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# *On the Discrepancy of the $\beta$-Adic van der Corput Sequence*

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**Abstract.** The $\beta$-adic van der Corput sequence is constructed. When $\beta$ satisfies some conditions, the order of discrepancy of the sequence become $O(\log M/M)$ or $O((\log M)^2/M)$.

## 1. Introduction

It is well known that low-discrepancy sequences and their discrepancy play essential roles in quasi-Monte Carlo methods [6]. The author constructed a new class of low-discrepancy sequences $N_\beta$ [7] by using the $\beta$-adic transformation [9][11]. Here, $\beta$ is a real number greater than 1; when $\beta$ is an integer greater than or equal to 2, $N_\beta$ becomes the classical van der Corput sequence in base $\beta$. Therefore, the class $N_\beta$ can be regarded as a generalization of the van der Corput sequence. $N_\beta$ also contains a new construction by Barat and Grabner [1] [7]. The principle of the construction of $N_\beta$ is that we can consider the van der Corput sequence to be a Kakutani adding machine [10]. Pagès [8] and Hellekalek [4] also considered the van der Corput sequence from this point of view. In [7], it is shown that when $\beta$ satisfies the following two conditions:

- Markov condition: $\beta$ is Markov, that is to say, for this $\beta$, the $\beta$-adic transformation becomes Markov,
- Pisot-Vijayaraghavan condition: All conjugates of $\beta$ with respect to its characteristic equation belong to $\{z \in \mathbf{C} \mid |z| < 1\}$,

the discrepancy of $N_\beta$ decreases in the fastest order $O(N^{-1}\log N)$. In this paper, we consider the case in which $\beta$ is not necessarily Markov. We introduce the function $\phi_\beta(z)$ from Ito and Takahashi [5]. It is shown that when $\beta$ satisfies the following condition **(PV)**:

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**(PV)** All zeroes of $1 - \phi_\beta(z)$ except for $z = 1$ belong to $\{z \in \mathbf{C} \mid |z| > \beta\}$,

which is a generalization of the above Pisot-Vijayaraghavan condition, the discrepancy of $N_\beta$ decreases in the order $O(N^{-1}(\log N)^2)$. We also remark that the condition **(PV)** is considered to be a condition for the second eigenvalue of the Perron-Frobenius operator associated with the $\beta$-adic transformation.

## 2. Low-discrepancy sequence

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points [6]. A sequence $x_1, x_2, \ldots$ in the $s$-dimensional unit cube $I^s = \prod_{i=1}^s [0, 1)$ is said to be uniformly distributed in $I^s$ when

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_J(x_n) = \lambda_s(J)$$

holds for all subintervals $J \subset I^s$, where $c_J$ is the characteristic function of $J$ and $\lambda_s$ is the $s$-dimensional Lebesgue measure. If $x_1, x_2, \ldots \in I^s$ is a uniformly distributed sequence, the formula

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{I^s} f(x)\, dx \tag{2.1}$$

holds for any Riemann integrable function on $I^s$. The discrepancy of the point set $P = \{x_1, x_2, \ldots, x_N\}$ in $I^s$ is defined as follows:

$$D_N(\mathcal{B}; P) = \sup_{B \in \mathcal{B}} \left| \frac{A(B; P)}{N} - \lambda_s(B) \right| \tag{2.2}$$

where $\mathcal{B} \subset \wp(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B; P)$ is the counting function that indicates the number of $n$, where $1 \leq n \leq N$, for which $x_n \in B$. When $\mathcal{J}^* = \{\prod_{i=1}^s [0, u_i), 0 \leq u_i < 1\}$, the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(J^*; P)$. When $S = \{x_1, x_2, \ldots\}$ is a sequence in $I^s$, we define $D_N^*(S)$ as $D_N^*(S_N)$, where $S_N$ is the point set $\{x_1, x_2, \ldots, x_N\}$. Let $S$ be a sequence in $I^s$. It is known that the following two conditions are equivalent:

1. $S$ is uniformly distributed in $I^s$;

2. $\lim_{N\to\infty} D_N^*(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy:

THEOREM 2.1 (Koksma-Hlawka [6]). *If $f$ has bounded variation $V(f)$ on $\bar{I}^s$ in the sense of Hardy and Krause, then for any $x_1, x_2, \ldots, x_N \in I^s$, we have*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_{I^s} f(x)\, dx \right| \leq V(f) D_N^*(x_1, \ldots, x_N).$$

Schmidt [12] showed that, when $s = 1$ or $2$, there exists a positive constant $C$ that depends only on $s$, and the following inequality holds for an arbitrary point set $P$ consisting of $N$ elements:

$$D_N^*(P) \geq C \frac{(\log N)^{s-1}}{N}. \tag{2.3}$$

If (2.3) holds, then there exists a positive constant $C$ that depends only on $s$, and any sequence $S \subset I^s$ satisfies

$$D_N^*(S) \geq C \frac{(\log N)^{s}}{N} \tag{2.4}$$

for infinitely many $N$. Taking account of (2.3) and (2.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

DEFINITION 2.1. Let $S$ be an one-dimensional sequence in $[0, 1)$. If $D_N^*(S)$ satisfies

$$D_N^*(S) = O(N^{-1} \log N)$$

then $S$ is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [2] [6].

DEFINITION 2.2. Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n) p^j, \qquad a_j(n) \in \{0, 1, \ldots, p-1\} \text{ for all } j \geq 0,$$

in base $p$. Let $\tau = \{\tau_j\}_{j\geq 0}$ be a set of permutations $\tau_j$ of $\{0, 1, \ldots, p-1\}$. Then the radical-inverse function $\phi_p^\tau$ is defined by

$$\phi_p^\tau(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n))p^{-j-1} \qquad \text{for all integers} \qquad n \geq 0.$$

The van der Corput sequence in base $p$ with digit permutations $\tau$ is the sequence $\{\phi_p^\tau(n)\}_{n=0}^\infty \subset [0, 1)$.

THEOREM 2.2 ([2][6]). *For an arbitrary integer $p \geq 2$, the van der Corput sequence in base $p$ is a low-discrepancy sequence.*

## 3. $\beta$-adic transformation

In this section we define the fibred system and the $\beta$-adic transformation, following [5] [13].

**C**, **R**, **Z**, and **N** are the sets of all complex numbers, all real numbers, all integers, and all natural numbers, respectively. We also set

$$\begin{aligned} \mathbf{R}_{>a} &= \{r \in \mathbf{R} \mid r > a\} \\ \mathbf{Z}_{\geq n} &= \{i \in \mathbf{Z} \mid i \geq n\} \\ &\vdots \end{aligned}$$

and so on. For $x \in \mathbf{R}$, $[x]$ denotes the integer part of $x$.

DEFINITION 3.1. Let $B$ be a set and $T : B \to B$ be a map. The pair $(B, T)$ is called a fibred system if the following conditions are satisfied:

1. There is a finite countable set $A$.
2. There is a map $k : B \to A$, and the sets

   $$B(i) = k^{-1}(\{i\}) = \{x \in B : k(x) = i\}$$

   form a partition of B.
3. For an arbitrary $i \in A$, $T|_{B(i)}$ is injective.

Definition 3.2. Let $\Omega = A^{\mathbf{N}}$ and $\sigma : \Omega \to \Omega$ be the one-sided shift operator. Let $k_j(x) = k(T^{j-1}x)$. We derive a canonical map $\varphi : B \to \Omega$ from

$$\varphi(x) = \{k_j(x)\}_{n=1}^{\infty}.$$

$\varphi$ is called the representation map.

We have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{T} & B \\ \varphi\big\downarrow & & \varphi\big\downarrow \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

Definition 3.3. If a representation map $\varphi$ is injective, $\varphi$ is called a valid representation.

Definition 3.4. Let $\omega \in \Omega$. If $\omega \in \mathrm{Im}(\varphi)$, $\omega$ is called an admissible sequence.

Definition 3.5. The cylinder of rank $n$ defined by $a_1, a_2, \ldots, a_n \in A$ is the set

$$B(a_1, a_2, \ldots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \ldots \cap T^{-n+1}B(a_n).$$

We define $B$ to be a cylinder of rank 0.

For a sequence $a \in \Omega$, we write the $i$-th element of $a$ as $a(i)$, that is, $a = (a(0), a(1), a(2), \ldots)$.

Definition 3.6. Let $\beta > 1$ and $\beta \in \mathbf{R}$. Let $f_\beta : [0,1) \to [0,1)$ be the function defined by

$$f_\beta(x) = \beta x - [\beta x].$$

Let $A = \mathbf{Z} \cap [0, \beta)$. Then we have the following fibred system $([0,1), f_\beta)$:

$$\begin{array}{ccc} [0,1) & \xrightarrow{f_\beta} & [0,1) \\ \varphi\big\downarrow & & \varphi\big\downarrow \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array} \tag{3.1}$$

The representation map $\varphi$ of this fibred system is defined as follows:

$$\varphi(x)(n) = k, \quad \text{if} \quad \frac{k}{\beta} \leq f_\beta^n(x) < \frac{(k+1)}{\beta}$$

where $f_\beta^0(x) = x$, and $f_\beta^{n+1}(x) = f_\beta(f_\beta^n(x))$. Let $X_\beta$ be the closure of $\mathrm{Im}(\varphi)$ in the product space $\Omega$ with the product topology. The lexicographical order $\prec$ (resp. $\succ$) is defined in $\Omega$ as follows: $\omega \prec \omega'$ (resp. $\omega \succ \omega'$) if and only if there exists an integer $n$ such that $\omega(k) = \omega'(k)$ for $k < n$ and $\omega(n) < \omega'(n)$ (resp. $\omega(n) > \omega'(n)$). We also define $\preceq$ (resp. $\succeq$) as $\prec$ (resp. $\succ$) or equal. In this situation, we set

$$f_\beta^n(1) = \lim_{x \nearrow 1} f_\beta^n(x),$$

$$\zeta_\beta = \max\{X_\beta\} = \varphi(1),$$

and

$$\rho_\beta(a) = \sum_{n=0}^{\infty} a(n)\beta^{-n-1}.$$

We have the following diagram:

$$\begin{array}{ccc} [0,1] & \xrightarrow{f_\beta} & [0,1] \\ \varphi\big\downarrow\big\uparrow\rho_\beta & & \varphi\big\downarrow\big\uparrow\rho_\beta \\ X_\beta & \xrightarrow{\sigma} & X_\beta \end{array} \tag{3.2}$$

This diagram is called a $\beta$-adic transformation.

We use the following notation for periodic sequences:

$$\begin{aligned}(a_1, a_2, \ldots, \dot{a}_n, \ldots \dot{a}_{n+m}) = (a_1, a_2, \ldots, \; & a_n, a_{n+1}, \ldots, a_{n+m}, \\ & a_n, a_{n+1}, \ldots, a_{n+m}, \\ & \vdots \\ & a_n, a_{n+1}, \ldots, a_{n+m}, \\ & \ldots)\end{aligned}$$

We introduce the following proposition from Ito and Takahashi [5].

PROPOSITION 3.1. *For an arbitrary $\beta \in \mathbf{R}_{>1}$ the following statements hold in (3.2).*

1. *$\sigma \circ \varphi = \varphi \circ f_\beta$ on $[0,1)$.*

2. *$\varphi : [0,1] \to X_\beta$ is an injection and is strictly order-preserving, i.e., $t < s$ implies that $\varphi(t) \prec \varphi(s)$.*

3. *$\rho_\beta \circ \varphi = \mathrm{id}$ on $[0,1]$.*

4. *$\rho_\beta \circ \sigma = f_\beta \circ \rho_\beta$ on $\mathrm{Im}(\varphi)$.*

5. *$\rho_\beta : X_\beta \to [0,1]$ is a continuous surjection and is order-preserving, i.e., $\omega \prec \omega'$ implies that $\rho_\beta(\omega) \le \rho_\beta(\omega')$.*

6. *For an arbitrary $t \in [0,1]$, $\rho_\beta^{-1}(t)$ consists either of a one point $\varphi(t)$ or of two points $\varphi(t)$ and $\sup\{\varphi(s) \mid s < t\}$. The latter case occurs only when $f_\beta^n(t) = (\dot{0})$ for some $n > 0$.*

We also remark that the following proposition holds:

Proposition 3.2.

$$X_\beta = \{\omega \in \Omega \mid \sigma^n \omega \preceq \zeta_\beta, \quad \textit{for all} \quad n \ge 0\}$$

Definition 3.7. Let $u \in X_\beta$. If there exist $n \in \mathbf{Z}_{\ge 1}$ which satisfies $u(i) = u(i+n)$ for any $i \in \mathbf{Z}$, $u$ is called a periodic sequence. When $u \in X_\beta$ is periodic, we define the period of $u$ as $\min\{n \in \mathbf{Z}_{\ge 1} \mid u(i) = u(i+n) \text{ for any } i \in \mathbf{Z}\}$.

The following definition is from Parry [9].

Definition 3.8. If $\zeta_\beta$ has periodic tail whose period is $m$, that is, $\sigma^l \zeta_\beta$ is periodic for some non-negative integer $l$ and the period of $\sigma^l \zeta_\beta$ is $m$, then $\beta$ and $\beta$-adic transformation (3.2) are called Markov. In this case, $\beta$ is the unique $z > 1$ solution of the following equation:

$$z^{m+l} - \sum_{i=1}^{m+l} a_{i-1} z^{m+l-i} = z^l - \sum_{i=1}^{l} a_{i-1} z^{l-i} \tag{3.3}$$

where

$$\zeta_\beta = (a_0, a_1, \ldots, a_{l-1}, \dot{a}_l, a_{l+1}, \ldots, a_{l+\dot{m}-1})$$

and

$$l = \min\{l \in \mathbf{Z}_{\geq 0} \mid \sigma^l \zeta_\beta \text{ is periodic}\}.$$

This equation is called the characteristic equation of $\beta$. When $l = 0$, $\beta$ is called simple. When $\beta$ is Markov, $p(\beta)$ denotes the length of the period of $\zeta_\beta$'s periodic tail.

When $\beta$ is not necessarily Markov, the notion of the characteristic equation is generalized as follows. This function was first studied in Takahashi [14][15] and Ito and Takahashi [5].

Definition 3.9.

$$\phi_\beta(z) = \sum_{n \geq 0} \zeta_\beta(n) \left(\frac{z}{\beta}\right)^{n+1}$$

We also have the following proposition from Ito and Takahashi [5].

Proposition 3.3. *$\phi_\beta(z)$ converges in a neighborhood of the unit disk $\{z \in \mathbf{C} \mid |z| \leq 1\}$ and the equation $1 - \phi_\beta(z) = 0$ has only one simple root at $z = 1$ in a neighborhood of the unit disk.*

Remark 3.1. When $\beta$ is Markov, $1 - \phi_\beta(\beta/z) = 0$ becomes the characteristic equation of $\beta$.

## 4. Constructing the sequence

In this section, a sequence $N_\beta \subset [0, 1)$ is defined by the use of $\beta$-adic transformation, following [7]. Let $\beta \in \mathbf{R}_{>1}$ and let $([0, 1], f_\beta, X_\beta, \sigma, \varphi, \rho_\beta)$ be a $\beta$-adic transformation (3.2). Let $B = [0, 1)$, and $A, \Omega, \zeta_\beta, B(a_1, \ldots, a_n)$ be the same as in the previous section.

Definition 4.1. Let $n \in \mathbf{Z}_{\geq 0}$. Define

$$\begin{aligned} X_\beta(n) &= \begin{cases} \{(\dot{0})\}, & n = 0 \\ \{\omega \in X_\beta \mid \sigma^{n-1}\omega \neq (\dot{0}) \quad \text{and} \quad \sigma^n \omega = (\dot{0})\}, & n \neq 0 \end{cases}, \\ Y_\beta(n) &= \{(\omega(0), \ldots, \omega(n-1)) \mid \omega \in X_\beta\}, \end{aligned}$$

and

$$Y^0_\beta(n) = \{(a_0, \ldots, a_{n-1}) \mid (a_0, \ldots, a_{n-2}, a_{n-1} + 1) \in Y_\beta(n)\}.$$

Let $k \in \mathbf{Z}_{\geq 0}$, $u \in Y_\beta(k)$, and $v \in Y_\beta(l)$. Define $Y_\beta(u;n)$, $Y^0_\beta(u;n)$, $Y_\beta(u;n;v)$, $Y^0_\beta(u;n;v)$, $G_\beta(n)$, $G_\beta(u;n)$, $G^0_\beta(n)$, $G^0_\beta(u;n)$, and $G^0_\beta(u;n;v)$ as follows:

$$\begin{aligned}
Y_\beta(u;n) &= \{u \cdot \omega \mid u \cdot \omega \in Y_\beta(k+n)\} \\
Y^0_\beta(u;n) &= \{u \cdot \omega \mid u \cdot \omega \in Y^0_\beta(k+n)\} \\
Y_\beta(u;n;v) &= \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y_\beta(k+n+l)\} \\
Y^0_\beta(u;n;v) &= \{u \cdot \omega \cdot v \mid u \cdot \omega \cdot v \in Y^0_\beta(k+n+l)\} \\
G_\beta(n) &= \sharp Y_\beta(n) \\
G^0_\beta(n) &= \sharp Y^0_\beta(n) \\
G_\beta(u;n) &= \sharp Y_\beta(u;n) \\
G^0_\beta(u;n) &= \sharp Y^0_\beta(u;n) \\
G_\beta(u;n;v) &= \sharp Y_\beta(u;n;v) \\
G^0_\beta(u;n;v) &= \sharp Y^0_\beta(u;n;v)
\end{aligned}$$

where $u \cdot v$ means the concatenation of $u$ and $v$, that is to say,

$$u \cdot v = (u(0), \ldots, u(n-1), v(0), v(1), \ldots).$$

Finally we set $Y_\beta(0) = Y^0_\beta(0) = \{\epsilon\}$ where $\epsilon$ is the empty word and satisfies $\epsilon \cdot u = u \cdot \epsilon = u$ for any $u \in Y_\beta(n)$.

Definition 4.2. Define the right-to-left lexicographical order $\overset{r-l}{\prec}$ in $\bigsqcup_{n=0}^{\infty} X_\beta(n)$ as follows: $\omega \overset{r-l}{\prec} \omega'$ if and only if $(\omega(n-1), \ldots, \omega(0)) \prec (\omega'(m-1), \ldots, \omega'(0))$ where $\omega \in X_\beta(n)$ and $\omega' \in X_\beta(m)$.

Definition 4.3 ($N_\beta$ [7]). Define $L_\beta = \{\omega_i\}_{i=0}^{\infty}$ as $\bigsqcup_{n=0}^{\infty} X_\beta(n)$ ordered in right-to-left lexicographical order, that is, $L_\beta$ is $\bigsqcup_{n=0}^{\infty} X_\beta(n)$ as a set and $\omega_i \overset{r-l}{\prec} \omega_j$ holds for all $i < j$. Then, the sequence $N_\beta$ is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_i)\}_{i=0}^{\infty}.$$

*Example* 4.1. If $\beta = \frac{1+\sqrt{5}}{2}$, then $\zeta_\beta = (\dot{1}, \dot{0})$ and elements of $N_\beta$ are calculated as follows:

$$\begin{aligned}
N_\beta(0) &= \rho_\beta(0) = 0 \\
N_\beta(1) &= \rho_\beta(1) = 0.618033988749895\ldots \\
N_\beta(2) &= \rho_\beta(01) = 0.381966011250106\ldots \\
N_\beta(3) &= \rho_\beta(001) = 0.23606797749979\ldots \\
N_\beta(4) &= \rho_\beta(101) = 0.854101966249686\ldots \\
N_\beta(5) &= \rho_\beta(0001) = 0.145898033750316\ldots \\
N_\beta(6) &= \rho_\beta(1001) = 0.763932022500212\ldots \\
N_\beta(7) &= \rho_\beta(0101) = 0.527864045000422\ldots \\
N_\beta(8) &= \rho_\beta(00001) = 0.0901699437494747\ldots \\
N_\beta(9) &= \rho_\beta(10001) = 0.70820393249937\ldots \\
N_\beta(10) &= \rho_\beta(01001) = 0.472135954999581\ldots \\
N_\beta(11) &= \rho_\beta(00101) = 0.326237921249265\ldots \\
N_\beta(12) &= \rho_\beta(10101) = 0.944271909999161\ldots \\
N_\beta(13) &= \rho_\beta(000001) = 0.0557280900008416\ldots \\
N_\beta(15) &= \rho_\beta(100001) = 0.673762078750737\ldots \\
N_\beta(16) &= \rho_\beta(010001) = 0.437694101250947\ldots \\
&\vdots
\end{aligned}$$

From this definition, we immediately have the following proposition:

PROPOSITION 4.1. *If $\beta$ is an integer greater than or equal to 2 then $N_\beta$ is the van der Corput sequence in base $\beta$ with all digit permutations $\tau_j = \mathrm{id}$.*

From Theorem 2.2 and Proposition 4.1, we see that if $\beta \in \mathbf{Z}_{\geq 2}$ then $N_\beta$ is a low-discrepancy sequence, that is to say, $D_M^*(N_\beta) = O(M^{-1} \log M)$ holds for all $\beta \in \mathbf{Z}_{\geq 2}$. We also have the following theorem:

THEOREM 4.1. *Let $\beta$ be a real number greater than 1, and let the following condition* **(PV)** *hold:*

**(PV)** *All zeroes of* $1 - \phi_\beta(z)$ *except for* $z = 1$ *belong to* $\{z \in \mathbf{C} \mid |z| > \beta\}$.

*Then,*

$$D^*_M(N_\beta) = O\left(\frac{(\log M)^2}{M}\right)$$

*holds. Moreover, if* $\beta$ *is Markov, then*

$$D^*_M(N_\beta) = O\left(\frac{\log M}{M}\right)$$

*holds.*

Remark 4.1. When $\beta$ is Markov, the condition **(PV)** is equivalent to the condition that all conjugates of $\beta$ with respect to its characteristic equation (3.3) belong to $\{z \in \mathbf{C} \mid |z| < 1\}$.

Remark 4.2. In [7], the case in which $\beta$ is Markov is proved.

To prove this theorem, we provide lemmas and definitions. We use the following notations:

$$\omega[i,j) = \begin{cases} (\omega(i), \dots, \omega(j-1)), & i < j \\ \epsilon, & i = j \end{cases},$$

where $\omega \in X_\beta$ and $i, j \in \mathbf{Z}_{\geq 0}$. $R_\beta(u) = \lambda(B(u))$ where, $\lambda$ is the one-dimensional Lebesgue measure, $u \in X_\beta(n)$, and $B(u)$ is the cylinder (3.5). For a sequence $S$, $S[N]$ denotes the point set consisting of the first $N$ elements of $S$, and $S[N; M] = S[N+M] \setminus S[N]$.

Definition 4.4. For any $k \geq 0$ and $u \in Y_\beta(k)$, define

$$e(u) = \{i \in \mathbf{Z}_{\geq 0} \mid \zeta_\beta[0, i+1) \cdot u \notin Y_\beta(k+i+1)\}.$$

Lemma 4.1 ([5]). *For an arbitrary* $k \geq 0$ *and* $u \in Y_\beta(k)$*, we have the following partitioning of* $Y_\beta(u; n)$*:*

$$Y_\beta(u; n) = \bigsqcup_{j=1}^{n} Y^0_\beta(u; j) \cdot \zeta_\beta[0, n-j) \bigsqcup \max\{Y_\beta(u; n)\}$$

Proof. It is trivial to show that the left-hand side includes the right-hand side.

If $v = (a_1, \ldots, a_{n+k}) \in Y_\beta(u; n) \setminus Y^0_\beta(u; n)$ and $v \neq \max\{Y_\beta(u; n)\}$, then there exists an integer $l$ that satisfies

$$k + 1 \leq l \leq n + k$$

and

$$\min\{w \in Y_\beta(u; n) \mid w \succ v\} = (a_1, \ldots, a_l + 1, 0, \ldots, 0).$$

This means that

$$(a_{l+1}, \ldots, a_{n+k}) = \zeta_\beta[0, n + k - l)$$

and

$$(a_1, \ldots, a_{l-1}, a_l + 1) \in Y^0_\beta(u; l - k)$$

hold. □

Taking account of Lemma 4.1, we give the following definition:

Definition 4.5. For an arbitrary $u \in Y_\beta(n)$, define an integer $d(u)$ as follows: $d(u) = k$ if

$$u \in Y^0_\beta(k) \cdot \zeta_\beta[0, n - k)$$

holds. Remark that $\max\{Y_\beta(n)\} = \zeta_\beta[0, n)$.

From Lemma 4.1, Definition 4.4, and Definition4.5 we have the following lemma:

Lemma 4.2. *For any $k, l, n \geq 0$, $u \in Y_\beta(k)$, and $v \in Y_\beta(l)$, we have the following partitioning of $Y_\beta(u; n; v)$:*

$$Y_\beta(u; n; v) \cong \begin{cases} \displaystyle\bigsqcup_{\substack{1 \leq j \leq n \\ n-j-1 \notin e(v)}} Y^0_\beta(u; j) \cdot \zeta_\beta[0, n - j), & \\ \qquad\qquad\qquad \textit{if} \ \ n + k - d(\max\{Y_\beta(u; n)\}) - 1 \in e(v) \\ \displaystyle\bigsqcup_{\substack{1 \leq j \leq n \\ n-j-1 \notin e(v)}} Y^0_\beta(u; j) \cdot \zeta_\beta[0, n - j) \bigsqcup \max\{Y_\beta(u; n)\}, & \textit{otherwise.} \end{cases}$$

LEMMA 4.3. *For any $n \geq 0$ and $u \in Y_\beta(n)$,*

$$R_\beta(u) = \frac{1}{\beta^{d(u)}}\left(1 - \sum_{i=0}^{n-d(u)-1} \frac{\zeta_\beta(i)}{\beta^{i+1}}\right)$$

*holds.*

PROOF. Let $u = u^0 \cdot \zeta_\beta[0, n-d(u))$ where $u^0 \in Y_\beta^0(d(u))$. From Definition 3.6,

$$\begin{aligned} R_\beta(u^0) &= \rho_\beta((u^0(0), \ldots, u^0(d(u)-1)+1) \\ &\quad - \rho_\beta((u^0(0), \ldots, u^0(d(u)-1)) = \frac{1}{\beta^{d(u)}} \end{aligned}$$

and

$$R_\beta(\zeta_\beta[0, n-d(u))) = 1 - \sum_{i=0}^{n-d(u)-1} \frac{1}{\beta^{i+1}}.$$

When $v \cdot w \in Y_\beta(m)$, it follows that $R_\beta(v \cdot w) = R_\beta(v) R_\beta(w)$. Then, the lemma holds. □

REMARK 4.3. From Definition 3.6, it follows that

$$f_\beta^n(x) = \beta^n\left(x - \sum_{i=0}^{n-1} \frac{\varphi(x)(i)}{\beta^{i+1}}\right)$$

for any $x \in [0,1]$ and $n \geq 0$. Then, we have

$$R_\beta(u) = \frac{1}{\beta^n} f_\beta^{n-d(u)}(1)$$

for any $u \in Y_\beta(n)$ and $n \geq 0$, from Lemma 4.3.

LEMMA 4.4 ([5]). *Let $r$ be the absolute value of the second smallest zero of $1 - \phi_\beta(z)$, that is, $r = \min\{|z| \mid z \in \mathbf{C},\ z \neq 1,\ 1 - \phi_\beta(z) = 0\}$. Then for any small $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ and*

$$\left| G_\beta^0(u; n) - \frac{\beta^{n+k} R_\beta(u)}{\phi'_\beta(1)} \right| \leq \frac{C_\varepsilon}{n}\left(\frac{\beta}{r-\varepsilon}\right)^n$$

*holds for any $n \geq 0$, $k \geq 0$ and $u \in Y_\beta(k)$.*

PROOF. Let $k \geq 0$ and $u \in Y_\beta(k)$. Remark that

$$R_\beta(u) = \sum_{u\cdot v \in Y_\beta(u;n)} R_\beta(u \cdot v) \tag{4.1}$$

holds. From (4.1), Lemma 4.1, and Remark 4.3, we have

$$\beta^{n+k} R_\beta(u) = \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) + f_\beta^{n+l}(1) \tag{4.2}$$

where $l = k - d(\max\{Y_\beta(u;n)\}) \geq 0$. Remark that the formal power series

$$\sum_{n\geq 1} z^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \beta^{-(n+k)}$$

converges for $|z| < 1$. We have the following equality from (4.2):

$$\begin{aligned} \beta^k \sum_{n\geq 1} z^n R_\beta(u) \quad &= \quad \sum_{n\geq 1} \left(\frac{z}{\beta}\right)^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \\ &\quad + \sum_{n\geq 1} \left(\frac{z}{\beta}\right)^n f_\beta^{n+l}(1) \end{aligned} \tag{4.3}$$

We also have

$$\begin{aligned} &\sum_{n\geq 1} \left(\frac{z}{\beta}\right)^n \sum_{j=0}^{n-1} f_\beta^j(1) G_\beta^0(u; n-j) \\ &= \sum_{j\geq 1} \sum_{n\geq j} f_\beta^{j-1}(1) G_\beta^0(u; n-j+1) \left(\frac{z}{\beta}\right)^n \\ &= \sum_{j\geq 0} f_\beta^j(1) \left(\frac{z}{\beta}\right)^j \sum_{n\geq 1} G_\beta^0(u; n) \left(\frac{z}{\beta}\right)^n \end{aligned}$$

and, from Remark 4.3,

$$(1-z) \sum_{n\geq 0} f_\beta^n(1) \left(\frac{z}{\beta}\right)^n$$

$$= (1-z) + (1-z)\sum_{n\geq 1}\left(1-\sum_{i=0}^{n-1}\frac{\zeta_\beta(i)}{\beta^{i+1}}\right)z^n$$
$$= 1-\sum_{n\geq 0}\zeta_\beta(n)\left(\frac{z}{\beta}\right)^{n+1} = 1-\phi_\beta(z).$$

By using these two equalities, we obtain from (4.3) that

$$\text{(4.4)}\quad \sum_{n\geq 1} G^0_\beta(u;n)\left(\frac{z}{\beta}\right)^n = \frac{z\beta^k R_\beta(u)}{1-\phi_\beta(z)} - \frac{(1-z)\sum_{n\geq 1} f_\beta^{n+l}(1)(z/\beta)^n}{1-\phi_\beta(z)}.$$

Consider the function

$$\text{(4.5)}\qquad h_u(z) = \sum_{n\geq 1}\left(G^0_\beta(u;n)\left(\frac{z}{\beta}\right)^n - \frac{\beta^k R_\beta(u)}{\phi'_\beta(1)}z^n\right)$$
$$= \frac{z\beta^k R_\beta(u)}{1-\phi_\beta(z)} - \frac{(1-z)\sum_{n\geq 1} f_\beta^{n+l}(1)(z/\beta)^n}{1-\phi_\beta(z)} - \frac{z\beta^k R_\beta(u)}{(1-z)\phi'_\beta(1)}.$$

The second equality comes from (4.4). From Proposition 3.3, we see that $h_u(z)$ is analytic in a neighborhood of $\{z \in \mathbf{C} \mid |z| \leq r-\varepsilon,\ z \neq 1\}$. We also see from (4.5) that $\lim_{z\to 1}(1-z)h_u(z) = 0$. Considering the fact that $\beta^k R_\beta(u) \leq 1$ for any $u \in Y_\beta(k)$, $k \geq 1$ and that the second term of the right-hand side of (4.4) and its derivative are bounded uniformly in $l$, we see that there exists a constant $C_\varepsilon$ and

$$\text{(4.6)}\qquad \sup_{\substack{k\geq 1,\ u\in Y_\beta(k)\\ |z|=r-\varepsilon}} \left|h'_u(z)\right| < C_\varepsilon$$

holds. Then we have

$$n!\left|\frac{G^0_\beta(u;n)}{\beta^n} - \frac{\beta^k R_\beta(u)}{\phi'_\beta(1)}\right| = \left|h_u^{(n)}(0)\right|$$
$$= \left|\frac{d^{n-1}h'_u}{dz^{n-1}}(0)\right|$$
$$= \left|\frac{(n-1)!}{2\pi(r-\varepsilon)^n}\int_{|z|=r-\varepsilon} h'_u(z)\,dz\right|$$

$$\leq \quad (n-1)!\frac{C_\varepsilon}{(r-\varepsilon)^n}$$

and the lemma follows. □

LEMMA 4.5. *If $\beta \in \mathbf{R}_{>1}$ is Markov and $\zeta_\beta = (a_0, \ldots, a_{l-1}, \dot{a}_l, \ldots, \dot{a}_{l+m-1})$, where $m = p(\beta)$ and $l = \min\{l \in \mathbf{Z}_{\geq 0} \mid \sigma^l \zeta_\beta$ is periodic$\}$, then we have the following statements:*

1. *For an arbitrary $v \in X_\beta$, $\{G^0_\beta(n)\}_{n=0}^\infty$ and $\{G_\beta(n)\}_{n=0}^\infty$ satisfy the following linear recurrent equation:*

$$\begin{aligned}(4.7)\qquad & G_\beta(\epsilon; n+m+l; v) - \sum_{i=0}^{m+l-1} a_i G_\beta(\epsilon; n+m+l-i-1; v)\\ &= G_\beta(\epsilon; n+l; v) - \sum_{i=0}^{l-1} a_i G_\beta(\epsilon; n+l-i-1; v)\\ &= G_\beta(\zeta_\beta[0,l); n; v).\end{aligned}$$

2. *For arbitrary $u \in Y_\beta(k)$, $k \geq m+l$ and $v \in X_\beta$, the following equations hold for any $n \geq m+l-k+d$:*

$$(4.8)\quad G_\beta(u; n; v) = \begin{cases} \displaystyle\sum_{i=1}^{m+l-k+d} a_{k-d-1+i} G_\beta(\zeta_\beta[0,l); n-i; v) & \text{when} \quad d > k-m-l \\ G_\beta(\zeta_\beta[0,l); n; v) & \text{when} \quad d = k-m-l \end{cases}$$

$$\begin{aligned}(4.9)\qquad G_\beta(\zeta_\beta[0,l); n; v) \quad &= \quad \sum_{i=1}^{m} a_{l+i-1} G_\beta(\epsilon; n-i; v)\\ &\quad + G_\beta(\zeta_\beta[0,l); n-m; v)\end{aligned}$$

*where $d = d(u[k-m-l,k)) + k-m-l$.*

PROOF. First, we remark that $u = u[0,d) \cdot \zeta_\beta[0, k-d)$. From Proposition 3.2, we have the following partitioning:

$$Y_\beta(\epsilon; n+l; v) \setminus \bigsqcup_{j=0}^{l-1} \bigsqcup_{i=0}^{a_j-1} \zeta_\beta[0,j) \cdot i \cdot Y_\beta(\epsilon; n+l-j-1; v)$$

$$= Y_\beta(\zeta_\beta[0,l);n;v)$$

$$= Y_\beta(\epsilon;n+m+l;v) \setminus \bigsqcup_{j=0}^{m+l-1} \bigsqcup_{i=0}^{a_j-1} \zeta_\beta[0,j)\cdot i\cdot Y_\beta(\epsilon;n+m+l-j-1;v).$$

Then, (4.7) holds. When $d = k-m-l$, it is trivial to obtain (4.8) from Proposition 3.2. When $d > k-m-l$, we obtain the following partitioning:

$$Y_\beta(u;n;v) = \bigsqcup_{j=1}^{m+l-(k-d)} \bigsqcup_{i=0}^{a_{k-d+j-1}-1} u[0,d)\cdot\zeta_\beta[0,k-d+j)\cdot i\cdot w\cdot v$$

where $\zeta_\beta[0,l)\cdot w\cdot v \in Y_\beta(\zeta_\beta[0,l);n-j;v)$. We also have

$$\begin{aligned} Y_\beta(\zeta_\beta[0,l);n;v) &= \bigsqcup_{j=1}^{m} \bigsqcup_{i=0}^{a_{l+j-1}-1} \zeta_\beta[0,l)\cdot\zeta_\beta[l,l+j-1)\cdot i\cdot Y_\beta(\epsilon;n-j;v) \\ &\qquad \bigsqcup \zeta_\beta[0,l+m)\cdot Y_\beta(\zeta_\beta[0,l);n-m;v). \end{aligned}$$

The lemma follows from these partitionings. □

Proof of Theorem 4.1. Let $k>0$, $u \in Y_\beta(k)$. Let $M \in \mathbf{N}$ and $b = (b_0, b_1, \ldots, b_{m-1}) = L_\beta(M)$. We assume $M$ to satisfy $m > k$. Define

$$\Delta(I;P) = A(I;P) - M\lambda(I),$$

where $I$ is an interval in $[0,1)$ and $P = \{x_1, x_2, \ldots, x_M\} \subset [0,1)$. For any finite sets of points $P, P'$ in $[0,1)$ and any intervals $I, I' \subset [0,1)$, $I \cap I' = \emptyset$,

$$\begin{aligned} \Delta(I;P\sqcup P') &= \Delta(I;P)+\Delta(I;P') \\ \Delta(I\sqcup I';P) &= \Delta(I;P)+\Delta(I';P) \end{aligned} \tag{4.10}$$

hold. Here, $P \sqcup P'$ is the disjoint union of $P$ and $P'$ or the union of $P$ and $P'$ with multiplicity. From Definition 4.3 and (4.10), we have

$$\begin{aligned} \Delta(B(u);N_\beta[M]) &= \Delta(B(u); \bigsqcup_{j=0}^{m-1} \bigsqcup_{i=0}^{b_j-1} Y_\beta(\epsilon;j;v_{ij})) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{b_j-1} \Delta(B(u);Y_\beta(\epsilon;j;v_{ij})) \end{aligned} \tag{4.11}$$

where $v_{ij} = i \cdot b[j+1, m)$. Consider the $0 \le j \le k$ part of the right hand side of (4.11).

$$\sum_{j=0}^{k} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| \le \sum_{j=0}^{k} ([\beta]+1) G_\beta(j) R_\beta(u) \tag{4.12}$$

holds from the definition of $\Delta$. From Lemma 4.1 and Lemma 4.4, there exists a constant $C'$ and $G_\beta(j) \le C' \beta^j$ holds for any $j$. From this and $R_\beta(u) \le \beta^{-k}$, there exists a constant $C_0$, and

$$\sum_{j=0}^{k} ([\beta]+1) G_\beta(j) R_\beta(u) < C_0$$

is satisfied for any $k$. Then, from (4.11) and (4.12), we have

$$\Delta(B(u); N_\beta[M]) \le C_0 + \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| . \tag{4.13}$$

Define

$$\begin{aligned} \delta(u; n) &= G^0_\beta(u; n) - \frac{\beta^{n+k} R_\beta(u)}{\phi'_\beta(1)} \\ \delta(n) &= G^0_\beta(n) - \frac{\beta^n}{\phi'_\beta(1)} \end{aligned}$$

for $u \in Y_\beta(k)$ and $k, n \ge 0$. From this definition,

$$\begin{aligned} |\Delta(B(u); Y^0_\beta(n))| &= |G^0_\beta(u; n) - R_\beta(u) G^0_\beta(k+n)| \\ &= |\delta(u; n) - R_\beta(u) \delta(k+n)| \end{aligned} \tag{4.14}$$

holds. Then, from Lemma 4.2 we have

$$\begin{aligned} &\sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} |\Delta(B(u); Y_\beta(\epsilon; j; v_{ij}))| \\ &\le \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} \left( \sum_{\substack{l=1,\dots,j \\ j-l-1 \notin e(v_{ij})}} \left| \Delta(B(u); Y^0_\beta(l) \cdot \zeta_\beta[0, j-l)) \right| + 1 \right) \\ &\le \sum_{j=k+1}^{m-1} \sum_{i=0}^{b_j - 1} \left( \sum_{l=1}^{j} \left| \Delta(B(u); Y^0_\beta(l)) \right| + 1 \right) . \end{aligned} \tag{4.15}$$

From the **(PV)** condition and Lemma 4.4, there exist $r > \beta$ and a constant $C_r$ that satisfy

$$|\delta(u;n)| \le \frac{C_r}{n}\left(\frac{\beta}{r}\right)^n \tag{4.16}$$

for any $n, k > 0$ and $u \in Y_\beta(k)$. From (4.13), (4.14), (4.15), (4.16), and $r > \beta$, we see that

$$\begin{aligned}\Delta(B(u); N_\beta[M]) &\le C_0 + C_r([\beta]+1)\\ &\quad\cdot \sum_{j=k+1}^{m-1}\left(\sum_{l=1}^{j}\left(\frac{1}{l}\left(\frac{\beta}{r}\right)^l + \frac{1}{k+l}\left(\frac{\beta}{r}\right)^{k+l} R_\beta(u)\right)+1\right)\\ &= O(m) = O(\log M)\end{aligned} \tag{4.17}$$

holds.

Choose an arbitrary $t \in [0,1)$. Let $M \in \mathbf{N}$ and $L_\beta(M) = (b_0, \ldots, b_{m-1})$. Let $B(t_0, \ldots, t_{m-1})$ be a cylinder of rank $m$ that satisfies $t \in B(t_0, \ldots, t_{m-1})$. Then we have

$$[0,t) = B_{s_1} \sqcup B_{s_2} \sqcup \ldots \sqcup B_{s_k} \sqcup R,$$

where $0 \le s_1 < s_2 < \ldots < s_k = m-1$, $B_{s_i}$ is a disjoint union of up to $[\beta]+1$ cylinders of rank $s_i$ and $\lambda(R) < \beta^{-m+1}$. Then from (4.10) and (4.17), we have

$$|\Delta([0,t); N_\beta[M])| = O((\log M)^2),$$

and therefore

$$D_M^*(N_\beta) = O\left(\frac{(\log M)^2}{M}\right).$$

In the following part, we consider the case in which $\beta$ is Markov. Let $\zeta_\beta = (a_0, \ldots, a_{l'-1}, \dot{a_{l'}}, \ldots, \dot{a_{l-1}})$ and $l - l' = p(\beta)$. Then, $\beta$ is the unique $z > 1$ solution of

$$z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} = z^{l'} - \sum_{i=0}^{l'-1} a_i z^{l'-1-i}. \tag{4.18}$$

Let $\alpha_1, \ldots, \alpha_q$ be the conjugates of $\beta$ with respect to the equation (4.18), that is,

$$z^l - \sum_{i=0}^{l-1} a_i z^{l-1-i} - z^{l'} + \sum_{i=0}^{l'-1} a_i z^{l'-1-i} = (z-\beta)\prod_{i=1}^{q}(z-\alpha_i)^{l_i}$$

where $l_i \geq 1$, $\alpha_i \neq \alpha_j$ for all $i \neq j$ and $\sum_{i=1}^{q} l_i = l - 1$. We also have

$$|\alpha_i| < 1, \quad \text{for all } i \in \{1, \ldots, q\} \tag{4.19}$$

from the **(PV)** condition. Let $v \in X_\beta$. From Lemma 4.5, there exist complex numbers $c$, $c_{ij}$ ($i = 1, \ldots, q$, $j = 0, \ldots l_i - 1$) that satisfy the following equation:

$$G_\beta(\epsilon; n; v) = c\beta^n + \sum_{i=1}^{r} \sum_{j=0}^{l_i - 1} c_{ij} n^j \alpha_i^n \quad \text{for all} \quad n \in \mathbf{N}. \tag{4.20}$$

From Lemma 4.3, Lemma 4.5, and (4.20), we have

$$\begin{aligned} &\Delta(B(u); N_\beta[G_\beta(\epsilon; k+n; v)]) \\ &= \begin{cases} \displaystyle\sum_{h=1}^{q} \sum_{j=0}^{l_h - 1} c_{hj} \left( n^j \alpha_h^n - \frac{1}{\beta^k} (k+n)^j \alpha_h^{k+n} \right), \\ \qquad \text{when} \quad d = k - l \\ \displaystyle\sum_{i=k-d}^{l-1} a_i \sum_{h=1}^{q} \sum_{j=0}^{l_h - 1} c_{hj} \\ \quad \cdot \left( (k+n-d)^j \alpha_h^{k+n-d-i} - \frac{1}{\beta^{d+i}} (k+n)^j \alpha_h^{k+n} \right), \\ \qquad \text{when} \quad d > k - l \end{cases} \end{aligned} \tag{4.21}$$

where $u \in Y_\beta(k)$, $n \in \mathbf{N}$, and $d = d(u[\max\{0, k-l+1\}, k+1)) + k - l$. From (4.10), (4.13), (4.15), (4.19), and (4.21), there exists a constant $C$ that satisfies the following inequality (4.22) for any cylinder $B(u)$ of any rank $k$ and $M > G_\beta(l + d)$.

$$|\Delta(B(u); N_\beta[M])| < C \tag{4.22}$$

Then, we obtain

$$D_M^*(N_\beta) = O\left(\frac{\log M}{M}\right)$$

by the above reasoning. □

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