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Three-Term Asymptotics of the Spectrum of Self-Similar Fractal Drums

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Abstract. In the present paper we consider the number $\mathcal{N}_\Omega(\lambda)$ of eigenvalues not exceeding λ of the negative Laplacian with homogeneous DIRICHLET boundary conditions in a domain $\Omega \subset \mathbb{R}^n$ with fractal boundary $\partial\Omega$. It is known that for $\lambda \to \infty$, $\mathcal{N}_\Omega(\lambda) = \mathcal{C}_n|\Omega|_n\lambda^{n/2} + O(\lambda^{D/2})$, where D is the MINKOWSKI dimension of $\partial\Omega$. For a certain class of domains with self–similar boundary, so-called "fractal drums", we obtain a second term of the form $-\mathcal{F}(\ln\lambda)\,\lambda^{D/2}$ with a bounded periodic function $\mathcal{F}$ and a third term. We investigate the function $\mathcal{F}$ which contains a generalized WEIERSTRASS function with a self–similar fractal graph. Exact estimates for the MINKOWSKI dimension for this graph will be presented.

1. Introduction

Can one hear the shape of a drum? asked M. KAC [Ka] in 1966, thereby comprising a whole mathematical research program into a suggestive headline. What he meant was the following inverse problem: Consider the spectrum of the Laplacian with DIRICHLET boundary conditions on a domain $\Omega \subset \mathbb{R}^n$. Which geometrical information concerning Ω could be recovered from only knowing this spectrum? For the cases of interest, the spectrum of the negative Laplacian is discrete and consists of an infinite sequence of positive eigenvalues, each with finite multiplicity, written in increasing order according to their multiplicity:

$$(1.1) \qquad 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots, \quad \text{with } \lambda_i \to \infty \text{ as } i \to \infty.$$

All information about the spectrum can be obtained from the counting function:

For $\lambda \geq 0$ let $\mathcal{N}_\Omega : \lambda \mapsto \mathcal{N}_\Omega(\lambda)$ be the counting function, that is the number of positive eigenvalues counted with multiplicity not exceeding λ:

$$(1.2) \qquad \mathcal{N}_\Omega(\lambda) := \#\{i \in \mathbb{N} : \lambda_i \leq \lambda\}.$$

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H. Weyl's classical asymptotic formula, obtained in this generality by G. Métivier in [Me], states that

$$(1.3)\qquad \mathcal{N}_\Omega(\lambda) \sim \Phi_\Omega(\lambda) := (2\pi)^{-n}\mathcal{B}_n|\Omega|_n\lambda^{n/2}, \quad \text{as } \lambda \to \infty,$$

where $\mathcal{B}_n = \pi^{n/2}(n/2)!$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $|A|_n$ is the n-dimensional Lebesgue measure or "volume" of $A \subset \mathbb{R}^n$. According to this formula, one can hear the "area" of a drum. By the way, this is equivalent to the semi–classical approximation to the energy spectrum of a quantum particle confined to Ω according to the rule: $\mathcal{N}_\Omega(\lambda)$ is roughly the volume of the classically available part of the phase–space over the volume of a Planck cell h^n. From this it is plausible that the contribution of those cells has to be subtracted which contribute by a fractional portion to the Weyl term but not to $\mathcal{N}_\Omega$. This part is proportional to $|\partial\Omega|_{n-1}$ and $\lambda^{(n-1)/2}$, the latter also following from dimensional analysis. For domains Ω with a smooth boundary (for details see below) we thus arrive at the following asymptotic formula:

$$(1.4)\qquad \mathcal{N}_\Omega(\lambda) = \Phi_\Omega(\lambda) - \mathcal{C}_{n-1}\lambda^{(n-1)/2} + O(\lambda^\kappa), \quad \text{as } \lambda \to \infty$$

with a suitable constant $\kappa \in [0,(n-1)/2]$ and $\mathcal{C}_{n-1} = \frac{1}{4}[\mathcal{B}_{n-1}/(2\pi)^{n-1}] \cdot |\partial\Omega|_{n-1}$.

In this case one can also hear the "circumference" of a drum. However, if Ω has a fractal boundary $\Gamma = \partial\Omega$, the second term must be modified since then $|\Gamma|_{n-1} = \infty$. But, following M.V. Berry [Be], one may argue as follows: A vibrational mode (i.e. an eigenfunction of $-\Delta$) with energy λ and wavelength $\epsilon = 2\pi/\sqrt{\lambda}$ cannot resolve details of the boundary of smaller scale than ϵ, hence it "sees" a boundary of volume $|\Gamma|_{n-1}(\varepsilon) \approx \mathcal{H}(H;\Gamma)\epsilon^{n-1-H}$, where H is the (fractal) Hausdorff dimension of Γ and $\mathcal{H}(H;\Gamma)$ its H–dimensional Hausdorff measure. Inserting this into the above second term, Berry arrived at his conjecture:

$$(1.5)\quad N_\Omega(\lambda) = \Phi_\Omega(\lambda) - \mathcal{C}_{n,H}\mathcal{H}(H;\Gamma)\lambda^{H/2} + o(\lambda^{H/2}), \quad \text{as } \lambda \to \infty,$$

where $\mathcal{C}_{n,H}$ is a positive constant depending only on n and H.

Later work clarified that this conjecture, however appealing it is, had to be modified at least in two respects. First, J. Brossard and R. Carmona showed in [BrCa] by means of a counter–example that the Hausdorff dimension in (1.5) must be replaced by the Minkowski dimension

D. Actually, M.L. LAPIDUS, in a more general context, proved the following asymptotics [La1]:

$$\mathcal{N}_\Omega(\lambda) = \Phi_\Omega(\lambda) + O(\lambda^{D/2}), \quad \text{as } \lambda \to \infty. \tag{1.6}$$

Further, J. FLECKINGER and D.G. VASSILIEV [FlVa1] gave an example, where the factor of $\lambda^{D/2}$ is not a constant but a complicated function of λ.

The investigation of this function is the main objective of this paper. To this end we consider a class of "self–similar" fractal drums which are constructed from a smooth basic domain ω by adding more and more scaled down copies of ω. Thus we obtain a domain Ω which is the disjoint union of ω and N copies of $r\Omega$, $r \in (0,1)$. For example, the SIERPIŃSKI gasket will be obtained for $N = 3, r = 1/2$ and ω being an equilateral triangle, see figure 1.

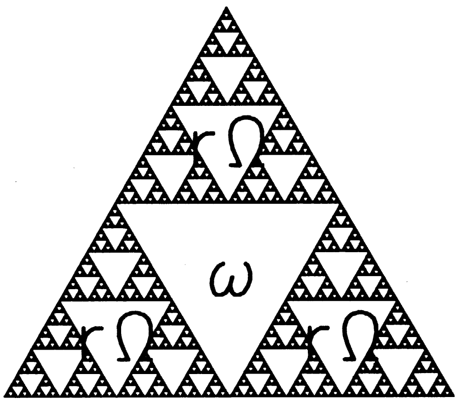

Fig. 1. This figure illustrates the kind of self–similar fractal drums considered in this paper by means of the SIERPIŃSKI gasket.

Of course, this will be a proper subclass of the class of all fractal drums, and it is not clear which of our results will be typical for the larger class. But it has the advantage that $\mathcal{N}_\Omega$ can be explicitly calculated in terms of $\mathcal{N}_\omega$.

Generally, the counting function has the following scaling and summation properties:

(i) Let be $r \in \mathbb{R}^+$. Then $\mathcal{N}_{r\Omega}(\lambda) = \mathcal{N}_\Omega(r^2\lambda),\ \lambda \geq 0$.

(ii) Let Ω, Ω_1, $\Omega_2 \subset \mathbb{R}^n$ be open bounded sets with $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Then [Me, p. 133]:

$$\mathcal{N}_\Omega(\lambda) = \mathcal{N}_{\Omega_1}(\lambda) + \mathcal{N}_{\Omega_2}(\lambda), \quad \lambda \geq 0. \tag{1.7}$$

Hence $\mathcal{N}_\Omega$ must satisfy the following functional equation, if Ω is a self–similar drum:

$$\mathcal{N}_\Omega(\lambda) = N\mathcal{N}_{r\Omega}(\lambda) + \mathcal{N}_\omega(\lambda) = N\mathcal{N}_\Omega(r^2\lambda) + \mathcal{N}_\omega(\lambda). \tag{1.8}$$

We assume that both counting functions, $\mathcal{N}_\Omega$ and $\mathcal{N}_\omega$ possess asymptotic expansions in λ,

$$\mathcal{N}_\omega(\lambda) \sim \sum_{\nu\in\sigma} a_\nu \lambda^\nu \quad \text{and} \quad \mathcal{N}_\Omega(\lambda) \sim \sum_{\mu\in\Sigma} A_\mu \lambda^\mu, \quad \text{as } \lambda \to \infty. \tag{1.9}$$

Here, a_ν and A_μ may be functions of λ satisfying

$$0 < \liminf_{\lambda\to\infty} A_\mu(\lambda) \leq \limsup_{\lambda\to\infty} A_\mu(\lambda) < \infty, \tag{1.10}$$

analogously for a_ν. Inserting these expansions in (1.8) yields a number of relations. First, $\sigma \subset \Sigma$ must hold. For $\mu \in \Sigma$, $\mu \notin \sigma$ we have $A_\mu(\lambda) = NA_\mu(r^2\lambda)r^{2\mu}$. If $A_\mu(\lambda) \neq A_\mu(r^2\lambda)$ this contradicts (1.10). Hence $A_\mu(\lambda) = \mathcal{F}(\ln\lambda)$, $\mathcal{F}$ a bounded and $2\ln r$–periodic function and $Nr^{2\mu} = 1$, i.e.

$$\mu = \frac{D}{2}, \quad \text{where} \quad D = \frac{\ln N}{\ln(1/r)}. \tag{1.11}$$

D is the Minkowski dimension of Ω. For $\mu \in \sigma \cap \Sigma$ we only consider the case of constant a_ν resp. A_μ. One easily obtains

$$A_\mu = \frac{a_\mu}{1 - Nr^{2\mu}}. \tag{1.12}$$

Some coefficients of the expansion in λ of $\mathcal{N}_\omega$ are well known (cf. Definition 2.2):

$$\begin{aligned} a_{n/2} &= \Phi_\omega(1) \quad \Rightarrow \quad A_{n/2} = \Phi_\Omega(1) \\ a_{(n-1)/2} &= -\mathcal{C}_{n-1} \quad \Rightarrow \quad A_{(n-1)/2} = \frac{\mathcal{C}_{n-1}}{Nr^{n-1} - 1}. \end{aligned} \tag{1.13}$$

Therefore we expect a priori the following asymptotic expansion of the counting function $\mathcal{N}_\Omega$ for domains with (strongly) self–similar boundary

$$\mathcal{N}_\Omega(\lambda) = \Phi_\Omega(\lambda) - \mathcal{F}(\ln\lambda)\lambda^{D/2} + \frac{\mathcal{C}_{n-1}}{Nr^{n-1} - 1}\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}), \quad \text{as } \lambda \to \infty. \tag{1.14}$$

For the class of self–similar drums we have considered, it is easy to see why the second term contains a proper function $\mathcal{F}(\ln\lambda)$ and not just a constant as Berry conjectured. From the scaling and summation properties of $\mathcal{N}_\Omega$ it follows that

$$\mathcal{N}_\Omega(\lambda) = \sum_{i=0}^{[I]} N^i \mathcal{N}_\omega(r^{2i}\lambda), \tag{1.15}$$

where I is defined by $r^{2I}\lambda = \lambda_0$ and λ_0 denotes the lowest eigenvalue in ω. It follows that $\ln\lambda \sim 2I\ln(1/r)$. Suppose that there are "noise–like" deviations of $\mathcal{N}_\omega$ from its two–term asymptotics of amplitude $\leq C\lambda^\kappa, \kappa > 0$. Since $\mathcal{N}_\omega$ is integer–valued there must be some deviations at least with $\kappa = 0$. These sum up to deviations of $\mathcal{N}_\Omega$ from its mean value which are of amplitude

$$D_\Omega(\lambda) \sim C\sum_{i=0}^{[I]} N^i (r^{2i}\lambda)^\kappa. \tag{1.16}$$

From the leading term of the geometric sum we obtain

$$D_\Omega(\lambda) \sim C_1\lambda^\kappa (Nr^{2\kappa})^I \sim C_2 N^I \sim C_3\lambda^{D/2}, \tag{1.17}$$

independent of κ.

The present paper is organized as follows: In the next section, after introducing some definitions and results, we state and prove our main theorem concerning the counting function of self–similar fractal drums (see Theorem 2.8). Section 3 is devoted to the discussion of the occurring generalized Weierstrass function, especially the sharp estimates of the Minkowski dimension of its fractal graph (see Theorem 3.8 and 3.9). The purpose of section 4 is to illustrate our results by several examples.

2. Definitions and Main Theorem

Let Ω be an arbitrary nonempty bounded open set in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with boundary $\Gamma := \partial\Omega$. We consider the following eigenvalue problem:

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{2.1}$$

where Δ denotes the Friedrichs extension of the n-dimensional Dirichlet Laplacian $\sum_{k=1}^n \partial^2/\partial x_k^2$ in Ω.

Let the counting function $\mathcal{N}_\Omega$ be defined as above, cf. (1.2). Since we are especially interested in the case when the boundary Γ is fractal, let us now recall the definition of the interior (resp. exterior) Minkowski dimension [La1, Tr]. This dimension is greater than or equal to the Hausdorff dimension defined for example in [Fa].

Definition 2.1 (Minkowski dimension). (a) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with boundary $\Gamma = \partial\Omega$. Given $\varepsilon > 0$, let $\Gamma_\varepsilon = \{x \in \mathbb{R}^n : d(x,\Gamma) < \varepsilon\}$ be the open ε-neighborhood of Γ, where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^n$. For $d \geq 0$, let

$$(2.2)\qquad \mathcal{M}^*(d;\Gamma) := \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-d)}|\Gamma_\varepsilon \cap \Omega|_n$$

be the d-dimensional upper Minkowski content of Γ, relative to Ω. Then

$$(2.3)\quad D(\Gamma) := \inf\{d \geq 0 : \mathcal{M}^*(d;\Gamma) = 0\} = \sup\{d \geq 0 : \mathcal{M}^*(d;\Gamma) = +\infty\}.$$

is called the Minkowski dimension of Γ, relative to Ω.

(b) Let $A \subset \mathbb{R}^n$ be bounded. Given $\varepsilon > 0$, let A_ε be the open ε-neighborhood of A as above. For $d \geq 0$ let

$$(2.4)\qquad \tilde{\mathcal{M}}^*(d;A) := \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-d)}|A_\varepsilon|_n$$

the d-dimensional Minkowski content of A. Then

$$(2.5)\qquad \begin{aligned} \tilde{D}(A) &:= \inf\{d \geq 0 : \tilde{\mathcal{M}}^*(d;A) = 0\} \\ &= \sup\{d \geq 0 : \tilde{\mathcal{M}}^*(d;A) = +\infty\}. \end{aligned}$$

is called the Minkowski dimension of A.

For the rest of this paper we will fix the following notations:

(a) Let $\omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded, open domain with corresponding counting function $\mathcal{N}_\omega : \lambda \mapsto \mathcal{N}_\omega(\lambda)$ satisfying

$$(2.6)\qquad \mathcal{N}_\omega(\lambda) = \Phi_\omega(\lambda) - \mathcal{C}_{n-1}\lambda^{(n-1)/2} + O(\lambda^\kappa), \quad \text{as } \lambda \to \infty$$

with a suitable constant $\kappa \in [0,(n-1)/2]$, the Weyl term

$$(2.7)\qquad \Phi_\omega(\lambda) = (2\pi)^{-n}\mathcal{B}_n|\omega|_n\lambda^{n/2}$$

and the constant $\mathcal{C}_{n-1} = \frac{1}{4}[\mathcal{B}_{n-1}/(2\pi)^{n-1}]|\partial\omega|_{n-1}$. The domain ω will be called the *basic domain.* Its lowest eigenvalue will be denoted by λ_0.

(b) Let be $N \in \{2, 3, \ldots\}$ and $r \in (0,1)$ such that $Nr^{n-1} > 1$ and $Nr^n < 1$, and let be $n_0 \in \mathbb{N}$. Further let $\Omega \subset \mathbb{R}^n$ be the open domain which consists of the union of all n_0N^i ($i \in \mathbb{N}_0$) mutual disjoint copies of $r^i\omega$:

$$\Omega := \bigcup_{i\in\mathbb{N}_0} \biguplus_{\nu=1}^{n_0N^i} r^i\omega. \tag{2.8}$$

(c) Finally, we define

$$D := \frac{\ln N}{\ln(1/r)} \in (n-1, n). \tag{2.9}$$

Remark. Pham The Lai has proved (2.6) with $\kappa = (n-1)/2$ if the boundary $\partial\omega$ is of class C^∞ [Ph, p. 5]. Under an additional condition (the manifold $\bar{\Omega}$ does not have too many multiply reflected closed geodesics) V.Ja. Ivrii showed (2.6) with $o(\lambda^{(n-1)/2})$ instead of $O(\lambda^\kappa)$ [Iv,p. 98], i.e. a boundary term exists. In the case $n = 1$ expansion (2.6) holds with $\mathcal{C}_0 = 0$ and $\kappa = 0$.

Definition 2.2 (Curly Bracket). We define for all $x \geq 0$:

$$\{x\}_\omega := x - [x]_\omega, \quad \text{where} \quad [x]_\omega := \mathcal{N}_\omega(\Phi_\omega(1)^{-2/n}x^{2/n}). \tag{2.10}$$

Remark. Obviously the curly bracket is independent of the size of ω, i.e. $\{x\}_{\alpha\omega} = \{x\}_\omega$, where $\alpha \in \mathbb{R}^+$, $x \geq 0$. In the one–dimensional case we have $[x]_{(0,1)} = \mathcal{N}_{(0,1)}(x^2\pi^2) = \#\{n \in \mathbb{N} : n^2\pi^2 \leq x^2\pi^2\} = [x]$, where $[x]$ denotes the integer part of x. Table 1 shows several basic domains ω where upper bounds of κ are known.

Since the spectrum of $-\Delta$ is discrete and consists only of eigenvalues with finite multiplicity, it is easy to show the following

Lemma 2.3. *There exists positive constants C_1 and C_2 such that for $n \geq 2$:*

$$|\{x\}_\omega - \mathcal{C}_{n-1}\Phi_\omega(1)^{-(n-1)/n}x^{(n-1)/n}| \leq C_1x^{2\kappa/n} + C_2, \quad x \geq 0. \tag{2.11}$$

In the one-dimensional case ($\omega = (0,1)$) *we have the estimate*

$$0 \le \{x\}_{(0,1)} =: \{x\} < 1, x \ge 0. \tag{2.12}$$

Now we are able to define the so-called generalized WEIERSTRASS function f_ω which plays an important role in our theory (For examples see section 4 and [Ge]):

DEFINITION 2.4 (Function f_ω). We define for all $\mu \ge 0$:

$$f_\omega(\mu) := \sum_{i=0}^{\infty} N^{-i}\{r^{-ni}\mu\}_\omega. \tag{2.13}$$

The usual WEIERSTRASS function obtains, if $\{\cdot\}_\omega$ in (2.13) is replaced by the sinus function. The sum in (2.13) converges absolutely and by Lemma 2.3 the following remainder estimate holds:

COROLLARY 2.5.

$$f_\omega(\mu) = \mathcal{C}_{n-1}\Phi_\omega(1)^{-(n-1)/n}\frac{Nr^{n-1}}{Nr^{n-1}-1}\mu^{(n-1)/n} + O(\mu^{2\kappa/n}), \quad \textit{as } \mu \to \infty. \tag{2.14}$$

DEFINITION 2.6 (Function g_ω). We define for all $x \ge 0$:

$$g_\omega(x) := \sum_{i=-\infty}^{\infty} N^{i-x}\{r^{n(i-x)}\}_\omega. \tag{2.15}$$

PROPOSITION 2.7. *For all* $\lambda > 0$ *we have:*

$$g_\omega\left(\frac{\ln \Phi_\omega(\lambda)}{\ln(1/r^n)}\right) = f_\omega(\mu)\mu^{-D/n} + \frac{Nr^n}{1-Nr^n}\mu^{1-D/n}, \tag{2.16}$$

where $\mu(\lambda) := r^{nI(\lambda)}\Phi_\omega(\lambda)$ *and*

$$I(\lambda) := \max\{i \in \mathbb{Z} : \mathcal{N}_\omega(r^{2i}\lambda) > 0\} = [\ln(\lambda_0/\lambda)/(2\ln r)]. \tag{2.17}$$

PROOF. Let $J(x) := \max\{i \in \mathbb{Z} : \mathcal{N}_\omega(\Phi_\omega(1)^{-2/n} r^{2(i-x)}) > 0\}$. Then we split the sum in (2.15) into two parts and write

$$\begin{aligned} g_\omega(x) &= \sum_{i=-\infty}^{J(x)} N^{i-x}\{r^{n(i-x)}\}_\omega + \sum_{i=J(x)+1}^{\infty} N^{i-x} r^{n(i-x)} \\ &= N^{J(x)-x} f_\omega(r^{n(J(x)-x)}) + \frac{Nr^n}{1-Nr^n}(Nr^n)^{J(x)-x}. \end{aligned} \tag{2.18}$$

We now set $\mu := r^{n(J(x)-x)}$ and $\Phi_\omega(1)^{-2/n} r^{-2x} =: \lambda$ to simplify the argument of $\mathcal{N}_\omega$ in the definition of J. The proposition follows with $J(\ln \Phi_\omega(\lambda)/\ln(1/r^n)) = I(\lambda)$. □

REMARK. We illustrate the relation between μ and λ. One has

$$\mu(\lambda) = r^{nI(\lambda)}\Phi_\omega(\lambda) = r^{-\{\ln(\lambda_0/\lambda)/(2\ln r)\}}\Phi_\omega(\lambda_0). \tag{2.19}$$

Therefore we have the estimate

$$\mu_{\min} := \Phi_\omega(\lambda_0) \le \mu < \Phi_\omega(\lambda_0/r^2) =: \mu_{\max}. \tag{2.20}$$

Figure 2 shows the relation between μ and λ.

Now we can state one of our main theorems as announced in [GeSc] (For further details see also [Ge]):

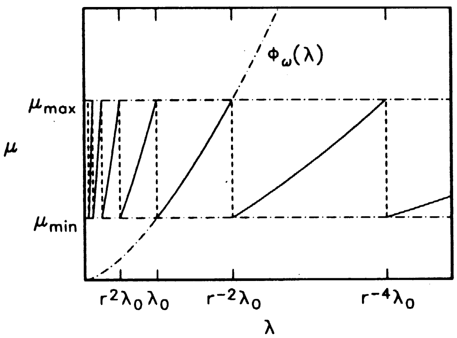

Fig. 2. Relation between μ and λ.

THEOREM 2.8 (Counting function). *Let ω and Ω be as above. Then:*

(a) *The upper* MINKOWSKI *content of* $\Gamma = \partial\Omega$*, relative to* Ω *is finite and the* MINKOWSKI *dimension of* Γ*, relative to* Ω *is*

$$D = D(\Gamma) = \frac{\ln N}{\ln(1/r)} \in (n-1, n). \tag{2.21}$$

(b) *For all* $\lambda > \lambda_0$ *the following identity holds:*

$$\begin{aligned} \mathcal{N}_\Omega(\lambda) &= \Phi_\Omega(\lambda) - n_0 \Phi_\omega(1)^{D/n} g_\omega \left(\frac{\ln \Phi_\omega(\lambda)}{\ln(1/r^n)} \right) \lambda^{D/2} \\ &\quad + \frac{n_0}{N} f_\omega(\Phi_\omega(\lambda/r^2)), \end{aligned} \tag{2.22}$$

where the functions f_ω *and* g_ω *are given by Definition* 2.4 *and* 2.6*, respectively.*

REMARKS. (a) Part (a) gives the well-known MINKOWSKI dimension of (strictly) self–similar fractals which coincides for this class of fractals with the HAUSDORFF dimension [Fa, p. 42 and p. 118].

(b) One easily shows the following identity

$$g_\omega \left(\frac{\ln \Phi_\omega(\lambda)}{\ln(1/r^n)} \right) = \Phi_\omega(1)^{-D/n} G\left(\frac{\ln \lambda}{2} \right), \tag{2.23}$$

where

$$G(t) = \sum_{i=-\infty}^{\infty} e^{-D(t+i\ln r)} \delta_\omega(e^{2(t+i\ln r)}) \tag{2.24}$$

and $\delta_\omega(x) = \Phi_\omega(x) - \mathcal{N}_\omega(x)$, which is compatible with part (ii) of Conjecture 3 given by M.L. LAPIDUS in [La2, pp. 163–164] for drums with strictly self–similar boundary.

(c) In [FlVa2] a special two–dimensional example is investigated. Choose ω as a square of side $s \in (1/3, 1/(1+\sqrt{2}))$, $N = 3$, $r = s$ and $n_0 = 4$ to get a slightly modified version of that example which exhibits the same second term in the asymptotic expansion.

(d) Notice that a slightly modified version of the famous counter–example of BROSSARD and CARMONA is included in our theorem too (cf. [Ge]).

By Corollary 2.5 we have:

Corollary 2.9.

$$\frac{1}{N} f_\omega(\Phi_\omega(\lambda/r^2)) = \frac{\mathcal{C}_{n-1}}{Nr^{n-1}-1}\lambda^{(n-1)/2} + O(\lambda^\kappa), \quad \text{as } \lambda \to \infty. \tag{2.25}$$

In the one–dimensional case we have the estimate

$$0 \le f_\omega(\Phi_\omega(\lambda/r^2)) < \frac{N}{N-1}, \quad \lambda \ge 0. \tag{2.26}$$

Corollary 2.10. *The counting function $\mathcal{N}_\Omega$ satisfies the following functional equation*

$$\mathcal{N}_\Omega(\lambda) = N\mathcal{N}_\Omega(r^2\lambda) + n_0\mathcal{N}_\omega(\lambda), \quad \lambda \ge 0. \tag{2.27}$$

Using the estimate for the curly bracket (Lemma 2.3) and the condition $Nr^{n-1} > 1$, one easily shows:

Lemma 2.11. *The function f_ω is a bounded function on the interval $[\mu_{\min}, \mu_{\max})$, where $\mu_{\min}$ and $\mu_{\max}$ are given by* (2.20).

Corollary 2.12. *The coefficient of the second term in the asymptotic expansion of $\mathcal{N}_\Omega$ is bounded for all $\lambda > 0$.*

Proof of Theorem 2.8. (a) Given $\varepsilon > 0$ sufficient small, let

$$\varepsilon_0 = \min\{\varepsilon > 0 : |(\partial\omega)_\varepsilon \cap \omega| = |\omega|\} \tag{2.28}$$

and

$$\begin{aligned} i_0(\varepsilon) \; &:= \max\{i \ge 1 : |(\partial(r^i\omega))_\varepsilon \cap \Omega| \le |r^i\omega|\} \\ &= \max\{i \ge 1 : |(\partial\omega)_{\varepsilon/r^i} \cap \omega| \le r^{ni}|\omega|\} \\ &= \max\{i \ge 1 : \varepsilon \le r^i\varepsilon_0\} = [\ln(\varepsilon/\varepsilon_0)/\ln r] \end{aligned} \tag{2.29}$$

using the two elementary facts $rB_\varepsilon = (rB)_\varepsilon$ and $|rB_\varepsilon| = r^n|B_\varepsilon|$ for the ε-neighborhood of a bounded set $B \subset \mathbb{R}^n$. Since

$$|(\partial(r^i\omega))_\varepsilon \cap \Omega| = \begin{cases} r^{ni}|(\partial\omega)_{\varepsilon/r^i} \cap \omega|, & i \le i_0(\varepsilon) \\ r^{ni}|\omega|, & i > i_0(\varepsilon) \end{cases} \tag{2.30}$$

we have for the interior ε-neighborhood of Γ relative to Ω:

$$(2.31)\quad |\Gamma_\varepsilon \cap \Omega| = n_0 \sum_{i=0}^{i_0(\varepsilon)} (Nr^n)^i |(\partial\omega)_{\varepsilon/r^i} \cap \omega| + n_0|\omega| \sum_{i>i_0(\varepsilon)} (Nr^n)^i.$$

Because of $D(\partial\omega) = n-1$ it follows that there exists an $\varepsilon_1 > 0$ and a constant $c > 0$ such that $|(\partial\omega)_\varepsilon \cap \omega| \le c\varepsilon,\ \varepsilon < \varepsilon_1$. Therefore $|(\partial\omega)_{\varepsilon/r^i} \cap \omega| \le c\varepsilon/r^i,\ \varepsilon \le r^i\varepsilon_1$. Now let

$$(2.32)\quad i_1(\varepsilon) := \max\{i \ge 1 : \varepsilon \le r^i \varepsilon_1\} = [\ln(\varepsilon/\varepsilon_1)/\ln r].$$

We suppose $\varepsilon_1 \le \varepsilon_0$, then $i_1(\varepsilon) \le i_0(\varepsilon),\ \varepsilon \le \varepsilon_1$. It is an elementary exercise to verify that there exist some constants $m_1, m_2 > 0$ and an $\varepsilon_2 > 0$ such that

$$(2.33)\quad m_1 \varepsilon^{n-D} \le |\Gamma_\varepsilon \cap \Omega| \le m_2 \varepsilon^{n-D}, \quad \varepsilon \le \varepsilon_2.$$

By Definition 2.1 we now conclude that $D(\Gamma) = \ln N/\ln(1/r)$ and

$$(2.34)\quad 0 < m_1 \le \mathcal{M}_*(D;\Gamma) \le \mathcal{M}^*(D;\Gamma) \le m_2 < +\infty.$$

(b) Referring to the summation law and the scaling property of the counting function we write:

$$(2.35)\quad \begin{aligned} \mathcal{N}_\Omega(\lambda) &= n_0 \sum_{i=0}^{\infty} N^i \mathcal{N}_\omega(r^{2i}\lambda) = n_0 \sum_{i=0}^{I(\lambda)} N^i \mathcal{N}_\omega(r^{2i}\lambda) \\ &= n_0(S_2(\lambda) - S_1(\lambda)), \end{aligned}$$

where we have set $I = I(\lambda) := \max\{i \in \mathbb{N}_0 : \mathcal{N}_\omega(r^{2i}\lambda) > 0\}$ for all $\lambda \ge \lambda_0$, i.e. $\mathcal{N}_\omega(r^{2i}\lambda)$ vanishes for all $i > I(\lambda)$. Furthermore we have introduced the two functions

$$(2.36)\quad S_1(\lambda) = \sum_{i=0}^{I(\lambda)} N^i \{\Phi_\omega(r^{2i}\lambda)\}_\omega \quad \text{and} \quad S_2(\lambda) = \sum_{i=0}^{I(\lambda)} (Nr^n)^i \Phi_\omega(\lambda).$$

After introducing the new summation variable $j = I - i$ and substitution

$$(2.37)\quad \mu := \Phi_\omega(r^{2I}\lambda) = r^{nI}\Phi_\omega(1)\lambda^{n/2},$$

i.e. $N^I = r^{-ID} = \Phi_\omega(1)^{D/n}\mu^{-D/n}\lambda^{D/2}$, we have:

$$S_1(\lambda) = N^I \sum_{j=0}^{I} N^{-j}\{r^{-nj}\mu\}_\omega = \Phi_\omega(1)^{D/n} f_\omega^I(\mu)\mu^{-D/n}\lambda^{D/2}, \tag{2.38}$$

where $f_\omega^I(\mu)$ denotes the first $I+1$ terms of f_ω, i.e. $f_\omega^I(\mu) = \sum_{i=0}^{I} \cdot N^{-i}\{r^{-ni}\mu\}_\omega$. Using Lemma 2.3 and the definition of f_ω it is an elementary calculation to verify the estimate

$$|f_\omega(\mu) - f_\omega^I(\mu)| = O\left(\frac{1}{(Nr^{n-1})^I} + \frac{1}{N^I}\right), \quad \text{as } \lambda \to \infty, \tag{2.39}$$

since μ is bounded and $I(\lambda)$ increases with λ. Therefore we replace f_ω^I by f_ω and write

$$S_1(\lambda) = \Phi_\omega(1)^{D/n} f_\omega(\mu)\mu^{-D/n}\lambda^{D/2} - A(\lambda), \tag{2.40}$$

where

$$A(\lambda) = N^I \sum_{i=I+1}^{\infty} N^{-i}\{r^{x-ni}\mu\}_\omega = \frac{1}{N} f_\omega(\Phi_\omega(\lambda/r^2)), \tag{2.41}$$

by the definition of μ. Transforming the second part $S_2(\lambda)$ with

$$(Nr^n)^I = r^{I(n-D)} = \Phi_\omega(1)^{D/n-1}\mu^{1-D/n}\lambda^{(D-n)/2} \tag{2.42}$$

yields:

$$\begin{aligned} S_2(\lambda) &= \frac{1-(Nr^n)^{I+1}}{1-Nr^n}\Phi_\omega(\lambda) \\ &= \frac{\Phi_\omega(\lambda)}{1-Nr^n} - \Phi_\omega(1)^{D/n}\frac{Nr^n}{1-Nr^n}\mu^{1-D/n}\lambda^{D/2}. \end{aligned} \tag{2.43}$$

Since $|\Omega|_n = n_0 \sum_{i=0}^{\infty}(Nr^n)^i|\omega|_n$ is finite ($Nr^n < 1$) the statement of our theorem follows by applying Proposition 2.7. □

3. Discussion of f_ω

3.1. The general case

Theorem 3.1. (a) *The graph of f_ω on $[0, \mu_{\max})$, cf.* (2.20) *is self–similar under the linear transformation*

$$A = \begin{pmatrix} r^n & 0 \\ r^n & \frac{1}{N} \end{pmatrix}. \tag{3.1}$$

(b) *The function* f_ω *satisfies the following functional equation for all* $\mu \geq 0$*:*

$$f_\omega(r^n\mu) = \{r^n\mu\}_\omega + \frac{1}{N} f_\omega(\mu). \tag{3.2}$$

PROOF. The statements follow immediately with Definition 2.4 from

$$f_\omega(r^n\mu) = \{r^n\mu\}_\omega + \sum_{i=1}^{\infty} N^{-i}\{r^{(1-i)n}\mu\}_\omega = r^n\mu + \frac{1}{N} f_\omega(\mu), \tag{3.3}$$

where the first identity is true for all $\mu \geq 0$, the second for $0 \leq \mu < \mu_{\max}$, since $\Phi_\omega(1)^{-2/n}(r^n\mu)^{2/n} < \lambda_0$, and $[r^n\mu]_\omega$ vanishes because of Definition 2.2. □

REMARK. The graph of f_ω cannot be determined uniquely by only using its functional equation, since $f^\bullet(\mu) = f_\omega(\mu) + A(\mu)\mu^{D/n}$ is a solution of (3.2) too, where $A : \mathbb{R}_0^+ \to \mathbb{R}$ is any function satisfying $A(r^n\mu) = A(\mu)$, $\mu \geq 0$.

LEMMA 3.2. *Let* f_ω *be the function of Definition* 2.4 *and* $k \in \mathbb{N}$. *Then:*

$$\left|f_\omega(\mu) - m_k\mu - b_k\mu^{(n-1)/n} + \sum_{i=0}^{k} N^{-i}[r^{-ni}\mu]_\omega\right| \leq d_k\mu^{2\kappa/n} + c_k, \quad \mu \geq 0, \tag{3.4}$$

where

$$m_k = \frac{(Nr^n)^{-k} - Nr^n}{1 - Nr^n}, \quad b_k = \frac{\mathcal{C}_{n-1}}{Nr^{n-1} - 1}\frac{\Phi_\omega(1)^{-(n-1)/n}}{(Nr^{n-1})^k}, \tag{3.5}$$
$$d_k = \frac{C_1}{Nr^{2\kappa} - 1}\frac{1}{(Nr^{2\kappa})^k} \quad \textit{and} \quad c_k = \frac{C_2}{N-1}\frac{1}{N^k},$$

and $C_1, C_2 > 0$ *are constants given by Lemma 2.3.*

PROOF. This Lemma follows easily by splitting the sum in (2.13) and applying Lemma 2.3. □

DEFINITION 3.3 (k-system of strips). Given $k \in \mathbb{N}$, let be

$$\tilde{f}_\omega^k(\mu) := \sum_{i=0}^{k} N^{-i}[r^{-ni}\mu]_\omega, \tag{3.6}$$

and $(a, b) \subset \mathbb{R}^+$ a bounded open interval. Denote with $\{\tilde{\mu}_i^k\}_{i=1}^{i_{\max}}$ the sequence of discontinuities of $\tilde{f}_\omega^k$ in (a, b), where $\tilde{\mu}_i^k < \tilde{\mu}_{i+1}^k,\ i = 1, \ldots, i_{\max}-1$. Further let be $\tilde{\mu}_0^k = a$ and $\tilde{\mu}_{i_{\max}+1}^k = b$. Now define for all $i = 1, \ldots, i_{\max}+1$:

$$\begin{aligned}(3.7)\quad S_k^i := \{(x,y) \in [a,b) \times \mathbb{R} : \ & \tilde{\mu}_{i-1}^k \leq x < \tilde{\mu}_i^k; \\ & |y - m_k x - b_k x^{(n-1)/n} + \tilde{f}_\omega^k(x)| \\ & \leq d_k b^{2\kappa/n} + c_k\},\end{aligned}$$

where m_k, b_k, d_k and c_k are given by Lemma 3.2. Furthermore we define the so-called k-system of strips on $[a, b)$ by

$$(3.8) \qquad S_k := \bigcup_{i=1}^{i_{\max}+1} S_k^i.$$

One easily shows the following three corollaries using Definition 3.3 and the definition of the function f_ω.

COROLLARY 3.4. *Given $k \in \mathbb{N}$, let S_k be the k-system of strips on a bounded interval $(a, b) \subset \mathbb{R}^+$ and $G := \{(\mu, f_\omega(\mu)) : \mu \in (a, b)\}$ the graph of f_ω on (a, b). Then $G \subset S_k,\ k \in \mathbb{N}$.*

COROLLARY 3.5. *For all $k \in \mathbb{N}$ we have $S_{k+1} \subset S_k$.*

COROLLARY 3.6. *The function f_ω is right continuous for all $\mu \geq 0$.*

Since the graph of f_ω is self–similar the question arises: What is its MINKOWSKI dimension? This problem is not completely solved yet. But we obtained exact estimates as announced in [GeSc]. A detailed analysis shows that the MINKOWSKI dimension of the graphs of the functions f_ω and g_ω on each bounded interval $(a, b) \subset \mathbb{R}^+$ are the same (cf. [Ge] for further details). Therefore we restrict ourselves to the computation of the MINKOWSKI dimension of the fractal graph of f_ω.

DEFINITION 3.7. Let G be the graph of f_ω restricted to any bounded interval $(a, b) \subset \mathbb{R}^+$. Then we denote by Σ the set of all points forming the jumps of f_ω on (a, b), i.e. vertical line segments at the discontinuities.

THEOREM 3.8 (Dimension of the connected graph, upper bound). *The* MINKOWSKI *dimension* $\tilde{D}(G^c)$ *of the connected graph* $G^c := G \cup \Sigma$ *of* f_ω *restricted to any bounded interval* $(a,b) \subset \mathbb{R}^+$ *satisfies*

$$\tilde{D}(G^c) \le 2 - \frac{D - 2\kappa}{n - 2\kappa}. \tag{3.9}$$

THEOREM 3.9 (Dimension of the connected graph, lower bound). *Let* $\{\Lambda_i\}_{i=1}^{\infty}$ *be the sequence of eigenvalues of* $-\Delta$ *in the basic domain* ω *counted* without *multiplicity. If there exists some constants* A, $i_0 > 0$ *and some* $\alpha < 1$ *such that*

$$\Lambda_{i+1} - \Lambda_i > A\Lambda_i^{\alpha}, \quad i \ge i_0, \quad (gapcondition) \tag{3.10}$$

then the MINKOWSKI *dimension* $\tilde{D}(G^c)$ *of the connected graph of* f_ω *restricted to any bounded interval* $(a,b) \subset \mathbb{R}^+$ *satisfies*

$$\tilde{D}(G^c) \ge 1 + \frac{1}{1-\alpha}\frac{n-D}{2}. \tag{3.11}$$

Especially this applies if the eigenvalues are integer multiples of a "unit" Λ_0:

COROLLARY 3.10. *If the spectrum of the basic domain satisfies*

$$\lambda_{i,\omega} = \nu_i \Lambda_0 \quad \text{for some } \nu_i \in \mathbb{N} \text{ with } i \in \mathbb{N} \tag{3.12}$$

and a constant $\Lambda_0 > 0$, *where* $\{\lambda_{i,\omega}\}_{i=1}^{\infty}$ *denotes the sequence of eigenvalues of* $-\Delta$ *in the basic domain counted* with *multiplicity, it follows that* $\alpha \ge 0$ *and*

$$\tilde{D}(G^c) \ge 1 + \frac{n-D}{2}. \tag{3.13}$$

THEOREM 3.11. *Let* G^c *be the connected graph of the "one–dimensional" function* $f := f_{(0,1)}$. *Then* $\tilde{D}(G^c) = 2 - D$.

PROOF. In the one–dimensional case we have $\kappa = 0$ and $\Lambda_i = i^2\pi^2$. Apply Theorem 3.9 with $A = 2\pi$, $i_0 = 1$ and $\alpha = 1/2$. □

Proof of Theorem 3.8. Given $k \in \mathbb{N}$, let S_k be the k-system of strips on (a,b) in sense of Definition 3.3. Then we have $G_\varepsilon \subset (S_k)_\varepsilon$, $\varepsilon > 0$, $k \in \mathbb{N}$. Furthermore let J_k be the set of indices of the discontinuities of the function $\tilde{f}^k_\omega : \mu \mapsto \sum_{i=0}^k N^{-i}[r^{-ni}\mu]_\omega$ in (a,b). Denote the height of the jump at $\tilde{\mu}^k_i$ by σ^k_i, $i \in J_k$ and the length of curve $B_k : \mu \mapsto m_k\mu + b_k\mu^{(n-1)/n}$ on (a,b) by ℓ_k. Then the following estimate of the two–dimensional Lebesgue measure of the ε-neighborhood of the connected graph holds (cf. figure 3):

$$|G^c_\varepsilon| \lesssim |S_k|_2 + 2\ell_k\varepsilon + 2|\Sigma^k|_1\varepsilon + 4(d_k b^{2\kappa/n} + c_k + \varepsilon)\varepsilon, \tag{3.14}$$

where Σ^k denotes the set of points forming the jumps of $\tilde{f}^k_\omega$ on (a,b). We have $|\Sigma^k|_1 = \sum_{i\in J_k}\sigma^k_i$, and the area of S_k is given by

$$|S_k| = 2(b-a)(d_k b^{2\kappa/n} + c_k) \approx (Nr^{2\kappa})^{-k}, \quad \text{as } k \to \infty, \tag{3.15}$$

since $\kappa \geq 0$. ($\approx$ is the short hand notation of weak asymptotic behavior, i.e we write $f(\lambda) \approx g(\lambda)$, as $\lambda \to \infty$, whenever their exist positive constants c_1, c_2 and λ_0 such that $c_1 g(\lambda) \leq f(\lambda) \leq c_2 g(\lambda)$, $\lambda \geq \lambda_0$.) The length ℓ_k is

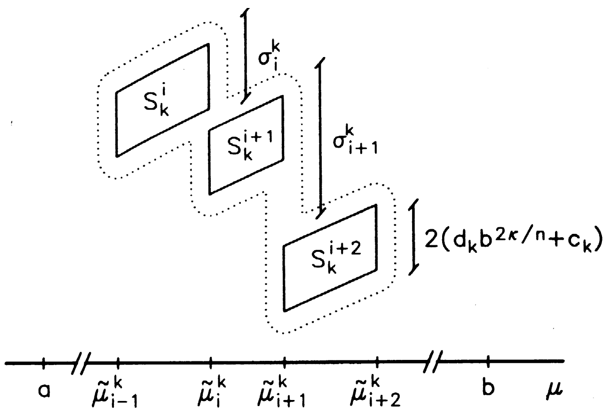

Fig. 3. k-system of strips. This figure illustrates estimate (3.14). The dotted line shows the ε-neighborhood of the system of strips including the ε-neighborhood at the jumps. G^c_ε is included in this ε-neighborhood.

given by

$$
\begin{aligned}
\ell_k &= \int_{\mu=a}^{\mu=b} ds = \int_a^b (1+[B_k'(\mu)]^2)^{1/2}\, d\mu \\
&\le \int_a^b \Big(1+(m_k+\tfrac{n-1}{n}b_k a^{-1/n})^2\Big)^{1/2}\, d\mu \\
&\sim (b-a)(m_k+\tfrac{n-1}{n}b_k a^{-1/n}) \approx (Nr^n)^{-k}, \quad \text{as } k\to\infty.
\end{aligned} \tag{3.16}
$$

For estimating $|\Sigma^k|$ — the sum of all sizes of jumps of $\tilde f^k_\omega$ on (a,b) — we have to know the number of discontinuities of $\{\cdot\}^i_\omega : \mu \mapsto \{r^{-ni}\mu\}_\omega = r^{-ni}\mu - [r^{-ni}\mu]_\omega$ $(i=0,\dots,k)$ in (a,b), say f_i. Let $\{\lambda_{j,\omega}\}_{j=1}^\infty$ be the sequence of eigenvalues according to $-\Delta$ in the basic domain ω. Then a discontinuity μ_j of $[\cdot]_\omega : \mu \mapsto [\mu]_\omega$ occurs if and only if $\mu_j = \Phi_\omega(\lambda_{j,\omega})$, see Definition 2.2, and now it follows easily that

$$
\begin{aligned}
f_i &:= \#\{j\in\mathbb{N} : \mu_j \in (r^{-ni}a, r^{-ni}b)\} \\
&= \#\{j\in\mathbb{N} : r^{-ni}a < \Phi_\omega(\lambda_{j,\omega}) < r^{-ni}b\}.
\end{aligned} \tag{3.17}
$$

Notice that $|\Sigma^k| = \sum_{i=0}^k N^{-i} f_i$, since jumps of $\tilde f^k_\omega$ only occur as multiples of N^{-i} $(i=0,\dots,k)$. Because of the properties of the spectrum according to $-\Delta$ in the basic domain and since $\Phi_\omega(\lambda_{j,\omega}) = j + O(j^{(n-1)/n})$, as $j\to\infty$ we can show that $|f_i - (b-a)r^{-ni}| \le c_1 r^{-(n-1)i} + c_2,\ i\in\mathbb{N}_0$ with suitable constants $c_1,\, c_2 > 0$, independent of i. Hence there exists two constants $s_2, k_2 > 0$ such that $|\Sigma^k| \le s_2 r^{k(D-n)},\ k\ge k_2$. Together with the above results it follows that we can choose some positive constants C and k_0 such that

$$
|G^c_\varepsilon| \le C(r^{k(D-2\kappa)} + r^{k(D-n)}\varepsilon), \quad k \ge k_0. \tag{3.18}
$$

Given $0<\varepsilon<r$, choose $k>0$ such that $r^{k+1} < \varepsilon^{1/(n-2\kappa)} \le r^k$. We then conclude that there exists some constants $m_2>0$ and $\varepsilon_0>0$ such that

$$
|G^c_\varepsilon| \le m_2 \varepsilon^{(D-2\kappa)/(n-2\kappa)}, \quad \varepsilon\le\varepsilon_0. \tag{3.19}
$$

For completing the proof remember the definition of the Minkowski dimension and note $\tilde{\mathcal{M}}^*(d;G^c) = 0$ for all $d > 2-(D-2\kappa)/(n-2\kappa)$. □

Proof of Theorem 3.9. Given $k\in\mathbb{N}$ $(k\ge i_0)$, let $\{\mu^k_i\}_{i=1}^\infty$ be the sequence of discontinuities of $[\cdot]^k_\omega : \mu\mapsto [r^{-nk}\mu]_\omega$ in (a,b) and I_k the set of the corresponding indices. Furthermore let

$$
\delta\mu_k = \min\{\mu^k_{i+1} - \mu^k_i : i \in I_k\}. \tag{3.20}
$$

Discontinuities μ_i^k of $[\cdot]_\omega^k$ are given by $\mu_i^k = \Phi_\omega(r^{2k}\Lambda_i) \overset{!}{\in} (a,b),\ i \in I_k$. Therefore we have

$$\begin{aligned} \delta\mu_k &= \Phi_\omega(r^{2k}) \min\{\Lambda_{i+1}^{n/2} - \Lambda_i^{n/2} : i \in I_k\} \\ &\geq \frac{n}{2}\Phi_\omega(r^{2k})(\Lambda_{i+1} - \Lambda_i)\Lambda_i^{n/2-1} > \frac{nA}{2}\Phi_\omega(r^{2k})\Lambda_i^{\alpha+n/2-1} \end{aligned} \tag{3.21}$$

by assumption. Noting $\Phi_\omega(r^{2k}\Lambda_i) \in (a,b)$ we obtain that there exist positive constants M_0 and k_0 independent of k such that

$$\delta\mu_k > M_0 r^{k(2-2\alpha)}, \quad k \geq k_0. \tag{3.22}$$

The connected graph consists at least of the discontinuities at $\mu_i^k \in (a,b)$ since there are only decreasing jumps. Hence, for given $\varepsilon > 0$ (sufficient small) we can choose $k \geq k_0$ such that

$$M_0 r^{k(2-2\alpha)} < 2\varepsilon \leq \delta\mu_k. \tag{3.23}$$

Therefore, since the ε-neighborhoods at the discontinuities do not overlap, we have

$$|G_\varepsilon^c| = |(G \cup \Sigma)_\varepsilon| \geq |\Sigma_\varepsilon| \geq 2\varepsilon f_k N^{-k}, \tag{3.24}$$

where f_k denotes the number of discontinuities in the corresponding λ-interval, $f_k \sim (b-a)r^{-nk}$, as $k \to \infty$. Combining with (3.23) we obtain

$$M_0 r^{k(2-2\alpha)} < 2\varepsilon \quad \Longleftrightarrow \quad r^{-k} > (M_0/2)^{1/(2-2\alpha)}\varepsilon^{1/(2\alpha-2)} \tag{3.25}$$

since $\alpha < 1$. So we can choose constants $m_1, \varepsilon_0 > 0$ such that

$$|G_\varepsilon^c| \geq m_1 \varepsilon^{1 + \frac{1}{1-\alpha}\frac{n-D}{2}}, \quad \varepsilon \leq \varepsilon_0. \tag{3.26}$$

For all $d < 1 + \frac{1}{1-\alpha}\frac{n-D}{2}$ we have $\tilde{\mathcal{M}}^*(d; G^c) = +\infty$. This fact completes our proof. □

We complete this paragraph by considering a couple of basic domains and employing the best known values for κ (cf. Table 1).

Table 1. This table shows several basic domains and their best known κ-values, where $\varepsilon > 0$ is arbitrary. It shows also lower and upper bounds for the Minkowski dimension of the connected graph G^c. The κ-values are taken from [IwMo] (square), [Vi] (cube), [L] (4-dimensional cube), [Wa] (n-dimensional cube with $n \geq 5$), [Hu] together with [Pi] and [Ge, Appendix A] (equilateral triangle and isosceles right triangle) and [KuFe] (circle). Confer also the review paper [Kr]. Notice that except for the circular membrane problem the calculation of the counting function for the above mentioned basic domains leads to the calculation of the number of integer lattice points in n-dimensional ellipsoids.

n	basic domain ω	upper bound for κ	lower bound for $\tilde{D}(G^c)$	upper bound for $\tilde{D}(G^c)$	maximal difference
1	interval	0	$2-D$	$2-D$	0
2	square	$\frac{7}{22}$	$2-\frac{D}{2}$	$\frac{37}{15}-\frac{11}{15}D$	$\frac{7}{30}$
3	cube	$\frac{2}{3}$	$\frac{5}{2}-\frac{D}{2}$	$\frac{14}{5}-\frac{3}{5}D$	$\frac{1}{10}$
4	4-dimensional cube	$1+\varepsilon$	$3-\frac{D}{2}$	$3-\frac{D}{2}$	0
≥ 5	n-dimensional cube	$\frac{n}{2}-1$	$1+\frac{n-D}{2}$	$1+\frac{n-D}{2}$	0
2	equilateral triangle	$\frac{7}{22}+\varepsilon$	$2-\frac{D}{2}$	$\frac{37}{15}-\frac{11}{15}D$	$\frac{7}{30}$
2	isosceles right triangle	$\frac{7}{22}+\varepsilon$	$2-\frac{D}{2}$	$\frac{37}{15}-\frac{11}{15}D$	$\frac{7}{30}$
2	circle	$\frac{1}{3}$	1	$\frac{5}{2}-\frac{3}{4}D$	$\frac{3}{4}$

3.2. The one–dimensional case

In the one–dimensional case more information about $f := f_{(0,1)}$ and $g := g_{(0,1)}$ is available (cf. figure 5 in section 4). For further details see [Ge].

Theorem 3.12 (Congruence property of f). *Let $1/r \in \mathbb{N}$, then the graph of function f on $[0,1)$ is invariant under the affine transformations $w_i : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $w_i\binom{x}{y} = A\binom{x}{y} + (i-1)\binom{r}{r}$ $(i = 1, \dots, 1/r)$, where $y := f(x)$. It is possible to reconstruct this graph on $[0,1)$ by merely using* *these transformations. Furthermore,* $f : \mu \mapsto f(\mu)$ *is* 1*-periodical in* μ*.*

The proof will be omitted.

THEOREM 3.13 (Nonlinear self–similarity of g). *Let be* $1/r \in \mathbb{N}$*. Then the graph of* g *given by*

$$g(x) = f(\mu)\mu^{-D} + \frac{Nr}{1-Nr}\mu^{1-D}, \quad \text{where} \quad \mu = r^{-\{x\}} \tag{3.27}$$

is self–similar for all $x \in [0,1)$ *according to the following nonlinear map:*

$$\begin{aligned} x \quad &\mapsto x' = \frac{\ln(r^{1-x}+1-r)}{\ln(1/r)} \\ g(x) \quad &\mapsto g(x') = \left(N^{x-1}g(x) + \frac{Nr-r}{1-Nr}\right)\left(r^{1-x}+1-r\right)^{-D}. \end{aligned} \tag{3.28}$$

REMARK. Notice that g is obviously a 1-periodic function in x, since $\{\cdot\} : \nu \mapsto \{\nu\}$ is 1-periodic in ν.

PROOF. Given $x \in [0,1)$, we have $\mu = r^{-x}$. An affine transformation

$$\mu \to \mu' = r\mu + 1 - r \tag{3.29}$$

corresponds with $\mu' = r^{-x'}$ to a nonlinear transformation

$$x \to x' = \ln(r^{1-x}+1-r)/\ln(1/r). \tag{3.30}$$

For all $x \in [0,1)$ we have therefore $x' \in [0, \ln(2-r)/\ln(1/r))$, so $\mu' = r^{-x'} \in [1, 2-r) \subset [1,2)$ follows, because of $r \in (0,1)$, definition of g and f's functional equation yields

$$\begin{aligned} g(x') &= f(\mu')(\mu')^{-D} + \frac{Nr}{1-Nr}(\mu')^{1-D} \\ &= \left(N\left(f(r\mu') - r\mu'\right) + \frac{Nr}{1-Nr}\mu'\right)(\mu')^{-D}. \end{aligned} \tag{3.31}$$

By evaluating of $f(r\mu')$ with applying f's functional equation a twice, and noting $\mu' \in [1,2)$ we have

$$\begin{aligned} f(r\mu') &= \{r\mu'\} + \frac{1}{N}\{\mu'\} + \frac{1}{N^2}f(\mu'/r) \\ &= r\mu' + \frac{1}{N}(\mu'-1) + \frac{1}{N^2}f(\mu), \end{aligned} \tag{3.32}$$

since $1/r - 1 \in \mathbb{N}$, and applying the previous Theorem. Inserting (3.32) in (3.31) and noting the assumption, the statement of our theorem follows after a few steps. □

4. Examples

This section is devoted to a few examples illustrating our results. For more examples see [Ge].

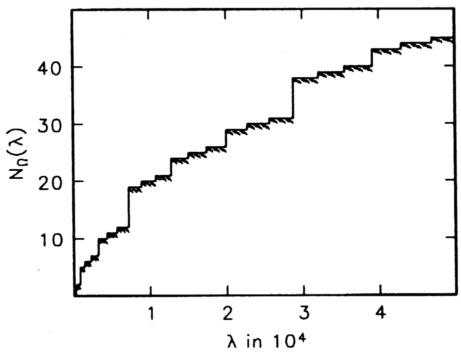

Fig. 4. The figure shows the counting function for the triadic CANTOR string and the two terms approximation given by Theorem 2.8 (see also Corollary 2.9).

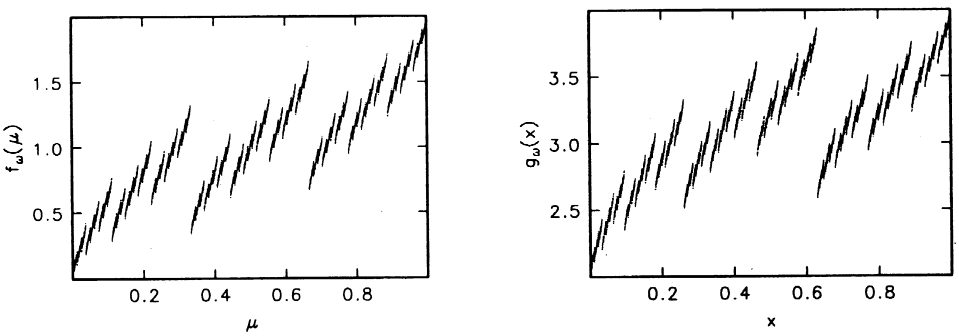

Fig. 5. This figure shows the generalized WEIERSTRASS function f_ω (left figure) and the function g_ω (right figure) for the triadic CANTOR string. Notice the linear self–similarity of f_ω and the nonlinear self–similarity of g_ω. Notice also the congruence property of f_ω given by Theorem 3.12.

Example. CANTOR String

In this example we consider the vibrations of the triadic CANTOR string, i.e. let Ω be the complement of the triadic CANTOR set with respect to the interval $(0,1)$. Figure 4 shows the counting function while figure 5 shows the functions f_ω and g_ω, respectively.

REMARK. The CANTOR string has also been studied in [LaPo2, pp. 65–67]. The authors show that the asymptotic expansions of $\mathcal{N}_\Omega$ does not admit

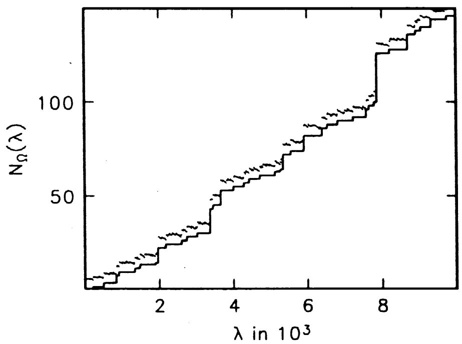

Fig. 6. The figure shows the counting function for the SIERPIŃSKI drum and the three terms approximation (shifted with 5 units) according to Theorem 2.8 and Corollary 2.9.

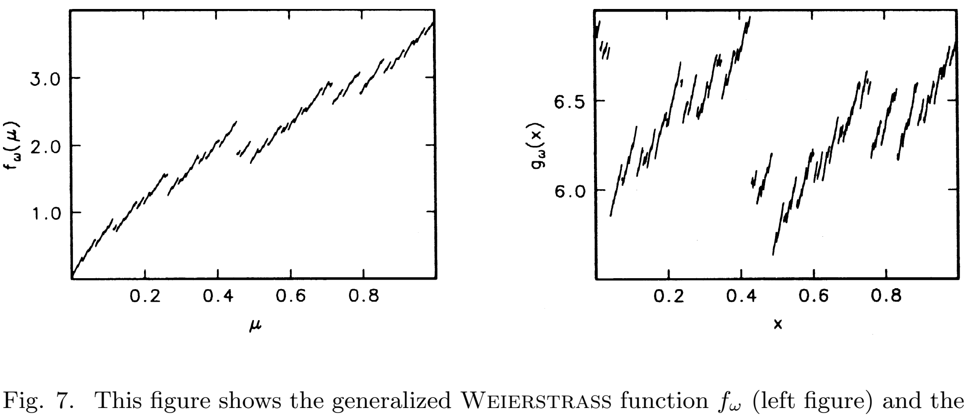

Fig. 7. This figure shows the generalized WEIERSTRASS function f_ω (left figure) and the function g_ω (right figure) for the SIERPIŃSKI drum.

a monotonic term (i.e. $\lambda^{-D/2}(\mathcal{N}_\Omega(\lambda) - \Phi_\Omega(\lambda))$ does not converge).

Example. Sierpiński Drum

We now consider the vibrations of the Sierpiński drum (see figure 1), i.e. let ω be an equilateral triangle with side $1/2$. Therefore we have $n_0 = 1$, $N = 3$ and $r = 1/2$, hence $D(\Gamma) = \ln 3/\ln 2$ (see figures 6 and 7).

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