Vertex operators and background solutions for ultradiscrete soliton equations
(超離散ソリトン方程式における頂点作用素と背景解)

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Chapter 1

Introduction

During the last two decades, the discovery of Box and Ball systems (BBS) [1] and of the ultradiscretization procedure [2] had a great impact on the study of the integrable systems. The BBS is a cellular automaton that consists of an infinite sequence of boxes and a finite amount of balls and distinguishes states by means of existence/non-existence of balls in each box. The time evolution of the BBS is described by the following procedure:

1. Starting from the left, pick up the ball in the box and put it into the nearest right empty box.

2. Skip the balls already moved in this time evolution.

3. Finish the time evolution when all balls have been moved.

Figure 1.1 shows an example of the time evolution of the BBS. Even though the states are binary-valued, blocks of balls behave like solitons. Cellular

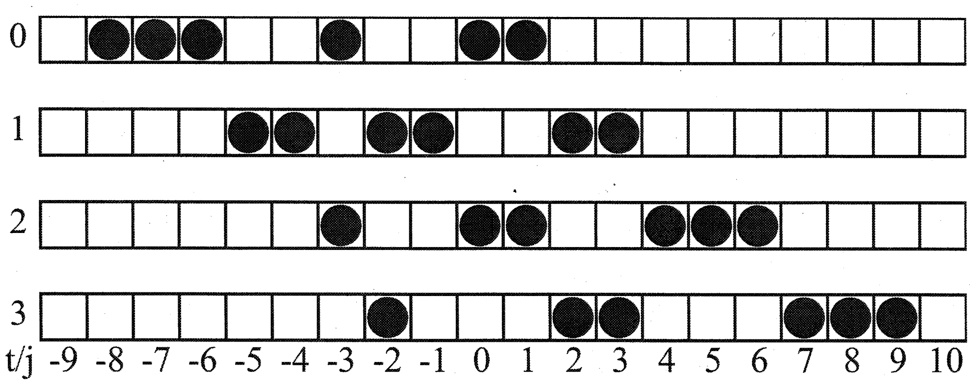

Figure 1.1: An example of time evolution of BBS

automata with such behavior are called soliton cellular automata. Furthermore, the BBS has an infinite amount of conserved quantities [3], like most ordinary soliton equations.

By introducing the dependent variable B_j^t to represent the state of the box at site j and time t and defining $B_j^t = 1$ when there is a ball in the box and $B_j^t = 0$ when not, the time evolution rule is rewritten as

$$B_j^{t+1} = \min\left(1 - B_j^t, \sum_{n=-\infty}^{j-1} (B_n^t - B_n^{t+1})\right) \tag{1.1}$$

and B_j^t is required to satisfy the following boundary conditions:

$$B_j^t = 0 \quad \text{for } |j| \gg 0, \tag{1.2}$$

which means that the number of balls is finite.

The ultradiscretization is a limiting procedure used to relate discrete soliton equations to soliton cellular automata and is defined as follows:

1. Transform the dependent variables and parameters by exponential functions, upon introduction of a parameter ε, for example $a = e^{A/\varepsilon}$.

2. Take the logarithm of each side of the equation and take the limit $\varepsilon \to 0$. Then, by means of the identity

$$\lim_{\varepsilon \to \infty} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B), \tag{1.3}$$

   the operators $+$ and $\times$ are replaced with max and $+$ respectively.

The time evolution rule of the BBS (1.1) is obtained by ultradiscretizing the discrete KdV equation [2].

The BBS has extensions such as variable capacity of boxes, several kinds of balls [4] and a carrier with limited capacity [5] and these varieties are also obtained by ultradiscretizing the discrete soliton equations. It is therefore assumed that the ultradiscrete systems preserve the main characteristics of integrability and that finding ultradiscrete analogues of the ideas for the ordinary soliton equations is a road map to reveal structures of the soliton cellular automata.

A procedure to obtain a new solution of a soliton equation from a given one is known under the name of a Bäcklund transformation and is generally expressed in the form of differential equations. The ultradiscrete analogue of the Bäcklund transformation for the KdV equation in the case of the ultradiscrete KdV equation is presented in [6].

The fact that soliton solutions are expressed as determinants and that the equations themselves are indeed determinantal identities, is the main paradigm of soliton theory. Takahashi and Hirota presented an approach based on so-called "permanent type solutions" [7] (which are expressed as signature-free Casorati determinants) to discuss particular solutions of ultradiscrete systems. Nagai presented identities for permanent type solutions, which can be considered as ultradiscrete analogues of Plücker relations for determinants in [8].

In this thesis, we propose another approach by means of the ultradiscrete analogue of the vertex operator, which is an operator representation of the Bäcklund transformation and maps $N-1$-soliton solutions to N-soliton ones. The approach is believed to be closely related to certain types of symmetries for this system because in fact, the vertex operator approach is closely related to the existence of certain symmetry algebras for integrable systems. In chapter 2, we propose the vertex operator for the ultradiscrete KdV equation. In chapter 3, we propose a vertex operator for the non-autonomous ultradiscrete KP equation, which is an extension of the ultradiscrete KdV equation.

Recently, ultradiscrete systems have drawn increasing interest due to the establishment of relationships to other mathematical topics. Mada et al. solved the initial value problem of the BBS by means of combinatorial techniques in [9]. The fundamental period of the BBS with the periodic boundary condition is found by means of an algebraic geometrical approach in [10]. The dynamics of the BBS is described by representation theory in [11]. It is therefore fruitful to clarify the symmetries and the algebraic structure of ultradiscrete soliton equations, as was done for the continuous ones.

However, by definition (1.1), we can obtain the time evolution of arbitrary initial states as long as they satisfy the boundary condition (1.2). Figures 1.2 and 1.3 show time evolutions for this system when the initial values are not limited to $\{0,1\}$. Travelling waves with various values, which are not traditional soliton solutions, are observed in these time evolutions, in addition to blocks of '1"s (solitons).

In chapter 4, we deal with such solutions. We first propose a class of solutions which can be "backgrounds" for solitons, an extension presented in [12]. We prove that we can apply the vertex operator to these solutions and give explicit formulae for solitons blended with backgrounds such as in Figure 1.2 and 1.3. This approach is expected to lead to a solution of the initial value problem for these equations because of the observation that most integer-valued initial states split into solitons and backgrounds under the time evolutions.

$$
\begin{array}{ll}
 & \xrightarrow{j} \\
 & .\mathtt{2}.................. \\
 & .\underline{\mathtt{1}}\mathtt{111}............... \\
t\downarrow & ..\underline{\mathtt{1}}..\mathtt{111}............ \\
 & ...\underline{\mathtt{1}}....\mathtt{111}......... \\
 & ....\underline{\mathtt{1}}......\mathtt{111}......
\end{array}
$$

Figure 1.2: time evolution of 2 on a site (an underscore means minus sign at this site and a dot means 0)

$$
\begin{array}{ll}
 & \xrightarrow{j} \\
 & .\mathtt{4}\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{12}}\mathtt{2}............. \\
 & .\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{3}}\mathtt{23}\underline{\mathtt{1}}\mathtt{11}........... \\
t\downarrow & ..\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{32}}\mathtt{2}..\mathtt{111111}..... \\
 & ...\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{32}}\mathtt{11}......\mathtt{11111} \\
 & ....\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{32}}.\mathtt{11}......... \\
 & .....\underline{\mathtt{3}}\mathtt{4}\underline{\mathtt{32}}..\mathtt{11}.......
\end{array}
$$

Figure 1.3: time evolution of an initial state

Chapter 2

Vertex operator for the ultradiscrete KdV equation

2.1 Introduction

In this chapter, we propose an ultradiscrete analogue of the vertex operator in the case of the ultradiscrete KdV equation. In section 2 this operator is introduced and in section 3 we prove that the functions generated by this operator indeed solve the ultradiscrete KdV equation. In section 4 we give some examples for the action of the vertex operator in the language of the Box and Ball System.

2.1.1 KdV hierarchy and the vertex operator

The KdV hierarchy is a series of partial differential equations for $u = u(x + t_1, t_3, t_5, \dots)$ given by

$$\frac{\partial}{\partial t_l} u = -[P, (P^{l/2})_{\geq 0}] \qquad (l = 1, 3, 5, \dots), \tag{2.1}$$

for

$$P = \delta^2 + u. \tag{2.2}$$

Here, $(A)_{\geq 0}$ is non-negative part of the pseudo-differential operator $A = \sum_{k=0}^{\infty} a_k \delta^{n-k}$, i.e.:

$$(A)_{\geq 0} = \sum_{k=0}^{n} a_k \delta^{n-k} \tag{2.3}$$

and $\delta = \frac{\partial}{\partial x}$.

By substituting

$$u = \frac{\partial^2}{\partial x^2} \log f \tag{2.4}$$

and setting some boundary conditions, the KdV hierarchy (2.1) is transformed into the series of bilinear forms, for example:

$$\left(D_1^4 - 4D_1D_3\right) f \cdot f = 0, \tag{2.5}$$

which is the bilinear form of the KdV equation. Here, $D_i^\alpha D_j^\beta \tau \cdot \tau$ (Hirota derivative) is defined as

$$D_i^\alpha D_j^\beta f \cdot f := \frac{\partial^\alpha}{\partial s_i^\alpha} \frac{\partial^\beta}{\partial s_j^\beta} f(\mathbf{t}+\mathbf{s}) f(\mathbf{t}-\mathbf{s}) \bigg|_{\mathbf{s}=\mathbf{0}} \tag{2.6}$$

for $\mathbf{s} = (s_1, s_3, s_5, \ldots)$.

The vertex operator of the KdV hierarchy is given in the form of an infinitesimal transformation:

$$X(p) = \exp\left(2\sum_{k=0}^{\infty} p^{2k+1} t_{2k+1}\right) \exp\left(-2\sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{1}{p^{2k+1}} \frac{\partial}{\partial t_{2k+1}}\right). \tag{2.7}$$

which was first presented in [13]. The exponent of this operator maps $N-1$-soliton solutions to N-soliton ones and all soliton solutions are generated by repeated application. It should be noted that $X(p_N)^2 = 0$ i.e. $e^{c_N X(p_N)} = 1 + c_N X(p_N)$ indeed.

2.1.2 Discrete KdV equation

The bilinear form of the discrete KdV equation [14] is written as

$$r^2 f_j^t f_{j+1}^{t+2} = f_j^{t+2} f_{j+1}^t + (r^2 - 1) f_j^{t+1} f_{j+1}^{t+1}. \tag{2.8}$$

By taking a continuum limit, (2.8) is transformed into the bilinear form of the KdV equation (2.5).

The N-soliton solution of (2.8) is expressed as the Casorati determinant [15]:

$$f_j^t = \begin{vmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,N} \end{vmatrix}, \tag{2.9}$$

where $\phi_{k,l}$ is the function of t, j written as

$$\phi_{k,l} = 1 + (-1)^{i+l} c_i \omega_i^{2j} q_i^{2t-4(l-1)} \tag{2.10}$$

and the parameters q_i, ω_i $(i = 1, \ldots, N)$ satisfy the dispersion relation

$$r^2 = \frac{q_i^2 - \omega_i^2}{q_i^2 \omega_i^2 - 1}. \tag{2.11}$$

2.1.3 Ultradiscrete KdV equation and Box and Ball System

By assuming $r > 1$, substituting

$$f_j^t = e^{F_j^t/\varepsilon}, \qquad r = e^{R/\varepsilon} \tag{2.12}$$

into f_j^t and r in (2.8) and taking the ultradiscrete limit $\varepsilon \to 0$, we obtain the ultradiscrete KdV equation:

$$F_j^t + F_{j+1}^{t+2} = \max(F_j^{t+2} + F_{j+1}^t - 2R, F_j^{t+1} + F_{j+1}^{t+1}) \qquad (R > 0). \tag{2.13}$$

By means of the dependent variable transformation

$$B_j^t = \frac{1}{2}\left(F_j^{t+1} + F_{j+1}^t - F_{j+1}^{t+1} - F_j^t\right) \tag{2.14}$$

and boundary condition

$$B_j^t = 0 \quad \text{for } |j| \gg 0, \tag{2.15}$$

the ultradiscrete KdV equation (2.13) is transformed into

$$B_j^{t+1} = \min\left(R - B_j^t, \sum_{n=-\infty}^{j-1} (B_n^t - B_n^{t+1})\right), \tag{2.16}$$

which is nothing but the time evolution of the BBS.

2.2 Vertex operator for the ultradiscrete KdV equation

First, we define the vertex operator and consider the functions generated by this operator.

Definition 2.1 *The function with* $2N$ *parameters* $F(Q_1, \ldots, Q_N; C_1, \ldots, C_N)$ *and the vertex operator* X *are defined as follows:*

1. *The function* $F(;)$ *is defined as:*
$$F(;) = 0. \tag{2.17}$$

2. *The vertex operator* X *depends on two parameters* $Q_1 \geq 0$ *and* C_1*, and maps the function* $F(;)$ *to*
$$X(Q_1, C_1)F(;) := \max(0, 2\eta_1). \tag{2.18}$$
where η_1 *is the function of* t, j *written as*
$$\eta_1 = C_1 - j\Omega_1 + tQ_1. \tag{2.19}$$
Accordingly, $F(Q_1; C_1)$ *is defined as:*
$$F(Q_1; C_1) := X(Q_1, C_1)F(;). \tag{2.20}$$

3. *For general* $N \geq 1$*, the vertex operator* X *maps a function* $F(Q_1, \ldots, Q_{N-1}; C_1, \ldots, C_{N-1})$ *(written as* $F(\boldsymbol{Q}'; \boldsymbol{C}')$ *for brevity) to the function written as:*
$$\begin{aligned} X(Q_N, C_N)F(\boldsymbol{Q}'; \boldsymbol{C}') &:= \max\left(F(\boldsymbol{Q}'; \boldsymbol{C}')), 2\eta_N + F(\boldsymbol{Q}'; \boldsymbol{C}' - \boldsymbol{A}'_N)\right) \\ &=: F(Q_1, \ldots, Q_{N-1}, Q_N; C_1, \ldots, C_{N-1}, C_N) \end{aligned} \tag{2.21}$$
where
$$\eta_i = C_i - j\Omega_i + tQ_i \tag{2.22}$$
$$\mathbf{A}'_N = {}^t(A_{1,N}, \ldots, A_{N-1,N}) \tag{2.23}$$
$$A_{i,j} = 2\min(Q_i, Q_j) \tag{2.24}$$
and the parameters Ω_i, Q_i *satisfy the dispersion relation:*
$$\Omega_i = \min(R, Q_i), \tag{2.25}$$
which is the ultradiscretization of (2.11).
We denote $F(Q_1, \ldots, Q_N; C_1, \ldots, C_N) = F(\boldsymbol{Q}; \boldsymbol{C})$ *for brevity again.*

The parameters Q_N and C_N in the vertex operator X are in fact the amplitude and phase parameter of the new soliton, inserted by the operator. The definition (2.21) indicates that all pre-existing solitons, described by $F(\boldsymbol{Q}';\boldsymbol{C}')$, have their phases shifted by inserting a new soliton. The phase shifts correspond to the contribution of the shift operator in (2.7).

Here we present the basic properties of the operator X and the corresponding function F.

Proposition 2.2 *The action of the vertex operators is commutative.*

Proof We calculate $X(Q_b, C_b)X(Q_a, C_a)F(\boldsymbol{Q};\boldsymbol{C})$ directly following the definition (2.21).

$$\begin{aligned} X(Q_b, C_b)X(Q_a, C_a)F(\boldsymbol{Q};\boldsymbol{C}) = \max\big(&F(\boldsymbol{Q};\boldsymbol{C}), 2\eta_b + F(\boldsymbol{Q};\boldsymbol{C}-\boldsymbol{A}_b),\\ &2\eta_a + F(\boldsymbol{Q};\boldsymbol{C}-\boldsymbol{A}_a), 2\eta_a + 2\eta_b - 2A_{b,a} + F(\boldsymbol{Q};\boldsymbol{C}-\boldsymbol{A}_a-\boldsymbol{A}_b)\big). \end{aligned} \tag{2.26}$$

From this relation it is clear that interchanging the subscripts a and b does not change its overall value.

Rewriting this proposition in the language of the function F, yields the following corollary:

Corollary 2.3 *The function $F(\boldsymbol{Q};\boldsymbol{C})$ is invariant under the permutation of their parameters, i.e.:*

$$\begin{aligned} &F(Q_1, \ldots, Q_N; C_1, \ldots, C_N)\\ &\quad = F(Q_{\sigma(1)}, \ldots, Q_{\sigma(N)}; C_{\sigma(1)}, \ldots, C_{\sigma(N)}) \quad (\sigma \in S_N). \end{aligned} \tag{2.27}$$

2.3 Recursive representation of solutions for the ultradiscrete KdV equation

Let us prove that these functions are indeed solutions of the ultradiscrete KdV equation, by means of the recursive form (2.21). This property indicates that the vertex operator defined in (2.21) is nothing but the operator, well known from soliton theory, which maps an N-soliton solution to an $N+1$-soliton solution.

Theorem 2.4 *The function $F(\boldsymbol{Q};\boldsymbol{C})$ solves the ultradiscrete KdV equation*

$$F_j^t + F_{j+1}^{t+2} = \max(F_j^{t+2} + F_{j+1}^t - 2R, F_j^{t+1} + F_{j+1}^{t+1}) \qquad (R > 0). \tag{2.28}$$

Proof By virtue of corollary 2.3, we can fix the labels of the parameters

$$Q_N \geq Q_{N-1} \geq \ldots \geq Q_1 \geq 0 \tag{2.29}$$

without loss of generality. By virtue of this ordering, the phase shifts in definition (2.21) simplify to

$$\min(Q_i, Q_N) = Q_i \quad (i = 1, \ldots, N-1). \tag{2.30}$$

It should be noted that the phase shifts $\boldsymbol{C} \to \boldsymbol{C} \pm \boldsymbol{Q}$ are equivalent to the time shifts $t \to t \pm 1$. Then, $F(\boldsymbol{Q}; \boldsymbol{C})$ is reduced to $F_j^{(N),t}$, written as

$$F_j^{(N),t} = \begin{cases} \max(F_j^{(N-1),t}, 2(C_N - j\Omega_N + tQ_N) + F_j^{(N-1),t-2}) & (N \geq 1) \\ 0 & (N = 0) \end{cases} \tag{2.31}$$

To prove this theorem, let us introduce a fundamental property of the max operator and prepare some lemmas.

Proposition 2.5 *The inequality*

$$\max(x, y) - \max(z, w) \leq \max(x - z, y - w) \tag{2.32}$$

holds for arbitrary $x, y, z, w \in \mathbb{R}$.

Proof The inequality

$$\max(a + c, b + d) \leq \max(a, b) + \max(c, d) \tag{2.33}$$

holds for $a, b, c, d \in \mathbb{R}$ because the right hand side in (2.33) can be expanded to yield $\max(a+c, a+d, b+c, b+d)$, which includes all candidates of the left hand side. Then, we obtain (2.32) by setting $a = z, b = w, c = x-z, d = y-w$ in (2.33) respectively. □

It should be noted the relation (2.33) is an extension of the triangle inequality because it reduces to the triangle inequality by setting $b = -a, d = -c$.

Lemma 2.6 *Let*

$$H_j^{(N),t} = F_j^{(N),t+m+2} - F_{j+l}^{(N),t+2} - F_j^{(N),t+m} + F_{j+l}^{(N),t}, \tag{2.34}$$

where parameters $l \geq 0$ and m satisfy $l + m \geq 0$. Then the relation

$$H_j^{(N),t} \leq 2(l\Omega_N + mQ_N) \tag{2.35}$$

holds.

Proof By employing the inequality (2.32), we obtain

$$
\begin{aligned}
F_j^{(N),t+m+2} - F_{j+l}^{(N),t+2} \leq & \max(F_j^{(N-1),t+m+2} - F_{j+l}^{(N-1),t+2}, \\
& 2(l\Omega_N + mQ_N) + F_j^{(N-1),t+m} - F_{j+l}^{(N-1),t}) \qquad (2.36)\\
F_{j+l}^{(N),t} - F_j^{(N),t+m} \leq & \max(F_{j+l}^{(N-1),t} - F_j^{(N-1),t+m}, \\
& -2(l\Omega_N + mQ_N) + F_{j+l}^{(N-1),t-2} - F_j^{(N-1),t+m-2}). \qquad (2.37)
\end{aligned}
$$

Adding the inequalities yields

$$
\begin{aligned}
H_j^{(N),t} \leq \max(& H_j^{(N-1),t}, \\
& 2(l\Omega_N + mQ_N), \\
& -2(l\Omega_N + mQ_N) + H_j^{(N-1),t} + H_j^{(N-1);t-2}, \\
& H_j^{(N-1),t-2}).
\end{aligned} \qquad (2.38)
$$

By taking into account the relation $Q_N \geq Q_{N-1}$, it can be shown inductively that the four arguments in this maximum are all less than $2(l\Omega_N + mQ_N)$. □

Lemma 2.7 *Let*

$$
H_j^{\prime(N),t} = F_j^{(N),t} + F_{j+1}^{(N),t+2} - F_j^{(N),t+1} - F_{j+1}^{(N),t+1}, \qquad (2.39)
$$

one then has:

$$
0 \leq H_j^{\prime(N),t} \leq 2(Q_N - \Omega_N) \qquad (2.40)
$$

when $F_j^{(i),t}$ solves the ultradiscrete KdV equation (2.28) for $i = 0, \ldots, N$.

Proof If $F_j^{(N),t}$ solves the equation (2.28), by virtue of the property of maximum operator, we obtain:

$$
\begin{aligned}
F_j^{(N),t} + F_{j+1}^{(N),t+2} &= \max(F_j^{(N),t+2} + F_{j+1}^{(N),t} - 2R, F_j^{(N),t+1} + F_{j+1}^{(N),t+1}) \\
&\geq F_j^{(N),t+1} + F_{j+1}^{(N),t+1},
\end{aligned} \qquad (2.41)
$$

which is nothing but the positivity of $H_j^{\prime(N),t}$.

By employing the inequality (2.32), we obtain

$$
\begin{aligned}
F_j^{(N),t-1} - F_{j+1}^{(N),t} \leq & \max(F_j^{(N-1),t-1} - F_{j+1}^{(N-1),t}, \\
& 2(Q_N - \Omega_N) + F_j^{(N-1),t-3} - F_{j+1}^{(N-1),t-2}) \qquad (2.42)\\
F_{j+1}^{(N),t+1} - F_j^{(N),t} \leq & \max(F_{j+1}^{(N-1),t+1} - F_j^{(N-1),t}, \\
& -2(Q_N - \Omega_N) + F_{j+1}^{(N-1),t-1} - F_j^{(N-1),t-2}) \qquad (2.43)
\end{aligned}
$$

Adding these inequalities, we obtain

$$
\begin{aligned}
H_j^{\prime(N),t} \leq \max(&H_j^{\prime(N-1),t}, \\
&2(Q_N - \Omega_N) - H_j^{\prime(N-1),t-1}, \\
&-2(Q_N - \Omega_N) + H_j^{\prime(N-1),t-2} + H_{j+1}^{(N-1),t}\Big|_{l=-1,m=1}, \\
&H_j^{\prime(N-1),t-2}),
\end{aligned} \tag{2.44}
$$

where $H_j^{(N),t}$ is given by Lemma 2.6. It can be shown inductively that all arguments are less than $2(Q_N - \Omega_N)$. □

In particular, $H_j^{\prime(N),t} = 0$ when $\Omega_N = Q_N$ by (2.40) because of the form of $F_j^{(N),t}$ and the conditions (2.29).

Let us return to the proof of the Theorem 2.4. We shall prove the theorem inductively. It is clear that $F_j^{(0),t} = 0$ solves the equation (2.28) because of the positivity of R. Now, let us assume that the theorem holds for $N-1$. By substituting (2.31) in the ultradiscrete KdV equation (2.28), the left hand side can be rewritten as

$$
\begin{aligned}
F_j^{(N),t} + F_{j+1}^{(N),t+2} = \max\Big(&F_j^{(N-1),t} + F_{j+1}^{(N-1),t+2}, \\
&2(2C_N - (2j+1)\Omega_N + (2t+2)Q_N) + F_j^{(N-1),t-2} + F_{j+1}^{(N-1),t}, \\
&2(C_N - j\Omega_N + tQ_N) + F_j^{(N-1),t-2} + F_{j+1}^{(N-1),t+2}, \\
&2(C_N - (j+1)\Omega_N + (t+2)Q_N) + F_j^{(N-1),t} + F_{j+1}^{(N-1),t}\Big)
\end{aligned} \tag{2.45}
$$

In this expression it looks as if the maximum in (2.45) has four arguments. However, the third argument cannot be the maximum because it is always less than the fourth one by virtue of Lemma 2.6, in case $l = -1, m = 2$.

By means of the same procedure and Lemma 2.6, in case $l = 1, m = 2$ and $l = 1, m = 0$, the right hand side of equation (2.28) is rewritten as

$$
\begin{aligned}
\max\Big(&\max\big(F_j^{(N-1),t+2} + F_{j+1}^{(N-1),t} - 2R, F_j^{(N-1),t+1} + F_{j+1}^{(N-1),t+1}\big), \\
&2(2C_N - (2j+1)\Omega_N + (2t+2)Q_N) \\
&\quad + \max\big(F_j^{(N-1),t} + F_{j+1}^{(N-1),t-2} - 2R, F_j^{(N-1),t-1} + F_{j+1}^{(N-1),t-1}\big), \\
&\max\big(2(C_N - (j+1)\Omega_N + (t+2)Q_N) + F_j^{(N-1),t} + F_{j+1}^{(N-1),t} - 2R, \\
&\quad 2(C_N - j\Omega_N + (t+1)Q_N) + F_j^{(N-1),t-1} + F_{j+1}^{(N-1),t+1}\big)\Big)
\end{aligned}
$$

There are three arguments in the principal maximum. The first and second arguments are the same as those on the left hand side because $F_j^{(N-1),t}$ solves equation (2.28) by assumption. It can also be shown that the last argument is also the same by employing the method which was used to prove Lemma 2.6.

The condition which expresses the equality of the last argument to that of the left hand side then reduces to:

$$0 = \max(2(\Omega_N - R), 2(\Omega_N - Q_N) - H_j^{\prime(N-1),t-1}) \tag{2.46}$$

In the case $\Omega_N = Q_N \leq R$, Ω_{N-1} has to be equal to Q_{N-1}, due to condition (2.29). Then, the first argument in (2.46) is non-positive and the second argument is 0, due to Lemma 2.7.

In the case $\Omega_N = R$, the first argument in the maximum in (2.46) is 0 and the second argument is non-positive by virtue of Lemma 2.7. Thus, (2.46) is equal to 0, in both possible cases (i.e., $\Omega_N = Q_N$ or $\Omega_N = R$).

We have therefore shown that all arguments of the maximum in (2.45) which constitutes the left hand side of (2.28), have an equivalent counterpart among the three arguments that contribute to the right hand side of (2.28). Hence, (2.28) is satisfied. □

By taking the ordering (2.29), the recursive representation (2.31) is equivalent to the ultradiscretization of the cofactor expansion by the N-th column of the Casorati determinant solution (2.9), because the terms with negative signs in the determinant do not contribute to the ultradiscrete limit. The permanent type solutions presented in [7] are equivalent to this ultradiscrete limit for the same reason. However, we would like to stress that these two approaches are quite different.

2.4 An example of the action of the vertex operator for the Box and Ball System

Finally, we give an explicit example of the action of the vertex operator X, in the case of the Box and Ball System.

It can be shown that the possible values of the variable B_j^t that correspond to the N-soliton solutions of theorem 2.4 (i.e. constructed by means of the vertex operator X) are restricted to the set $\{0, 1\}$.

The lattices depicted in Figure 2.1 are the vacuum lattice, and the lattices that are obtained from vacuum by successive application of the vertex operator $X_i = X_i(\Omega_i, C_i)$ with parameter values $(\Omega_1, C_1) = (1, 2)$, $(\Omega_2, C_2) = (2, 3)$, and $(\Omega_3, C_3) = (4, 2)$. By repeated application of the vertex operators, it is observed that the number of the blocks of '1"s (solitons) is increasing. Figure 2.2 represents lattices that are obtained from vacuum by successive application of the same vertex operators X_i $(i = 1, 2, 3)$ in the order X_3, X_2, X_1. By Proposition 2.2, the final state is the same but the middle states are different.

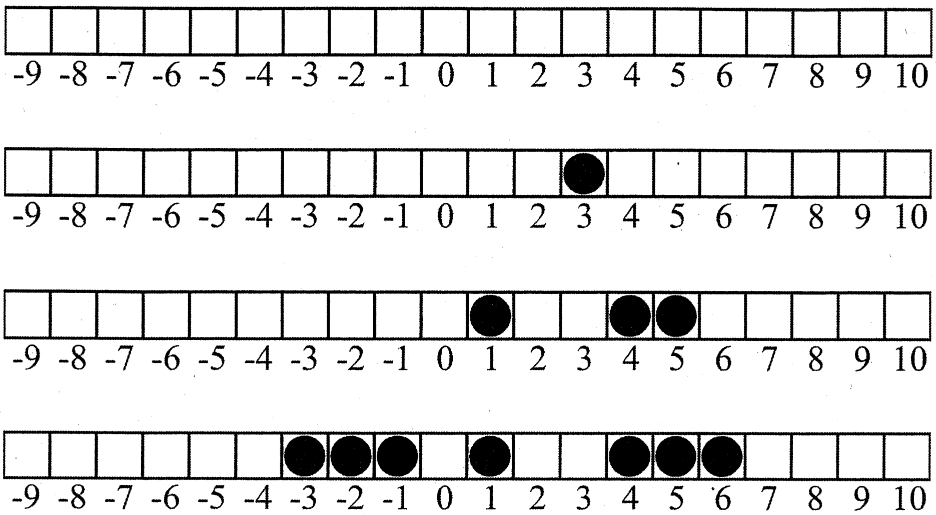

Figure 2.1: BBS obtained by successive application of the vertex operator in order of X_1, X_2, X_3.

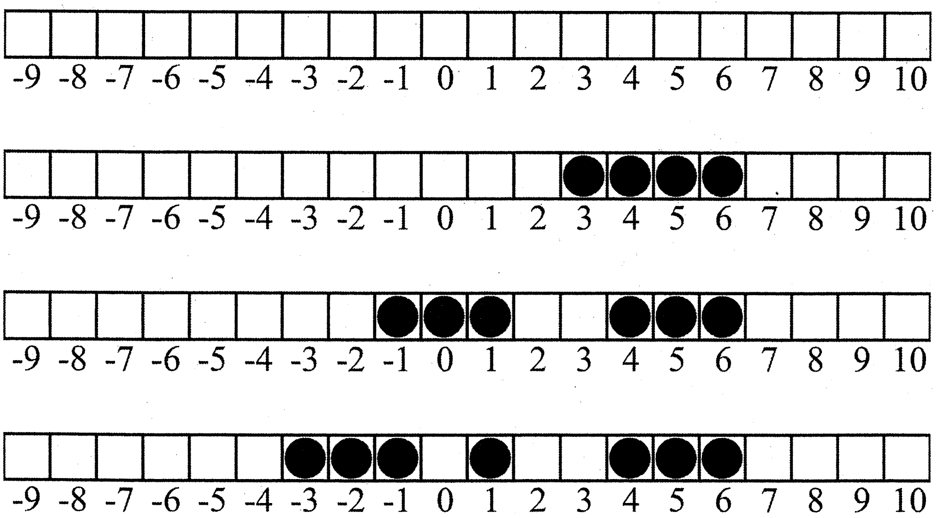

Figure 2.2: BBS obtained by successive application of the vertex operator in order of X_3, X_2, X_1.

Chapter 3

Vertex operator for the non-autonomous ultradiscrete KP equation

3.1 Introduction

In this chapter, we propose a vertex operator for the non-autonomous ultradiscrete KP equation and various ultradiscrete soliton equations obtained from it by reduction. In section 2 we first propose a recursive representation of the soliton solutions of the non-autonomous ultradiscrete KP equation. In section 3, we propose the vertex operator as an operator representation of the recursive one. In section 4, we present various reductions of this equation and discuss their vertex operators and solutions.

3.1.1 KP hierarchy and its vertex operators

The bilinear identity of the KP hierarchy is given by [13]

$$\operatorname*{Res}_{\lambda=\infty}\left[\tau\left(\mathbf{t}-\boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right)\tau\left(\mathbf{t}'+\boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right)e^{\xi(\mathbf{t}-\mathbf{t}',\lambda)}\right]=0 \tag{3.1}$$

where $\mathbf{t}=(t_1,t_2,t_3,\ldots)$ are an infinite number of independent variables and

$$\boldsymbol{\epsilon}\left(u\right)=\left(u,\frac{1}{2}u^2,\frac{1}{3}u^3,\ldots\right) \tag{3.2}$$

$$\xi(\mathbf{t},\lambda)=\sum_{k=1}^{\infty}t_k\lambda^k. \tag{3.3}$$

By denoting $\mathbf{t} = \mathbf{x}+\mathbf{y}$, $\mathbf{t}' = \mathbf{x}-\mathbf{y}$ and expanding for y_i, a series of partial differential equations (the KP hierarchy) is obtained. It includes the bilinear form of the KP equation:

$$\left(D_1^4 + 3D_2^2 - 4D_1D_3\right)\tau\cdot\tau = 0, \tag{3.4}$$

where $D_i^\alpha D_j^\beta \tau\cdot\tau$ is defined as

$$D_i^\alpha D_j^\beta \tau\cdot\tau := \frac{\partial^\alpha}{\partial z_i^\alpha}\frac{\partial^\beta}{\partial z_j^\beta}\tau(\mathbf{x}+\mathbf{z})\tau(\mathbf{x}-\mathbf{z})\Bigg|_{\mathbf{z}=0} \tag{3.5}$$

for $\mathbf{z} = (z_1, z_2, \ldots)$.

The N-soliton solution of the KP hierarchy (3.1) is expressed as

$$\tau = \sum_{S\subset[N]} \exp\Big(\sum_{i\in S}\eta_i + \sum_{\substack{i,j\in S\\ i<j}} A_{i,j}\Big) \tag{3.6}$$

for η_i and $A_{i,j}$ given by

$$\eta_i = \sum_{k=1}^{\infty}(p_i^k - q_i^k)x_k \tag{3.7}$$

$$\exp A_{i,j} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(p_j - q_i)} \tag{3.8}$$

for parameters p_i, q_i. Here, $\sum_{S\subset U}(\ldots)$ stands for the summation of the argument for all subsets of U and $[N] = \{1, 2, \ldots, N\}$.

The vertex operator of the KP hierarchy is given in the form of an infinitesimal transformation:

$$X(p,q) = \exp\left(\sum_{k=1}^{\infty}(p^k - q^k)x_k\right)\exp\left(-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{p^k} - \frac{1}{q^k}\right)\frac{\partial}{\partial x_k}\right) \tag{3.9}$$

The exponent of this operator maps $N-1$-soliton solutions and N-soliton one and all soliton solutions are generated by repeated application, i.e. (3.6) is equal to

$$\tau = e^{c_N X(p_N,q_N)}\cdots e^{c_1 X(p_1,q_1)}\cdot 1. \tag{3.10}$$

3.1.2 Non-autonomous discrete/ultradiscrete KP equation

By denoting

$$\mathbf{t} = \sum_{0}^{l} \epsilon(a_d) + \sum_{0}^{m} \epsilon(b_d) + \sum_{0}^{n} \epsilon(c_d) \tag{3.11}$$

$$\mathbf{t}' = \sum_{0}^{l-1} \epsilon(a_d) + \sum_{0}^{m-1} \epsilon(b_d) + \sum_{0}^{n-1} \epsilon(c_d) \tag{3.12}$$

and $\tau(t) = \tau_{l,m,n}$, the bilinear identity (3.1) is transformed into the non-autonomous discrete KP equation [4]:

$$\begin{aligned} a_l(b_m - c_n)\tau_{l+1,m,n}\tau_{l,m+1,n+1} + b_m(c_n - a_l)\tau_{l,m+1,n}\tau_{l+1,m,n+1} \\ +c_n(a_l - b_m)\tau_{l,m,n+1}\tau_{l+1,m+1,n} = 0 \end{aligned} \tag{3.13}$$

Here, $\sum_i^j \epsilon(a_d)$ stands for

$$\sum_{i}^{j} \epsilon(a_d) = \begin{cases} \sum_{d=i+1}^{j} \epsilon(a_d) & (i < j) \\ 0 & (i = j) \\ -\sum_{d=j+1}^{i} \epsilon(a_d) & (i > j) \end{cases} \tag{3.14}$$

and similarly for $\sum_i^j \epsilon(b_d)$ and $\sum_i^j \epsilon(c_d)$.

The discrete KP equation (or Hirota-Miwa equation), which is a discretized version of the KP equation, is also regarded as a fundamental discrete soliton equation. By restricting its solutions, it reduces to many well-known discrete soliton equations, as for example the discrete KdV equation or the discrete Toda equation.

Taking $a_l = a(\text{const.})$, $b_m = b(\text{const.})$ and

$$r_n^2 = -\frac{b(c_n - a)}{a(b - c_n)}, \tag{3.15}$$

(3.13) is rewritten as

$$r_n^2\tau_{l,m+1,n}\tau_{l+1,m,n+1} = \tau_{l+1,m,n}\tau_{l,m+1,n+1} + (r_n^2 - 1)\tau_{l,m,n+1}\tau_{l+1,m+1,n}. \tag{3.16}$$

By assuming that $r_n > 1$, substituting

$$\tau_{l,m,n} = e^{T_{l,m,n}/\varepsilon}, \qquad r_n = e^{R_n/\varepsilon} \tag{3.17}$$

and taking the ultradiscrete limit $\varepsilon \to 0$ of (3.16), we obtain the non-autonomous ultradiscrete KP equation in bilinear form [4]

$$T_{l,m+1,n} + T_{l+1,m,n+1} = \\ \max\left(T_{l+1,m,n} + T_{l,m+1,n+1} - 2R_n, T_{l,m,n+1} + T_{l+1,m+1,n} \right) \quad (R_n > 0). \tag{3.18}$$

3.2 Recursive representation solution for the non-autonomous ultradiscrete KP equation

In this section, we propose a class of solutions for the non-autonomous ultradiscrete KP equation, with parameters $P_1, \ldots, P_N$, $Q_1, \ldots, Q_N$ and $C_1, \ldots, C_N$.

Theorem 3.1 *The function $T^{(N)}_{l,m,n}$ expressed as*

$$T^{(N)}_{l,m,n} = \begin{cases} \max\left(T^{(N-1)}_{l,m,n}, 2\eta_N + T^{(N-1)}_{l-1,m+1,n}\right) & (N \geq 1) \\ 0 & (N = 0) \end{cases} \tag{3.19}$$

solves equation (3.18) for η_N given by

$$\eta_N = C_N + lP_N - mQ_N - \sum_0^n \Omega_{N,d}. \tag{3.20}$$

Here, $\sum_i^j \Omega_{N,d}$ stands for

$$\sum_i^j \Omega_{N,d} = \begin{cases} \sum_{d=i+1}^{j} \Omega_{N,d} & (i < j) \\ 0 & (i = j) \\ -\sum_{d=j+1}^{i} \Omega_{N,d} & (i > j) \end{cases}, \tag{3.21}$$

and the parameters P_i, Q_i and $\Omega_{i,n} (i = 1, \ldots, N)$ satisfy the relations:

$$P_N \geq P_{N-1} \geq \ldots \geq P_1 \geq 0 \tag{3.22}$$
$$Q_N \geq Q_{N-1} \geq \ldots \geq Q_1 \geq 0 \tag{3.23}$$
$$\Omega_{i,n} = \min(Q_i, R_{n-1}). \tag{3.24}$$

Before starting the proof, we shall prepare some lemmas.

Lemma 3.2 *Let*

$$H^{(N),i,j,k}_{l,m,n} = T^{(N)}_{l,m+j+1,n+k} + T^{(N)}_{l+i+1,m,n} - T^{(N)}_{l+1,m+j,n+k} - T^{(N)}_{l+i,m+1,n} \tag{3.25}$$

for i, j, k such that

$$iP_N + jQ_N + \sum_{n}^{n+k} \Omega_{N,d} \geq \ldots \geq iP_1 + jQ_1 + \sum_{n}^{n+k} \Omega_{1,d} \geq 0. \tag{3.26}$$

Then it holds that

$$H^{(N),i,j,k}_{l,m,n} \leq 2(iP_N + jQ_N + \sum_{n}^{n+k} \Omega_{N,d}). \tag{3.27}$$

for $N \geq 1$.

Proof Since i, j, k do not change in this proof, we denote $H^{(N),i,j,k}_{l,m,n} = H^{(N)}_{l,m,n}$ for brevity.

By employing the inequality (proven in proposition 2.5)

$$\max(a, b) - \max(c, d) \leq \max(a - c, b - d), \tag{3.28}$$

we obtain

$$\begin{aligned} T^{(N)}_{l,m+j+1,n+k} - T^{(N)}_{l+i,m+1,n} \leq \max\big(&T^{(N-1)}_{l,m+j+1,n+k} - T^{(N-1)}_{l+i,m+1,n}, \\ &- 2(iP_N - jQ_N - \sum_{n}^{n+k} \Omega_{N,d}) + T^{(N-1)}_{l-1,m+j+2,n+k} - T^{(N-1)}_{l+i-1,m+2,n}\big) \end{aligned} \tag{3.29}$$

$$\begin{aligned} T^{(N)}_{l+i+1,m,n} - T^{(N)}_{l+1,m+j,n+k} \leq \max\big(&T^{(N-1)}_{l+i+1,m,n} - T^{(N-1)}_{l+1,m+j,n+k}, \\ &2(iP_N + jQ_N + \sum_{n}^{n+k} \Omega_{N,d}) + T^{(N-1)}_{l+i,m+1,n} - T^{(N-1)}_{l,m+j+1,n+k}\big) \end{aligned} \tag{3.30}$$

Adding the inequalities yields

$$\begin{aligned} H^{(N)}_{l,m,n} \leq \max\big(&H^{(N-1)}_{l,m,n},\ 2(iP_N + jQ_N + \sum_{n}^{n+k} \Omega_{N,d}), \\ &- 2(iP_N + jQ_N + \sum_{n}^{n+k} \Omega_{N,d}) + H^{(N-1)}_{l,m,n} + H^{(N-1)}_{l-1,m+1,n},\ H^{(N-1)}_{l-1,m+1,n}\big). \end{aligned} \tag{3.31}$$

Taking into account the relations (3.26) and

$$H^{(0),i,j,k}_{l,m,n} = T^{(0)}_{l,m+j+1,n+k} + T^{(0)}_{l+i+1,m,n} - T^{(0)}_{l+1,m+j,n+k} - T^{(0)}_{l+i,m+1,n} = 0, \quad (3.32)$$

it can be shown inductively that the four arguments in this maximum are all less than $2(iP_N + jQ_N + \sum_n^{n+k} \Omega_{N,d})$. □

Lemma 3.3 *Let*

$$H'^{(N)}_{l,m,n} = T^{(N)}_{l,m,n+1} + T^{(N)}_{l,m+2,n} - T^{(N)}_{l,m+1,n} - T^{(N)}_{l,m+1,n+1}. \quad (3.33)$$

One then has

$$H'^{(N)}_{l,m,n} \leq 2(Q_N - \Omega_{N,n+1}) \quad (3.34)$$

for $N \geq 1$ when one requires that the $T^{(i)}_{l,m,n}(i = 1, \ldots, N)$ are solutions of (3.18). Especially when $\Omega_{N,n} = Q_N$, the inequality (3.34) becomes an equality, i.e: $H'^{(N)}_{l,m,n} = 0$.

Proof When $\Omega_{N,n} = R_{n+1}$, we obtain by virtue of the inequality (3.28):

$$\begin{aligned} H'^{(N)}_{l,m,n} \leq \max \big(& H'^{(N-1)}_{l,m,n}, H'^{(N-1)}_{l-1,m+1,n}, \\ & 2(Q_N - \Omega_{N,n+1}) + T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l,m+2,n} - T^{(N-1)}_{l-1,m+2,n} - T^{(N-1)}_{l,m+1,n+1}, \\ & - 2(Q_N - \Omega_{N,n+1}) + H^{(N-1),0,1,-1}_{l-1,m,n+1} + H'^{(N-1)}_{l-1,m+1,n}\big). \end{aligned} \quad (3.35)$$

However, $T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l,m+2,n} - T^{(N-1)}_{l-1,m+2,n} - T^{(N-1)}_{l,m+1,n+1} \leq 0$ because $T^{(N-1)}_{l,m,n}$ satisfies (3.18), and

$$H'^{(0)}_{l,m,n} = T^{(0)}_{l,m,n+1} + T^{(0)}_{l,m+2,n} - T^{(0)}_{l,m+1,n} - T^{(0)}_{l,m+1,n+1} = 0. \quad (3.36)$$

It can then be shown (again inductively) that all arguments in the maximum are less than $2(Q_N - \Omega_{N,n+1})$, by virtue of the relation (3.23) and (3.24).

On the other hand, when $\Omega_{N,n} = Q_N$, by virtue of (3.23), $T^{(N-1)}_{l,m+1,n}$ is equal to $T^{(N-1)}_{l,m,n+1}$ for all l, m because

$$C_i + lP_i - (m+1)Q_N - \sum_0^n \Omega_{N,d} = C_N + lP_N - mQ_N - \sum_0^{n+1} \Omega_{N,d} \quad (3.37)$$

for all $i = 1, \ldots, N$. We thus obtain that

$$H'^{(N)}_{l,m,n} = (T^{(N)}_{l,m,n+1} - T^{(N)}_{l,m+1,n}) + (T^{(N)}_{l,m+2,n} - T^{(N)}_{l,m+1,n+1}) = 0. \quad (3.38)$$

□

Lemma 3.4 *Let*

$$H''^{(N)}_{l,m,n} = T^{(N)}_{l,m,n+1} + T^{(N)}_{l+2,m,n} - T^{(N)}_{l+1,m,n} - T^{(N)}_{l+1,m,n+1}. \tag{3.39}$$

One then has

$$H''^{(N)}_{l,m,n} \leq 2P_N \tag{3.40}$$

for $N \geq 1$ when all of $T^{(i)}_{l,m,n} (i = 1, \ldots, N)$ are solutions of (3.18).

Proof By virtue of the inequality (3.28), we obtain

$$\begin{aligned} H''^{(N)}_{l,m,n} \leq \max \big(& H''^{(N-1)}_{l,m,n}, H''^{(N-1)}_{l-1,m+1,n} \\ & 2P_N + T^{(N-1)}_{l,m,n+1} + T^{(N-1)}_{l+1,m+1,n} - T^{(N-1)}_{l+1,m,n+1} - T^{(N-1)}_{l,m+1,n}, \\ & -2P_N + H^{(N-1),1,0,0}_{l-1,m,n+1} + H''^{(N-1)}_{l+1,m,n} \big) \end{aligned} \tag{3.41}$$

Now, $T^{(N-1)}_{l+1,m,n+1} + T^{(N-1)}_{l,m+1,n} - T^{(N-1)}_{l,m,n+1} - T^{(N-1)}_{l+1,m+1,n} \leq 0$ because $T^{(N-1)}_{l,m,n}$ satisfies (3.18) and

$$H''^{(0)}_{l,m,n} = T^{(0)}_{l,m,n+1} + T^{(0)}_{l+2,m,n} - T^{(0)}_{l+1,m,n} - T^{(0)}_{l+1,m,n+1} = 0 \tag{3.42}$$

Then it can again be shown inductively that all arguments are less than $2P_N$, by virtue of the relation (3.22). □

It should be noted that Lemmas 3.2 and 3.3 correspond respectively to Lemmas 2.6 and 2.7 in the chapter 2 and that Lemma 3.4 is a new necessary condition. In some special cases of parameters, for example $P_i = Q_i$, we need not to use this lemma to prove this theorem by virtue of Lemma 3.2.

We now have all the necessary lemmas at our disposal and proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1 We shall prove the theorem inductively. It is clear that $T^{(0)}_{l,m,n}$ solves equation (3.18) because of the non-negativity of R_n. Now, let us assume that the theorem holds at $1, \ldots, N-1$. By substituting (3.19) in equation (3.18), each contribution can be written as

$$\begin{aligned} T^{(N)}_{l,m+1,n} + T^{(N)}_{l+1,m,n+1} = \max \big(& T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l+1,m,n+1}, \\ & 2(P_N - \Omega_{N,n+1}) + 2\eta_N + T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l,m+1,n+1}, \\ & -2Q_N + 2\eta_N + T^{(N-1)}_{l-1,m+2,n} + T^{(N-1)}_{l+1,m,n+1}, \\ & 4\eta_N + 2(P_N - Q_N - \Omega_{N,n+1}) + T^{(N-1)}_{l-1,m+2,n} + T^{(N-1)}_{l,m+1,n+1} \big), \end{aligned} \tag{3.43}$$

for the left hand side of (3.18), and

$$
\begin{aligned}
T^{(N)}_{l+1,m,n} + T^{(N)}_{l,m+1,n+1} = \max\big(&T^{(N-1)}_{l,m,n} + T^{(N-1)}_{l,m+1,n+1},\\
&2P_N + 2\eta_N + T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l-1,m+1,n+1},\\
&-2(Q_N + \Omega_{N,n+1}) + T^{(N-1)}_{l+1,m,n} + T^{(N-1)}_{l,m+2,n+1},\\
&4\eta_N + 2(P_N - Q_N - \Omega_{N,n+1}) + T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l-1,m+2,n+1}\big)
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
T^{(N)}_{l,m,n+1} + T^{(N)}_{l+1,m+1,n} = \max\big(&T^{(N-1)}_{l,m,n+1} + T^{(N-1)}_{l+1,m,n},\\
&2(P_N - Q_N) + 2\eta_N + T^{(N-1)}_{l,m,n+1} + T^{(N-1)}_{l,m+2,n},\\
&-2\Omega_{N,n+1} + T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l+1,m+1,n},\\
&4\eta_N + 2(P_N - Q_N - \Omega_{N,n+1}) + T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l,m+2,n}\big)
\end{aligned}
\tag{3.45}
$$

for the right hand side. In these expressions it looks as if each of the maximum operations in (3.43)–(3.45) has four arguments. However, by virtue of Lemma 3.2, the third argument in (3.43) and (3.44) cannot yield the maximum because it is always less than the second argument.

Then, the relevant arguments of the maximum in (3.43) are in fact

$$T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l+1,m,n+1} \tag{3.46}$$

$$2\eta_N + 2(P_N - \Omega_{N,n+1}) + T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l,m+1,n+1} \tag{3.47}$$

$$4\eta_N + 2(P_N - Q_N - \Omega_{N,n+1}) + T^{(N-1)}_{l-1,m+2,n} + T^{(N-1)}_{l,m+1,n+1} \tag{3.48}$$

and those in the maximum of the contributions in (3.44), (3.45), as they appear in the right hand side of equation (3.18):

$$\max(T^{(N-1)}_{l+1,m,n} + T^{(N-1)}_{l,m+1,n+1} - 2R_n, T^{(N-1)}_{l,m,n+1} + T^{(N-1)}_{l+1,m,n}) \tag{3.49}$$

$$
\begin{aligned}
&2\eta_N + \max(2P_N - 2R_n + T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l,m+1,n+1},\\
&\quad 2(P_N - Q_N) + T^{(N-1)}_{l,m,n+1} + T^{(N-1)}_{l,m+2,n}, -2\Omega_{N,n+1} + T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l+1,m+1,n})
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
&4\eta_N + 2(P_N - Q_N - \Omega_{N,n+1})\\
&\quad + \max(T^{(N-1)}_{l,m+1,n} + T^{(N-1)}_{l-1,m+2,n+1} - 2R_n, T^{(N-1)}_{l-1,m+1,n+1} + T^{(N-1)}_{l,m+2,n}).
\end{aligned}
\tag{3.51}
$$

Here, (3.46) and (3.48) are identical to (3.49) and (3.51) because by assumption, $T^{(N-1)}_{l,m,n}$ solves the equation (3.18).

By subtracting (3.47) from (3.50), we obtain

$$\max\big(2(\Omega_{N,n+1} - R_n), 2(\Omega_{N,n+1} - Q_N) + H'^{(N-1)}_{l,m,n}, -2P_N + H''^{(N-1)}_{l,m,n}\big). \tag{3.52}$$

The third argument of this maximum is non-positive by virtue of Lemma 3.4.

In the case $\Omega_{N,n+1} = Q_N \le R_n$, Ω_{N-1} has to be equal to Q_{N-1}, due to condition (3.23). Then, the first argument in (3.52) is non-positive and the second argument is 0, due to Lemma 3.3.

In the case $\Omega_{N,n+1} = R_n$, the first argument in the maximum in (3.52) is 0 and the second argument is non-positive by virtue of Lemma 3.3. Thus, (3.52) is equal to 0, in both possible cases (i.e., $\Omega_{N,n+1} = Q_N$ or $\Omega_{N,n+1} = R_n$).

We have therefore shown that all arguments of the maximum in (3.43) which constitutes the left hand side of (3.18), have an equivalent counterpart among (3.49), (3.50), (3.51), i.e. among the three arguments that contribute to the right hand side of (3.18). Hence, (3.18) is satisfied. $\square$

Please note that the proof allows for the possibility that, at different values of n, $\Omega_{N,n+1}$ satisfies different equalities ($\Omega_{N,n+1} = R_n$ or $\Omega_{N,n+1} = Q_N$ for different n), because the shift of the independent variables induced by (3.19) affects only l and m, not n.

3.3 Vertex operator for the non-autonomous ultradiscrete KP equation

In this section we propose an alternative representation of the soliton solutions, generated by a vertex operator X and we prove that these solutions are equivalent to the recursive solutions we proposed in the previous section.

By [4], all of the N-soliton solutions of the non-autonomous ultradiscrete KP equation (3.18) are written as

$$T_{l,m,n} = \max_{S\subset[N]} \Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j} \Big) \tag{3.53}$$

for η_i and $A_{i,j}$ given by

$$\eta_i = C_i + lP_i - mQ_i - \sum_{0}^{n} \min(Q_i, R_{d-1}) \tag{3.54}$$

$$A_{i,j} = \min(P_i, P_j) + \min(Q_i, Q_j). \tag{3.55}$$

Here, $\max_{S\subset U}(\ldots)$ stands for the maximum of the argument for all subsets of U and the parameters P_i, Q_i satisfy

$$(P_i - P_j)(Q_i - Q_j) \ge 0. \tag{3.56}$$

3.4.1 The Box and Ball System and its varieties

By restricting $T_{l,m,n}$ to

$$T_{l,m,n} = F_n^{l-Mm} \tag{3.69}$$

and denoting $s = l - Mm$ and $n = j$, the non-autonomous ultradiscrete KP equation (3.18) is reduced to the so-called non-autonomous ultradiscrete hungry KdV equation:

$$F_{j+1}^{s+M+1} + F_j^s = \max(F_j^{s+M+1} + F_{j+1}^s - 2R_j, F_j^{s+1} + F_{j+1}^{s+M}). \tag{3.70}$$

By means of the dependent variable transformation

$$B_{i,j}^t = \frac{1}{2}(F_j^{s+1} + F_{j+1}^s - F_{j+1}^{s+1} - F_j^s), \tag{3.71}$$

and denoting $s = Mt + i$, (3.70) is transformed into

$$B_{i,j}^{t+1} = \min\left(R_j - \sum_{k=1}^{i-1} B_{k,j}^{t+1} - \sum_{k=i}^{M} B_{k,j}^t, \sum_{n=-\infty}^{j-1} (B_{i,n}^t - B_{i,n}^{t+1})\right), \tag{3.72}$$

which describes the dynamics of a Box and Ball System with M kinds of balls as presented in [4]. This system is required to satisfy the following boundary conditions:

$$B_{i,j}^t = 0 \quad \text{for} \quad j \ll 0 \tag{3.73}$$

In particular, in the case of $M = 1$, it reduces to an extension of the standard BBS [1], with variable size of boxes at each site.

In our representation (3.62), the reduction (3.69) is equivalent to the parameter restriction:

$$MP_N = Q_N. \tag{3.74}$$

It should be noted that our representation satisfies the boundary condition (3.73) because the first argument of max in (3.62) is never chosen for sufficiently small j.

Then, the vertex operator for (3.70) can be written as

$$\begin{aligned} X(P_{N+1}, C_{N+1})T(\boldsymbol{P}; \boldsymbol{C}) :=& \\ &\max(T(\boldsymbol{P}; \boldsymbol{C}), 2\eta_{N+1} + T(\boldsymbol{P}; \boldsymbol{C} - \boldsymbol{A}_{N+1})), \end{aligned} \tag{3.75}$$

where the phase factor η_{N+1} is

$$\eta_{N+1} = C_{N+1} + sP_{N+1} - \sum_0^j \Omega_{N+1,d}, \tag{3.76}$$

For $N = 1$, we define $T(\boldsymbol{P}'; \boldsymbol{Q}'; \boldsymbol{C}') = T(;;) = 0$, which is the vacuum solution of this equation.

Here, we define the operator $X(P_N, Q_N, C_N)$ as

$$X(P_N, Q_N, C_N)T(\boldsymbol{P}'; \boldsymbol{Q}'; \boldsymbol{C}') := \max(T(\boldsymbol{P}'; \boldsymbol{Q}'; \boldsymbol{C}'), 2\eta_N + T(\boldsymbol{P}'; \boldsymbol{Q}'; \boldsymbol{C}' - \boldsymbol{A}'_N)) \tag{3.64}$$

i.e.

$$X(P_N, Q_N, C_N)T(\boldsymbol{P}'; \boldsymbol{Q}'; \boldsymbol{C}') = T(\boldsymbol{P}; \boldsymbol{Q}; \boldsymbol{C}). \tag{3.65}$$

The operator $X(P_N, Q_N, C_N)$ maps an $N-1$-soliton solution to an N-soliton one, which is nothing but the vertex operator. By the form of the N-soliton solutions, they are expressed as the repeated application of vertex operators to $T(;;)$ i.e.

$$T(\boldsymbol{P}; \boldsymbol{Q}; \boldsymbol{C}) = X(P_N, Q_N, C_N) \cdots X(P_1, Q_1, C_1)T(;;). \tag{3.66}$$

By virtue of definition (3.53), the N-soliton solution $T(\boldsymbol{P}; \boldsymbol{Q}; \boldsymbol{C})$ is invariant under the permutation of its parameters, i.e.:

$$\begin{aligned} &T(P_1, \ldots, P_N; Q_1, \ldots, Q_N; C_1, \ldots, C_N) \\ &= T(P_{\sigma(1)}, \ldots, P_{\sigma(N)}; Q_{\sigma(1)}, \ldots, Q_{\sigma(N)}; C_{\sigma(1)}, \ldots, C_{\sigma(N)}) \quad (\sigma \in S_N), \end{aligned} \tag{3.67}$$

which is equivalent to the commutativity of the action of the vertex operator.

Due to this property, we can fix the labels of the parameters (3.22) (3.23) without loss of generality. Hence, given a specific ordering, the phase shifts in $A_{i,j}$ in the definition (3.62) simplify to

$$\min(P_i, P_N) = P_i, \ \min(Q_i, Q_N) = Q_i \quad (i = 1, \ldots, N-1). \tag{3.68}$$

Then, the phase shifts $\boldsymbol{C} \to \boldsymbol{C} + \boldsymbol{P}$ and $\boldsymbol{C} \to \boldsymbol{C} + \boldsymbol{Q}$ are equivalent to shifts on the independent variables $l \to l + 1$ and $m \to m - 1$, which shows that $T(\boldsymbol{P}; \boldsymbol{Q}; \boldsymbol{C})$ is equivalent to $T^{(N)}_{l,m,n}$.

3.4 Reduction to various ultradiscrete soliton equations

In this section we present some examples of reductions of the ultradiscrete KP equation to $1 + 1$ dimensional ultradiscrete equations and we give the vertex operators for these equations.

In this discussion, we denote

$$T_{l,m,n} = T(P_1, \dots, P_N; Q_1, \dots, Q_N; C_1, \dots, C_N) \tag{3.57}$$

(written as $T(\boldsymbol{P};\boldsymbol{Q};\boldsymbol{C})$ for brevity), because the parameters P_i, Q_i, C_i are more important than the independent variables l, m, n. By separating conditions in the max of (3.53), where S includes N or not, we rewrite (3.53) as

$$T(\boldsymbol{P};\boldsymbol{Q};\boldsymbol{C}) = \max\left(\max_{\substack{S\subset[N]\\ N\notin S}}\Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j}\Big), \max_{\substack{S\subset[N]\\ N\in S}}\Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j}\Big)\right). \tag{3.58}$$

The former argument of the maximum is rewritten as

$$\begin{aligned} \max_{S\subset[N], N\notin S}\Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j}\Big) &= \max_{S\subset[N-1]}\Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j}\Big) \\ &= T(\boldsymbol{P}';\boldsymbol{Q}';\boldsymbol{C}'). \end{aligned} \tag{3.59}$$

Here, we denote

$$T(P_1, \dots, P_{N-1}; Q_1, \dots, Q_{N-1}; C_1, \dots, C_{N-1}) = T(\boldsymbol{P}';\boldsymbol{Q}';\boldsymbol{C}') \tag{3.60}$$

for brevity. By virtue of the identity $\{S | S \subset [N], N \in S\} = \{\{N\} \cup S' | S' \subset [N-1]\}$, the latter argument is

$$\begin{aligned} &\max_{S\subset[N], N\in S}\Big(\sum_{i\in S} 2\eta_i - \sum_{\substack{i,j\in S\\ i<j}} 2A_{i,j}\Big) \\ &= \max_{S'\subset[N-1]}\Big(\sum_{i\in S'} 2\eta_i + 2\eta_N - \sum_{\substack{i,j\in S'\\ i<j}} 2A_{i,j} - \sum_{i\in S'} 2A_{i,N}\Big) \\ &= 2\eta_N + \max_{S'\subset[N-1]}\Big(\sum_{i\in S'} 2(\eta_i - A_{i,N}) - \sum_{\substack{i,j\in S'\\ i<j}} 2A_{i,j}\Big) \\ &= 2\eta_N + T(\boldsymbol{P}';\boldsymbol{Q}';\boldsymbol{C}' - \boldsymbol{A}'_N). \end{aligned} \tag{3.61}$$

Then, we obtain

$$T(\boldsymbol{P};\boldsymbol{Q};\boldsymbol{C}) = \max(T(\boldsymbol{P}';\boldsymbol{Q}';\boldsymbol{C}'), 2\eta_N + T(\boldsymbol{P}';\boldsymbol{Q}';\boldsymbol{C}' - \boldsymbol{A}'_N)), \tag{3.62}$$

where $\boldsymbol{A}'_N$ is

$$\boldsymbol{A}'_N = {}^t(A_{1,N}, \dots, A_{N-1,N}). \tag{3.63}$$

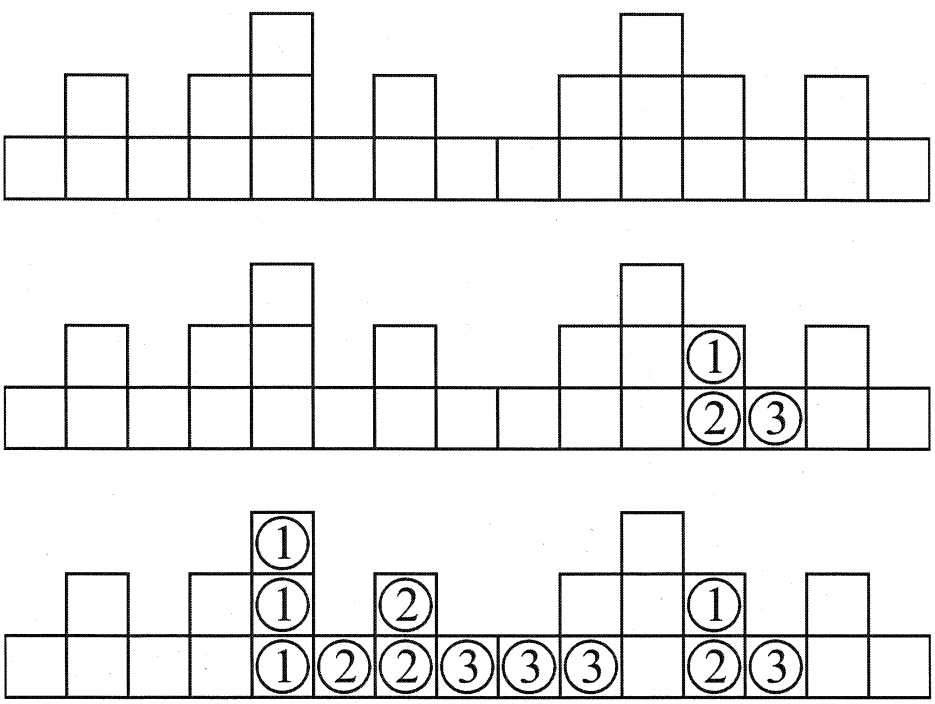

Figure 3.1: An example of BBS with 3-kinds of balls and variable size of boxes

and $\Omega_{N,j}$ and the interaction terms $A_{i,j}$ are expressed as

$$\Omega_{N,j} = \min(R_{j-1}, MP_N), \quad A_{i,j} = (M+1)\min(P_i, P_j). \tag{3.77}$$

To end this subsection, let us show an example of a soliton solution of the ultradiscrete hungry KdV equation for $M = 3$. The lattices depicted in Figure 3.1 are the lattices that obtained from vacuum by successive application of the vertex operator X with parameter values $(P_1, C_1) = (1, 8)$ and $(P_2, C_2) = (3, 1)$.

3.4.2 The ultradiscrete Toda equation

By restricting $T_{l,m,n}$ to

$$T_{l,m,n} = F_{m+n}^{l+n} \tag{3.78}$$

and denoting $t = l + n$, (3.18) is reduced to the ultradiscrete Toda equation:

$$F_{s+1}^{t} + F_{s+1}^{t+2} = \max(F_{s+2}^{t+1} + F_{s}^{t+1} - 2R, 2F_{s+1}^{t+1}) \tag{3.79}$$

By means of the dependent variable transformation

$$U_s^t = \frac{1}{2}(F_{s+2}^t - 2F_{s+1}^t + F_s^t), \tag{3.80}$$

(3.79) is transformed into

$$U_{s+1}^{t+2}-2U_{s+1}^{t+1}+U_{s+1}^{t} = \max(U_{s+2}^{t+1}-R,0)-2\max(U_{s+1}^{t+1}-R,0)+\max(U_s^{t+1}-R,0), \tag{3.81}$$

which describes the dynamics of the Toda type cellular automaton presented in [16].

In our representation (3.62), the reduction (3.78) is equivalent to the parameter restriction:

$$\Omega_N = Q_N - P_N \qquad \text{i.e.} \qquad P_N = Q_N - \Omega_N = \max(Q_N - R, 0) \tag{3.82}$$

The vertex operator of (3.79) can be expressed as

$$\begin{aligned} X(P_{N+1}, C_{N+1})T(\boldsymbol{Q};\boldsymbol{C}) \\ := \max(T(\boldsymbol{Q};\boldsymbol{C}), 2\eta_{N+1} + T(\boldsymbol{Q};\boldsymbol{C} - \boldsymbol{A}_{N+1})), \end{aligned} \tag{3.83}$$

where the phase factor η_{N+1} is

$$\eta_{N+1} = C_{N+1} + t\max(Q_{N+1} - R, 0) - sQ_{N+1}, \tag{3.84}$$

and the interaction term $A_{i,j}$ is written as

$$A_{i,j} = \min(Q_i, Q_j) + \max(\min(Q_i, Q_j) - R, 0) \tag{3.85}$$

We end by showing an example of the soliton solution for the ultradiscrete Toda equation for $R = 1$. Figure 3.3 shows the time evolution of the corresponding Toda-type cellular automaton for $Q_1 = 2, C_1 = 0, Q_2 = 5, C_2 = 5$.

$\xrightarrow{s}$

```
.....2...............
.....11..............
......2..............
......11.............
.......2.............
```

$t\downarrow$

Figure 3.2: time evolution of Toda-type CA (1)

$\xrightarrow{s}$

```
..32.2...............
...4111..............
....5.2..............
....1411.............
.....232.............
......331............
......124............
.......223...........
.......1132..........
........2.41.........
........11.5.........
.........2.14........
.........11.23.......
```

$t\downarrow$

Figure 3.3: time evolution of Toda-type CA (2)

Chapter 4

Background solutions of ultradiscrete soliton equation

4.1 Introduction

In this chapter, we propose a wide class of solutions of the ultradiscrete KP equation, and prove that we can inject solitons into these solutions by means of the vertex operator. At the end of the chapter, we introduce some examples of such solutions for the reductions of this equation, which corresponds to soliton cellular automata.

4.2 Background solutions for the ultradiscrete KP equation

By assuming that $R_n = R$ (const.), the non-autonomous ultradiscrete KP equation (3.18) is reduced to the ultradiscrete KP equation:

$$\begin{aligned} &T_{l,m+1,n} + T_{l+1,m,n+1} = \\ &\qquad \max\left(T_{l+1,m,n} + T_{l,m+1,n+1} - 2R, T_{l,m,n+1} + T_{l+1,m+1,n} \right) \quad (R > 0) \end{aligned} \tag{4.1}$$

Now, let us consider the solution of this equation such that

$$T_{l,m,n} = T^0_{l,m+n}. \tag{4.2}$$

By subtracting the left hand side of (4.1) from the right hand side, we obtain the proposition:

Proposition 4.1 *$T^0_{l,m+n}$ solves (4.1) iff $T^0_{l,s}$ satisfies*

$$T^0_{l+1,s} + T^0_{l,s+2} - T^0_{l,s+1} - T^0_{l+1,s+1} \leq 2R. \tag{4.3}$$

This type of the solutions contains the soliton solutions such that $\Omega_N = Q_N$ because they are depending only l and $m+n$ by the form of these solutions. However, it contains, for example, $T_{l,s} = l^2 + s^2$, which is obviously out of the soliton solutions.

Theorem 4.2 *If there exist $0 \leq \exists K \leq R$ and $0 \leq \exists L$ such that $T^0_{l,x}$ satisfies*

$$H^0_{l,s} = T^0_{l+1,s} + T^0_{l,s+2} - T^0_{l,s+1} - T^0_{l+1,s+1} \leq 2K \tag{4.4}$$
$$H'^0_{l,s} = T^0_{l,s+1} + T^0_{l+2,s} - T^0_{l+1,s} - T^0_{l+1,s+1} \leq 2L, \tag{4.5}$$

the function $T^{(N)}_{l,m,n}$ expressed as

$$T^{(N)}_{l,m,n} = \begin{cases} \max\left(T^{(N-1)}_{l,m,n}, 2\eta_N + T^{(N-1)}_{l-1,m+1,n}\right) & (N \geq 1) \\ T^0_{l,m+n} & (N = 0) \end{cases} \tag{4.6}$$

solves equation (4.1) for η_N given by

$$\eta_N = C_N + lP_N - mQ_N - n\Omega_N. \tag{4.7}$$

Here, the parameters P_i, Q_i and $\Omega_{i,n}(i = 1, \ldots, N)$ satisfy the relations:

$$P_N \geq P_{N-1} \geq \ldots \geq P_1 \geq L \tag{4.8}$$
$$Q_N \geq Q_{N-1} \geq \ldots \geq Q_1 \geq K \tag{4.9}$$
$$\Omega_i = \min(Q_i, R). \tag{4.10}$$

Proof We have proven Theorem 3.1 inductively by starting from the following conditions:

1. $T^{(0)}_{l,m,n}$ is a solution of (3.18).

2. $H^{(0)}_{l,m,n}$ given in Lemma 3.2 satisfies $H^{(0)}_{l,m,n} \leq 2(iP_1 + jQ_1 + k\Omega_1)$ for $(i,j,k) = (1,1,-1)$, $(1,1,1)$, $(0,1,-1)$ and $(1,0,0)$.

3. $H'^{(0)}_{l,m,n}$ given in Lemma 3.3 satisfies $H'^{(0)}_{l,m,n} \leq 2(Q_1 - \Omega_1)$.

4. $H''^{(0)}_{l,m,n}$ given in Lemma 3.4 satisfies $H''^{(0)}_{l,m,n} \leq 2P_1$.

Thus, it suffices to prove that $T^{(0)}_{l,m,n} = T^0_{l,m+n}$ satisfies these conditions. It is clear that $T^{(0)}_{l,m,n}$ solves (4.1) by the condition (4.4) and Proposition 4.1. By the form of $T^{(0)}_{l,m,n}$, conditions 2–4 reduce to (4.4) and (4.5) by virtue of the the relations:

$$H^{(0),1,1,-1}_{l,m,n} = H^{(0),1,0,0}_{l,m,n} = H''^{(0)}_{l,m,n} = H'^0_{l,m+n} \tag{4.11}$$

$$H^{(0),0,1,-1}_{l,m,n} = H'^{(0)}_{l,m,n} = 0 \tag{4.12}$$

$$H^{(0),1,1,1}_{l,m,n} = H^0_{l+1,m+n} + H^0_{l,m+n+1} + H'^0_{l,m+n+1} \tag{4.13}$$

By employing the relations (4.8) and (4.9), we can prove this theorem again inductively. □

It is known that this class of the solutions is observed in the time evolution of the soliton cellular automata. We give explicit formulae for these solutions in the following section.

4.3 Integer-valued solution for the Box and Ball system

4.3.1 Background solution for the ultradiscrete KdV equation

By restricting $T_{l,m,n}$ in the ultradiscrete KP equation to

$$T_{l,m,n} = F^{l-m}_n \tag{4.14}$$

and denoting $t = l - m$ and $n = j$, the ultradiscrete KP equation (4.1) is reduced to the ultradiscrete KdV equation

$$F^{t+2}_{j+1} + F^t_j = \max(F^{t+2}_j + F^t_{j+1} - 2R, F^{t+1}_j + F^{t+1}_{j+1}), \tag{4.15}$$

which is the same as (2.28).

Since the reduction (4.14) is equivalent to the parameter restriction

$$P_N = Q_N, \tag{4.16}$$

the solution containing both N-solitons and background is written as

$$F^{(N),t}_j = \begin{cases} \max(F^{(N-1),t}_j, 2(C_N - j\Omega_N + tQ_N) + F^{(N-1),t-2}_j) & (N \geq 1) \\ F_0(t-j) & (N = 0). \end{cases} \tag{4.17}$$

Here, the parameters P_i, Q_i and $\Omega_{i,n}(i = 1, \ldots, N)$ satisfy the relations:

$$Q_N \geq Q_{N-1} \geq \ldots \geq Q_1 \geq K \tag{4.18}$$
$$\Omega_i = \min(Q_i, R). \tag{4.19}$$

$F_0(x)$ corresponds to the reduction of $T^0_{l,m+n}$ as

$$T^0_{l,m+n} = F_0(l-m-n) = F_0(t-j) \tag{4.20}$$

and satisfies the condition

$$F_0(x+2) - F_0(x+1) - F_0(x) + F_0(x-1) \leq 2K \qquad (\forall x \in \mathbb{Z}), \tag{4.21}$$

which corresponds to the reduction of (4.4) and (4.5) (We rewrite $\min(L, K)$ as K).

It should be noted that we can prove directly that $F_j^{(N),t}$ given by (4.17) solves (4.15) inductively by starting the conditions:

1. $F_j^{(0),t}$ is a solution of (2.28).

2. $H_j^{(0),t}$ given in Lemma 2.6 satisfies $H_j^{(0),t} \leq 2(l\Omega_1 + mQ_1)$.

3. $H_j'^{(0),t}$ given in Lemma 2.7 satisfies $0 \leq H_j'^{(0),t} \leq 2(Q_1 - \Omega_1)$.

for $F_j^{(0),t} = F_0(t-j)$ satisfying the condition (4.21)

4.3.2 Travelling wave solution for the Box and Ball System (BBS)

By means of the dependent variable transformation

$$B_j^t = \frac{1}{2}\left(F_j^{t+1} + F_{j+1}^t - F_{j+1}^{t+1} - F_j^t\right), \tag{4.22}$$

the ultradiscrete KdV equation (4.15) is transformed into the BBS with a capacity of R balls.

The background solutions which satisfy boundary conditions $B_j^t = 0$ for $|j| \gg 0$ are the "travelling wave solutions" which are presented in [12]. Travelling wave solutions are explicitly expressed as

$$F_0(x) = \sum_{x_0 \in \mathbb{Z}} B_0(x_0) G(x + x_0), \tag{4.23}$$

$$
\begin{array}{ll}
 & \xrightarrow{j} \\
 & \texttt{.2..................} \\
 & \texttt{.}\underline{\texttt{1}}\texttt{111...............} \\
t\big\downarrow & \texttt{..}\underline{\texttt{1}}\texttt{..111............} \\
 & \texttt{...}\underline{\texttt{1}}\texttt{....111.........} \\
 & \texttt{....}\underline{\texttt{1}}\texttt{......111......}
\end{array}
$$

Figure 4.1: time evolution of 2 on a cite (underscore represents a minus sign at this site and dot means 0)

where $G(x)$ stands for

$$G(x) = \max(0, 2x), \tag{4.24}$$

and $B_0(x)$ satisfies the conditions:

$$B_0(x) + B_0(x+1) \le K \qquad (\forall x \in \mathbb{Z}) \tag{4.25}$$

$$B_0(x) = 0 \qquad (\text{for } |x| \gg 1). \tag{4.26}$$

Here, it should be noted that the summation in (4.23) is indeed finite by the boundary conditions (4.26).

By means of the property

$$G(x + x_0 + 1) + G(x + x_0 - 1) - 2G(x + x_0) = 2\delta_{x,x_0} \quad (\text{for } x \in \mathbb{Z}), \tag{4.27}$$

the relation (4.22) for travelling waves is equivalent to

$$F_0(x+1) + F_0(x-1) - 2F_0(x) = 2B_0(x). \tag{4.28}$$

By virtue of the relation, the condition (4.25) is nothing but the condition (4.21).

4.3.3 Finding solutions from arbitrary initial states

In this subsection, let us give some initial states and try to find their explicit solutions.

Figure 4.1 depicts the time evolution of the initial state such that $B_0^0 = 2$ and $B_j^0 = 0$ for $j \neq 0$. By the time evolution, we can observe that there are a block of '1' with the length 3 (soliton) and a travelling wave '-1'. Indeed, the function F_j^t which gives the state of this figure is expressed as

$$F_j^t = \max(-G(t-j), 2(-3 - j + 3t) - G(t - j - 2)). \tag{4.29}$$

```
  j →
   .434122.............
   .34323111...........
t ↓ ..34322..111111.....
   ...343211......11111
   ....3432.11.........
   .....3432..11.......
```

Figure 4.2: time evolution of an initial state

Figure 4.2 also depicts the time evolution of an initial state. We observe two solitons with length 6 and 2 and a travelling wave '-3,4,-3,-2'. Then, the explicit solution is expressed as (4.17) for $N = 2, Q_1 = 2, C_1 = 0, Q_2 = 6, C_2 = -8$ and

$$F_0(x) = -3G(x-1) + 4G(x) - 3G(x+1) - 2G(x+2). \tag{4.30}$$

Observing the time evolution of the system, one finds the building blocks (travelling waves and solitons) into which the initial data split. If one can thus obtain all soliton data, it is possible to reconstruct an analytic expression for the solution of the system that corresponds to this initial state.

4.3.4 Split of travelling waves

Unlike the ordinary state of the BBS, we can observe some strange phenomena by appending the solitons to travelling waves. For example in Figure 4.3, we append a soliton with length $Q_1 = 3$ to a travelling wave '-5'. By increasing C_1, which indicates the location of the soliton, we can observe that travelling wave '-5' is transformed into another travelling wave.

Furthermore, we can append the soliton with length '0' to the travelling waves. Figure 4.4 depicts appending soliton $Q_1 = 0$ to travelling waves. As C_1 increasing, travelling waves are split by the soliton located by C_1 but no soliton appears there. It should be noted that appending the soliton with length '0' do nothing for the ordinary state of the BBS.

$$
\begin{array}{rl}
 & \xrightarrow{j} \\
 & \texttt{...111.\underline{5}.......} \\
 & \texttt{....111\underline{5}.......} \\
 & \texttt{.....12\underline{5}.......} \\
 & \texttt{......3\underline{5}.......} \\
 & \texttt{.....\underline{1}4\underline{5}.......} \\
 & \texttt{.....\underline{2}5\underline{5}.......} \\
 & \texttt{.....\underline{3}6\underline{5}.......} \\
C_1\big\downarrow & \texttt{.....\underline{4}6\underline{4}.......} \\
 & \texttt{.....\underline{5}6\underline{3}.......} \\
 & \texttt{.....\underline{5}5\underline{2}.......} \\
 & \texttt{.....\underline{5}4\underline{1}.......} \\
 & \texttt{.....\underline{5}3........} \\
 & \texttt{.....\underline{5}21.......} \\
 & \texttt{.....\underline{5}111......} \\
 & \texttt{.....\underline{5}.111.....}
\end{array}
$$

Figure 4.3: Appending soliton with length 3 and variable location C_1 to background solution

$$
\begin{array}{rl}
 & \xrightarrow{j} \\
 & \texttt{....\underline{3}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{1}1\underline{3}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{2}2\underline{3}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{3}3\underline{3}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{3}2\underline{2}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{3}1\underline{1}..\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{3}....\underline{1}..\underline{3}..} \\
C_1\big\downarrow & \texttt{..\underline{3}..\underline{1}1\underline{1}..\underline{3}..} \\
 & \texttt{..\underline{3}..\underline{1}....\underline{3}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{1}1\underline{3}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{2}2\underline{3}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{3}3\underline{3}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{3}2\underline{2}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{3}1\underline{1}..} \\
 & \texttt{..\underline{3}..\underline{1}..\underline{3}....}
\end{array}
$$

Figure 4.4: Appending 0-length soliton to background solution

4.4 Integer valued solution for Toda-type cellular automaton

4.4.1 Background solution for the ultradiscrete Toda equation

By restricting $T_{l,m,n}$ to

$$T_{l,m,n} = F_{m+n}^{l+n} \tag{4.31}$$

and denoting $t = l + n$ and $s = m + n$, (4.1) is reduced to the ultradiscrete Toda equation:

$$F_{s+1}^{t} + F_{s+1}^{t+2} = \max(F_{s+2}^{t+1} + F_{s}^{t+1} - 2R, 2F_{s+1}^{t+1}) \tag{4.32}$$

In our representation (4.6), the reduction (4.31) is equivalent to the parameter restriction:

$$\Omega_N = Q_N - P_N \qquad \text{i.e.} \qquad P_N = Q_N - \Omega_N = \max(Q_N - R, 0) \tag{4.33}$$

and the solution appending an N-soliton to a background solution is written as

$$F_s^{(N),t} = \begin{cases} \max(F_j^{(N-1),t}, 2(C_N - sQ_N + tP_N) + F_{s+1}^{(N-1),t-1}) & (N \geq 1) \\ F_0(s) & (N = 0). \end{cases} \tag{4.34}$$

Here, the parameters P_i, Q_i and $\Omega_{i,n} (i = 1, \ldots, N)$ satisfy the relation:

$$Q_N \geq Q_{N-1} \geq \ldots \geq Q_1 \geq M \tag{4.35}$$

$F_0(x)$ corresponds to the reduction of $T_{l,m+n}^0$ as

$$T_{l,m+n}^0 = F_0(m+n) = F_0(s) \tag{4.36}$$

and satisfies the condition

$$F_0(s+1) - F_0(s-1) - 2F_0(s) \leq 2M, \tag{4.37}$$

which corresponds to the restriction of (4.4). The condition (4.5) is automatically satisfied by the form of the solutions.

```
 s
--->
 .1...............
 ..1..............
t| .12..............
 .111.............
 .1.2.............
```

Figure 4.5: Appending solitons to static solution of Toda-type CA (1)

4.4.2 Toda-type cellular automaton and its static solutions

By means of the dependent variable transformation

$$U_s^t = \frac{1}{2}(F_{s+2}^t - 2F_{s+1}^t + F_s^t), \tag{4.38}$$

(4.32) is transformed into

$$U_{s+1}^{t+2} - 2U_{s+1}^{t+1} + U_{s+1}^t = \max(U_{s+2}^{t+1} - R, 0) - 2\max(U_{s+1}^{t+1} - R, 0) + \max(U_s^{t+1} - R, 0), \tag{4.39}$$

which describes the dynamics of the Toda type cellular automaton presented in [16].

These background solutions are so-called "static solutions" (also presented in [12]), which do not depend on the (time) independent variable t. By setting boundary conditions: $U_j^t = 0$ for $|j| \gg 1$, static solutions are explicitly expressed as

$$F_0(s) = \sum_{x_0 \in \mathbb{Z}} U_0(x_0) G(x - x_0), \tag{4.40}$$

where $G(x)$ is defined in (4.24) and $U_0(x)$ satisfies

$$U_0(x) \leq R. \tag{4.41}$$

By means of the relation (4.27), the condition (4.41) is equivalent to (4.37).

To finish this section, we give an explicit example of a blended soliton and non-soliton solution for the Toda-type cellular automaton. Since the time evolution of Toda-type cellular automaton (4.39) is a second order difference equation, all states U_j^t are determined by the initial values U_j^0 and U_j^1. Figure

$$
\begin{array}{cl}
 & \xrightarrow{s} \\
 & \mathtt{.1\underline{1}21.............} \\
 & \mathtt{.1\underline{1}.21............} \\
 & \mathtt{.1\underline{11}13............} \\
t\downarrow & \mathtt{.1\underline{12}212...........} \\
 & \mathtt{.1\underline{12}21............} \\
 & \mathtt{.1\underline{12}1121..........} \\
 & \mathtt{.1\underline{12}.2.3..........}
\end{array}
$$

Figure 4.6: Appending solitons to static solution of Toda-type CA (2)

4.5 depicts the time evolution of the initial state such that $U_j^0 = \delta_{0,j}$ and $U_j^1 = \delta_{1,j}$. The function F_j^t which gives this state is expressed as

$$F_j^t = \max(-G(s), 2(-1-2s+t) - G(s+1)). \tag{4.42}$$

The function which gives the state of Figure 4.6 is expressed as (4.34) for $N = 2, Q_1 = 2, C_1 = 1, Q_2 = 3, C_2 = -1$ and

$$F_0(s) = G(s+1) - G(s) - 2G(s-1). \tag{4.43}$$

Chapter 5

Concluding Remarks

In this thesis, we have proposed vertex operators for the ultradiscrete KdV equation and the non-autonomous ultradiscrete KP equation. We have proved that we can apply these operators to not only soliton solutions but also to a wide class of solutions called backgrounds.

In chapter 2, we proposed an ultradiscrete analogue of the vertex operator for the ultradiscrete KdV equation and discussed its properties. We presented a recursive representation of the N-soliton solution generated by this operator and proved that it indeed solves the ultradiscrete KdV equation.

In chapter 3, we proposed a recursive representation of the N-soliton solutions and vertex operators for the ultradiscrete KP equation. We also proposed expressions for various ultradiscrete equations, obtained by reduction from the KP equation.

In chapter 4, we proposed a wider class of solutions for the ultradiscrete KP equation, which can be a base when appending solitons–backgrounds. We also proposed the solutions to cellular automata, equivalent to the reduction of this class and we can append solitons to these solutions by means of the vertex operator.

The discussion in section 2 of chapter 2 and section 3 of chapter 3 does not depend on the explicit forms of η_i and $A_{i,j}$. Thus, the vertex operator for other ultradiscrete soliton equations can be obtained as long as the N-soliton solutions are written in the form (3.53). Conversely, we can obtain a recursive representation which consists of only max and $\pm$ operators from the vertex operators.

The solitons in the case of $0 < \Omega_N = Q_N \leq R$ can be considered as background solutions, by the form of the solutions. However, we can append these solitons to backgrounds by the vertex operator. It is an interesting problem to describe the border between solitons and backgrounds. We believe that the difficulty of the initial value problem for the BBS with multi-capacity boxes is caused by this problem.

The relationship between ultradiscrete systems and geometry is an interesting and important problem. For example, the conditions for background solutions (4.4) and (4.5) correspond to the curvature of $T_{l,s}$ for two different orientations.

In fact, the vertex operator approach is closely related to the existence of certain symmetry algebras for integrable systems and the exact relation of our ultradiscrete operator to the symmetries of ultradiscrete systems is an especially interesting problem we want to address in the future.

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