J. Math. Sci. Univ. Tokyo
5 (1998), 41–74.

Laplace Distributions and Hyperfunctions on $\overline{\mathbb{R}}^n_+$

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The paper presents foundations of the theory of Laplace distributions in several variables. Laplace distributions are investigated from the point of view of two different frameworks: of functional analysis and hyperfunction theory. The main results are the Martineau–Harvey type theorems establishing topological isomorphism between the spaces of Laplace distributions regarded as the dual space of Laplace test functions and those regarded as certain quotient spaces of holomorphic functions of exponential growth.

Automatically these results lead to the imbedding of Laplace distributions in the space of Laplace hyperfunctions. We consider only a canonical realization of hyperfunctions as the sum of boundary values from wedges modelled over coordinate orthants in $\mathbb{R}^n$ and avoid introducing coordinate independent versions based on suitable relative cohomologies. The realizations are convenient for the applications to PDE's. Namely solutions to a large class of constant coefficient PDE's can be represented at infinity as sums of Laplace integrals of the form $T[e^{x \cdot z}]$ where T is a Laplace distribution whose support is related to the complex geometry of the characteristic set (see [S2–Z1], [Z1], [Z2]). Similar results have also been established for semilinear Laplace equations [P–Z].

Finally let us note that Laplace hyperfunctions considered in this paper can be regarded as a special case of Fourier hyperfunctions (cf. [K], [Ka], [S–Mo]) and are closely related to those introduced by Komatsu [Ko] in the case of one variable.

Notation. Throughout this paper we shall deal with polytubular open

1991 *Mathematics Subject Classification.* 46F12, 46F15, 46F20.
Key words: Laplace distributions, Laplace hyperfunctions, boundary values.
Supported by KBN grant No. 2 P03 A00608.

sets $W \subset \mathbb{C}^n$ defined as follows: $W = W_1 \times \cdots \times W_n$ and there exists $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ such that W_j contains a tubular neighbourhood of the halfline $\zeta_j + \mathbb{R}_+$, i.e.

$$W_j \supset (\zeta_j + \overline{\mathbb{R}}_+)_\varepsilon \overset{\mathrm{df}}{=} \{z : \operatorname{dist}(z, \zeta_j + \overline{\mathbb{R}}_+) < \varepsilon\}$$

for some $\varepsilon > 0$ $(j = 1, \ldots, n)$. If $\zeta = 0$ and $W_j \supset (\overline{\mathbb{R}}_+)_\varepsilon$, $j = 1, \ldots, n$, we say that W is an open polytubular neighbourhood of $\overline{\mathbb{R}}^n_+$. We shall also need the following sets

$$\begin{aligned} W \mathbin{\backslash\!\!\!\backslash} \overline{\mathbb{R}}^n_+ &= (W_1 \setminus \overline{\mathbb{R}}_+) \times \cdots \times (W_n \setminus \overline{\mathbb{R}}_+), \\ W \mathbin{\backslash\!\!\!\backslash}_j \overline{\mathbb{R}}^n_+ &= (W_1 \setminus \overline{\mathbb{R}}_+) \times \cdots \times W_j \times \cdots \times (W_n \setminus \overline{\mathbb{R}}_+) \quad (j = 1, 2, \ldots, n). \end{aligned}$$

We write $V \Subset W$ if V is a subset of W such that $\operatorname{dist}(V, \operatorname{bd} W)$ is strictly positive. We denote $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$. If Γ' is a proper subcone of an open cone $\Gamma \subset \dot{\mathbb{R}}^n$, we write $\Gamma' \Subset \Gamma$. By $\Gamma|_r$ we denote the intersection of the cone Γ with a ball of radius r with centre at zero. For $x, y \in \mathbb{R}^n$ we denote by $x \cdot y$ the scalar product $x \cdot y = \sum_{j=1}^n x_j y_j$.

1. Basic Spaces and Their Properties

We define the following spaces.

The space $\mathfrak{L}_{(\omega)}(W)$ of holomorphic functions on W having exponential growth of type $\omega \in (\mathbb{R} \cup \{\infty\})^n$ at infinity is defined by

$$\mathfrak{L}_{(\omega)}(W) = \{H \in \mathcal{O}(W) : \ q_{\delta, \widetilde{W}}(H) < \infty \qquad \text{for every } \delta \in \mathbb{R}^n,\ \delta < \omega$$
$$\text{and every closed (in } \mathbb{C}^n\text{) polytubular subset } \widetilde{W} \Subset W\},$$

where $q_{\delta, \widetilde{W}}(H) = \sup_{\zeta \in \widetilde{W}} |e^{\delta \cdot \zeta} H(\zeta)|$ are seminorms defining the topology in $\mathfrak{L}_{(\omega)}(W)$.

Let for $k \in \mathbb{N}_0$, $\kappa \in \mathbb{R}^n$

$$\mathfrak{L}^k_\kappa(W \mathbin{\backslash\!\!\!\backslash} \overline{\mathbb{R}}^n_+) = \{\psi \in \mathcal{O}(W \mathbin{\backslash\!\!\!\backslash} \overline{\mathbb{R}}^n_+) : \ \theta^k_{\kappa, V}(\psi) < \infty$$
$$\text{for every polytubular subset } V \Subset W\},$$

with the convergence defined by the family of norms

$$\theta^k_{\kappa,V}(\psi) = \sup_{\alpha+i\beta\in V} e^{\alpha\cdot\kappa}|\psi(\alpha+i\beta)|\,\|\beta\|^k$$

for any $V \Subset W$ and denote $\mathfrak{L}^k_{(\omega)}(W\#\overline{\mathbb{R}}^n_+) = \varprojlim_{\kappa<\omega} \mathfrak{L}^k_\kappa(W\#\overline{\mathbb{R}}^n_+)$.

Now we define the spaces $\underset{\sim}{L}_a(W)$, $a \in \mathbb{R}^n$:

$$\underset{\sim}{L}_a(W) = \{\sigma \in \mathcal{O}(W) :\ \rho_{a,V}(\sigma) < \infty \quad \text{for every polytubular} \ \ V \Subset W\},$$

with the convergence defined by the family of norms

$$\rho_{a,V}(\sigma) = \sup_{\zeta\in V}|e^{-a\cdot\zeta}\sigma(\zeta)| \quad \text{for any } V \Subset W,$$

and denote $\underset{\sim}{L}_a(\overline{\mathbb{R}}^n_+) \overset{\text{df}}{=} \varinjlim_{W\supset\overline{\mathbb{R}}^n_+} \underset{\sim}{L}_a(W)$, where W ranges over open polytubular neighbourhoods of $\overline{\mathbb{R}}^n_+$. We put for any $\omega \in (\mathbb{R}\cup\{\infty\})^n$ $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+) \overset{\text{df}}{=} \varinjlim_{a<\omega} \underset{\sim}{L}_a(\overline{\mathbb{R}}^n_+)$. By $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$[(1)] we denote the dual space to $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$.

Note that:

1. $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ if and only if there exists a polytubular neighbourhood $W \Supset \overline{\mathbb{R}}^n_+$ and $a < \omega$ such that $\sigma \in \mathcal{O}(W)$ and $\rho_{a,V}(\sigma) < \infty$ for every $V \Subset W$.
2. $f \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ if and only if for every $a < \omega$, f is linear on $\underset{\sim}{L}_a(\overline{\mathbb{R}}^n_+)$ and for every open polytubular set $V \supset \overline{\mathbb{R}}^n_+$ there exists $C_{a,V} < \infty$ such that[(2)]

$$|f[\sigma]| \le C_{a,V}\cdot\rho_{a,V}(\sigma) \quad \text{for} \ \ \sigma \in \underset{\sim}{L}_a(W),$$

where W is an arbitrary open polytubular set such that $V \Subset W$.

Next for $a \in \mathbb{R}^n$ we define the space

$$L_a(\overline{\mathbb{R}}^n_+) = \{\varphi \in C^\infty(\overline{\mathbb{R}}^n_+) :\ \gamma_{a,\nu}(\varphi) < \infty \quad \text{for every } \nu \in \mathbb{N}^n_0\},$$

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(1) The elements of $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ may have support also at infinity, cf. [Mo–Y].

(2) This follows from the fact that $\underset{\sim}{L}_a(\overline{\mathbb{R}}^n_+) = \varinjlim_{V\supset\overline{\mathbb{R}}^n_+} \underset{\sim}{L}_a(\overline{V})$, where $\underset{\sim}{L}_a(\overline{V}) = \{\sigma \in \mathcal{O}(V)\cap C(\overline{V}) :\ \rho_{a,V}(\sigma) < \infty\}$ is a Banach space.

with the convergence defined by the seminorms

$$\gamma_{a,\nu}(\varphi) = \sup_{x\in\overline{\mathbb{R}}^n_+} |e^{-a\cdot x}\big(\frac{\partial}{\partial x}\big)^{\nu}\varphi(x)|,\ \nu\in\mathbb{N}_0^n.$$

For $k\in\mathbb{N}_0,\ a\in\mathbb{R}^n$ we denote

$$L^k_a(\overline{\mathbb{R}}^n_+) = \{\varphi\in C^k(\overline{\mathbb{R}}^n_+):\ \gamma_{a,\nu}(\varphi)<\infty \quad \text{for} \quad |\nu|\le k\}$$

and for an $\omega\in(\mathbb{R}\cup\{\infty\})^n$ we define

$$L_{(\omega)}(\overline{\mathbb{R}}^n_+) = \varinjlim_{a<\omega} L_a(\overline{\mathbb{R}}^n_+), \qquad L^k_{(\omega)}(\overline{\mathbb{R}}^n_+) = \varinjlim_{a<\omega} L^k_a(\overline{\mathbb{R}}^n_+)$$

equipped with the inductive limit topology.

The spaces defined above are counterparts of those introduced in [S2–Z1] in the case of Mellin distributions. They are isomorphic to the Mellin spaces under the logarithmic change of variable and hence the following their properties can be derived therefrom.

Since the set $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ of restrictions to $\overline{\mathbb{R}}^n_+$ of functions in $C^\infty_0(\mathbb{R}^n)$ is dense in $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ (cf. [S2–Z1]), the dual space $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ is a subspace of $D'(\overline{\mathbb{R}}^n_+)$ (=the dual space of $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$). We call it the *space of Laplace distributions on* $\overline{\mathbb{R}}^n_+$.

Note that:

3. The spaces $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ are complete (cf. [S2–Z1]), $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ is dense in $L^k_{(\omega)}(\overline{\mathbb{R}}^n_+)$ for any $k\in\mathbb{N}$ and $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)_{|\overline{\mathbb{R}}^n_+}$ is dense in $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$.
4. There exist natural imbeddings $\underset{\sim}{L}_c(\overline{\mathbb{R}}^n_+)\hookrightarrow L_c(\overline{\mathbb{R}}^n_+)$ and $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)\hookrightarrow L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ given by the restrictions to $\mathbb{R}^n$. By duality they induce a natural imbedding
$$L'_{(\omega)}(\overline{\mathbb{R}}^n_+)\hookrightarrow \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+).$$

The following theorem characterizes the space of Laplace distributions on $\mathbb{R}^n_+$ (see [S2–Z1] Theorem 8.2 and p. 164):

Theorem 1 [Ły]. *Let* $\omega\in(\mathbb{R}\cup\{\infty\})^n$. *A distribution* $T\in D'(\overline{\mathbb{R}}^n_+)$ *is in* $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ *if and only if for every* $\kappa<\omega$ *there exist* $m_\kappa\in\mathbb{N}_0$ *and measurable functions* $T_{\nu,\kappa}$ *on* $\mathbb{R}^n$, $|\nu|\le m_\kappa$, *with support in* $\overline{\mathbb{R}}^n_+$ *such that*

$$T = \sum_{|\nu|\le m_\kappa} \Big(\frac{\partial}{\partial y}\Big)^{\nu} T_{\nu,\kappa} \quad in \quad L'_{(\kappa)}(\overline{\mathbb{R}}^n_+),$$

where

$$|T_{\nu,\kappa}(y)| \le C_\kappa e^{-\kappa\cdot y} \quad \textit{for} \;\; 0 \le y < \infty$$
$$\textit{almost everywhere with some } C_\kappa < \infty.$$

In the sequel besides the space $L_a(\overline{\mathbb{R}}^n_+)$ we shall deal with the space $L_a(\overline{\mathbb{R}}^n_+ + \mathring{x})$, where $\mathring{x} \in \mathbb{R}^n$:

$$L_a(\overline{\mathbb{R}}^n_+ + \mathring{x}) = \{\varphi \in C^\infty(\overline{\mathbb{R}}^n_+ + \mathring{x}) : \sup_{x\in\overline{\mathbb{R}}^n_+ + \mathring{x}} \Big| e^{-a\cdot x}\Big(\frac{\partial}{\partial x}\Big)^\nu \varphi(x)\Big| < \infty, \quad \nu \in \mathbb{N}^n_0\}.$$

Let $A \in GL(n,\mathbb{R})$, $\mathring{\xi} = A\mathring{x}$, $\alpha = Ax$. Then $a \cdot x = a \cdot A^{-1}\alpha = b \cdot \alpha$, where $b = (A^{\mathrm{tr}})^{-1}a$. If $\varphi \in C^\infty(\overline{\mathbb{R}}^n_+ + \mathring{x})$ then $\psi \overset{\mathrm{df}}{=} \varphi \circ A^{-1} \in C^\infty(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi})$. It is natural to define

$$L_b(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi}) = \{\psi : \psi \circ A \in L_{A^{\mathrm{tr}}b}(\overline{\mathbb{R}}^n_+ + \mathring{x})\}$$

and we easily see that

$$L_b(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi}) = \{\psi \in C^\infty(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi}) : \sup_{\alpha\in A(\overline{\mathbb{R}}^n_+ + \mathring{x})} |e^{-b\cdot\alpha}\Big(\frac{\partial}{\partial \alpha}\Big)^\nu \psi(\alpha)| < \infty, \quad \nu \in \mathbb{N}^n_0\}.$$

Introduce further the space $L_{(\kappa)}(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi})$ in any of the following equivalent ways:

$$L_{(\kappa)}(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi}) = \{\psi : \psi \circ A \in L_{(A^{\mathrm{tr}}\kappa)}(\overline{\mathbb{R}}^n_+ + \mathring{x})\} = \varinjlim_{A^{\mathrm{tr}}b < A^{\mathrm{tr}}\kappa} L_b(A(\overline{\mathbb{R}}^n_+) + \mathring{\xi}).$$

Now, let $\Omega \subset \mathbb{R}^n$ be an open set and let $a \in \mathbb{R}^n$. We define

$$L_a(\Omega) = \{\varphi \in C^\infty(\Omega) : \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > 0, \;\; \gamma_{a,\nu}(\varphi) < \infty, \;\; \nu \in \mathbb{N}^n_0\}.$$

If for some open set $\Omega \subset \mathbb{R}^n$ $L_a(\Omega) \subset L_b(\Omega)$ for $a < b$, we put as usual $L_{(\omega)}(\Omega) = \varinjlim_{a<\omega} L_a(\Omega)$ and denote by $L'_{(\omega)}(\Omega)$ the dual space, called the space of *Laplace distributions in* Ω (indexed by ω).

2. Characterization of Holomorphic Functions Whose Boundary Values are Laplace Distributions

In this section we provide conditions under which the boundary value of a holomorphic function is a Laplace distribution. We start with the following lemma.

LEMMA 1. *Let Ω be an open set in $\mathbb{R}^n$ for which there exists $b \in \mathbb{R}^n$ such that the function*

$$\Omega \ni \alpha \longmapsto e^{\alpha \cdot b} \quad \textit{is integrable.} \tag{1}$$

Let $\kappa \in \mathbb{R}^n, r \in \mathbb{R}_+, \boldsymbol{r} = (r, \dots, r), \boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ and let a function $F \in \mathcal{O}(\Omega + i(0, \boldsymbol{r}))$ be of growth type $\boldsymbol{k}$ near Ω, i.e. such that

$$|F(\alpha + i\beta)| \le C \frac{e^{-\alpha \cdot \kappa}}{\beta_1^{k_1} \cdot \ldots \cdot \beta_n^{k_n}} \quad \textit{for } \alpha + i\beta \in \Omega + i(0, \boldsymbol{r}), \quad C < \infty. \tag{2}$$

Let $a = b + \kappa$. Then there exists $T \in L'_a(\Omega)$ such that

$$\lim_{\beta \to 0_+} \int_\Omega F(\alpha + i\beta)\varphi(\alpha) d\alpha = T[\varphi] \quad \textit{for } \varphi \in L_a(\Omega). \tag{3}$$

More precisely, there exist functions $H_\nu \in C^0(\Omega)$ such that $|H_\nu(\alpha)| e^{\alpha \cdot \kappa} \le C < \infty$ for $\alpha \in \Omega$, $\nu \in \mathbb{N}_0^n$, $\nu \le \boldsymbol{k} + \mathbb{2}$ (i.e. $\nu_s \le k_s + 2,\ s = 1, \dots, n$) and $T = \sum_{\nu \le \boldsymbol{k} + \mathbb{2}} \left(\frac{\partial}{\partial \alpha}\right)^\nu H_\nu$ in $L'_a(\Omega)$.

REMARK 1. Let $H \in C^0(\Omega)$, $|H(\alpha)| e^{\alpha \cdot \kappa} \le C < \infty$ for $\alpha \in \Omega$ and let $\nu \in \mathbb{N}_0^n$. Then by assumption (1) $u \stackrel{\text{df}}{=} \left(\frac{\partial}{\partial \alpha}\right)^\nu H \in L'_a(\Omega)$, since $a = b + \kappa$.

PROOF OF LEMMA 1. We shall prove Lemma 1 for $n = 2$ and introduce to this aim the operators[(3)]

$$J_1 F(z) = \int_{z_1}^{z_1 + i\delta} F(\zeta_1, z_2) d\zeta_1, \qquad J_2 F(z) = \int_{z_2}^{z_2 + i\delta} F(z_1, \zeta_2) d\zeta_2 \tag{4}$$

[(3)] To prove the general case one can take n operators $J_1, \dots, J_n$ defined analogously and proceed in the same way.

for $z = (z_1, z_2) \in \Omega + i(0, \boldsymbol{r})$, $0 < \delta < r$, $0 < \beta_k = \operatorname{Im} z_k < r - \delta$, $k = 1, 2$.

If $z = \alpha + i\beta$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, we obtain by (4)

$$\begin{aligned} \frac{\partial J_1 F}{\partial z_1}(z) &= F(z_1 + i\delta, z_2) - F(z), \\ (J_1 F)(z) &= i \int_{\beta_1}^{\beta_1+\delta} F(\alpha_1 + is, z_2) ds, \end{aligned} \tag{5}$$

and analogous formulae for $J_2 F$. Hence

$$(J_2(J_1 F))(z) = -\int_{\beta_2}^{\beta_2+\delta} \Big\{ \int_{\beta_1}^{\beta_1+\delta} F(\alpha_1 + is, \alpha_2 + it) ds \Big\} dt. \tag{6}$$

We prove Lemma 1 for F of the following growth types: (i) case $(0, 0)$; (ii) case $(1, 0)$; (iii) case $(1, 1)$; (iv) case $(2, 0)$ (neglecting the obvious symmetrical cases $(0, 1)$ and $(0, 2)$) and hence establish that it is true for $k_1, k_2 \in \mathbb{N}_0$, $k_1 + k_2 \leq 2$. Next we proceed by induction.

(i) Case $(0, 0)$. Then by (2) and (6) $J_2 J_1 F(\alpha + i\beta)$ is locally uniformly convergent as $\beta \to 0_+$ to a continuous function F^{11}. By (5) we get

$$\begin{aligned} F(z_1, z_2) = \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} J_2 J_1 F(z) &- F(z_1 + i\delta, z_2 + i\delta) \\ &+ F(z_1, z_2 + i\delta) + F(z_1 + i\delta, z_2) \end{aligned} \tag{7}$$

and from this formula we shall find the limit (3). It is easy to see that $|J_2 J_1 F(\alpha + i\beta)| \leq C\delta^2 e^{-\alpha \cdot \kappa}$ hence also $|F^{11}(\alpha)| \leq C\delta^2 e^{-\alpha \cdot \kappa}$. So for $\varphi \in L_a(\Omega)$ $(a = b + \kappa)$ we get by assumption (1)

$$\lim_{\beta \to 0_+} \int_\Omega \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2} J_2 J_1 F(z) \cdot \varphi(\alpha) d\alpha = \int_\Omega F^{11}(\alpha) \frac{\partial^2 \varphi}{\partial \alpha_1 \partial \alpha_2} d\alpha \tag{8}$$

since $|J_2 J_1 F(\alpha + i\beta) \frac{\partial^2 \varphi}{\partial \alpha_1 \partial \alpha_2}| \leq M e^{\alpha \cdot b}$ with some constant M. Observe that by (5) the third summand on the right-hand side of (7) can be written in the form:

$$F(z_1, z_2 + i\delta) = F(z_1 + i\delta, z_2 + i\delta) - \frac{\partial J_1 F}{\partial z_1}(z_1, z_2 + i\delta) \tag{9}$$

and by (2) we get assuming $0 \le \beta_1' - \beta_1'' \le \delta$

$$(10)\quad \begin{aligned} &|J_1F(\alpha_1 + i\beta_1', \alpha_2 + i(\beta_2' + \delta)) - J_1F(\alpha_1 + i\beta_1'', \alpha_2 + i(\beta_2'' + \delta))| \\ &\le 2Ce^{-\alpha\cdot\kappa}|\beta_1'' - \beta_1'| + \delta \sup_{\beta_1' \le s \le \beta_1' + \delta} |F(\alpha_1 + is, z_2') - F(\alpha_1 + is, z_2'')|, \end{aligned}$$

where $z_2' = \alpha_2 + i(\beta_2' + \delta)$, $z_2'' = \alpha_2 + i(\beta_2'' + \delta)$.

To estimate the second summand in (10) we consider $F(z_1, z_2)$ as a function of z_2 depending on a parameter z_1: $G_{z_1}(z_2) \overset{\text{df}}{=} F(z_1, z_2)$. By the Cauchy integral formula we get:

$$\frac{\partial G_{z_1}}{\partial z_2}(z_2) = \frac{1}{2\pi i} \int_{|\zeta - z_2| = \delta} \frac{G_{z_1}(\zeta)}{(\zeta - z_2)^2} d\zeta$$

and by (2): $|G_{z_1}(\zeta)| \le Ce^{-\alpha\cdot\kappa + |\kappa_2|\delta}$ for $|\zeta - z_2| = \delta$. Hence we obtain the estimate

$$|G_{\alpha_1 + is}(z_2') - G_{\alpha_1 + is}(z_2'')| \le Ce^{-\alpha\cdot\kappa} \frac{1}{\delta} e^{|\kappa_2|\delta} |\beta_2'' - \beta_2'|$$

independent of $s \in [\beta_1', \beta_1' + \delta]$ and thus by (10) $\lim_{\beta \to 0_+} J_1F(\alpha_1 + i\beta_1, \alpha_2 + i(\beta_2 + \delta))$ exists locally uniformly and defines a continuous function $F^1(\alpha)$ on Ω, $|F^1(\alpha)| \le C\delta e^{-\alpha\cdot\kappa}$ for $\alpha \in \Omega$. Thus by (9) and (1) we get

$$(11)\quad \begin{aligned} &\lim_{\beta \to 0_+} \int_\Omega F(\alpha_1 + i\beta_1, \alpha_2 + i(\beta_2 + \delta))\varphi(\alpha) d\alpha \\ &\qquad = \int_\Omega F(\alpha_1 + i\delta, \alpha_2 + i\delta)\varphi(\alpha) d\alpha + \int_\Omega F^1(\alpha) \frac{\partial \varphi}{\partial \alpha_1} d\alpha, \end{aligned}$$

since $|J_1F(\alpha_1 + i\beta_1, \alpha_2 + i(\beta_2 + \delta)) \frac{\partial \varphi}{\partial \alpha_1}| \le M_1 e^{\alpha\cdot b}$ with some constant M_1 independent of β. We also prove that there exists a continuous function F^2 on Ω such that $|F^2(\alpha)| \le C\delta e^{-\alpha\cdot\kappa}$ and

$$(12)\quad \begin{aligned} &\lim_{\beta \to 0_+} \int_\Omega F(\alpha_1 + i(\beta_1 + \delta), \alpha_2 + i\beta_2)\varphi(\alpha) d\alpha \\ &\qquad = \int_\Omega F(\alpha_1 + i\delta, \alpha_2 + i\delta)\varphi(\alpha) d\alpha + \int_\Omega F^2(\alpha) \frac{\partial \varphi}{\partial \alpha_2} d\alpha. \end{aligned}$$

Thus by (7), (8), (11), (12) we get

$$\lim_{\beta\to 0_+}\int_\Omega F(\alpha+i\beta)\varphi(\alpha)d\alpha = \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}F^{11}[\varphi]-\frac{\partial}{\partial\alpha_1}F^1[\varphi]-\frac{\partial}{\partial\alpha_2}F^2[\varphi]+F^3[\varphi]$$

for $\varphi \in L_a(\Omega)$, where $F^3(\alpha) = F(\alpha_1+i\delta, \alpha_2+i\delta)$, $|F^3(\alpha)| \le Ce^{-\alpha\cdot\kappa}$ for $\alpha \in \Omega$. Hence assertion (3) follows with $T = \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}F^{11} - \frac{\partial}{\partial\alpha_1}F^1 - \frac{\partial}{\partial\alpha_2}F^2 + F^3$ and by Remark 1 $T \in L'_a(\Omega)$ is a Laplace distribution of multiorder[(4)] $(1,1)$ (hence also of multiorder $(2,2)$). The proof of Lemma 1 in the case (i) is thus complete.

For the proof of (3) in the cases (ii)–(iv) the following remark will be useful: If $\boldsymbol{k} = (k_1, 0)$, $k_1 \ge 1$ and $0 < \delta < r(k_1+1)^{-1}$, then by (4) and (2) we get with some constant $C_{k_1+1} < \infty$

$$(13)\qquad |J_1^{k_1+1}F(\alpha+i\beta)| < C_{k_1+1}e^{-\alpha\cdot\kappa} \quad \text{for } 0<\beta_1<r-(k_1+1)\delta,\ 0<\beta_2<r.$$

Thus $J_1^{k_1+1}F$ is of growth type $(0,0)$ for $\alpha+i\beta \in \Omega + i\big((0, r-(k_1+1)\delta)\times(0,r)\big)$ and there exists a Laplace distribution $T_{k_1} \in L'_a(\Omega)$ of multiorder $(1,1)$ such that

$$(14)\qquad \lim_{\beta\to 0_+}\int_\Omega J_1^{k_1+1}F(\alpha+i\beta)\varphi(\alpha)d\alpha = T_{k_1}[\varphi] \quad \text{for } \varphi\in L_a(\Omega).$$

If F is of growth type (k_1, k_2) with $k_1 > 1$ then J_1F is of growth type (k_1-1, k_2).

(ii) Case $(1,0)$. By (5) we get

$$(15)\qquad \frac{\partial^2}{\partial z_1^2}J_1^2F(z) = F(z_1+2i\delta, z_2) - 2F(z_1+i\delta, z_2) + F(z_1, z_2).$$

Since $|F(z_1+ip\delta, z_2)| \le \frac{C}{p\delta}e^{-\alpha\cdot\kappa}$ $(p=1,2)$, by the case (i) there exist Laplace distributions $T_2, T_3 \in L'_a(\Omega)$ of multiorder $(1,1)$ such that

$$\lim_{\beta\to 0_+}\int_\Omega F(\alpha_1+i(\beta_1+p\delta), \alpha_2+i\beta_2)\varphi(\alpha)d\alpha = T_{p+1}[\varphi] \quad \text{for } \varphi\in L_a(\Omega),\ p=1,2,$$

(4) For the use of the proof we introduce the notion of the multiorder of a Laplace distribution T; namely we say that $T \in L'_a(\Omega)$ is of multiorder $\boldsymbol{p}\in\mathbb{N}_0^n$ if $T = \sum_{\nu\le\boldsymbol{p}}\big(\frac{\partial}{\partial\alpha}\big)^\nu H_\nu$ with H_ν such as H in Remark 1.

and hence by (14), (15) we get for $\varphi \in L_a(\Omega)$:

$$\lim_{\beta\to 0_+}\int_\Omega F(\alpha+i\beta)\varphi(\alpha)d\alpha = T_1\Big[\frac{\partial^2\varphi}{\partial\alpha_1^2}\Big]+2T_2[\varphi]-T_3[\varphi]=T[\varphi],$$

where $T=\frac{\partial^2 T_1}{\partial\alpha_1^2}+2T_2-T_3$ is a Laplace distribution of multiorder $(3,1)$, $T\in L_a'(\Omega)$.

(iii) Case $(1,1)$. Then with some constant $C<\infty$ $|J_2^2J_1^2F(\alpha+i\beta)|\le Ce^{-\alpha\cdot\kappa}$ for $0<\beta_j<r-2\delta$, $j=1,2$, and by (i) there exists a Laplace distribution $T_1\in L_a'(\Omega)$ of multiorder $(1,1)$ such that

$$\lim_{\beta\to 0_+}\int_\Omega J_2^2J_1^2F(\alpha+i\beta)\varphi(\alpha)d\alpha = T_1[\varphi]\quad\text{for}\quad \varphi\in L_a(\Omega). \tag{16}$$

For the proof of assertion (3) one derives $F(z)$ from the formula analogous to (15) with $\frac{\partial^2}{\partial z_1^2}\frac{\partial^2}{\partial z_2^2}J_2^2J_1^2F(z)$ on the left-hand side, this time. There is no problem with the terms in which both arguments of F are translated by $i\delta$ or $2i\delta$. If only one of the arguments is translated, for example the second one, we deduce by (13) and (i) that there exists $T_2\in L_a'$ of multiorder $(1,1)$ such that

$$\lim_{\beta\to 0_+}\int_\Omega J_1^2F(\alpha_1+i\beta_1,\alpha_2+i(\beta_2+\delta))\varphi(\alpha)d\alpha = T_2[\varphi]\quad\text{for}\quad \varphi\in L_a(\Omega). \tag{17}$$

By (15) and (17) we get for $\varphi\in L_a(\Omega)$:

$$\begin{aligned}\lim_{\beta\to 0_+}\int_\Omega & F(\alpha_1+i\beta_1,\alpha_2+i(\beta_2+\delta))\varphi(\alpha)d\alpha\\ &= T_2\Big[\frac{\partial^2\varphi}{\partial\alpha_1^2}\Big]+T_3[\varphi]+T_4[\varphi]=T_1^*[\varphi],\end{aligned} \tag{18}$$

where $T_3(\alpha)=2F(\alpha_1+i\delta,\alpha_2+i\delta)$, $T_4(\alpha)=-F(\alpha_1+2i\delta,\alpha_2+i\delta)$, $T_1^*=\frac{\partial^2}{\partial\alpha_1^2}T_2+T_3+T_4$ is a Laplace distribution of multiorder $(3,1)$. Thus by (16), (18) and analogous formulae for all the terms involving translation of one variable only we find a distribution $T^*\in L_a'(\Omega)$ of multiorder $(3,3)$ such that formula (3) holds with $T=(\frac{\partial}{\partial\alpha_1})^2(\frac{\partial}{\partial\alpha_2})^2T_1+T^*+\widetilde{T}\in L_a'(\Omega)$ of multiorder $(3,3)$, where $\widetilde{T}(\alpha)=-F(\alpha_1+2i\delta,\alpha_2+2i\delta)+2F(\alpha_1+i\delta,\alpha_2+2i\delta)+2F(\alpha_1+2i\delta,\alpha_2+i\delta)-4F(\alpha_1+i\delta,\alpha_2+i\delta)$.

(*iv*) Case $(2,0)$. The proof of (3) with T of multiorder $(4,1)$ follows by (*i*), (*ii*) and the formula

$$F(z_1,z_2) = -\frac{\partial}{\partial z_1} J_1F(z_1,z_2) + F(z_1+i\delta, z_2). \tag{19}$$

Thus we have proved Lemma 1 for F of growth type (k_1,k_2) with $k_1 \in \mathbb{N}_0$, $k_2 \in \mathbb{N}_0$, $k_1+k_2 \le 2$. For the proof by induction fix arbitrarily $p \in \mathbb{N}$, $p \ge 2$ and suppose that Lemma 1 is true for all (k_1,k_2) with $k_1 \in \mathbb{N}_0$, $k_2 \in \mathbb{N}_0$, $k_1+k_2 \le p$. We have to prove it for F having the following growth types (near Ω):

$$(p+1,0),\ (p,1),\ \dots,\ (2,p-1) \text{ and } (1,p),\ (0,p+1).$$

If F is of growth type belonging to the first group then J_1F is of growth type $(p,0),(p-1,1),\dots,(1,p-1)$ correspondingly and $F(z_1+i\delta,z_2)$ is of growth type $(0,0),(0,1),\dots,(0,p-1)$. Applying formula (19) and the induction assumption we get (3).

If F is of growth type $(1,p)$ or $(0,p+1)$ then J_2F is of growth type $(1,p-1)$ or $(0,p)$ correspondingly and by the induction assumption and the formula for J_2F analogous to (5) we get (3).

This ends the proof of Lemma 1 for $n=2$. □

The set $\Omega \subset \mathbb{R}^n$ in Lemma 1 was an arbitrary open set satisfying condition (1). In Proposition 1 we shall assume that $\Omega \subset \mathring{\xi} + \Gamma$ where $\mathring{\xi} \in \mathbb{R}^n$ and Γ is an open cone in $\dot{\mathbb{R}}^n$, whose *dual cone* $\Gamma^\perp \overset{\text{df}}{=} \{y \in \mathbb{R}^n : y\cdot x < 0 \text{ for every } x \in \Gamma\}$ has a non-empty interior.

Note that

(*i*) $(\mathbb{R}^n_+)^\perp = \mathbb{R}^n_-$; if $\Gamma \subset \mathbb{R}^n_+$ then $\Gamma^\perp \supset \mathbb{R}^n_-$; if for some $x \in \Gamma$, $-x \in \Gamma$ then $\Gamma^\perp$ is empty;

(*ii*) if $\operatorname{Int}\Gamma^\perp$ is not empty then $\int_\Gamma e^{b\cdot x}dx < \infty$ for every $b \in \operatorname{Int}\Gamma^\perp$.

(*iii*) Let Γ be a cone in $\dot{\mathbb{R}}^n$, $A \in GL(n,\mathbb{R})$ and let $\Delta = A\Gamma$. Then $\Delta^\perp = (A^{\text{tr}})^{-1}\Gamma^\perp$ and hence in particular $(A\mathbb{R}^n_+)^\perp$ is open and not empty.

Proof of (*iii*). Let $h \in \Delta^\perp$. For every $a \in \Gamma$ we have $h\cdot Aa < 0$ and hence also $A^{\text{tr}}h\cdot a < 0$. This means that $A^{\text{tr}}h \in \Gamma^\perp$, hence $h \in (A^{\text{tr}})^{-1}\Gamma^\perp$. The proof of the inclusion $(A^{\text{tr}})^{-1}\Gamma^\perp \subset \Delta^\perp$ is analogous. □

PROPOSITION 1. *Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega \subset \mathring{\xi} + \Gamma$, where $\mathring{\xi} \in \mathbb{R}^n$ and Γ is an open cone, $\operatorname{Int}\Gamma^\perp \neq \emptyset$. Let $F \in \mathcal{O}(\Omega + i\mathbb{R}^n_+|_r)$ satisfy condition (2). Then for every $a \in \operatorname{Int}\Gamma^\perp + \kappa$ there exists $T \in L'_a(\Omega)$ such that (3) holds.*

PROOF. Let $a \in \operatorname{Int}\Gamma^\perp + \kappa$. Then by *(ii)* $\int_\Gamma e^{(a-\kappa)\cdot x} dx < \infty$ and hence also $\int_\Omega e^{(a-\kappa)\cdot x} dx \le \int_{\mathring{\xi}+\Gamma} e^{(a-\kappa)\cdot x} dx = e^{(a-\kappa)\cdot \mathring{\xi}} \int_\Gamma e^{(a-\kappa)\cdot y} dy < \infty$. Thus our assertion follows by Lemma 1. □

Let $\Omega = \mathring{\xi} + \mathbb{R}^n_+$, $\mathring{\xi} \in \mathbb{R}^n$ and let Γ be an open connected cone in $\dot{\mathbb{R}}^n$. We denote by $[\Omega + i\Gamma]$ the *germ of the set* $\Omega + i\Gamma$ near $\mathbb{R}^n$, i.e. the class of open sets $V \subset \mathbb{C}^n$ for which there exists a complex neighbourhood W of $\mathbb{R}^n$ such that $V \cap W = (\Omega + i\Gamma) \cap W$. We write $F \in \mathcal{O}([\Omega + i\Gamma])$ if for some W as above $F \in \mathcal{O}((\Omega + i\Gamma) \cap W)$.

DEFINITION 1. Let $F \in \mathcal{O}\big([\Omega + i\Gamma]\big)$. Assume that there exists $\kappa \in \mathbb{R}^n$ such that for every $\Gamma' \Subset \Gamma$ and any $\beta \in \Gamma'$ close to zero the functional $L_{(\kappa)}(\Omega) \ni \varphi \mapsto u_\beta[\varphi] = \int_\Omega F(\alpha + i\beta)\varphi(\alpha) d\alpha$ belongs to $L'_{(\kappa)}(\Omega)$ and that there exists $\lim_{\Gamma' \ni \beta \to 0} u_\beta$. We say that $u \overset{\mathrm{df}}{=} \lim_{\Gamma' \ni \beta \to 0} u_\beta$ (belonging to $L'_{(\kappa)}(\Omega)$) is a *Laplace distributional boundary value* (LDBV in short) of F on Ω (from the wedge $\Omega + i\Gamma$) and write $u = b_\Gamma(F)$,

$$u[\varphi] = \lim_{\Gamma' \ni \beta \to 0} \int_\Omega F(\alpha + i\beta)\varphi(\alpha) d\alpha \quad \text{for} \quad \varphi \in L_{(\kappa)}(\Omega). \tag{20}$$

Next we define the space $\mathfrak{L}^h_{\widetilde{\kappa}}([\Omega + i\Gamma])$ ($h \in \mathbb{N}_0$, $\widetilde{\kappa} \in \mathbb{R}^n$) of all functions $F \in \mathcal{O}\big([\Omega + i\Gamma]\big)$ such that for every $\Gamma' \Subset \Gamma$, $\Omega' \Subset \Omega$ there exist $r > 0$ and $C_{\Gamma',\Omega'} < \infty$ such that

$$|F(\alpha + i\beta)| \le C_{\Gamma',\Omega'} \frac{e^{-\alpha\cdot\widetilde{\kappa}}}{\|\beta\|^h} \quad \text{for} \quad \alpha + i\beta \in \Omega' + i\Gamma'|_r. \tag{21}$$

For $\kappa \in \mathbb{R}^n$ we define[5] $\mathfrak{L}^{(\infty)}_{(\kappa)}([\Omega + i\Gamma]) = \varprojlim_{\widetilde{\kappa} < \kappa} \varinjlim_{h \in \mathbb{N}_0} \mathfrak{L}^h_{\widetilde{\kappa}}([\Omega + i\Gamma])$.

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[5] Sometimes, for convenience, we use the notation $\mathfrak{L}^h_\kappa(\Omega + i\Gamma)$, $\mathfrak{L}^{(\infty)}_{(\kappa)}(\Omega + i\Gamma)$ instead of $\mathfrak{L}^h_\kappa([\Omega + i\Gamma])$, $\mathfrak{L}^{(\infty)}_{(\kappa)}([\Omega + i\Gamma])$.

Proposition 2. *Let* Γ *be an open connected cone in* $\dot{\mathbb{R}}^n$, $\mathring{\xi} \in \mathbb{R}^n$, $\kappa \in \mathbb{R}^n$, $\Omega = \mathring{\xi} + \mathbb{R}^n_+$. *Then every function* $F \in \mathfrak{L}^{(\infty)}_{(\kappa)}(\Omega + i\Gamma)$ *has an LDBV:* $u = b_\Gamma(F) \in L'_{(\kappa)}(\Omega)$.

Proof. Take an arbitrary cone $\Gamma' \Subset \Gamma$ and a covering $\Gamma' \Subset \bigcup_{j=1}^p \Gamma'_j \Subset \Gamma$, where Γ'_j are open simplexes $\Gamma'_j \Subset \Gamma$ $(j = 1, \ldots, p)$ such that $\Gamma'_j \bigcap \Gamma'_{j+1} \neq \emptyset$ $(j = 1, \ldots, p-1)$. Take $A_j \in GL(n, \mathbb{R})$ such that $A_j(\mathbb{R}^n_+) = \Gamma'_j$. Define $F_j = F|_{\Omega + i\Gamma'_j}$, $G_j = F_j \circ A_j$ $(j = 1, \ldots, p)$. Then $G_j \in \mathcal{O}(A_j^{-1}\Omega + iA_j^{-1}\Gamma'_j)$ and since $\mathbb{R}^n_+ = A_j^{-1}\Gamma'_j \Subset A_j^{-1}\Gamma$ and $A_j^{-1}\Omega = \mathring{x}_j + \Lambda_j$ with $\mathring{x}_j = A_j^{-1}\mathring{\xi}$, $\Lambda_j = A_j^{-1}(\mathbb{R}^n_+)$, we have $G_j \in \mathcal{O}(\mathring{x}_j + \Lambda_j + i\mathbb{R}^n_+)$ $(j = 1, \ldots, p)$.

Moreover for every $\widetilde{\kappa} < \kappa$ there exists $h \in \mathbb{N}_0$ such that by (21)

$$|G_j(x + iy)| = |G_j(z)| = |F_j(A_j(z))| \leq C \frac{e^{-\widetilde{\kappa} \cdot A_j x}}{\|A_j y\|^h} \leq C_1 \frac{e^{-x \cdot A_j^{\mathrm{tr}} \widetilde{\kappa}}}{y_1^k \cdot \ldots \cdot y_n^k}$$

for $x \in \mathring{x}_j + \Lambda_j$, $y \in \mathbb{R}^n_+$, where $k = \frac{h}{n}$ if $\frac{h}{n} \in \mathbb{N}$, and $k = [\frac{h}{n}] + 1$ if $\frac{h}{n} \notin \mathbb{N}$. Hence by (iii) and Proposition 1 for every $a_j \in \mathrm{Int}(A_j^{-1}(\mathbb{R}^n_+))^\perp + A_j^{\mathrm{tr}}\widetilde{\kappa}$ there exists $T_j \in L'_{a_j}(\mathring{x}_j + A_j^{-1}(\mathbb{R}^n_+))$ such that

$$\lim_{y \to 0_+} \int_{\mathring{x}_j + A_j^{-1}(\mathbb{R}^n_+)} G_j(x + iy)\psi(x)dx = T_j[\psi] \quad \text{for} \quad \psi \in L_{a_j}(\mathring{x}_j + A_j^{-1}(\mathbb{R}^n_+)).$$

Then

$$\begin{aligned} T_j[\psi] &= \lim_{y \to 0_+} \int_{A_j^{-1}(\mathring{\xi} + \mathbb{R}^n_+)} (F_j \circ A_j)(x + iy)\psi(x)dx \\ &= \lim_{\Gamma'_j \ni \beta \to 0} \int_{\mathring{\xi} + \mathbb{R}^n_+} F(\alpha + i\beta)\psi(A_j^{-1}\alpha)|\det A_j^{-1}|d\alpha. \end{aligned}$$

Let $\varphi = \psi \circ A_j^{-1}$. Clearly $\varphi \in L_{(A_j^{\mathrm{tr}})^{-1}a_j}(\mathring{\xi} + \mathbb{R}^n_+)$ and

$$(22) \quad \lim_{\Gamma'_j \ni \beta \to 0} \int_{\mathring{\xi} + \mathbb{R}^n_+} F(\alpha + i\beta)\varphi(\alpha)d\alpha = |\det A_j|\ T_j[\varphi \circ A_j] = T_j \circ A_j^{-1}[\varphi].$$

Since $a_j \in \mathrm{Int}(A_j^{-1}(\mathbb{R}^n_+))^\perp + A_j^{\mathrm{tr}}\widetilde{\kappa}$, we get easily by (iii) that $(A_j^{\mathrm{tr}})^{-1}a_j \in \widetilde{\kappa} + \mathbb{R}^n_-$, hence (22) holds for $\varphi \in L_{(\widetilde{\kappa})}(\mathring{\xi} + \mathbb{R}^n_+)$. Since $\Gamma'_j \cap \Gamma'_{j+1} \neq \emptyset$,

$$\lim_{\substack{\beta\to 0\\ \beta\in\Gamma'_j\cap\Gamma'_{j+1}}} \int_{\mathring{\xi}+\mathbb{R}^n_+} F(\alpha+i\beta)\varphi(\alpha)d\alpha = T_j \circ A_j^{-1}[\varphi] = T_{j+1}\circ A_{j+1}^{-1}[\varphi]$$

for $\varphi \in L_{(\widetilde{\kappa})}(\Omega)$ and hence there exists $\lim_{\Gamma'\ni\beta\to 0}\int_\Omega F(\alpha+i\beta)\varphi(\alpha)d\alpha$ for $\varphi \in L_{(\widetilde{\kappa})}(\Omega)$. This ends the proof, since $\widetilde{\kappa} < \kappa$ was arbitrary. □

Define for $\sigma = (\sigma_1, \dots, \sigma_n) \in \{+,-\}^n$: $\operatorname{sgn}\sigma = \sigma_1 \cdot \ldots \cdot \sigma_n$ and a cone $\Gamma^\sigma = \{\beta \in \mathbb{R}^n : \sigma_j\beta_j > 0,\ 1 \le j \le n\}$, called the *n-th orthant.*

Now we shall deduce from Proposition 2 the following important corollary.

COROLLARY 1. *Let* $F \in \mathfrak{L}^{(\infty)}_{(\kappa)}(W\setminus\!\!\setminus\overline{\mathbb{R}}^n_+)$, $\kappa \in \mathbb{R}^n$, *and define*

$$bF = \sum_\sigma \operatorname{sgn}\sigma\, b_{\Gamma^\sigma}F$$

where $b_{\Gamma^\sigma}F$ *is defined by* (20). *Then* $bF \in L'_{(\kappa)}(\overline{\mathbb{R}}^n_+)$.

PROOF. Let $F \in \mathfrak{L}^{(\infty)}_{(\kappa)}(W\setminus\!\!\setminus\overline{\mathbb{R}}^n_+)$, $\kappa \in \mathbb{R}^n$. Then for every $\widetilde{\kappa} < \kappa$ there exists $h \in \mathbb{N}_0$ such that $F \in \mathfrak{L}^h_{\widetilde{\kappa}}(W\setminus\!\!\setminus\overline{\mathbb{R}}^n_+)$. Hence $F \in \mathcal{O}((\mathbb{R}^n + i\Gamma^\sigma)\cap W)$ for every $\sigma \in \{+,-\}^n$ and we can choose $\Omega = \mathring{\xi} + \mathbb{R}^n_+$ with $\mathring{\xi} < 0$ such that for every $\Omega' \Subset \Omega$, $\Gamma' \lessdot \Gamma^\sigma$ estimate (21) holds. Thus by Proposition 2, $bF \in L'_{(\kappa)}(\Omega)$. Due to the cancellation of boundary values $\sum_\sigma \operatorname{sgn}\sigma\, b_{\Gamma^\sigma}F$ from the wedges $\Omega + i\Gamma^\sigma$ with Ω in the complement of $\overline{\mathbb{R}}^n_+$ it follows that $\operatorname{supp} bF \subset \overline{\mathbb{R}}^n_+$. Finally we apply Theorem 3 from [S2–Z2] (or Theorem 8.3 in [S2–Z1]) in logarithmic coordinates. □

PROPOSITION 3. *Let* Ω, Γ, W *be as in Definition* 1. *If* $F \in \mathcal{O}\big((\Omega + i\Gamma)\cap W\big)$ *has LDBV* $u \in L'_{(\kappa)}(\Omega)$ $(\kappa \in \mathbb{R}^n)$, *then* $F \in \mathfrak{L}^{(\infty)}_{(\kappa)}(\Omega + i\Gamma)$.

PROOF. Fix $\Omega' \Subset \Omega$, $\Gamma' \lessdot \Gamma$ and take $\Omega' \Subset \Omega'' \Subset \Omega$, $\Gamma' \lessdot \Gamma'' \lessdot \Gamma$, $0 < r < 2$ such that $F \in \mathcal{O}(\Omega'' + i\Gamma|_r)$. Fix arbitrarily $\mathring{\alpha} \in \Omega'$, $\mathring{\beta} \in \Gamma'|_{r/2}$.

Then there exists $0 < c \le 1$ such that $F \in \mathcal{O}\big(\{\alpha : \|\alpha - \mathring{\alpha}\| \le c\|\mathring{\beta}\|\} + i\{\beta : \|\beta - \mathring{\beta}\| \le c\|\mathring{\beta}\|\}\big)$. Since $b^L_\Gamma(F) \in L'_{(\kappa)}(\Omega)$, by the Banach-Steinhaus theorem for every $\widetilde{\kappa} < \kappa$ there exist constants $\widetilde{c}$, $\widetilde{k} \in \mathbb{N}_0$ such that

$$\Big|\int_\Omega F(\alpha + i\beta)\varphi(\alpha)d\alpha\Big| \le \widetilde{c} q_{\widetilde{k},\widetilde{\kappa}}(\varphi) \quad \text{for} \quad \varphi \in L^{\widetilde{k}}_{\widetilde{\kappa}}(\Omega),\ \beta \in \Gamma''|_r,$$
$$\text{where} \quad q_{\widetilde{k},\widetilde{\kappa}}(\varphi) = \max_{|\nu| \le \widetilde{k}} \sup_\alpha \Big|e^{-\widetilde{\kappa}\cdot\alpha}\Big(\frac{\partial}{\partial\alpha}\Big)^\nu \varphi(\alpha)\Big|.$$

Let $\psi \in C^\infty_0\big(\{\alpha : \|\alpha - \mathring{\alpha}\| \le c\|\mathring{\beta}\|\}\big)$ with $q_{\widetilde{k}}(\psi) \overset{\text{df}}{=} \max_{|\nu|\le\widetilde{k}} \sup\big|\big(\frac{\partial}{\partial\alpha}\big)^\nu\psi(\alpha)\big| \le \frac{1}{\widetilde{c}}$. Then $q_{\widetilde{k},\widetilde{\kappa}}(\psi) \le \big(\sup_{\|\alpha-\mathring{\alpha}\|\le c\|\mathring{\beta}\|} e^{-\widetilde{\kappa}\cdot\alpha}\big) q_{\widetilde{k}}(\psi) \le \frac{1}{\widetilde{c}} e^{|\widetilde{\kappa}|} e^{-\widetilde{\kappa}\cdot\mathring{\alpha}}$, where $|\widetilde{\kappa}| = \sum_{j=1}^n \widetilde{\kappa}_j$. Hence

$$\Big|\int_\Omega F(\alpha + i\beta)\psi(\alpha)d\alpha\Big| \le e^{|\widetilde{\kappa}|} e^{-\widetilde{\kappa}\cdot\mathring{\alpha}} \quad \text{if} \quad \|\beta - \mathring{\beta}\| \le c\|\mathring{\beta}\|. \tag{23}$$

In the further proof we shall use the function(6) $\rho \in C^\infty_0(\mathbb{C}^n)$ supported by $\{z \in \mathbb{C}^n : \|z\| \le \frac{1}{2}\}$ such that $\int \rho(\alpha + i\beta) f(\alpha + i\beta)d\alpha d\beta = f(0)$ for $f \in \mathcal{O}\big(\{z : \|z\| \le 1\}\big)$. Let $\mathring{z} = \mathring{\alpha} + i\mathring{\beta}$, $g(z) \overset{\text{df}}{=} F(z + \mathring{z})$, $\mu(\alpha) \overset{\text{df}}{=} \psi(\mathring{\alpha} + \alpha)$. Then by (23) $\Big|\int g(\alpha + i\beta)\mu(\alpha)d\alpha\Big| \le e^{|\widetilde{\kappa}|} e^{-\widetilde{\kappa}\cdot\mathring{\alpha}}$ for $\|\beta\| \le c\|\mathring{\beta}\|$. Let $g_{c\|\mathring{\beta}\|}(\alpha + i\beta) = g(c\|\mathring{\beta}\|\alpha + ic\|\mathring{\beta}\|\beta)$. Then

$$\begin{aligned} F(\mathring{z}) = g(0) = g_{c\|\mathring{\beta}\|}(0) &= \int \rho(\alpha + i\beta) g(c\|\mathring{\beta}\|\alpha + ic\|\mathring{\beta}\|\beta)d\alpha d\beta \\ &= \frac{1}{(c\|\mathring{\beta}\|)^{2n}} \int \rho\Big(\frac{\xi}{c\|\mathring{\beta}\|} + i\frac{\eta}{c\|\mathring{\beta}\|}\Big) g(\xi + i\eta)d\xi d\eta. \end{aligned} \tag{24}$$

Observe that since $c\|\mathring{\beta}\| \le 1$, we get for every $|\nu| \le \widetilde{k}$:

$$\Big|\Big(\frac{\partial}{\partial\xi}\Big)^\nu \rho\Big(\frac{\xi}{c\|\mathring{\beta}\|} + i\frac{\eta}{c\|\mathring{\beta}\|}\Big)\Big| \le \frac{1}{(c\|\mathring{\beta}\|)^{|\nu|}} \sup\Big|\Big(\frac{\partial}{\partial\alpha}\Big)^\nu \rho(\alpha + i\beta)\Big| \le \frac{M}{(c\|\mathring{\beta}\|)^{\widetilde{k}}}$$

with some $M < \infty$ (depending on $\widetilde{k}$ but independent of ξ, η). Now fix $\eta : \|\eta\| < \frac{c\|\mathring{\beta}\|}{2} < \frac{1}{2}$ and let $\sigma(\xi, \eta) = \frac{(c\|\mathring{\beta}\|)^{\widetilde{k}}}{M\widetilde{c}} \rho\big(\frac{\xi}{c\|\mathring{\beta}\|} + i\frac{\eta}{c\|\mathring{\beta}\|}\big)$. Then

(6) See e.g. [L].

$\int g(\xi+i\eta)\sigma(\xi,\eta)d\xi \le e^{|\widetilde{\kappa}|}e^{-\widetilde{\kappa}\cdot\mathring{\alpha}}$. Hence we deduce from (24) that $|F(\mathring{z})| \le \frac{c}{\|\mathring{\beta}\|^{n+\widetilde{k}}}e^{-\widetilde{\kappa}\cdot\mathring{\alpha}}$, where the constants $c<\infty$, $\widetilde{k}\in\mathbb{N}_0$ do not depend on the choice of $\mathring{\alpha}+i\mathring{\beta}\in\Omega+i\Gamma|_{r/2}$. □

By Propositions 2 and 3 we get

THEOREM 2. *Let* Γ *be an open cone in* $\dot{\mathbb{R}}^n$, $\mathring{\alpha}\in\mathbb{R}^n$, $\kappa\in\mathbb{R}^n$, $\Omega=\mathring{\alpha}+\mathbb{R}^n_+$ *and let* W *be a complex neighbourhood of* Ω. *Let* $F\in\mathcal{O}\big((\Omega+i\Gamma)\cap W\big)$. *Then the following assertions are equivalent:*

(*i*) *There is* $u\in L'_{(\kappa)}(\Omega)$ *with* $u=b_\Gamma(F)$.

(*ii*) $F\in\mathfrak{L}^{(\infty)}_{(\kappa)}\big([\Omega+i\Gamma]\big)$.

3. Laplace Hyperfunctions and Distributions

Throughout this section $W=W_1\times\cdots\times W_n$ is a polytubular neighbourhood of $\overline{\mathbb{R}}^n_+$ such that $\operatorname{Im}\zeta_j$ is bounded for $\zeta_j\in W_j$ $(j=1,\dots,n)$. We shall denote by γ_j a regular curve in $W_j\setminus\overline{\mathbb{R}}_+$ encircling $\overline{\mathbb{R}}_+$ once in the anticlockwise direction $(j=1,\dots,n)$ and put $\gamma=\gamma_1\times\cdots\times\gamma_n$.

Let $G\in\mathfrak{L}_{(\omega)}(W\#\overline{\mathbb{R}}^n_+)$, $\varphi\in\underset{\approx}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$. By the definition of such functions given in Section 1, in every polytubular set $V=V_1\times\cdots\times V_n$ in which both of them are defined there exist $a<\kappa<\omega$ and $C<\infty$ such that

$$|\varphi(z)G(z)|\le Ce^{-(\kappa-a)\cdot\operatorname{Re}z}. \tag{25}$$

When considering the integral $\int_\gamma G(z)\varphi(z)dz$ we shall always assume that $\gamma\subset V$.

Let $\Lambda(\zeta,w)=\prod_{j=1}^n\Lambda_j(\zeta_j,w_j)$ with $\Lambda_j(\zeta_j,w_j)=e^{-(\zeta_j-w_j)^2}/(\zeta_j-w_j)$ $(j=1,\dots,n)$. In vector notation we write

$$\Lambda(\zeta,w)=e^{-(\zeta-w)^2}(\zeta-w)^{-\mathbb{1}}$$

and call it a modified Cauchy kernel.

PROPOSITION 4. *Fix a polytubular neighbourhood* $W\supset\overline{\mathbb{R}}^n_+$ *and a polytubular set* $V^1\Subset W\#\overline{\mathbb{R}}^n_+$. *Choose a polytubular set* $V^2:\overline{\mathbb{R}}^n_+\subset V^2\Subset W$

such that $\operatorname{dist}(V_j^1, V_j^2) = \eta_j > 0$ $(j = 1, \ldots, n)$ *and let* $a \in \mathbb{R}^n$. *Then there exists* $C < \infty$ *such that*

$$\sup_{w \in V^2} \sup_{\zeta \in V^1} |e^{a\cdot(\zeta - w)}\Lambda(\zeta, w)| < C. \tag{26}$$

In particular for a fixed $\zeta \in W \mathbin{\#} \overline{\mathbb{R}}^n_+$ *with* $\operatorname{dist}(\zeta_j, V_j^2) \geq \eta_j > 0$ $(j = 1, \ldots, n)$ *we have* $\sup_{w \in V^2} |e^{-a\cdot w}\Lambda(\zeta, w)| < \infty$ *which means that* $\Lambda(\zeta, \cdot) \in \underset{\sim}{L}_a(V^2)$.

The proof follows from the estimate:

$$\begin{aligned}\sup_{w \in V^2} \sup_{\zeta \in V^1} |e^{a\cdot(\zeta - w)}\Lambda(\zeta, w)| &\leq C \sup_{w \in V^2} \sup_{\zeta \in V^1} e^{\operatorname{Re}(\zeta - w)\cdot(a - \operatorname{Re}(\zeta - w))} \\ &\leq C \prod_{j=1}^n \sup_{\xi_j \in \mathbb{R}} e^{\xi_j\cdot(a_j - \xi_j)} < \infty.\end{aligned}$$

Lemma 2. *Let* $G \in \mathfrak{L}_{(\omega)}(W \mathbin{\#} \overline{\mathbb{R}}^n_+)$. *Then* $G \in \sum_{j=1}^n \mathfrak{L}_{(\omega)}(W \mathbin{\#}_j \overline{\mathbb{R}}^n_+)$ *if and only if*

$$\int_\gamma G(z)\varphi(z)dz = 0 \quad \textit{for} \quad \varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+). \tag{27}$$

Proof. To simplify the notation take $G \in \mathfrak{L}_{(\omega)}(W \mathbin{\#}_n \overline{\mathbb{R}}^n_+)$ and put $z^1 = (z_1, \ldots, z_{n-1})$, $\gamma^1 = \gamma_1 \times \cdots \times \gamma_{n-1}$. Then for all $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ the function $G(z^1, \cdot)\varphi(z^1, \cdot)$ is holomorphic in the domain bounded by γ_n, hence by (25) $\int_{\gamma_n} G(z^1, z_n)\varphi(z^1, z_n)dz_n = 0$ and consequently we get (27). For the proof of the second part of Lemma 2 let $P_{\gamma_j^-, \gamma_j^+} \subset W_j \setminus \overline{\mathbb{R}}_+$ be a domain bounded by an inner curve γ_j^- and an outer curve γ_j^+, both rectifiable and encircling $\overline{\mathbb{R}}_+$, $\operatorname{dist}(\gamma_j^-, \gamma_j^+) > 0$, $\operatorname{dist}(\gamma_j^-, \overline{\mathbb{R}}_+) > 0$, $\gamma_j = \partial P_{\gamma_j^-, \gamma_j^+} = \gamma_j^+ - \gamma_j^-$ $(j = 1, \ldots, n)$. By (27), the Cauchy formula and Proposition 4 we have

$$\begin{aligned} &G(z) = \sum_{\substack{\sigma \in \{+,-\}^n \\ \sigma \neq (-,\ldots,-)}} H_\sigma(z) \quad \text{for} \quad z \in P = P_{\gamma_1^-, \gamma_1^+} \times \cdots \times P_{\gamma_n^-, \gamma_n^+}, \\ &\quad \text{where} \quad H_\sigma(z) = \frac{1}{(2\pi i)^n} \operatorname{sgn}\sigma \int_{\gamma_n^{\sigma_n}} \cdots \int_{\gamma_1^{\sigma_1}} G(\zeta)\Lambda(\zeta, z)d\zeta_1 \cdots d\zeta_n. \end{aligned} \tag{28}$$

Observe that every summand in (28) extends holomorphically to the cartesian product of a number $p \geq 1$ of sets $V_{\gamma_j^+}$ and $n-p$ of sets $W_k \setminus \overline{V}_{\gamma_k^-}$, where $V_{\gamma_k^\pm}$ denotes the domain bounded by the curve $\gamma_k^\pm$. Since the curves γ_j^- can be taken arbitrarily close to $\overline{\mathbb{R}}_+$ and γ_j^+ arbitrarily close to ∂W_j $(j=1,\dots,n)$, we see that $G \in \sum_{j=1}^n \mathcal{O}(W \mathbin{\backslash\!\!\!=}_j \overline{\mathbb{R}}_+^n)$. To prove the desired estimates we observe that by Proposition 4 the function $G(\zeta)\Lambda(\zeta,z)$ satisfies estimates (25) and hence applying the Cauchy formula we can write H_σ as a linear combination of integrals taken only over the curves γ_j^+. Consider for instance the integral

$$(29) \quad G_n(z) = \int_{\gamma_n^+} G(z^1,\zeta_n)\Lambda_n(\zeta_n,z_n)d\zeta_n \quad \text{for} \quad (z^1,z_n) = z \in W \mathbin{\backslash\!\!\!=}_n \overline{\mathbb{R}}_+^n.$$

Clearly $G_n \in \mathcal{O}(W \mathbin{\backslash\!\!\!=}_n \overline{\mathbb{R}}_+^n)$. To prove that $G_n \in \mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=}_n \overline{\mathbb{R}}_+^n)$ we take a polytubular set $V = V^1 \times V_n$, where V^1 is a polytubular set in $(W_1 \setminus \overline{\mathbb{R}}_+) \times \dots \times (W_{n-1} \setminus \overline{\mathbb{R}}_+)$ and V_n is contained inside γ_n^+, $\operatorname{dist}(V_n, \gamma_n^+) > 0$. Take arbitrarily $a < \omega$. Let $a = (a^1, a_n)$, $b = (a^1, b_n)$, $a_n < b_n < \omega_n$. By the assumption on G we have $|G(z^1,\zeta_n)| \leq C|e^{-a^1\cdot z^1 - b_n\zeta_n}|$ for $z^1 \in V^1$, $\zeta_n \in \gamma_n^+$ and some $C < \infty$. Hence by (29) and (26) we get, with some $\widetilde{C} < \infty$, $\sup_{z\in V} |e^{a\cdot z}G_n(z)| \leq \widetilde{C}\big|\int_{\gamma_n^+} e^{-(b_n-a_n)\operatorname{Re}\zeta_n} d\zeta_n\big| < \infty$, which proves that $G_n \in \mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=}_n \overline{\mathbb{R}}_+^n)$. The proof of Lemma 2 follows by the consecutive application of the above reasoning. $\square$

Proposition 5. *The space* $\sum_{j=1}^n \mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=}_j \overline{\mathbb{R}}_+^n)$ *is a closed subspace of* $\mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=} \overline{\mathbb{R}}_+^n)$.

Proof. Let $\sum_{j=1}^n \mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=}_j \overline{\mathbb{R}}_+^n) \ni G^\nu \xrightarrow[\nu\to\infty]{} G$ in $\mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=} \overline{\mathbb{R}}_+^n)$. Hence by Lemma 2 for $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+^n)$, $\gamma \subset W \mathbin{\backslash\!\!\!=} \mathbb{R}_+^n$ we have $\int_\gamma G^\nu(z)\varphi(z)dz = 0$ $(\nu = 1,2,\dots)$, and to prove that $G \in \sum_{j=1}^n \mathfrak{L}_{(\omega)}(W \mathbin{\backslash\!\!\!=}_j \overline{\mathbb{R}}_+^n)$ it suffices to show that the same is true for G. Take any $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+^n)$ and $a < \omega$ such that $\varphi \in \underset{\sim}{L}_a(\overline{\mathbb{R}}_+^n)$. Then for $0 < \delta < \omega - a$ we get the estimate

$$\Big|\int_\gamma G(z)\varphi(z)dz\Big| = \Big|\int_\gamma (G(z) - G^\nu(z))\varphi(z)dz\Big|$$
$$\leq C \sup_{z\in\gamma}\big|e^{(\omega-\delta)\cdot z}(G(z) - G^\nu(z))\big| \cdot \Big|\int_\gamma e^{-(\omega-a-\delta)\cdot z}dz\Big|,$$

in which the right-hand side converges to zero as $\nu \to \infty$. Hence $\int_\gamma G(z)\varphi(z)dz = 0$. □

Definition 2. The quotient space[(7)]

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+) = \mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)\Big/\sum_{j=1}^{n}\mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus_j\overline{\mathbb{R}}^n_+)$$

is called the *space of Laplace hyperfunctions* on $\overline{\mathbb{R}}^n_+$ of type $\omega \in \mathbb{R}^n$. By Proposition 5 it is a Hausdorff topological space. A function $F \in \mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$ is called a *defining function* for the Laplace hyperfunction $f = F + \sum_{j=1}^{n}\mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus_j\overline{\mathbb{R}}^n_+)$ denoted shortly $f = [F]$.

Definition 3. We say that a sequence $f_\nu \in \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ $(\nu = 1, 2, \ldots)$ is *convergent* if there exist defining functions F_ν such that $\{F_\nu\}$ converges in $\mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$ to some F. We set $\lim_{\nu\to\infty} f_\nu = f \overset{\mathrm{df}}{=} [F]$.

We intend to provide an n-dimensional version of the well-known Köthe theorem [Kö] and Martineau–Harvey theorem [M], [H] for the case of Laplace hyperfunctions. To this aim we need Lemma 3 below.

Lemma 3. *Let* $\Psi \in \mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$, $W \supset \overline{\mathbb{R}}^n_+$, $\gamma = \gamma_1 \times \cdots \times \gamma_n$, *and define* $\Psi^*(z) = \left(\frac{-1}{2\pi i}\right)^n \int_\gamma \Psi(w)\Lambda(w,z)dw$, *where* γ_j *leaves* z_j *on the right. Then* $\Psi^* \in \mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$ *and*

$$\Psi - \Psi^* \in \sum_{j=1}^{n}\mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus_j\overline{\mathbb{R}}^n_+). \tag{30}$$

Proof. Observe first that $\Psi^* \in \mathcal{O}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$. To prove that $\Psi^* \in \mathfrak{L}_{(\omega)}(W\setminus\!\!\!\setminus\overline{\mathbb{R}}^n_+)$ take a polytubular set $\widetilde{W}_j \Subset W_j \setminus \overline{\mathbb{R}}_+$ $(j = 1, \ldots, n)$, an $a < \omega$ and choose $0 < \rho < \omega - a$ and a curve γ_j encircling $\overline{\mathbb{R}}_+$ in the

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(7) The correctness of this symbol (i.e. the independence from W) will be clear from Theorem 3 below.

anticlockwise direction and leaving $\widetilde{W}_j$ on the right $(j = 1, \dots, n)$. Then by assumption on Ψ and Proposition 4 we get

$$\sup_{z\in\widetilde{W}} |e^{a\cdot z}\Psi^*(z)| \le C \sup_{z\in\widetilde{W}} \sup_{w\in\gamma} \big|e^{a\cdot(z-w)}\Lambda(z,w)\big| \cdot \Big|\int_\gamma e^{-\rho\cdot \operatorname{Re} w} dw\Big| < \infty$$

and thus $\Psi^* \in \mathfrak{L}_{(\omega)}(W \nparallel \overline{\mathbb{R}}^n_+)$. For the proof of (30) we apply Lemma 2. To this aim take $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and the curves γ and $\widetilde{\gamma}$ verifying the usual conditions and moreover such that for every $j = 1, \dots, n$ the curve $\widetilde{\gamma}_j$ leaves γ_j on the left, $\operatorname{dist}(\widetilde{\gamma}_j, \gamma_j) = \eta_j > 0$.

Fix arbitrarily a point $w \in \gamma$ and let $M > \operatorname{Re} w_j$ for $j = 1, \dots, n$, split the curve $\widetilde{\gamma}_j$ into two curves: a bounded $\widetilde{\gamma}^{1,M}_j$ and an unbounded $\widetilde{\gamma}^{2,M}_j$ having a common bounded segment with the line $\operatorname{Re} z_j = M$ $(j = 1, \dots, n)$. Clearly by the Cauchy formula applied to the function $f_w(z) = \varphi(z)e^{-(z-w)^2}$ we have $\varphi(w) = \frac{1}{(2\pi i)^n}\int_{\widetilde{\gamma}^{1,M}} \varphi(z)\Lambda(z,w)dz$ for every $M > \operatorname{Re} w_j$, $j = 1, \dots, n$. By the standard estimation $|\varphi(z)\Lambda(z,w)| \le Ce^{-\rho\cdot\operatorname{Re} z}$ with some $\rho \in \overline{\mathbb{R}}^n_+$, $C = C(w) < \infty$ and hence $\int_{\widetilde{\gamma}^{2,M}_j} \varphi(z)\Lambda(z,w)dz_j \xrightarrow[M\to\infty]{} 0$, $j = 1, \dots, n$. Thus

$$\varphi(w) = \frac{1}{(2\pi i)^n}\int_{\widetilde{\gamma}} \varphi(z)\Lambda(z,w)dz \tag{31}$$

and, by the standard estimation we get $\int_{\widetilde{\gamma}} \Psi^*(z)\varphi(z)dz = \frac{1}{(2\pi i)^n}\int_\gamma \Psi(w) \times \Big(\int_{\widetilde{\gamma}} \varphi(z)\Lambda(z,w)dz\Big)dw = \int_\gamma \Psi(w)\varphi(w)dw = \int_{\widetilde{\gamma}} \Psi(z)\varphi(z)dz$ for every $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$, which by Lemma 2 yields (30). $\square$

THEOREM 3. *There exists a natural topological isomorphism*

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+) \cong \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+), \quad \omega \in \mathbb{R}^n,$$

given by the assignment

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+) \ni f = [F] \longmapsto \mathcal{I}f \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+),$$

where $F \in \mathfrak{L}_{(\omega)}(W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+)$ *and the functional* $\mathcal{I}f$ *is given by* $\mathcal{I}f[\varphi] = (-1)^n \int_\gamma F(z)\varphi(z)dz$ *for* $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$. *The inverse mapping* $\mathcal{J}$ *is the assignment*

$$\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+) \ni T \overset{\mathcal{J}}{\longmapsto} [\mathcal{C}_\Lambda T] = \Psi + \sum_{j=1}^{n} \mathfrak{L}_{(\omega)}(W \setminus\!\!\setminus_j \overline{\mathbb{R}}^n_+),$$

where

$$(\mathcal{C}_\Lambda T)(\zeta) = \Psi(\zeta) \overset{\text{df}}{=} \Big(\frac{-1}{2\pi i}\Big)^n T\Big[\frac{e^{-(\zeta - w)^2}}{(\zeta - w)^{\mathbb{1}}}\Big] \quad \textit{for} \ \ \zeta \in \mathbb{C}^n \setminus\!\!\setminus \overline{\mathbb{R}}^n_+$$

belongs to $\mathfrak{L}_{(\omega)}(W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+)$ *for every tubular neighbourhood* W *of* $\overline{\mathbb{R}}^n_+$.

Proof. By the assumptions on F, φ, γ there exists a polytubular neighbourhood $V \supset \overline{\mathbb{R}}^n_+$ and $a < \omega$ such that $\big|\int_\gamma F(z)\varphi(z)dz\big| \le C\rho_{a,V}(\varphi)$ and $\mathcal{I}f(\varphi)$ is independent of the choice of γ encircling $\overline{\mathbb{R}}_+$ in V. Thus the functional $\mathcal{I}f \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and by Lemma 2 it does not depend on the choice of a defining function F.

Let $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and let W, V_1, V_2 be defined as in Proposition 4. Take $a < \omega$. Then $T \in \underset{\sim}{L}'_a(V_2)$, $\Lambda(\zeta,\cdot) \in \underset{\sim}{L}_a(V_2)$ for every $\zeta \in V_1$ and hence $\Psi(\zeta) = (\frac{-1}{2\pi i})^n T[\Lambda(\zeta,\cdot)]$ is well defined for $\zeta \in V_1$. By point 2 in Section 1 and (26) we get

$$\begin{aligned}\sup_{\zeta\in V_1} |e^{a\cdot\zeta}\Psi(\zeta)| &\le (2\pi)^{-n} C_{a,V_2} \sup_{\zeta\in V_1} \Big| e^{a\cdot\zeta} \sup_{w\in V_2} |e^{-a\cdot w}\Lambda(\zeta,w)|\,\Big| \\ &\le (2\pi)^{-n} C_{a,V_2} \sup_{\zeta\in V_1}\sup_{w\in V_2} |e^{a\cdot(\zeta-w)}\Lambda(\zeta,w)| \le \widetilde{C} < \infty.\end{aligned}$$

Recall that V_1 was an arbitrary polytubular set $\Subset W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+$ and $a < \omega$ was also arbitrary. Thus $\Psi \in \mathfrak{L}_{(\omega)}(W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+)$ since (as it can be shown directly) it is holomorphic on $W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+$. Thus the transformation $\mathcal{J}$ in Theorem 3 is well defined and $\mathcal{C}_\Lambda T \in \mathfrak{L}_{(\omega)}(W \setminus\!\!\setminus \overline{\mathbb{R}}^n_+)$. The equality $\mathcal{J} = \mathcal{I}^{-1}$ can be shown by (31) in the following way: $(\mathcal{I} \circ \mathcal{J}T)[\varphi] = \big(\frac{1}{2\pi i}\big)^n \int_\gamma \varphi(z) T[\Lambda(z,w)]dz = T\big[\big(\frac{1}{2\pi i}\big)^n \int_\gamma \varphi(z)\Lambda(z,w)dz\big] = T[\varphi]$ for $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$. To prove that $\mathcal{J} \circ \mathcal{I}f = f$ for $f = [F] \in \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ observe that by Lemma 3: $(\mathcal{C}_\Lambda(\mathcal{I}f))(\zeta) = \big(\frac{1}{2\pi i}\big)^n \int_\gamma F(z)\Lambda(\zeta,z)dz = F^*(\zeta)$ and $\mathcal{J} \circ \mathcal{I}f = [\mathcal{C}_\Lambda(\mathcal{I}f)] = [F^*] = f$.

To prove the continuity of $\mathcal{I}$ assume that $\lim_{\nu\to\infty} f_\nu = f$ in $\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ (cf. Definition 3), note that $|\mathcal{I}f_\nu[\varphi]-\mathcal{I}f[\varphi]| = |\int_\gamma (F_\nu(z)-F(z))\varphi(z)dz|$ for $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and end the proof as in Proposition 5. The continuity of the mapping $\mathcal{J}$ follows from the Banach-Steinhaus and the Vitali theorems. □

By Theorem 3 and point 4. in Section 1 we deduce immediately:

COROLLARY 2 (Imbedding of Laplace distributions in Laplace hyperfunctions). *There exists a natural topological imbedding:*

$$L'_{(\omega)}(\overline{\mathbb{R}}^n_+) \subset\!\!\!\!\!\rightarrow \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+).$$

Now we pass to the description of the image of $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ under the imbedding.

The space $\mathfrak{L}^{(\infty)}_{(k)}(\Omega + i\Gamma)$ of Section 2 turns out however to be unsuitable for our purpose, namely we need to control the way we approach the boundary of the cone Γ. Therefore we proceed as follows. We consider local *wedges at infinity* $Q = \Omega + i\Gamma|_r$ with profile $\Gamma \subset \dot{\mathbb{R}}^n$ (cf. Definition 1) and edge $\Omega \subset \mathbb{R}^n$ having (up to a permutation) one of the following forms: $(M_1,\infty)\times\cdots\times(M_n,\infty)$ (i.e. as in Definition 1 if $\Omega = M + \mathbb{R}^n_+$, $M = (M_1,\dots,M_n) \in \mathbb{R}^n$) or $\omega_{\alpha^j}\times(M_{n-j+1},\infty)\times\cdots\times(M_n,\infty)$, $j = 1,\dots,n-1$; here $\alpha^j = (\alpha_1,\dots,\alpha_{n-j})$, $\omega_{\alpha^j} = \omega_{\alpha_1}\times\cdots\times\omega_{\alpha_{n-j}}$, ω_{α_k} are open bounded neighbourhoods of α_k in $\mathbb{R}$.

DEFINITION 4. Let $V \subset \mathbb{C}^n$ be open and $[V]$ be its germ near $\mathbb{R}^n$ (cf. Definition 1). Let $k \in \mathbb{N}_0$, $\kappa \in \mathbb{R}^n$. We define the space $\mathfrak{L}^{|k}_\kappa([V])$ by

$$\mathfrak{L}^{|k}_\kappa([V]) = \{H \in \mathcal{O}(V) : q_Q(H) < \infty$$
$$\text{for every local wedge } Q = \Omega + i\Gamma|_r \subset V \text{ and}$$
$$q_Q(H) = \sup_{\alpha+i\beta\in Q} |H(\alpha+i\beta)|\cdot(\mathrm{dist}(\beta,\mathrm{bd}\,\Gamma))^k \exp(\sum_{j=1}^p \alpha_{l_j}\kappa_{l_j})\}.$$

The exponential factor in the definition of $q_Q(H)$ appears every time the cartesian product Ω contains unbounded intervals $(M_{l_1},\infty),\dots,$

(M_{l_p}, ∞).[8] By $\mathfrak{L}^{|k}_{(\kappa)}([V])$ we denote $\varinjlim_{\widetilde{\kappa}<\kappa} \mathfrak{L}^{|k}_{\kappa}([V])$.

We shall often write $\mathfrak{L}^{|k}_{\kappa}(V)$ instead of $\mathfrak{L}^{|k}_{\kappa}([V])$.

Lemma 4. *Let $k \in \mathbb{N}_0$, $\kappa \in \mathbb{R}^n$, and $G \in \mathfrak{L}^{|k}_{(\kappa)}(W \# \overline{\mathbb{R}}^n_+)$. Fix j, $1 \leq j \leq n$, take $z_j \in W_j$ and let $\gamma_j \subset W_j \setminus \mathbb{R}_+$ be a regular curve encircling $\mathbb{R}_+$ in the anticlockwise direction and leaving the point z_j on the left. Define*

$$G_j(z) = \int_{\gamma_j} G(z_1, \dots, z_{j-1}, \zeta_j, z_{j+1}, \dots, z_n) \Lambda_j(\zeta_j, z_j) d\zeta_j \quad \text{for } z \in W \#_j \overline{\mathbb{R}}^n_+.$$

Then $G_j \in \mathfrak{L}^{|k}_{(\kappa)}(W \#_j \overline{\mathbb{R}}^n_+)$.

Proof. We have to control at the same time the behaviour of the estimates as $\operatorname{Re} z \to \infty$ as in Lemma 3 and moreover the way we approach $\mathbb{R}^n$. To simplify the formulae assume $j = n$ and thus G_n is given by (29) with $z_n \in W_n$, $z^1 = (z_1, \dots, z_{n-1}) \in (W_1 \setminus \mathbb{R}) \times \dots \times (W_{n-1} \setminus \mathbb{R})$. Let $\alpha^1 = \operatorname{Re} z^1$, $\beta^1 = \operatorname{Im} z^1$. We have to show that for any local wedge $Q \subset W \#_n \mathbb{R}^n$ and any $\widetilde{\kappa} < \kappa$ the inequality $q_Q(G_n) < \infty$ holds. We distinguish some types of local wedges:

(i) Let $\mathring{\alpha}^1 \in \overline{\mathbb{R}}^{n-1}_+$, $\mathring{\alpha}_n \in W_n \cap \mathbb{R}$, and $\omega_{\mathring{\alpha}^1}$, $\omega_{\mathring{\alpha}_n}$ be their bounded neighbourhoods. Take $\Omega = \omega_{\mathring{\alpha}^1} \times \omega_{\mathring{\alpha}_n}$ and[9] $Q_\varepsilon = \Omega + i(\mathbb{R}^{n-1}_+ \times \mathbb{R})|_\varepsilon \subset W \#_n \overline{\mathbb{R}}^n_+$.

In this case we have to show that for some $C < \infty$ the following inequality holds for ε sufficiently small:

$$|G_n(\alpha + i\beta)| \leq C(\min_{1 \leq j \leq n-1} \beta_j)^{-k} \quad \text{for } \alpha + i\beta \in Q_\varepsilon.$$

Take $\alpha^- < c^- < 0$ such that $[\alpha^-, +\infty) \subset W_n \cap \mathbb{R}$, $\omega_{\mathring{\alpha}_n} \subset (c^-, +\infty)$. Let $\{\omega_1, \omega_2\}$ be an open covering of $[\alpha^-, +\infty)$ in $\mathbb{R}$, where ω_1 is a bounded neighbourhood of α^-. Let ω_{α^1} be an open bounded neighbourhood of $\alpha^1 \in$

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[8] We may equivalently define the topology assuming q_Q with the exponential factor $\sum_{j=1}^n \alpha_j \kappa_j$ in all the cases.

[9] To simplify the notation we select the case $\Gamma = \mathbb{R}^{n-1}_+ \times \mathbb{R}$ instead of the general one: $\Gamma = \mathbb{R}_{\sigma_1} \times \dots \times \mathbb{R}_{\sigma_{n-1}} \times \mathbb{R}$, $\sigma_q \in \{+, -\}$ for $q = 1, \dots, n-1$.

$\mathbb{R}^{n-1}$ and denote $\Omega^1 = \omega_{\alpha^1} \times \omega_1$, $\Omega^2 = \omega_{\alpha^1} \times \omega_2$. Let $\Gamma^1 = \mathbb{R}^{n-1}_+ \times \mathbb{R}$, $(\Gamma^2)^\pm = \mathbb{R}^{n-1}_+ \times \mathbb{R}_\pm$ and $r^* > 0$ small enough that $\Omega^1 + i\Gamma^1|_{r^*} \subset W\#_n\overline{\mathbb{R}}^n_+$, $\Omega^2 + i(\Gamma^2)^\pm|_{r^*} \subset W\#_n\overline{\mathbb{R}}^n_+$. Take $0 < \varepsilon < r < r^*$ and consider two strips:

1) P^+ bounded by the half-lines $l^\pm = (\alpha^- + s) \pm ir$ $(0 < s < \infty)$ and the segment $[\alpha^- - ir, \alpha^- + ir]$,
2) P^- bounded by $(c^- + s) \pm i\varepsilon$ $(0 < s < \infty)$ and $[c^- - i\varepsilon, c^- + i\varepsilon]$.

Let $\gamma_n = \partial P^+$ and let $z_n \in \omega_{\&_n} + i\mathbb{R}|_\varepsilon$. Then there exists $\rho > 0$ such that $|z_n - \zeta_n| \geq \rho$ if $\zeta_n \in \gamma_n$ and, by the estimate verified by G on the wedge $\Omega^1 + i\Gamma^1|_{r^*}$ there exists $C < \infty$ such that

$$|G(z^1, \zeta_n)| \leq C(\min_{1\leq j\leq n-1} \beta_j)^{-k}$$
$$\text{for } \ z^1 \in \omega_{\&^1} + i\mathbb{R}^{n-1}_+|_{r^*}, \ \zeta_n = \alpha^- + it, \ -r \leq t \leq r.$$

Hence with $C_1 < \infty$ we get the estimate:

$$\Big|\int_{\alpha^- - ir}^{\alpha^- + ir} G(z^1, \zeta_n)\Lambda_n(\zeta_n, z_n)d\zeta_n\Big| \leq C_1(\min_{1\leq j\leq n-1} \beta_j)^{-k}. \tag{32}$$

If $\zeta_n \in l^\pm$ and $\|\beta^1\| \leq \varepsilon$ we have $\operatorname{dist}((\beta^1, \operatorname{Im}\zeta_n), \operatorname{bd}(\mathbb{R}^{n-1}_+ \times \mathbb{R}_\pm)) = \min_{1\leq j\leq n-1} \beta_j$. Note that by the estimate verified by G on the wedge $\Omega^2 + i(\Gamma^2)^\pm|_{r^*}$ there exists $C < \infty$ such that

$$|G(z^1, \zeta_n)| \leq Ce^{-\operatorname{Re}\zeta_n \cdot \widetilde{\kappa}_n}(\min_{1\leq j\leq n-1} \beta_j)^{-k} \tag{33}$$
$$\text{for } \ \zeta_n \in l^\pm, \ z^1 \in \omega_{\&^1} + i\mathbb{R}^{n-1}_+|_\varepsilon,$$

and hence by the assumption that $\operatorname{Re} z_n$ ranges over the bounded set $\omega_{\&_n}$ we get with some $C_2 < \infty$: $\left|\int_{l^\pm} G(z^1, \zeta_n)\Lambda_n(\zeta_n, z_n)d\zeta_n\right| \leq C_2(\min_{1\leq j\leq n-1} \beta_j)^{-k}$, since $e^{-\operatorname{Re}\zeta_n \cdot \widetilde{\kappa}_n - (\operatorname{Re}(\zeta_n - z_n))^2} \leq e^{-\operatorname{Re}\zeta_n(\operatorname{Re}\zeta_n + \widetilde{\kappa}_n - 2\operatorname{Re} z_n)}$ is integrable over $l^\pm$.

Thus by (29), (32) we get the desired assertion for $\|\beta\| \leq \varepsilon$.

Consider now the following case:

(ii) $\&^1 \in \overline{\mathbb{R}}^{n-1}_+$, $M_n \in \mathbb{R}$, $\Omega = \omega_{\&^1} \times (M_n, +\infty)$, $Q_\varepsilon = \Omega + i(\mathbb{R}^{n-1}_+ \times \mathbb{R})|_\varepsilon \subset W\#_n\overline{\mathbb{R}}^n_+$ (or more generally in the spirit of foot-note[9]).

Thus we have to show that to every $\widetilde{\kappa}_n < \kappa_n$ there exist $C < \infty$, $\varepsilon > 0$ such that $|G_n(\alpha + i\beta)| \leq Ce^{-\alpha_n \widetilde{\kappa}_n}(\min_{1\leq j\leq n-1} \beta_j)^{-k}$ for $z \in \Omega + i(\mathbb{R}^{n-1}_+ \times \mathbb{R})|_\varepsilon$.

To this aim fix $\widetilde{\kappa}_n < \kappa_n$ and take $\widetilde{M}_n < M_n$, $\varepsilon < r$ such that $\widetilde{Q}_\pm = (\omega_{\mathring{\alpha}^1} + i\mathbb{R}^{n-1}_+|_\varepsilon) \times ((\widetilde{M}_n, +\infty) + i\mathbb{R}_\pm|_r) \subset W \#_n \overline{\mathbb{R}}^n_+$. Let P^+, P^- be defined as in the case (i) with $\alpha^- = \widetilde{M}_n$, $c^- = M_n$ and let $\gamma_n = \partial P^+$. Let $\widetilde{\kappa}_n < b_n < \kappa_n$. By the assumption on G there exists a constant $C_1 < \infty$ such that

$$|G(z^1, \zeta_n)| \le C_1 (\min_{1 \le j \le n-1} \beta_j)^{-k} e^{-b_n \operatorname{Re} \zeta_n}$$
$$\text{for} \ \ \zeta_n \in l^+ \cup l^-, \ z^1 \in \omega_{\mathring{\alpha}^1} + i\mathbb{R}^{n-1}_+|_\varepsilon$$

and similarly as in the proof of Proposition 4, we get the following estimates with some new constants $C_2 < \infty$, $C_3 < \infty$:

$$\begin{aligned} &\Big| \int_{l^\pm} G(z^1, \zeta_n) \Lambda_n(\zeta_n, z_n) e^{\widetilde{\kappa}_n \operatorname{Re} z_n} d\zeta_n \Big| \\ &\qquad \le C_2 (\min_{1 \le j \le n-1} \beta_j)^{-k} \\ &\qquad\quad \times \Big| \int_{l^\pm} e^{-\operatorname{Re}(z_n - \zeta_n)(\operatorname{Re}(z_n - \zeta_n) - \widetilde{\kappa}_n)} \cdot e^{-(b_n - \widetilde{\kappa}_n)\zeta_n} d\zeta_n \Big| \\ &\qquad \le C_3 (\min_{1 \le j \le n-1} \beta_j)^{-k}. \end{aligned} \tag{34}$$

To estimate the integral over the interval $[\widetilde{M}_n - ir, \widetilde{M}_n + ir]$ we observe that by the assumption on G there exists $C_4 < \infty$ such that $|G(z^1, \zeta_n)| \le C_4 (\min_{1 \le j \le n-1} \beta_j)^{-k}$ on $(\omega_{\mathring{\alpha}^1} + i\mathbb{R}^{n-1}|_r) \times (\omega_{\widetilde{M}_n} + i\mathbb{R}|_r) \subset W \#_n \mathbb{R}^n_+$. Hence we get the following estimates with some new constants $C_5 < \infty$, $C_6 < \infty$:

$$\begin{aligned} &\Big| \int_{\widetilde{M}_n - ir}^{\widetilde{M}_n + ir} G(z^1, \zeta_n) \Lambda_n(\zeta_n, z_n) e^{\widetilde{\kappa}_n \operatorname{Re} z_n} d\zeta_n \Big| \\ &\qquad \le 2rC_5 (\min_{1 \le j \le n-1} \beta_j)^{-k} e^{-\alpha_n(\alpha_n - 2\widetilde{M}_n - \widetilde{\kappa}_n)} \le C_6 (\min_{1 \le j \le n-1} \beta_j)^{-k}. \end{aligned}$$

This together with (34) and (29) gives the desired estimate.

(iii) $\mathring{\alpha}_1 \ge 0, \dots, \mathring{\alpha}_m \ge 0$, $\mathring{\alpha}_{m+1} < 0, \dots, \mathring{\alpha}_{n-1} < 0$. Write $z^* = (z_{m+1}, \dots, z_{n-1})$, $\mathring{\alpha}^* = (\mathring{\alpha}_{m+1}, \dots, \mathring{\alpha}_{n-1})$ and observe that the function $G_{z^*}(z_1, \dots, z_m, z_n) = G(z_1, \dots, z_m, z^*, z_n)$ is holomorphic with respect to the parameter z^* in a complex neighbourhood of $\mathring{\alpha}^* \in \mathbb{R}^{n-m-1}$. Select the cases (i) or (ii) for the function G_{z^*} of $m+1$ variables $z_1, \dots, z_m, z_n$ and prove the adequate estimates uniformly with respect to the parameter z^*.

(iv) The case where Ω is a cartesian product of more than one unbounded intervals, for instance $\Omega = \omega_{\overset{\&}{\alpha}^2} \times (M_{n-1}, \infty) \times (M_n, \infty)$. Then we have to show that for every $\widetilde{\kappa}_{n-1} < \kappa_{n-1}$, $\widetilde{\kappa}_n < \kappa_n$ there exists $C < \infty$ such that $|G_n(\alpha + i\beta)| \le C(\min_{1\le j\le n-1} \beta_j)^{-k} \exp(-\alpha_{n-1}\widetilde{\kappa}_{n-1} - \alpha_n\widetilde{\kappa}_n)$ for $z = \alpha + i\beta \in \Omega + i(\mathbb{R}_+^{n-1} \times \mathbb{R})|_\varepsilon$. This can be derived from the estimate $|G(z^2, z_{n-1}, \zeta_n)| \le C(\min_{1\le j\le n-1} \beta_j)^{-k} \exp(-\alpha_{n-1}\widetilde{\kappa}_{n-1} - \alpha_n b_n)$ for $\zeta_n \in l^+ \cup l^-$, $z^2 \in \omega_{\overset{\&}{\alpha}^2} + i\mathbb{R}_+^{n-2}|_\varepsilon$, $z_{n-1} \in (M_{n-1}, +\infty) + i\mathbb{R}_+|_\varepsilon$, where l^+, l^-, b_n are defined as in (ii), and from the estimate $|G(z^2, z_{n-1}, \zeta_n)| \le C(\min_{1\le j\le n-1} \beta_j)^{-k} \exp(-\alpha_{n-1}\widetilde{\kappa}_{n-1})$ on $(\omega_{\overset{\&}{\alpha}^2} \times (M_{n-1}, +\infty) \times \omega_{\widetilde{M}_n}) + i(\mathbb{R}_+^{n-1} \times \mathbb{R})|_\varepsilon$ where $\widetilde{M}_n < M_n$. $\square$

LEMMA 5. *If $u \in L'_{(\omega)}(\mathbb{R}_+^n)$ then the function $\mathcal{C}_\Lambda u(z) \overset{\mathrm{df}}{=} \left(\frac{-1}{2\pi i}\right)^n u[\Lambda(z,\cdot)]$ for $z \in \mathbb{C}^n \Cap \overline{\mathbb{R}}_+^n$ belongs to $\mathfrak{L}_{(\omega)}^{|\infty}(\mathbb{C}^n \Cap \overline{\mathbb{R}}_+^n) \overset{\mathrm{df}}{=} \varprojlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}_0} \mathfrak{L}_\kappa^{|k}(\mathbb{C}^n \Cap \overline{\mathbb{R}}_+^n)$ (cf. Definition 4).*

PROOF. By point 4. of Section 1 and by Theorem 3 $\mathcal{C}_\Lambda u \in \mathcal{O}(\mathbb{C}^n \Cap \overline{\mathbb{R}}_+^n)$. On the other hand for any $\widetilde{\kappa} < \omega$ there exist $C = C(\widetilde{\kappa}) < \infty$, $m = m(\widetilde{\kappa}) \in \mathbb{N}_0$ such that (with $\gamma_{\widetilde{\kappa},\nu}$ defined in Section 1) $|u[\varphi]| \le C \sum_{|\nu|\le m} \gamma_{\widetilde{\kappa},\nu}(\varphi)$ for $\varphi \in L_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+^n)$. Hence by Proposition 4 we get the estimate

$$(35)\qquad |\mathcal{C}_\Lambda u(z)| \le C_1 \sum_{|\nu|\le m} \sup_{x\in\overline{\mathbb{R}}_+^n} \Big| e^{-\widetilde{\kappa}\cdot x} \Big(\frac{\partial}{\partial x}\Big)^\nu \frac{e^{-(z-x)^2}}{(z-x)^{\mathbb{1}}} \Big|.$$

Take first $\Omega = \omega_1 \times \dots \times \omega_n$, where ω_j are open bounded intervals in $\mathbb{R}$ with $\mathrm{dist}(\omega_j, \overline{\mathbb{R}}_+) \ge \rho > 0$ $(j = 1, \dots, n)$. Let $Q = \Omega + i\dot{\mathbb{R}}^n|_r$, $0 < r < \infty$. Then by (35) there exist $C_2 = C_2(r)$, $C_3 = C_3(r)$ such that $|\mathcal{C}_\Lambda u(z)| \le C_2\rho^{-m-n} \le C_3 < \infty$ for $z \in Q$ since Ω is bounded.

Let now $\Omega = \omega_1 \times \dots \times \omega_{n-1} \times (M_n, +\infty)$ where ω_j are open bounded intervals in $\mathbb{R}_+$, $j = 1, \dots, n-1$, $M_n \in \mathbb{R}$, $\Gamma = \mathbb{R}_+^n$, $Q = \Omega + i\Gamma|_r$. Then $\mathrm{dist}(\beta, \mathrm{bd}\,\Gamma) = \min_{1\le j\le n} \beta_j$. Take an arbitrary $\kappa < \omega$ and let $\kappa < \widetilde{\kappa} < \omega$. Then by the standard estimation (e.g. as in (34)) we derive from (35)

$$\left|e^{\kappa_n \operatorname{Re} z_n} \mathcal{C}_\Lambda u(z)\right| \le \frac{C}{(\mathrm{dist}(\beta, \mathrm{bd}\,\Gamma))^{m+n}} \quad \text{for } z \in \Omega + i\Gamma|_r$$

with some $C < \infty$, $m = m(\kappa)$, $0 < r < \infty$. In an analogous way we establish the pertinent estimates in other wedges and hence deduce that the function $\mathcal{C}_\Lambda u \in \mathfrak{L}^{|\infty}_{(\omega)}(\mathbb{C}^n \# \mathbb{R}^n_+)$. $\square$

LEMMA 6. *Under the notation of Lemma 4, a function* $G \in \mathfrak{L}^{|k}_{(\kappa)}(W \# \overline{\mathbb{R}}^n_+)$, $k \in \mathbb{N}_0$, $\kappa \in \mathbb{R}^n$, *is such that* $\int_\gamma G(z)\varphi(z)dz = 0$ *for* $\varphi \in \underset{\sim}{L}_{(\kappa)}(\overline{\mathbb{R}}^n_+)$ *if and only if* $G \in \sum_{j=1}^n \mathfrak{L}^{|k}_{(\kappa)}(W \#_j \overline{\mathbb{R}}^n_+)$. *Hence if* $G \in \mathfrak{L}^{|\infty}_{(\kappa)}(W \# \overline{\mathbb{R}}^n_+)$ *then* G *belongs to* $\sum_{j=1}^n \mathfrak{L}^{|\infty}_{(\kappa)}(W \#_j \overline{\mathbb{R}}^n_+)$ *if and only if* $\int_\gamma G(z)\varphi(z)dz = 0$ *for* $\varphi \in \underset{\sim}{L}_{(\kappa)}(\overline{\mathbb{R}}^n_+)$.

PROOF. Let $G \in \mathfrak{L}^{|k}_{(\kappa)}(W \# \overline{\mathbb{R}}^n_+)$, $\int_\gamma G(z)\varphi(z)dz = 0$ for $\varphi \in \underset{\sim}{L}_{(\kappa)}(\overline{\mathbb{R}}^n_+)$. Then following the proof of Lemma 2 we can write G as a linear combination of integrals taken only over the curves γ_j^+. By Lemma 4 the function G_n given by (29) belongs to $\mathfrak{L}^{|k}_{(\kappa)}(W \#_n \overline{\mathbb{R}}^n_+)$. $\square$

LEMMA 7. *Let* $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W \# \overline{\mathbb{R}}^n_+)$, $\omega \in \mathbb{R}^n$. *Then the functional* v *given by* $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+) \ni \varphi \longmapsto (-1)^n \int_\gamma \psi(z)\varphi(z)dz$, *where* $\gamma = \gamma_1 \times \cdots \times \gamma_n$ *is as in Theorem 3, extends uniquely to a distribution* $b\psi \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$.

PROOF. By Theorem 3 $v \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}^n_+)$. The further proof is divided into two steps.

Step I. $n = 1$. For $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ and $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ there exists $\varepsilon > 0$ such that $\varphi \in \mathcal{O}((\overline{\mathbb{R}}_+)_\varepsilon)$, $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}((\overline{\mathbb{R}}_+)_\varepsilon \setminus \overline{\mathbb{R}}_+)$. Moreover for some $c < \omega$ $\sup_{\zeta \in (\overline{\mathbb{R}}_+)_\varepsilon} |e^{-c\zeta}\varphi(\zeta)| < \infty$ and for any $\kappa \in \omega$ there exists $k(\kappa) \in \mathbb{N}_0$ such that $\psi \in \mathfrak{L}^k_\kappa((\overline{\mathbb{R}}_+)_\varepsilon \setminus \overline{\mathbb{R}}_+)$. Using the estimates satisfied by ψ and φ one can prove the relation

$$-\int_\gamma \psi(z)\varphi(z)dz = \lim_{\beta \to 0_+} \int_{-\varepsilon/2}^{+\infty} \psi(\alpha + i\beta)\varphi(\alpha)d\alpha - \lim_{\beta \to 0_+} \int_{-\varepsilon/2}^{+\infty} \psi(\alpha - i\beta)\varphi(\alpha)d\alpha \tag{36}$$

for $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. Take now $\varphi \in L_{(\omega)}(\overline{\mathbb{R}}_+)$ and its extension $\widetilde{\varphi} \in C^\infty(\mathbb{R})$, $\widetilde{\varphi}(\alpha) = 0$ for $\alpha \leq -\varepsilon/2$. We shall prove that the right-hand side of (36) makes sense for such $\widetilde{\varphi}$ and defines a functional $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$, in fact $T = b\psi$. Assume first $\omega \le 0$ and fix arbitrarily $a < \omega$. Next take $a < \kappa < \omega$, choose a base point $\mathring{\zeta} = -\rho$, where $0 < \rho < \varepsilon$, and define an operation $\mathcal{J}\psi(\zeta) = \int_{\gamma_\zeta} \psi(w)dw$, where γ_ζ is a curve joining $\mathring{\zeta}$ with $\zeta \in (\overline{\mathbb{R}}_+)_\varepsilon \setminus \overline{\mathbb{R}}_+$. After $k+1$ iterations of the operation $\mathcal{J}$ we arrive at the function $\mathcal{J}^{k+1}\psi \in \mathcal{O}((\overline{\mathbb{R}}_+)_\varepsilon \setminus \overline{\mathbb{R}}_+)$, $|\mathcal{J}^{k+1}\psi(\alpha+i\beta)| \le Ce^{-\alpha\kappa}$ for $\alpha + i\beta \in (\overline{\mathbb{R}}_+)_\varepsilon$, $\frac{d^{k+1}}{dz^{k+1}}\mathcal{J}^{k+1}\psi = \psi$ and such that $\lim_{\beta\to 0_+} \mathcal{J}^{k+1}\psi(\alpha \pm i\beta)$ exist locally uniformly and define continuous functions on $(-\varepsilon, \infty)$: $\psi_\pm(\alpha) \stackrel{\mathrm{df}}{=} \lim_{\beta\to 0_+} \mathcal{J}^{k+1}\psi(\alpha \pm i\beta)$, $\psi_+(\alpha) = \psi_-(\alpha)$ for $\alpha < 0$, $|\psi_\pm(\alpha)| \le Ce^{-\alpha\kappa}$ for $\alpha > -\varepsilon$. Now if $\varphi \in L_a(\overline{\mathbb{R}}_+)$ (and $\widetilde{\varphi}$ is its extension), we get easily by integrating by parts

$$
\begin{aligned}
&\lim_{\beta\to 0_+} \int_{-\varepsilon/2}^{+\infty} (\psi(\alpha+i\beta) - \psi(\alpha - i\beta))\widetilde{\varphi}(\alpha)d\alpha \\
&\qquad = (-1)^{k+1} \int_0^\infty (\psi_+(\alpha) - \psi_-(\alpha)) \frac{d^{k+1}}{d\alpha^{k+1}}\varphi(\alpha)d\alpha = T[\varphi], \\
&\qquad \text{where } T = \frac{d^{k+1}}{d\alpha^{k+1}}(\psi_+(\alpha) - \psi_-(\alpha)) \in L'_a(\overline{\mathbb{R}}_+).
\end{aligned}
\tag{37}
$$

Since $a < \omega \le 0$ was arbitrary, we have $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$.

Assume now that $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}((\overline{\mathbb{R}}_+)_\varepsilon \setminus \overline{\mathbb{R}}_+)$ with $\omega > 0$. Then T^* defined by (37), with $\psi^*(\zeta) = \psi(\zeta)e^{\zeta\omega}$ instead of ψ, belongs to $L'_{(0)}(\overline{\mathbb{R}}_+)$ and $\widetilde{T} \stackrel{\mathrm{df}}{=} e^{-\omega\alpha}T^*$ belongs to $L'_{(\omega)}$. Moreover for $\varphi \in L_a(\overline{\mathbb{R}}_+)$, $a < \omega$, $\widetilde{T}[\varphi] = T[\varphi]$ given by the right-hand side of (36).

Step II. Let $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W \mathbin{\backslash\!\!\!\backslash} \overline{\mathbb{R}}^n_+)$. Consider v on functions $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$ in the product form $\varphi(z) = \varphi_1(z_1) \cdot \ldots \cdot \varphi_n(z_n)$ with $\varphi_j \in \underset{\sim}{L}_{(\omega_j)}(\overline{\mathbb{R}}_+)$, $j = 1, \ldots, n$, and apply a parameter version of the one-dimensional assertion (36) proved above:

$$
\begin{aligned}
&-\int_{\gamma_1} \psi(z_1, z_2, \ldots, z_n)\varphi_1(z_1)dz_1 \\
&\qquad = \sum_{\sigma_1 \in \{+,-\}} \operatorname{sgn}\sigma_1 \lim_{\beta_1\to 0_+} \int \psi(\alpha_1 + i\sigma_1\beta_1, z_2, \ldots, z_n)\varphi(\alpha_1)d\alpha_1
\end{aligned}
$$

with $\gamma_1 \subset W_1$ encircling $\overline{\mathbb{R}}_+$. Let $W' = W_2 \times \cdots \times W_n$. We note the following result:

the function

$$W' \setminus\!\!\!\setminus \overline{\mathbb{R}}_+^{n-1} \ni (z_2, \ldots, z_n) \mapsto \psi_1(z_2, \ldots, z_n) = -\int_{\gamma_1} \psi(z_1, z_2, \ldots, z_n)\varphi_1(z_1)dz_1$$

belongs to $\mathfrak{L}^{|\infty}_{(\omega')}(W' \setminus\!\!\!\setminus \overline{\mathbb{R}}_+^{n-1}),\ \omega' = (\omega_2, \ldots, \omega_n)$

whose proof is done in the spirit of the proof of Lemma 4.

Hence we deduce that

$$\begin{aligned} v[\varphi] &= (-1)^n \int_{\gamma_1 \times \cdots \times \gamma_n} \psi(z)\varphi_1(z_1) \cdot \ldots \cdot \varphi_n(z_n)dz \\ &= \sum_{\sigma \in \{+,-\}^n} \operatorname{sgn}\sigma \lim_{\beta_n \to 0_+} \int \Big(\ldots \big(\lim_{\beta_1 \to 0_+} \int \psi(\alpha + i\sigma\beta)\varphi(\alpha)d\alpha_1 \big) \ldots \Big) d\alpha_n. \end{aligned}$$

Next we prove that

$$\begin{aligned} &\sum_{\sigma \in \{+,-\}^n} \operatorname{sgn}\sigma \lim_{\beta_n \to 0_+} \int \Big(\ldots \big(\lim_{\beta_1 \to 0_+} \int \psi(\alpha + i\sigma\beta)\varphi(\alpha)d\alpha_1 \big) \ldots \Big) d\alpha_n \\ &\quad = \sum_{\sigma \in \{+,-\}^n} \operatorname{sgn}\sigma \lim_{\beta \to 0_+} \int \psi(\alpha + i\sigma\beta)\varphi(\alpha)d\alpha. \end{aligned} \tag{38}$$

This is clear for ψ which extends continously to the boundary from every local wedge $\Omega + i\sigma\overline{\mathbb{R}}_+^n \subset W$. In the general case we use the fact that every $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W \setminus\!\!\!\setminus \overline{\mathbb{R}}_+^n)$ can be represented (cf. e.g. (7)) as a finite sum over multiindices $\boldsymbol{k} = (k_1, \ldots, k_n)$

$$\psi(z) = \sum_{\boldsymbol{k}} \Big(\frac{\partial}{\partial z}\Big)^{\boldsymbol{k}} F_{\boldsymbol{k}}(z),$$

where $F_{\boldsymbol{k}}$ have the above property. Thus the proof reduces to proving a series of identities (38) with ψ replaced by $F_{\boldsymbol{k}}$ and φ by $\big(\frac{\partial}{\partial z}\big)^{\boldsymbol{k}}\varphi$.

The result follows by the density of the space (cf. [Mi]) $\underset{\sim}{L}(\omega_1)(\overline{\mathbb{R}}_+) \otimes \cdots \otimes \underset{\sim}{L}(\omega_n)(\overline{\mathbb{R}}_+)$ in $\underset{\sim}{L}(\omega)(\overline{\mathbb{R}}^n_+)$. $\square$

THEOREM 4. *The isomorphism* $\mathcal{I}$ *of Theorem* 3 *extends to a topological isomorphism of the spaces*

$$\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)\Big/\sum_{j=1}^{n}\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}_j\overline{\mathbb{R}}^n_+) \cong L'_{(\omega)}(\overline{\mathbb{R}}^n_+), \quad \omega \in \mathbb{R}^n.$$

PROOF. Let $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)$. Then by Lemma 7 the functional $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+) \ni \varphi \mapsto (-1)^n \int_\gamma \psi(z)\varphi(z)dz$ extends uniquely to a distribution $b\psi \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and in view of Lemma 6 the mapping $\mathcal{I}$:

$$\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)\Big/\sum_{j=1}^{n}\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}_j\overline{\mathbb{R}}^n_+) \ni [\psi] \stackrel{\mathcal{I}}{\longmapsto} b\psi \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+) \tag{39}$$

is well defined.

On the other hand Lemma 5 provides a mapping $\mathcal{J}$

$$L'_{(\omega)}(\overline{\mathbb{R}}^n_+) \ni u \stackrel{\mathcal{J}}{\longmapsto} [\mathcal{C}_\Lambda u] \in \mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)\Big/\sum_{j=1}^{n}\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}_j\overline{\mathbb{R}}^n_+), \tag{40}$$

which turns out to be the inverse of $\mathcal{I}$.

Indeed, take $u \in L'_{(\omega)}(\mathbb{R}^n)$ and observe that by (39), (40) $\mathcal{I}\circ\mathcal{J}u - u \in L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$. By point 4. from Section 1 and by Theorem 3 $(\mathcal{I}\circ\mathcal{J}u - u)[\varphi] = 0$ for $\varphi \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}^n_+)$. Thus by point 3. from Section 1 $(\mathcal{I}\circ\mathcal{J}u - u)[\varphi] = 0$ for $\varphi \in L_{(\omega)}(\overline{\mathbb{R}}^n_+)$ and hence $\mathcal{I}\circ\mathcal{J}u = u$.

Next take $\psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)$. By (39) and Lemma 5 $\mathcal{C}_\Lambda(\mathcal{I}[\psi]) \in \mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)$ and hence $F \stackrel{\mathrm{df}}{=} \mathcal{C}_\Lambda(\mathcal{I}[\psi]) - \psi \in \mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)$. Since $\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+) \subset \mathfrak{L}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}\overline{\mathbb{R}}^n_+)$ by Theorem 3 $F \in \sum_{j=1}^{n}\mathfrak{L}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}_j\overline{\mathbb{R}}^n_+)$ and hence by Lemmas 2 and 6 $F \in \sum_{j=1}^{n}\mathfrak{L}^{|\infty}_{(\omega)}(W\mathbin{\backslash\!\!\backslash}_j\overline{\mathbb{R}}^n_+)$. $\square$

Note that the last fragment of the proof amounts in fact to the statement:

Remark 2. We have the canonical imbedding

$$\mathfrak{L}^{|\infty}_{(\omega)}(W\#\overline{\mathbb{R}}^n_+)\Big/\sum_{j=1}^{n}\mathfrak{L}^{|\infty}_{(\omega)}(W\#_j\overline{\mathbb{R}}^n_+)\hookrightarrow\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}^n_+).$$

4. Martineau–Harvey Theorems

Let $K_1,\dots,K_n$ be compact sets in $\mathbb{R}$ and let $K=K_1\times\cdots\times K_n$. For every open bounded set V in $\mathbb{C}^n$ containing K denote by $H(\overline{V})$ the space of continuous functions in $\overline{V}$ which are holomorphic in V with $\|F\|_V=\sup_{z\in V}|F(z)|$. Let $A(K)=\varinjlim_{V\supset K}H(\overline{V})$. The elements of the dual space of $A(K)$, denoted by $A'(K)$, are called *analytic functionals carried by K*.

Denote by $B_K(\mathbb{R}^n)$ the space of hyperfunctions with support contained in K. The standard realization of $B_K(\mathbb{R}^n)$ can be represented in the form

$$B_K=B_K(\mathbb{R}^n)\cong\mathcal{O}(U\#K)\Big/\sum_{j=1}^{n}\mathcal{O}(U\#_jK),$$

where $U=U_1\times\cdots\times U_n$, U_j—a connected domain in $\mathbb{C}$ containing K_j $(j=1,\dots,n)$, $U\#_jK=(U_1\setminus K_1)\times\cdots\times U_j\times\cdots\times(U_n\setminus K_n)$ $(j=1,\dots,n)$.

Theorem 5 (Martineau–Harvey, [M], [H]). *There exists a natural topological isomorphism*

$$B_K\cong A'(K)$$

given by the assignment

$$B_K\ni f=[F]\mapsto\mathcal{I}f\in A'(K),$$

where $F\in\mathcal{O}(U\#K)$. The functional $\mathcal{I}f$ is given by

$$\mathcal{I}f[\varphi]=(-1)^n\int_{\gamma_1\times\cdots\times\gamma_n}F(z)\varphi(z)dz\quad\textit{for}\ \ \varphi\in A(K)$$

and for $j=1,\dots,n$, γ_j is a closed curve in $U_j\setminus K_j$ encircling K_j in the anticlockwise direction and contained in an open set $V_j\supset K_j$ (provided φ extends holomorphically to $V_1\times\ldots\times V_n$). The inverse mapping $\mathcal{I}^{-1}$ is the *assignment* $A'(K) \ni g \mapsto \mathcal{I}^{-1}g = [\mathcal{C}g]$ *where* $\mathcal{C}g(z) = \left(\frac{-1}{2\pi i}\right)^n g[(z-\zeta)^{-\mathbb{1}}] \in \mathcal{O}(\mathbb{C}^n \# K)$.

Theorem 5 leads to a distributional version of Martineau–Harvey theorem in a way similar to that of deriving Theorem 4 from Theorem 3. The situation now is however simpler since it does not involve the estimates at infinity. Therefore we only restrict ourselves to introducing pertinent spaces and formulating final results.

Ω will stand now for a bounded open set in $\mathbb{R}^n$. As before, Γ denotes the non-empty open cone in $\dot{\mathbb{R}}^n$, $\Gamma|_r$—its intersection with a ball of radius r, $0 < r < \infty$, and $Q = \Omega + i\Gamma|_r$—the corresponding local wedge.

Let $V \subset \mathbb{C}^n$ be an open set and $[V]$ its germ near $\mathbb{R}^n$. Let $k \in \mathbb{N}_0$. Define the space $\mathcal{O}^{|k}([V])$ by

$$\mathcal{O}^{|k}([V]) = \Big\{H \in \mathcal{O}(V):\ q_Q(H) < \infty \ \text{ for every local wedge } Q \subset V \\ \text{where } q_Q(H) \overset{\mathrm{df}}{=} \sup_{z\in Q} |H(z)| \cdot \big(\operatorname{dist}(\operatorname{Im} z, \operatorname{bd}\Gamma)\big)^k\Big\}$$

and let

$$\mathcal{O}^{|\infty}([V]) = \varinjlim_{k\in\mathbb{N}_0} \mathcal{O}^{|k}([V]).$$

Finally, we denote by $D'_K(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$ with support in K.

THEOREM 6 (cf. [M]). *The isomorphism* $\mathcal{I}$ *of Theorem 5 extends to a topological isomorphism of the spaces*

$$\mathcal{O}^{|\infty}(U \# K) \Big/ \sum_{j=1}^{n} \mathcal{O}^{|\infty}(U \#_j K) \cong D'_K(\mathbb{R}^n). \tag{41}$$

Observe that for $k, p \in \mathbb{N}_0$, $k < p$, $F \in \mathcal{O}^{|k}(U \# K)$ the mapping[(10)] $\mathcal{E}$:

$$F + \sum_{j=1}^{n} \mathcal{O}^{|k}(U \#_j K) \overset{\mathcal{E}}{\longmapsto} F + \sum_{j=1}^{n} \mathcal{O}^{|p}(U \#_j K)$$

[(10)] For the proof observe that a function $F \in \mathcal{O}^{|k}(U \# K)$ satisfies $\int_{\gamma_1\times\cdots\times\gamma_n} \times F(\zeta)\varphi(\zeta)d\zeta = 0$ for $\varphi \in A(K)$ and $\gamma_1 \times \cdots \times \gamma_n$ as in Theorem 5, if and only if $F \in \sum_{j=1}^{n} \mathcal{O}(U \#_j K)$ (cf. Lemma 6).

is a well defined 1–1 mapping and hence we can write (41) in the following form

$$\text{(41}'\text{)}\qquad \varinjlim_{k\in\mathbb{N}_0} \Big(\mathcal{O}^{\lfloor k}(U\#K)\Big/\sum_{j=1}^{n}\mathcal{O}^{\lfloor k}(U\#_j K)\Big) \cong D'_K(\mathbb{R}^n).$$

Similarly, if $F\in\mathcal{O}^{\lfloor k}(U\#K)$, $k\in\mathbb{N}_0$, the mapping $\widetilde{\mathcal{E}}$

$$F+\sum_{j=1}^{n}\mathcal{O}^{\lfloor k}(U\#_j K) \overset{\widetilde{\mathcal{E}}}{\longmapsto} F+\sum_{j=1}^{n}\mathcal{O}(U\#_j K)$$

is 1–1, which gives the imbedding

$$\varinjlim_{k\in\mathbb{N}_0} \Big(\mathcal{O}^{\lfloor k}(U\#K)\Big/\sum_{j=1}^{n}\mathcal{O}^{\lfloor k}(U\#_j K)\Big) \hookrightarrow \frac{\mathcal{O}(U\#K)}{\sum_{j=1}^{n}\mathcal{O}(U\#_j K)}.$$

Hence (41′) leads to a natural imbedding of distributions in hyperfunctions:

$$D'_K(\mathbb{R}^n)\hookrightarrow B_K(\mathbb{R}^n),\quad K \text{ compact in } \mathbb{R}^n.$$

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(Received February 21, 1996)

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