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Agmon-Type Exponential Decay Estimates for Pseudodifferential Operators

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Abstract. We study generalizations of Agmon-type estimates on eigenfunctions for Schrödinger operators. In the first part, we prove an exponential decay estimate on eigenfunctions for a class of pseudodifferential operators. In the second part, we study the semiclassical limit of $\hbar$-pseudodifferential operators, and exponential decay estimates on eigenfunctions and Briet-Combes-Duclos-type resolvent estimates are proved.

1. Introduction

In this paper, we study exponential decay estimates of eigenfunctions and semiclassical resolvent estimates for a class of pseudodifferential operators. At first we study exponential decay of eigenfunctions at infinity. We consider operators of the following form:

$$H = h(x, D_x) + V(x) \quad \text{on } L^2(\mathbb{R}^n),$$

where $n \geq 1$, $h(x, D_x)$ is a pseudodifferential operator with a symbol $h(x, \xi)$:

$$h(x, D_x)\varphi(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} h(\tfrac{x+y}{2}, \xi)\varphi(y)dyd\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

and $V(x)$ is a multiplication operator. We always use the Weyl-quantization to define pseudodifferential operators in this paper (cf. [13]). Let $\mathcal{S}_\tau$ be a strip in $\mathbb{C}^n$ with the width $\tau > 0$:

$$\mathcal{S}_\tau = \{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z| < \tau\}.$$

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Definition 1.1. Let $m, k, \delta > 0$ and $\tau > 0$. $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is said to be an element of the symbol class $S_\delta^{m,k}$ if for any multi-indices α, β,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{k-\delta|\alpha|} \langle \xi\rangle^{m-|\beta|}, \quad x, \xi \in \mathbb{R}^n.$$

Similarly, $a(x,\xi) \in S_{\delta,\tau}^{m,k}$ if $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathcal{S}_\tau)$, a is analytic in ξ, and for any multi-indices α, β,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{k-\delta|\alpha|} \langle \xi\rangle^{m-|\beta|}, \quad x \in \mathbb{R}^n, \xi \in \mathcal{S}_\tau.$$

Assumption A. (i) *For some $m, \delta > 0$ and $\tau > 0$, $h(x,\xi) \in S_{\delta,\tau}^{m,0}$. Moreover, there is $h_0(\xi) \in S_{\delta,\tau}^{m,0}$ (depending only on ξ) such that $h(x,\xi) - h_0(\xi) \in S_{\delta,\tau}^{m,-\varepsilon}$ for some $\varepsilon > 0$.*
(ii) *h is elliptic, i.e., there are $c, C > 0$ such that*

$$|h(x,\xi)| \ge c|\xi|^m - C, \quad x \in \mathbb{R}^n, \xi \in \mathcal{S}_\tau.$$

(iii) *Let $p \ge 2$ and $p > n/m$. $V \in L^p_{loc}(\mathbb{R}^n)$ and*

$$\lim_{|x|\to\infty} V(x) = 0.$$

We note that under the above assumptions, H is a well-defined closed operator with $\mathfrak{D}(H) = H^m(\mathbb{R}^n)$. Here we denote the definition domain of an operator A by $\mathfrak{D}(A)$. Moreover, it is easy to observe that the essential spectrum is given by

$$\sigma_{\text{ess}}(H) = \{h_0(\xi) \mid \xi \in \mathbb{R}^n\}.$$

We also note that we do not suppose H is self-adjoint.

Definition 1.2. Let $E \in \mathbb{C}$. The Agmon metric for $h_0(\xi)$ at the energy E is defined by

$$g^E = \min(\tau, \inf\{|\mathrm{Im}\,\xi| \mid \xi \in \mathcal{S}_\tau,\ h_0(\xi) = E\}).$$

Our first result is the following:

THEOREM 1.1. *Suppose $H = h(x, D_x)+V$ satisfies Assumption* A, *and let $E \in \sigma_d(H)$. Let $\psi \in \mathfrak{D}(H)$ be an E-eigenfunction: $H\psi = E\psi$. Then for any $\kappa < g^E$, there are $C, R > 0$ such that*

$$|\psi(x)| \le Ce^{-\kappa|x|}, \quad x \in \mathbb{R}^n, |x| > R. \tag{1.1}$$

If we apply the above theorem to the Schrödinger operator $H = -\triangle + V(x)$ with real-valued potential $V(x)$ which decays at infinity, we recover the well-known exponential decay estimate due to Agmon (cf. [1]): If $H\psi = E\psi$ with $E < 0$, then for any $\kappa < \sqrt{-E}$,

$$|\psi(x)| \le Ce^{-\kappa|x|}, \quad x \in \mathbb{R}^n,$$

since $g^E = \sqrt{-E}$. This estimate is known to be optimal. More applications are given in Section 3.

Next, we study the semiclassical limit of a class of pseudodifferential operators with a parameter $\hbar > 0$, sometimes called $\hbar$-pseudodifferential operators: $H = h(\hbar; x, \hbar D_x)$. The quantization of a symbol $a(\hbar; x, \xi)$ is the same as before:

$$\begin{aligned} a(\hbar; x, \hbar D_x)\varphi(x) &= (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a(\hbar; \tfrac{x+y}{2}, \hbar\xi)\varphi(y)dyd\xi \\ &= (2\pi\hbar)^{-n} \iint e^{i(x-y)\cdot\xi/\hbar} a(\hbar; \tfrac{x+y}{2}, \xi)\varphi(y)dyd\xi, \end{aligned}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Following the textbook of Robert [21], we use the following symbol class. We can employ other classes of symbols with little modifications.

DEFINITION 1.3. Let $k \in \mathbb{R}$ and $\tau > 0$. We write $a(\hbar; x, \xi) \in S^k$ if $a(\hbar; x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for each $\hbar > 0$, and for any multi-indices α, β,

$$|\partial_x^\alpha \partial_\xi^\beta a(\hbar; x, \xi)| \le C_{\alpha\beta}\hbar^k, \quad x, \xi \in \mathbb{R}^n.$$

Similarly, $a(\hbar; x, \xi) \in S_\tau^k$ if $a(\hbar; x, \xi) \in C^\infty(\mathbb{R}^n \times \mathcal{S}_\tau)$ for each $\hbar > 0$, $a(\hbar; x, \xi)$ is analytic in ξ, and for any multi-indices α, β,

$$|\partial_x^\alpha \partial_\xi^\beta a(\hbar; x, \xi)| \le C_{\alpha\beta}\hbar^k, \quad x \in \mathbb{R}^n, \xi \in \mathcal{S}_\tau.$$

We also write $A = A(\hbar) \in OPS^k$ if $A(\hbar) \in B(L^2(\mathbb{R}^n))$ for each $\hbar > 0$, and there is $a \in S^k$ such that for any $N > 0$,

$$\|A(\hbar) - a(\hbar; x, \hbar D_x)\| \leq C_N \hbar^N, \quad \hbar > 0.$$

Assumption B. *$h(\hbar; x, \xi) \in S^0_\tau$, and there is a principal symbol $h_0(x, \xi) \in S^0_\tau$ such that $h(\hbar; x, \xi) - h_0(x, \xi) \in S^1_\tau$. Moreover, $h_0(x, \xi)$ is elliptic in the following sense: there are $c > 0$ and $R > 0$ such that*

$$|h_0(x, \xi)| \geq c, \quad \text{if } |x| + |\xi| \geq R,\ x \in \mathbb{R}^n,\ \xi \in \mathcal{S}_\tau.$$

We consider zero-energy eigenfunctions of $H = h(\hbar; x, \hbar D_x)$. Let $\mathcal{G}$ be the accessible area for the zero-energy classical particle governed by the principal symbol $h_0(x, \xi)$, i.e.,

$$\mathcal{G} = \{x \in \mathbb{R}^n \mid h_0(x, \xi) = 0 \ \text{ for some } \xi \in \mathbb{R}^n\}.$$

$\mathcal{F} = \mathbb{R}^n \setminus \mathcal{G}$ is the classically forbidden area. We will prove that the zero-energy eigenfunction decays exponentially in $\hbar^{-1}$ as $\hbar \to 0$ in the classically forbidden area $\mathcal{F}$. Note that $\mathcal{G}$ is compact by Assumption B.

Definition 1.4. Let $g(x)$ be a function on $\mathbb{R}^n$ defined by

$$g(x) = \min(\tau, \inf\{|\mathrm{Im}\, \xi| \mid \xi \in \mathcal{S}_\tau, h_0(x, \xi) = 0\}).$$

The Agmon metric ds^2 for $h_0(x, \xi)$ (at energy zero) is the (pseudo-)metric defined by

$$ds^2 = g(x)^2 dx^2.$$

The Agmon distance $\boldsymbol{d}(x, y)$ is the distance generated by ds^2, i.e.,

$$\boldsymbol{d}(x, y) = \inf\left\{ \int_\gamma ds \;\middle|\; \gamma : \text{ piecewise } C^1\text{-path such that } \gamma(0) = x, \gamma(1) = y \right\}.$$

We also write

$$\boldsymbol{d}(x) = \boldsymbol{d}(x, \mathcal{G}) = \inf\{\boldsymbol{d}(x, y) \mid y \in \mathcal{G}\}.$$

Theorem 1.2. *Suppose $h(\hbar; x, \xi)$ satisfies Assumption B, and let $K \subset \mathcal{F}$ be a compact set. Let $\psi \in D(H)$ be a normalized zero-energy eigenfunction of $H = h(\hbar; x, \hbar D_x)$, i.e., $H\psi = 0$ and $\|\psi\| = 1$. Then for any $\varepsilon > 0$ there is $C > 0$ such that*

$$\|e^{(\boldsymbol{d}(x)-\varepsilon)/\hbar}\psi\|_{L^2(K)} \le C, \quad \hbar > 0. \tag{1.2}$$

In particular,

$$\|\psi\|_{L^2(K)} \le Ce^{-(\boldsymbol{d}(K,\mathcal{G})-\varepsilon)/\hbar}, \quad \hbar > 0. \tag{1.3}$$

The next result is a resolvent estimate, that is useful in the analysis of resonances in the semiclassical limit.

Theorem 1.3. *Suppose $h(\hbar; x, \xi)$ satisfies Assumption B, and suppose*

$$|h_0(x, \xi)| \ge c, \quad x, \xi \in \mathbb{R}^n, \tag{1.4}$$

with some $c > 0$. Let K and L be compact subsets of $\mathbb{R}^n$ such that $K \cap L = \emptyset$. Then for any $\varepsilon > 0$, there is $C > 0$ such that

$$\|\chi_K H^{-1} \chi_L\| \le Ce^{-(\boldsymbol{d}(K,L)-\varepsilon)/\hbar} \tag{1.5}$$

if $\hbar$ is sufficiently small, where χ_Ω is the characteristic function of Ω.

The idea of the proof of these theorems is as follows: Let $\rho(x)$ be a real-valued smooth function and we compute $H_\rho = e^{\rho(x)}h(x, D_x)e^{-\rho(x)}$. The main step of the proof is to show that

$$\chi|H_\rho - E|^2\chi \ge c\chi^2, \quad c > 0,$$

in the operator sense, where χ is a suitable characteristic function. Since the principal symbol of $|H_\rho - E|^2$ is given by $|h(x, \xi - i(\nabla\rho)(x)) - E|^2$, we can show our assumptions imply the inequality using the sharp Gårding inequality.

The Agmon method for the exponential decay of eigenfunctions was introduced by Agmon to give precise exponential decay estimates for N-body Schrödinger operators ([1]). The idea was then applied to the semiclassical analysis of Schrödinger operators by Simon, Helffer, Sjöstrand and others ([22], [9], [10], [6], [5]. See also [12] and references therein). A generalization to other operators (in the semiclassical analysis) was studied by Helffer and Sjöstrand in their papers on the Harper operator ([11]), and it turned out that the method is quite effective tool in the study of phase space tunneling phenomena ([15], [3], [17], [18], [19], [20]). Note that the method employed in [15] and [18] might appear to be different from the Agmon method, but the essential idea is closely related. In the applications to phase space tunneling phenomena, it is often the case that we have to study the behavior of eigenfunctions (or resolvents) for pseudodifferential operators quite different from Schrödinger operators, and generalizations of the Agmon method to such operators are necessary (see, e.g., [17], [19], [20]).

One purpose of this paper is to present a general theory of the Agmon method in the semiclassical limit, which is applicable to the analysis of phase space tunneling phenomena. We study exponential estimates on eigenfunctions and resolvents in the semiclassical limit in Section 4. The same idea applies to obtain exponential decay estimates of eigenfunctions to a rather large class of elliptic pseudodifferential operators, and they are discussed in Section 3. We prepare several abstract (operator-theoretical) theorems in Section 2.

2. Abstract Theory

In this section, we prove several simple operator-theoretical results, which are generalizations of calculus of the Agmon method. Throughout this section we suppose the following:

Assumption C. *$\mathfrak{H}$ is a Hilbert space, and H is a closed operator on $\mathfrak{H}$. A and X (or X_1, X_2) are bounded self-adjoint operators on $\mathfrak{H}$ and they commute each other, i.e., $[A, X] = 0$. Moreover, $[H, X]$ is H-bounded.*

Note that $[H, X]$ is defined at first as a quadratic form on $\mathfrak{D}(H)$. Operators A and X model multiplication operators by bounded smooth functions. H models pseudodifferential operator we are interested in. The last condition of Assumption C implies that X maps $\mathfrak{D}(H)$ into itself. For $\theta \in \mathbb{C}$, we denote

$$H_\theta = e^{i\theta A} H e^{-i\theta A}.$$

In applications, usually we set $\theta = -i\tau$ with $\tau > 0$.

The following theorem is a straightforward generalization of the original Agmon estimate.

THEOREM 2.1. *Let $\theta \in \mathbb{C}$ and let $H\varphi = 0$ with $\varphi \in \mathfrak{D}(H)$. If*

$$\mathrm{Re}\,[\chi H_\theta \chi] \geq c_1^2\,\chi^2 \tag{2.1}$$

for some $c_1 > 0$, then

$$\|e^{i\theta A}\chi\varphi\| \leq c_1^{-2}\|e^{i\theta A}[H,\chi]\varphi\|. \tag{2.2}$$

PROOF. If we set $\psi_\theta = e^{i\theta A}\chi\varphi$, then $\psi_\theta \in \mathfrak{D}(H_\theta)$. By the assumption, we have

$$\begin{aligned}\mathrm{Re}\,\langle\psi_\theta, H_\theta\psi_\theta\rangle &= \mathrm{Re}\,\langle e^{i\theta A}\chi\varphi, H_\theta e^{i\theta A}\chi\varphi\rangle = \mathrm{Re}\,\langle e^{i\theta A}\varphi, (\chi H_\theta\chi)e^{i\theta A}\varphi\rangle \\ &\geq c_1^2\langle e^{i\theta A}\varphi, \chi^2 e^{i\theta A}\varphi\rangle = c_1^2\|\psi_\theta\|^2.\end{aligned}$$

On the other hand, since $H\varphi = 0$, we also have

$$H_\theta\psi_\theta = e^{i\theta A}H\chi\varphi = e^{i\theta A}[H,\chi]\varphi, \tag{2.3}$$

and hence

$$\mathrm{Re}\,\langle\psi_\theta, H_\theta\psi_\theta\rangle \leq \|\psi_\theta\|\cdot\|e^{i\theta A}[H,\chi]\varphi\|.$$

Combining these inequalities, we obtain (2.2). $\square$

The next variation of the Agmon estimate seems to be more versatile in applications.

THEOREM 2.2. *Let $\theta \in \mathbb{C}$ and let $H\varphi = 0$ with $\varphi \in \mathfrak{D}(H)$. If*

$$\chi|H_\theta|^2\chi \geq c_2^2\,\chi^2 \tag{2.4}$$

for some $c_2 > 0$, then

$$\|e^{i\theta A}\chi\varphi\| \leq c_2^{-1}\|e^{i\theta A}[H,\chi]\varphi\|. \tag{2.5}$$

Proof. We set $\psi_\theta = e^{i\theta A}\chi\varphi$ as in the last proof. Then

$$\|H_\theta\psi_\theta\|^2 = \langle e^{i\theta A}\varphi, \chi|H_\theta|^2\chi e^{i\theta A}\varphi\rangle \geq c_2^2\langle e^{i\theta A}\varphi, \chi^2 e^{i\theta A}\varphi\rangle = \|\psi_\theta\|^2.$$

Combining this with (2.3), we obtain (2.5). $\square$

The next simple lemma is useful to estimate the right hand side terms of (2.2) and (2.5).

Assumption D. *$0 \leq \chi \leq 1$ and $A\overline{\chi} = 0$, where $\overline{\chi} = 1 - \chi$.*

Lemma 2.3. *Suppose Assumptions C and D. Then*

$$\|e^{i\theta A}[H, \chi]\varphi\| \leq \|H_\theta\overline{\chi}\varphi\| + \|H\varphi\|, \quad \varphi \in \mathfrak{D}(H). \tag{2.6}$$

Proof. Since $e^{i\theta A}\overline{\chi} = \overline{\chi}$, we have

$$\begin{aligned} e^{i\theta A}[H, \chi] &= -e^{i\theta A}[H, \overline{\chi}] = -e^{i\theta A}H\overline{\chi} + e^{i\theta A}\overline{\chi}H \\ &= -(e^{i\theta A}He^{-i\theta A})(e^{i\theta A}\overline{\chi}) + \overline{\chi}H = -H_\theta\overline{\chi} + \overline{\chi}H. \ \square \end{aligned}$$

Summarizing these, we have the following result, which will be used later.

Theorem 2.4. *Suppose Assumptions C and D. Let $\tau > 0$ and let $H\varphi = 0$ with $\varphi \in \mathfrak{D}(H)$. If*

$$\chi|H_{(-i\tau)}|^2\chi \geq c_2^2\,\chi^2 \tag{2.7}$$

for some $c_2 > 0$, then

$$\|e^{\tau A}\chi\varphi\| \leq c_2^{-1}\|H_{(-i\tau)}\overline{\chi}\varphi\|. \tag{2.8}$$

Now we consider generalizations of resolvent estimates due to Briet, Combes and Duclos [5]. We call them *BCD-type resolvent estimates.* We use three bounded self-adjoint operators A, χ_1 and χ_2, that commute each other. We write $\rho(H)$ for the resolvent set of H.

THEOREM 2.5. *Let $\tau > 0$ and suppose $0 \in \rho(H)$. Suppose moreover*

$$\chi_1 A \chi_1 \geq d\chi_1^2, \quad \chi_2 A = 0, \tag{2.9}$$

$$|H_{(-i\tau)}|^2 \geq c_3^2, \quad (i.e., \|(H_{(-i\tau)})^{-1}\| \leq c_3^{-1}). \tag{2.10}$$

Then

$$\|\chi_1 H^{-1} \chi_2\| \leq c_3^{-1} e^{-\tau d}. \tag{2.11}$$

PROOF. At first we note $e^{\tau A}\chi_2 = \chi_2$. On the other hand,

$$\begin{aligned} \frac{d}{dt}(\chi_1 e^{tA} \chi_1) &= \chi_1 A e^{tA} \chi_1 = e^{tA/2}(\chi_1 A \chi_1) e^{tA/2} \\ &\geq d e^{tA/2} \chi_1^2 e^{tA/2} = d\chi_1 e^{tA} \chi_1, \end{aligned}$$

and hence

$$\chi_1 e^{tA} \chi_1 \geq e^{td} \chi_1^2, \quad \chi_1 e^{-tA} \chi_1 \leq e^{-td} \chi_1^2$$

for $t \geq 0$. We also have

$$\chi_1 H^{-1} \chi_2 = \chi_1 e^{-\tau A} (H_{(-i\tau)})^{-1} e^{\tau A} \chi_2 = \chi_1 e^{-\tau A} (H_{(-i\tau)})^{-1} \chi_2.$$

Combining these, we compute

$$\begin{aligned} \|\chi_1 H^{-1} \chi_2 \varphi\|^2 &= \langle \chi_1 e^{-\tau A} (H_{(-i\tau)})^{-1} \chi_2 \varphi, \chi_1 e^{-\tau A} (H_{(-i\tau)})^{-1} \chi_2 \varphi \rangle \\ &= \langle (H_{(-i\tau)})^{-1} \chi_2 \varphi, (\chi_1 e^{-2\tau A} \chi_1)(H_{(-i\tau)})^{-1} \chi_2 \varphi \rangle \\ &\leq e^{-2\tau d} \|(H_{(-i\tau)})^{-1} \chi_2 \varphi\|^2 \leq c_3^{-2} e^{-2\tau d} \|\varphi\|^2, \end{aligned}$$

where $\varphi \in \mathfrak{H}$. This completes the proof. □

THEOREM 2.6. *Let $\tau > 0$ and suppose $0 \in \rho(H)$. Suppose moreover*

$$\chi_1 A \chi_1 \geq d\chi_1^2, \quad \chi_2 A = 0, \tag{2.12}$$

$$\operatorname{Re} H \geq c_4 - c_5 \chi_2, \quad \operatorname{Re} H_{(-i\tau)} \geq c_4 - c_5 \chi_2, \tag{2.13}$$

$$\|H^{-1}\| \leq c_6^{-1} e^{\tau d}. \tag{2.14}$$

for some $c_4, c_5, c_6 > 0$. Then

$$\|\chi_1 H^{-1}\| \leq c_4^{-1}(1 + c_5 c_6^{-1}). \tag{2.15}$$

Proof. By (2.13), we have

$$\|(H+c_5\chi_2)^{-1}\| \le c_4^{-1}, \quad \|(H_{(-i\tau)}+c_5\chi_2)^{-1}\| \le c_4^{-1}.$$

By the second resolvent equation, we learn

$$\chi_1 H^{-1} = \chi_1(H+c_5\chi_2)^{-1} + c_5\chi_1(H+c_5\chi_2)^{-1}\chi_2 H^{-1}.$$

Now we apply Theorem 2.5 to $H+c_5\chi_2$ to obtain

$$\begin{aligned} \|\chi_1 H^{-1}\| &\le \|\chi_1(H+c_5\chi_2)^{-1}\| + c_5\|\chi_1(H+c_5\chi_2)^{-1}\chi_2\|\,\|H^{-1}\| \\ &\le c_4^{-1} + c_5\cdot c_4^{-1}e^{-\tau d}\cdot c_6^{-1}e^{\tau d} = c_4^{-1}(1+c_5c_6^{-1}). \ \square \end{aligned}$$

3. Exponential Decay Estimates at Infinity

In this section, we give a proof of Theorem 1.1 and discuss several examples. Let $H = h(x,D_x)+V(x)$ as in Section 1, and we assume that Assumption A is satisfied throughout this section. We compute $e^{\rho(x)}He^{-\rho(x)}$ for a smooth function $\rho(x)$. Since $e^{\rho(x)}V(x)e^{-\rho(x)} = V(x)$, it is sufficient to consider $e^{\rho(x)}h(x,D_x)e^{-\rho(x)}$.

Let $\rho(x)$ be a C^∞-class real-valued bounded function such that for any multi-index α,

$$|\partial_x^\alpha \rho(x)| \le C_\alpha \langle x\rangle^{1-|\alpha|}, \quad x\in\mathbb{R}^n, \tag{3.1a}$$

$$\sup|\nabla\rho(x)| < \tau. \tag{3.1b}$$

Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ (Schwartz class), and we compute $e^{\rho(x)}h(x,D_x)e^{-\rho(x)}\varphi$ as follows.

$$\begin{aligned} &e^{\rho(x)}h(x,D_x)e^{-\rho(x)}\varphi(x) \\ &\quad = (2\pi)^{-n}\iint e^{i(x-y)\cdot\xi+(\rho(x)-\rho(y))}h(\tfrac{x+y}{2},\xi)\varphi(y)\,dyd\xi \\ &\quad = (2\pi)^{-n}\iint e^{i(x-y)\cdot(\xi-i\Phi(x,y))}h(\tfrac{x+y}{2},\xi)\varphi(y)\,dyd\xi, \end{aligned} \tag{3.2}$$

where

$$\Phi(x,y) = \int_0^1 (\nabla\rho)(y+t(x-y))\,dt, \quad x,y\in\mathbb{R}^n.$$

Note that $|\Phi(x,y)| \le \sup|\nabla\rho(x)| < \tau$. Since $h(x,\xi)$ is analytic with respect to ξ in $\mathcal{S}_\tau$, we can use Cauchy's theorem to change the integral plane in $\mathcal{S}_\tau$ to observe

$$(\text{The RHS of (3.2)}) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} h(\tfrac{x+y}{2}, \xi + i\Phi(x,y))\varphi(y)dyd\xi.$$

Thus $e^{\rho(x)}h(x,D_x)e^{-\rho(x)}$ is a pseudodifferential operator with a double symbol defined by

$$\tilde{h}_\rho(x,\xi,y) = h(\tfrac{x+y}{2}, \xi + i\Phi(x,y)).$$

It is easy to see that for any multi-indices α, β,

$$|\partial_x^\alpha \partial_y^\beta \Phi(x,y)| \le C_{\alpha\beta}\langle x\rangle^{-\delta|\alpha|}\langle y\rangle^{-\delta|\beta|}\langle x-y\rangle^{\delta|\alpha+\beta|}, \quad x,y\in\mathbb{R}^n,$$

where $0<\delta\le 1$ is the constant in Definition 1.1. We note that the constant $C_{\alpha\beta}$ depends only on the constants in (3.1a). Hence the double symbol $\tilde{h}_\rho(x,\xi,y)$ satisfies

$$\begin{aligned}&|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma \tilde{h}_\rho(x,\xi,y)|\\ &\qquad \le C_{\alpha\beta\gamma}\langle x\rangle^{-\delta|\alpha|}\langle y\rangle^{-\delta|\beta|}\langle x-y\rangle^{\delta|\alpha+\beta|}\langle\xi\rangle^{m-|\gamma|}, \quad x,y,\xi\in\mathbb{R}^n\end{aligned}$$

for any α, β and γ. It follows from this estimate that there is a simplified symbol $h_\rho(x,\xi)\in S_\delta^{m,0}$ such that $h_\rho(x,D_x) = e^{\rho(x)}h(x,D_x)e^{-\rho(x)}$. The asymptotic expansion of $h_\rho(x,\xi)$ is given by

$$h_\rho(x,\xi) \sim \sum_{k=0}^\infty \frac{1}{k!}[(-i/2)\partial_\xi\cdot(\partial_x-\partial_y)]^k \tilde{h}_\rho(x,\xi,y)|_{y=x}.$$

The first term is $h(x,\xi+i\nabla\rho(x))$ since $\Phi(x,x)=\nabla\rho(x)$, and the second term vanishes since $\tilde{h}_\rho(x,\xi,y)=\tilde{h}_\rho(y,\xi,x)$. Thus we have proved the following:

Lemma 3.1. *Suppose $h(x,\xi)\in S_{\delta,\tau}^{m,0}$ and suppose $\rho(x)\in C^\infty(\mathbb{R}^n)$ satisfies (3.1). Then $e^{\rho(x)}h(x,D_x)e^{-\rho(x)}$ is a pseudodifferential operator with the symbol $h_\rho(x,\xi)\in S_\delta^{m,0}$ and*

$$h_\rho(x,\xi) - h(x,\xi+i\nabla\rho(x)) \in S_\delta^{m-2,-2\delta}. \tag{3.3}$$

Remark. The seminorms of $h_\rho(x,\xi)$, remainder estimates of (3.3), etc., depend on $\rho(x)$ only through constants C_α in (3.1a). In particular, these estimates are independent of $\sup|\rho(x)|$.

Let $E \in \mathbb{C}$ be as in Theorem 1.1. Now we choose $\rho(x)$ so that

$$\sup|\nabla\rho(x)| = \kappa < g^E. \tag{3.4}$$

Then by the definition of g^E (Definition 1.2) and the ellipticity of $h_0(\xi)$, there is $a > 0$ such that

$$|h_0(\xi + i\nabla\rho(x)) - E| \geq a\langle\xi\rangle^m, \quad x,\xi \in \mathbb{R}^n.$$

Since $h(x,\xi) - h_0(\xi) \in S^{m,-\varepsilon}_{\delta,\tau}$, this and Lemma 3.1 imply

$$|h_\rho(x,\xi) - E| \geq (a - C\langle x\rangle^{-\varepsilon'})\langle\xi\rangle^m, \quad x,\xi \in \mathbb{R}^n$$

with some $C > 0$, where $\varepsilon' = \min(\varepsilon, 2\delta)$. We apply the sharp Gårding inequality (and the composition formula of pseudodifferential operators), and we have

$$|e^{\rho(x)}(h(x,D_x) - E)e^{-\rho(x)}|^2 = |h_\rho(x,D_x) - E|^2 \geq b\langle D_x\rangle^{2m} - f(x) \tag{3.5}$$

with some $b > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$.

Let $\chi^0(x)$ be a C^∞-function on $\mathbb{R}$ such that

$$0 \leq \chi^0(x) \leq 1, \quad \chi^0(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 & \text{if } x \geq 2, \end{cases}$$

and for $R > 0$, we set

$$\chi_R(x) = \chi^0(|x|/R), \quad x \in \mathbb{R}^n.$$

It follows from (3.5) that if R is sufficiently large,

$$\chi_R|e^{\rho(x)}(h(x,D_x) - E)e^{-\rho(x)}|^2\chi_R \geq b\,\chi_R\langle D_x\rangle^{2m}\chi_R.$$

Moreover, by the assumption: $\lim V(x) = 0$, we also have

$$\chi_R|e^{\rho(x)}(H - E)e^{-\rho(x)}|^2\chi_R \geq c\,\chi_R\langle D_x\rangle^{2m}\chi_R \geq c\,\chi_R^2$$

taking R larger if necessary, with some $c > 0$. Thus we have shown:

LEMMA 3.2. *Suppose $\rho \in C^\infty(\mathbb{R}^n)$ satisfies* (3.1) *and* (3.4). *If $R > 0$ is sufficiently large, there is $c > 0$ such that*

$$\chi_R |e^{\rho(x)}(H - E)e^{-\rho(x)}|^2 \chi_R \geq c\,\chi_R^2. \tag{3.6}$$

The constant c depends on ρ only through the constants C_α in (3.1a) *and κ in* (3.4).

We then construct $\rho(x)$ as follows (cf. [1]). We set

$$\mu_R(x) = \int_0^{|x|} \chi^0(t - 2R)dt,$$

so that

$$|\nabla \mu_R(x)| \leq 1, \quad \mu_R(x) = \begin{cases} 0 & \text{if } |x| \leq 2R + 1, \\ |x| - a & \text{if } |x| \geq 2R + 2, \end{cases}$$

where $a > 0$ is a constant. We also set

$$\nu(x) = \int_0^{|x|} (1 - \chi^0(t))dt,$$

so that

$$|\nabla \nu(x)| \leq 1, \quad \nu(x) = \begin{cases} |x| & \text{if } |x| \leq 1, \\ b & \text{if } |x| \geq 2, \end{cases}$$

where $1 < b < 2$ is another constant. We note that μ_R and ν are smooth functions. Let $R > 0$ be the constant in Lemma 3.2, and fix $0 < \kappa < g^E$ arbitrarily. For $M >> R$ we set

$$\rho_M(x) = \kappa M \cdot \nu(M^{-1}\mu_R(x)), \quad x \in \mathbb{R}^n.$$

Then $\rho_M(x)$ is a bounded smooth function. Moreover, it is simple computation to see that ρ_M satisfies (3.1) uniformly in M. We note

$$\begin{aligned} &|\nabla \rho_M(x)| \leq \kappa < g^E, \\ &\rho_M(x) = \kappa(|x| - a) \quad \text{if } 2R + 2 \leq |x| \leq M - a, \\ &\rho_M(x) \cdot (1 - \chi_R(x)) = 0. \end{aligned}$$

We apply Theorem 2.4 to $H-E$ with $\chi=\chi_R(x)$, $A=\rho_M(x)$ and $\tau=1$. If $H\psi=E\psi$, $\psi\in\mathfrak{D}(H)$, we learn

$$\|e^{\rho_M(x)}\chi_R(x)\psi\|\le c^{-1}\|(h_{\rho_M}(x,D_x)+V)(1-\chi_R)\psi(x)\|\le C\|\psi\| \tag{3.7}$$

by virtue of Lemma 3.2, where $C>0$ is independent of M. As $M\to\infty$, $\rho_M(x)\nearrow\kappa\mu_R(x)$, and hence $e^{\kappa(|x|-a)}\chi_R\varphi\in L^2$. Summing up these, we have proved the following:

LEMMA 3.3. *Let $H\psi=E\psi$, $\psi\in\mathfrak{D}(H)$, and let $0<\kappa<g^E$. Then $e^{\kappa|x|}\psi(x)\in L^2(\mathbb{R}^n)$.*

The above lemma implies $\psi(x)=o(e^{-\kappa|x|})$ in L^2-sense. Now Theorem 1.1 follows by the standard elliptic estimate, which utilizes the L^p-boundedness of the parametrix of H.

Example 3.1. Let $H=-\triangle+V(x)$ on $L^2(\mathbb{R}^n)$, and suppose $V(x)$ is a complex-valued L^p_{loc}-function where $p\ge 2$, $p>n/2$. Suppose moreover that $V(x)\to 0$ as $|x|\to\infty$. Let $E\in\sigma_{\mathrm{d}}(H)$. Then $g^E=|\mathrm{Im}\,\sqrt{E}|$. Hence, by Theorem 1.1, if $H\psi=E\psi$, for any $\kappa<|\mathrm{Im}\,\sqrt{E}|$ we have

$$|\psi(x)|\le Ce^{-\kappa|x|},\quad x\in\mathbb{R}^n.$$

Example 3.2. Let $H=\triangle^2+V(x)$ on $L^2(\mathbb{R}^n)$, and suppose $V(x)$ is a real-valued L^p_{loc}-function, where $p\ge 2$, $p>n/4$. $V(x)$ is supposed to decay at infinity as before. Let $E<0$ be an eigenvalue. Then by simple computations, we have $g^E=\mathrm{Im}\,E^{1/4}=(-E)^{1/4}/\sqrt{2}$.

Example 3.3. (cf. [8]) Let $H=\sqrt{-\triangle+m^2}+V(x)$ be the so-called relativistic Schrödinger operator with mass $m>0$. We suppose $V\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with $p\ge 2$, $p>n$, and $\lim V(x)=0$. The essential spectrum of H is given by $[m,\infty)$. Let $E<m$ be an eigenvalue. Then the Agmon metric is given by

$$g^E=\begin{cases}\sqrt{m^2-E^2} & \text{if } 0<E<m,\\ m & \text{if } E\le 0.\end{cases}$$

4. Exponential Decay Estimates in the Semiclassical Limit

In this section we consider the $\hbar$-pseudodifferential operator

$$H\varphi(x) = h(\hbar; x, \hbar D_x)\varphi(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

and suppose $h(\hbar; x, \xi)$ satisfies Assumption B throughout this section.

Let $\rho(x)$ be a real-valued C^∞-function such that for any multi-index α,

$$|\partial_x^\alpha \rho(x)| \le C_\alpha, \quad x \in \mathbb{R}^n, \tag{4.1a}$$

$$\sup |\nabla \rho(x)| \le \tau, \tag{4.1b}$$

and we compute $H_\rho = e^{\rho(x)/\hbar} H e^{-\rho(x)/\hbar}$. By almost exactly the same computation as in the proof of Lemma 3.1, we can prove the following.

Lemma 4.1. *Suppose $h(\hbar; x, \xi) \in S^0_\tau$ and suppose $\rho(x) \in C^\infty(\mathbb{R}^n)$ satisfies (4.1). Then $H_\rho = e^{\rho(x)/\hbar} H e^{-\rho(x)/\hbar} \in OPS^0$. Moreover, if we let $h_\rho(\hbar; x, \xi)$ be the symbol of H_ρ, then*

$$h_r(\hbar; x, \xi) - h_0(x, \xi + i\nabla\rho(x)) \in S^1. \tag{4.2}$$

For $\beta > 0$, we set

$$\mathcal{G}_\beta = \{x \in \mathbb{R}^n \mid \boldsymbol{d}(x, \mathcal{G}) \le \beta\}.$$

In order to prove Theorem 1.2, we construct $\rho(x)$ so that

$$\rho(x) = 0 \qquad \text{if } x \in \mathcal{G}_{3\delta}, \tag{4.3a}$$

$$\rho(x) \ge \boldsymbol{d}(x, \mathcal{G}) - \varepsilon, \qquad \text{if } x \in K, \tag{4.3b}$$

$$|\nabla\rho(x)| \le g(x) - \gamma, \qquad \text{if } x \notin \mathcal{G}_\delta, \tag{4.3c}$$

$$\rho(x) = \text{const.} \qquad \text{if } |x| > R, \tag{4.3d}$$

with some $\gamma, \delta > 0$ and $R > 0$. We sketch the construction of $\rho(x)$. At first, we set

$$\rho_1(x) = \begin{cases} 0 & \text{if } x \in \mathcal{G}_{4\delta} \\ (1-\delta_1)(\boldsymbol{d}(x) - 4\delta) & \text{if } x \notin \mathcal{G}_{4\delta} \text{ and } (1-\delta_1)(\boldsymbol{d}(x) - 4\delta) < M \\ M & \text{if } (1-\delta_1)(\boldsymbol{d}(x) - 3\delta) \ge M. \end{cases}$$

with small $\delta_1, \delta > 0$ and large $M > 0$. $\rho_1(x)$ satisfies

$$\begin{aligned} \rho_1(x) &= 0 \qquad \text{if } x \in \mathcal{G}_{4\delta}, \\ \rho_1(x) &\geq \boldsymbol{d}(x, \mathcal{G}) - \varepsilon/2, \qquad \text{if } x \in K, \\ |\nabla \rho_1(x)| &\leq g(x) - 2\gamma, \qquad \text{if } x \notin \mathcal{G}_{\delta}, \\ \rho_1(x) &= \text{const.} \qquad \text{if } |x| > R, \end{aligned}$$

provided γ, δ, δ_1 and R are chosen suitably. Note that since $\boldsymbol{d}(x)$ is Lipshiz continuous, $\nabla \rho_1(x)$ is well-defined almost everywhere. Then we use mollifier to find $\rho(x) \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} &\operatorname{supp} \rho(x) \subset (\delta\text{-neighborhood of } \operatorname{supp} \rho_1(x)), \\ &|\rho(x) - \rho_1(x)| \leq \varepsilon/2, \\ &|\nabla \rho(x) - \nabla \rho_1(x)| \leq \gamma. \end{aligned}$$

Then it is easy to see that $\rho(x)$ satisfies (4.3). Note that (4.3d) is satisfied since $\boldsymbol{d}(x) \to \infty$ as $|x| \to \infty$ under Assumption B.

We next choose a smooth cut-off function $\chi(x)$. Let $\chi_1(x) \in C^\infty(\mathbb{R}^n)$ such that

$$0 \leq \chi_1(x) \leq 1, \quad \chi_1(x) = \begin{cases} 0 & \text{if } x \in \mathcal{G}_\delta, \\ 1 & \text{if } x \notin \mathcal{G}_{2\delta}, \end{cases}$$

and let $\chi(x) \in C^\infty(\mathbb{R}^n)$ such that

$$0 \leq \chi(x) \leq 1, \quad \chi(x) = \begin{cases} 0 & \text{if } x \in \mathcal{G}_{2\delta}, \\ 1 & \text{if } x \notin \mathcal{G}_{3\delta}. \end{cases}$$

We note $\chi_1(x)\chi(x) = \chi(x)$.

By virtue of Lemma 4.1, the principal symbol of $\chi_1 |H_\rho|^2 \chi_1$ is given by

$$\chi_1(x) |h_0(x, \xi + i\nabla \rho(x))|^2 \chi_1(x).$$

If $x \in \operatorname{supp} \chi_1$, then $h_0(x, \xi + i\nabla\rho(x)) \neq 0$ by the definition of the Agmon metric and (4.3c). Moreover, $|h_0(\xi + i\nabla\rho(x))| \geq c > 0$ if $|x|$ is large, by Assumption B. Hence there is $c_1 > 0$ such that

$$\chi_1(x) |h_0(x, \xi + i\nabla \rho(x))|^2 \chi_1(x) \geq c_1 \chi_1(x)^2, \quad x, \xi \in \mathbb{R}^n.$$

Combining this with the sharp Gårding inequality (see, e.g., [13] Theorem 18.6.7 or [16] Theorem 3.5.2), we learn

$$\chi_1|H_\rho|^2\chi_1 \geq c_1\chi_1^2 - C\hbar, \quad \hbar > 0,$$

with some $C > 0$. This implies

$$\chi|H_\rho|^2\chi \geq (c_1 - C\hbar)\chi^2 \geq (c_1/2)\chi^2$$

if $\hbar$ is sufficiently small. Thus we have shown:

LEMMA 4.2. *Let $\rho(x)$ and $\chi(x)$ as above. Then there is $c > 0$ and $\hbar_0 > 0$ such that*

$$\chi|e^{\rho/\hbar}He^{-\rho/\hbar}|^2\chi \geq c\,\chi^2, \tag{4.4}$$

if $0 < \hbar \leq \hbar_0$.

PROOF OF THEOREM 1.2. We set A be the multiplication operator by $\rho(x)/\hbar$. Then it is easy to show A and χ satisfy Assumptions C and D in Section 2. Now we can apply Theorem 2.4 by virtue of Lemma 4.2 to obtain

$$\|e^{\rho(x)/\hbar}\psi\| \leq \|H_\rho(1-\chi)\psi\|, \quad 0 < \hbar \leq \hbar_0, \tag{4.5}$$

where ψ is an eigenfunction of H, i.e., $H\psi = 0$. Since $H_\rho \in OPS^0$, it is bounded uniformly in $\hbar$, and hence the right hand side of (4.5) is bounded by $C\|\psi\|$. This implies (1.2) since $\rho(x) \geq \boldsymbol{d}(x) - \varepsilon$. $\square$

PROOF OF THEOREM 1.3. The proof is similar to that of Theorem 1.2, and we only sketch it. In fact, the proof is slightly simpler than Theorem 1.2. For $\beta > 0$, we write

$$K_\beta = \{x \mid \boldsymbol{d}(x, K) \leq \beta\}, \quad L_\beta = \{x \mid \boldsymbol{d}(x, L) \leq \beta\}.$$

We construct $\rho(x)$ such that

$$\begin{aligned} \rho(x) &= 0, && \text{if } x \in L_\delta \\ \rho(x) &\geq \boldsymbol{d}(x, L) - \varepsilon - \delta, && \text{if } x \in K_\delta \\ |\rho(x)| &\leq g(x) - \gamma, && \text{for } x \in \mathbb{R}^n \\ \rho(x) &= \text{const.} && \text{if } |x| > R \end{aligned}$$

with some γ, δ and $R > 0$. We use χ_1 and $\chi_2 \in C^\infty(\mathbb{R}^n)$ such that

$$0 \le \chi_1(x) \le 1, \quad \chi_1(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K_\delta, \end{cases}$$

and

$$0 \le \chi_2(x) \le 1, \quad \chi_2(x) = \begin{cases} 1 & \text{if } x \in L, \\ 0 & \text{if } x \notin L_\delta. \end{cases}$$

We set A is the multiplication operator by $\rho(x)/\hbar$ as before. Then it is easy to see that

$$\chi_1 A \chi_1 \ge (\boldsymbol{d}(K,L) - \varepsilon)\chi_1^2, \quad \chi_2 A \chi_2 = 0.$$

On the other hand, by using the sharp Gårding inequality again, we learn

$$|e^{\rho/\hbar} H e^{-\rho/\hbar}|^2 \ge c > 0$$

if $\hbar$ is sufficiently small. Now we apply Theorem 2.5 to obtain

$$\|\chi_1 H^{-1} \chi_2\| \le C e^{-(\boldsymbol{d}(K,L)-\varepsilon)/\hbar}$$

with some $C > 0$. Theorem 1.3 now follows immediately. □

As an application of Theorem 2.6 to $\hbar$-pseudodifferential operators, we can also prove the following theorem, which is applicable to semiclassical resonance theory.

THEOREM 4.3. *Suppose $h(\hbar; x, \xi)$ satisfies Assumption B, and suppose*

$$\mathrm{Re}\, h_0(x,\xi) \ge c_1 - c_2 \chi_K(x), \quad x, \xi \in \mathbb{R}^n,$$

where K is a compact set in $\mathbb{R}^n$, and $c_1, c_2 > 0$. Let $L \subset \mathbb{R}^n$ be another compact set such that $K \cap L = \emptyset$, and let $\varepsilon > 0$. If $0 \in \rho(H)$ and

$$\|H^{-1}\| \le C e^{(\boldsymbol{d}(K,L)-\varepsilon)/\hbar}, \quad \hbar > 0,$$

then $\|\chi_L H^{-1}\|$ is uniformly bounded for $\hbar > 0$.

The proof is similar to Theorem 1.3, and we omit it.

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