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Remarks on Gorenstein Terminal Fourfold Flips

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Abstract. We prove that for any flipping contraction from a Gorenstein terminal 4-fold, a general hyperplane section which contains the exceptional locus has only canonical singularities. Based on this fact and using Ran's theorem, we prove the existence of the flip of a flipping contraction from a Gorenstein terminal 4-fold whose general hyperplane section has only isolated canonical singularities and exceptional locus is irreducible. Furthermore we classify such flipping contractions and flips.

0. Introduction

To proceed the Minimal Model Program (in short MMP), an elementary transformation called flip is very important (see [KMM] for detail).

Definition 0.1. Let X be a normal algebraic variety (resp. normal analytic variety) with only terminal singularities and Y a normal algebraic variety (resp. (Y, S) a pair of an analytic space and its compact subspace). A projective morphism $f : X \to Y$ is called a (terminal) flipping contraction if

(1) $-K_X$ is f-ample;
(2) $\rho(X/Y) = 1$ (resp. $\rho(X/Y, f^{-1}(S)) = 1$);
(3) f is an isomorphism in codimension 1.

If there exists a normal algebraic variety (resp. normal analytic variety) X^+ with only terminal singularities and a projective morphism $f^+ : X^+ \to Y$

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such that

(1) K_{X^+} is f^+-ample;
(2) $\rho(X^+/Y) = 1$ (resp. $\rho(X^+/Y, (f^+)^{-1}(S)) = 1$);
(3) f^+ is an isomorphism in codimension 1,

we call f^+ the (terminal) flip of f. We call the following diagram a flipping diagram:

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ f \searrow & & \swarrow f^+ \\ & Y & \end{array} .$$

The existence of the flip is a very hard problem. In dimension 3, Shigefumi Mori proved it in [M4]. As a test case of 4-dimensional flips, we consider a flipping contraction from an algebraic 4-fold with only Gorenstein terminal singularities. Let X be an algebraic 4-fold with only Gorenstein terminal singularities and $f\colon X \to Y$ be a flipping contraction. Let E be the exceptional locus. Since there is no flipping contraction from an algebraic (or analytic) 3-fold with only Gorenstein terminal singularities, we find that $f(E)$ is a set of finite points. Hence replacing Y by a small Stein neighborhood of a point in $f(E)$, we can proceed in the analytic category. Precisely speaking, we consider the following object below (we call this $(*)$).

$(*)$ Let X be an analytic 4-fold with only Gorenstein terminal singularities and (Y, P) a pair of a contractible 4-dimensional Stein space and a point in it such that Y has only ccDV singularities (i.e., singularities whose general hyperplane sections have only cDV singularities) outside P. Let $f\colon X \to Y$ be a flipping contraction and $E := f^{-1}(P)$, i.e., the exceptional locus of f.

In [Kaw3], Yujiro Kawamata considered the case where X is smooth. He proved the following:

Theorem 0.1. *Assume that X is smooth. Then the flip exists and $E \simeq \mathbb{P}^2$ and $\mathcal{N}_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$. In particular we obtain the flip by blowing up E (the exceptional locus of the blowing up is $\mathbb{P}^2 \times \mathbb{P}^1$) and blowing down this $\mathbb{P}^2 \times \mathbb{P}^1$ to $\mathbb{P}^1$.*

Quite recently Yasuyuki Kachi proved in his preprint [Kac2] the following:

Theorem 0.2. *Assume that X is singular and has only isolated complete intersection terminal singularities. Suppose that there is a member of $|-2K_Y|$ through P which has only a rational singularity at P.*

Then the flip exists and $E \simeq \mathbb{P}^2$ and $\mathcal{N}_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. Furthermore X has only one singularity on E, which is analytically isomorphic to $o \in (xy + zw + t^m = 0) \subset \mathbb{C}^5$.

He proved the existence of such a flip by induction and constructed the desired flip very explicitly (see [Kac2, §8] for detail). He also investigated some special semistable 4-fold flipping contractions in [Kac1].

First we consider the above object $(*)$ with no additional assumption. Our starting points are the following two theorems:

Theorem 1.2 (Rough classification of the exceptional locus). *Assume that the exceptional locus E contains 2-dimensional components. Let $E = \cup E_i$ be the irreducible decomposition of E. Then E is purely 2-dimensional and $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{F}_{n,0}, nl)$, where l is a ruling of $\mathbb{F}_{n,0}$.*

Theorem 1.3. *Let B a general hyperplane section through P. Then the strict transform $A := f^*B$ has only canonical singularities.*

Our main result is the following:

Main Theorem. *We use also the notation as in Theorem 1.3. Assume that A has only isolated singularities. Then E is 2-dimensional. Furthermore assume that E is irreducible and hence isomorphic to $\mathbb{F}_{n,0}$ for some natural number n.*

Then the flip exists.

We give the description of the flipping diagram for such an f as in the main theorem. (See Corollary 2.3.)

A key to the main theorem is the theorem of Ziv Ran (Theorem 1.5 below). By this, we can construct the flip as János Kollár and Shigefumi Mori did for 3-dimensional terminal flipping contractions by using the deformation theory of rational singularities of surfaces. (see [KM, Theorem 11.7]). We hope that by generalizing Ran's Theorem we can prove the existence of the flip for any flipping contraction from a Gorenstein terminal 4-fold as we do in this paper.

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Notation and Convention.

(1) In this paper, we will work over $\mathbb{C}$, the complex number field and in the analytic category;
(2) We denote by $\mathbb{F}_n$, the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ and by $\mathbb{F}_{n,0}$ the normal surface which is obtained from the Hirzebruch surface $\mathbb{F}_n$ by contracting the negative section.

1. Preliminaries

Theorem 1.1. *Let X and Y be normal log terminal varieties and $f : X \to Y$ a projective morphism. Let L an f-ample line bundle on X and F a fiber of f. Assume that $f : X \to Y$ is the adjoint contraction supported by $K_X + rL$ and either* $\dim F < r + 1$ *if* $\dim Y < \dim X$ *or* $\dim F \leq r + 1$ *if* $\dim Y = \dim X$.

Then $f^* f_* L \to L$ is surjective at every point of F.

Proof. See [AW1]. They assume that L is ample but their proof works also for the case that X is analytic and L is relatively ample. □

Theorem 1.2 (Rough classification of the exceptional locus). *We consider the object* $(*)$. *Assume that the exceptional locus E contains* 2-*dimensional components. Let $E = \cup E_i$ be the irreducible decomposition of E. Then E is purely* 2-*dimensional and $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{F}_{n,0}, nl)$, where l is a ruling of $\mathbb{F}_{n,0}$.*

Proof. By Theorems 1.10 and 1.19 of [AW3], it is sufficient to exclude the following possibilities: $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ or $(\mathbb{F}_n, C_0 + ml)$, where C_0 is the negative section and l is a ruling and $m \geq$

$n+1$. By following the argument of [W, Theorem 1.1, claim] with Theorem 2.13 in [Ko3], we can prove

CLAIM. Let X be a variety with only log terminal singularities and R an extremal ray of X. Let F be an irreducible component of a non-trivial fiber of the contraction of R. Assume that for a general point $x \in F$, there is a rational curve $M \subset F$ through x with the following condition:

(1) its intersection with $-K_X$ is minimal among all rational curves in F through x.
(2) X has only local complete intersection singularities along M and M is not contained in the singular locus of X.

Then

$$\dim F + \dim(\text{locus of } R) \geq \dim X + l(R) - 1, \tag{1.2.1}$$

where $l(R)$ is the length of R. Furthermore if the equality holds, the dimension of the deformation of M which through a fixed point x is $\dim F - 1$.

If $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, a general line satisfies the assumption of M in Claim. So we can use Claim and derive a contradiction to the inequality (1.2.1). If $(E_i, -K_X|_{E_i})$ is isomorphic to $(\mathbb{F}_n, C_0 + ml)$, a general ruling satisfies the assumption of M in Claim. So by using Claim, we obtain the equality in (1.2.1). But a ruling cannot move if a general point on it is fixed, a contradiction to the second part of Claim. □

By this Theorem, the exceptional locus of a Gorenstein terminal 4-fold flipping contraction is either purely 1-dimensional or purely 2-dimensional. In the former case, we call it a flipping contraction of type $(1,0)$. In the latter case, we call it a flipping contraction of type $(2,0)$.

THEOREM 1.3. *We consider the object* $(*)$. *Let* B *be a generic hyperplane section through* P. *Then the strict transform* $A := f^*B$ *has only canonical singularities.*

PROOF. We take a general member $C \in |-K_X|$ and let $D := f(C)$. By the freeness of $|-K_X|$ (Theorem 1.1), we can assume that C is Gorenstein terminal. D is Gorenstein by the Serre-Grothendieck duality (cf. [Kaw

2, the Proof of Theorem 8.7]), which in turn shows that D is normal and $C \to D$ has then only connected fibers by the Zariski Main Theorem. Hence if f is of type $(1,0)$, $C \to D$ is an isomorphism or if f is of type $(2,0)$, $f|_C$ is a flopping contraction. So in any case D has also Gorenstein terminal singularity at P, i.e., cDV singularity. Then we may assume that $B|_D$ is canonical by replacing B if necessary. So $A|_C$ must be also normal and canonical since $A|_C \to B|_D$ is isomorphism if f is of type $(1,0)$ or $A|_C \to B|_D$ is crepant if f is of type $(2,0)$. We know that A is canonical along $C|_A$ by the above argument. So it suffices to prove that A is canonical outside $C|_A$. The argument below is inspired by the proof of [Kaw 2, Theorem 8.5]. Let C' be a general member of $|-2K_X|$ and $D' := f(C')$. We know that $K_B + D|_B$ is canonical by the inversion of adjunction as follows:

$K_B + D|_B|_{D|_B} = K_{D|_B}$ is canonical and in particular kawamata log terminal. Hence by [Utah, 17.6 Theorem], $K_B + D|_B$ is purely log terminal. But $K_B + D|_B$ is Cariter so it is canonical.

Hence

(1.3.1) $K_B + \frac{1}{2}D'|_B$ is also canonical since D' is more general than D.

We take the double cover $\tilde{A} \to A$ (resp. $\tilde{B} \to B$) whose branch locus is $C'|_A$ (resp. $D'|_B$). Let $g : \tilde{A} \to \tilde{B}$ be the natural morphism. It is sufficient to prove that $\tilde{A}$ is canonical since $\tilde{A} \to A$ is etale outside $C'|_A$. By (1.3.1), $\tilde{B}$ is Gorenstein canonical. So $\tilde{A}$ is also Gorenstein canonical since g is crepant and we are done. □

Remark. By this Theorem, we see that the object $(*)$ is a very special example of a semistable 4-fold flipping contraction. (See [C] for the definition of a semistable flipping contraction.)

Proposition 1.4. *Consider the situation of Theorem 1.3 and take A and B as there. Then*

(1) *a general element of $|-K_B|$ has only Du Val singularity at P;*
(2) *for any i, E_i is not $\mathbb{Q}$-Cartier divisor in A.*

Proof.

(1) $D|_B \in |-K_B|$ in the proof of Theorem 1.3 satisfies (1).

(2) (cf. the argument of [Kac1, 4.3]) We assume that for some i, E_i is $\mathbb{Q}$-Cartier. Assume further that E has another component. Let E_j be a component such that $E_i \cap E_j \neq \phi$. Then by the assumption that E_i is $\mathbb{Q}$-Cartier, $E_i \cap E_j$ is 1-dimensional. Then since the Picard numbers of such E_j's and E_i are 1, the union of E_i and E_j's is covered by one extremal ray in $\overline{NE}(A/B)$. For a ruling m of E_j (not contained in E_i), $E_i.m > 0$. But for a ruling l in E_i, $E_i.l < 0$, a contradiction. Hence E is irreducible and $\mathbb{Q}$-Cartier. So B has only canonical singularities by [KMM, Lemma 5-1-7]. Note that B is smooth outside P and that $|-K_B|$ has a Du Val element through P. So in fact B is terminal by [St, Section 5]. Since B can deform to a 3-fold with only cDV singularities in Y, B also has only cDV singularity (cf. [Ko1] or [Ra2, Theorem 2.3]). In particular B has only hypersurface singularity so Y has also only hypersurface singularity, a contradiction. We establish the proposition. □

The next theorem due to Z. Ran is another key to the proof of the main theorem.

THEOREM 1.5. *Let U be a smooth 3-fold and $\mu : U \to V$ a projective bimeromorphic morphism. Let C be the exceptional locus of μ. Assume that C is $\mathbb{P}^1$ and $K_U.C > 0$. Then μ is target stable, i.e., if there is a small deformation $\mathcal{V} \to (S, o)$ of V over S, then we have a small deformation $\mathcal{U} \to \mathcal{V}$ of $\mu : U \to V$ over (S, o).*

PROOF. See [Ra2, Theorem 3.2]. □

PROPOSITION 1.6 (H. Laufer). *Let S be normal Gorenstein surface and $f : S \to T$ a projective bimeromorphic morphism to a normal suface T. Let C be the exceptional curve. Suppose that C is irreducible, isomorphic to $\mathbb{P}^1$ and $K_S.C = -1$. Then $f(C)$ is a smooth point of T, S has only one singular point on C which is of type A_{n-1} for some $n \in \mathbb{N}$. Furthermore $C^2 = -\frac{1}{n}$.*

PROOF. See [LS, Theorem 0.1]. □

THEOREM 1.7 (Length of an extremal ray). *Let X be a variety with only canonical singularities and R an extremal ray of X. Let F be a 1-*

dimensional irreducible component of the fiber of the contraction of R which contains Gorenstein points of X. Then $K_X.F \geq -1$.

PROOF. See [M4, 1.3 and 2.3.2] or [I, Lemma 1]. □

PROPOSITION 1.8. *Let U be a 3-fold with only Gorenstein canonical singularities and (V, P) a pair of a 3-dimensional normal Stein space and a point in it. Let $f : U \to V$ be a flipping contraction whose exceptional locus l is connected. Then l is irreducible and $l \subset$ Sing U.*

PROOF. The irreducibility of l can be proved by the same argument as the first part of the proof of Theorem 1.3. Assume that

(1.8.1) Sing $U \cap l$ consists of finite points.

Let $g : U' \to U$ be a partial resolution such that g is crepant and U' has only Gorenstein terminal singularities (cf. [M3] and [Re2]). Since $K_{U'}$ is not $f \circ g$-nef, we can find an extremal ray $R \in \overline{NE}(U'/V)$. Let l' be an irreducible curve such that $[l'] \in R$. Then l' is the strict transform of l by (1.8.1) and the fact that $K_{U'}$ is g-nef. So R is a flipping ray. But this contradicts the fact that there is no flipping contraction from a Gorenstein terminal 3-fold. □

2. Proof of the Main Theorem

We will also use the notation in Theorem 1.3 freely.

PROOF. First we prove that E is 2-dimensional. Assume that E is 1-dimensional. Then we obtain a flipping contraction from a Gorenstein canonical 3-fold A such that the exceptional locus E is not contained in the singular locus of A, a contradiction to Proposition 1.8.

In the following we assume that E is irreducible and is isomorphic to $\mathbb{F}_{n,0}$. Let $q : A_q \to A$ be a small morphism such that the inverse image E_q of E is q-anti-ample (i.e., $A_q := \textbf{\textit{Projan}} \bigoplus_{m=0}^{\infty} \mathcal{O}_A(-mE)$ and q is the natural projection). We can take such a small morphism by [Kaw2, Theorem 6.1]. Since E is not $\mathbb{Q}$-Cartier by Proposition 1.4, A_q is not isomorphic to A.

Let $\Phi : A_q \to A^+$ be the contraction of an extremal ray in $\overline{NE}(A_q/B)$ and $g^+ : A^+ \to B$ the natural morphism. We obtain the following diagram:

$$\begin{array}{ccccc} & & A_q & & \\ & q \swarrow & & \searrow \Phi & \\ A & & & & A^+ \\ & f|_A \searrow & & \swarrow g^+ & \\ & & B & & \end{array} .$$

Claim 2.1. Φ is a divisorial contraction which contracts E_q to a curve.

Proof. Since q is not an isomorphism and $-E_q$ is q-ample, E_q contains all q-exceptional curves. If Φ is a divisorial contraction which contracts E_q to a point, such q-exceptional curves are contracted by Φ. But this is absurd since K_{A_q} is q-trivial but Φ-negative. If Φ is a flipping contraction, then the flipping curve m is contained in the curve singularity of A_q by Proposition 1.8. So by the assumption that A has only isolated singularities, m must be contained in the q-exceptional curve, a contradiction. $\square$

Let E^+ be the curve $\Phi(E_q)$.

Claim 2.2.

(1) A^+ is smooth along E^+. $E^+ \simeq \mathbb{P}^1$ and $E_q \simeq \mathbb{F}_n$;

(2) A_q is smooth outside the negative section M of E_q and A_q has only (locally trivial) cA_{m-1} singularity along M, where m is an positive integer (later we will show that $m = n$ in the proof of Corollary 2.3);

(3) $K_{A^+}.E^+ = 2m - n > 0$.

Proof.

(1) By Theorem 1.1, $|-K_{A_q}|$ is free near the fiber over any point Q of E^+, so we can take a smooth member $D \in |-K_{A_q}|$ near the fiber since A_q has only isolated singularities. Since D maps isomorphically to $\Phi(D) \in |-K_{A^+}|$ (cf. [Kaw2, the Proof of Theorem 8.7]), we see that there is a smooth member of $|-K_{A^+}|$ through Q. Note that Q is a canonical singularity of A^+. By these, we can see that Q is a smooth point of A^+ as follows:

It is sufficient to prove that K_{A^+} is Cartier at Q. Assume the contrary. Let $\pi : \tilde{A^+} \to A^+$ be the index 1 cover for K_{A^+} near Q.

Then $\tilde{A}^+$ is Gorenstein canonical at $\pi^{-1}(Q)$. Since π is ramified only at Q and $\Phi(D)$ is smooth,

(2.2.1) $\pi^{-1}\Phi(D)$ has at least 2 components and they intersect mutually only at $\pi^{-1}(Q)$.

Furthermore they are all smooth. In praticular $\pi^{-1}\Phi(D)$ satisfies R_1 condition. On the other hand $\pi^{-1}\Phi(D)$ satisfies S_2 condition since this is a Cartier divisor of a canonical singularity. Hence $\pi^{-1}\Phi(D)$ is normal by the Serre's criterion. But this is a contradiction to (2.2.1).

Since B has only rational singularities and E^+ is an irreducible curve, E^+ must be $\mathbb{P}^1$. Since a general fiber n of Φ is irreducible and reduced and $-K_{A_q}.n = 1$ (Theorem 1.7), any fiber is irreducible and reduced. So E_q is $\mathbb{F}_n$.

(2) Let Q be any point on E^+, G a general (smooth) hyperplane section of A^+ through Q such that $A^+|_{E^+}$ is one point and F the pull back of G (we consider analytically locally near Q). Then F is normal. In fact, since the fiber $(E_q)_Q$ over Q of Φ is not contained in the singular locus of A_q, E_q is generically Cartier in A_q near $(E_q)_Q$, which in turn shows $(E_q)_Q$ is generically Cartier divisor on F. Since $(E_q)_Q$ is smooth, F is generically smooth along $(E_q)_Q$, i.e., F is normal. Furthermore we have $K_F.(E_q)_Q = -1$ and F is Gorenstein. So we know by Proposition 1.6 that F has only one A_{m-1} singularity for some integer m and $((E_q)_Q)^2_F = -\frac{1}{m}$. On the other hand, $((E_q)_Q)^2_F = (E_q.(E_q)_Q)_{A_q}$ and the value of the right side of this equality is independent of Q, so m is also independent of Q. Consequently, we find that A_q has the locally trivial cA_{m-1} curve singularity along M and outside M, A_q is smooth.

(3) By $((E_q)_Q)^2_F = -\frac{1}{m}$, we obtain the subadjunction formula $K_{E_q} + \frac{m-1}{m}M = K_{A_q} + E_q|_{E_q}$ and $K_{A_q} = \Phi^* K_{A^+} + mE_q$. Intersecting these with M, we can see (3). Remark that $E_q.M$ is negative since E_q is q-anti-ample. Hence $K_{A_+}.E^+$ is positive. □

By Claim 2.2 (1) and (3), $g^+ : A^+ \to B$ satisfies the assumption of Theorem 1.5 and hence is target stable. We can take Y as the total space of a 1-parameter deformation of B. Let $f^+ : X^+ \to Y$ be the deformation of g^+ associated to this deformation. Then this is the flip of f. (We can easily check the conditions of the definition of flip.) □

COROLLARY 2.3. *Consider the situation as in the main theorem. Let $f^+ \colon X^+ \to Y$ be the flip of f and E^+ the exceptional locus of f^+. Then*

(1) *A is singular only at the vertex v of E (if $n = 1$, the vertex means a point on E). Near v, $(v \in E \subset A \subset X)$ is analytically isomorphic to $(o \in (x = z = t = 0) \subset (xy + zw = t = 0) \subset (xy + zw + t^k = 0))$ in $\mathbb{C}^5/\mathbb{Z}_n(1, -1, 1, -1, 0)$;*

(2) *X^+ is smooth, E^+ is $\mathbb{P}^1$ and $\mathcal{N}_{E^+/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n)$ or $\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-n)$. Furthermore the former case occurs if and only if X has only $\frac{1}{n}(1, -1, 1, -1)$ singularity at v;*

(3) *there exists a Weil divisor H on X such that $-K_X \sim nH$. Let C be a general member of $|nH|$ such that C has only Gorenstein terminal singularities. By using C, define ring structures to*

$$\bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jH) \text{ and } \bigoplus_{j=0}^{n-1} \mathcal{O}_Y(-jf(H))$$

and set

$$\tilde{X} := \textbf{\textit{Specan}} \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jH) \text{ and } \tilde{Y} := \textbf{\textit{Specan}} \bigoplus_{j=0}^{n-1} \mathcal{O}_X(-jf(H)).$$

Let $\tilde{E} \subset \tilde{X}$ be the pull back of E by the natural morphism $\tilde{X} \to X$. Then the natural morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a flipping contraction which satisfies the same assumption as f, and $\tilde{E}$ is the exceptional locus of $\tilde{f}$ and is isomorphic to $\mathbb{P}^2$.

PROOF. First we will show $m = n$ in Claim 2.2. If we restrict the flip to a general member $C \in |-K_X|$ and its strict transform $C^+ \subset X^+$, we obtain the flop $C \dashrightarrow C^+$. (Remark that C^+ is normal because dim Sing $C^+ \leq 1$ and C^+ is a Cartier divisor on a smooth 4-fold. Hence by the uniqueness of the flop, $C \dashrightarrow C^+$ is the flop.) Since C is smooth, the analytic structure along the exceptional curves is unchanged by the flop (cf. [Ko2, Theorem 2.4]). So C^+ is also smooth. Let E' be the exceptional curve of $f|_C$ and

$C' \in |-K_X|_C|$. Then we have $C'.E' = n$. Let $C'^+ \in |-K_{X^+}|_{C^+}|$ be the strict transform of C'. We see that $C'^+.E^+ = -(2m-n)$ by Claim 2.2 (3). Since $C \dashrightarrow C^+$ is a terminal flop, we must have $2m - n = n$, i.e., $m = n$.

We will prove (3). Let H^+ be a Cartier divisor on X^+ such that $H^+.E^+ = -1$ and $H \subset X$ be the strict transform of H^+. Then $K_{X^+} + nH^+$ is linearly f^+-trivial by Claim 2.2 (3) since we consider locally analytically along E^+. Hence $K_X + nH$ is linearly f-trivial since the linear triviality is preserved by an anti-flip. Let $\tilde{X}$ and $\tilde{Y}$ be as in the statement of (3). We check that the natural morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ and the pull backs $\tilde{A}$ of A, $\tilde{E}$ of E satisfy the same assumption as f, A and E and that $\tilde{E}$ is $\mathbb{P}^2$. Let $\pi : \tilde{X} \to X$ be the covering morphism and $\tilde{C} := (\pi^*(C))_{\mathrm{red}}$. Note that $n\tilde{C} = \pi^*(C)$, $\tilde{C} \simeq C$ and $\tilde{C}$ is a Cartier divisor since C is contained in the branch locus. Then by the ramification formula $K_{\tilde{X}} = \pi^* K_X + (n-1)\tilde{C}$, $\tilde{C} \in |-K_{\tilde{X}}|$. Since $\tilde{C}$ is a Cartier divisor, we see that $\tilde{X}$ is Gorenstein. We also know that $\tilde{X}$ is terminal since codimension 1 ramification locus $\tilde{C}$ of π is smooth. The rest are clear except that $\tilde{E} \simeq \mathbb{P}^2$. The restriction of π to $\tilde{E}$ is $\pi|_{\tilde{E}} : \tilde{E} = \boldsymbol{Specan} \bigoplus_{j=0}^{n-1} \mathcal{O}_E(-jl) \to E$, where l is a ruling of E. (Note that $H|_E \sim l$.) So it coincides with the quotient $\mathbb{P}^2 \to \mathbb{F}_{n,0}$ by the action of $\mathbb{Z}_n$, $(X : Y : Z) \to (\eta X : \eta Y : Z)$, where X, Y and Z is the homogeneous coordinate of $\mathbb{P}^2$ and η is a primitive n-th root of unity. So $\tilde{E}$ is $\mathbb{P}^2$.

To prove (1), we apply the same method to these $\tilde{f}$, $\tilde{A}$ and $\tilde{E}$ that we used in Claim 2.2. (we express all things with $\sim$ in this case). Then we can show that $\tilde{A}_{\tilde{q}}$ and $\tilde{E}_{\tilde{q}}$ are smooth, $\tilde{E}_{\tilde{q}}.\tilde{M} = -1$. By considering the normal bundle sequence

$$0 \to \mathcal{N}_{\tilde{M}/\tilde{E}_{\tilde{q}}} \to \mathcal{N}_{\tilde{M}/\tilde{A}_{\tilde{q}}} \to \mathcal{N}_{\tilde{E}_{\tilde{q}}/\tilde{A}_{\tilde{q}}}|_{\tilde{M}} \to 0,$$

we see that $\mathcal{N}_{\tilde{M}/\tilde{A}_{\tilde{q}}} \simeq \mathcal{O}_{\tilde{M}}(-1) \oplus \mathcal{O}_{\tilde{M}}(-1)$. So $\tilde{M}$ is contracted to an ordinary double point by $\tilde{q}$. Denote this point by $\tilde{v} (\in \tilde{A})$. We note that $\tilde{X}$ is singular at worst only at $\tilde{v}$ since so is $\tilde{A}$ and $\tilde{X}^+$ is smooth. Then $\tilde{v}$ is the unique isolated ramification point of π and hence $\tilde{v} = \pi^{-1}(v)$. (Recall that v is the vertex of E.) So we can write locally analytically $(\tilde{v} \in \tilde{E} \subset \tilde{A} \subset \tilde{X}) \simeq (o \in (x = z = t = 0) \subset (xy + zw = t = 0) \subset (xy + zw + t^k = 0))$, where x, y, z, w are the semi-invariant coordinates and $xy + zw$ is semi-invariant with respect to the action of $\mathbb{Z}_n$. When we restrict the action to $\tilde{E}$, the action is $(y, w) \to (\eta y, \eta w)$, where η is a primitive n-th root of unity by the explicit description of $\pi|_{\tilde{E}}$. Hence the action is

$(x, y, z, w) \to (\eta^a x, \eta y, \eta^a z, \eta w)$, where a is an integer. By the necessary condition for the quotient to be canonical ([M3, Theorem 2]), a must be -1. This is also sufficient.

Next we will prove (2). To determine the normal bundle $\mathcal{N}_{E^+/X^+}$, we consider the normal bundle sequence

$$0 \to \mathcal{N}_{E^+/C^+} \to \mathcal{N}_{E^+/X^+} \to \mathcal{N}_{C^+/X^+}|_{E^+} \to 0. \tag{2.1}$$

Since $A|_C$ is smooth, we see that $\mathcal{N}_{E|_C/C} = \mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by using the normal bundle sequence

$$0 \to \mathcal{N}_{E|_C/A|_C} \to \mathcal{N}_{E|_C/C} \to \mathcal{N}_{A|_C/C}|_{E|_C} \to 0.$$

Since $C \dashrightarrow C^+$ is the flop, we have $\mathcal{N}_{E^+/C^+} \simeq \mathcal{N}_{E|_C/C}$. On the other hand, $\mathcal{N}_{C^+/X^+}|_{E^+} = \mathcal{O}(-n)$, so the sequence (2.1) is split. Hence we obtained the first part of (2).

To prove the second part of (2), we consider the covering described in the statement of (3). We use the notation there. Recall that $\tilde{C} \simeq C$ and $\tilde{C} \in |-K_{\tilde{X}}|$. By the argument above together with this, we see that

$$\mathcal{N}_{E^+/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n) \quad \text{if and only if } \mathcal{N}_{\tilde{E}|_{\tilde{C}}/\tilde{C}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1). \tag{2.2}$$

So 'if' part of (2) follows from Kawamata's determination of a flipping contraction from a smooth 4-fold (see Theorem 0.1) and (1). Finally we prove 'only if' part. Assume that $\mathcal{N}_{E^+/X^+} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-n)$. Then by (2.2), $\mathcal{N}_{\tilde{E}|_{\tilde{C}}/\tilde{C}} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then locally analytically there is a smooth surface $\tilde{S}$ such that $\tilde{S} \subset \tilde{C}$ and $\tilde{S}.(\tilde{E}|_{\tilde{C}}) = -1$ (note that $\tilde{S} \in |K_{\tilde{X}}|_{\tilde{C}}|$.) Let $\tilde{S}^+ \in |K_{\tilde{X}^+}|_{\tilde{C}^+}|$ be the strict transform of $\tilde{S}$ on $\tilde{C}^+$. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}^+}(2K_{\tilde{X}^+}) \to \mathcal{O}_{\tilde{X}^+}(K_{\tilde{X}^+}) \to \mathcal{O}_{\tilde{C}^+}(K_{\tilde{X}^+}) \to 0.$$

By the Kodaira-Kawamata-Viehweg vanishing theorem, $H^1(\tilde{X}^+, \mathcal{O}_{\tilde{X}^+}(2K_{\tilde{X}^+})) = 0$. So there is an element $\tilde{V}^+ \in |K_{\tilde{X}^+}|$ such that $\tilde{V}^+|_{\tilde{C}^+} = \tilde{S}^+$. Let $\tilde{V} \in |K_{\tilde{X}}|$ be the strict transform of $\tilde{V}^+$. Then $\tilde{V}|_{\tilde{C}} = \tilde{S}$. We claim that $\tilde{V}$ is smooth. This implies $\tilde{X}$ is also smooth, which completes the proof of the ‘only if’ part whence (1). First we note that $\tilde{V}$ is normal since $\tilde{V}|_{\tilde{C}}$ is smooth. Let x be any point of $\tilde{V}$ and $\tilde{C}_x$ a normal general member of $|-K_{\tilde{X}}|_{\tilde{V}}|$. Let $\tilde{E}_x := \tilde{E}|_{\tilde{C}_x}$. $\tilde{E}_x$ is the exceptional curve of $\tilde{f}|_{\tilde{C}_x}$ and $K_{\tilde{C}_x}.\tilde{E}_x = -1$. Hence by Proposition 1.6, $\tilde{C}_x$ has only one singular point on $\tilde{E}_x$ which is of type A_{m-1} for some m and $(\tilde{E}_x)^2_{\tilde{C}_x} = -\frac{1}{m}$. Since $(\tilde{E}_x)^2_{\tilde{C}_x} = (\tilde{E}^2.\tilde{C}_x)_{\tilde{V}}$, m is independent of x. Hence $(\tilde{E}_x)^2_{\tilde{C}_x} = (\tilde{E}|_{\tilde{S}})^2 = -1$ and m is 1, i.e., $\tilde{C}_x$ is smooth. Consequently $\tilde{V}$ is found to be smooth at any point x and we are done. Now we finished the proof of Corollary 2.3. □

Remark 2.4. Let $f : X \to (Y, P)$ be a flipping contraction from a Gorenstein terminal 4-fold. We use the notation of Theorem 1.3 and Proposition 1.4. By Proposition 1.5, a general member of $|-K_B|$ has only Du Val singularity at P. So $\bigoplus_{j=0}^{\infty} \mathcal{O}_X(-jK_B)$ and $\bigoplus_{j=0}^{\infty} \mathcal{O}_X(jK_B)$ are finitely generated. Set $A^+ :=$ ***Projan*** $\bigoplus_{j=0}^{\infty} \mathcal{O}_X(jK_B)$ and let $g^+ : A^+ \to B$ be the natural morphism. Then A^+ has only terminal singularities, K_{B^+} is g^+-ample and g^+ is a small morphism (see [KM, Theorem 3.1]). So if we can generalize Theorem 1.5, we may construct the flip of f by deforming g^+.

3. Some Examples

We construct examples of flipping contractions from Gorenstein terminal 4-folds.

Example 3.1 (Toric example). Let $\boldsymbol{e}_i$ be the vector $(0, \dots, \overset{\overset{i}{\vee}}{1}, \dots, 0)$ in $\mathbb{R}^4$ for $i = 1, 2, 3$, $\boldsymbol{e}_4 = (-1, -1, n-1, n)$ and $\boldsymbol{e}_5 = (0, 0, -1, -1)$. Let C_i be the cone $< \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \check{\boldsymbol{e}}_i, \dots, \boldsymbol{e}_5 >$ for $i \geq 0$ and C_0 the cone $< \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_5 >$. We denote the toric variety associated to the fan $*$ by $V(*)$. Set $X := V(C_3 \cup C_4 \cup C_5)$, $X^+ := V(C_1 \cup C_2)$ and $Y := V(C_0)$. Let $f : X \to Y$ and $f^+ : X^+ \to Y$ be the natural morphisms. Then it is easy to check that they define a flipping diagram. (See [Re].)

Example 3.2 (Y. Kachi, M. Gross). For the above example we can easily find A (as in the main theorem) with only isolated canonical singularity as determined in Corollary 2.3. We can consider that X is locally a 1-parameter family of A over the unit disk $\Delta(t)$. Take the cyclic coverings $\hat{X} \to X$, $\hat{Y} \to Y$ and $\hat{X^+} \to X^+$ associated to the cyclic covering

$\Delta(s) \to \Delta(t)$ defined by $t = s^m$. Then the natural morphisms $\hat{X} \to \hat{Y}$ and $\hat{X^+} \to \hat{Y}$ give a flipping diagram.

Example 3.3. For the example 3.1 with $n = 1$, we can find A whose singularity is the curve singularity of generically cA_1 type along a line of $\mathbb{P}^2$. For this A, we make the similar construction to Example 3.2. We obtain a flipping contraction from a Gorenstein terminal 4-fold which has a 1-dimensional singular locus. Furthermore if we take $q : A_q \to A$ as in the proof of the main theorem for this A, the first extremal contraction of A_q over B is a flipping contraction and after the flip, we can contract the strict transform of E to a Gorenstein terminal point.

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