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The First Eigenvalue of the Laplacian on p-Forms and Metric Deformations

By Junya Takahashi

Abstract. We prove that the limits of the first eigenvalues of functions and 1-forms for modified Gentile-Pagliara's metric deformation are both 0. It essentially means that this deformation is not a counter example of Berger's problem for 1-forms.

1. Introduction

Let (M, g) be an m-dimensional connected compact oriented Riemannian manifold without boundary. The spectrum of the Laplacian $\Delta = d\delta + \delta d$ acting on p-forms on M consists of only non-negative eigenvalues. We denote the *positive* eigenvalues by

$$0 < \lambda_1^{(p)}(M, g) \leq \lambda_2^{(p)}(M, g) \leq \cdots \leq \lambda_i^{(p)}(M, g) \leq \cdots.$$

The existence of 0-eigenvalue on p-forms is determined only by the p-th Betti number $\beta_p(M)$, independently of the metric g by the Hodge theory. If 0-eigenvalue exists, we set $\lambda_0^{(p)} = 0$. As usual for $p = 0$ i.e. for functions we write $\lambda_i = \lambda_i^{(0)}$.

In 1970 J. Hersch [H-70] proved that for every Riemannian metric g on 2-sphere S^2 with volume $= 1$, we have

$$\lambda_1(S^2, g) \leq 8\pi.$$

In 1980 P. Yang and S. T. Yau [YY-80] extended it for a connected closed oriented surface S with genus γ. That is, for every Riemannian metric g on S with volume $= 1$, we have

$$\lambda_1(S, g) \leq 8\pi(\gamma + 1).$$

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In 1973 from Hersch's inequality M.Berger ([Be-73] p. 138) rased the question whether the following statement holds or not. That is,

"Does there exist a constant $C(M) > 0$ such that for every Riemannian metric g on M with volume $= 1$, $\lambda_1(M,g) \leq C(M)$ follows?"

After many negative examples were found between 1979 and 1983 (e.g. [U-79], [T-79], [M-80], [MU-80], [BB-82], [Bl-83]), B.Colbois and J.Dodziuk [CD-94], [D-94] proved that for $m \geq 3$ there does not exist such constant $C(M)$ in 1994. Moreover, for $2 \leq p \leq m-2$, $m \geq 4$ G. Gentile and V. Pagliara [GP-95] also showed that the similar statement is false in 1995. Namely, they constructed a metric deformation $\{\bar{g}_t\}_{t\geq 1}$ with volume $= 1$ such that $\lambda_1^{(p)}(\bar{g}_t) \to \infty$ as $t \to \infty$ for all $p = 2, \cdots, m-2$. Their metric deformation is as follows. We take a connected sum of a given manifold and a sphere and lengthen the sphere part like a cylinder. Finally we regularize the volume to be 1.

For 1-forms, however, we have not yet known whether the above statement is affirmative or not. Notice that the Poincaré duality implies $\lambda_1^{(1)} = \lambda_1^{(m-1)}$.

By modifying the above $\bar{g}_t$ (cf. Sect. 2), we have

THEOREM 1.1. *Let M be an m-dimensional connected compact oriented manifold without boundary. If $m \geq 2$, there exists a metric deformation $\{\bar{g}_t\}_{t\geq 1}$ such that* $\mathrm{vol}(M, \bar{g}_t) \equiv 1$ *and*

$$\lim_{t\to\infty} \lambda_1^{(p)}(M, \bar{g}_t) = \begin{cases} 0 & (p = 0, 1, m-1, m), \\ \infty & (p = 2, 3, \cdots, m-2,\ m \geq 4). \end{cases}$$

We call the metric deformation $\bar{g}_t$ in Theorem 1.1 modified Gentile-Pagliara's metric deformation. To construct it, the following is essential. We take m, n-dimensional $(m, n \geq 1)$ connected compact oriented Riemannian manifolds (M, g), (N, h) without boundaries. Then on the product manifold $L = M \times N$ set

$$G_t := t^{\frac{1}{m}} g \oplus t^{-\frac{1}{n}} h \ (t > 0).$$

THEOREM 1.2. *We have* $\mathrm{vol}(L, G_t) \equiv$*constant for every t, and*

$$\lim_{t\to\infty} \lambda_1^{(p)}(L, G_t) = \begin{cases} \infty\ , & \text{if } m < p < n,\ H^k(N; \mathbf{R}) = 0\ (p - m \leq k \leq p), \\ 0\ , & \text{otherwise.} \end{cases}$$

Especially we remark that $\lim_{t\to\infty} \lambda_1^{(p)}(L, G_t) = 0$ for $p = 0, 1$. And if we take $M = S^1$ and $N = S^{m-1}$, then we find that $\lim_{t\to\infty} \lambda_1^{(p)}(L, G_t) = \infty$ for $p = 2, \ldots, m-2$, because of $H^k(S^{m-1}; \mathbf{R}) = 0$ $(1 \le k \le m-2, m \ge 4)$.

In Theorem 1.2 it is interested in treating eigenvalues for p-forms because $\lim_{t\to\infty} \lambda_1^{(p)}(L, G_t)$ depends on the topological property $H^k(N; \mathbf{R}) = 0$ $(p-m \le k \le p)$.

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2. Modified Gentile-Pagliara's metric deformation

Let M be an m-dimensional connected compact oriented manifold without boundary. First we prepare the cylinder $C = [-1, 2] \times S^{m-1}$ and glue the m-hemisphere H to the one side of the boundary ∂C. Next we remove an m-disk from M and glue it to the other side of the boundary ∂C. We denote by $\bar{M}$ this new manifold which is diffeomorphic to the original M (see Fig. 1).

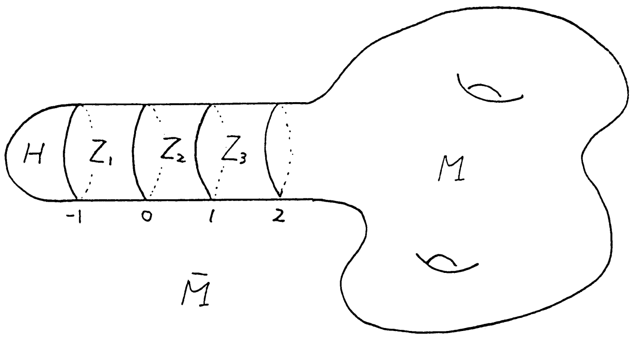

Fig. 1. $\bar{M}$

We divide C into the three parts, $Z_1 = [-1,0]\times S^{m-1}$, $Z_2 = [0,1]\times S^{m-1}$ and $Z_3 = [1,2]\times S^{m-1}$. We take any metric g on $\bar{M}$ such that $g = dr^2 \oplus h$ on Z_2, where r is the canonical coordinate of $[0,1]$ and h the canonical metric on $S^{m-1}(1)$. Then we define the metric deformation g_t of g by

$$g_t := \begin{cases} g & \text{on } \bar{M}\backslash Z_2, \\ f_t(r)dr^2 \oplus h & \text{on } Z_2. \end{cases}$$

Here for $t \geq 1$, $f_t(r)$ is a C^∞-function on $[0,1]$ such that $1 \leq f_t(r) \leq t^2$ and

$$f_t(r) = \begin{cases} 1 & (r = 0, 1), \\ t^2 & (\frac{1}{3} \leq r \leq \frac{2}{3}), \end{cases}$$

(see Fig. 2).

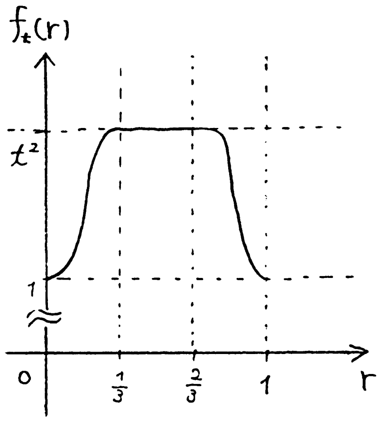

Fig. 2. $f_t(r)$

Finaly, we set

$$\bar{g}_t := \mathrm{vol}(\bar{M}, g_t)^{-\frac{2}{m}} g_t,$$

then $\mathrm{vol}(\bar{M}, \bar{g}_t) \equiv 1$.

LEMMA 2.1. *We have* $a + \frac{b}{3}t \leq \mathrm{vol}(\bar{M}, g_t) \leq a + bt$ *for some constants* $a, b > 0$.

PROOF. From the definition of $f_t(r)$, we have $\mathrm{vol}(S^{m-1},h)\,\frac{t}{3} \leq \mathrm{vol}(Z_2,g_t) \leq \mathrm{vol}(S^{m-1},h)\,t$. Hence if we set $a = \mathrm{vol}(\bar{M}\backslash Z_2, g_t)$ and $b = \mathrm{vol}(S^{m-1},h)$, we obtain Lemma 2.1. □

REMARK 2.2. We note that the difference between original Gentile-Pagliara's metric deformation and our modified Gentile-Pagliara's one comes from the choice of $f_t(r)$. Their $f_t(r)$ satisfies $\mathrm{vol}(\bar{M},g_t) = a + bt$ for some constants $a, b > 0$.

The next two lemmas are well-known.

LEMMA 2.3. *Let (M,g) be as above. For a constant $a > 0$,*

$$\begin{aligned}(1)\lambda_1^{(p)}(M,ag) &= a^{-1}\lambda_1^{(p)}(M,g),\\ (2)\ vol(M,ag) &= a^{\frac{m}{2}} vol(M,g).\end{aligned}$$

LEMMA 2.4. *Let $(M,g),(N,h)$ be m,n-dimensional connected compact oriented Riemannian manifolds. Then the spectrum of the product Riamannian manifold $(M\times N, g\oplus h)$ is given as follows:*

(a) for $\partial M,\ \partial N = \phi$,

$$\begin{aligned}\mathrm{Spec}^{(p)}(M\times N, g\oplus h) &= \{\lambda_i^{(r)}(M,g) + \lambda_j^{(s)}(N,h) \,|\, r+s=p,\\ &\qquad 0\leq r\leq m,\ 0\leq s\leq n,\ i,j=(0),1,2,\cdots\},\end{aligned}$$

(b) for $\partial M \neq \phi,\ \partial N = \phi$ with the Dirichlet boundary condition,

$$\mathrm{Spec}_D(M\times N, g\oplus h) = \{\mu_i(M,g) + \lambda_j(N,h) \,|\, i=1,2,\cdots,\ j=0,1,\cdots\},$$

where $\mu_i(M,g)$ is the i-th eigenvalue with the Dirichlet boundary condition.

3. Proof of Theorem 1.1

LEMMA 3.1. *Let (M,g) be a connected compact oriented Riemannian manifold without boundary. Then,*

$$\lambda_1^{(1)} \leq \lambda_1^{(0)}.$$

Proof. Let f be the first eigenfunction of (M,g). Since Δ commutes with d, we have $\Delta(df) = d(\Delta f) = \lambda_1^{(0)} df$. By $df \neq 0$, $\lambda_1^{(0)}$ is a non-zero eigenvalue of the Laplacian on 1-forms, hence $\lambda_1^{(1)} \leq \lambda_1^{(0)}$. $\square$

Remark 3.2. We note that for a 2-dimensional connected compact oriented Riemannian manifold without boundary we have $\lambda_1^{(0)} = \lambda_1^{(1)} = \lambda_1^{(2)}$. This follows from $\lambda_1^{(1)} = \min\{\lambda_1^{(0)},\ \lambda_1^{(2)}\}$ and the Poincaré duality $\lambda_1^{(0)} = \lambda_1^{(2)}$.

Next, we estimate the k-th eigenvalue from above.

Lemma 3.3. *Let (M,g) be a connected compact oriented Riemannian manifold without boundary. We take $k+1$ disjoint domains $U_1, U_2, \cdots, U_{k+1}$ with piece-wise C^∞-boundaries. Then, we obtain*

$$\lambda_k(M,g) \leq \max\{\mu_1(U_1), \mu_1(U_2), \cdots, \mu_1(U_{k+1})\}.$$

Here, each $\mu_1(U_i)$ $(i = 1, \cdots, k+1)$ is the first eigenvalue of the Laplacian on $(U_i, \text{induced metric})$ with the Dirichlet boundary condition.

Proof. We use Cheng's argument ([Ch-75], p. 292).
Let φ_i be an eigenfunction for $\mu_1(U_i)$ $(i = 1, \cdots, k+1)$. We set

$$\widetilde{\varphi_i} := \begin{cases} \varphi_i & \text{on} \quad U_i\ , \\ 0 & \text{on } M \backslash U_i\ . \end{cases}$$

Because φ_i satisfies the Dirichlet boundary condition, $\widetilde{\varphi_i}$ is C^0 on M and C^∞ almost everywhere. Since $\|d\varphi_i\|_{L^2(U_j)} < \infty$, therefore $\widetilde{\varphi_i} \in {L^2}_1(M,g)$.

Now let $u_0, u_1, \cdots, u_{k-1}$ be orthonormal eigenfunctions on (M,g) for $\lambda_0 = 0, \lambda_1, \cdots, \lambda_{k-1}$, i.e. $(u_i, u_j)_{L^2(M,g)} = \delta_{ij}\,(i \neq j)$, $\Delta u_i = \lambda_i u_i$. Then there are some constants $a_1, \cdots, a_{k+1}$ (one of them is not zero) such that $(\Phi, u_i)_{L^2(M,g)} = 0$ $(i = 0, 1, \cdots, k-1)$, where $\Phi = \sum_{i=1}^{k+1} a_i \widetilde{\varphi_i}$. In fact, as $\mathrm{supp}(\widetilde{\varphi}_i) \cap \mathrm{supp}(\widetilde{\varphi}_j) = \phi$ $(i \neq j)$, $(\widetilde{\varphi}_i, \widetilde{\varphi}_j)_{L^2} = \int_M \widetilde{\varphi}_i \widetilde{\varphi}_j\ v_g = 0$. So the linear spans $V := \langle \widetilde{\varphi}_1, \widetilde{\varphi}_2, \cdots, \widetilde{\varphi}_{k+1} \rangle_{\mathbf{R}}$, $W := \langle u_0, u_1, \cdots, u_{k-1} \rangle_{\mathbf{R}}$ are the $k+1$, k-dimensional linear subspaces of $L^2(M,g)$. We define that the linear operator

P from V to W is $P(\varphi) = \sum_{i=0}^{k-1} (\varphi, u_i)_{L^2} u_i$ $(\forall \varphi \in V)$. Since dim Ker(P) = dim V − dim Im$(P) \geq$ dim V − dim $W = (k+1) - k = 1$, Ker$(P) \neq 0$. Hence we can take a non-zero element Φ in Ker(P).

Then using the above Φ as a test function of the min-max principle, we obtain

$$\lambda_k(M, g) \quad \leq \quad \frac{\|d\Phi\|_{L^2}^2}{\|\Phi\|_{L^2}^2}.$$

Now we estimate the right-hand side from above.

$$\begin{aligned}
\|d\Phi\|_{L^2}^2 &= \int_M \langle d\Phi, d\Phi \rangle \, v_g \\
&= \sum_{i=1}^{k+1} a_i^2 \int_M \langle d\widetilde{\varphi}_i, d\widetilde{\varphi}_i \rangle \, v_g \\
&\quad (\text{ by } \operatorname{supp}(\widetilde{\varphi}_i) \cap \operatorname{supp}(\widetilde{\varphi}_i) = \phi \ (i \neq j) \ ) \\
&= \sum_{i=1}^{k+1} a_i^2 \int_{U_i} \Delta\varphi_i \cdot \varphi_i \, v_g \\
&\quad (\text{ by Stokes' theorem}) \\
&= \sum_{i=1}^{k+1} a_i^2 \mu_1(U_i) \int_{U_i} \varphi_i^2 \, v_g \\
&\quad (\text{ as } \varphi_i \text{ is an eigenfunction}) \\
&\leq \max_{i=1,\cdots,k+1} \{\mu_1(U_i)\} \sum_{i=1}^{k+1} a_i^2 \int_M \widetilde{\varphi}_i^2 \, v_g \\
&= \max_{i=1,\cdots,k+1} \{\mu_1(U_i)\} \int_M \Phi^2 \, v_g \\
&\quad (\text{ by } \operatorname{supp}(\widetilde{\varphi}_i) \cap \operatorname{supp}(\widetilde{\varphi}_j) = \phi \quad (i \neq j)) \ .
\end{aligned}$$

Therefore we get $\lambda_k(M, g) \leq \max_{i=1,2,\cdots,k+1} \{\mu_1(U_i)\}$. □

Proof of Theorem 1.1. When $p = 2, 3, \cdots, m-2$ and $m \geq 4$, there exists some constant $C > 0$ independent of t such that $\lambda_1^{(p)}(\bar{M}, g_t) \geq C$ from the proof of [G-P95] p. 3857. Hence from Lemma 2.1 and 2.3 we have

$$\lambda_1^{(p)}(\bar{M}, \bar{g}_t) \quad = \quad \operatorname{vol}(\bar{M}, g_t)^{\frac{2}{m}} \lambda_1^{(p)}(\bar{M}, g_t)$$

$$\geq \quad (a+\frac{b}{3}t)^{\frac{2}{m}}C.$$

Therefore $\lambda_1^{(p)}(\bar{M},\bar{g}_t)\to\infty$ as $t\to\infty$.

Then we have only the cases $p=0,1$. We remark from Lemma 3.1 that we only have to prove the case $p=0$, i.e. it is enough to prove that $\lambda_1(\bar{M},\bar{g}_t)\to 0$ as $t\to\infty$.

We take the two domains U_1,U_2 in $Z_2=[0,1]\times S^{m-1}$ as follows:

$$\begin{aligned} U_1 &\equiv (\alpha_1,\beta_1)\times S^{m-1},\\ U_2 &\equiv (\alpha_2,\beta_2)\times S^{m-1},\\ &\quad\left(\frac{1}{3}<\alpha_1<\beta_1<\alpha_2<\beta_2<\frac{2}{3}\right). \end{aligned}$$

By the choice of $f_t(r)$, the metric g_t on U_i is $t^2dr^2\oplus h$. Hence by Lemma 2.4, we have for $i=1,2$

$$\begin{aligned} \mu_1(U_i,g_t|_{U_i}) &= \min_{k\geq 1,\ l\geq 0}\{\mu_k(\,(\alpha_i,\beta_i),t^2dr^2)+\lambda_l(S^{m-1},h)\}\\ &\leq \min_{k\geq 1}\{\frac{1}{t^2}\mu_k((\alpha_i,\beta_i),dr^2)\}\\ &\quad \text{(by Lemma 2.3)}\\ &= \frac{1}{t^2}\mu_1((\alpha_i,\beta_i),dr^2). \end{aligned}$$

We obtain

$$\begin{aligned} \lambda_1(\bar{M},\bar{g}_t) &= \mathrm{vol}(\bar{M},g_t)^{\frac{2}{m}}\lambda_1(\bar{M},g_t)\\ &\quad \text{(by Lemma 2.3)}\\ &\leq (a+bt)^{\frac{2}{m}}\max\{\ \frac{1}{t^2}\,\mu_1((\alpha_1,\beta_1),dr^2),\ \frac{1}{t^2}\,\mu_1((\alpha_2,\beta_2),dr^2)\}\\ &\quad \text{(by Lemma 2.1, 3.3 and the above)}\\ &= \frac{(a+bt)^{\frac{2}{m}}}{t^2}\max\{\ \mu_1((\alpha_1,\beta_1),dr^2),\ \mu_1((\alpha_2,\beta_2),dr^2)\ \}. \end{aligned}$$

Since $m\geq 2$ and $\lambda_1(\bar{M},\bar{g}_t)\geq 0$, we have $\lambda_1(\bar{M},\bar{g}_t)\to 0$ as $t\to\infty$. $\square$

4. Proof of Theorem 1.2

First we note that $\mathrm{vol}(L, G_t)$ is constant in t because of the construction of G_t. We obtain by Lemma 2.3 and 2.4

$$
\begin{aligned}
&\lambda_1^{(p)}(L, G_t) \\
&\quad = \min_{i,j \geq 0,\ r,s \geq 0} \{\lambda_i^{(r)}(M, t^{\frac{1}{m}} g) + \lambda_j^{(s)}(N, t^{-\frac{1}{n}} h) \,|\, i^2 + j^2 \neq 0,\ r + s = p\ \} \\
&\quad = \min_{i,j \geq 0,\ r,s \geq 0} \{t^{-\frac{1}{m}} \lambda_i^{(r)}(M, g) + t^{\frac{1}{n}} \lambda_j^{(s)}(N, h) \,|\, i^2 + j^2 \neq 0,\ r + s = p\ \}.
\end{aligned}
$$

From the above we observe that $\lim_{t\to\infty} \lambda_1^{(p)}(L, G_t)$ apparently depends on the existence of harmonic forms i.e. 0-eigenvalues on (M, g) and (N, h). So we divide the proof into the following cases.

(I) $m \leq n$;

$$
\begin{aligned}
&(1) \quad 0 \leq p \leq m, \\
&(2) \quad m < p < n, \\
&(3) \quad n \leq p \leq l = m + n,
\end{aligned}
$$

(II) $n < m$;

$$
\begin{aligned}
&(4) \quad 0 \leq p < n, \\
&(5) \quad n \leq p \leq l = m + n.
\end{aligned}
$$

Case (1). $(0 \leq p \leq m)$:

Because we can take the pair $(r, s) = (p, 0)$ in the above formula, it follows that

$$
\begin{aligned}
\lambda_1^{(p)}(L, G_t) &\leq t^{-\frac{1}{m}} \lambda_1^{(p)}(M, g) + t^{\frac{1}{n}} \lambda_0^{(0)}(N, h) \\
&= t^{-\frac{1}{m}} \lambda_1^{(p)}(M, g) \\
&\to 0 \quad (\text{ as } t \to \infty\ ).
\end{aligned}
$$

Case (2). $(m < p < n)$:

If there is a k_0 $(p-m \leq k_0 \leq p)$ such that $H^{k_0}(N;\mathbf{R}) \neq 0$, we have $\lambda_0^{(k_0)}(N,h)=0$. Then, because we can take the pair $(r,s)=(p-k_0,k_0)$ in the above formula, it follows that

$$\begin{aligned}
\lambda_1^{(p)}(L,G_t) &\leq t^{-\frac{1}{m}}\lambda_1^{(p-k_0)}(M,g)+t^{\frac{1}{n}}\lambda_0^{(k_0)}(N,h)\\
&= t^{-\frac{1}{m}}\lambda_1^{(p-k_0)}(M,g)\\
&\to 0 \quad (\text{ as } t\to\infty\ ).
\end{aligned}$$

On the other hand if $H^k(N;\mathbf{R})=0$ ($p-m\leq k\leq p$) there is no harmonic k-form $(p-m\leq k\leq p\ )$. Because the possible pairs (r,s) are $(0,p),\ (1,p-1),\cdots$ and $(m,p-m)$, it follows that

$$\begin{aligned}
\lambda_1^{(p)}(L,G_t) &= \min_{p-m\leq k\leq p}\{t^{-\frac{1}{m}}\lambda_i^{(p-k)}(M,g)+t^{\frac{1}{n}}\lambda_j^{(k)}(N,h)\mid i^2+j^2\neq 0\}\\
&\geq t^{\frac{1}{n}}\min_{p-m\leq k\leq p}\{\lambda_1^{(k)}(N,h)\}\\
&\to \infty \quad (\text{ as } t\to\infty\ ).
\end{aligned}$$

Case (3). $(n\leq p\leq l=m+n)$:

Because we can take the pair $(r,s)=(p-n,n)$ in the above formula, it follows that

$$\begin{aligned}
\lambda_1^{(p)}(L,G_t) &\leq t^{-\frac{1}{m}}\lambda_1^{(p-n)}(M,g)+t^{\frac{1}{n}}\lambda_0^{(n)}(N,h)\\
&= t^{-\frac{1}{m}}\lambda_1^{(p-n)}(M,g)\\
&\to 0 \quad (\text{ as } t\to\infty).
\end{aligned}$$

Case (4). $(0\leq p<n<m)$:

Because we can take the pair $(r,s)=(p,0)$ in the above formula, it follows that

$$\begin{aligned}
\lambda_1^{(p)}(L,G_t) &\leq t^{-\frac{1}{m}}\lambda_1^{(p)}(M,g)+t^{\frac{1}{n}}\lambda_0^{(0)}(N,h)\\
&= t^{-\frac{1}{m}}\lambda_1^{(p)}(M,g)\\
&\to 0 \quad (\text{ as } t\to\infty).
\end{aligned}$$

Case (5). $(n \leq p \leq l = m + n)$:

Because we can take the pair $(r, s) = (p - n, n)$ in the above formula, it follows that

$$\begin{aligned}\lambda_1^{(p)}(L, G_t) &\leq t^{-\frac{1}{m}}\lambda_1^{(p-n)}(M, g) + t^{\frac{1}{n}}\lambda_0^{(n)}(N, h) \\ &= t^{-\frac{1}{m}}\lambda_1^{(p-n)}(M, g) \\ &\to 0 \quad (\text{ as } \ t \to \infty).\end{aligned}$$

Therefore we have just finished the proof of Theorem 1.2. □

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Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914, Japan
E-mail: junya@ms.u-tokyo.ac.jp