9, 1998)

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