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Deformations of $\mathbb{Q}$-Calabi-Yau 3-Folds and $\mathbb{Q}$-Fano 3-Folds of Fano Index 1

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Abstract. In this article, we prove that any $\mathbb{Q}$-Calabi-Yau 3-fold with only ordinary terminal singularities and any $\mathbb{Q}$-Fano 3-fold of Fano index 1 with only terminal singularities have $\mathbb{Q}$-smoothings.

0. Introduction

Let X be a normal $\mathbb{Q}$-Gorenstein projective variety over $\mathbb{C}$ of dimension 3 which has only terminal singularities. The index i_p of a singular point p is defind by

$$i_p := min\{m \in \mathbb{N} | mK_X \text{ is a Cartier divisor near p}\}.$$

A singular point of index 1 is called a Gorenstein singularity. The singularity index $i(X)$ of X is defined by

$$i(X) := min\{m \in \mathbb{N} | mK_X \text{ is a Cartier divisor}\}.$$

Definition 0.1. Let X be a normal $\mathbb{Q}$-Gorenstein projective variety of dimension 3 over $\mathbb{C}$ which has only terminal singularities. Let $(\Delta, 0)$ be a germ of the 1-parameter unit disk. Let $\mathfrak{g} : \mathfrak{X} \to (\Delta, 0)$ be a small deformation of X over $(\Delta, 0)$. We call $\mathfrak{g}$ a $\mathbb{Q}$-smoothing of X when the fiber $\mathfrak{X}_s = \mathfrak{g}^{-1}(s)$ has only cyclic quotient singurarities for each $s \in (\Delta, 0) \setminus \{0\}$.

Problem. Let X be a normal $\mathbb{Q}$-Gorenstein projective variety of dimension 3 over $\mathbb{C}$ which has only terminal singularities.
When X has a $\mathbb{Q}$-smoothing ?

We treat this problem for $\mathbb{Q}$-Calabi-Yau 3-folds and $\mathbb{Q}$-Fano 3-folds in this paper.

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DEFINITION 0.2. Let X be a normal $\mathbb{Q}$-Gorenstein projective variety over $\mathbb{C}$ of dimention 3 which has only terminal singularities.

(1) If there exists $m \in \mathbb{N}$ such that $mK_X \sim 0$, we call X a $\mathbb{Q}$-Calabi-Yau 3-fold.
(2) If $-K_X$ is ample, we call X a $\mathbb{Q}$-Fano 3-fold.

For the case of a $\mathbb{Q}$-Calabi-Yau 3-fold X, we define the Global index $I(X)$ of X by

$$I(X) := min\{m \in \mathbb{N} | mK_X \sim 0\}.$$

Let $\pi : Y = Spec \bigoplus_{m=0}^{I-1} \mathcal{O}_X(-mK_X) \to X$ be the Global canonical cover of X. If $h^1(Y, \mathcal{O}_Y) \neq 0$, then Y is smooth by [Ka 2]. Thus our interest are the case that $h^1(Y, \mathcal{O}_Y) = 0$. From here, we assume the additional condition for our $\mathbb{Q}$-Calabi-Yau 3-fold

$$h^1(Y, \mathcal{O}_Y) = 0.$$

For the case that $i(X) = I(X) = 1$, i.e., a Calabi-Yau 3-fold, is studied in [Fr], [Na 1], [Na 2], [Na-St], and [Gr]. Summing up those results, we know the following :

THEOREM 0.3 (Friedman, Namikawa, Steenbrink, Gross)(Cf. [Na 2]). *Let X be a Calabi-Yau 3-fold with only isolated rational Gorenstein singularities. Assume That:*

(1) *X is $\mathbb{Q}$-factorial.*
(2) *Every singularity of X is locally smoothable*
(3) *The semi-universal deformation space $Def(X, p)$ of each singularity (X, p) is smooth.*

then X is smoothable by a flat deformation.

We consider a generalization of the theorem 0.3 in the case $I(X) \neq 1$ and especially the case X has non-Gorenstein sinagularities if possible.

MAIN THEOREM 1. *Let X be a $\mathbb{Q}$-Calabi-Yau 3-folds with only terminal singularities and $\pi : Y \to X$ the global canonical cover of X. Assume that Y is $\mathbb{Q}$-factorial and X has only ordinary terminal singularities. Then X has a $\mathbb{Q}$-smoothing.*

We remark that the word *ordinary* is the same meaning in [Morrison, §2].

Y. Namikawa studied $\mathbb{Q}$-smoothings of $\mathbb{Q}$-Calabi-Yau 3-folds in his personal notes, and he had similar results using the invariant μ as in [Na 1], [Na 2], and [Na-St, §2]. But our proof is different and based on [Na-St, §1].

For the case of a $\mathbb{Q}$-Fano 3-fold X, there is a positive integer r and a Cartier divisor H such that $-i(X)K_X \sim rH$. Taking the largest number of such r, we call $\frac{r}{i(X)}$ the Fano index of X. The case $i(X) = 1$, i.e., X is a Gorenstein Fano 3-fold, X has a smoothing by Namikawa and Mukai ([Na 3], [Mukai]). The case that the Fano index > 1, these 3-folds are classified by Sano ([Sa 2]), and they have $\mathbb{Q}$-smoothings by these classifications. The case that the Fano indices $= 1$, we have a next theorem.

MAIN THEOREM 2. *Let X be a $\mathbb{Q}$-Fano 3-folds of Fano index 1 with only terminal singularities. Then X has a $\mathbb{Q}$-smoothing.*

Sano classified $\mathbb{Q}$-Fano 3-folds of Fano index 1 with only cyclic quotient singularities. So any $\mathbb{Q}$-Fano 3-fold of Fano index 1 is a degeneration of a 3-fold classified by Sano ([Sa 1]).

In §1 of this paper, we treat the unobstructedness of deformation functors.

In §2, we investigete deformations of isolated complete intersection singularities with cyclic finite group actions, and we define an ordinary complete intersection quotient singularity for an isolated $\mathbb{Q}$-Gorenstein normal singularity.

In §3, we prove our main theorem 1 using the results in §2.

In §4, we prove our main theorem 2.

In §5, we give some examples of $\mathbb{Q}$-smoothings.

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Notation.
$\mathbb{C}$: the complex number field.
$\sim$: linear equivalence.
$\sim_{\mathbb{Q}}$: $\mathbb{Q}$-linear equivalence.
K_X : canonical divisor of X.
Let G be a group acting on a set S. We set

$$S^G := \{s \in S | gs = s \text{ for any } g \in G\}.$$

In this paper, $(\Delta, 0)$ means a germ of a 1-parameter unit disk.

Let X be a compact complex space or a good representative for a germ, and $\mathfrak{g} : \mathfrak{X} \to (\Delta, 0)$ a 1-parameter small deformation of X. We denote the fiber $\mathfrak{g}^{-1}(s)$ for $s \in (\Delta, 0)$ by $\mathfrak{X}_s$.

(Ens) : the category of sets.

Let k be a field. We set (Art_k) : the category of Artin local k-algeblas with residue field k.

1. Unobstructedness

DEFINITION 1.1.

Let k be a field, and $D : (Art_k) \to (Ens)$ be a covariant functor such that $D(k) = \{X\}$ (=: a single point).

We call D unobstructed if for any surjection $\alpha : B \to A$ in (Art_k), $D(B) \to D(A)$ is a surjection.

REMARK ([Sch 1, Remark 2.10]).

If D has a hull R, then D is unobstructed if and only if R is a power series ring over k, i.e., R is smooth.

Let X be a normal algebraic variety over $\mathbb{C}$. Let D_X be a functor from $(Art_{\mathbb{C}})$ to (Ens) defined by

$$D_X(A) := \{ \text{ Isomorphic classes of deformations of } X \text{ over } A\}$$

for $A \in (Art_{\mathbb{C}})$.

We will show the following theorem in this section.

THEOREM 1.2.

(1) *Let X be a $\mathbb{Q}$-Calabi-Yau 3-fold with only isolated log-terminal singularities, and $\pi : Y \to X$ the global canonical cover of X. If D_Y is unobstructed, then D_X is unobstructed.*
(2) *Let X be a $\mathbb{Q}$-Fano 3-fold with only isolated log-terminal singularities. If Fano index of X is 1, then D_X is unobstructed.*

We will prove this theorm by the method in [Na 1, §4] using T^1-lifting criterion. We prove only (2), because we can prove (1) by the same method of (2) and [Na 1, §4].

Let X be a $\mathbb{Q}$-Fano 3-fold of Fano index 1 with only isolated log-terminal singularities, $i(X) = r$, and $G = \mathbb{Z}/r\mathbb{Z}$. Then there exist a Cartier divisor H such that $rH \sim -rK_X$. Let $\pi : Y = Spec(\oplus_{i=0}^{r-1}\mathcal{O}_X(-m(H+K_X))) \to X$ be a canonical cover, then Y is a Gorenstein Fano 3-fold with only isolated rational Gorenstein singularities, and π is a Galois covering of Galois group G which is étale outside finite number of points. Let $A \in (Art_{\mathbb{C}})$ and set $S = SpecA$. Assume that an infinitesimal deformation $\mathfrak{f} : \mathfrak{X} \to S$ of X over S is given. Let $Sing(X) = \{p_1, p_2, \dots, p_n\}$. Set $U = X - \{p_1, p_2, \dots, p_n\}$ and $\mathfrak{U} = \mathfrak{X} - \{p_1, p_2, \dots, p_n\}$. Denote by the same j the inclusions $U \to X$ and $\mathfrak{U} \to \mathfrak{X}$. Set $\omega_{\mathfrak{X}/S}^{[i]} = j_*\omega_{\mathfrak{U}/S}^{\otimes i}$. Then there is an invertible sheaf $\mathfrak{H}$ on $\mathfrak{X}$ such that $\mathfrak{H}|_X \simeq \mathcal{O}_X(H)$, and we have that $\omega_{\mathfrak{X}/S}^{[-r]} \simeq \mathfrak{H}^{\otimes r}$ because $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ as in [Na 1, §4]. We have the relative canonical cover $\pi_A : \mathfrak{Y} = Spec(\oplus_{i=0}^{r-1}\omega_{\mathfrak{X}/S}^{[-i]} \otimes \mathfrak{H}^{\otimes(-i)}) \to \mathfrak{X}$ of $\mathfrak{X}$ which is a deformation of π over S as in [Na 1,§4]. Then π_A is a Galois covering with Galois group G which is étale outside finite number of points. Let $A_n := \mathbb{C}[t]/(t^{n+1})$ and $S_n = SpecA_n$.

PROPOSITION 1.3 ([Na 1, Proposition 4.1]). *Let X_n be an infinitesimal deformation of X over S_n, $\pi_n : Y_n \to X_n$ be a relative canonical cover of X_n. Then we have a followig isomorphism for all $n \geq 0$:*

$$Ext^1_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/S_n}, \mathcal{O}_{X_n}) \simeq Ext^1_{\mathcal{O}_{Y_n}}(\Omega^1_{Y_n/S_n}, \mathcal{O}_{Y_n})^G.$$

PROOF. See [Na 1, Proposition 4.1]. □

We have an important theorem on unobstructedness called T^1-lifting criterion. Let X be a normal algebraic variety. Let $B_n = \mathbb{C}[x,y]/(x^{n+1}, y^2)$, and $[X_n] \in D_X(A_n)$. We define $T^1_{D_X}(X_n/A_n)$ to be the set of isomorphic classes of pairs $(\mathfrak{X}_n, \psi_n)$ consisting of deformations $\mathfrak{X}_n$ of X over B_n with marking isomorphisms $\psi : \mathfrak{X}_n \otimes_{B_n} A_n \simeq X_n$. Let $\varepsilon_n : A_n \to B_{n-1}$ be a homomorphism defined by $t \mapsto x + y$. We have an A_n-module structure of B_{n-1} by ε_n. Let $\alpha_n : A_{n+1} \to A_n$ be a natural homomorphism, $T^1(\alpha_n) : T^1_{D_X}(X_{n+1}/A_{n+1}) \to T^1_{D_X}(X_n/A_n)$ which is introduced by α_n.

THEOREM 1.4 (Ran, Kawamata, Deligne) (Cf. [Ka 3], [Ka 5]).
Let D_X be a deformation functor of X. D_X is unobstructed if and only if the following condition holds : Assume that $[X_n] \in D_X(A_n)$ is given. Set $X_{n-1} = X_n \otimes_{A_n} A_{n-1}$, $\mathfrak{X}_{n-1} = X_n \otimes_{A_n} B_{n-1}$, and $\psi_{n-1} : \mathfrak{X}_{n-1} \otimes_{B_{n-1}} A_{n-1} \simeq X_{n-1}$ which is a natural isomorphism of X_{n-1}, then $(\mathfrak{X}_{n-1}, \psi_{n-1}) \in T^1_{D_X}(X_{n-1}/A_{n-1})$ is in the image of $T^1(\alpha_{n-1})$.

PROOF OF THEOREM 1.2,(2). Let $i(X) = r$, $G = \mathbb{Z}/r\mathbb{Z}$, and $\pi : Y \to X$ a canonical cover of X. Then Y is a Gorenstein Fano 3-fold with only isolated rational Gorenstein singularities. By [Na 3, Proposition 3], D_Y is unobstructed. Let $[X_n] \in D_X(A_n)$, $X_{n-1} = X_n \otimes_{A_n} A_{n-1}$, $\mathfrak{X}_{n-1} = X_n \otimes_{A_n} B_{n-1}$, and $\varphi_{n-1} : \mathfrak{X}_{n-1} \otimes_{B_{n-1}} A_{n-1} \simeq X_{n-1}$ which is a natural isomorphism of X_{n-1}. Let $\pi_n : Y_n \to X_n$ be the relative canonical cover of X_n which is a deformation of π. Set $Y_{n-1} = Y_n \otimes_{A_n} A_{n-1}$, $\mathfrak{Y}_{n-1} = Y_n \otimes_{A_n} B_{n-1}$, and $\psi_{n-1} : \mathfrak{Y}_{n-1} \otimes_{B_{n-1}} A_{n-1} \simeq Y_{n-1}$ which is a natural isomorphism of Y_{n-1}. Then there exists $(\mathfrak{Y}_n, \psi_n) \in T^1_{D_Y}(Y_n/A_n)$ such that $T^1(\alpha_{n-1})(\mathfrak{Y}_n, \psi_n) = (\mathfrak{Y}_{n-1}, \psi_{n-1})$, and we remark that $(\mathfrak{Y}_{n-1}, \psi_{n-1}) \in T^1_{D_Y}(Y_{n-1}/A_{n-1})^G$. Here we can take the trace of $(\mathfrak{Y}_n, \psi_n)$ with respect to G :

$$tr(\mathfrak{Y}_n, \psi_n) = \frac{1}{|G|} \sum_{g \in G} g(\mathfrak{Y}_n, \psi_n) \in [T^1_{D_Y}(Y_n/A_n)]^G = T^1_{D_Y^G}(Y_n/A_n).$$

Then $T^1(\alpha_{n-1})(tr(\mathfrak{Y}_n, \psi_n)) = (\mathfrak{Y}_{n-1}, \psi_{n-1})$.
By the isomorphism in Proposition 1.3, there is a $(\mathfrak{X}_n, \varphi_n) \in T^1(X_n/A_n)$ such that $T^1(\alpha_{n-1})((\mathfrak{X}_n, \varphi_n)) = (\mathfrak{X}_{n-1}, \varphi_{n-1})$. Thus we have that D_X is unobstructed by the Theorem 1.4. □

COROLLARY 1.5. *Let X be a $\mathbb{Q}$-Calabi-Yau 3-fold with only isolated log-terminal singularities, and $\pi : Y \to X$ be the global canonical cover of*

X. For any $q \in Sing(Y)$, we denote by $D_{(Y,q)}$ the deformation functor of a germ (Y,q). If $D_{(Y,q)}$ is unobstracted for any $q \in Sing(Y)$, then D_X is unobstructed.

In particular, if Y has only isolated complete interesction singularities, then D_X is unobstructed.

Proof. By [Gr, Theorem 2.2], if $D_{(Y,q)}$ is unobstructed for any $q \in Sing(Y)$, then D_Y is unobstructed. Thus D_X is unobstructed by Theorem 1.2, (1). □

2. Isolated Complete Intersection Singularities with Cyclic Finite Group Actions

At first, we investigate deformations of isolated complete intersection singularities and secondly with cyclic finite group actions.

Let (X,p) be a germ of an n-dimensional isolated singularity and $\mathfrak{m}_{X,p}$ the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

Definition 2.1. We call $dim_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p})$ the minimal enbedding dimension and it is denoted by $e(X,p)$.

For example, (X,p) is smooth if and only if $e(X,p) = n$.

We denote the set of isomorphic classes of first order deformations of (X,p) by $T^1_{(X,p)}$, and we remark that $T^1_{(X,p)}$ has a natural $\mathcal{O}_{X,p}$-module stracture. (Cf. [A, §6])

Definition 2.2.

(1) We call $\eta \in T^1_{(X,p)}$ is a good direction if η satisfies following 2 conditions:

(2.2.1) There is a 1-parameter small deformation of (X,p) which is a realization of η.

(2.2.2) For any realization of η ; $\mathfrak{g} : (\mathfrak{X},p) \to (\Delta,0)$, we have $e(\mathfrak{X}_s,p') < e(X,p)$ for any $s \in (\Delta,0)\backslash\{0\}$ and any $p' \in Sing(\mathfrak{X}_s)$.

(2) Let $M \subsetneq T^1_{(X,p)}$ be a proper $\mathcal{O}_{X,p}$ submodole of $T^1_{(X,p)}$. We call $\eta \in T^1_{(X,p)}$ is a good direction for M if $\eta + m$ is a good direction for any $m \in M$.

For example, if (X,p) is smooth, then there are no good directions.

PROPOSITION 2.3. *Let (X,p) be a germ of an n-dimensional isolated complete intersection singualrity and $M \subsetneq T^1_{(X,p)}$ a proper $\mathcal{O}_{X,p}$-submodule of $T^1_{(X,p)}$.*

If (X,p) is not smooth, then there is a good direction for M.

PROOF. Let $e = e(X,p)$ and $d = e - n$, then there are elements $f_1, f_2, \ldots, f_d \in \mathbb{C}\{x_1, x_2, \ldots, x_e\}$ which define (X,p) in $(\mathbb{C}^e, 0)$. We have an $\mathcal{O}_{X,p}$-module isomorphism

$$T^1_{(X,p)} \cong \mathcal{O}^d_{X,p}/J$$

where J is the submodule generated by $(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_d}{\partial x_{\underline{i}}})$, $1 \leq i \leq e$. For $a = (a_1, a_2, \ldots, a_d) \in \mathcal{O}^d_{X,p}$, $\bar{a} = (a_1, a_2, \ldots, a_d)$ denote $(a_1, a_2, \ldots, a_d)(mod J)$ which is an element of $\mathcal{O}^d_{X,p}/J$. and we think it as an element of $T^1_{(X,p)}$ by the isomorphism $T^1_{(X,p)} \cong \mathcal{O}^d_{X,p}/J$. If we choose elements $g_1 = (g_{11}, g_{12}, \ldots, g_{1d}), g_2 = (g_{21}, g_{22}, \ldots, g_{2d}), \ldots, g_m = (g_{m1}, g_{m2}, \ldots, g_{md}) \in \mathfrak{m}_{X,p}\mathcal{O}^d_{X,p}$ which along with $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ form a bisis of $\mathbb{C}$-vector space $T^1_{(X,p)}$ after reducing module J, then for any $\eta \in T^1_{(X,p)}$ there are unique $a_1, a_2, \ldots, a_d, b_1, b_2, \ldots, b_m \in \mathbb{C}$ such that $\eta = a_1\bar{e_1} + a_2\bar{e_2} + \cdots + a_d\bar{e_d} + b_1\bar{g_1} + b_2\bar{g_2} + \cdots + b_m\bar{g_m}$.

At first, we prove the following claim

CLAIM 2.4. If $a_i \neq 0$ for some i, then η is a good direction.

PROOF OF CLAIM 2.4. For any realization of η, $\mathfrak{f} : (\mathfrak{X}, p) \to (\Delta, 0)$, there are $h_{12}, h_{13}, \ldots, h_{22}, h_{23}, \ldots, h_{d2}, h_{d3}, \cdots \in \mathbb{C}\{x_1, x_2, \ldots, x_e\}$ such that $\mathfrak{f} : (\mathfrak{X}, p) \to (\Delta, 0)$ can be described by:

$F_1 = f_1 + s(a_1 + b_1g_{11} + \cdots + b_mg_{m1}) + s^2h_{12} + s^3h_{13} + \cdots = 0$

$F_2 = f_2 + s(a_2 + b_1g_{12} + \cdots + b_mg_{m2}) + s^2h_{22} + s^3h_{23} + \cdots = 0$

$\vdots$

$F_d = f_d + s(a_d + b_1g_{1d} + \cdots + b_mg_{md}) + s^2h_{d2} + s^3h_{d3} + \cdots = 0$

in $(\mathbb{C}^e \times \mathbb{C}, (0,0))$ and its second projection, because (X,p) is a complete intersection singularity. Then, $\{F_i = 0 | (\mathbb{C}^e \times \mathbb{C}), (0,0)\}$ is smooth by the assumption. Thus $\{F_i(x_1, x_2, \ldots, x_e, s) = 0 | (\mathbb{C}^e \times \mathbb{C}, (0,0))_s\}$ is smooth for any $s \in (\varDelta, 0) \setminus \{0\}$ by the theorem of Bertini. This shows that $e(\mathfrak{X}_s, p') < e(X, p)$ for any $p' \in Sing(\mathfrak{X}_s)$. □

Back to the proof of proposition 2.3, suppose that $\bar{e_i} \in T^1_{(X,p)}$ is not a good direction for any i, then there are elements $\bar{m_i} \in M$ where $m_i \in \mathcal{O}^d_{X,p}$ such that $e_i - m_i \in \mathfrak{m}_{X,p}\mathcal{O}^d_{X,p}$. This shows that $M = T^1_{(X,p)}$, this is a contradiction. Thus we can choose a good direction for M among $\bar{e_1}, \bar{e_2}, \ldots, \bar{e_d}$. □

Let G be a cyclic finite group of order r, and (Y, q) be an n-dimensional isolated complete intersection singularity with a G-action over $\mathbb{C}$. Then $T^1_{(Y,q)}$ has a natural G-action. Let $M \subsetneq T^1_{(Y,q)}$ a proper $\mathcal{O}_{Y,q}$-submodule of $T^1_{(Y,q)}$. We want to know a sufficient condition for an existence of G-invariant good element for M. We present here one sufficient condition. Let $e = e(Y, q)$. G-action on (Y, q) can be extended to $(\mathbb{C}^e, 0)$, and we can choose coordinates on $(\mathbb{C}^e, 0)$, $x_i \mapsto \xi^{a_i} x_i$, where ξ is a primitive r-th root of the unity and $0 \le a_i < r$. Let $d = e - n$.

Definition 2.5.

(1) We call (Y, q) ordinary, if we can choose $f_1, f_2, \ldots, f_d \in \mathbb{C}\{x_1, x_2, \ldots, x_e\}$ to be G-invariant which define (Y, q) in $(\mathbb{C}^e, 0)$; $(Y, q) \simeq \{f_1 = f_2 = \cdots = f_d = 0 | (\mathbb{C}^e, 0)\}$.
(2) Let (X, p) be a $\mathbb{Q}$-Gorenstein normal isolated singularity, and (Y, q) its canonical cover. Thus (Y, q) has a cyclic finite group action. We call (X, p) an ordinary complete intersection singularity if (Y, q) is an ordinary isolated complete intersection singularity.

Corollary 2.6. *Let G be a cyclic finite group, and (Y, q) a germ of an isoleted complete intersection singularity with G-action over $\mathbb{C}$. Let $M \subsetneq T^1_{(Y,q)}$ be a proper $\mathcal{O}_{Y,q}$-submodule of $T^1_{(Y,q)}$. If (Y, q) is ordinary and not smooth, then there is a G-invariant good element for M.*

Proof. Let r be the order of G, $e = e(Y, q)$, and $d = e - n$. We can choose coordinates on $(\mathbb{C}^e, 0)$, $x_i \mapsto \xi^{a_i} x_i$, where ξ is a primitive r-th root and $0 \le a_i < r$. We can take $f_1, f_2, \ldots, f_d \in \mathbb{C}\{x_1, x_2, \ldots, x_e\}$ to be

G-invariant which defines (Y,q) in $(\mathbb{C}^e,0)$; $(Y,q) \simeq \{f_1, = f_2 = \cdots = f_d = 0 | (\mathbb{C}^e,0)\}$ by assumption. We have an $\mathcal{O}_{Y,q}$-module isomorphism

$$T^1_{(Y,q)} \cong \mathcal{O}^d_{Y,q}/J$$

where J is the submodule generated by $(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_d}{\partial x_i})$, $1 \leq i \leq e$ as in the proof of Proposition 2.3. Then G-action of $T^1_{(Y,q)}$ is the same as the natural G-action of $\mathcal{O}^d_{Y,q}/J$ because f_i is G-invariant for all i. For $a = (a_1, a_2, \ldots, a_d) \in \mathcal{O}^d_{Y,q}$, $\bar{a} = \overline{(a_1, a_2, \ldots, a_d)}$ denote $(a_1, a_2, \ldots, a_d)(modJ)$ which is an element of $\mathcal{O}^d_{Y,q}/J$. Let $e_1 = (1,0,\ldots,0), e_2 = (0,1,0,\ldots,0)$, $\ldots, e_d = (0,\ldots,0,1)$. Then $\bar{e_1}, \bar{e_2}, \ldots, \bar{e_d}$ are G-invariant. As in the proof of Proposition 2.3, we can choose a good direction among $\bar{e_1}, \bar{e_2}, \ldots, \bar{e_d}$. $\square$

DEFENITION 2.7. Let $(S,0)$ be a pointed analytic space,G a finite group. Let Y be an analytic space with G-action Let $\mathfrak{f} : \mathfrak{Y} \to (S,0)$ be a deformation of Y over $(S,0)$. We call $\mathfrak{f}$ G-equivariant if $\mathfrak{Y}$ has a G-action over $(S,0)$ compatible with the G-action on Y.

Next, we give one property of the ordinarity under G-equivariant deformations.

LEMMA 2.8. *Let G be a cyclic finite group, (Y,q) a germ of an n-dimentional isolated complete intersection singularity with G-action over $\mathbb{C}$. Let $\mathfrak{f} : (\mathfrak{Y},q) \to (\Delta,0)$ be a G-equivariant 1-parameter small deformation of (Y,q).*

If (Y,q) is ordinary, then so is $\mathfrak{Y}_s$ for any $s \in (\Delta,0) \setminus \{0\}$.

PROOF. Let r be the order of G, $e = e(Y,q)$, and $d = e - n$. We can choose coordinates on $(\mathbb{C}^e,0)$, $x_i \mapsto \xi^{a_i} x_i$, where ξ is a primitive r-th root and $0 \leq a_i < r$. We can take $f_1, f_2, \ldots, f_d \in \mathbb{C}\{x_1, x_2, \ldots, x_e\}$ to be G-invariant which defines (Y,q) in $(\mathbb{C}^e,0)$; $(Y,q) \simeq \{f_1, = f_2 = \cdots = f_d = 0 | (\mathbb{C}^e,0)\}$ by assumption. We define G-action on $(\mathbb{C}^e \times \mathbb{C}, (0,0))$ via $x_i \mapsto \xi^{a_i} x_i$ and $s \mapsto s$. Since $\mathfrak{f}$ is a G-equivariant deformation, f_i are G-invariant and (Y,q) is complete intersection singularity, there are G-invariant elements $h_1, h_2, \ldots, h_d \in (s)\mathbb{C}\{x_1, x_2, \ldots, x_e, s\}$ such that $\mathfrak{f} : (\mathfrak{Y},q) \to (\Delta,0)$ is G-equivariantly isomorphic to $(\mathfrak{Y}',q') \simeq \{f_1 + h_1 = f_2 + h_2 = \cdots = f_d + h_d = 0 | (\mathbb{C}^e \times \mathbb{C}, (0,0))\}$ and the second projection $(\mathbb{C}^e \times \mathbb{C}, (0,0)) \to (\mathbb{C},0)$. After looking fiberwise, we have the result. $\square$

3. Proof of the Main Theorem 1

We use 2 facts in [Na-St, §1] to prove our main theorem. At first, we recall them. Let (Y, q) be an isolated singularity, Y a good representative for the germ, and $V = Y \setminus q$. Let $\nu : \tilde{Y} \to Y$ be a good resolution of Y and $E = \nu^{-1}(q)$. ("good" means the restriction of $\nu : \nu^{-1}(V) \to V$ is an isomorphism and its exceptional divisor E has simple nomal crossings.) Identifing $\nu^{-1}(V)$ with V, we have a natural homomorphism of $\mathcal{O}_{Y,q}$-modules:

$$\tau : H^1(V, \Omega^2_V) \to H^2_E(\tilde{Y}, \Omega^2_{\tilde{Y}})$$

Lemma 3.1 ([Na-St, Theorem 1.1]). *Suppose that (Y, q) is a 3-dimensional isolated normal Gorenstein Du Bois (e.g. rational) singularity for which τ is the zero map. Then (Y, q) is rigid.*

Let Y be a Calabi-Yau 3-fold with only isolated rational Gorenstein singularities, $\{q_1, q_2, \ldots, q_n\} = Sing(Y)$, Let Y_i be a sufficiently small neighborhood of q_i, and $V_i = Y_i \setminus \{q_i\}$. Let $\nu : \tilde{Y} \to Y$ be a good resolution of Y and set $E_i = \nu^{-1}(q_i)$. Then there is natural maps $\tau_i : H^1(V_i, \Omega^2_{V_i}) \to H^2_{E_i}(\tilde{Y}, \Omega^2_{\tilde{Y}})$ and $\iota : H^2_{E_i}(\tilde{Y}, \Omega^2_{\tilde{Y}}) \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})$.

Proposition 3.2 ([Na-St, Proposition 1.2]). *Assume that Y is $\mathbb{Q}$-factorial. Then the composition map $\iota \circ \tau_i : H^1(V_i, \Omega^2_{V_i}) \to H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})$ is the zero map.*

Next theorem shows the main theorem 1 by [Mori].

Theorem 3.3. *Let X be a $\mathbb{Q}$-Calabi-Yau 3-fold with only isolated log-terminal singularities and $\pi : Y \to X$ the global canonical cover of X. Assume that Y is $\mathbb{Q}$-factorial and X has only oridinary complete-intersection quoitient singularities. Then X has a $\mathbb{Q}$-smoothing.*

Proof. Let $i(X) = r$ and $G = \mathbb{Z}/r\mathbb{Z}$ the Galois group of π. Let $\{p_1, p_2, \ldots, p_n\} = \{p \in Sing(X) | p \text{ is not a quotient singularity.}\}$, and $\{q_{i1}, q_{i2}, \ldots, q_{ik_i}\} = \pi^{-1}(\{p_i\})$ for $1 \le i \le n$. Then $Sing(Y) = \{q_{ij} | i, j\}$. Let Y_{ij} be a sufficiently small neighborhood of q_{ij}, $V = Y \setminus \{q_{ij} | i, j\}$ and $V_{ij} = Y_{ij} \setminus \{q_{ij}\}$. Let $\nu : \tilde{Y} \to Y$ be a G-equivariant good resolution of Y

and set $E_{ij} = \nu^{-1}(q_{ij})$. We set $G_i = \{g \in G | g(q_{ij}) = q_{ij}$ for any $j\}$. We consider the following commutative diagram:

$$
\begin{array}{ccccc}
H^1(V, \Omega_V^2) & \xrightarrow{\alpha'} & \oplus_{i,j} H^2_{E_{ij}}(\tilde{Y}, \Omega^2_{\tilde{Y}}) & \xrightarrow{\iota} & H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}) \\
\uparrow \wr & & \uparrow \oplus_{i,j}\tau_{ij} & & \\
H^1(V, \Theta_V) & \xrightarrow{\alpha} & \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}}) & &
\end{array} \tag{3.3.1}
$$

where α is the map determined by the map from global deformations to local deformations, α' is the coboundary map of the exact sequence of local cohomology, and τ_{ij} is also the coboundary map of the exact sequence of local cohomology. We remark that τ_{ij} is an homomorphism of $O_{Y_{ij},q_{ij}}$-modules.

Let ξ be a primitive r-th root, $\omega \in H^0(Y, K_Y)$ a non-where vanishing section. Let $g \in G$ be a genarator of G. Because G is a finite cyclic group, We have that $g(\omega) = \xi^a\omega$ for some positive integer a, and we have the commutative diagram:

$$
\begin{array}{ccccc}
[H^1(V, \Omega_V^2)]^{[\xi^a]} & \xrightarrow{\alpha'} & [\oplus_{i,j} H^2_{E_{ij}}(\tilde{Y}, \Omega^2_{\tilde{Y}})]^{[\xi^a]} & \xrightarrow{\iota} & [H^2(\tilde{Y}, \Omega^2_{\tilde{Y}})]^{[\xi^a]} \\
\uparrow \wr & & \uparrow \oplus_{i,j}\tau_{ij} & & \\
H^1(V, \Theta_V)^G & \xrightarrow{\alpha} & \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}})^G & &
\end{array} \tag{3.3.2}
$$

where $F^{[\xi^a]} = \{x \in F | g(x) = \xi^a x\}$ for a $\mathbb{C}$-vector space F with a G-action.

As all singularities q_{ij} are Gorenstein rational by [Ka 1, Proposition 1.7] and non smooth complete intersection singularities, they are not rigid. We have that τ_{ij} is not the zero map by lemma 3.1 , i.e., $Ker(\tau_{ij})$ is a proper $\mathcal{O}_{Y_{ij},q_{ij}}$-submodule of $h^1(V_{ij}, \Theta_{V_{ij}}) = T^1_{(Y_{ij},q_{ij})}$ for each i, j. By Corollary 2.6, there exist G_i-invariant good directions $\eta_{ij} \in (T^1_{(Y_{ij},q_{ij})})^G$ for $Ker(\tau_{ij})$ such that $g(\eta_{ij}) = \eta_{il}$ for $g \in G$ satisfing $g(q_{ij}) = q_{il}$. Thus $(\eta_{ij}|i,j) \in \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}})$ is a G-invariant element.

By Proposition 3.2, $\iota \circ \tau_{ij}$ is the zero map for any i, j because of $\mathbb{Q}$-factoriality of Y. Considering the commutative diagram (3.3.2), there exists an G-invariant element $\eta \in H^1(V, \Theta_V)$ such that $\alpha'(\eta)_{ij} = \tau_{ij}(\eta_{ij})$ for any i, j. Then $\alpha(\eta)_{ij} - \eta_{ij} \in Ker(\tau'_{ij})$, and we have that $\alpha(\eta)_{ij}$ is a good direction for any i, j. By Corollary 1.4, we have a G-equivariant 1-parameter small deformation of Y, $\mathfrak{f} : \mathfrak{Y} \to (\Delta, 0)$ determined by $\eta \in T^1_{D^G_Y}(Y/\mathbb{C})$. Let $M.e(\mathfrak{Y}_s) = max\{e(\mathfrak{Y}_s, q)|q \in Sing(\mathfrak{Y}_s)\}$ for $s \in (\Delta, 0)$, then we have $M.e(Y) > M.e(\mathfrak{Y}_s)$ for any $s \in (\Delta, 0) \setminus \{0\}$ by the choice of η. $\mathfrak{Y}_s$ is also a Calabi-Yau 3-fold with G-action which acts freely outside finite points, with only isolated rational Gorenstein complete intersection singularities which are ordinary by Lemma 2.8, and $\mathfrak{Y}_s$ is $\mathbb{Q}$-factorial by [K-M, 12.1.10]. Thus we can continue the same process as above for $\mathfrak{Y}_s$. Finally we reach a smooth Calabi-Yau 3-fold by G-equivariant deformations. $\Box$

4. Proof of the Main Theorem 2

Let (Y, q) be an isolated singularity, Y a good representative for the germ, and $V = Y \setminus q$. Let $\nu : \tilde{Y} \to Y$ be a good resolution of Y and $E = \nu^{-1}(q)$. Identifing $\nu^{-1}(V)$ with V, we have a natural homomorphism of $\mathcal{O}_{Y,q}$-modules :

$$\tau' : H^1(V, \Omega^2_V) \to H^2_E(\tilde{Y}, \Omega^2_{\tilde{Y}}(logE)(-E))$$

as the coboundary map of the exact sequence of local cohomology. By theorem (3.1), we have the following lemma.

LEMMA 4.1. *Suppose that (Y, q) is a 3-dimensional isolated normal Gorenstein Du Bois (e.g. rational) singularity for what τ' is the zero map. Then (Y, q) is rigid.*

THEOREM 4.2. *Let X be a $\mathbb{Q}$-Fano-3-fold of Fano index 1 which has only isolated log terminal singularities. Assume that X has only ordinary complete intersection quotient singularities. Then X has a $\mathbb{Q}$-smoothing.*

PROOF. Let $i(X) = r$ and $G = \mathbb{Z}/r\mathbb{Z}$, and $\pi : Y \to X$ be a canonical cover of X with Galois group G. Let $\{p_1, p_2, \ldots, p_n\} = \{p \in Sing(X)|p$ is not a quotient singularity.$\}$, and $\{q_{i1}, q_{i2}, \ldots, q_{ik_i}\} = \pi^{-1}(\{p_i\})$ for $1 \le i \le n$. Then $Sing(Y) = \{q_{ij}|i, j\}$. Let Y_{ij} be a sufficiently small neighborhood of q_{ij}, $V = Y \setminus \{q_{ij}|i, j\}$ and $V_{ij} = Y_{ij} \setminus \{q_{ij}\}$. Let $\nu : \tilde{Y} \to Y$ be a G-equivariant good resolution of Y and set $E_{ij} = \nu^{-1}(q_{ij})$. We set $G_i = \{g \in$

$G|g(q_{ij}) = q_{ij}$ for any $j\}$. We consider the following commutative diagram:

$$
\text{(4.2.1)} \qquad
\begin{array}{ccc}
H^1(V,\Theta_V) & \xrightarrow{\alpha'} & \oplus_{i,j} H^2_{E_{ij}}(\tilde{Y}, \Omega^2_{\tilde{Y}}(logE)(-E)(\pi^* - K_Y)) \\
\| & & \uparrow {\scriptstyle \oplus_{i,j}\tau'_{ij}} \\
H^1(V,\Theta_V) & \xrightarrow{\alpha} & \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}})
\end{array}
$$

where α is the map determined by the map from global deformations to local deformations, α' is the coboundary map of the exact sequence of local cohomology, and τ'_{ij} is also the coboundary map of the exact sequence of local cohomology. We remark that τ'_{ij} is an homomorphism of $O_{Y_{ij},q_{ij}}$-modules which is compatible with the G_i-actions.

As all singularities q_{ij} are Gorenstein rational by [Ka 1, Proposition 1.7] and non smooth complete intersection singularities, they are not rigid. We have that τ'_{ij} is not the zero map by lemma 4.1 , i.e., $Ker(\tau'_{ij})$ is a proper $\mathcal{O}_{Y_{ij},q_{ij}}$-submodule of $h^1(V_{ij}, \Theta_{V_{ij}}) = T^1_{(Y_{ij},q_{ij})}$ for each i,j. By corollary 2.6, there exist G_i-invariant good directions $\eta_{ij} \in (T^1_{(Y_{ij},q_{ij})})^G$ for $Ker(\tau'_{ij})$ such that $g(\eta_{ij}) = \eta_{il}$ for $g \in G$ satisfing $g(q_{ij}) = q_{il}$. Thus $(\eta_{ij}|i,j) \in \oplus_{i,j} H^1(V_{ij}, \Theta_{V_{ij}})$ is a G-invariant element.

By the vanishing theorem of Guillén, Navarro Aznar and Puerta (cf. [St]), $H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(logE)(-E)(\pi^* - K_Y)) = 0$, thus α' is a surjection. By the commutative diagram (4.2.1), there exists an element $\eta' \in H^1(V, \Theta_V)$ such that $\alpha'(\eta')_{ij} = \tau'_{ij}(\eta_{ij})$ for any i,j. Let $\eta = tr(\eta') = \frac{1}{r}\sum_{g\in G} g(\eta')$. Then $\alpha(\eta)_{ij} - \eta_{ij} \in Ker(\tau'_{ij})$, and we have that $\alpha(\eta)_{ij}$ is a good direction for any i,j. By Theorem 1.2.(2), we have a G-equivariant 1-parameter small deformation of Y, $\mathfrak{f} : \mathfrak{Y} \to (\Delta, 0)$ determined by $\eta \in T^1_{D^G_Y}(Y/\mathbb{C})$. Let $M.e(\mathfrak{Y}_s) = max\{e(\mathfrak{Y}_s, q)|q \in Sing(\mathfrak{Y}_s)\}$ for $s \in (\Delta, 0)$, then we have $M.e(Y) > M.e(\mathfrak{Y}_s)$ for any $s \in (\Delta, 0) \setminus \{0\}$ by the choice of η. $\mathfrak{Y}_s$ is also a Fano 3-fold with G-action which acts freely outside finite points, with only isolated rational Gorenstein complete intersection singularities which are ordinary by lemma 2.8. Thus we can continue the same process as above for $\mathfrak{Y}_s$. Finally we reach a smooth Fano 3-fold by G-equivariant deformations. □

Let X be a $\mathbb{Q}$-Fano 3-fold of Fano index 1 which has only terminal singularities. Takagi telled me that:

THEOREM 4.3 (Sano, Takagi) (cf [Sa 1]).
Let X be a $\mathbb{Q}$-Fano 3-fold of Fano index 1 which has only terminal singularities. Then $i(X) = 2$. In particular, X has only ordinary terminal singularities.

Thus theorem 4.2 shows the main theorem 2.

5. Examples of $\mathbb{Q}$-Smoothing

In this section, we give some examples of $\mathbb{Q}$-smoothings of $\mathbb{Q}$-Calabi-Yau 3-folds and $\mathbb{Q}$-Fano 3-folds of Fano index 1.

Example 1. We construct a $\mathbb{Q}$-Calabi-Yau 3-fold of $I(X) = 5$ with one non-quotient terminal singularity. Let Y be the quintic hypersurface in $\mathbb{P}^4$ defined by the equation $F = X_0^3X_1X_2 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0$, then we have $h^1(Y, \mathcal{O}_Y) = 0$, and Y is a $\mathbb{Q}$-factorial Calabi-Yau 3-fold singular only $(1 : 0 : 0 : 0 : 0)$. We define an action of $G = \mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^4$ by $(X_0 : X_1 : X_2 : X_3 : X_4) \mapsto (X_0 : \xi^2X_1 : \xi^3X_2 : X_3 : \xi X_4)$ where ξ is a primitive 5-th root of unity. Then it acts also on Y and it fixes only at $(1 : 0 : 0 : 0 : 0)$ in Y. Let $X = Y/G$. Then it is a $\mathbb{Q}$-Calabi-Yau 3-fold with only terminal singularities and which has a non-quotient terminal singularity. X has a $\mathbb{Q}$-smoothing by Theorem 3.3, for example, $F+sX_0^5 = 0$ in $\mathbb{P}^4/G \times (\Delta, 0)$ gives a $\mathbb{Q}$-smoothing of X.

Example 2. We construct a $\mathbb{Q}$-Calabi-Yau 3-fold of $I(X) = 2$ with one non-quotient terminal singularity and 13 quotient terminal singularities. Let $(X_0 : X_1) \times (Y_0 : Y_1 : Y_2 : Y_3)$ be homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^3$ and Y a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^3$ defined by the bi-homogeneous equation of bi-degree $(2, 4)$ $F = \{Y_0Y_1^3+Y_2(2Y_2^3+Y_3^3)\}X_0^2+\{Y_0^4+Y_1^4+Y_2^4+Y_3^4\}X_1^2 = 0$. One can check that Y is a $\mathbb{Q}$-factorial Calabi-Yau 3-fold which is singular only at $(1 : 0) \times (1 : 0 : 0 : 0)$. We define $G = \mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{P}^1 \times \mathbb{P}^3$ by $(X_0 : X_1) \times (Y_0 : Y_1 : Y_2 : Y_3) \mapsto (X_0 : -X_1) \times (Y_0 : Y_1 : -Y_2 : -Y_3)$. Then it acts also on Y, and it fixes only finite points.Let $X = Y/G$. Then X is a $\mathbb{Q}$-Calabi-Yau 3-fold with only terminal singularities which has one non-qotient terminal singularity at $(1 : 0) \times (1 : 0 : 0 : 0)$. X has a $\mathbb{Q}$-smoothing by Theorem 3.3, for example, $F + sX_0^2Y_0^4 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^3/G \times (\Delta, 0)$ gives a $\mathbb{Q}$-smoothing of X.

Example 3. We construct a $\mathbb{Q}$-Calabi-Yau 3-fold of $I(X)=3$ with one non-quotient terminal singularity and 7 quotient terminal singularities. Let $(X_0:X_1:X_2)\times(Y_0:Y_1:Y_2)$ be homogeneous coordinates on $\mathbb{P}^2\times\mathbb{P}^2$ and Y a hypersurface in $\mathbb{P}^2\times\mathbb{P}^2$ defined by the bi-homogeneous equation of bi-degree $(3,3)$ $F=(Y_0Y_1^2+Y_2^3)X_0^3+(Y_0^3+2Y_1^3+Y_2^3)X_1^3+(Y_0^3+Y_1^3+2Y_2^3)X_2^3=0$. One can check that Y is a $\mathbb{Q}$-factorial Calabi-Yau 3-fold which is only singular at $(1:0:0)\times(1:0:0)$. We define $G=\mathbb{Z}/3\mathbb{Z}$ action on $\mathbb{P}^2\times\mathbb{P}^2$ by $(X_0:X_1:X_2)\times(Y_0:Y_1:Y_2)\mapsto(X_0:\xi^2X_1:\xi X_2)\times(Y_0:Y_1:\xi Y_2)$ where ξ is a primitive third root of unity. Then it acts also on Y, and it fixes only finite points. Let $X=Y/G$. Then X is a $\mathbb{Q}$-Calabi-Yau 3-fold with only terminal singularities which has one non-qotient terminal singularity at $(1:0:0)\times(1:0:0)$. X has a $\mathbb{Q}$-smoothing by Theorem 3.3, for example, $F+sX_0^3Y_0^3=0$ in $\mathbb{P}^2\times\mathbb{P}^2/G\times(\Delta,0)$ gives a $\mathbb{Q}$-smoothing of X.

Example 4. We construct a $\mathbb{Q}$-Fano 3-fold of Fano index 1 which has one non-cyclic quotient singularity and 4 qotient terminal singularities. Let $G=\mathbb{Z}/2\mathbb{Z}$. G acts on $\mathbb{P}(1,1,1,1,2)$ by $(X_0:X_1:X_2:X_3:X_4)\to(X_0:X_1:-X_2:-X_3:-X_4)$. Let Y be a $\{F=X_0^2X_2X_3+X_1^4+X_2^4+X_3^4+X_4^2=0\}\subset\mathbb{P}(1,1,1,1,2)$. Then G acts on Y. Let $X=Y/G$, then X is a $\mathbb{Q}$-Fano 3-fold of Fano index 1 with only terminal singularities. X has a non Gorenstein terminal singularity which is not a cyclic quotient singularity at $(1:0:0:0:0)$ and its $\mathbb{Q}$-smoothing is given by $\mathfrak{X}:F+sX_0^4=0\subset\mathbb{P}(1,1,1,1,2)/G\times(\Delta,0)$.

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