J. Math. Sci. Univ. Tokyo
6 (1999), 415–436.

Analogue of Flat Basis and Cohomological Intersection Numbers for General Hypergeometric Functions

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Dedicated to Professor Kazuhiko Aomoto on the occasion of his 60-th birthday

Abstract. The general hypergeometric functions of confluent type given by 1-dimensional integral are studied. To such functions, the rational de Rham cohomology group is associated and cohomological intersection numbers for a good basis are computed explicitly, using the property of the basis analogous to the flat basis of simple singularity of A-type.

1. Introduction

This paper concerns the explicit computation of intersection numbers for the de Rham cohomology classes associated with the general hypergeometric functions (GHF, for short) introduced in [1], [6] and [12]. According to [12], one can define, for any given partition λ of any positive integer n, general hypergeometric functions as solutions of a holonomic system on a Zariski open set of the space of complex matrices $M(r, n; \mathbb{C})$ or by integrals of Euler-Laplace type of $(r-1)$-form. See Sec. 2 for the details. For the partition $\lambda = (1, \dots, 1)$, GHF, which was introduced by K. Aomoto [1] and I.M. Gelfand [6], gives a generalization of the famous Gauss hypergeometric function. In fact, Gauss hypergeometric function corresponds to the case $(r, n) = (2, 4)$. For the hypergeometric function of Aomoto and Gelfand, an intersection theory is developped in [4], [16] and the explicit computation of the cohomological intersection numbers is carried out for the de Rham cohomology classes represented by logarithmic forms in the case $r = 2$.

For partitions λ containing parts greater than or equal to 2, GHF gives generalizations to several variables of the classical hypergeometric functions

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1991 *Mathematics Subject Classification.* 33C45, 33C50, 33C80.

Key words: hypergeometric function, de Rham cohomology, intersection number, flat basis.

of confluent type, say, Kummer's confluent hypergeometric function, Bessel function, Hermite function and Airy function. The de Rham cohomology group associated to GHF is calculated explicitly in the case $r = 2$ in [10]. To compute the intersection numbers for this case, the definition of intersection numbers given in [19] can be applied. We choose a "good basis" for the de Rham cohomology group which turns out to be an analogue of flat basis for the Jacobi ring for the simple singularity of A-type ([20], [21]) at several points in $\mathbb{P}^1$. Using this good basis, we obtain the matrix of intersection numbers which is independent of the variables of the GHF as was the case for Aomoto-Gelfand hypergeometric function when the logarithmic form are taken as a basis of the cohomology group. The contents of this paper are as follows.

§2 : General hypergeometric integral.
§3 : Twisted de Rham cohomology.
§4 : Cohomological intersection number.
§5 : Main theorem.
§6 : Invariance of intersection pairing by the group action.
§7 : Flatness of the basis $\varphi_i^{(k)}$.
§8 : Proof of Theorem 5.1.

2. General Hypergeometric Integral

Let $(n_1, \dots, n_l)$ be a partition of $n \geq 3$, namely a nonincreasing sequence of positive integers such that

$$n = \sum_{k=1}^{l} n_k.$$

To this partition we associate the abelian complex Lie subgroup of dimension n:

$$H = J(n_1) \times \cdots \times J(n_l),$$

where $J(n_k)$ is the Jordan group of size n_k defined by

$$J(n_k) = \left\{ h^{(k)} = \sum_{0 \leq i \leq n_k - 1} h_i^{(k)} \Lambda_{n_k}^i \mid h_0^{(k)} \neq 0, h_i^{(k)} \in \mathbb{C} \right\} \subset GL(n_k, \mathbb{C}),$$

$\Lambda_{n_k} = (\delta_{i+1,j})_{0\le i,j<n_k}$ being the shift matrix.

Let Z be the set of $2\times n$ complex matrices $z = (z^{(1)},\dots,z^{(l)}), z^{(k)} = (z_0^{(k)},\dots,z_{n_k-1}^{(k)}) \in M(2,n_k,\mathbb{C})$, satisfying the condition:

$$\begin{aligned}&\det(z_0^{(k)}, z_1^{(k)}) \neq 0 \quad \text{for any } k \text{ such that } n_k \ge 2.\\ &\det(z_0^{(k)}, z_0^{(k')}) \neq 0 \quad \text{for any } k \neq k' \ .\end{aligned} \tag{2.1}$$

The general hypergeometric integral (GHI) is defined as follows. Let $\tilde{H}$ be the universal covering group of H and let $\chi : \tilde{H} \to \mathbb{C}^\times$ be a character of $\tilde{H}$, that is, a complex analytic homomorphism from $\tilde{H}$ to the complex torus $\mathbb{C}^\times$. Define the functions $\theta_i(x)$ of $x = (x_0, x_1, x_2, \dots)$ by the generating function

$$\sum_{m=0}^{\infty} \theta_m(x) T^m = \log(x_0 + x_1 T + x_2 T^2 + \cdots).$$

Expanding the right hand side as

$$\begin{aligned}\log(&x_0 + x_1T + x_2T^2 + \cdots)\\ &= \log x_0 + \log\left(1 + \frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right)\\ &= \log x_0 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\left(\frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right)^m,\end{aligned}$$

we have

$$\theta_0(x) = \log x_0$$

and the weighted homogeneous polynomials in $x_1/x_0, x_2/x_0, \dots$

$$\begin{aligned}\theta_m(x) = &\sum_{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m} (-1)^{\lambda_1+\cdots+\lambda_m-1} \frac{(\lambda_1+\cdots+\lambda_m-1)!}{\lambda_1!\cdots\lambda_n!}\\ &\times \left(\frac{x_1}{x_0}\right)^{\lambda_1} \cdots \left(\frac{x_m}{x_0}\right)^{\lambda_m} .\end{aligned}$$

For example we have

$$\theta_0(x) = \log x_0,$$

$$\begin{aligned}
\theta_1(x) &= \frac{x_1}{x_0},\\
\theta_2(x) &= \frac{x_2}{x_0} - \frac{1}{2}\left(\frac{x_1}{x_0}\right)^2\\
\theta_3(x) &= \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) + \frac{1}{3}\left(\frac{x_1}{x_0}\right)^3\\
\theta_4(x) &= \frac{x_4}{x_0} - \frac{1}{2}\left(\frac{x_2}{x_0}\right)^2 - \left(\frac{x_1}{x_0}\right)\left(\frac{x_3}{x_0}\right) + \left(\frac{x_1}{x_0}\right)^2\left(\frac{x_2}{x_0}\right) - \frac{1}{4}\left(\frac{x_1}{x_0}\right)^4.
\end{aligned}$$

Then, the character $\chi : \tilde{H} \to \mathbb{C}^\times$ is explicitly written as

$$\chi(h;\alpha) = \prod_{k=1}^{l} \exp\left(\sum_{i=0}^{n_k-1} \alpha_i^{(k)} \theta_i(h^{(k)})\right)$$

for appropriate complex constants $\alpha = (\alpha^{(1)}, \dots, \alpha^{(l)}) \in \mathbb{C}^n$, $\alpha^{(k)} = (\alpha_0^{(k)}, \dots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$. Define a biholomorphic map

$$\iota : \quad \tilde{H} \to \prod_{k=1}^{l} \left(\tilde{\mathbb{C}}^\times \times \mathbb{C}^{n_k-1}\right) \subset \mathbb{C}^n$$

by

$$\iota(h) = (h_0^{(1)}, \dots, h_{n_1-1}^{(1)}, \dots, h_0^{(l)}, \dots, h_{n_l-1}^{(l)})$$

for $h = (h^{(1)}, \cdots, h^{(l)}) \in \tilde{H}$.

ASSUMPTION. *For the character* $\chi(\cdot\ ;\alpha)$ *of* $\tilde{H}_\lambda$, *we assume*

$$\sum_{k=1}^{l} \alpha_0^{(k)} = 0. \tag{2.2}$$

For $z \in Z$, we consider the n polynomials in t:

$$\mathbf{t}z = (\mathbf{t}z_0^{(0)}, \dots, \mathbf{t}z_{n_1-1}^{(0)}, \dots, \mathbf{t}z_0^{(l)}, \dots, \mathbf{t}z_{n_l-1}^{(l)})$$

defined by the multiplication of matrices $\mathbf{t} = (1, t)$ and $z_j^{(k)}$:

$$\mathbf{t}z_j^{(k)} = z_{0j}^{(k)} + tz_{1j}^{(k)}$$

and substitute these polynomials to the character $\chi(\cdot\,;\alpha)$ to obtain the function $\chi(\iota^{-1}(\mathbf{t}z);\alpha)$. By the assumption (2.2), $\chi(\iota^{-1}(\mathbf{t}z);\alpha)$ is a multivalued function of $(t,z) \in \mathbb{P}^1 \times Z$ having the branch locus

$$\bigcup_{k=1}^{l} \{(t,z) \mid \mathbf{t}z_0^{(k)} = 0\}.$$

Definition 2.1. The general hypergeometric integral is defined by

$$F(z;\alpha) = \int_{\Delta(z)} \chi(\iota^{-1}(\mathbf{t}z);\alpha)dt$$

where $\Delta(z)$ is some 1-dimensional cycle in $\mathbb{P}^1$ depending on $z \in Z$.

3. Twisted de Rham Cohomology

The hypergeometric integral is naturally regarded as a dual pairing of some cocycle of de Rham cohomology and the twisted cycle. We recall the definition of the de Rham cohomology.

For the moment we fix $z \in Z$, and consider the 1-form in t

$$\omega := d\log\chi(\iota^{-1}(\mathbf{t}z);\alpha) = \left(\sum_{k=1}^{l}\sum_{j=0}^{n_k-1} \alpha_j^{(k)}\partial_t\theta_j(\mathbf{t}z)\right)dt$$

obtained as the logarithmic derivative of $\chi(\iota^{-1}(\mathbf{t}z);\alpha)$. The 1-form ω has poles at

$$p_k = -z_{00}^{(k)}/z_{01}^{(k)}, \quad (k = 1,\dots,l)$$

of order n_k and these poles are distinct each other by virtue of the assumption (2.1). Let D be the divisor of the meromorphic 1-form ω in $\mathbb{P}^1$, i.e.,

$$D = \sum_{k=1}^{l} n_k\,p_k.$$

Let $\Omega^\bullet(*D)$ be the sheaf of meromorphic 1-forms on $\mathbb{P}^1$ having poles at most on $|D| = \{p_1, \dots, p_l\}$. Consider the de Rham complex

$$(\Omega^\bullet(*D), \nabla_\omega): \quad 0 \longrightarrow \Omega^0(*D) \xrightarrow{\nabla_\omega} \Omega^1(*D) \longrightarrow 0,$$

where ∇_ω is the connection defined by

$$\nabla_\omega f = df + \omega f, \quad f \in \Omega^0(*D).$$

The cohomology group of the complex of the global sections of the above complex of sheaves

$$H^p(\Gamma(\mathbb{P}^1, \Omega^\bullet(*D)), \nabla_\omega)$$

is called the *twisted rational de Rham cohomology group.* We simply denote this group by $H^p(\Omega^\bullet(*D), \nabla_\omega)$.

In [10] we proved the following.

PROPOSITION 3.1. *Let the parameters α in the connection form ω satisfy*

$$\alpha^{(k)}_{n_k-1} \begin{cases} \notin \mathbb{Z} & \text{if } n_k = 1, \\ \neq 0 & \text{if } n_k \geq 2. \end{cases} \tag{3.1}$$

Then we have

1. $H^i(\Omega^\bullet(*D), \nabla_\omega) = 0, \quad (i \neq 1)$,
2. $H^1(\Omega^\bullet(*D), \nabla_\omega) \simeq \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C}\cdot\omega$, *where $\Omega^1(D)$ is the sheaf of meromorphic 1-forms η such that*

$$(\eta) + D \geq 0,$$

3. $\dim_{\mathbb{C}} H^1(\Omega^\bullet(*D), \nabla_\omega) = n - 2$.

As a $\mathbb{C}$-basis of the vector space $\Gamma(\mathbb{P}^1, \Omega^1(D))$ we can take, for example, the 1-forms

$$\begin{array}{ll} (\mathbf{t}z_0^{(k)})^{-i}dt, & (k = 1, \dots, l; i = 2, \dots, n_k) \\ d\log(\mathbf{t}z_0^{(k)}) - d\log(\mathbf{t}z_0^{(k+1)}) & (k = 1, \dots, l-1), \end{array}$$

which were chosen in [10], [19]. In this paper we take the following 1-forms as a basis.

$$\begin{aligned} &\varphi_i^{(k)} = d\theta_i(\mathbf{t}z^{(k)}), && (k = 1, \dots, l; i = 1, \dots, n_k - 1) \\ &\varphi_0^{(k)} = d\theta_0(\mathbf{t}z^{(k)}) - d\theta_0(\mathbf{t}z^{(k+1)}), && (k = 1, \dots, l-1). \end{aligned} \tag{3.2}$$

For later use, we also prepare the 1-form

$$\varphi_0^{(l)} = d\theta_0(\mathbf{t}z^{(l)}) - d\theta_0(\mathbf{t}z^{(1)})$$

Note that, by virtue of the conditions (2.2), the 1-form ω is a linear combination of $\varphi_i^{(k)}$'s listed in (3.2). The reason for the choice of the forms $\varphi_i^{(k)}$'s will become clear in Sections 7 and 8.

4. Cohomological Intersection Number

We recall the definition of intersection numbers for the de Rham cohomology classes. For the details we refer to [19]. Consider two complexes of sheaves of meromorphic differential forms

$$\begin{aligned} (\Omega^\bullet(D), \nabla_\omega): &\quad 0 \longrightarrow \Omega^0 \xrightarrow{\nabla_\omega} \Omega^1(D) \longrightarrow 0, \\ (\Omega^\bullet(-D), \nabla_\omega): &\quad 0 \longrightarrow \Omega^0(-D) \xrightarrow{\nabla_\omega} \Omega^1 \longrightarrow 0. \end{aligned}$$

Then computing the associated hypercohomologies, we get the isomorphisms

$$\begin{aligned} j_\omega: &\ \mathbb{H}^1(\mathbb{P}^1, (\Omega^\bullet(D), \nabla_\omega)) \longrightarrow \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C}\cdot\omega \\ k_\omega: &\ \mathbb{H}^1(\mathbb{P}^1, (\Omega^\bullet(-D), \nabla_\omega)) \longrightarrow \mathrm{Ker}(\nabla_\omega : H^1(\mathbb{P}^1, \Omega^0(-D)) \to H^1(\mathbb{P}^1, \Omega^1)). \end{aligned}$$

On the otherhand there exists an isomorphism

$$\iota_\omega: \quad \mathbb{H}^\bullet(\mathbb{P}^1, (\Omega^\bullet(D), \nabla_\omega)) \longrightarrow \mathbb{H}^\bullet(\mathbb{P}^1, (\Omega^\bullet(-D), \nabla_\omega)).$$

This follows from the following exact sequence of complexes of sheaves and from the fact that the complex represented by the third column is exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^0(-D) \arrow[r, "\iota_\omega"] \arrow[d, "\nabla_\omega"'] & \Omega^0 \arrow[r] \arrow[d, "\nabla_\omega"'] & \bigoplus_{k=1}^{l} \left(\sum_{i=1}^{n_k} b_{ki}(t-p_k)^{i-1}\right)_{p_k} \arrow[r] \arrow[d, "\bar{\nabla}_\omega"'] & 0 \\
0 \arrow[r] & \Omega^1 \arrow[r, "\iota_\omega"] \arrow[d] & \Omega^1(D) \arrow[r, "\pi"] \arrow[d] & \bigoplus_{k=1}^{l} \left(\sum_{i=1}^{n_k} c_{ki}(t-p_k)^{-i}\right)_{p_k} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where π is defined by taking the principal part of a meromorphic 1-form in $\Omega^1(D)$ at each point p_k and $\bar{\nabla}_\omega$ is defined by applying ∇_ω to an element $\sum_{i=1}^{n_k} b_{ki}(t-p_k)^{i-1}$ and then taking the principal part of the resulted germ of meromorphic 1-form at p_k. Put $i_\omega := k_\omega \circ \iota_\omega$.

Now consider the de Rham complex $(\Omega^\bullet(*D), \nabla_{-\omega})$ defined by the connection $\nabla_{-\omega}$ with the connection form $-\omega$ which is dual to ∇_ω:

$$0 \longrightarrow \Omega^0 \xrightarrow{\nabla_{-\omega}} \Omega^1(D) \longrightarrow 0.$$

Assuming the condition (3.1), we have

$$H^p(\Omega^\bullet(*D), \nabla_{-\omega}) \simeq \begin{cases} \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C}\cdot(-\omega) & \text{if } p=1 \\ 0 & \text{otherwise.} \end{cases}$$

We define the intersection pairing between the de Rham cohomologies

$$H^1(\Omega^1(*D), \nabla_\omega) \times H^1(\Omega^1(*D), \nabla_{-\omega}) \longrightarrow \mathbb{C}$$

as follows. Take $[\varphi^+] \in H^1(\Omega^\bullet(*D), \nabla_\omega)$ and $[\varphi^-] \in H^1(\Omega^\bullet(*D), \nabla_{-\omega})$ represented by the forms $\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))$. Then $i_\omega \circ j_\omega^{-1}([\varphi^+]) \in \mathrm{Ker}(\nabla_\omega : H^1(\mathbb{P}^1, \Omega^0(-D)) \to H^1(\mathbb{P}^1, \Omega^1))$ and $[\varphi^-] \in \Gamma(\mathbb{P}^1, \Omega^1(D))/\mathbb{C}\cdot(-\omega)$. Then by the Serre duality $H^1(\mathbb{P}^1, \Omega^0(-D)) \times \Gamma(\mathbb{P}^1, \Omega^1(D)) \to H^1(\mathbb{P}^1, \Omega^1)$, we have an element of $H^1(\mathbb{P}^1, \Omega^1)$, which is represented by a global $(1,1)$-form by virtue of Dolbeault theorem. Integrating this 2-form over $\mathbb{P}^1$ we get a complex number, well defined for the classes $i_w \circ j_\omega^{-1}([\varphi^+]), [\varphi^-]$, which is denoted by $\langle[\varphi^+],[\varphi^-]\rangle$ and is called the intersection number of the classes $[\varphi^+]$ and $[\varphi^-]$.

For the 1-forms $\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ and $\omega \in \Gamma(\mathbb{P}^1, \Omega^1(D))$, we set

$$\varphi^+ = g^+(t)dt, \quad \varphi^- = g^-(t)dt, \quad \omega = h(t)dt.$$

Put

$$\frac{\varphi^+ * \varphi^-}{\omega} := \frac{g^+(t)g^-(t)}{h(t)}dt.$$

By following carefully the argument in [19], we see the following, the proof of which we omit.

Proposition 4.1. *The intersection number of the cohomology classes* $[\varphi^+] \in H^1(\Omega^1(*D), \nabla_\omega)$ *and* $[\varphi^-] \in H^1(\Omega^1(*D), \nabla_{-\omega})$ *with the representatives* $\varphi^+, \varphi^- \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ *is given by summing up the residues of the form at each point of* $|D|$*:*

$$\langle[\varphi^+],[\varphi^-]\rangle = 2\pi\sqrt{-1}\sum_{k=1}^{l} \mathrm{Res}_{t=p_k}\frac{\varphi^+ * \varphi^-}{\omega}.$$

5. Main Theorem

As in Section 3, we consider the elements of $\Gamma(\mathbb{P}^1, \Omega^1(D))$:

$$\varphi_0^{(1)}, \dots, \varphi_{n_1-1}^{(1)}, \dots, \varphi_0^{(l)}, \dots, \varphi_{n_l-1}^{(l)}. \tag{5.1}$$

If one omits one of $\varphi_0^{(1)}, \dots, \varphi_0^{(l)}$, the $n-1$ remaining 1-forms give a $\mathbb{C}$-basis of $\Gamma(\Omega^1(D))$. The classes in $H^1(\Omega^\bullet(*D), \nabla_\omega)$ and in $H^1(\Omega^\bullet(*D), \nabla_{-\omega})$ represented by the 1-form $\varphi_i^{(k)}$ is denoted by $[\varphi_i^{(k)+}]$ and $[\varphi_i^{(k)-}]$ respectively. Although we can obtain a basis of $H^1(\Omega^\bullet(*D), \nabla_\omega)$ by omitting one of the classes $[\varphi_{n_k-1}^{(k)}]$ $(k = 1, \dots, l)$, in order to present the matrix of intersection numbers $(\langle[\varphi_i^{(k)+}], [\varphi_j^{(k')-}]\rangle)$ in a symmetric manner, we compute these numbers for the classes given by the forms (5.1).

Introduce a series of polynomials $e_0(x) = 1, e_1(x), e_2(x), \dots$ of $x = (x_1, x_2, \dots)$ by using the generating function

$$(1 + x_1 T + x_2 T^2 + \cdots)^{-1} = \sum_{k=0}^{\infty} e_k(x) T^k$$

and put

$$\beta^{(k)} = (1, \beta_1^{(k)}, \dots, \beta_{n_k-1}^{(k)}) := \left(\frac{\alpha_{n_k-1}^{(k)}}{\alpha_{n_k-1}^{(k)}}, \frac{\alpha_{n_k-2}^{(k)}}{\alpha_{n_k-1}^{(k)}}, \dots, \frac{\alpha_0^{(k)}}{\alpha_{n_k-1}^{(k)}} \right), \qquad (k = 1, \dots, l).$$

Theorem 5.1. *The matrix of intersection numbers*

$$I = (I_{kk'})_{k,k'=1,\dots,l}, I_{k,k'} = (\langle \varphi_i^{(k)}, \varphi_j^{(k')} \rangle)_{0 \le i < n_k, 0 \le j < n_{k'}}$$

is symmetric and have the form

$$I = \begin{pmatrix} I_{11} & I_{12} & 0 & \dots & 0 & I_{1l} \\ I_{21} & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & I_{l-1,l} \\ I_{l1} & 0 & \dots & 0 & I_{l,l-1} & I_{ll} \end{pmatrix}$$

where

$$\begin{aligned} I_{kk} = {} & \frac{2\pi\sqrt{-1}}{\alpha_{n_k-1}^{(k)}} \begin{pmatrix} & & & e_0(\beta^{(k)}) \\ & & \iddots & e_1(\beta^{(k)}) \\ & \iddots & \iddots & \vdots \\ e_0(\beta^{(k)}) & e_1(\beta^{(k)}) & \dots & e_{n_k-1}(\beta^{(k)}) \end{pmatrix} \\ & + \delta_{n_{k+1},1} \frac{2\pi\sqrt{-1}}{\alpha_0^{(k+1)}} \begin{pmatrix} 1 & & \\ & & \\ & & \end{pmatrix} \end{aligned}$$

$$I_{k-1,k} = \frac{2\pi\sqrt{-1}}{\alpha^{(k)}_{n_k-1}} \begin{pmatrix} & & -1 \\ & & \\ & & \end{pmatrix} \qquad (k = 1, \dots, l)$$

Here when $k = 1$, *we understand* $(k-1, k)$ *as* $(l, 1)$ *by convention.*

Remark 5.2. The intersection numbers computed in the above theorem are independent of the variables $z \in Z$ of general hypergeometric functions. This fact relies on the choice of representatives of the cohomology classes. Here we took $\varphi_i^{(k)}$ as representatives, which, as will be seen in Sec. 7, can be regarded as an analogue of the flat basis of the Jacobi ring for the simple singularity of A-type at each point p_k of $|D|$. As for the flat basis, we refer the reader to [20], [21].

6. Invariance of Intersection Numbers by the Group Action

Let us consider the action of $G = GL(2, \mathbb{C})$ and of H on Z defined by

$$\rho_{g,h} : Z \longrightarrow Z, \quad z \mapsto gzh$$

and let X be the subset of Z consisting of the matrices

$$x = (x^{(1)}, \dots, x^{(l)}), \quad x^{(k)} \in M(2, n_k, \mathbb{C})$$

with

$$x^{(k)} = \begin{pmatrix} x_0^{(k)} & x_1^{(k)} & \dots & x_{n_k-1}^{(k)} \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

satisfying

1. $x_0^{(1)}, \dots, x_0^{(l)}$ are distinct complex numbers,
2. $x_1^{(k)} \neq 0$ for k such that $n_k \geq 2$,
3. in the case $l = 1$, $x_0^{(1)}, x_2^{(1)}$ are fixed to arbitrary prescribed numbers and $x_1^{(1)}$ to an arbitrary prescribed nonzero number, say, $x_0^{(1)} = 0, x_1^{(1)} = 1, x_2^{(1)} = 0$,
4. in the case $l = 2$, $x_0^{(1)}, x_0^{(2)}$ are fixed to arbitrary prescribed distinct numbers and $x_1^{(1)}$ to an arbitrary prescribed nonzero number, say, $x_0^{(1)} = 0, x_1^{(1)} = 1, x_2^{(1)} = 1$,
5. in the case $l \geq 3$, three among $x_0^{(1)}, \dots, x_0^{(l)}$, say $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}$, are fixed to some prescribed 3 distinct numbers.

Note that X is a closed submanifold of Z of dimension $n-3$.

PROPOSITION 6.1. *The subset X gives a realization of the quotient space $G\backslash Z/H$:*

$$\begin{array}{ccc} X & \longrightarrow & G\backslash Z/H \\ x & \mapsto & [x] \end{array}$$

is a homeomorphism.

By the proposition, we see that for any $z \in Z$ there are $g \in G$ and $h \in H$ such that

$$x = gzh \in X.$$

The forms $\varphi_i^{(k)} \in \Gamma(\mathbb{P}^1, \Omega^1(D))$ depend on $z \in Z$. When we want to make apparent the dependence of these forms on z we write $\varphi_i^{(k)}(z)$ instead of writing $\varphi_i^{(k)}$. We want to reduce the computation of the intersection numbers for $\varphi_i^{(k)}(z)$ to those for $\varphi_i^{(k)}(x)$ with $x \in X$. The first step is the following.

LEMMA 6.2. *The 1-forms $\varphi_i^{(k)}$ and ω are invariant under the action of H.*

PROOF. Since ω is a linear combination of $\varphi_i^{(k)}$'s, it suffices to show that $\varphi_i^{(k)}$ are invariant under the action of H. We prove in the case $i \geq 1$, since the case $i = 0$ is similarly proved. In this case, $\varphi_i^{(k)}(z) = d_t(\theta_i(\mathbf{t}z^{(k)}))$. By the definition of the functions $\theta_i(x)$, we have

$$\theta_i(\iota(hh')) = \theta_i(\iota(h)) + \theta_i(\iota(h')) \quad (h, h' \in J(n_k)).$$

Thus

$$\theta_i(\mathbf{t}z^{(k)}h^{(k)}) = \theta_i(\mathbf{t}z^{(k)}) + \theta_i(\iota(h^{(k)})).$$

Taking the exterior derivative of the both sides with respect to t, we get

$$d\bigl(\theta_i(\mathbf{t}z^{(k)}h^{(k)})\bigr) = d\bigl(\theta_i(\mathbf{t}z^{(k)})\bigr).$$

This implies the invariance $\varphi_i^{(k)}(z) = \varphi_i^{(k)}(zh) \quad (h \in H)$. □

Next we consider the action of G on Z.

Lemma 6.3. *We have*

$$\langle[\varphi_i^{(k)^+}(z)],[\varphi_i^{(k)^-}(z)]\rangle=\langle[\varphi_i^{(k)^+}(gz)],[\varphi_i^{(k)^-}(gz)]\rangle,\quad (g\in G).\tag{6.1}$$

Proof. Consider the projective transformation

$$P_g:\ \mathbb{P}^1\ni t\mapsto s:=t\cdot g=\frac{b+dt}{a+ct}\in\mathbb{P}^1\quad\text{for}\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G.$$

In view of Proposition 4.1, the intersection number for $\psi^+,\psi^-\in\Gamma(\mathbb{P}^1,\Omega^1(D))$ satisfies

$$\langle[\psi^+],[\psi^-]\rangle=\langle[P_g^*\psi^+],[P_g^*\psi^-]\rangle.\tag{6.2}$$

On the otherhand, for the forms $\varphi_i^{(k)}(z)$, we have

$$P_g^*\varphi_i^{(k)}(z)=\varphi_i^{(k)}(gz).\tag{6.3}$$

In fact, for the case $i\geq 1$,

$$\begin{aligned}P_g^*\varphi_i^{(k)}(z)&=P_g^*d\big(\theta_i(sz^{(k)})\big)\\&=d\big(\theta_i((1,t\cdot g)z^{(k)})\big)\\&=d\big(\theta_i((1,t)gz^{(k)})\big)\\&=\varphi_i^{(k)}(gz).\end{aligned}$$

The case $i=0$ can be shown similarly. Combining (6.2) and (6.3), we have the desired identity (6.1). □

Summing up we have shown the following.

Proposition 6.4. *The intersection number* $\langle[\varphi_i^{(k)^+}],[\varphi_j^{(k')^-}]\rangle$ *is invariant by the action of* $G\times H$ *on* Z, *namely we have*

$$\langle[\varphi_i^{(k)^+}(z)],[\varphi_j^{(k')^-}(z)]\rangle=\langle[\varphi_i^{(k)^+}(\rho_{g,h}(z))],[\varphi_j^{(k')^-}(\rho_{g,h}(z))]\rangle\quad \textit{for all}\ \ (g,h)\in G\times H.$$

7. Flatness of the Basis $\varphi_i^{(k)}$

As is seen in Section 6, for the aim of computing intersection numbers for the forms $\varphi_i^{(k)}$'s, it is sufficient to consider $\varphi_i^{(k)}(x)$ for $x \in X$. In this section we fix $x \in X$ and write simply $\varphi_i^{(k)}$ for $\varphi_i^{(k)}(x)$. We look into in detail the property of these forms which permit us to regard these forms as analogues of flat basis of the Jacobi ring of simple singularity of A-type.

Let $x \in X$ be as in Section 6. Note that the pole divisor of the 1-form $\omega = d\log\chi(\mathbf{t}x;\alpha)$ is

$$D = \sum_{k=1}^{l} n_k p_k, \quad p_k = -x_0^{(k)}.$$

We consider the forms

$$\varphi_0^{(k)}, \dots, \varphi_{n_k-1}^{(k)}. \tag{7.1}$$

having poles at p_k. Take a local coordinate u at p_k defined by

$$u = \frac{1}{x_1^{(k)}}(t + x_0^{(k)}) \tag{7.2}$$

and put

$$y_i = x_i^{(k)}/x_1^{(k)}, \quad (i = 1, \dots, n_k - 1)$$

Note that $y_1 = 1$. Then the forms (7.1) are expressed as

$$\begin{aligned} \varphi_i^{(k)} &= d\big(\theta_i(1, y_1 u^{-1}, \dots, y_{n_k-1}u^{-n_k+1})\big), \quad (i = 1, \dots, n_k - 1) \\ \varphi_0^{(k)} &= d\log(u) - d\log(u - p_{k+1} + p_k). \end{aligned}$$

This situation motivates to introduce the polynomials $h_m(u)$ in u^{-1} depending on the parameters $(y_1, y_2, \dots), y_1 = 1$, by substituting

$$x_0 = 1, x_1 = y_1 u^{-1}, x_2 = y_2 u^{-1}, \cdots \quad (y_1 = 1).$$

in the functions $\theta_m(x)$ $(m = 1, 2 \dots)$:

$$\begin{aligned}h_m(u) &= \theta_m(1, y_1 u^{-1}, y_2 u^{-1}, \dots)\\ &= \sum_{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m} (-1)^{\lambda_1+\cdots+\lambda_m-1}(\lambda_1+\cdots+\lambda_m-1)!\\ &\quad \times \frac{y_1^{\lambda_1}\cdots y_m^{\lambda_m}}{\lambda_1!\cdots\lambda_m!}u^{-(\lambda_1+\cdots+\lambda_m)}.\end{aligned} \tag{7.3}$$

Note that $h_m(u)$ is a polynomial of u^{-1} of degree m without constant term whose top term is

$$(-1)^{m+1}u^{-m}/m$$

and the coefficients of u^{-1} is equal to y_m. Consider a Laurent series in u:

$$f = -u^{-1}(1 + s_1 u + s_2 u^2 + \cdots)$$

with parameters $s = (s_1, s_2, \dots)$. Then the power f^m is a Laurent series in u whose principal part $(f^m)_-$ is a polynomial of u^{-1} of degree m with the top term $(-1)^m u^{-m}$. Note that the coefficients of u^{-1} of $(f^m)_-$ has the form

$$(-1)^m m s_{m-1} + (\text{a polynomial in } s_1, \dots, s_{m-2}\ ).$$

Then the property we want to establish for $h_m(u)$ is the following.

Proposition 7.1. *Determine $s_1, s_2, \dots$ by the condition:*

$$y_m = \textit{the coefficient of } u^{-1} \textit{ of } \ -\frac{1}{m}f^m, \qquad (m = 1, 2, \dots). \tag{7.4}$$

Then the identities

$$h_m(u) = -\frac{1}{m}(f^m)_- \qquad (m = 1, 2, \dots) \tag{7.5}$$

hold as polynomials in u^{-1}.

To prove the proposition, it is convenient to use the Schur functions $p_0(t), p_1(t), p_2(t), \dots$ defined by the generating function:

$$\exp(t_1 T + t_2 T^2 + \cdots) = \sum_{m=0}^{\infty} p_m(t) T^m,$$

where $p_0(t) = 1$. For the parameters $s = (s_1, s_2, \dots)$ in f, we define $t = (t_1, t_2, \dots)$ by

$$s_m = p_m(t), \quad (m = 1, 2, \dots).$$

Then

$$\begin{aligned} f^m &= (-1)^m u^{-m} (1 + s_1 u + s_2 u^2 + \cdots)^m \\ &= (-1)^m u^{-m} \exp(t_1 u + t_2 u^2 + \cdots)^m \\ &= (-1)^m u^{-m} \exp(m t_1 u + m t_2 u^2 + \cdots) \\ &= (-1)^m u^{-m} \sum_{k=0}^{\infty} p_k(mt) u^k. \end{aligned}$$

Hence we have

$$(f^m)_- = (-1)^m \sum_{k=1}^{m} p_{m-k}(mt) u^{-k}.$$

The condition (7.4) is then written as

$$y_m = \frac{(-1)^{m+1}}{m} p_{m-1}(mt), \quad (m = 1, 2, \dots). \tag{7.6}$$

Putting the expression (7.6) into (7.3), we see that $h_m(u)$ is written as

$$\begin{aligned} h_m(u) = (-1)^{m+1} &\sum_{\lambda_1 + 2\lambda_2 + \cdots + m\lambda_m = m} \frac{(\lambda_1 + \cdots + \lambda_m - 1)!}{\lambda_1! \cdots \lambda_m!} \\ &\times (p_0(t))^{\lambda_1} \left(\frac{1}{2} p_1(2t) \right)^{\lambda_2} \cdots \left(\frac{1}{m} p_{m-1}(mt) \right)^{\lambda_m} u^{-(\lambda_1 + \cdots + \lambda_m)}. \end{aligned}$$

Thus the verification of the identity (7.5) is reduced to showing the following identities for the Schur functions.

LEMMA 7.2. *We have the identities*

$$\begin{aligned} \frac{1}{m} p_{m-k}(mt) = &\sum_{\substack{\lambda_1 + 2\lambda_2 + \cdots + m\lambda_m = m \\ \lambda_1 + \cdots + \lambda_m = k}} \frac{(\lambda_1 + \cdots + \lambda_m - 1)!}{\lambda_1! \cdots \lambda_m!} \\ &\times (p_0(t))^{\lambda_1} \left(\frac{1}{2} p_1(2t) \right)^{\lambda_2} \cdots \left(\frac{1}{m} p_{m-1}(mt) \right)^{\lambda_m} \end{aligned} \tag{7.7}$$

for $m = 1, 2, \dots$ *and* $k = 1, 2, \dots, m$.

Proof. The proof is carried out by induction on m and k. In the case $m = 1$ or the case $k = 1$, the identities (7.7) trivially hold. Assume that (7.7) holds for m replaced by $1, 2, \dots, m-1$. Moreover, for m fixed, the identity (7.7) holds for k replaced by $1, 2, \dots, k-1$ We will prove (7.7) still holds for the case where k is replaced by $k+1$. We may assume $k \geq 2$. In this case the possible n-tuple of indices $\lambda = (\lambda_1, \dots, \lambda_n)$ appearing in the sum of the right hand side of (7.7) satisfies $\lambda_n = 0$. Differentiate the both sides of the identity (7.7). Then we get

$$\text{L.H.S} = p_{m-k-1}(mt).$$

and

$$\begin{aligned}
\text{R.H.S} = & \sum_{\substack{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m,\\ \lambda_1+\cdots+\lambda_m=k}} \frac{(k-1)!}{\lambda_1!\cdots\lambda_m!} \sum_{j=1}^{m-1} \frac{\left(\frac{1}{j}p_{j-1}(jt)\right)^{\lambda_j-1}}{(\lambda_j-1)!} p_{j-2}(jt) \\
& \times \prod_{i\neq j} \frac{\left(\frac{1}{i}p_{i-1}(it)\right)^{\lambda_i}}{\lambda_i!} \\
= & \sum_{\substack{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m,\\ \lambda_1+\cdots+\lambda_m=k}} \frac{(k-1)!}{\lambda_1!\cdots\lambda_m!} \sum_{j=1}^{m-1} \sum_{\substack{\mu_1+2\mu_2+\cdots+(j-1)\mu_{j-1}=j\\ \mu_1+\cdots+\mu_{j-1}=2}} j \\
& \times \prod_{1\leq i<j} \frac{\left(\frac{1}{i}p_{i-1}(it)\right)^{\lambda_i+\mu_i}}{\lambda_i!\mu_i} \times \\
& \times \frac{\left(\frac{1}{j}p_{j-1}(jt)\right)^{\lambda_j-1}}{(\lambda_j-1)!} \prod_{j\leq i\neq m-1} \frac{\left(\frac{1}{i}p_{i-1}(it)\right)^{\lambda_i}}{\lambda_i!}.
\end{aligned}$$

We want to show that this right hand side is equal to

$$m \sum_{\substack{\nu_1+2\nu_2+\cdots+m\nu_m=m\\ \nu_1+\cdots+\nu_m=k+1}} \frac{k!}{\nu_1!\cdots\nu_m!} \prod_{1\leq i\leq m} \left(\frac{1}{i}p_{i-1}(it)\right)^{\nu_i}. \tag{7.8}$$

Now we fix the indices $\nu = (\nu_1, \cdots, \nu_m)$ such that $\nu_1 + 2\nu_2 + \cdots + m\nu_m = m, \nu_1 + \cdots + \nu_m = k + 1$. Then, in the sum R.H.S, the contribution to the coefficients of $\prod_i \left(\frac{1}{i} p_{i-1}(it)\right)^{\nu_i}$ comes from the following cases of indices λ and μ. Take any index $1 \leq \alpha, \beta \leq m - 1$ such that $\alpha + \beta \leq m - 1$. If $\alpha < \beta$, we put

$$\begin{aligned} \lambda &= (\nu_1, \ldots, \nu_\alpha - 1, \ldots, \nu_\beta - 1, \ldots, \nu_m) \\ \mu &= (0, \ldots, 1, \ldots, 1, \ldots, 0), \quad j = \alpha + \beta. \end{aligned}$$

If $\alpha = \beta$, we put

$$\begin{aligned} \lambda &= (\nu_1, \ldots, \nu_\alpha - 2, \ldots, \nu_m) \\ \mu &= (0, \ldots, 2, \ldots, 0), \quad j = 2\alpha. \end{aligned}$$

Summing up all the contribution, we have

$$\begin{aligned} &\frac{(k-1)!}{\nu_1! \cdots \nu_m!} \left\{ \sum_{1 \leq \alpha < \beta, \alpha + \beta \leq m-1} (\alpha + \beta) \mu_\alpha \mu_\beta + \sum_{1 \leq \alpha, 2\alpha \leq m-1} 2\alpha \frac{\mu_\alpha(\mu_\alpha - 1)}{2} \right\} \\ &= \frac{(k-1)!}{\nu_1! \cdots \nu_m!} \left\{ \sum_{1 \leq \alpha, \beta \leq m-1} \alpha \mu_\alpha \mu_\beta - \sum_{1 \leq \alpha \leq m-1} \alpha \mu_\alpha \right\} \\ &= \frac{(k-1)!}{\nu_1! \cdots \nu_m!} km \end{aligned}$$

Thus R.H.S is written as (7.8) as is desired. □

As a corollary, wehave

COROLLARY 7.3. *In the above situation, we have*

$$\varphi_i^{(k)}(x) = -(\partial f \cdot f^{-1})_- du \qquad (i = 1, \ldots, n_k - 1).$$

8. Proof of Theorem 5.1

In view of the invariance of the intersection numbers $\langle[\varphi_i^{(k)^+}(z)], [\varphi_j^{(k')^-}(z)]\rangle$ by the action $G\times H$ (Sec.6), it is sufficient to prove the theorem for $z\in X$. In this case the flatness of the basis $\varphi_i^{(k)}$'s plays a crucial role. Recall that

$$\langle[\varphi_i^{(k)^+}], [\varphi_j^{(k')^-}]\rangle = 2\pi\sqrt{-1}\sum_{k=1}^{l}\mathrm{Res}_{t=p_k}\frac{\varphi_i^{(k)} * \varphi_j^{(k')}}{\omega}.$$

Take the local coordinate u at p_k as in (7.2) and choose the Laurent series f at $u=0$ of the form

$$f = -u^{-1}(1+s_1u+s_2u^2+\cdots)$$

as in Section 7. Then Corollary 7.3 says that, at $u=0$, the 1-forms $\varphi_i^{(k)}$ can be expressed as

$$\varphi_i^{(k)} = -(\partial f\cdot f^{i-1})_- du, \quad (i=1,\dots,n_k-1). \tag{8.1}$$

Similarly the 1-form ω is expressed as

$$\begin{aligned}\omega &= \alpha_0^{(k)} d\log u + \sum_{m=1}^{n_k-1}\alpha_m^{(k)}\varphi_m^{(k)} + (\text{1-form holomorphic at } u=0\ )\\ &= -\sum_{m=0}^{n_k-1}\alpha_m^{(k)}(\partial f\cdot f^{m-1})_- du + (\text{1-form holomorphic at } u=0).\end{aligned}$$

Then we can prove the following.

Lemma 8.1. *We have*

$$\mathrm{Res}_{u=0}\frac{\varphi_i^{(k')} * \varphi_j^{(k')}}{\omega} = \begin{cases} \frac{1}{\alpha_{n_k-1}^{(k)}} e_{i+j-n_k+1}(\beta^{(k)}) & k'=k\\ \frac{1}{\alpha_0^{(k)}} & k'=k-1, n_k=1, (i,j)=(0,0),\\ 0 & \textit{otherwise.}\end{cases}$$

PROOF. We prove only the case $k' = k$ and $n_k \geq 2, i \geq 1, j \geq 1$. Using the expression (8.1) for $\varphi_i^{(k)}$, we have

$$\begin{aligned}
&\operatorname{Res}_{u=0} \frac{\varphi_i^{(k)} * \varphi_j^{(k)}}{\omega} \\
&= - \operatorname{Res}_{u=0} \frac{(\partial f \cdot f^{i-1})_-(\partial f \cdot f^{j-1})_-}{\sum_{m=0}^{n_k-1} \alpha_m^{(k)} (\partial f \cdot f^{m-1})_- + (\text{holo. function at } u=0)} du \\
&= -\frac{1}{\alpha_{n_k-1}^{(k)}} \operatorname{Res}_{u=0} \frac{\partial f \cdot f^{i+j-n_k}}{1 + \beta_1 f^{-1} + \cdots + \beta_{n_k-1} f^{-(n_k-1)}} du \\
&= -\frac{1}{\alpha_{n_k-1}^{(k)}} \operatorname{Res}_{u=0} \partial f \cdot f^{i+j-n_k} \sum_{m=0}^{\infty} e_m(\beta) f^{-m} du \\
&= \frac{1}{\alpha_{n_k-1}^{(k)}} e_{i+j-n_k+1}(\beta).
\end{aligned}$$

Here we have used the fact

$$\operatorname{Res}_{u=0} \partial f \cdot f^{i+j-n_k-m} du = \begin{cases} -1 & i+j-n_k-m = -1 \\ 0 & \text{otherwise,} \end{cases}$$

For the other cases $i = 0$ or $j = 0$ or $n_k = 1$, the assertion is similary proved. □

The above computation in the proof of Lemma 8.1 shows that

$$\operatorname{Res}_{u=0} \frac{\varphi^+ * \varphi^-}{\omega} = 0$$

if the sum of the orders of pole of φ^+ and φ^- at $u = 0$ is less than or equal to n_k. This remark implies the following.

LEMMA 8.2.

$$\operatorname{Res}_{u=0} \frac{\varphi_i^{(k)} * \varphi_j^{(k')}}{\omega} = 0 \quad \textit{if} \quad |k-k'| \geq 2, (k,k') \neq (1,l), (l,1) \tag{8.2}$$

$$\operatorname{Res}_{u=0} \frac{\varphi_i^{(k-1)} * \varphi_j^{(k)}}{\omega} = \begin{cases} -1/\alpha_{n_k-1}^{(k)}, & (i,j) = (0, n_k-1) \\ 0 & \textit{otherwise} \end{cases} \tag{8.3}$$

When $k = 1$, we understand the second formula as that for the case $(k-1, k) = (l, 1)$.

Combining these lemmas we have the following lemma which complete the proof of Theorem 5.1.

LEMMA 8.3. *We have the following equality.*

$$\begin{aligned}
\langle [\varphi_i^{(k)^+}], [\varphi_j^{(k)^-}] \rangle &= \frac{2\pi\sqrt{-1}}{\alpha^{(k)}_{n_k-1}} e_{i+j-n_k+1}(\beta^{(k)}) + \frac{2\pi\sqrt{-1}}{\alpha^{(k)}_0} \delta_{n_{k+1},1}\delta_{i,0}\delta_{j,0}, \\
\langle [\varphi_i^{(k-1)^+}], [\varphi_j^{(k)^-}] \rangle &= -\frac{2\pi\sqrt{-1}}{\alpha^{(k)}_{n_k-1}} \delta_{i,0}\delta_{j,n_k-1}, \\
\langle [\varphi_i^{(k)^+}], [\varphi_j^{(k')^-}] \rangle &= 0 \qquad \textit{if} \quad |k-k'| \geq 2, (k,k') \neq (1,l), (l,1).
\end{aligned}$$

In the second equality, we used the same convention as in Lemma 8.2.

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(Received September 2, 1998)

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