Exact Solutions of Ultradiscrete Integrable Systems

（和訳：超離散可積分系の厳密解）

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Abstract

The initial value problem of periodic box-ball systems is solved.

Through a limiting procedure called ultradiscretisation, the box-ball system is obtained from a reduced discrete KP equation. Our main theme is *ultradiscretising the inverse scattering method for discrete KP.*

Two key theorems in this thesis are listed below:

a) The calculating method for the ultradiscrete limit of Abelian integrals over Riemannian surfaces is established. A graphical method concerning tropical geometry is the essential tool for the calculation.

b) The explicit expression for the general solution of the reduced KP equation is obtained. The method bases on the inverse sacattering method. It is important to say that our method does not depend on the Fay identity, or on any transcendental equations. In other words, our solutions are constructible.

The conbination of these results allows us to analyse the periodic box-ball systems. Our strategy to solve the initial value problem is summerised as follows: (i) Lift the initial value of the BBSs to the initial value of the KP equation. (ii) Solve the initial value problem of the KP equation (Key theorem b). The general solution is given as the combination of Riemann theta functions. (iii) Ultradiscretise the solution of KP (Key theorem a).

Contents

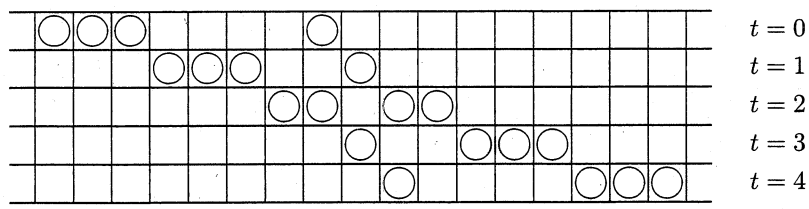
Chapter 1

Introduction

1.1 What is the Ultradiscrete Integrable System?

Our main theme in this thesis is the *Ultradiscrete Integrable System.* Naively speaking, the ultradiscrete integrable system is a discrete-time evolution equation involving addition and min-operator which possesses "integrability" in some sense.

Most important and simplest example of the ultradiscrete integrable system is the *box-ball system* (BBS) which have been discovered in 1990 [20] by D.Takahashi and J.Satsuma. The following is a typical example of the original version of BBS (Takahashi-Satsuma BBS):



Here, t is the time coordinate, a circle represents a ball and a square represents a box. One would find two "solitons" of lengths 3 and 1 in this picture. The BBS possesses many characteristic behaviour analogous to soliton equations such as the solitonical behaviour, the existence of a sufficient number of conserved quantities, *etc.*

In 1996, the most direct method to construct the BBS from soliton equations was established by T.Tokihiro, D.Takahashi, J.Matsukidaira and J.Satsuma [22]. They discovered the method of *ultradiscretisation*, and demonstrated that the Takahashi-Satsuma BBS is a ultradiscrete limit of the discrete KdV equation. This is the origin of the term "ultradiscrete integrable system".

Up to the present, many variants of BBS have been studied by various researchers. For example, they found the BBS with multi kind of balls from the hungry Lotka-Volterra equation[], the BBS with a career from the modified KdV equation[] *etc.* In general, the discrete KP equation and its reductions are ultradiscretisable and create various kinds of BBSs.

The merit of the ultradiscrete systems is that they can be easily calculated and shown graphically. This enable us to classify the discrete/ultradiscrete systems by more detailed information. Put another way, the ultradiscretisation provides us with a quantitative method for studying the integrable systems.

1.2 How to Obtain Ultradiscrete Systems?

The definition of the ultradiscretisation is simply expressed by the formula:

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log(\bullet). \tag{1.2.1}$$

Let $a, b \in \mathbb{R}$. The following is the most fundamental formula for studying the ultradiscrete systems:

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log(e^{-\frac{a}{\varepsilon}} + e^{-\frac{b}{\varepsilon}}) = \min[a,b], \quad -\lim_{\varepsilon\to 0^+} \varepsilon \log(e^{-\frac{a}{\varepsilon}} \cdot e^{-\frac{b}{\varepsilon}}) = a+b. \tag{1.2.2}$$

Simply stated, the ultradiscretisation transforms the addition to the min-operator, and the multiplication to the addition.

Let $\mathbb{T} = \mathbb{R} \cup \{+\infty\}$ be the tropical semifield, that is a semifield with the addition "$a \oplus b$" $:= \min[a,b]$ and the multiplication "$a \otimes b$" $:= a+b$. Suppose a polynomial $f = \sum_{w\in\Lambda} a_w(\varepsilon) \cdot x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$ such that $a_w(\varepsilon)$ is a positive function of $\varepsilon > 0$ and $\Lambda := \{w = (w_1, \ldots, w_n) \in \mathbb{Z}^n \,|\, a_w \not\equiv 0\} < \infty$. The *associated tropical polynomial* with f is a piecewise linear function $Tf(X_1, \ldots, X_n)$ defined by the formula

$$Tf := \min_{w\in\Lambda}[A_w + w_1X_1 + \cdots + w_nX_n], \quad A_w := -\lim_{\varepsilon\to 0^+} \varepsilon \log a(\varepsilon)$$

Our starting point is the following fact:

Theorem 0 *Let $\varepsilon > 0$ and $\{x_\lambda(\varepsilon)\}_{\lambda\in\mathbb{Z}}$ be a one-parameter family satisfying the following recursive formula*

$$x_\mu = \mathfrak{X}_\mu(\{x_\lambda\}_{\lambda<\mu}),$$

where $\mathfrak{X}_\mu$ is a totally-positive polynomial in x_λ ($\lambda \in \mathbb{Z}$), that is a polynomial without subtraction. Suppose that there exists the limits $X_\mu := -\lim_{\varepsilon\to 0^+} \varepsilon \log x_\mu$. Then, the sequence $\{X_\lambda\}_{\lambda\in\mathbb{Z}}$ satisfies

$$X_\mu = T\mathfrak{X}_\mu(\{X_\lambda\}_{\lambda<\mu}).$$

This theorem describes how an ultradiscrete system is derived from a discrete system. We show an example. Let us consider the Takahashi-Satsuma BBS, which is expressed by the recursive formula:

$$u_n^{t+1} = \min\left[1 - u_n^t,\ \sum_{k=-\infty}^{n-1} u_k^t - \sum_{k=-\infty}^{n-1} u_k^{t+1}\right], \tag{1.2.3}$$

$$u_n^t = \begin{cases} 1 & : \text{there exists a ball in } n\text{-th box from the left at time } t \\ 0 & : n\text{-th box is empty at time } t \end{cases}.$$

Changing variables, we rewrite (1.2.3) as

$$X_{n+1}^{t+1} + X_n^{t-1} = \min[X_{n+1}^{t-1} + X_n^{t+1} + 1,\ X_n^t + X_n^{t+1}],$$
$$X_n^t := -\sum_{k=n}^{\infty}\sum_{s=-\infty}^{t} u_k^s.$$

This tropical recursive equation is the ultradiscretisation of the discrete KdV equation:

$$(1+\delta)\cdot x_{n+1}^{t+1} x_n^{t-1} = \delta \cdot x_{n+1}^{t-1} x_n^{t+1} + x_n^t x_n^{t+1}, \quad \delta = e^{-\frac{1}{\varepsilon}}.$$

As is well known, discrete KdV equation is one of the reduced discrete KP equations. We will find that various kind of BBSs are created from various reduced discrete KP equations.

Our aim in this thesis is to solve the initial value problem of various BBSs. The strategy is summarised as follows: i) solve the initial value problem of reduced discrete KP equations, ii) ultradiscretise the solution.

1.3 Contents of the thesis

In this thesis, we deal with reduced KP equations with periodic boundary condition (Chapter 3). For studying discrete periodic systems, the knowledge of *Abelian integrals over Rieman surfaces* is necessary. In analogy with this, when studing ultradiscrete systems, the knowledge of "ultradiscrete version" of the Abelian integral is necessary.

In Chapter 2, we discuss the ultradiscrete limit of algebraic curves and Abelian integrals. Main thema in this section is the *tropical geometry.* We prove that the ultradiscrete limit of an Abelian integral can be calculated from a tropical curve.

In Chapter 3, we give the general solutions of reduced KP equations with periodic boundery condition.

In Chapter 4, we give the general solutions of initial value preblem of various ultradiscrete integrable systems. One would find that the results of Chapters 2 and 3 are applied for analysis of ultradiscrete integrable systems.

Chapter 2

Integrations over a Tropical Curve

2.1 Introduction

A *Tropical curve* is a kind of algebraic curve defined over the tropical semifield $\mathbf{T} = \mathbb{R} \cup \{\infty\}$ equipped with the min-plus operation:

$$\text{“}x + y\text{”} = \min\{x, y\}, \quad \text{“}xy\text{”} = x + y.$$

The geometry over tropical curves is introduced by several authors [13, 16]. Among these works, the integral theory over tropical curves was introduced by Mikhalkin and Zharkov in [14]. According to their works, a holomorphic differential on a tropical curve is defined as a global section of the real cotangent sheaf (Definition 4.1 [14]). Using the concept of tropical differentials, they derive the definition of tropical holomorphic integral.

As one of the applications of tropical geometry, a number of authors attempt to solve problems concerning integrable systems or dynamical systems by using the method of tropical geometry [2, 11].

The bridge between integrable systems and tropical geometry is the method of *ultradiscretization*. Ultradiscretization is a kind of limiting procedure, which is usually described as $-\lim_{\varepsilon\to 0^+} \varepsilon \log \cdot$. The two equations

$$-\lim_{\varepsilon\to 0^+} \varepsilon \log (e^{-a/\varepsilon} \cdot e^{-b/\varepsilon}) = a + b, \quad -\lim_{\varepsilon\to 0^+} \varepsilon \log (e^{-a/\varepsilon} + e^{-b/\varepsilon}) = \min\,[a, b], \qquad (2.1.1)$$

$(a, b \in \mathbb{R})$ are fundamental formulae. Through ultradiscretization, we translate the objects over $\mathbb{C}$ into the min-plus algebra.

In this paper we study the tropicalization (or ultradiscretization) of holomorphic integrals over complex plane curves. Loosely speaking, our problem is: “Why is it that tropical integrals can tell us something about the behaviour of complex integrals?”

Many researchers have been studied on the relationship between analytic curves and tropical curves. Katz, E., Markwig, H. and Markwig, T. studied the j-invariant of cubic curves and its tropicalization in [8]. For genus zero and one case, Speyer, D. proved the existence of analytic curve of which tropicalization coincides with given tropical curve in any ambient space [19]. Helm, D. and Kats, E. discussed the relationship between the tropical curve and the monodromy action on the Hodge structure [10].

In order to make use of the established results for hypersurfaces (for example Viro's approximation theorem [3, 23]), we restrict ourselves to the tropical integral calculus over plane curves instead of considering more general tropical curves which have been studied by many researchers. The main theorem (theorem 2.4.3) gives us the exact relation between them. (Figure 2.1).

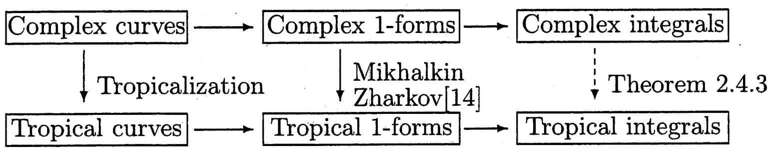


Figure 2.1: Theorem 2.4.3 shows us the direct relation between two kinds of "integrals".

Note : Throughout this paper, ε is a small real number $0 < \varepsilon < 1$. The symbol $\mathbf{e}$ denotes the real number $e^{-1/\varepsilon}$. A *formal Puiseux series with respect to* $\mathbf{e}$ is a formal sum of the form $\sum_{i=-n}^{\infty} a_i \mathbf{e}^{i/d}$, where a_i is a complex number and d is a positive integer. If a formal Puiseux series converges for ε sufficiently small, it is called *convergent Puiseux series.* K denotes the field of convergent Puiseux series. The field K has the standard non-archimedean valuation val : $K \to \mathbb{Q} \cup \{+\infty\}$. (val $(0) = +\infty$). In this paper, 'Puiseux series' means only convergence series unless otherwise is stated.

2.2 Approximation of hypersurfaces

2.2.1 PL-polynomials and tropical hypersurfaces

Our purpose of this section is to give a brief review of the method for the *approximation of hypersurfaces* of algebraic tori according to [23, §6].

Let $\mathbb{C}$ be the complex number field and $\mathbb{C}^n$ be the complex n-space. Throughout this paper, U denotes the unit circle $\{x \in \mathbb{C} \mid |x| = 1\}$, and $\mathbb{CR}^n$ denotes the algebraic torus $\{(x_1, x_2, \ldots, x_n) \mid x_1 x_2 \cdots x_n \neq 0\}$.

For a small positive parameter $0 < \varepsilon < 1$, define the maps $l(\varepsilon) : \mathbb{CR}^n \to \mathbb{R}^n$ and $a : \mathbb{CR}^n \to U^n$ $(:= U \times \cdots \times U)$ by the formulae

$$l(\varepsilon)(x_1, \ldots, x_n) = (-\varepsilon \log|x_1|, \ldots, -\varepsilon \log|x_n|), \quad a(x_1, \ldots, x_n) = \left(\frac{x_1}{|x_1|}, \ldots, \frac{x_n}{|x_n|}\right).$$

It is clear that the map $la(\varepsilon) : \mathbb{CR}^n \to \mathbb{R}^n \times U^n$ defined by $x \mapsto (l(\varepsilon)(x), a(x))$ is a diffeomorphism for any ε.

For $w \in \mathbb{R}$ and $\varepsilon > 0$, denote by $\mathcal{Q}_{w,\varepsilon}$ the transformation $\mathbb{CR}^n \to \mathbb{CR}^n$ defined by

$$\mathcal{Q}_{w,\varepsilon}(x_1, \ldots, x_n) = (e^{-w_1/\varepsilon} x_1, \ldots, e^{-w_n/\varepsilon} x_n), \qquad \text{where } w = (w_1, \ldots, w_n).$$

We abbreviate the symbol $\mathcal{Q}_{w,\varepsilon}$ as $\mathcal{Q}_w$ if there is no confusion.

Let $T_w : \mathbb{R}^n \to \mathbb{R}^n$ be the translation $x \mapsto x + w$. By definition, we can derive the relation

$$la(\varepsilon) \circ \mathcal{Q}_w \circ la(\varepsilon)^{-1} = T_w \times \mathrm{id}_{U^n}. \tag{2.2.1}$$

Our main object is an algebraic hypersurface of $\mathbb{CR}^n$ defined by a Laurent polynomial, which is an element of the ring $\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. Denote by $V_{\mathbb{CR}^n}(f)$ the algebraic set in $\mathbb{CR}^n$ defined by the Laurent polynomial $f(x_1, \dots, x_n)$, and let $V_W(f) := V_{\mathbb{CR}^n}(f) \cap W$ for a subset $W \subset \mathbb{CR}^n$.

For $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ and ordered n variables $x = (x_1, \dots, x_n)$, we abbreviate the monomial $x_1^{w_1} \cdots x_n^{w_n}$ as x^w. Let $\{V_{\mathbb{CR}^n}(f_\varepsilon)\}_\varepsilon$ be an one-parameter family of algebraic hypersurfaces, where ε is a positive real parameter and $f_\varepsilon = f_\varepsilon(x_1, \dots, x_n)$ is a Laurent polynomial of which coefficients depend on ε. Especially, in this paper, we mainly consider the polynomials f_ε of the form:

$$f_\varepsilon(x) = \sum_{w \in \mathbb{Z}^n} a_w(\varepsilon)\, x^w, \qquad x = (x_1, \dots, x_n), \qquad a_w(\varepsilon) \in K.$$

We call this type of polynomial *parameterised L-polynomial*, or *pL-polynomial.*

Define a *tropical polynomial* $\mathrm{Val}\,(X; f_\varepsilon)$ *associated with* f_ε by the formula

$$\mathrm{Val}\,(X; f_\varepsilon) := \min_{w \in \mathbb{Z}^n} [\mathrm{val}\,(a_w) + w_1 X_1 + \dots + w_n X_n], \qquad X = (X_1, \dots, X_n).$$

A *tropical hypersurface defined by* f_ε is a subset of $\mathbb{R}^n$ defined by

$$\left\{ P = (A_1, \dots, A_n) \in \mathbb{R}^n \;\middle|\; \text{the function } \mathrm{Val}\,(X; f_\varepsilon) \text{ is not smooth at } X = P \right\}. \tag{2.2.2}$$

(For explicit examples for $n = 2$, see section 2.3.1). We denote this tropical hypersurface by $TV_{\mathbb{R}^n}(f_\varepsilon)$.

Let $\mathfrak{a} = \{(X, \mathrm{Val}\,(X; f_\varepsilon)) \mid X \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of $\mathrm{Val}\,(X; f_\varepsilon)$. Clearly, $\mathfrak{a}$ is a skeleton of an $(n+1)$-dimensional convex (unbounded) polytope. The tropical hypersurface $TV_{\mathbb{R}^n}(f_\varepsilon)$ is the image of collection of $(n-1)$-faces in $\mathfrak{a}$ by the natural projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

2.2.2 Canonical expressions of pL-polynomials

We define $||f|| := \max_w |a_w|$ for a Laurent polynomial $f(x_1, \dots, x_n) = \sum_w a_w x^w$.

Let $P = (A_1, \dots, A_n)$ be a point in $\mathbb{R}^n$. There exist finitely many integer vectors $w^{(1)}, \dots, w^{(\alpha)} \in \mathbb{Z}^n$ such that the equations $\mathrm{Val}\,(P; f_\varepsilon) = \mathrm{val}\,(a_{w^{(i)}}) + w_1^{(i)} A_1 + \dots + w_n^{(i)} A_n$, $(i = 1, 2, \dots, \alpha)$ hold. Let $\Theta(P)$ be a set of these vectors. Define the polynomial $\widetilde{f}_\varepsilon^P$ by the formula $\widetilde{f}_\varepsilon^P = \sum_{w \in \Theta(P)} a_w(\varepsilon)\, x^w$.

Remark 2.2.1 *Because* $\Theta(P) = \{w\}$ *implies the formula* $\mathrm{Val}\,(X; f_\varepsilon) = \mathrm{val}\,(a_w) + w_1 X_1 + \dots + w_n X_n$ *around* $X = P$, *the number of elements of* $\Theta(P)$ *is always greater than one for* $P \in TV_{\mathbb{R}^n}(f_\varepsilon)$.

We note that the following easy two lemmas:

Lemma 2.2.1 *Let* $O = (0, \dots, 0)$ *be the origin of* $\mathbb{R}^n$. *Then,*

$$\mathrm{Val}\,(P; f_\varepsilon) = \mathrm{Val}\,(O; f_\varepsilon \circ \mathcal{Q}_P), \qquad \textit{where} \qquad f_\varepsilon \circ \mathcal{Q}_P(x) = f_\varepsilon(\mathcal{Q}_P(x)).$$

Proof. It is clear by the definition of $\mathrm{Val}\,(P; f_\varepsilon)$.

Lemma 2.2.2 $(\widetilde{f_\varepsilon \circ \mathcal{Q}_P})^O = \widetilde{f}_\varepsilon^P \circ \mathcal{Q}_P$.

Proof. Let $f_\varepsilon = \sum_w a_w(\varepsilon)x^w$ and $P = (A_1, \dots, A_n)$. Then,

$$(\widetilde{f_\varepsilon \circ \mathcal{Q}_P})^O = \sum_\sharp a_w e^{A_1 w_1 + \cdots + A_n w_n} x^w = \sum_{w \in \Theta(P)} a_w e^{A_1 w_1 + \cdots + A_n w_n} x^w = \widetilde{f}_\varepsilon^P \circ \mathcal{Q}_P,$$

where $\sharp$ means '$\{w \mid \mathrm{val}\,(a_w) + A_1 w_1 + \cdots + A_n w_n = \mathrm{Val}\,(O; f_\varepsilon \circ \mathcal{Q}_P)\}$ '.

Next we consider the decomposition:

$$f_\varepsilon = \left(\sum_{w \in \Theta(O)} + \sum_{w \notin \Theta(O)}\right)(a_w(\varepsilon)\, x^w) = \widetilde{f}_\varepsilon^O + \sum_{w \notin \Theta(O)} a_w(\varepsilon)\, x^w.$$

By definition, the elements of the set $\Theta(O)$ satisfy the following relation: $w \in \Theta(O), v \notin \Theta(O) \Rightarrow \mathrm{Val}\,(O; f_\varepsilon) = \mathrm{val}\,(a_w) < \mathrm{val}\,(a_v)$. Therefore the pL-polynomial $e^{-\mathrm{Val}\,(O;f_\varepsilon)} f_\varepsilon$ can be decomposed as $e^{-\mathrm{Val}\,(O;f_\varepsilon)} f_\varepsilon = f_1 + f_2$, where $f_1 = e^{-\mathrm{Val}\,(O;f_\varepsilon)} \widetilde{f}_\varepsilon^O = \sum_w b(\varepsilon)\, x^w$ s.t. $\mathrm{val}\,(b(\varepsilon)) = 0$, and $f_2 = \sum_w b'(\varepsilon)\, x^w$ s.t. $\mathrm{val}\,(b'(\varepsilon)) > 0$. Seeing the facts that i) $\mathrm{val}\,(a(\varepsilon)) = 0 \Leftrightarrow \lim_{\varepsilon \to 0^+} a(\varepsilon) \in \mathbb{C} \setminus \{0\}$, ii) $\mathrm{val}\,(a(\varepsilon)) > 0 \Leftrightarrow \lim_{\varepsilon \to 0^+} a(\varepsilon) = 0$ for $a(\varepsilon) \in K$, we can decompose uniquely the pL-polynomial $e^{-\mathrm{Val}\,(O;f_\varepsilon)} f_\varepsilon$ as:

$$e^{-\mathrm{Val}\,(O;f_\varepsilon)} f_\varepsilon = f^O + \Delta(f_\varepsilon), \tag{2.2.3}$$

where f^O is the Laurent polynomial $\lim_{\varepsilon \to 0^+} e^{-\mathrm{Val}\,(O;f_\varepsilon)} \widetilde{f}_\varepsilon^O$ and $||\Delta(f_\varepsilon)||$ becomes to zero when $\varepsilon \to 0^+$. Of course, f^O does not depend on ε.

More general formula we can derive soon:

Proposition 2.2.3 *Let f_ε be a pL-polynomial and P be a point in $\mathbb{R}^n$. Then the pL-polynomial $e^{-\mathrm{Val}\,(P;f_\varepsilon)} f_\varepsilon \circ \mathcal{Q}_P$ is uniquely decomposed as:*

$$e^{-\mathrm{Val}\,(P;f_\varepsilon)} f_\varepsilon \circ \mathcal{Q}_P = f^P + \Delta(f_\varepsilon \circ \mathcal{Q}_P), \tag{2.2.4}$$

where f^P is the Laurent polynomial $\lim_{\varepsilon \to 0^+} e^{-\mathrm{Val}\,(P;f_\varepsilon)} \widetilde{f}_\varepsilon^P \circ \mathcal{Q}_P$ and $||\Delta(f_\varepsilon \circ \mathcal{Q}_P)||$ becomes zero when $\varepsilon \to 0^+$.

Proof. To prove this, it is sufficient to substitute $f_\varepsilon \mapsto f_\varepsilon \circ \mathcal{Q}_P$ to (2.2.3) and to use lemma 2.2.1 and 2.2.2.

Hearafter we denote $\mathcal{R}^P(f_\varepsilon) := e^{-\mathrm{Val}\,(P;f_\varepsilon)} f_\varepsilon \circ \mathcal{Q}_P$ and $\Delta^P := \Delta(f_\varepsilon \circ \mathcal{Q}_P)$ simply. Then equation (2.2.4) is expressed as $\mathcal{R}^P(f_\varepsilon) = f^P + \Delta^P$. We call this expression *canonical expression of f_ε at P*. The Laurent polynomial f^P is considered as a 'main part' of $\mathcal{R}^P(f_\varepsilon)$. We call it *$P$-truncation of f_ε*. Note that $\mathcal{R}^P(f_\varepsilon)$ is continuous with respect to ε and P, but f^P and Δ^P are continuous with respect to ε only.

2.2.3 Approximation theorem (local version)

Let M be a smooth submanifold of a smooth manifold X. A *tubular neighbourhood* of M in X is a submanifold $N \subset X$ such that (i) $M \subset \mathrm{Int}N$, (ii) there exists a smooth retraction $p : N \to M$ such that $p^{-1}(x)$ is diffeomorphic to the $(\dim X - \dim M)$-dimensional ball for any $x \in M$. If X is equipped with a metric, a tubular neighbourhood N of M is called *tubular μ-neighbourhood* when any fibre $p^{-1}(x)$ is contained in a ball of radius μ centred in x.

The space $\mathbb{R}^n \times U^n$ is equipped with a flat metric defined by Euclidean metric of $\mathbb{R}^n$ and the standard flat metric of torus U.

The main role of tubular neighbourhoods is to formalise the approximation of hypersurfaces of $\mathbb{R}^n \times U^n$. For later arguments, it is convenient to consider some special class of tubular neighbourhoods. The tubular neighbourhood $p : N \to M$ is called *normal* if (i) any fibre $p^{-1}(x)$ consists of segments of geodesics, (ii) any fibre $p^{-1}(x)$ intersects with M orthogonally at x. Note that two normal tubular neighbourhoods $p : N \to M$ and $p' : N' \to M$ coincides with each other on $N \cap N'$.

Denote the subset $\{x = (x_1, \ldots, x_n) \in \mathbb{CR}^n \mid 1/r < |x_i| < r, \ \forall i\}$ by $D(r)$ for a positive number $r > 1$. Let Λ be a finite subset of $\mathbb{Z}^n$, and $f = \sum_{w \in \Lambda} \alpha_w x^w$ and $g = \sum_{w \in \Lambda} \beta_w x^w$ be Laurent polynomials. Define $F := f + g$. We consider the behaviour of two algebraic sets $V_{\mathbb{CR}^n}(F)$ and $V_{\mathbb{CR}^n}(f)$ when $||g||$ goes to zero without changing f. Assume that $V_{\mathbb{CR}^n}(f)$ is a smooth hypersurface of $\mathbb{CR}^n$. Standard arguments based on Implicit Function Theorem give us the following lemma:

Lemma 2.2.4 *Fix $r > 1$ arbitrarily. Then, for arbitrary $\mu_0 > 0$, there exists a positive number δ_0 such that:*

$||g|| < \delta_0 \ \Rightarrow$
$V_{D(r)}(f)$ *is a smooth section of a normal tubular μ_0-neighbourhood $N \to V_{D(r)}(F)$.*

We call f_ε *non-singular* if there exists a positive number $\delta > 0$ such that '$\varepsilon \in (0, \delta] \Rightarrow V_{\mathbb{CR}^n}(f_\varepsilon)$ is non-singular', and we call f_ε *totally non-singular* if, for all $P \in TV_{\mathbb{R}^n}(f_\varepsilon)$, f^P is non-singular (as usual meaning).

Remark 2.2.2 *Totally non-singularity does not imply non-singularity.*

Recall that $la(\varepsilon) : \mathbb{CR}^n \to \mathbb{R}^n \times U^n$ is a diffeomorphism between two topological spaces. The following theorem is an essential part of the method of approximation.

Theorem 2.2.5 (The approximation theorem at the origin)
Assume that f_ε is a non-singular and totally non-singular pL-polynomial, and that the origin $O = (0, 0, \ldots, 0)$ of $\mathbb{R}^n$ is contained in $TV_{\mathbb{R}^n}(f_\varepsilon)$. Let $\mathcal{R}^O f_\varepsilon = f^O + \Delta^O$ be the canonical expression of f_ε at O. Then for arbitrary $\mu > 0$, there exists a positive number $\delta > 0$ such that

$$||\Delta^O|| < \delta \Rightarrow \left(\begin{array}{l} \textit{There exists an open neighbourhood } W_\varepsilon \textit{ of } O \in \mathbb{R}^n \textit{ such that} \\ (W_\varepsilon \times U^n) \cap la(\varepsilon)(V_{\mathbb{CR}^n}(f^O)) \textit{ is a smooth section of} \\ \textit{a tubular } \mu\textit{-neighbourhood } N \to (W_\varepsilon \times U^n) \cap la(\varepsilon)(V_{\mathbb{CR}^n}(f_\varepsilon)). \end{array} \right).$$

Moreover, we can assume that the preimage of this tubular neighbourhood by $la(\varepsilon)$ is normal.

Proof. Fix a positive number $r > 1$ arbitrarily and let $\mu_0 := (2\sqrt{n}r)^{-1} \cdot \mu$. To begin with, note that we can derive soon the following relation by direct calculations:

$$\mathrm{dist}_{\mathbb{CR}^n}(\alpha, \beta) < \mu_0, \quad \alpha, \beta \in D(r) \quad \Rightarrow \quad \mathrm{dist}_{\mathbb{R}^n \times U^n}(la(\varepsilon)(\alpha), la(\varepsilon)(\beta)) < \mu.$$

By lemma 2.2.4, there exists a small number $\delta > 0$ such that '$||\Delta^O|| < \delta \Rightarrow V_{D(r)}(f^O)$ is a smooth section of a normal tubular μ_0-neighbourhood $\mathcal{N} \to V_{D(r)}(f_\varepsilon)$ $(= V_{D(r)}(\mathcal{R}^O f_\varepsilon))$'.

Therefore, it is sufficient to define $W_\varepsilon := l(\varepsilon)(D(r))$ and $N := la(\varepsilon)(\mathcal{N})$. Clearly, we have $W_\varepsilon \ni O$. The slight extension of theorem 2.2.5 can be shown:

Corollary 2.2.6 (The approximation theorem (local version)) *Assume f_ε is a pL-polynomial satisfying the same condition stated in theorem 2.2.5. Let P be a point in $TV_{\mathbb{R}^n}(f_\varepsilon)$ and $\mathcal{R}^P f_\varepsilon = f^P + \Delta^P$ be the canonical expression of f_ε at P. Then for arbitrary $\mu > 0$, there exists a positive number $\delta > 0$ such that*

$$||\Delta^P|| < \delta \Rightarrow \left(\begin{array}{l} \textit{There exists an open neighbourhood } W_\varepsilon \textit{ of } P \in \mathbb{R}^n \textit{ such that} \\ (W_\varepsilon \times U^n) \cap (T_P \times \mathrm{id}_{U^n}) \circ la(\varepsilon)(V_{\mathbb{CR}^n}(f^P)) \textit{ is a smooth section of} \\ \textit{a tubular } \mu\textit{-neighbourhood } N \to (W_\varepsilon \times U^n) \cap la(\varepsilon)(V_{\mathbb{CR}^n}(f_\varepsilon)). \end{array} \right).$$

Moreover, if $f^P = f^Q$ $(P, Q \in TV_{\mathbb{R}^n}(f_\varepsilon))$, we can take same δ for these two points.

Proof. Let $\widetilde{T} := T_P \times id_{U^n}$. To prove the first statement, it is sufficient to to apply theorem 2.2.5 to the canonical expression $\mathcal{R}^P(f_\varepsilon) = f^P + \Delta^P$ and to use the following equation:

$$\widetilde{T} \circ la(\varepsilon)(V(\mathcal{R}^P f_\varepsilon)) = \widetilde{T} \circ la(\varepsilon)(V(f_\varepsilon \circ \mathcal{Q}_P)) = \widetilde{T} \circ la(\varepsilon) \circ \mathcal{Q}_P^{-1}(V(f_\varepsilon)) = la(\varepsilon)(V(f_\varepsilon)).$$

The second statement is follows from the fact that $\widetilde{T}$ does not change the metric.

2.2.4 Approximation theorem (global version)

Theorem 2.2.5 and corollary 2.2.6 show us the existence of a small region in which two varieties $V_{\mathbb{C}\mathbb{R}^n}(f_\varepsilon)$ and $V_{\mathbb{C}\mathbb{R}^n}(f^P)$ are 'similar'. We extend this region by *gluing* tubular neighbourhoods.

As mentioned above, the tropical hypersurface $TV_{\mathbb{R}^n}(f_\varepsilon)$ is an union of finitely many (not necessarily bounded) $(n-1)$-faces. This tropical hypersurface is an image of $(n-1)$-faces of a convex $(n+1)$-polytope by the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Therefore, naturally $TV_{\mathbb{R}^n}(f_\varepsilon)$ has a cell decomposition. We express this decomposition as $TV_{\mathbb{R}^n}(f_\varepsilon) = \bigcup_{\lambda \in \Lambda} \mathcal{X}_\lambda$ formally. The index set Λ is finite.

Lemma 2.2.7 *For $P_1, P_2 \in \mathcal{X}_\lambda$, the truncations f^{P_1} and f^{P_2} coincide each other.*

Proof. Let $\mathfrak{a} \in \mathbb{R}^n \times \mathbb{R}$ be the graph of $\mathrm{Val}\,(X; f_\varepsilon)$ and $p : \mathfrak{a} \xrightarrow{\sim} \mathbb{R}^n$ be a restriction of the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ to $\mathfrak{a}$.

Assume $f^{P_1} \neq f^{P_2}$. This means $\Theta(P_1) \neq \Theta(P_2)$. We can assume $w = (w_1, \ldots, w_n) \in \Theta(P_1)$ and $w \notin \Theta(P_2)$. It follows that $p^{-1}(P_1)$ is contained in the face $\mathfrak{a} \cap \{\mathrm{Val}\,(X; f_\varepsilon) = \mathrm{val}\,(a_w) + w_1 X_1 + \cdots + w_n X_n\}$, and $p^{-1}(P_2)$ is not. Then P_1 and P_2 is not contained in a same cell.

We say that a smooth submanifold $V_1 \subset X$ is *μ-approximated by* another smooth submanifold $V_2 \subset X$ *around* a point $P \in X$ if there exists an open neighbourhood $W \ni P$ such that $W \cap V_2$ is a section of a tubular μ-neighbourhood $N \to W \cap V_1$. By corollary 2.2.6 and lemma 2.2.7, there exists positive δ (that does not depend on P because Λ is finite!) such that

$$||\Delta^P|| < \delta \quad \Rightarrow \quad la(\varepsilon)(V(f_\varepsilon)) \text{ is } \mu\text{-approximated by } (T_P \times \mathrm{id}) \circ la(\varepsilon)(V(f^P)) \text{ around } P.$$

Lemma 2.2.8 *There exist a positive number ζ such that $\varepsilon \in (0, \zeta] \Rightarrow ||\Delta^P|| < \delta$, $\forall P \in TV_{\mathbb{R}^n}(f_\varepsilon)$.*

Proof. Fix a small $0 < \varepsilon < 1$. (Then $e = e^{-1/\varepsilon} < 1$). It is sufficient to prove that $\{||\Delta^P|| \mid P \in \mathcal{X}_\lambda\}$ has an upper bound for each $\mathcal{X}_\lambda$. (Note that Λ is finite). Clearly, $||\Delta^P||$ is continuous with respect to $P \in \mathcal{X}_\lambda$.

(i) When the closure of $\mathcal{X}_\lambda$ is compact, it is sufficient to confirm the fact that

$$\lim_{P \to \partial \mathcal{X}_\lambda} ||\Delta^P|| \text{ is finite.}$$

(ii) When $\mathcal{X}_\lambda$ is unbounded, we should consider the behaviour of $||\Delta^P||$ when $|P| \to \infty$. The pL-polynomial Δ^P is of the form $\sum'_w e^{\mathrm{val}\,(a_w) + w_1 P_1 + \cdots + w_n P_n - \mathrm{Val}\,(P; f_\varepsilon)} x^w$, where w runs over all the element of $\mathbb{Z}^n$ such that $\mathrm{val}\,(a_w) + w_1 P_1 + \cdots + w_n P_n - \mathrm{Val}\,(P; f_\varepsilon) > 0$. Therefore, when P goes infinity along $\mathcal{X}_\lambda$, the function $\mathrm{val}\,(a_w) + w_1 P_1 + \cdots + w_n P_n - \mathrm{Val}\,(P; f_\varepsilon)$ should grow to infinity or be constant. Then the limit $\lim_{|P| \to \infty, P \in \mathcal{X}_\lambda} ||\Delta^P||$ should be finite.

Recall that two normal tubular neighbourhoods (or their images by $la(\varepsilon)$) coincides with each other on their intersection. By gluing these local tubular neighbourhoods, we obtain a global tubular neighbourhood which gives us the approximation theorem:

Theorem 2.2.9 (approximation theorem (global version)) *Let f_ε be a non-singular and totally non-singular pL-polynomial. Denote the canonical expression of f_ε at P by $\mathcal{R}^P f_\varepsilon = f^P + \Delta^P$. Then for arbitrary $\mu > 0$, there exists a positive number $\zeta > 0$ such that*

$$\varepsilon \in (0, \zeta] \Rightarrow \left(\begin{array}{l} \text{There exists a tubular } \mu\text{-neighbourhood } N \to la(\varepsilon)(V_{\mathbb{CR}^n}(f_\varepsilon)) \text{ such that} \\ (T_P \times \mathrm{id}_{U^n}) \circ la(\varepsilon)(V_{\mathbb{CR}^n}(f^P)) \text{ is a smooth section of it around } P. \end{array} \right).$$

2.2.5 surjectivity theorem

In this section, we introduce the *surjectivity theorem* without proof. For details, it is advisable to consult the following references: Einsiedler, M. Kapranov, M. and Lind, D. [9]; Payne, S. [18].

Let $f_\varepsilon(x)$ be a pL-polynomial in n valuables $x_1, \ldots, x_n$ and $\mathcal{R}^P f_\varepsilon = f^P + \Delta^P$ be its canonical expression at a point $P = (A_1, \ldots, A_n)$ in $TV_{\mathbb{R}^n}(f_\varepsilon)$. Then we have the following theorem.

Theorem 2.2.10 (surjectivity theorem) *Let $p = (p_1, \ldots, p_n)$ be a point in $V_{\mathbb{CR}^n}(f^P)$. Then there exist n Puiseux series $\widetilde{p}_1 = p_1 e^{A_1} + p'_1 e^{A'_1} + p''_1 e^{A''_1} + \cdots, \ \cdots, \ \widetilde{p}_n = p_n e^{A_n} + p'_n e^{A'_n} + p''_n e^{A''_n} + \cdots$ such that $(\widetilde{p}_1, \ldots, \widetilde{p}_n) \in V_{\mathbb{CR}^n}(f_\varepsilon)$.*

This theorem displays an information of pointwise convergence of $V_{\mathbb{CR}^n}(f_\varepsilon)$. In a word, the approximation theorem 2.2.9 deals with global information of $V_{\mathbb{CR}^n}(f_\varepsilon)$ and the surjectivity theorem 2.2.10 deals with local information.

2.3 Plane Curves over K and Tropical Curves

In the rest of this paper, we consider two-dimensional case. In this case the varieties $V_{\mathbb{CR}^2}(f_\varepsilon)$ and $V_{\mathbb{CR}^2}(f^P)$ are complex curves (or Riemannian surfaces) contained in the algebraic torus $\mathbb{CR}^2$. We assume $V_{\mathbb{CR}^2}(f_\varepsilon)$ is non-singular and totally non-singular unless otherwise is stated. Now we denote $V_{\mathbb{CR}^2}(f_\varepsilon)$, $V_{\mathbb{CR}^2}(f^P)$, $TV_{\mathbb{R}^2}(f_\varepsilon)$, ... *etc.* by $V(f_\varepsilon)$, $V(f^P)$, $TV(f_\varepsilon)$, ... *etc.* simply.

Recall that K is the Puiseux series field with the standard non-archimedean valuation val. Define the multiplicative group $R^\times := \{x \in K \mid \mathrm{val}\,(x) = 0\}$. Any element x of $R^\times$ goes to a finite non-zero complex number when ε goes to zero. Denote the limit by $\mathrm{top}\,(x)$. Let

$$f_\varepsilon(x, y) = \textstyle\sum_{i=0}^{N} a_i(x) y^{N-i} = a_0(x)\, y^N + a_1(x)\, y^{N-1} + \cdots + a_N(x) \qquad (2.3.1)$$

be a polynomial over K. As K is algebraically closed, a_i $(i = 0, 1, \ldots, N)$ has the expression:

$$a_i(x) = c_i e^{A_i} x^{m_i} \textstyle\prod_{j=1}^{d_i} (x - u_{i,j} e^{B_{i,j}}), \quad c_i, u_{i,j} \in R^\times,\ A_i, B_{i,j} \in \mathbb{Q},\ m_i \in \mathbb{N}. \qquad (2.3.2)$$

Define the algebraic curve $C_\varepsilon := V(f_\varepsilon)$. By changing the variables x, y by $x \mapsto x e^{-R}$ and $y \mapsto y e^{-R'}$ $(R, R' \gg 0)$, we can assume $A_i, B_{i,j} > 0$ without loss of generality. In the present paper, we investigate algebraic curves over K which satisfy the following genericness condition:

Genericness condition. The numbers $\mathrm{top}(u_{i,j})$ $\forall i, j$ are all distinct.

For the proof of our main theorem 2.4.3, we will impose a slightly stricter conditions on the curve. We discuss these conditions in the appendix.

2.3.1 Examples

We first give some examples of plane curves over K and their tropicalization. These examples show us how to approximately reconstruct the plane curve from its tropicalization.

Example (I)

Let C_ε be the curve defined by the pL-polynomial

$$f_\varepsilon(x,y) = (x+\mathbf{e})y^2 + (x+\mathbf{e}^2)(x+\mathbf{e}^3)y + \mathbf{e}^8 = 0. \tag{2.3.3}$$

The tropicalization of C_ε is described as

$$TV_{\mathbb{R}^2}(f_\varepsilon) := \left\{ (X,Y) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} \min\,[X+2Y, 2Y+1, 2X+Y, X+Y+2, Y+5, 8] \\ \text{is not smooth.} \end{array} \right\}. \tag{2.3.4}$$

Denote this variety by Trop C simply. Figure 2.2 shows Trop C. Trop C has four vertices $\alpha = (1,1), \beta = (2,4), \gamma = (2,3)$ and $\delta = (2.5, 3.5)$, and has one closed loop. We use the term "*edge*" only when it means a segment of finite length. Edges of infinite length shall be called *leaves* hereafter. The *genus of* Trop C is the number of independent closed cycles over Trop C. In this case, genus(Trop C) = 1.

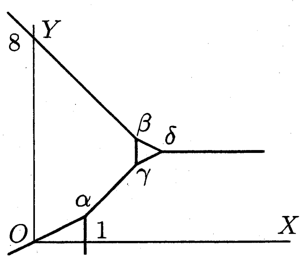


Figure 2.2: The figure of Trop C. It consists of four vertices, four edges and four leaves. The genus of Trop C is one.

The canonical expression of f_ε at α is $\mathcal{R}^\alpha f_\varepsilon = (xy^2 + y^2 + x^2y) + (\mathbf{e}xy + \mathbf{e}^2y + \mathbf{e}^3y + \mathbf{e}^5)$. By the approximation theorem 2.2.9, the curve $la(\varepsilon)(C_\varepsilon)$ is approximated by the translation of $la(\varepsilon)(V(f^\alpha)) = la(\varepsilon)(V(xy^2+y^2+x^2y)) = la(\varepsilon)(V_{\mathbb{CR}^2}(xy+y+x^2))$ around α. Similarly, around the points $\beta, \gamma, \delta \in$ Trop C, $la(\varepsilon)(C_\varepsilon)$ is approximated by $la(\varepsilon)(V(x^2y+xy+1))$, $la(\varepsilon)(V(y+x^2+x))$ and $la(\varepsilon)(V(y^2+xy+1))$ respectively.

Figure 2.3 shows four 'local' Riemannian surfaces $V(f^\alpha), V(f^\beta), V(f^\gamma)$ and $V(f^\delta)$.

According to the approximation theorem, we can approximately draw the form of the Riemannian surface C_ε for small $\varepsilon > 0$. Gluing the local data in figure 2.3 along Trop C (figure 2.2), we can sketch $la(\varepsilon)(C_\varepsilon)$ (or C_ε which is diffeomorphic to $la(\varepsilon)(C_\varepsilon)$) as in figure 2.4: Four small spheres, four long cylinders and five horns make up the figure. For later arguments, we take the completion of C_ε that is a compact Riemannian surface.

Example (II)

Let us consider

$$C_\varepsilon : y^3 + (x+\mathbf{e}^4)y^2 + \mathbf{e}^2(x+\mathbf{e})(x+2\mathbf{e})y + \mathbf{e}^{10} = 0 \tag{2.3.5}$$

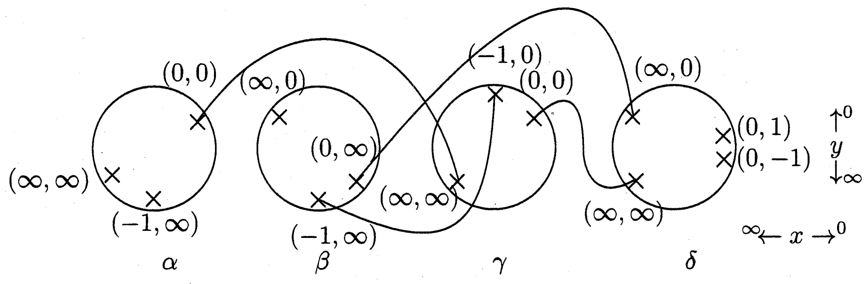


Figure 2.3: The figure of 'local' Riemannian surfaces. They are of genus 0. The points (x, y) satisfying $x = \infty, 0$ or $y = \infty, 0$ are described in the figure.

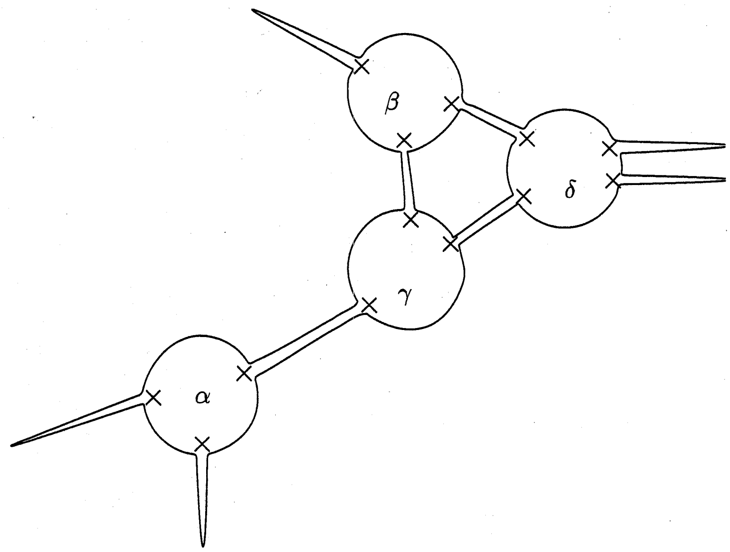


Figure 2.4: The sketch of (the completion of) C_ε. The figure consists of four spheres, four cylinders and five horns.

and its tropicalization:

$$\operatorname{Trop} C : \min \begin{bmatrix} 3Y, X+2Y, 2Y+4, 2X+Y+2, \\ X+Y+3, Y+4, 10 \end{bmatrix} \text{ is not smooth.}$$

$\operatorname{Trop} C$ has three vertices $\alpha = (1,6)$, $\beta = (1,3)$ and $\gamma = (2,2)$. The truncations associated with these points are: $f^{\alpha} = (x+1)(x+2)y+1$, $f^{\beta} = xy^2+(x+1)(x+2)$ and $f^{\gamma} = y^3+xy^2+2y$ respectively. Figure 2.5 displays an approximate sketch of (the completion of) C_{ε}. Although C_{ε} is of genus one, the genus of $\operatorname{Trop} C$ is zero. This difference comes from the edge which connects α and β in $\operatorname{Trop} C$. Two long cylinders are associated with this one edge.

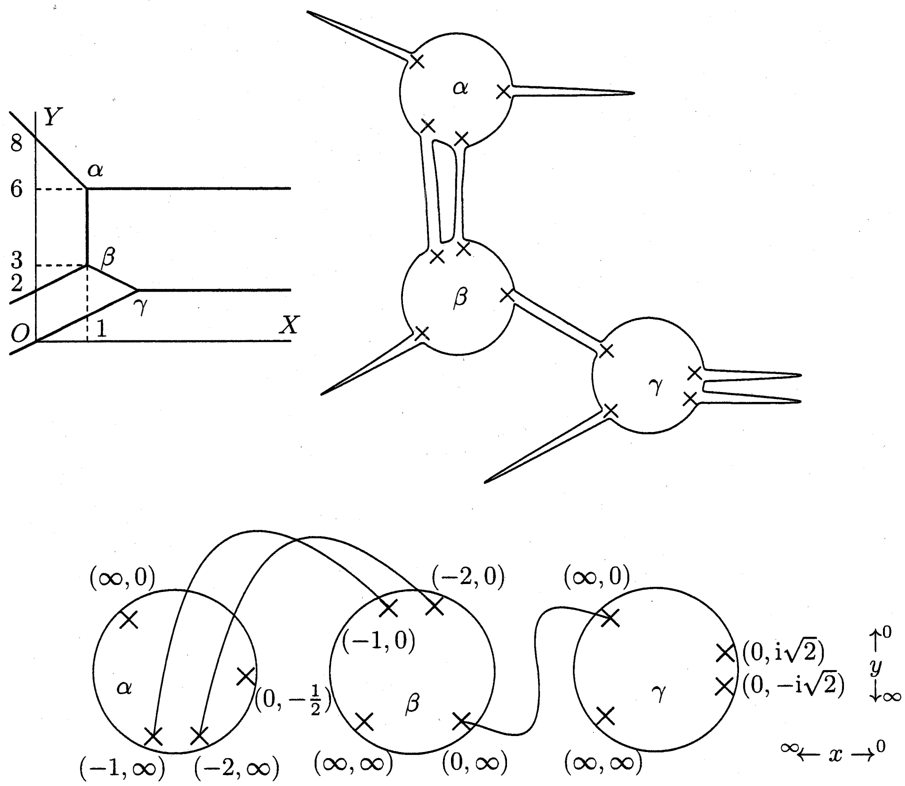


Figure 2.5: The sketch of C_{ε}.

In both examples above, the shapes of Riemannian surfaces $V(f^P)$ (P is a vertex of $\operatorname{Trop} C$) are essential for drawing C_{ε}. We denote these Riemannian surfaces by *Riemannian sub-surfaces.*

In example (II), there exists an edge associated to two cylinders. This property reflects the fact that: For $P = (1, Y) \in \operatorname{Trop} C \quad (3 < Y < 6)$, the variety $V(f^P) = V((x+1)(x+2)) = V(x+1) \coprod V(x+2)$ consists of two irreducible component. In such a case, we call the edge $\overline{\alpha\beta}$ *of multiplicity* 2.

Here we remark on the behaviour of sub-surfaces. It may happen that a sub-surface is reducible or of genus more than 0. In these cases, the genus of $\operatorname{Trop} C$ becomes inferior to the genus of C_{ε}.

The following definition is given for section 2.4.

Definition 2.3.1 *A curve over K is called non-degenerate if all of its Riemannian sub-surfaces are irreducible and of genus* 0.

2.3.2 Multiplicity of tropical edges

In this section, we define the *vertical thickness*, the *horizontal thickness* and the *multiplicity* of tropical edges.

Vertical and horizontal thickness

The 'Val' function associated with the pL-polynomial $a_i(x)$ (2.3.2) is expressed as

$$\mathrm{Val}\,(X; a_i) = A_i + m_i X + \textstyle\sum_{j=1}^{d_i} \min\,[X, B_{i,j}], \quad (i = 0, 1, \ldots, N). \tag{2.3.6}$$

Let $F_i(X) := \mathrm{Val}\,(X; f_\varepsilon)$. Then the defining condition of $\mathrm{Trop}\, C$ is expressed as

$$\mathrm{Trop}\, C: \quad \min_{i=0,\ldots,N}[F_i(X) + (N-i)Y] \quad \text{is not smooth.} \tag{2.3.7}$$

Of course, we have $\mathrm{Val}\,(X, Y; f_\varepsilon) = \min_{i=0,\ldots,N}[F_i(X) + (N-i)Y]$. Define the domain $\mathfrak{D}_i \subset \mathbb{R}^2$ $(i = 0, 1, \ldots, N)$ by

$$\mathfrak{D}_i := \{(X, Y) \,|\, \mathrm{Val}\,(X, Y; f_\varepsilon) = F_i(X) + (N-i)Y\}.$$

The domains $\mathfrak{D}_i$ separate the plane $\mathbb{R}^2$ into at most $N+1$ pieces: $\mathbb{R}^2 = \bigcup_{i=0}^N \mathfrak{D}_i$. If $i < j$, $(x, y_i) \in \mathfrak{D}_i$ and $(x, y_j) \in \mathfrak{D}_j$ then $y_i \leq y_j$. ($\because$ From the definition of $\mathfrak{D}_i$ and $\mathfrak{D}_j$, we have $F_i(x) + (N-i)y_i \leq F_j(x) + (N-j)y_i$ and $F_j(x) + (N-j)y_j \leq F_i(x) + (N-i)y_j$. It follows that $(j-i)y_i \leq (j-i)y_j$.) Let us define the piecewise linear function $\mathcal{N}_i(X)$ $(i = 1, 2, \ldots, N)$ defined by the relation $\mathcal{N}_i(X) = \min_{j \geq i}\{Y \,|\, (X, Y) \in \mathfrak{D}_i\}$. Note that $\mathcal{N}_{i+1}(X) \geq \mathcal{N}_i(X)$ for all X.

Using the function $\mathcal{N}_i(X)$, we obtain an another expression of $\mathrm{Trop}\, C$. We formally regard $\mathcal{N}_{N+1}(X) := +\infty$ and $\mathcal{N}_0(X) := -\infty$ for any X.

Proposition 2.3.1 *Let $L_{i,j}$ be the vertical edge which connects*

$$(B_{i,j}, \mathcal{N}_i(B_{i,j})) \qquad \textit{and} \qquad (B_{i,j}, \mathcal{N}_{i+1}(B_{i,j})).$$

Then, the set $\left(\bigcup_{i=1}^N \{Y = \mathcal{N}_i(X)\}\right) \cup \left(\bigcup_{i,j} L_{i,j}\right)$ *coincides with* $\mathrm{Trop}\, C$.

Remark 2.3.1 *If $\mathcal{N}_i(B_{i,j}) = \mathcal{N}_{i+1}(B_{i,j})$, then $L_{i,j} = \{a\ point\}$.*

Proof. Let $G_i := \{(X, Y) \in \mathbb{R}^2 \,|\, Y = \mathcal{N}_i(X)\}$. Because it is obvious, by definition, that $\mathrm{Trop}\, C \supset \left(\bigcup_{i=1}^N G_i\right)$, it is sufficient to prove $\mathrm{Trop}\, C \setminus \left(\bigcup_{i=1}^N G_i\right) = \left(\bigcup_{i,j} L_{i,j}\right)$. Choose a connected component O of $\mathbb{R}^2 \setminus \left(\bigcup_{i=1}^N G_i\right)$. The domain O is contained in $\mathfrak{D}_i^\circ$ for some i, where $\mathfrak{D}_i^\circ$ is the set of inner points of $\mathfrak{D}_i$. For any point (X, Y) in O, it follows that $\mathrm{Val}\,(X, Y; f_\varepsilon) = F_i(X) + (N-i)Y$. Then, a point $(X, Y) \in \mathrm{Trop}\, C \cap O$ must be a point at which $F_i(X)$ is not smooth. Because the function $F_i(X)$ (2.3.6) is not smooth iff $X = B_{i,j}$, we conclude that $\mathrm{Trop}\, C \setminus \left(\bigcup_{i=1}^N G_i\right)$ consists of the sets $\{X = B_{i,j}\} \cap O = L_{i,j}$.

By use of proposition 2.3.1, we introduce the vertical thickness of edges.

Definition 2.3.2 *Let $E \subset \operatorname{Trop} C$ be an edge. We call the number $\sharp\{i \,|\, E \subset G_i\}$ vertical thickness of E. In other words, the vertical thickness of an edge E is a difference of the maximum element and the minimum element of the set*

$$\{b \in \{0, \ldots, N\} \,|\, \mathrm{Val}\,(X, Y; f_\varepsilon) = aX + bY + c, \ \forall (X, Y) \in E\}.$$

For example, the vertical thickness of a vertical edge is 0.

Lemma 2.3.2 *The vertical thickness of E equals to the degree of the projection*

$$V(f^P) \to \mathbb{C} \setminus \{0\}; \ (x, y) \mapsto x, \qquad P \in \mathrm{Int}\, E.$$

Proof. First note that the truncation f^P $(P \in \mathrm{Int}\, E)$ is determined uniquely by lemma 2.2.7. The Laurent polynomial f^P is of the form $f^P = \sum c_{(a,b)} x^a y^b$, where (a, b) runs over the set $\{(a, b) \,|\, \mathrm{Val}\,(X, Y; f_\varepsilon) = aX + bY + c \ \forall (X, Y) \in E\}$ and $c_{(a,b)}$ is a non-zero complex number for any (a, b). Because the degree of the projection $x : V(f^P) \to \mathbb{C} \setminus \{0\}$ equals to the difference between the maximum degree and the minimum degree w.r.t. y consisted in f^P, the desired result is obtained.

Let $C_\varepsilon^T := V(f_\varepsilon(y, x))$ be the curve obtained from C_ε by switching the x and y coordinates. For an edge $E \subset \operatorname{Trop} C$, we define the *horizontal thickness* of E by the vertical thickness of $E^T \subset \operatorname{Trop} C^T$ which is the image of E by the morphism $(X, Y) \mapsto (Y, X)$. For example, the horizontal thickness of horizontal edges is 0.

Multiplicity

As example (II) in section 2.3.1 above, it may happen that more than one cylinders (or horns) are associated with one edge (or one leaf).

Definition 2.3.3 *Let $E \subset \operatorname{Trop} C$ be an edge (resp. a leaf). The multiplicity of E is the number of cylinders (resp. horns) associated with E.*

Let $E \subset \operatorname{Trop} C$ be an edge (resp. a leaf) of multiplicity m, and P be a point in $\mathrm{Int}\, E$. By the approximation theorem 2.2.9, the variety $V(f^P)$ must be decomposed into m irreducible components: $V(f^P) = V(f_1^P) \coprod \cdots \coprod V(f_m^P)$, where $f^P = f_1^P \cdots f_m^P$. We often regard the edge E as the union of distinguished m edges: $E = E_1 \amalg \cdots \amalg E_m$ (See figure 2.8 in section 2.4), where each E_i corresponds to $V(f_i^P)$ $(P \in \mathrm{Int}\, E)$ and is taken to be of multiplicity one.

Define the vertical thickness of E_i as the degree of the projection: $V(f_i^P) \to \mathbb{C} \setminus \{0\}$; $(x, y) \mapsto x$. Denote the vertical thickness of E_i by q_i. Naturally, $q_1 + \cdots + q_m$ equals to the vertical thickness of E. Similarly, we can define the horizontal thickness of E_i as the vertical thickness of E_i^T.

The following definition is given for the next section.

Definition 2.3.4 *Let $L_{i,j}$ be the vertical edge which connects*

$$(B_{i,j}, \mathcal{N}_i(B_{i,j})) \text{ and } (B_{i,j}, \mathcal{N}_{i+1}(B_{i,j})).$$

The ceiling of $L_{i,j}$ is the set $G_{i+1} = \{(X, Y) \,|\, Y = \mathcal{N}_{i+1}(X)\}$ and the floor of $L_{i,j}$ is the set $G_i = \{(X, Y) \,|\, Y = \mathcal{N}_i(X)\}$.

2.3.3 Regularity of tropical curves

Let $P = (X_0, Y_0)$ be a point in $\mathrm{Trop}\, C = TV(f_\varepsilon)$, where $f_\varepsilon = \sum_{w=(w_1,w_2)\in\mathbb{Z}^2} a_w x^{w_1} y^{w_2}$. Considering the set $\Theta(P) = \{w \in \mathbb{Z}^2 \,|\, \mathrm{Val}\,(X_0, Y_0; f_\varepsilon) = \mathrm{val}\,(a_w) + w_1 X_0 + w_2 Y_0\}$ (section 2.2), we have $\sharp\Theta(P) \geq 2$ (remark 2.2.1). More precisely, the number of elements of $\Theta(P)$ satisfies the following inequalities:

$$\begin{array}{lll} \text{i) } P \text{ is an inner point of some edge of } \mathrm{Trop}\, C & \Rightarrow & \sharp\Theta(P) \geq 2, \\ \text{ii) } P \text{ is a vertex of } \mathrm{Trop}\, C & \Rightarrow & \sharp\Theta(P) \geq 3. \end{array} \tag{2.3.8}$$

These inequalities reflect the fact that the intersection of (generic) two planes is an line and the intersection of (generic) three planes is a point in $\mathbb{R}^3$.

In the present paper, we often assume some generic condition on the defining polynomial of $\mathrm{Trop}\, C$.

Definition 2.3.5 *The tropical plane curve $TV(f_\varepsilon)$ is regular if the equalities in equation* (2.3.8) *hold.*

2.4 Integration Theory

The integration theory over tropical curves was first introduced in [14]. Hereafter we will show that the ultradiscrete limit of holomorphic integrals over C_ε coincides with the holomorphic integral over $\mathrm{Trop}\, C$ (for C_ε of some type).

2.4.1 Definition of the holomorphic integral over tropical curves

In this section, we give a brief introduction to the integration theory over tropical curves, following [2, 14].

We first equip $\mathrm{Trop}\, C$ with the structure of a *metric graph.* Let E be an edge of $\mathrm{Trop}\, C$. E has the expression $E = \{(X_0, Y_0) + t(u, v) \,|\, 0 \leq t \leq \ell\}$, $(u, v \in \mathbb{Z})$. It can be assumed that u and v are coprime without loss of generality. We call the vector (u, v) the *primitive vector* of E. We define a tropical length ℓ_T of E by $\ell_T(E) := \ell$. With this length the tropical curve $\mathrm{Trop}\, C$ becomes a metric graph.

The metric on $\mathrm{Trop}\, C$ defines a symmetric bilinear form $\ell_T(\cdot, \cdot)$ on the space of paths in $\mathrm{Trop}\, C$. For this, we define $\ell_T(\Gamma, \Gamma) := \ell_T(\Gamma)$ for non-self-intersecting path Γ, and extend it to any pairs of paths bilinearly. Figure 2.6 shows an example of $\ell_T(\cdot, \cdot)$. Note that the number $|\ell_T(\Gamma_1, \Gamma_2)|$ equals the tropical length $\ell_T(\Gamma_1 \cap \Gamma_2)$. This bilinear form gives "$\pm$\{the tropical length of intersection of two paths\}".

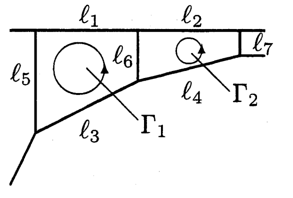


Figure 2.6: Example of a metric graph. We have $\ell_T(\Gamma_1, \Gamma_1) = \ell_1 + \ell_3 + \ell_5 + \ell_6$, $\ell_T(\Gamma_2, \Gamma_2) = \ell_2 + \ell_4 + \ell_6 + \ell_7$ and $\ell_T(\Gamma_1, \Gamma_2) = \ell_T(\Gamma_2, \Gamma_1) = -\ell_6$.

Let g be the genus of $\mathrm{Trop}\, C$ and choose a homology basis $T_{\beta_1}, \ldots, T_{\beta_g} \in H_1(\mathrm{Trop}\, C\,; \mathbb{Z})$. A *tropical period matrix* B_T is the $g \times g$ matrix defined by $B_T := (\ell_T(T_{\beta_i}, T_{\beta_j}))_{i,j}$. Since ℓ_T is non-degenerate, B_T is symmetric and positive definite.

Here, we note the relation between the tropical length, the multiplicity and the vertical thickness of the edge in $\mathrm{Trop}\, C$. Let

$$E := \{((1-t)X_0 + tX_1, (1-t)Y_0 + tY_1) \mid 0 \le t \le 1, X_0 \lneq X_1\}$$

be a non-vertical edge of vertical thickness q and of horizontal thickness w. From the definition of vertical thickness and horizontal thickness, it follows that E is part of the line defined by

$$aX + bY + c = (a+w)X + (b \pm q)Y + c'. \qquad (2.4.1)$$

Lemma 2.4.1 *Let* $\xi := \text{g.c.d.}(q, w)$. *Then the tropical length of* E *is expressed as:*

$$\ell_T(E) = \frac{\xi}{q}(X_1 - X_0).$$

Proof. We can assume $(X_0, Y_0) = (0,0)$ by translation. Then we obtain $wX_1 = \pm qY_1$. Let $\eta := \text{g.c.d.}(X_1, Y_1)$ and

$$\begin{cases} X_1 = x\eta \\ Y_1 = y\eta \end{cases}, \quad \begin{cases} q = \mu\xi \\ w = \nu\xi \end{cases}.$$

Then, we conclude $\mu = x$ and $\nu = y$ by elementary arguments. From the definition of the tropical length we obtain

$$\ell_T(E) = \text{g.c.d.}(X_1, Y_1) = \eta = \frac{X_1}{x} = \frac{\xi}{q} X_1.$$

Next, we introduce *affine transformations* of tropical curves. Let $\mathcal{T}$ be a tropical curve in $\mathbb{R}^2$. For a 2×2 matrix $\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $(\alpha\delta - \beta\gamma = 1)$, we have a new set

$$\mathcal{U} := \{\theta(X,Y)^T \in \mathbb{R}^2 \mid (X,Y) \in \mathcal{T}\}.$$

For the complex curve $\{P(x,y) = 0\}$, this transformation associates with the translation $x \mapsto x^\delta y^{-\beta}$ and $y \mapsto x^{-\gamma} y^\alpha$, which is invertible and holomorphic. Especially, $\mathcal{U}$ is also a tropical curve associated with $P(x^\delta y^{-\beta}, x^{-\gamma} y^\alpha) = 0$.

Concerning such affine translations, we have the following fundamental result.

Proposition 2.4.2 *Let* $\theta \in M_2(\mathbb{Z})$ *be a* 2×2 *matrix with* $\det \theta = 1$. *Then*

(*i*) *The length of an edge is* θ*-invariant.*

(*ii*) *For an edge* $L \in \mathbb{R}^2$, *there exists* $\theta \in M_2(\mathbb{Z})$ *such that* $\theta \cdot L$ *is vertical.*

Proof. (i) Let (u, v) be a primitive vector of the edge $L \in \mathbb{R}^2$. It is sufficient to prove that the image $\theta(u,v)^T = (\alpha u + \beta v, \gamma u + \delta v)$ is also primitive. For this, we have only to prove $\text{g.c.d.}(\alpha u + \beta v, \gamma u + \delta v) = 1$. This can be proved by elementary methods and we omit the proof.

(ii) For the primitive vector (u, v) of L, it is enough to define $\theta = \begin{pmatrix} v & -u \\ w & z \end{pmatrix}$ such that $vz + wu = 1$.

Remark 2.4.1 *The vertical and the horizontal thickness of an edge depend on the coordinate function* X *and* Y.

2.4.2 Main theorem

Now we proceed to the integration theory over $C_\varepsilon = V(f_\varepsilon)$. In order to make the problem easier, we deal only with the case where:

i) C_ε is smooth for $0 < \varepsilon < 1$,

ii) C_ε is non-degenerate,

iii) $\operatorname{Trop} C$ is regular.

Conditions i)–iii) and the *genericness condition* (section 2.3) lead to the following properties (see Appendix):

iv) for each edge E, $m = \text{g.c.d.}(q, w)$, where m, q, w are respectively the multiplicity, the vertical thickness and the horizontal thickness of E,

v) for each edge $E = E_1 \amalg \cdots \amalg E_m$, $q_1 = \cdots = q_m$, $w_1 = \cdots = w_m$, where q_i, w_i are the vertical thickness and the horizontal thickness of E.

Remark 2.4.2 *The curves C introduced in examples* (I) *and* (II) *in section* 2.3.1 *satisfy these conditions.*

We say that C_ε has a *good tropicalization* if C_ε and $\operatorname{Trop} C$ satisfy conditions i)–iii) above.

Remark 2.4.3 *The conditions i) and ii) are necessary conditions to construct the integration theory. The condition iii) is required for simplicity of the calculations. (The author is not sure whether the condition iii) can be omitted.)*

By the approximation theorem 2.2.9, there exists small $\zeta > 0$ such that all C_ε $(\varepsilon \in (0, \zeta))$ are homotopic. Hereafter ε denotes a small real number which satisfies $0 < \varepsilon < \zeta$ unless otherwise is stated.

Let g be the genus of C_ε. Define homology cycles $\alpha_1, \alpha_2, \ldots, \alpha_g \in H_1(C_\varepsilon; \mathbb{Z})$ as in figure 2.7. Any cycle is associated with a long cylinder connects two sub-surfaces. Next define the homology cycles $\beta_1, \beta_2, \ldots, \beta_g \in H_1(C_\varepsilon; \mathbb{Z})$ such that the intersection index $\alpha_i \circ \beta_j$ is $\delta_{i,j}$, in a canonical way. We assume the cycles $\alpha_i = \alpha_i(\varepsilon)$ and $\beta_i = \beta_i(\varepsilon)$ are continuous with respect to ε. Denote the normalised holomorphic differentials over C_ε by $\omega_1, \omega_2, \ldots, \omega_g$ $(\int_{\alpha_i} \omega_j = \delta_{i,j})$. The *period matrix* B_ε of the Riemannian surface C_ε is the $g \times g$ matrix defined by $B_\varepsilon := (\int_{\beta_i} \omega_j)_{i,j}$.

Now we state the main theorem of this paper. Take a homology basis $T_{\beta_1}, \ldots, T_{\beta_g} \in H_1(\operatorname{Trop} C\,; \mathbb{Z})$ associated with a homology basis $\beta_1, \ldots, \beta_g \in H_1(C\,; \mathbb{Z})$. If more than one cylinder can be associated with one tropical edge, we say that hidden edges and cycles exist here (see figure 2.8). When we define homology cycles in $H_1(\operatorname{Trop} C; \mathbb{Z})$, these edges must be distinguished. Recall that we regard an edge E of multiplicity m as the union: $E = E_1 \amalg E_2 \amalg \cdots \amalg E_m$. Let $B_T := (\ell_T(T_{\beta_i}, T_{\beta_j}))_{i,j}$ be a *period matrix of* $\operatorname{Trop} C$.

Theorem 2.4.3 *If C_ε has a good tropicalization, then*

$$B_\varepsilon \sim \frac{-1}{2\pi i \varepsilon} B_T \qquad (\varepsilon \to 0).$$

The rest of this paper is devoted to the proof of this theorem.

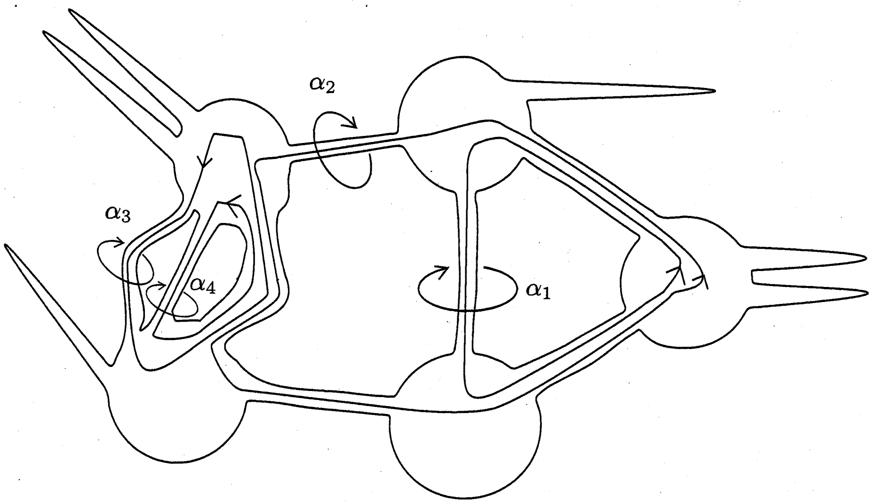


Figure 2.7: An example of the definition of $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$ in $H^1(C\,;\mathbb{Z})$. We always consider that an α-cycle surrounds a cylinder.

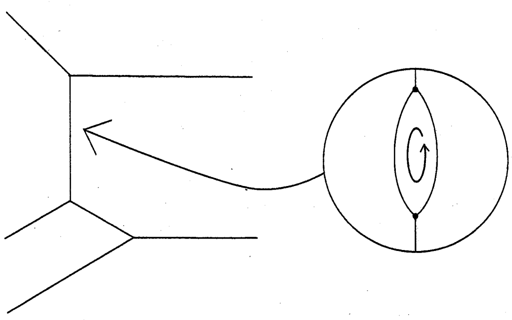


Figure 2.8: If there exist an edge with multiplicity $m > 1$, we regard this edge as union of m edges of multiplicity one.

Preliminaries for an integration theory

In this section, we study the integral calculus over $V(f_\varepsilon)$ and the asymptotic behaviours of integrals when ε goes to zero. For a pL-polynomial f_ε, let us define the variety $\widetilde{V}(f) := \{(x,y,\varepsilon) \in \mathbb{CR}^2 \times (0,1) \mid f_\varepsilon(x,y) = 0\}$. Denote the natural embedding $\mathbb{CR}^2 \hookrightarrow \mathbb{CR}^2 \times (0,1)$; $(x,y) \mapsto (x,y,\varepsilon)$ by j_ε. Naturally it follows that $j_\varepsilon^{-1}(\widetilde{V}(f)) = V(f_\varepsilon)$.

Let $\mathcal{U} \subset \widetilde{V}(f)$ be a simply connected domain and ω_ε be a 1-form over $\mathcal{U}$ such that i) ω_ε is a holomorphic differential over $j_\varepsilon^{-1}(\mathcal{U}) = \mathcal{U} \cap V(f_\varepsilon)$, and ii) ω_ε is continuous with respect with ε. By elementary arguments on the complex analysis, we can prove the existence of a primitive function Ω_ε of ω_ε. The integration of ω_ε along a smooth path $[0,1] \to j_\varepsilon^{-1}(\mathcal{U})$; $\theta \mapsto \gamma_\varepsilon(\theta)$ is defined by the formula $\int_{\gamma_\varepsilon} \omega_\varepsilon := \Omega_\varepsilon(\gamma_\varepsilon(1)) - \Omega_\varepsilon(\gamma_\varepsilon(0))$.

Let $(0,1) \times [0,1] \mapsto \widetilde{V}(f)$; $(\varepsilon,\theta) \mapsto \gamma_\varepsilon(\theta)$ be a smooth map such that $\gamma_\varepsilon(\theta) \in V(f_\varepsilon)$ for all ε and θ. Our aim in this section is to evaluate the asymptotic behaviour of the value $\int_{\gamma_\varepsilon} \omega_\varepsilon$ when ε goes to zero.

Due to the surjectivity theorem 2.2.10, more detailed information can be added to the definition of a path on $\widetilde{V}(f)$. Let $v_1 = (X_1, Y_1)$ and $v_2 = (X_2, Y_2)$ be two points in $\operatorname{Trop} C = TV(f_\varepsilon)$ and $\mathcal{R}^{v_i} f_\varepsilon = f^{v_i} + \Delta^{v_i}$ $(i = 1,2)$ be the canonical expressions of f_ε at v_i.

Consider a path $\gamma' : [0,1] \to V(f^{v_1})$ on the variety $V(f^{v_1})$. By the surjectivity theorem, there exists the smooth map $(0,1) \times [0,1] \to \widetilde{V}(f)$; $(\varepsilon,\theta) \mapsto \gamma_\varepsilon(\theta)$ such that

$$\gamma_\varepsilon(\theta) = (x_\varepsilon(\theta) e^{X_1}, y_\varepsilon(\theta) e^{Y_1}) \in V(f_\varepsilon), \qquad \text{and} \qquad \lim_{\varepsilon \to 0^+} (x_\varepsilon(\theta), y_\varepsilon(\theta)) \to \gamma'(\theta).$$

Moreover, we have $x_\varepsilon(\theta), y_\varepsilon(\theta) \in R^\times$ for any θ. We often abbreviate the above notation as $\gamma(\theta) = (xe^{X_1}, ye^{Y_1})$ $x, y \in R^\times$ if there is no fear of confusion.

Similarly, for two points $(x_1', y_1') \in V(f^{v_1})$ and $(x_2', y_2') \in V(f^{v_2})$, there exists a smooth map $(0,1) \times [0,1] \to \widetilde{V}(f)$; $(\varepsilon,\theta) \mapsto \gamma_\varepsilon(\theta)$ such that

$$\gamma_\varepsilon(0) = (x_{1,\varepsilon} e^{X_1}, y_{1,\varepsilon} e^{Y_1}), \quad \gamma_\varepsilon(1) = (x_{2,\varepsilon} e^{X_2}, y_{2,\varepsilon} e^{Y_2}), \quad \text{and} \quad \lim_{\varepsilon \to 0^+} (x_{i,\varepsilon}, y_{i,\varepsilon}) \to (x_i', y_i').$$

We often use the notation $\int_{\gamma(0)}^{\gamma(1)} \omega$ in stead of $\int_{\gamma_\varepsilon} \omega_\varepsilon$ if the meaning is clear.

Approximation of integral calculus

In the rest of this paper, $b_{i,j}$ denotes $\int_{\beta_j} \omega_i$.

Let $\mathfrak{S}$ be the set of 1-forms over $\widetilde{V}(f_\varepsilon)$ such that i) $\omega_\varepsilon \in \mathfrak{S}$ is a meromorphic differential over $j_\varepsilon^{-1}(\widetilde{V}(f)) = V(f_\varepsilon)$, ii) $\omega_\varepsilon \in \mathfrak{S}$ is continuous with respect to ε. Define the subsets $\mathcal{M}$ and $\mathcal{F}$ by the formulae

$$\mathcal{M} := \{\omega_\varepsilon \in \mathfrak{S} \mid \textstyle\int_{\beta_k} \omega_\varepsilon = \sum_{i,j=1}^{g} c_{i,j}^k(\varepsilon)\, b_{i,j}, \quad \text{where } -\lim_{\varepsilon \to 0^+} \varepsilon \log c_{i,j}^k > 0, \quad \forall i,j,k\},$$
$$\mathcal{F} := \{\omega_\varepsilon \in \mathfrak{S} \mid \lim_{\varepsilon \to 0^+} |\textstyle\int_{\beta_i} \omega_\varepsilon| < +\infty, \quad \forall i\}.$$

It is clear that they are $R^\times$-vector spaces.

Remark 2.4.4 *If the main theorem 2.4.3 is true, is follows that $\mathcal{M} \subset \mathcal{F}$.*

For the proof of the main theorem, we start with the differential of the third kind over C_ε. Let P_+, P_- be two points on C_ε. A smooth curve C_ε has the normalised differential of the third kind $\omega_{P_+ - P_-} = \omega_{P_+ - P_-}(\varepsilon)$, possessing simple poles with residue $+1/(2\pi i)$ at P_+

and $-1/(2\pi \mathrm{i})$ at P_-, and holomorphic over $C_\varepsilon \setminus \{P_+, P_-\}$ satisfying $\int_{\alpha_i} \omega_{P_+-P_-} = 0$ $(i = 1, 2, \ldots, g)$. Generally, for n points $P_1, \ldots, P_n \in C_\varepsilon$ and complex numbers $c_1 + \cdots + c_n = 0$, there is a unique normalised differential $\omega_{c_1 P_1 + \cdots + c_n P_n}$, with residue $c_i/(2\pi \mathrm{i})$ at P_i $(i = 1, \ldots, n)$.

Recall that a point in $\{x = \infty, 0\} \cup \{y = \infty, 0\}$ is associated with a leaf in $\operatorname{Trop} C$. (cf. Section 2.3.1).

Lemma 2.4.4 *Let $P_+, P_- \in C_\varepsilon$ are two points in $\{x = \infty\} \cup \{x = 0\} \cup \{y = \infty\} \cup \{y = 0\}$ which are associated with the same leaf in* $\operatorname{Trop} C$. *Then it follows that $\omega_{P_+-P_-} \in \mathcal{M} + \mathcal{F}$.*

Proof. Let $L \subset \operatorname{Trop} C$ be the leaf which includes $P_\pm$. By rotation, L can be assumed to run toward $Y \to +\infty$ vertically: $L = \{(B, Y) \mid Y \geq \mathcal{N}_N(B)\}$, where $\mathcal{N}_N(X)$ is a tropical function as defined in section 2.3. Hereafter, we denote this function by $\mathcal{N}(X)$.

Let $f_\varepsilon(x, y) = \sum_{i=0}^{N} a_i(x) y^{N-i}, a_N(x) = c e^A x^m \prod_{j=1}^{d}(x - u_j e^{B_j})$, $(c, u_j \in R^\times, m \in \mathbb{N}_{\geq 0}, A, B_j \in \mathbb{Q}_{\geq 0})$ be the defining polynomial of C_ε. By the above assumption the y-coordinate of $P_\pm$ equals 0. Then the x-coordinate of $P_\pm$ can be written as $x(P_+) = u_{k_1} e^B$, $x(P_-) = u_{k_2} e^B$ for some $k_1, k_2 \in \{1, 2, \ldots, d\}$ such that $B = B_{k_1} = B_{k_2}$.

Now we proceed to the integration theory. Consider the polynomial $\phi(x)$ defined by

$$\frac{1}{2\pi \mathrm{i}} \left\{ \frac{1}{x - u_{k_2} e^B} - \frac{1}{x - u_{k_1} e^B} \right\} = \frac{\phi(x)}{a_N(x)}. \tag{2.4.2}$$

Define the new differential ω_{f} defined by

$$\omega_{\mathrm{f}} := \frac{\phi(x)\, dx}{y\, f_y(x, y)}, \qquad \text{where } f_y(x, y) := \partial_y f_\varepsilon(x, y) = \textstyle\sum_{i=0}^{N-1} (N - i)\, a_i(x)\, y^{N-i-1}. \tag{2.4.3}$$

The singularity of ω_{f} must be contained in $\{x = \infty\} \cup \{y = \infty, 0\}$. ($\because dx/f_y$ is always holomorphic over smooth plane curves).

The following sublemmas describes the behaviour of the differential ω_{f}.

The distribution of residues of ω_{f} is given as follows:

(i) $\operatorname{Res}_{P_+} \omega_{\mathrm{f}} = +1/(2\pi \mathrm{i}) + o(e^0)$,

(ii) $\operatorname{Res}_{P_-} \omega_{\mathrm{f}} = -1/(2\pi \mathrm{i}) + o(e^0)$,

(iii) $\operatorname{Res}_P \omega_{\mathrm{f}} = o(e^0), \quad P \neq P_\pm$.

Proof. i)–ii) Let $v \in \operatorname{Trop} C$ be the vertex which is at the foot of L and let Ω be the subsurface associated with v. We take small cycles $\gamma_\pm \subset \Omega$ which loop around $P_\pm$ anti-clockwise and which satisfy

$$(x, y) \in \gamma_\pm \;\Rightarrow\; x = r e^B, y = s e^{\mathcal{N}(B)}, \quad \exists r, s \in R^\times. \tag{2.4.4}$$

Denote the vertical thickness of the floor of L by q'. On $\gamma_\pm \subset \Omega$, the dominant terms of $f_\varepsilon(x, y) = \sum a_i(x)\, y^{N-i}$ are a_N and $a_{N-q'} y^{q'}$. Then the dominant term of $f_y(x, y) = \sum (N - i) a_i(x) y^{N-i-1}$ is $q' a_{N-q'} y^{q'-1}$. Hence, on $\gamma_\pm$, one has

$$y\, f_y \sim q' a_{N-q'}\, y^{q'} = -q' a_N + \cdots. \tag{2.4.5}$$

For the second equation in (2.4.5), we used $f_\varepsilon(x, y) = 0$. Then, we obtain

$$\int_{\gamma_\pm} \omega_{\mathrm{f}} \sim \int_{\gamma_\pm} \frac{\phi(x)\, dx}{-q' a_N(x)}. \tag{2.4.6}$$

We claim that the integral on the right hand side takes the value ∓ 1. To prove this, we recall the relation

$$(x,y)\in\Omega \quad\Rightarrow\quad 0=f_\varepsilon(x,y)=a_{N-q'}y^{q'}+a_N+o(e^{q'\mathcal{N}(B)+\mathrm{val}(f_{N-q'})}).$$

Hence q' of the zeros of $a_N(x)$ satisfy the equation $(x,y)=(u_{k_1}e^B, o(e^{\mathcal{N}(B)}))$, which implies these points (x,y) are in the horn containing P_+. (There also exist q' points satisfying $x=u_{k_2}e^B, y=o(e^{\mathcal{N}(B)})$ in the horn containing P_-). The circles γ_+ and γ_- encircle these q' points respectively. (Figure 2.9). By definition of $\phi(x)$ (2.4.2), the residues of $-\phi(x)/(q'a_N(x))$ equal $\pm 1/q'$ at each pole. Therefore, by the residue theorem, we obtain

$$\int_{\gamma_+}\omega_{\mathrm{f}}=1+o(e^0),\qquad \int_{\gamma_-}\omega_{\mathrm{f}}=-1+o(e^0). \tag{2.4.7}$$

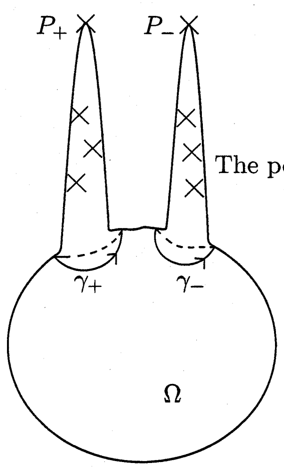


Figure 2.9: The sketch of Ω. The sphere Ω has finitely many horns. Each horn is associated with E_i.

(iii) Let $P\in(\{x=\infty,0\}\cup\{y=\infty,0\})\setminus\{P_+,P_-\}$. Defining a circle γ in the appropriate sub-surface, we can assume that γ surrounds one horn containing P. Let L' be a leaf and q'' be the vertical thickness of L'. The leaf L' :(a) contained in $G_N=\{Y=\mathcal{N}(X)\}$ and γ does not surround any pole of ω_{f} or (b) contained in $\{Y<\mathcal{N}(X)\}$. In case (b), the dominant term of f_ε is not a_N nor $a_{N-q'}y^{q'}$, then there exists positive δ such that $|a_N|<e^\delta|f_y|$. Therefore we have

$$\int_\gamma\omega_{\mathrm{f}}\sim\begin{cases}\int\{\phi(x)/(-q''a_N(x))\}\,dx & (L'\subset G_N)\\ o(e^0) & (\text{else})\end{cases}, \tag{2.4.8}$$

which yields $\int_\gamma\omega_{\mathrm{f}}=o(e^0)$ (cf. (2.4.3)).

i) $\int_{\alpha_i}\omega_{\mathrm{f}}=o(e^0)\quad\forall i,$

ii) $\omega_{\mathrm{f}}\in\mathcal{F}$.

i) Let E be the edge associated with α_i. The edge E :(a) contained in $G_N = \{Y = \mathcal{N}(X)\}$ and α_i does not surround any pole of ω_{f} or (b) contained in $\{Y < \mathcal{N}(X)\}$. A similar calculation to that in the proof of sublemma 1 iii) immediately leads to the evaluation: $\int_{\alpha_i} \omega_{\mathrm{f}} = o(\mathbf{e}^0)$.

ii) Let us decompose $\beta_i = \beta_i(\varepsilon)$ into finitely many parts. Denote one of these portions by $\gamma_\varepsilon(\theta)$. It is sufficient to prove the finiteness of the limit of integral for arbitrary simply connected region $\mathcal{U} \in \widetilde{V}(f)$ and arbitrary path $\gamma_\varepsilon(\theta) : (0,1) \times [0,1] \to \mathcal{U}$. Moreover, we can assume $j_\varepsilon^{-1}(\mathcal{U})$ is contained in some cylinder. Let E be the associated edge. If $E \not\subset G_N$, we can soon derive that $\int_{\gamma_\varepsilon} \omega_{\mathrm{f}} = o(\mathbf{e}^0)$ through a similar calculation to that in the proof of sublemma 1 iii). Let $E \subset G_N$. Denote the vertical thickness of E by q. Let $x = r_1\mathbf{e}^{X_1}$ be the x-coordinate of $\gamma_\varepsilon(0)$ and $x = r_2\mathbf{e}^{X_2}$ the x-coordinate of $\gamma_\varepsilon(1)$. Then,

$$\begin{aligned}
\int_{\gamma_\varepsilon} \omega &= \int_{r_1\mathbf{e}^{X_1}}^{r_2\mathbf{e}^{X_2}} \{\phi(x)/(-q\, a_N(x))\}\, dx + o(\mathbf{e}^0) \\
&= (-2q\pi\mathrm{i})^{-1} \int_{r_1\mathbf{e}^{X_1}}^{r_2\mathbf{e}^{X_2}} \{(x - u_{k_1}\mathbf{e}^B)^{-1} - (x - u_{k_2}\mathbf{e}^B)^{-1}\}\, dx + o(\mathbf{e}^0) \\
&= (-2q\pi\mathrm{i})^{-1} \log\left\{\frac{r_2\mathbf{e}^{X_2} - u_{k_1}\mathbf{e}^B}{r_1\mathbf{e}^{X_1} - u_{k_1}\mathbf{e}^B} \cdot \frac{r_1\mathbf{e}^{X_1} - u_{k_2}\mathbf{e}^B}{r_2\mathbf{e}^{X_2} - u_{k_2}\mathbf{e}^B}\right\} + o(\mathbf{e}^0) \\
&= (-2q\pi\mathrm{i})^{-1} \log\left\{\frac{r_1\mathbf{e}^{X_1} - u_{k_2}\mathbf{e}^B}{r_1\mathbf{e}^{X_1} - u_{k_1}\mathbf{e}^B} \cdot \frac{r_2\mathbf{e}^{X_2} - u_{k_1}\mathbf{e}^B}{r_2\mathbf{e}^{X_2} - u_{k_2}\mathbf{e}^B}\right\} + o(\mathbf{e}^0).
\end{aligned}$$

Recalling that $\mathbf{e} = e^{-1/\varepsilon}$, we conclude that the expression in the last line converges to the finite number

$$(-2q\pi\mathrm{i})^{-1},\ (-2q\pi\mathrm{i})^{-1}\log(u_{k_1}/u_{k_2}),\ \text{ or }\ (-2q\pi\mathrm{i})^{-1}\log(u_{k_2}/u_{k_1})$$

when $\varepsilon \to 0^+$.

We say a differential over Riemannian surface is *of first kind* if it has no singularity, *of second kind* if it has poles without residue and *of third kind* if it has poles with non-zero residue. In the final step of the proof of lemma 2.4.4, we use the Riemann bilinear relation [1]:

$$\begin{gathered}
\textstyle\sum_{i=1}^g (A_i'B_i - A_iB_i') = 2\pi\mathrm{i} \cdot \sum_j \mathrm{Res}_{P_j}(\omega^{(3)}) \cdot \int^{P_j} \omega^{(1)} \qquad (2.4.9)\\
A_i = \int_{\alpha_i} \omega^{(3)},\quad B_i = \int_{\beta_i} \omega^{(3)},\quad A_i' = \int_{\alpha_i} \omega^{(1)},\quad B_i' = \int_{\beta_i} \omega^{(1)} \\
\omega^{(3)} \text{ is of the third kind.} \quad \omega^{(1)} \text{ is of the first kind.}
\end{gathered}$$

Applying this formula for ω_{f} (of third kind) and ω_i (of first kind) $(i = 1, \ldots, g)$, we obtain

$$\int_{\beta_i} \omega_{\mathrm{f}} - o(\mathbf{e}^0) \cdot \textstyle\sum_{l=1}^g (\int_{\beta_l} \omega_i) = (1 + o(\mathbf{e}^0))(\int_{P_-}^{P_+} \omega_i). \qquad (2.4.10)$$

due to Sublemma 1 and 2 ($\because A_i' = \delta_{i,j},\ A_i = o(\mathbf{e}^0)$). On the other hand, applying the formula for $\omega_{P_+-P_-}$ and ω_i, we obtain

$$\int_{\beta_i} \omega_{P_+-P_-} = \int_{P_-}^{P_+} \omega_i. \qquad (2.4.11)$$

Thus, we derive

$$\int_{\beta_i} (\omega_{\mathrm{f}} - \{1 + o(\mathbf{e}^0)\} \cdot \omega_{P_+-P_-}) = o(\mathbf{e}^0) \times \textstyle\sum_{l=1}^g (\int_{\beta_l} \omega_i) \quad (\forall i),$$

which implies $\omega_{\mathrm{f}} - \{1 + o(\mathbf{e}^0)\} \cdot \omega_{P_+-P_-} \in \mathcal{M}$. This relation can be rewritten as $(1 + o(\mathbf{e}^0))\omega_{P_+-P_-} \in \mathcal{M} + \mathcal{F}$, or $\omega_{P_+-P_-} \in \mathcal{M} + \mathcal{F}$.

Differentials associated with edges

Next we consider *differentials associated with edges.* Let $E' \subset \operatorname{Trop} C$ be an edge of multiplicity m. By rotation, it can be assumed that E' is vertical without loss of generality. Denote the horizontal thickness of the vertical edge E' by w.

Let $E' = \{(B, (1-t)Y_0 + tY_1) \,|\, 0 \le t \le 1, Y_0 < Y_1\}$ and

$$\{\text{the ceiling of } E'\} \subset G_{I+1} = \{Y = \mathcal{N}_{I+1}(X)\}, \quad \{\text{the floor of } E'\} \subset G_I = \{Y = \mathcal{N}_I(X)\}.$$

By definition, it follows that $Y_1 = \mathcal{N}_{I+1}(B)$ and $Y_0 = \mathcal{N}_I(B)$: Because E' is of finite length, it follows that $1 \le I \le N-1$. The defining polynomial $f_\varepsilon(x,y)$ of C_ε is of the form

$$f_\varepsilon(x,y) = \textstyle\sum_{i=0}^N a_i(x) y^{N-i}, \quad a_I(x) = c e^A x^n \prod_j (x - u_j e^{B_j}),$$

where $c, u_j \in R^\times$, $A, B_j \in \mathbb{Q}_{>0}$, $n \in \mathbb{N}$. We rewrite the polynomial $a_I(x)$ as

$$a_I(x) = c e^A x^n \textstyle\prod_{j=1}^w (x - u_j e^B) \cdot \prod_{j>w} (x - u_j e^{B_j}),$$

where w is the horizontal thickness of E' and $B_j \neq B$ $(j > w)$. Let $E' = E_1 \amalg \cdots \amalg E_w$ be the decomposition into edges of multiplicity one. Note that the horizontal thickness of E_i equals 1.

We define *the differential associated with the edge* $E \equiv E_1$ by:

$$\omega_E := \frac{\phi(x) \cdot y^{N-I-1}\, dx}{f_y}, \qquad \frac{\phi(x)}{a_I(x)} := \frac{1}{2\pi \mathrm{i}} \frac{-1}{(x - u_1 e^B)}. \tag{2.4.12}$$

Let $(x,y) = (re^X, se^Y)$ be a point on C_ε. Assume $(X,Y) \in G_J = \{Y = \mathcal{N}_J(X)\}$ on the edge of vertical thickness q. If $J = I, I+1$, we can use the following estimation:

$$\begin{aligned} y f_y &= \begin{cases} (N-I+q)\, a_{I-q} y^{N-I+q} + (N-I)\, a_I y^{N-I} + \cdots & (J = I) \\ (N-I-q)\, a_{I+q} y^{N-I-q} + (N-I)\, a_I y^{N-I} + \cdots & (J = I+1) \end{cases} \\ &= \begin{cases} -q\, a_I y^{N-I} + \cdots & (J = I) \\ q\, a_I y^{N-I} + \cdots & (J = I+1) \end{cases}. \end{aligned}$$

This implies the following:

$$\omega_E \sim \begin{cases} \{-\phi(x)/q\, a_I\}\, dx & \text{if } (X,Y) \in G_I \\ \{\phi(x)/q\, a_I\}\, dx & \text{if } (X,Y) \in G_{I+1} \end{cases}. \tag{2.4.13}$$

If $J \neq I, I+1$, f_I cannot be a dominant term of yP_y, and it follows that

$$\omega_E = o(e^0)\, dx \qquad \text{if} \qquad (X,Y) \in G_J \ (J \neq I, I+1). \tag{2.4.14}$$

Next we study the estimation of ω_E on vertical edges. Let $M \subset \operatorname{Trop} C$ be a vertical edge whose ceiling is contained in G_{J+1} and whose floor is contained in G_J. Let us rewrite the expression of ω_E into

$$\omega_E = -\phi(x) y^{N-I-1} \frac{dy}{f_x}, \quad \phi(x) = \frac{-1}{2\pi \mathrm{i}} \cdot c e^A x^n \textstyle\prod_{j \neq 1} (x - u_j e^{B_j}).$$

($\because \frac{dx}{f_y} = -\frac{dy}{f_x}$ for smooth curve C_ε).

Due to the definition of M, the dominant term of $f_\varepsilon(x,y)$ on

$$\{(x,y) \,|\, (\operatorname{val}(x), \operatorname{val}(y)) \in M\}$$

is $a_J\, y^{N-J}$. We claim that the dominant term of f_x is $a'_J\, y^{N-J}$, where $a'_i(x) := \frac{d}{dx} a_i(x)$ and $f_x = \sum_i a'_i y^{N-i}$.

Lemma 2.4.5 *The dominant term of f_x on M is $a'_J\, y^{N-J}$.*

Proof. We consider the difference $\mathrm{val}\,(a_i) - \mathrm{val}\,(a'_i)$ $(1 \le i \le N)$. Let

$$M = \{(B_0, (1-t)Y_2 + tY_3) \,|\, 0 \le t \le 1, Y_2 < Y_3\},$$

and $a_i = c\mathbf{e}^{A_i} x^{n_i} \prod_j (x - u_{i,j}\mathbf{e}^{B_{i,j}})$. Then, a'_i is of the form:

$$a'_i = cn_i\mathbf{e}^{A_i} x^{n_i - 1} \textstyle\prod_j (x - u_{i,j}\mathbf{e}^{B_{i,j}}) + c\mathbf{e}^{A_i} x^{n_i} \sum_k \prod_{j \ne k} (x - u_{i,j}\mathbf{e}^{B_{i,j}}).$$

From this expression we see that

$$\mathrm{val}\,(a_i(x)) - \mathrm{val}\,(a'_i(x)) \le \mathrm{val}\,(x). \tag{2.4.15}$$

On the other hand, if $i = J$, $a_J(x)$ is of the form

$$a_J = c_J\mathbf{e}^{A_J} x^{n_J} \textstyle\prod_j (x - u_{J,j}\mathbf{e}^{B_{J,j}}), \quad \sharp\{j \,|\, B_0 = B_{J,j}\} > 0.$$

Let $\Lambda := \{j \,|\, B_0 = B_{J,j}\}$. We can assume $\Lambda = \{1, 2, \ldots, w'\}$ by exchanging the indices if necessary. Again we consider the derivative a'_J. Among the factors

$$x - u_{J,1}\mathbf{e}^{B_0},\ x - u_{J,2}\mathbf{e}^{B_0},\ \ldots,\ x - u_{J,w'}\mathbf{e}^{B_0},$$

$(x - u_{J,k}\mathbf{e}^{B_0})$ is the only one that becomes very small when we take $x = u_{J,k}\mathbf{e}^{B_0} + o(\mathbf{e}^{B_0})$ because of the genericness condition in section 2.3. Thus, for fixed $k \in \Lambda$, it follows that

$$x = u_{J,k}\mathbf{e}^{B_0} + o(\mathbf{e}^{B_0}) \ \Rightarrow\ a'_J(x) \sim c_J\mathbf{e}^{A_J} x^{n_J} \prod_{j \ne k} (x - u_{J,j}\mathbf{e}^{B_{J,j}}) \tag{2.4.16}$$

which implies

$$\mathrm{val}\,(a_J(x)) - \mathrm{val}\,(a'_J(x)) = B_0 = \mathrm{val}\,(x). \tag{2.4.17}$$

Therefore, (2.4.15) and (2.4.17) give rise to

$$\begin{aligned}
&(\mathrm{val}\,(x), \mathrm{val}\,(y)) \in M \ \Rightarrow\ \left\{ \begin{array}{l} x = u_{J,k}\mathbf{e}^{B_0} + o(\mathbf{e}^{B_0}) \quad (\exists k) \\ \mathrm{val}\,(a_J y^{N-J}) \le \mathrm{val}\,(a_i y^{N-i}) \end{array} \right. \\
&\Rightarrow\ \mathrm{val}\,(a'_J y^{N-J}) - \mathrm{val}\,(a'_i y^{N-i}) \\
&\qquad \le -\mathrm{val}\,(x) + \mathrm{val}\,(a'_J y^{N-J}) + \mathrm{val}\,(x) - \mathrm{val}\,(a'_i y^{N-i}) \le 0
\end{aligned}$$

for each $i = 0, 1, \ldots, N$. In particular, we can conclude that $f_x \sim a'_J y^{N-J}$ on M.

Recall that the floor of E is contained in G_I, and that the floor of M is contained in G_J. From the explicit form of a'_J given in (2.4.16) and the definition of $\phi(x)$ (2.4.12), we obtain

$$\begin{aligned}
&- \phi(x) y^{N-J} / f_x\big|_{(\mathrm{val}\,(x), \mathrm{val}\,(y)) \in M} \\
&= \left\{ \begin{array}{ll} (2\pi \mathrm{i})^{-1} + o(\mathbf{e}^0) & (J = I \text{ and } x = u_1\mathbf{e}^B + o(\mathbf{e}^B)) \\ o(\mathbf{e}^0) & (J = I \text{ and } x = u_j\mathbf{e}^B + o(\mathbf{e}^B),\ j > 1) \\ o(\mathbf{e}^0) & (J \ne I) \end{array} \right. .
\end{aligned}$$

These relations lead to the following:

$$\omega_E \sim \left\{ \begin{array}{ll} (2\pi \mathrm{i})^{-1} (dy/y) & (\mathrm{val}\,(x), \mathrm{val}\,(y)) \in E \\ o(\mathbf{e}^0)\, dx & (\mathrm{val}\,(x), \mathrm{val}\,(y)) \in M \ne E \end{array} \right. . \tag{2.4.18}$$

The three equations (2.4.13), (2.4.14) and (2.4.18) gives us the singularities of ω_E. Let L_1 be the leftmost leaf of G_I and L_2 be the leftmost leaf of G_{I+1}. Denote the vertical thickness of L_i $(i = 1, 2)$ by q_i and the multiplicity by m_i, respectively. Let us consider the decomposition $L_i = L_{i,1} \amalg L_{i,2} \amalg \cdots \amalg L_{i,m_i}$ and denote the vertical thickness of $L_{i,j}$ by $q_{i,j}$ $(q_i = q_{i,1} + \cdots + q_{i,m_i})$, the horn associated with $L_{i,j}$ by $\Sigma_{i,j}$, the vertex that is the end point of L_i by v_i and the sphere associated with v_i by Ω_i.

The set $(\{x = \infty, 0\} \cup \{y = \infty, 0\}) \cap \Sigma_{i,j}$ has only one element, which we denote by $P_{i,j}$. Consider cycles $\gamma_{i,j} \in \Omega_i$ which loop around the point $P_{i,j}$ anti-clockwise. By (2.4.13) it then follows that

$$\int_{\gamma_{1,j}} \omega_E = +(q_{1,j}/q_1) + o(e^0), \tag{2.4.19}$$

$$\int_{\gamma_{2,j}} \omega_E = -(q_{2,j}/q_2) + o(e^0). \tag{2.4.20}$$

Hence, we obtain the following:

Proposition 2.4.6 *The differential ω_E has a pole with residue $+q_{1,j}/(2q_1\pi \mathrm{i})$ at $P_{1,j}$ and a pole with residue $-q_{2,j}/(2q_2\pi \mathrm{i})$ at $P_{2,j}$.*

Let $\gamma \in H_1(C_\varepsilon; \mathbb{Z})$ be a cycle which loops a cylinder Σ. We fix the direction of γ as figure 2.10. The integral $\int_\gamma \omega_E$ takes various values depending on the position of Σ in C_ε. Let M

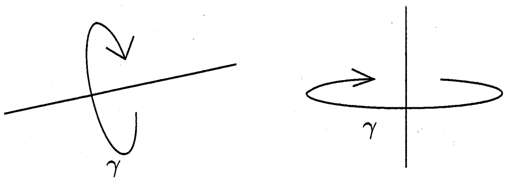


Figure 2.10: We fix the direction of γ as these figures. The figure on the left shows the direction of γ for non-vertical Σ, and the one on the right shows that for vertical Σ.

be a tropical edge associated with a cylinder Σ.

(i) The case $M = E$.

When one runs around a cylinder Σ, the y-coordinate runs around the origin. Then, by (2.4.18), we derive

$$\begin{aligned} \int_\gamma \omega_E &= \oint_{0+)} (2\pi \mathrm{i})^{-1} \frac{dy}{y} + o(e^0) = \int_0^{2\pi} (2\pi \mathrm{i})^{-1} \mathrm{i}\, d\theta + o(e^0) \\ &= 1 + o(e^0). \end{aligned} \tag{2.4.21}$$

(ii) In case M is vertical and $M \neq E$, it follows that $\int_\gamma \omega_E = o(e^0)$.

(iii) When $M \subset G_I \setminus G_{I+1}$, we consider the edge $M' = M_1 \amalg M_2 \amalg \cdots \amalg M_m$ $(M = M_1)$ and denote the vertical thickness of M_i by q_i $(q_i = q/m)$. From (2.4.3) and (2.4.13) it follows that

$$\int_\gamma \omega_E = \begin{cases} q_1/q + o(e^0) = 1/m + o(e^0) & (M \subset \{X < B\}) \\ o(e^0) & (M \subset \{X > B\}) \end{cases}. \tag{2.4.22}$$

We used the assumption that C has a good tropicalization for the first equality.

(iv) When $M \subset G_{I+1} \setminus G_I$, one has that

$$\int_\gamma \omega_E = \begin{cases} -1/m + o(e^0) & (M \subset \{X < B\}) \\ o(e^0) & (M \subset \{X > B\}) \end{cases}. \tag{2.4.23}$$

(v) When $M \subset G_i$ $(i \neq I, I+1)$, one has that $\int_\gamma \omega_E = o(e^0)$.

The remaining case is the degenerate case: $G_I \cap G_{I+1} \neq \emptyset$. (Figure 2.11)

(vi) The case $M \subset G_I \cap G_{I+1}$.
Since $a_I y^{N-I}$ is not a dominant term in f_y, it follows that

$$\int_\gamma \omega_E = o(e^0). \tag{2.4.24}$$

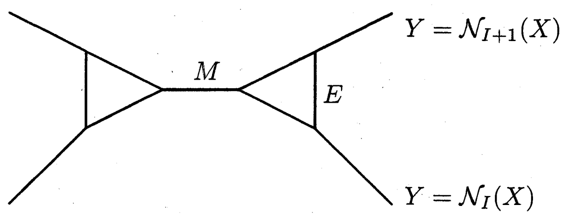


Figure 2.11: Two subsets $G_I = \{Y = \mathcal{N}_I(X)\}$ and $G_{I+1} = \{Y = \mathcal{N}_{I+1}(X)\}$ may have intersection. The edge M is in the intersection.

Now we proceed for the β-cycle of ω_E. For this, we first calculate the integrals along the cylinders. Let $\Sigma \subset C_\varepsilon$ be a cylinder. Take a path ρ which runs along Σ. We calculate the integral $\int_\rho \omega_E$ by using (2.4.13) and (2.4.18).

Lemma 2.4.7 *Let $\mathfrak{I} := \int_\rho \omega_E$. ($\rho$ runs along Σ).*

1. *If Σ is associated with $E = \{(B, (1-t)Y_0 + tY_1) \,|\, 0 \leq t \leq 1, Y_0 < Y_1\}$, then $\mathfrak{I} = -(2\pi i \varepsilon)^{-1}(Y_1 - Y_0) + o(e^0)$.*
2. *If Σ is associated with a vertical edge except E, $\mathfrak{I} = o(e^0)$.*
3. *If Σ is associated with a non-vertical edge L of multiplicity m and of vertical thickness q in G_I:*

   $$L = \{((1-t)X_0 + tX_1, (1-t)Y_0 + tY_1) \,|\, 0 \leq t \leq 1, X_0 < X_1, Y_0 < Y_1\} \subset G_I,$$

   then $\mathfrak{I} = -(2q\pi i \varepsilon)^{-1}(\min[B, X_1] - \min[B, X_0]) + o(e^0)$.
4. *If Σ is associated with a non-vertical edge L of multiplicity m and of vertical thickness q in G_{I+1}:*

   $$L = \{((1-t)X_0 + tX_1, (1-t)Y_0 + tY_1) \,|\, 0 \leq t \leq 1, X_0 < X_1, Y_0 < Y_1\} \subset G_{I+1},$$

   then $\mathfrak{I} = +(2q\pi i \varepsilon)^{-1}(\min[B, X_1] - \min[B, X_0]) + o(e^0)$.
5. *It Σ is associated with a non-vertical edge in G_J $(J \neq I, I+1)$, then $\mathfrak{I} = o(e^0)$.*

6. If Σ is associated with an edge in $G_I \cap G_{I+1}$, then $\mathfrak{I} = o(\mathrm{e}^0)$.

Proof. 1. From (2.4.18) it follows that

$$\begin{aligned}\mathfrak{I} &\sim (2\pi \mathrm{i})^{-1} \cdot \int_{s_0 \mathrm{e}^{Y_0}}^{s_1 \mathrm{e}^{Y_1}} (dy/y) = (2\pi \mathrm{i})^{-1} \log \{(s_1/s_0)\mathrm{e}^{Y_1 - Y_0}\} \\ &= -(2\pi \mathrm{i}\varepsilon)^{-1}(Y_1 - Y_0) + \cdots .\end{aligned}$$

2. This can be obtained from (2.4.18).
3. From (2.4.13),

$$\begin{aligned}\mathfrak{I} &\sim (2\pi \mathrm{i})^{-1} \cdot \int_{r_0 \mathrm{e}^{X_0}}^{r_1 \mathrm{e}^{X_1}} \{1/q(x - u_{j_0}\mathrm{e}^B)\}dx \\ &= (2q\pi \mathrm{i})^{-1} \log \{(r_1 \mathrm{e}^{X_1} - u_{j_0}\mathrm{e}^B)/(r_0 \mathrm{e}^{X_0} - u_{j_0}\mathrm{e}^B)\} \\ &= -(2q\pi \mathrm{i}\varepsilon)^{-1}(\min[B, X_1] - \min[B, X_0]) + \cdots .\end{aligned}$$

4. This can be obtained in the same way as the case 3.
5. This follows from (2.4.14).
6. This follows from (2.4.24).

The result of lemma 2.4.7 is easily understood by means of the tropical bilinear form. Let $\Gamma_E \subset \operatorname{Trop} C$ be a path with direction defined by the following route:

$$(X = -\infty) \overset{\text{on } G_I}{\to} (X = B, Y = Y_0) \overset{\text{on } E}{\to} (X = B, Y = Y_1) \overset{\text{on } G_{I+1}}{\to} (X = -\infty).$$

Using lemma 2.4.1, we can restate the claim of lemma 2.4.7 as:

$$\mathfrak{I} = \int_\rho \omega_E = -(2\pi \mathrm{i}\varepsilon)^{-1} \cdot m_L^{-1} \cdot \ell_T(L, \Gamma_E), \tag{2.4.25}$$

where m_L is the multiplicity of L. Recall that the bilinear form $\ell_T(\cdot, \cdot)$ gives the tropical length with signature of intersection.

In fact, the equations (2.4.21–2.4.24) can be restated by means of the *intersection number*. Let $\gamma \subset C_\varepsilon$ be a closed path surrounding some cylinder and $E \subset \operatorname{Trop} C$ be a directed edge. Denote the cylinder associated with E by Σ_E. And define the intersection number $(\gamma \circ E)$ by

$$(\gamma \circ E) := \begin{cases} +1 & (\gamma \text{ surrounds the cylinder } \Sigma_E \text{ by positive direction}) \\ -1 & (\gamma \text{ surrounds the cylinder } \Sigma_E \text{ by negative direction}) \\ 0 & (\text{else}) \end{cases} .$$

(Cf. figure 2.12).

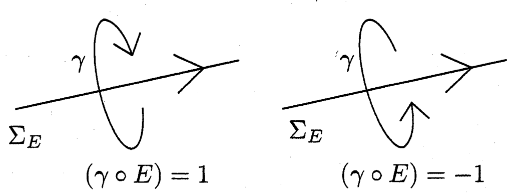


Figure 2.12: The definition of the intersection number $(\gamma \circ E)$.

When $\gamma \subset C_\varepsilon$ is a closed path loops a cylinder Σ, it follows that

$$\int_\gamma \omega_E = m^{-1}(\gamma \circ \Gamma_E) + o(\mathrm{e}^0), \tag{2.4.26}$$

where m is the multiplicity of the edge associated with Σ.

poles without residue

The differential ω_E also has poles without residue. However, we can neglect the influence of these poles. In fact, by (2.4.21–2.4.24), we can derive $\int_\gamma x^k \omega_E = o(\mathrm{e}^0)$ $(k \le -1, \gamma \subset \{\mathrm{val}\,(x) \ll 0\})$, which implies

$$\omega_E = \left(\frac{c_{-n}}{z^n} + \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots\right) dz \quad \text{at } z \in \{x = \infty\}$$
$$\Rightarrow \; c_k = o(\mathrm{e}^0) \quad (k < -1).$$

A modified differential

Let $E = E_1 \amalg \cdots \amalg E_m$ be an edge of multiplicity m. Define the differential $v := \omega_{E_i} - \omega_{E_j}$, where ω_{E_i} is the differential associated with the edge E_i. Then, v satisfies the followings:
(i) v has no pole with residue,
(ii) For a closed path γ surrounding a cylinder Σ in C_ε, one has that

$$\int_\gamma v = (\gamma \circ (E_i - E_j)) + o(\mathrm{e}^0).$$

(iii) For a path $\rho \subset C_\varepsilon$ which runs along a cylinder Σ, it follows that

$$\int_\rho v = -(2\pi \mathrm{i} \varepsilon)^{-1} \cdot \ell_T(L, E_i - E_j),$$

where L is the edge which is associated with Σ and which is of multiplicity one.

Now we define a new differential which is a modification of ω_E. Let $\Gamma_E = E \amalg E^{(1)} \amalg E^{(2)} \amalg \cdots \amalg E^{(n)}$ be the decomposition into edges. We decompose each $E^{(i)}$ into edges of multiplicity one: $E^{(i)} = E^{(i)}_1 \amalg \cdots \amalg E^{(i)}_{m_i}$, where m_i is the multiplicity of $E^{(i)}$.

Now we define a new modified differential associated with E. Let

$$\widetilde{\omega}_E := \omega_E + v^{(1)} + v^{(2)} + \cdots + v^{(n)},$$
$$v^{(i)} = m_i^{-1} \left\{ (\omega_{E^{(i)}_1} - \omega_{E^{(i)}_2}) + (\omega_{E^{(i)}_1} - \omega_{E^{(i)}_3}) + \cdots + (\omega_{E^{(i)}_1} - \omega_{E^{(i)}_{m_i}}) \right\}.$$

For the path $\widetilde{\Gamma}_E$ which is defined by

$$\widetilde{\Gamma}_E := E \amalg E^{(1)}_1 \amalg E^{(2)}_1 \amalg \cdots \amalg E^{(n)}_1,$$

we can rewrite the equation (2.4.25) into

$$\int_\rho \widetilde{\omega}_E = -(2\pi \mathrm{i} \varepsilon)^{-1} \ell_T(L, \widetilde{\Gamma}_E), \quad (\rho \text{ runs along } L), \tag{2.4.27}$$

and we also rewrite the equation (2.4.26) into

$$\int_\gamma \widetilde{\omega}_E = (\gamma \circ \widetilde{\Gamma}_E) + o(\mathrm{e}^0). \tag{2.4.28}$$

Proof of the theorem

We have finished all the preparations necessary to complete the proof of the main theorem. Let $\mathfrak{X}$ be a set of edges contained in $\mathrm{Trop}\, C$. Denote the free additive abelian group which is generated by the elements of $\mathfrak{X}$ by $\mathbb{Z}_{\mathfrak{X}}$.

For a closed path Γ contained in $\mathrm{Trop}\, C$, choose edges $E_1, \ldots, E_n; F_1, \ldots, F_m$ of multiplicity one such that $\widetilde{\Gamma}_{E_1} + \cdots + \widetilde{\Gamma}_{E_n} - \widetilde{\Gamma}_{F_1} - \cdots - \widetilde{\Gamma}_{F_m} = \Gamma \in \mathbb{Z}_{\mathfrak{X}}$. Define the differential

$$\omega'_\Gamma := \widetilde{\omega}_{E_1} + \cdots + \widetilde{\omega}_{E_n} - \widetilde{\omega}_{F_1} - \cdots - \widetilde{\omega}_{F_m}.$$

By prop 2.4.6, (2.4.27) and (2.4.28), ω'_E has the following properties:

(i) ω'_E has singularities in $\{x = \infty, 0\} \cup \{y = \infty, 0\}$.

(ii) Let $P_1, \ldots, P_q$ be points in $\{x = \infty, 0\} \cup \{y = \infty, 0\}$ and suppose that these are associated with the same leaf of $\operatorname{Trop} C$. Then, $\sum_{i=1}^{q} \operatorname{Res}_{P_i}(\omega'_\Gamma) = 0$.

(iii) Let α be a closed path surrounding a cylinder Σ which is associated with the edge $E \subset \operatorname{Trop} C$. Then,

$$\int_\alpha \omega'_\Gamma = (\alpha \circ \Gamma) + o(e^0). \tag{2.4.29}$$

For a leaf Γ_∞ of infinite length in $\operatorname{Trop} C$, we define the differential ω_{Γ_∞} by:

$$\omega_{\Gamma_\infty} := \omega_{\sum_i \operatorname{Res}_{P_i}(\omega'_\Gamma)\cdot P_i},$$

where $\omega_{c_1 P_1 + \cdots + c_n P_n}$ is the normalised differential of the third kind. Summing these differentials of the third kind for all edges of infinite length:

$$\omega_\infty := \sum_{|\Gamma_\infty| = \infty} \omega_{\Gamma_\infty}.$$

Then, the new differential $\omega_\Gamma := \omega'_\Gamma - \omega_\infty$ satisfies: i) $\int_\alpha \omega_\Gamma = \int_\alpha \omega'_\Gamma$ and ii) ω_Γ has no singularity with non-zero residue (ω_Γ is of the second kind). Moreover, by adding the normalised differentials of the second kind, we can assume ω_Γ is of first kind. (Recall that we can neglect the poles without residue.)

Proof of the theorem. Let $\Gamma = T_{\beta_i}$, that is the closed path on $\operatorname{Trop} C$ associated with the β-cycle β_i on C_ε. By (2.4.29), it follows that

$$\int_{\alpha_j} \omega_{T_{\beta_i}} = \int_{\alpha_j} \omega'_{T_{\beta_i}} = \begin{cases} 1 + o(e^0) & (i = j) \\ o(e^0) & (i \neq j) \end{cases}.$$

We apply the Riemann bilinear relation (2.4.9) for $\omega_{T_{\beta_i}}$ and ω_j (recall ω_j is the j-th normalised holomorphic differential), to obtain

$$\int_{\beta_i} \omega_j + o(e^0) \textstyle\sum_k \int_{\beta_k} \omega_j = \int_{\beta_j} \omega_{T_{\beta_i}}, \tag{2.4.30}$$

which yields $\omega_i - \omega_{T_{\beta_i}} = \omega_i - \omega'_{T_{\beta_i}} + \omega_\infty \in \mathcal{M}$. (Because of $\int_{\beta_i} \omega_j = \int_{\beta_j} \omega_i$ [1]). Due to lemma 2.4.4 we obtain

$$\omega_i - \omega'_{T_{\beta_i}} \in \mathcal{M} + \mathcal{F}. \tag{2.4.31}$$

Consider $g \times g$ matrices $B = (\int_{\beta_j} \omega_i)_{i,j}$ and $B' = (\int_{\beta_j} \omega'_{T_{\beta_i}})_{i,j}$. The equation (2.4.31) can be rewritten as

$$B - B' = o(e^0)B + B^\dagger \qquad \lim_{\varepsilon \to 0^+} |B^\dagger| < \infty, \tag{2.4.32}$$

where I is the identity matrix and Δ is a $g \times g$ matrix. On the other hand, from (2.4.25) one concludes that B' goes to infinity when $\varepsilon \to 0^+$. We thus obtain

$$B \sim B' \qquad (\varepsilon \to 0^+), \tag{2.4.33}$$

by taking a limit $\varepsilon \to 0^+$ of (2.4.32).

To conclude the proof of theorem, it is sufficient to prove that $\int_{\beta_j} \omega'_{T_{\beta_i}} = -(2\pi i \varepsilon)^{-1} \cdot \ell(T_{\beta_j}, T_{\beta_i})$, which is a mere linear combination of (2.4.27). ∎

2.5 Genericness Condition

In this paper, we introduced some conditions on C_ε and $\operatorname{Trop} C$ to make the problem easier. Let $f_\varepsilon(x,y) = \sum_i a_i(x)y^{N-i}$ be the defining polynomial of C_ε, where

$$a_i(x) = c_i e^{A_i} x^{m_i} \prod_{j=1}^{d_i} (x - u_{i,j}e^{B_{i,j}}), \quad c_i, u_{i,j} \in R^\times,\ A_i, B_{i,j} \in \mathbb{Q},\ m_i \in \mathbb{N}.$$

Let $\theta \in SL_2(\mathbb{Z})$ be a rotation of $\operatorname{Trop} C$. The translation θ naturally acts on C_ε by $x \mapsto x^\delta y^{-\beta}$; $y \mapsto x^{-\gamma}y^\alpha$. Define the new polynomial

$$f^\theta(x,y) = f_\varepsilon(x^\delta y^{-\beta}, x^{-\gamma}y^\alpha) = \textstyle\sum_i a_i^\theta(x) y^{N'-i}.$$
$$a_i^\theta(x) = c_i^\theta e^{A_i^\theta} x^{m_i^\theta} \textstyle\prod_j (x - u_{i,j}^\theta e^{B_{i,j}^\theta}).$$

To be precise, we assumed three conditions:

Genericness condition. For fixed $\theta \in SL_2(\mathbb{Z})$, $\operatorname{top}(u_{i,j}^\theta) \in \mathbb{C}\setminus\{0\}$ $(\forall i,j)$ are all distinct.

Condition I. For each edge E, $m = \text{g.c.d.}(q,w)$, where m, q, w respectively are the multiplicity, the vertical thickness and the horizontal thickness of E.

Condition II. For each edge $E = E_1 \amalg \cdots \amalg E_m$, $q_1 = \cdots = q_m$, $w_1 = \cdots = w_m$, where q_i, w_i respectively are the vertical thickness and the horizontal thickness of E.

The following relation exists between these conditions.

Proposition 2.5.1 *Genericness condition $\Rightarrow$ Condition II $\Rightarrow$ Condition I.*

Proof. (Genericness cond. $\Rightarrow$ Cond. II) We first prove the case when $E = E_1 \amalg \cdots \amalg E_m$ is vertical. Then it is clear that $q_1 = \cdots = q_m = 0$. The defining equation of vertical edge E is of the form $(a+w)X + bY + c = aX + bY + c'$. When we substitute $(x,y) = (re^X, se^Y)$, $(X,Y) \in E$ into $f_\varepsilon(x,y)$, the polynomial $a_{N-b}(x)y^b$ is dominant. Moreover, we can derive the relation $\operatorname{top}(a_{N-b}(x)) = 0$, which gives us:

$$\operatorname{top}(r - u_{i,j_1})(r - u_{i,j_2}) \dots (r - u_{i,j_w}) = 0, \quad j = N-b,\ j_k \in \{j \mid B_{i,j} = (c-c')/w\}.$$

By the genericness condition, this equation implies that w distinct cylinders in C are associated with the edge E. Then $m = w$, which implies $w_1 = w_2 = \cdots = w_m = 1$.

In the general case, we consider $\theta \in SL_2(\mathbb{Z})$ such that $\theta \cdot E = (\theta \cdot E_1) \amalg \cdots \amalg (\theta \cdot E_m)$ is vertical. In fact, the vertical thickness and the horizontal thickness of E_i satisfy the relation

$$q_i = \alpha q_i^\theta + \beta w_i^\theta, \qquad q_i = \gamma q_i^\theta + \delta w_i^\theta, \tag{2.5.1}$$

where q_i^θ and w_i^θ are the vertical thickness and the horizontal thickness of $\theta \cdot E_i$. This equation implies $q_1 = \cdots = q_m$ and $w_1 = \cdots = w_m$.

(Cond. II $\Rightarrow$ Cond. I) From the equation $w_1 = \cdots = w_m = 1$ for the vertical edge $E = E_1 \amalg \cdots \amalg E_m$ we conclude that it is enough to prove that $\text{g.c.d.}(q,w)$ is invariant under the rotation θ. This fact follows immediately from (2.5.1).

Due to the above proposition, we can claim that the only essential assumption for our theory is the genericness condition.

Chapter 3

Reduced Discrete KP Equations

3.1 Introduction

We start with the discrete KP equation:

$$-\delta f^{t+1}_{m,n} f^{t}_{m+1,n+1} + (1+\delta) f^{t}_{m+1,n} f^{t+1}_{m,n+1} - f^{t}_{m,n+1} f^{t+1}_{m+1,n} = 0. \tag{3.1.1}$$

Multiplying $\dfrac{f^{t+1}_{m+1,n+1}}{f^{t}_{m+1,n+1} f^{t+1}_{m+1,n} f^{t+1}_{m,n+1}}$, we have

$$-\delta \frac{f^{t+1}_{m,n} f^{t+1}_{m+1,n+1}}{f^{t+1}_{m+1,n} f^{t+1}_{m,n+1}} + (1+\delta) \frac{f^{t}_{m+1,n} f^{t+1}_{m+1,n+1}}{f^{t}_{m+1,n+1} f^{t+1}_{m+1,n}} - \frac{f^{t}_{m,n+1} f^{t+1}_{m+1,n+1}}{f^{t}_{m+1,n+1} f^{t+1}_{m,n+1}} = 0. \tag{3.1.2}$$

Let $I^t_{m,n} := (1+\delta) \dfrac{f^{t}_{m+1,n} f^{t+1}_{m+1,n+1}}{f^{t}_{m+1,n+1} f^{t+1}_{m+1,n}}$ and $V^t_{m,n} := \delta \dfrac{f^{t}_{m,n} f^{t}_{m+1,n+1}}{f^{t}_{m+1,n} f^{t}_{m,n+1}}$. Then (3.1.2) becomes

$$I^t_{m-1,n} + V^{t+1}_{m,n-1} = I^t_{m,n-1} + V^t_{m,n}, \tag{3.1.3}$$

$$I^t_{m-1,n} V^{t+1}_{m,n} = I^t_{m,n} V^t_{m,n}. \tag{3.1.4}$$

In this article, we study the discrete system (3.1.3–3.1.4) with the periodic boundary condition:

$$I^t_{n+N} \equiv I^t_n, \qquad V^t_{n+N} \equiv V^t_n. \tag{3.1.5}$$

Let us consider the following extra condition:

$$f^t_{m,n} = f^{t-D}_{m-E,n-F}, \qquad D, E \in \mathbb{Z} \setminus \{0\},\ F \in \mathbb{Z}. \tag{3.1.6}$$

Many researchers have been studying the discrete KP equation with the reduction condition (3.1.6). We call the system (3.1.3–3.1.5) with (3.1.6) *the reduced periodic discrete KP equation* (rpdKP). Because the discrete KP equation (3.1.1) is invariant under the transformation $t \mapsto -t$, $m \mapsto -m$, $n \mapsto -n$, we can assume $D > 0$ without loss of generality. Moreover, we can also assume $0 < E + D + F \leq N$ by virtue of the periodicity (3.1.5).

The rpdKP system is a generalisation of many periodic discrete integrable systems. In fact, this system contains the *periodic discrete Toda equation* (pdToda ; $D = E = 1, F = -1$), *pdKdV* ($D = E = 1, F = 0$), *hungry pdToda* ($E = 1, F = -1$), *etc.*

The author have studied the case of $E = 1$, $F = -1$ in [4, 5] and $E > 0$, $F = 0$ in [6]. This paper is a continuation of these studies.

Important remark (i): One would find that the discrete system (3.1.3–3.1.6) is decomposed into independent g.c.d.(D,E) systems. By changing variables, we can assume g.c.d.$(D,E)=1$ without loss of generality.
(ii):Aside from this, in this paper, we only consider the case

$$\text{g.c.d.}(D+E+F,N)=1.$$

One would find that this assumption make the problem easier well. For general cases, we should consider some limiting procedure. See [6], §3.

3.2 Inverse Scattering Method

The rpdKP system (3.1.3–3.1.5) is equivalent to the following matrix form:

$$L_m^{t+1}(y)R_{m-1}^t(y)=R_m^t(y)L_m^t(y), \tag{3.2.1}$$

where

$$L_m^t(y)=\begin{pmatrix} V_{m,1}^t & 1 & & \\ & V_{m,2}^t & \ddots & \\ & & \ddots & 1 \\ y & & & V_{m,N}^t \end{pmatrix},\quad R_m^t(y)=\begin{pmatrix} I_{m,1}^t & 1 & & \\ & I_{m,2}^t & \ddots & \\ & & \ddots & 1 \\ y & & & I_{m,N}^t \end{pmatrix},$$

and y is a complex parameter. On the other hand, the equation (3.1.6) is equivalent to

$$L_m^t=S^{-F}L_{m-E}^{t-D}S^F,\qquad R_m^t=S^{-F}R_{m-E}^{t-D}S^F, \tag{3.2.2}$$

where $S=S(y)$ is the $N\times N$ matrix defined by $S:=\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ y & & & 0 \end{pmatrix}$. Define the new $N\times N$ matrix $X_m^t=X_m^t(y)$ by the formula

$$X_m^t:=\begin{cases} S^F\cdot L_{m+E}^{t+D}\cdots L_{m+2}^{t+D}L_{m+1}^{t+D}\cdot R_m^{t+D-1}\cdots R_m^{t+1}R_m^t & (E>0) \\ S^F\cdot (L_m^{t+D}L_{m-1}^{t+D}\cdots L_{m+E+1}^{t+D})^{-1}\cdot R_m^{t+D-1}\cdots R_m^{t+1}R_m^t & (E<0) \end{cases}. \tag{3.2.3}$$

Therefore, (3.2.1–3.2.2) becomes

$$X_m^{t+1}R_m^t=R_m^tX_m^t, \tag{3.2.4}$$

or equivalently,

$$L_m^tX_{m-1}^t=X_m^tL_m^t. \tag{3.2.5}$$

These equations imply that the characteristic polynomial of $X_m^t(y)$ does not depend on t and m. Let $\widetilde{C}:=\{(x,y)\in\mathbb{C}^2 \mid \det(X_m^t(y)-xE)=0\}$. Denote by C the complete algebraic curve such that $\widetilde{C}\subset C$. We call C *the spectral curve.*

Remark 3.2.1 *By* (3.2.1)*, equation* (3.2.3) *can be rewritten as*

$$X_m^t=\begin{cases} S^F\cdot R_{m+E}^{t+D-1}\cdots R_{m+E}^{t+1}R_{m+E}^t\cdot L_{m+E}^t\cdots L_{m+2}^tL_{m+1}^t & (E>0) \\ S^F\cdot R_{m+E}^{t+D-1}\cdots R_{m+E}^{t+1}R_{m+E}^t\cdot (L_m^tL_{m-1}^t\cdots L_{m+E+1}^t)^{-1} & (E<0) \end{cases}. \tag{3.2.6}$$

3.2.1 Preliminaries

Hereafter, we assume g.c.d.$(D+E+F, N) = 1$. (See *Important remark* in page 36).

Proposition 3.2.1 *The spectral curve C has following special points:*

(i) *Let* $I_t := (-1)^N \cdot \prod_{n=1}^{N} I_{0,n}^t$. *On C, there exist D points $A_t : (x,y) = (0, I_t)$, $t = 0, 1, \ldots, D-1$.*

(ii) *Let* $V_m := \begin{cases} (-1)^N \cdot \prod_{n=1}^{N} V_{m,n}^D, & m = 1,2,\ldots,E \quad E>0 \\ (-1)^N \cdot \prod_{n=1}^{N} V_{m,n}^D, & m = 0,-1,\ldots,E+1 \quad E<0 \end{cases}$. *On C, there exist E points* $B_m : (x,y) = \begin{cases} (0, V_m), & m = 1,2,\ldots,E \\ (\infty, V_m), & m = 0,-1,\ldots,E+1 \end{cases}$.

(iii) *A unique point* $Q : (x,y) = \begin{cases} (\Lambda, 0) & F=0 \\ (0,0) & F>0 \end{cases}$, $\Lambda := (1+\delta)^D \cdot \delta^E$.

(iv) *A unique point* $P : (x,y) = (\infty, \infty)$.

Proof. The part (i) follows from $\det R_0^t = (-1)^{N+1}(y - I_t)$ and the part (ii) follows from $\det L_m^D = (-1)^{N+1}(y - V_m)$.

(iii): a) (The case $E > 0$) If $F = 0$, the n-th diagonal component of the matrix $X_m^t(y)$ is : $V_{m+E,n}^{t+D} \cdots V_{m+2,n}^{t+D} V_{m+1,n}^{t+D} \times I_{m,n}^{t+D-1} \cdots I_{m,n}^{t+1} I_{m,n}^t + y \cdot f(y)$ where $f(y)$ is a polynomial in y. By the definition of $I_{m,n}^t$, $V_{m,n}^t$ (p. 35) and the periodicity (3.1.6), the diagonal component is equal to $(1+\delta)^D \cdot \delta^E + yf$. Because the matrix $X_m^t(0)$ is upper triangular, its eigenvalues are $(1+\delta)^D \cdot \delta^E$. If $F > 0$, $X_m^t(0)$ is strictly upper triangular. Then its eigenvalues must be 0.

b) (The case $E < 0$) By (3.2.6), we have $X_m^t(0) = \Gamma_1 \Gamma_2^{-1}$ where Γ_1, Γ_2 are upper triangular. Its eigenvalues can be calculated from the diagonal components of these matrices. Similar calculations to (a) leads the conclusion.

(iv) a) (The case $E > 0$) For $(x,y) \in C$, there exists a eigenvector $\boldsymbol{v}(x,y)$ such that $X(y)\boldsymbol{v}(x,y) = x \cdot \boldsymbol{v}(x,y)$, which is expressed as

$$(S^{D+E+F} + \gamma_1 S^{D+E+F-1} + \cdots + \gamma_{D+E} S^F) \cdot \boldsymbol{v} = x \cdot \boldsymbol{v}, \tag{3.2.7}$$

where $\forall \gamma_j \in M_N(\mathbb{C})$ is a diagonal matrix. (See the definition of $X_m^t(y)$). Let us introduce a new parameter k by the formula $x = k^{-(D+E+F)}$. k is expected to be 0 at the point P. Multiplied by k^{D+E+F}, (3.2.7) becomes

$$(T^{D+E+F} + \gamma_1 k T^{D+E+F-1} \cdots + \gamma_{D+E} k^{D+E} T^F) \cdot \boldsymbol{v} = \boldsymbol{v}, \qquad T = kS.$$

If this vector equation has a solution $\boldsymbol{v} \neq \boldsymbol{0}$ on condition that $k \to 0$, $y \to \infty$, the relations $T^{D+E+F}\boldsymbol{v} \sim \boldsymbol{v}$ and $y \sim k^{-N}$ must be satisfied. Let $y = k_0^{-N}$. Then, the equation $T^{D+E+F}\boldsymbol{v}_0 = \boldsymbol{v}_0$ has unique solution $\boldsymbol{v}_0 = (k_0^{N-1}, \ldots, k_0^2, k_0, 1)^T$ up to constant multiple. The uniqueness is followed from g.c.d.$(D+E+F, N) = 1$. Therefore, there exist unique point $P \in C$ such that $(x,y) \sim (k^{-(D+E+F)}, k^{-N})$ around P.

b) (The case $E < 0$) For $(x,y) \in C$, there exists a vector $\boldsymbol{v}'(x,y)$ such that $X(y)\boldsymbol{v}'(x,y) = x \cdot \boldsymbol{v}'(x,y)$, which is expressed as

$$(S^{D+F} + \gamma_1 S^{D+F-1} + \cdots + \gamma_D S^F)(S^{-E} + \beta_1 S^{-E-1} + \cdots + \beta_{-E})^{-1} \cdot \boldsymbol{v}' = x \cdot \boldsymbol{v}',$$

or equivalently,

$$(S^{D+F} + \gamma_1 S^{D+F-1} + \cdots + \gamma_D S^F) \cdot \boldsymbol{v} = x(S^{-E} + \beta_1 S^{-E-1} + \cdots + \beta_{-E}) \cdot \boldsymbol{v},$$

where $\forall\gamma_j, \beta_j$ are constant diagonal matrices and $\boldsymbol{v}$ is a non-zero vector. We can soon obtain the desired result from the similar calculation to (a). ■

By the proof of the proposition, we have the following:

Corollary 3.2.2 *Around P, there exists a local coordinate k, $(k(P) = 0)$ such that $(x, y) \sim (k^{-(D+E+F)}, k^{-N})$ $((x, y) \to P)$ and*

$$X_m^t(x)\boldsymbol{v}(x,y) = x \cdot \boldsymbol{v}(x,y) \quad \Rightarrow \quad \boldsymbol{v} \sim (k^{N-1}, \dots, k^2, k, 1) \qquad ((x,y) \to P),$$

up to a constant multiple. ■

We can express the eigenvalue $\boldsymbol{v}$ also around the point $Q \in C$:

Corollary 3.2.3 *Let $\Lambda := (1+\delta)^D\delta^E$. Around Q, there exists a local coordinate k, $(k(Q) = 0)$ such that $(x, y) \sim (\Lambda k^F, \Lambda^N k^N)$ $((x, y) \to Q)$ and*

$$X_m^t(x)\boldsymbol{v}(x,y) = x \cdot \boldsymbol{v}(x,y) \quad \Rightarrow \quad \boldsymbol{v} \sim (1, \Lambda k, (\Lambda k)^2, \dots, (\Lambda k)^{N-1})$$

$((x, y) \to Q)$, up to a constant multiple. ■

Let us assume C to be smooth. Denote by $\mathcal{C}$ the algebraic curve with $D + |E|$ points: $\mathcal{C} := \begin{cases} (C; A_0, A_1, \dots, A_{D-1}; B_1, B_2, \dots, B_E) & E > 0 \\ (C; A_0, A_1, \dots, A_{D-1}; B_0, B_{-1} \dots, B_{E+1}) & E < 0 \end{cases}$. For $\mathcal{C}$, define the *isolevel set* $\mathcal{T}_\mathcal{C}$ by the formula: if (i) $E > 0$,

$$\mathcal{T}_\mathcal{C} := \left\{ (R_0^\alpha; L_\beta^D)_{0\le\alpha<D}^{0<\beta\le E} \;\middle|\; \begin{array}{l} \bullet \det(X_m^t - xE) \text{ is the defining function of } C, \\ \bullet \prod_n I_{m,n}^t \neq \prod_n V_{m,n}^{t+1},\ \prod_n I_{m-1,n}^t \neq \prod_n V_{m,n}^t, \\ \bullet A_t : (0, I_t),\ B_m : (0, V_m). \end{array} \right\},$$

if (ii) $E < 0$,

$$\mathcal{T}_\mathcal{C} := \left\{ (R_0^\alpha; L_\beta^D)_{0\le\alpha<D}^{E<\beta\le 0} \;\middle|\; \begin{array}{l} \bullet \det(X_m^t - xE) \text{ is the defining function of } C, \\ \bullet \prod_n I_{m,n}^t \neq \prod_n V_{m,n}^{t+1},\ \prod_n I_{m-1,n}^t \neq \prod_n V_{m,n}^t, \\ \bullet A_t : (0, I_t),\ B_m : (\infty, V_m). \end{array} \right\}.$$

The inequalities $\prod_n I_{m,n}^t \neq \prod_n V_{m,n}^{t+1}$ and $\prod_n I_{m-1,n}^t \neq \prod_n V_{m,n}^t$ are assumed to avoid the non-interesting solution $I_{m-1,n}^t = V_{m,n}^t$, $V_{m,n}^{t+1} = I_{m,n}^t$ of (3.1.3–3.1.4). By (3.2.1), this is equivalent to $\prod_n I_{m-1,n}^t = \prod_n I_{m,n}^t$ and $\prod_n V_{m,n}^t = \prod_n V_{m,n}^{t+1}$, which imply

$$I_t = \prod_n I_{m,n}^t, \quad (\forall m), \qquad V_m = \prod_n V_{m,n}^t, \quad (\forall t). \tag{3.2.8}$$

Moreover by the periodicity (3.1.6), we have:

Proposition 3.2.4 $I_t = I_{t\pm D} = I_{t\pm 2D} = \cdots, \quad V_m = V_{m\pm E} = V_{m\pm 2E} = \cdots.$ ■

Remark 3.2.2 *Hereafter, the indices of the points $A_t, B_m \in C$ are interpreted as elements of $\mathbb{Z}/D\mathbb{Z}$, $\mathbb{Z}/E\mathbb{Z}$ respectively, i.e. $A_t = A_{t\pm D} = A_{t\pm 2D} = \cdots$.*

We introduce here the *eigenvector mapping* which is a mapping from the isolevel set $\mathcal{T}_\mathcal{C}$ to the Picard group of the curve C. For a point (x, y) of C, there exists a eigenvector $\boldsymbol{v}_m^t(x, y)$ of the matrix $X_m^t = X_m^t(y)$ belonging to the eigenvector x. This defines a mapping

$$\Psi_{X_m^t} : \quad C^\circ \ni (x, y) \mapsto \boldsymbol{v}_m^t(x, y) \in \mathbb{P}^{N-1},$$

where C° is a Zariski open set of C. On condition that C is smooth, the domain of Ψ extends to C uniquely. The eigenvector mapping is a morphism $\varphi : \mathcal{T}_C \to \mathrm{Pic}^d(C)$ such that

$$\mathcal{O}_C(\varphi(X^t_m)) = (\Psi_{X^t_m})^* \mathcal{O}(1), \qquad X^t_m \in \mathcal{T}_C \tag{3.2.9}$$

where d is some integer and $\mathcal{O}(1)$ is the invertible sheaf of hyperplane sections of $\mathbb{P}^{N-1}$.

Although the definition of the eigenvector mapping is abstract, we can try to express the element $\varphi(X) \in \mathrm{Pic}^d(C)$, $(X \in \mathcal{T}_C)$ more explicitly. Let $g_1, g_2, \ldots, g_N$ be meromorphic functions on C such that $\Psi_X(p) = \boldsymbol{v}(p) = (g_1(p) : g_2(p) : \cdots : g_N(p))^T \in \mathbb{P}^{N-1}$ for $p \in C$. Define the divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ to be minimal positive divisors on C which satisfy

$$(g_j/g_N) + \mathcal{D}_1 \geq -(N-j) \cdot Q, \qquad \forall j, \tag{3.2.10}$$

$$(g_j/g_1) + \mathcal{D}_2 \geq -(j-1) \cdot P, \qquad \forall j. \tag{3.2.11}$$

On the other hand, we have calculated the order of poles and zeros of g_j at P and Q. These data imply that the pull-buck of $\{X_N = 0\} \cap \{\mathrm{Image}(\Psi_X)\}$ is $\mathcal{D}_1 + (N-1)Q$ and the pull-buck of $\{X_1 = 0\} \cap \{\mathrm{Image}(\Psi_X)\}$ is $\mathcal{D}_2 + (N-1)P$. In other words, it follows that:

$$\varphi(X) = [\mathcal{D}_1 + (N-1) \cdot Q] = [\mathcal{D}_2 + (N-1) \cdot P]. \tag{3.2.12}$$

Now we introduce the fundamental result which has been established in [15].

Theorem 3.2.5 *The divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are general positive divisors of degree $g = \mathrm{genus}(C)$.*

Proof. See [15], Lemma 4. ■

Corollary 3.2.6 *If $X \in \mathcal{T}_C$, then $\varphi(X) \in \mathrm{Pic}^{g+N-1}(C)$.* ■

Denote the positive divisor $\mathcal{D}_2$ in (3.2.12) by $\mathfrak{d}(X)$. Especially, we have

$$\varphi(X) = [\mathfrak{d}(X) + (N-1) \cdot P]. \tag{3.2.13}$$

3.2.2 Linearisation theorem for $E > 0$

The *linearisation theorem* for $E > 0$ has been essentially given in [6]. Here we review these results briefly.

In this section, we set $H := D + E + F$.

Define maps σ, μ_t and μ_m from $\mathcal{T}_C$ to itself by the formulae:

$$\sigma(X^t_m) := S X^t_m S^{-1}, \tag{3.2.14}$$

$$\mu_t(X^t_m) := R^t_m X^t_m (R^t_m)^{-1}, \tag{3.2.15}$$

$$\mu_m(X^t_m) := L^t_{m+1} X^t_m (L^t_{m+1})^{-1}. \tag{3.2.16}$$

In terms of KP equation, these maps associate with the maps: $n \mapsto n+1$, $t \mapsto t+1$, $m \mapsto m+1$ respectively (see (3.2.4–3.2.5)).

We start with the linear problem:

$$X^t_m(y) \cdot \boldsymbol{v}^t_m(x,y) = x \cdot \boldsymbol{v}^t_m(x,y). \tag{3.2.17}$$

This matrix equation can be transformed into the following form:

$$Y^t_m(x) \cdot \boldsymbol{w}^t_m(x,y) = y \cdot \boldsymbol{w}^t_m(x,y), \tag{3.2.18}$$

where $Y^t_m(x)$ is a $H \times H$ matrix and $\boldsymbol{w}^t_m(x,y)$ is a H-vector. We call the matrix equation (3.2.17) the *x-form*, and the matrix equation (3.2.18) the *y-form*.

Example 3.2.7 *For an x-form* $\begin{pmatrix} a_1 & b_1 & 1 \\ y & a_2 & b_2 \\ b_3 y & y & a_3 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = x\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$, *the associated y-form is:*

$$\begin{pmatrix} 0 & 1 \\ x-a_3 & -b_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ x-a_2 & -b_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ x-a_1 & -b_1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = y\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Now we calculate the actions of the shift operators σ, μ_t, μ_m on the x-form/y-form eigenvectors.

(i) Shift operator action on the x-form

By (3.2.14–3.2.16), the action of the shift operators on the x-form eigenvector is expressed as

$$\sigma : \boldsymbol{v} \mapsto S\boldsymbol{v}, \qquad \mu_t : \boldsymbol{v} \mapsto R^t_m \boldsymbol{v}, \qquad \mu_m : \boldsymbol{v} \mapsto L^t_{m+1}\boldsymbol{v}.$$

The following is given for later calculations.

Lemma 3.2.8 *By direct calculations, we have*
$\det S = (-1)^{N+1}\cdot y$, $\det R^t_m = (-1)^{N+1}(y - I_t)$, $\det L^t_{m+1} = (-1)^{N+1}(y - V_{m+1})$. ■

(ii) Shift operator action on the y-form

The shift operator action on the y-form is more complicated than the action on x-form. Let $\boldsymbol{v}^t_m(x,y) = (g_1(x,y), g_2(x,y), \ldots, g_N(x,y))^T$ be the x-form eigenvector. By (3.2.17), there exist H complex numbers $G_1, G_2, \ldots, G_H$ such that $g_{H+1} = \sum_{n=1}^{H} G_n g_n$. Let us define three matrices $\widehat{S}$, $\widehat{R}^t_m$, $\widehat{L}^t_{m+1}$ by the formulae:

$$\widehat{S} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ G_1 & G_2 & \cdots & G_H \end{pmatrix}, \quad \widehat{R}^t_m = \begin{pmatrix} I^t_{m,1} & 1 & & \\ & I^t_{m,2} & \ddots & \\ & & \ddots & 1 \\ G_1 & G_2 & \cdots & I^t_{m,H} + G_H \end{pmatrix},$$

$$\widehat{L}^t_{m+1} = \begin{pmatrix} V^t_{m+1,1} & 1 & & \\ & V^t_{m+1,2} & \ddots & \\ & & \ddots & 1 \\ G_1 & G_2 & \cdots & V^t_{m+1,H} + G_H \end{pmatrix}. \tag{3.2.19}$$

Therefore, the shift operator action on the y-form eigenvector is expressed as

$$\sigma : \boldsymbol{w} \mapsto \widehat{S}\boldsymbol{w}, \qquad \mu_t : \boldsymbol{w} \mapsto \widehat{R}^t_m \boldsymbol{w}, \qquad \mu_m : \boldsymbol{w} \mapsto \widehat{L}^t_{m+1}\boldsymbol{w}.$$

Lemma 3.2.9 *By direct calculations, we have*
$\det \widehat{S} = \begin{cases} (-1)^{H+1}\cdot\{x - \Lambda\} & F = 0 \\ (-1)^{H+1}\cdot x & F > 0 \end{cases}$, $\det \widehat{R}^t_m = \det \widehat{L}^t_{m+1} = (-1)^{H+1}\cdot x$.

Proof. Although lemma 3.2.9 have been proved in [6] Appendix A, we give another quicker proof here for further applications..In this proof, the N-vector "$(x_1, x_2, \ldots, x_H, 1, 0, \ldots, 0)$" is interpreted as the vector $(x_1, x_2, \ldots, x_H) + y\cdot(1, 0, \ldots, 0)$ if $H = N$.

Let $v = (g_1, g_2, \ldots, g_N)^T$. The matrix equation (3.2.17) is equivalent to infinitely many linear simultaneous equations expressed as follows:

$$\begin{cases} a_{j,1} \cdot g_{j+F} + a_{j,2} \cdot g_{j+F+1} + \cdots + a_{j,D+E} \cdot g_{j+H-1} + g_{j+H} = x \cdot g_j \\ g_{j+N} = y \cdot g_j \end{cases} \tag{3.2.20}$$

$(j \in \mathbb{Z})$. On the other hand, by (3.2.19), we have

$$(G_1, G_2, \ldots, G_H) = \begin{cases} -(-x,\, 0, \ldots, 0,\, a_{1,1}, a_{1,2}, \ldots, a_{1,D+E}) & F > 0 \\ -(a_{1,1} - x,\, a_{1,2}, \ldots, a_{1,D+E}) & F = 0 \end{cases}.$$

Therefore, by (3.2.20), it follows that

$$(1, 0, \ldots, 0) \cdot X^t_m(y) = -(G_1 - x, G_2, \ldots, G_H, 1, 0, \ldots, 0)$$

Especially, we have $a_{1,1} = \Lambda\,(= (1+\delta)^D \delta^E)$ if $F = 0$.

Due to the equation $\det \widehat{S} = (-1)^{H+1} G_1$, $\det \widehat{S}$ is soon calculated.

Define the N-vector $z := \left(\prod_{k=1}^{n-1} (-I^t_{m,k})\right)_{n=1,2,\ldots,N}$. By the cofactor expansion w.r.t. H-th row of the matrix $\widehat{R}^t_m$, we have

$$\begin{aligned}(-1)^{H+1} \cdot \det \widehat{R}^t_m &= \Big[(G_1, G_2, \ldots, G_H, 1, 0, \ldots, 0) \cdot z\Big]_{y=I_t} \\ &= -(1, 0, \ldots, 0) \cdot X^t_m(I_t) \cdot z + (x, 0, \ldots, 0) \cdot z = x\end{aligned}$$

because $R^t_m(I_t) z = \mathbf{0}$. See (3.2.3).

In a similar manner, we soon obtain $\det \widehat{L}^t_{m+1} = (-1)^{H+1} x$. ■

Combining the result of lemmas 3.2.8, 3.2.9, we can derive the conclusion:

Theorem 3.2.10 (Linearisation theorem for $E > 0$) (I): *Let $\mathcal{D}$ be the divisor $\mathcal{D} = P - Q$. Then the following diagram is commutative.*

$$\begin{array}{ccccc} & \mathcal{T}_C & \to & Pic^d(C) & \\ \sigma & \downarrow & & \downarrow & +[\mathcal{D}] \\ & \mathcal{T}_C & \to & Pic^d(C) & \end{array}.$$

(II): *Let $\mathcal{E}_j$ be the divisor $\mathcal{E}_j = P - A_j$ and $j = t$. The following diagram is commutative.*

$$\begin{array}{ccccc} & \mathcal{T}_C & \to & Pic^d(C) & \\ \mu_t & \downarrow & & \downarrow & +[\mathcal{E}_j] \\ & \mathcal{T}_C & \to & Pic^d(C) & \end{array}.$$

(III): *Let $\mathcal{F}_j$ be the divisor $\mathcal{F}_j = P - B_j$ and $j = m+1$. The following diagram is commutative.*

$$\begin{array}{ccccc} & \mathcal{T}_C & \to & Pic^d(C) & \\ \mu_m & \downarrow & & \downarrow & +[\mathcal{F}_j] \\ & \mathcal{T}_C & \to & Pic^d(C) & \end{array}.$$

Proof. Now we review the outline of the proof following to [6] §2.4.

The definition of the eigenvector mapping (3.2.9) can be rewritten as:

$$X(x,y)\boldsymbol{v}(x,y) = x\boldsymbol{v}(x,y) \;\Rightarrow\; \varphi(X) = -\left[\sum_{p\in C}(\operatorname{ord}\boldsymbol{v}(p))\cdot p\right], \text{ where}$$
$$\operatorname{ord}\boldsymbol{v}(p) = \min\{r\in\mathbb{Z}\mid \lim_{q\to p}|k^r\boldsymbol{v}(q)| < \infty, k \text{ is local coordinate s.t. } k(p)=0\}.$$

Hence force, the movement of the divisor $\varphi(X)$ is observed by the order of zeros of $\boldsymbol{v}$. Lemma 3.2.8 expresses the movement in the "x-direction" and Lemma 3.2.9 expresses the movement in the "y-direction". Combining these data, we obtain the desired conclusion. ■

3.2.3 Linearisation theorem for $E < 0$

The present method in the previous section is also valid essentially in the case of $E < 0$, but some calculations should be changed.

In this section, we set $H := D + F$ $(> -E)$. (See page 36.)

Recall (3.2.3):

$$X_m^t(y) = S^F\cdot(L_m^{t+D}L_{m-1}^{t+D}\cdots L_{m+E+1}^{t+D})^{-1}\cdot R_m^{t+D-1}\cdots R_m^{t+1}R_m^t \tag{3.2.3}$$
$$= (L_{m-E}^t\cdots L_{m+2}^tL_{m+1}^t)^{-1}\cdot S^F\cdot R_m^{t+D-1}\cdots R_m^{t+1}R_m^t. \tag{3.2.21}$$

Then, the linear problem $X_m^t(y)\cdot\boldsymbol{v}(x,y) = x\cdot\boldsymbol{v}(x,y)$ is rewritten as

$$S^F\cdot R_m^{t+D-1}\cdots R_m^{t+1}R_m^t\cdot\boldsymbol{v}_m^t(x,y) = x\cdot L_{m-E}^t\cdots L_{m+2}^tL_{m+1}^t\cdot\boldsymbol{v}_m^t(x,y). \tag{3.2.22}$$

Let $\boldsymbol{v}_m^t = (g_1, g_2, \ldots, g_N)^T$. The matrix equation (3.2.22) is equivalent to the following infinitely many linear simultaneous equation:

$$\begin{cases} \sum_{n=0}^{D} a_{j,n}\cdot g_{j+F+n} = x\cdot\sum_{n=0}^{|E|} b_{j,n}\cdot g_{j+n} \\ g_{i+N} = y\cdot g_i \end{cases}, \qquad i,j\in\mathbb{Z}. \tag{3.2.23}$$

Then, there exists a matrix $Y_m^t(x)$ of size H such that the following matrix equation is equivalent to (3.2.23):

$$Y_m^t(x)\cdot\boldsymbol{w}(x,y) = y\cdot\boldsymbol{w}(x,y), \qquad \boldsymbol{w} = (g_1, g_2, \ldots, g_H)^T. \tag{3.2.24}$$

Example 3.2.11 *For the matrix equation*

$$\begin{pmatrix} a_1 & b_1 & 1 \\ y & a_2 & b_2 \\ b_3y & y & a_3 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = x\begin{pmatrix} c_1 & 1 & 0 \\ 0 & c_2 & 1 \\ y & 0 & c_3 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

the associated y-form is :

$$\begin{pmatrix} 0 & 1 \\ xc_3 - a_3 & x - b_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ xc_2 - a_2 & x - b_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ xc_1 - a_1 & x - b_1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = y\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

In a similar fashion to the case of $E > 0$, we calculate the actions of σ, μ_t, μ_m on the x-form/y-form equation (3.2.22/3.2.24).

(i) Shift operator action on the x-form

Note that (3.2.14–3.2.16) is still valid for the case of $E < 0$. The action of the shift operators on the x-form eigenvector is expressed as

$$\sigma : \boldsymbol{v} \mapsto S\boldsymbol{v}, \qquad \mu_t : \boldsymbol{v} \mapsto R_m^t \boldsymbol{v}, \qquad \mu_m : \boldsymbol{v} \mapsto L_{m+1}^t \boldsymbol{v}.$$

Lemma 3.2.8 is also valid in this case.

(ii) Shift operator action on the y-form

By (3.2.17), there exist H complex numbers $G_1, G_2, \dots, G_H$ such that $g_{H+1} = \sum_{n=1}^{H} G_n g_n$. Let us define three matrices $\widehat{S}$, $\widehat{R}_m^t$, $\widehat{L}_{m+1}^t$ by the formulae (3.2.19). Therefore, the shift operator action on the y-form eigenvector is expressed as

$$\sigma : \boldsymbol{w} \mapsto \widehat{S}\boldsymbol{w}, \qquad \mu_t : \boldsymbol{w} \mapsto \widehat{R}_m^t \boldsymbol{w}, \qquad \mu_m : \boldsymbol{w} \mapsto \widehat{L}_{m+1}^t \boldsymbol{w}.$$

Lemma 3.2.12 *By direct calculations, we have*

$$\det \widehat{S} = \begin{cases} (-1)^{H+1} \cdot \{x - \Lambda\} & F = 0 \\ (-1)^{H+1} \cdot x & F > 0 \end{cases},$$

$$\det \widehat{R}_m^t = C_1 x, \quad \det \widehat{L}_{m+1}^t = C_2, \qquad \textit{where } C_1, C_2 \textit{ is const. } (\neq 0)$$

Proof. By (3.2.23), we have

$$\begin{aligned} &(G_1, G_2, \dots, G_H, 1, 0, \dots, 0) \\ &\quad = (1, 0, \dots, 0) \cdot \left(x L_{m-E}^t \cdots L_{m+2}^t L_{m+1}^t - S^F R_m^{t+D-1} \cdots R_m^{t+1} R_m^t \right). \end{aligned}$$

Let $\boldsymbol{z}_1 = \left(\prod_{k=1}^{n-1} (-I_{m,k}^t) \right)_{n=1}^{N}$ and $\boldsymbol{z}_2 = \left(\prod_{k=1}^{n-1} (-V_{m+1,k}^t) \right)_{n=1}^{N}$. Note that $R_m^t(I_t)\boldsymbol{z}_1 = \boldsymbol{0}$ and $L_{m+1}^t(V_{m+1})\boldsymbol{z}_2 = \boldsymbol{0}$. Therefore we have

$$\begin{aligned} \det \widehat{R}_m^t &= \Big[(G_1, G_2, \dots, G_H, 1, 0, \dots, 0) \cdot \boldsymbol{z}_1 \Big]_{y=I_t} = C_1 x \\ \det \widehat{L}_{m+1}^t &= \Big[(G_1, G_2, \dots, G_H, 1, 0, \dots, 0) \cdot \boldsymbol{z}_2 \Big]_{y=V_{m+1}} = C_2, \end{aligned}$$

where C_1, C_2 are non-zero constants. ■

Combining the result of lemmas 3.2.8 and 3.2.12, we derive the conclusion:

Theorem 3.2.13 (Linearisation theorem for $E < 0$) *The linearisation theorem (theorem 3.2.10) is also valid in the case of $E < 0$.* ■

3.2.4 Inverse scattering

The linearisation theorem 3.2.10, 3.2.13 states the *movement* of the rpdKP (Section 3.1) as a discrete dynamical system. In terms of the spectral curve C, this theorem describes the movement of the divisor $\mathfrak{d}(X) \in \mathrm{Div}^g(C)$.

In this section, we construct theta function solutions of the initial value problem of rpdKP by *inverse scattering*. The procedure we introduce here has been studied in [5, 6].

Let C be the (smooth) spectral curve associated with $X_m^t \in \mathcal{T}_C$. Fix a symplectic basis $\alpha_1, \dots, \alpha_g; \beta_1, \dots, \beta_g$ of C and the normalised holomorphic differential $\omega_1, \dots, \omega_g$ such that $\int_{\alpha_i} \omega_j = \delta_{i,j}$. The $g \times g$ matrix $\Omega := (\int_{\beta_i} \omega_j)_{i,j}$ is called the *period matrix* of C. For a fixed point $p_0 \in C$, the *Abel-Jacobi mapping* $\boldsymbol{A} : \mathrm{Div}(C) \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ is a homomorphism defined by:

$$\sum Y_i - \sum Z_j \;\mapsto\; \sum(\int_{p_0}^{Y_i} \omega_1, \cdots, \int_{p_0}^{Y_i} \omega_g) - \sum(\int_{p_0}^{Z_j} \omega_1, \cdots, \int_{p_0}^{Z_j} \omega_g).$$

Consider the universal covering $\pi : \mathfrak{U} \to C$ and fix an inclusion $\iota : C \hookrightarrow \mathfrak{U}$. For simplicity, we use the symbols "π" and "ι" to express the derived maps $\mathrm{Div}(\mathfrak{U}) \to \mathrm{Div}(C)$ and $\mathrm{Div}(C) \hookrightarrow \mathrm{Div}(\mathfrak{U})$ respectively. Naturally, there exists a continuous lift $\widetilde{\boldsymbol{A}} : \mathrm{Div}(\mathfrak{U}) \to \mathbb{C}^g$ such that $\widetilde{\boldsymbol{A}} \circ \iota(p_0) = 0$. For the projection $\rho : \mathbb{C}^g \to \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, it follows that $\rho \circ \widetilde{\boldsymbol{A}} = \boldsymbol{A} \circ \pi$.

Now we start with the general positive divisor $\mathfrak{d}(X_m^t)$ ($X_m^t \in \mathcal{T}_C$) on C. By the linearisation theorem 3.2.10, 3.2.13, we have the following.

Proposition 3.2.14 *Let $X \in \mathcal{T}_C$ and $\boldsymbol{v}(p) = (g_1, g_2, \dots, g_N)^T$ be the vector-valued meromorphic function on $C \ni p$ such that $X(p)\boldsymbol{v}(p) = x\boldsymbol{v}(p)$. Then,*

$$(g_1/g_N) = \mathfrak{d}(X) + (N-1)P - \mathfrak{d}(\sigma^{-1}X) - (N-1)Q.$$

Proof. By (3.2.12), the general divisor $\mathcal{D}_1 \in \mathrm{Div}^g(C)$ satisfies

$$[\mathcal{D}_1] = [\mathfrak{d}(X) + (N-1)\cdot(P-Q)] = [\mathfrak{d}(\sigma^{N-1}X)] = [\mathfrak{d}(\sigma^{-1}X)].$$

However, the divisors $\mathcal{D}_1$ and $\mathfrak{d}(\sigma^{-1}X)$ are general, positive and of degree g, it follows that $\mathcal{D}_1 = \mathfrak{d}(\sigma^{-1}X)$. Then (3.2.10) and (3.2.12) imply the desired result. ∎

By the linearisation theorem, the movement of the divisor $\mathfrak{d}(X)$ is given explicitly. For further arguments, we should "lift" the result of the theorem to the universal covering $\mathfrak{U}$.

Suppose that the lifted divisor $\mathfrak{D}(X_m^t) \in \mathrm{Div}^g(\mathfrak{U})$ such that $\pi(\mathfrak{D}(X_m^t)) = \mathfrak{d}(X_m^t)$ have been fixed. First define the lifted points $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{Q}$, $\widetilde{P}$ of $\mathfrak{U}$ ($\pi(\widetilde{A}_i) = A_i$, $\pi(\widetilde{B}_i) = B_i$,*etc.*) which satisfy the following condition:

$$F \cdot \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{Q}) + \sum_{j=1}^{D} \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{A}_j) + \sum_{j=1}^{E} \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{B}_j) = 0. \qquad (3.2.25)$$

(This condition comes from (3.1.6).) Then, define the lifted divisors $\mathfrak{D}(\sigma(X_m^t))$, $\mathfrak{D}(X_m^{t+1})$, $\mathfrak{D}(X_{m+1}^t) \in \mathrm{Div}^g(\mathfrak{U})$ by the formula:

$$\widetilde{\boldsymbol{A}}(\mathfrak{D}(\sigma(X_m^t))) = \widetilde{\boldsymbol{A}}(\mathfrak{D}(X_m^t)) + \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{Q}), \qquad \pi(\mathfrak{D}(\sigma(X_m^t))) = \mathfrak{d}(\sigma(X_m^t)), \qquad (3.2.26)$$

$$\widetilde{\boldsymbol{A}}(\mathfrak{D}(X_m^{t+1})) = \widetilde{\boldsymbol{A}}(\mathfrak{D}(X_m^t)) + \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{A}_t), \qquad \pi(\mathfrak{D}(X_m^{t+1})) = \mathfrak{d}(X_m^{t+1}), \qquad (3.2.27)$$

$$\widetilde{\boldsymbol{A}}(\mathfrak{D}(X_{m+1}^t)) = \widetilde{\boldsymbol{A}}(\mathfrak{D}(X_m^t)) + \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{B}_{m+1}), \qquad \pi(\mathfrak{D}(X_{m+1}^t)) = \mathfrak{d}(X_{m+1}^t). \qquad (3.2.28)$$

Let τ_m^t be a holomorphic function over $\mathfrak{U}$ defined by the formula:

$$\tau_m^t(p) = \theta\left(\widetilde{\boldsymbol{A}}\{\mathfrak{D}(X_m^t) - p - \Delta\}\right), \qquad p \in \mathfrak{U}, \qquad (3.2.29)$$

where $\theta(\bullet) = \theta(\bullet;\Omega)$ is the Riemann theta function and $\Delta \in \mathrm{div}^{g-1}(\mathfrak{U})$ is the lifted theta characteristic divisor of C ([17], Chap. II, cor. 3.11). To avoid cumbersome notations, we often omit the letter "$\widetilde{\boldsymbol{A}}$" and use a simpler expression $\tau^t(p) = \theta(\mathfrak{D}(X_t) - p - \Delta)$, when there is no confusion possible.

Although being defined over $\mathfrak{U}$, $\tau^t_m(p)$ is considered to be a multi-valued holomorphic function over C. By the Riemann vanishing theorem ([17], Chap. II, thm. 3.11), the zero divisor of $\tau^t_m(p)$ corresponds with $\mathfrak{d}(X^t_m)$.

Let $\tau^t_{m,n}(p) := \theta(\mathfrak{D}(\sigma^{n-1}X^t_m) - p - \Delta)$. Then, by the linearisation theorem, the function

$$\Psi^t_m(p) := \frac{\tau^t_{m,2}\cdot\tau^{t+1}_{m,1}}{\tau^t_{m,1}\cdot\tau^{t+1}_{m,2}}(p) = \frac{\theta(\mathfrak{D}(\sigma X^t_m) - p - \Delta)\cdot\theta(\mathfrak{D}(X^{t+1}_m) - p - \Delta)}{\theta(\mathfrak{D}(X^t_m) - p - \Delta)\cdot\theta(\mathfrak{D}(\sigma X^{t+1}_m) - p - \Delta)},$$

satisfies [(the zeros of denominator)] = [(the zeros of numerator)] $\in \mathrm{Pic}^{2g}(C)$ and therefore it is a single-valued and meromorphic function over C.

On the other hand, using the meromorphic function $g^t_{m1}, g^t_{m2}, \ldots, g^t_{mN}$ with

$$X^t_m(y)\cdot(g^t_{m1}, g^t_{m2}, \ldots, g^t_{mN})^T = x\cdot(g^t_{m1}, g^t_{m2}, \ldots, g^t_{mN})^T,$$

we can derive another expression for $\Psi(p)$. In fact, by proposition 3.2.14 and Liouville's theorem, there exists a complex number c such that

$$\Psi^t_m(p) = c\times\frac{g^t_{m,2}(p)\cdot g^{t+1}_{m,1}(p)}{g^t_{m,1}(p)\cdot g^{t+1}_{m,2}(p)}. \tag{3.2.30}$$

Due to this formula, we can calculate some special values of $\Psi^t(p)$:

Lemma 3.2.15 *If* g.c.d.$(D+E+F, N) = 1$, *we have* (i) $\Psi^t_m(P) = c$, (ii) $\Psi^t_m(Q) = c\times\dfrac{I^t_{m,1}}{I^t_{m,2}}$.

Proof. Because $(g^{t+1}_{m,1}, g^{t+1}_{m,2}, \ldots, g^{t+1}_{m,N}) = R^t_m\cdot(g^t_{m1}, g^t_{m2}, \ldots, g^t_{mN})$, we have

$$\Psi^t_m = c\times\frac{g^t_{m,2}\cdot(I^t_{m,1}g^t_{m,1} + g^t_{m,2})}{g^t_{m,1}\cdot(I^t_{m,2}g^t_{m,2} + g^t_{m,3})}.$$

By corollaries 3.2.2, 3.2.3, the values $\Psi^t_m(P)$ $\Psi^t_m(Q)$ are soon calculated. ∎

Because $\theta(\mathfrak{D}(X) - \widetilde{Q} - \Delta) = \theta(\mathfrak{D}(X) + (\widetilde{P} - \widetilde{Q}) - \widetilde{P} - \Delta) = \theta(\mathfrak{D}(\sigma X) - \widetilde{P} - \Delta)$, it follows that

$$\Psi^t_m(Q) = \Psi^t_{m,+}(P), \qquad \text{where} \quad \Psi^t_{m,+}(p) = \frac{\tau^t_{m,2}\cdot\tau^{t+1}_{m,1}}{\tau^t_{m,1}\cdot\tau^{t+1}_{m,2}}(p)$$

Then, lemma 3.2.15 implies $I^t_{m,2}\cdot\Psi^t_{m,+}(P) = I^t_{m,1}\cdot\Psi^t_m(P)$.

Repeating the same arguments with $\Psi^t_{m,+}(p)$, we derive $I^t_{m,3}\Psi^t_{m,++}(P) = I^t_{m,2}\Psi^t_{m,+}(P)$, and inductively, we have

$$I^t_{m,1}\Psi^t_m(P) = I^t_{m,2}\Psi^t_{m,+}(P) = I^t_{m,3}\Psi^t_{m,++}(P) = I^t_{m,4}\Psi^t_{m,+++}(P) = \cdots.$$

Let $\Psi^t_{m,n} := \Psi^t_{m,++\cdots+}(P)$ ($n-1$ "+"s). Finally we obtain the equations $\Psi^t_{m,n+N} = \Psi^t_{m,n}$ and $I^t_{n,m}\Psi^t_{n,m} = \epsilon^t_m$, where the number ϵ^t_m does not depend on n.

Next, consider the following single-valued meromorphic function over C:

$$\Phi_m^t(p) := \frac{\tau_{m+1,1}^t \cdot \tau_{m,2}^t}{\tau_{m,1}^t \cdot \tau_{m+1,2}^t}(p) = \frac{\theta(\mathfrak{D}(X_{m+1}^t) - p - \Delta) \cdot \theta(\mathfrak{D}(\sigma X_m^t) - p - \Delta)}{\theta(\mathfrak{D}(X_m^t) - p - \Delta) \cdot \theta(\mathfrak{D}(\sigma X_{m+1}^t) - p - \Delta)}.$$

By proposition 3.2.14 and Liouville's theorem, we derive the following expression:

$$\Phi_m^t(p) = c' \times \frac{g_{m,2}^t(p) \cdot g_{m+1,1}^t(p)}{g_{m,1}^t(p) \cdot g_{m+1,2}^t(p)}, \qquad c' : \text{constant}, \tag{3.2.31}$$

which allows us to compute some special values of $\Phi_m^t(p)$.

Lemma 3.2.16 *If* g.c.d.$(D+E+F, N) = 1$, *we have* (i) $\Phi_m^t(P) = c'$, (ii) $\Phi_m^t(Q) = c' \times \dfrac{V_{m+1,1}^t}{V_{m+1,2}^t}$.

Proof. By $(g_{m+1,1}^t, \ldots, g_{m+1,N}^t) = L_{m+1}^t \cdot (g_{m,1}^t, \ldots, g_{m,N}^t)$, it follows that

$$\Phi_m^t(p) = c' \times \frac{g_{m,2}^t \cdot (V_{m+1,1}^t g_{m,1}^t + g_{m,2}^t)}{g_{m,1}^t \cdot (V_{m+1,2}^t g_{m,2}^t + g_{m,3}^t)}.$$

By virtue of corollaries 3.2.2, 3.2.3, we soon derive the desired conclusion. ∎

Due to $\Phi_m^t(Q) = \Phi_{m,+}^t(P)$ and lemma 3.2.16, we have $V_{m+1,2}^t \Phi_{m,+}^t(P) = V_{m+1,1}^t \Phi_m^t(P)$, which implies

$$V_{m+1,1}^t \Phi_m^t(P) = V_{m+1,2}^t \Phi_{m,+}^t(P) = V_{m+1,3}^t \Phi_{m,++}^t(P) = \cdots.$$

Let $\Phi_{m,n}^t := \Phi_{++\cdots+}^t(P)$ ($n-1$ "+"s). Therefore we obtain $\Phi_{n+N}^t = \Phi_n^t$ and $V_{m+1,n}^t \Phi_{m,n}^t = \gamma_{m+1}^t$, where the number γ_{m+1}^t does not depend on n.

Define

$$\tau_{m+1,n}^t := \tau_{m,n}^t(\widetilde{P}).$$

By the above arguments, I_n^t and V_n^t have following expressions:

$$I_{m,n}^t = \epsilon_m^t \times \frac{\tau_{m+1,n}^t \tau_{m+1,n+1}^{t+1}}{\tau_{m+1,n+1}^t \tau_{m+1,n}^{t+1}}, \qquad V_{m,n}^t = \gamma_m^t \times \frac{\tau_{m,n}^t \tau_{m+1,n+1}^t}{\tau_{m+1,n}^t \tau_{m,n+1}^t}. \tag{3.2.32}$$

3.2.5 solution of rpdKP

For g-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ denotes $\boldsymbol{a}^T \boldsymbol{b} \in \mathbb{C}$.

Due to the periodicity $\mathfrak{d}(\sigma^N X_m^t) = \mathfrak{d}(X_m^t)$, there exist integer vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$ such that $\widetilde{\boldsymbol{A}}(N(\widetilde{P} - \widetilde{Q})) = \boldsymbol{n} + \Omega \boldsymbol{m}$. Considering the definition of the Riemann theta function (see [17], §II.1, for example), we have

$$\tau_{m,n+N}^t = \tau_{m,n}^t \times \exp(-2\sqrt{-1}\pi \cdot \langle \boldsymbol{m}, \boldsymbol{z} \rangle - \sqrt{-1}\pi \cdot \langle \boldsymbol{m}, \Omega \boldsymbol{m} \rangle),$$

where $z = z(n,m,t) = \widetilde{A}(\mathfrak{D}(\sigma^{n-1}X^t_{m-1}) - \widetilde{P} - \Delta)$. By (3.2.32), we have

$$\begin{aligned} I^t_{m,1}I^t_{m,2}\cdots I^t_{m,N} &= \{\epsilon^t_m\}^N \times \frac{\tau^t_{m+1,1}\tau^{t+1}_{m+1,N+1}}{\tau^t_{m+1,N+1}\tau^{t+1}_{m+1,1}} \\ &= \{\epsilon^t_m\}^N \times \exp(-2\sqrt{-1}\,\pi \cdot \langle \boldsymbol{m}, \widetilde{A}(\widetilde{P} - \widetilde{A}_t)\rangle), \end{aligned} \tag{3.2.33}$$

$$\begin{aligned} V^t_{m,1}V^t_{m,2}\cdots V^t_{m,N} &= \{\gamma^t_m\}^N \times \frac{\tau^t_{m,1}\tau^t_{m+1,N+1}}{\tau^t_{m+1,1}\tau^t_{m,N+1}} \\ &= \{\gamma^t_m\}^N \times \exp(-2\sqrt{-1}\,\pi \cdot \langle \boldsymbol{m}, \widetilde{A}(\widetilde{P} - \widetilde{B}_m)\rangle). \end{aligned} \tag{3.2.34}$$

Recall $\prod_n I^t_{m,n} = I_{t+1}$ (resp. $\prod_n V^t_{m,n} = V_m$) depends only on $t \bmod D$ (resp. $m \bmod |E|$). (Proposition 3.2.4). Finally, we conclude that the numbers ϵ^t and γ_m are determined only from the initial values by the formulae:

$$\epsilon^t = \{I_t\}^{1/N} \cdot \exp(2\sqrt{-1}\,N^{-1}\pi \cdot \langle \boldsymbol{m}, \widetilde{A}(\widetilde{P} - \widetilde{A}_t)\rangle), \tag{3.2.35}$$

$$\gamma_m = \{V_m\}^{1/N} \cdot \exp(2\sqrt{-1}\,N^{-1}\pi \cdot \langle \boldsymbol{m}, \widetilde{A}(\widetilde{P} - \widetilde{B}_m)\rangle). \tag{3.2.36}$$

Finally we obtain the conclusion:

Theorem 3.2.17 *On condition that* g.c.d.$(D+E+F, N) = 1$, (3.2.32, 3.2.35, 3.2.36) *solve the rpdKP equation* (3.1.3–3.1.6).

Chapter 4

Ultradiscrete Systems

4.1 Introduction

Let us start with the Takahashi-Satsuma BBS, which is the simplest example of the box-ball systems. In this thesis, the periodic boundary condition is assumed. The following is a typical example of BBS with the periodic boundary condition.

```
.......111..11...1.....          t=0
..........11..111.1....          t=1
............11...1.111.          t=2
11............11..1...1          t=3
..111...........11.1...          t=4
.....111..........1.11.          t=5
```

In the above figure, a dot '.' expresses an empty box and '1' expresses a ball. The rightmost box is connected to the leftmost one.

The box-ball system shows solitonic behaviour like KdV solitons and, in fact, is obtained from the discrete KdV equation through ultradiscretisation. Before explaining this, we introduce a variant of the original BBS here:

```
....123..22...1........          t=0
.....231.22....1.......          t=1
......312..22..1.......          t=2
.......123.22..1.......          t=3
........23122...1......          t=4
.........31..2221......          t=5
..........13.2221......          t=6
...........31222.1.....          t=7
...........31...2122...          t=8
............13..2122...          t=9
```

The numbers '1', '2',... express the species of the balls. The above figure shows the pBBS with 3 kinds of balls. The time evolution rule of this pBBS is defined as follows.

i) For each filled box, create a copy of the ball with index 1.

ii) Move each copy respectively to the nearest empty box on the right of it.

iii) Delete the original balls with index 1. At this point, the time coordinate proceeds: $t \mapsto t+1$.

iv) Do the same procedure i)–iii) for the balls with index 2.

v) Repeat these procedures for the balls with $3, 4, \ldots$ until all the balls have moved.

We call the sequence of nondecreasing positive integers (for example "123", "122") *soliton*. Note that the number of solitons does not change under the time evolution [21]. A state of the pBBS with N solitons is called *N-soliton system*.

Here, let us consider the pBBS with multi kinds of balls. Denote the number of kinds of balls by M. Define the non-negative integers Q_n^t, W_n^t $(n, t \in \mathbb{Z})$ by the formula:

$$Q_n^t := \sharp\{\text{balls of kind } t \bmod M \text{ in the } n\text{-th soliton from the left at time } t\},$$
$$W_n^t := \sharp\{\text{empty boxes between } n\text{-th soliton and } (n+1)\text{-th soliton from the left at time } t\}.$$

Strictly speaking, there exists no leftmost soliton because of the periodic boundary condition. Then we should choose the leftmost soliton arbitrary in advance [12].

For $n, t \in \mathbb{Z}$, the time evolution rule of pBBS is expressed as follows [21]:

$$Q_n^{t+M} = W_n^t + \min[0,\ X_n^t], \quad W_n^{t+1} = Q_{n+1}^t + W_n^t - Q_n^{t+M}, \tag{4.1.1}$$
$$(X_n^t := \max_{k=0}^{N-1}[\textstyle\sum_{l=0}^{k}(Q_{n-l}^t - W_{n-l}^t)].)$$
$$\textstyle\sum_{n=1}^N Q_n^t < \sum_{n=1}^N W_n^t, \quad \forall t. \tag{4.1.2}$$

The inequality (4.1.2) reflects the fact that the number of empty boxes must be larger than the number of moving balls.

Now we proceed for the theory of ultradiscretisation. The following proposition describes the relation between reduced KP equations and pBBS:

Proposition 4.1.1 *Let $\varepsilon > 0$ be a positive parameter. Assume that there exists a set of functions $\{I_n^t(\varepsilon), V_n^t(\varepsilon)\}_{n,t}$ such that*

$$I_n^{t+M} = I_n^t + V_n^t - V_{n-1}^{t+1}, \quad V_n^{t+1} = \frac{I_{n+1}^t}{I_n^{t+M}} \cdot V_n^t, \quad (I_{n+N}^t \equiv I_n^t,\ V_{n+N}^t \equiv V_n^t) \tag{4.1.3}$$
$$\textstyle\prod_{n=1}^N I_n^t > \prod_{n=1}^N V_n^t. \tag{4.1.4}$$

If the limits

$$Q_n^t := -\lim_{\varepsilon\to 0^+}(\varepsilon \log I_n^t), \quad W_n^t := -\lim_{\varepsilon\to 0^+}(\varepsilon \log V_n^t)$$

exist, then the set $\{Q_n^t, W_n^t\}_{n,t}$ satisfies (4.1.1) and (4.1.2). ■

The discrete system (4.1.3)–(4.1.4) is called *hungry periodic Toda equation*. As mentioned in the previous chapter (Chapter 3), the hungry Toda equation is a one of the reduced discrete KP equations. Proposition 4.1.1 is naturally generalised as follows:

Proposition 4.1.2 *Assume that there exists a set of functions $\{I_{m,n}^t(\varepsilon), V_{m,n}^t(\varepsilon)\}_{m,n,t}$ such that*

$$I_{m-1,n}^t + V_{m,n-1}^{t+1} = I_{m,n-1}^t + V_{m,n}^t, \quad I_{m-1,n}^t V_{m,n}^{t+1} = I_{m,n}^t V_{m,n}^t, \tag{4.1.5}$$
$$I_{m,n+N}^t \equiv I_{m,n}^t, \quad V_{m,n+N}^t \equiv V_{m,n}^t. \tag{4.1.6}$$
$$\textstyle\prod_{n=1}^N I_{m,n}^t \neq \prod_{n=1}^N V_{m,n}^t \quad \forall m, t. \tag{4.1.7}$$

Suppose that the limits

$$Q^t_{m,n} := -\lim_{\varepsilon\to 0^+}(\varepsilon \log I^t_{m,n}), \quad W^t_{m,n} := -\lim_{\varepsilon\to 0^+}(\varepsilon \log V^t_{m,n})$$

exists, then the set $\{Q^t_{m,n}, W^t_{m,n}\}_{m,n,t}$ *satisfies the following tropical recursive formula:*

$$\begin{cases} Q^t_{m-1,n} = W^t_{m,n} + \min\left[0,\ X^t_{m,n}\right] \\ W^{t+1}_{m,n} = Q^t_{m,n} + W^t_{m,n} - Q^t_{m-1,n} \\ X^t_{m,n} := \max_{k=0}^{N-1}\left[\sum_{l=0}^{k}(Q^t_{m,n-l-1} - W^t_{m,n-l})\right] \end{cases}, \ \textit{if } \sum_{n=1}^N Q^t_{m,n} \le \sum_{n=1}^N W^t_{m,n}, \tag{4.1.8}$$

$$\begin{cases} Q^t_{m-1,n} = W^t_{m,n} + Q^t_{m,n} - W^{t+1}_{m,n} \\ W^{t+1}_{m,n} = Q^t_{m,n} + \min\left[0, Y^t_{m,n}\right] \\ Y^t_{m,n} := \max_{k=0}^{N-1}\left[\sum_{l=0}^{k}(W^t_{m,n+l+1} - Q^t_{m,n+l})\right] \end{cases}, \ \textit{if } \sum_{n=1}^N Q^t_{m,n} > \sum_{n=1}^N W^t_{m,n}. \tag{4.1.9}$$

Proof. This proposition is essentially proved by T.Kimijima and T.Tokihiro [12]. Here we review their proof briefly. We regard equation (4.1.5) as the recursion formula which gives the revolution: $\{I^t_{m,n}, V^t_{m,n}\}_{n=1}^N \mapsto \{I^t_{m-1,n}, V^{t+1}_{m,n}\}_{n=1}^N$. Although (4.1.5) seemingly fails to define I, Vs at the next step explicitly, the periodic condition (4.1.6) and the inequality (4.1.7) allow us to determine these values uniquely. In fact, the recursion formula (4.1.5) can be written as

$$I^t_{m-1,n} = V^t_{m,n}\left\{1 + \frac{1 - \frac{V^t_{m,1}V^t_{m,2}\cdots V^t_{m,N}}{I^t_{m,1}I^t_{m,2}\cdots I^t_{m,N}}}{\frac{V^t_{m,n}}{I^t_{m,n-1}} + \frac{V^t_{m,n}V^t_{m,n-1}}{I^t_{m,n-1}I^t_{m,n-2}} + \cdots + \frac{V^t_{m,n}V^t_{m,n-1}\cdots V^t_{m,n+1}}{I^t_{m,n-1}I^t_{m,n-2}\cdots I^t_{m,n}}}\right\},$$

$$V^{t+1}_{m,n} = I^t_{m,n}\left\{1 + \frac{1 - \frac{I^t_{m,1}I^t_{m,2}\cdots I^t_{m,N}}{V^t_{m,1}V^t_{m,2}\cdots V^t_{m,N}}}{\frac{I^t_{m,n}}{V^t_{m,n+1}} + \frac{I^t_{m,n}I^t_{m,n+1}}{V^t_{m,n+1}V^t_{m,n+2}} + \cdots + \frac{I^t_{m,n}I^t_{m,n+1}\cdots I^t_{m,n-1}}{V^t_{m,n+1}V^t_{m,n+2}\cdots V^t_{m,n}}}\right\}.$$

(See [12]). Ultradiscretising this equation, we derive the desired result. ■

Under the assumptions $I^t_{m,n} \equiv I^{t-M}_{m-1,n+1} (:= I^t_n)$, $V^t_{m,n} \equiv V^{t-M}_{m-1,n+1} (:= V^t_{n+1})$ and $\prod_n I^t_n > \prod_n V^t_n$ $(\forall t)$, proposition 4.1.2 reduces to proposition 4.1.1.

In this chapter, we research the method to calculate the general solutions of pBBSs.

4.2 The general solution of pBBS

In this section, $\varepsilon > 0$ is a positive parameter and $\boldsymbol{e} := e^{-1/\varepsilon}$.

Consider the tropical recursive formula (4.1.8,4.1.9) with the reduction condition

$$Q^t_{m,n} \equiv Q^{t-D}_{m-E,n-F}, \quad W^t_{m,n} \equiv W^{t-D}_{m-E,n-F}, \qquad D, E \in \mathbb{Z}\setminus\{0\},\ F \in \mathbb{Z}.$$

We can assume $D > 0$, $0 < D + E + F \le N$ and g.c.d.$(D, E) = 1$ without loss of generality (Chapter 3).

The initial value of the system is the set of real numbers:

$$\begin{matrix} \{Q^0_{0,n}, Q^1_{0,n}, \ldots, Q^{D-1}_{0,n}; W^0_{0,n}, W^0_{-1,n}, \ldots, W^0_{-E+1,n}\}_n & E > 0 \\ \{Q^0_{0,n}, Q^1_{0,n}, \ldots, Q^{D-1}_{0,n}; W^D_{0,n}, W^D_{-1,n}, \ldots, W^D_{E,n}\}_n & E < 0 \end{matrix}. \tag{4.2.1}$$

Because the formula (4.1.8) is invariant under the translation $(Q^t_{m,n}, W^t_{m,n}) \mapsto (Q^t_{m,n} + 1, W^t_{m,n} + 1)$, we can assume that the real numbers in (4.2.1) are non-negative.

Hearafter, assume that all the initial values are non-negative. Then we have the following lemmas.

Lemma 4.2.1 *For all m, n, t, $Q^t_{m,n}$, $W^t_{m,n}$ are non-negative.*

Proof. By definitions of $X^t_{m,n}$ and $Y^t_{m,n}$, we have $-W^t_{m,n} \le \min[0, X^t_{m,n}] \le 0$ and $-Q^t_{m,n} \le \min[0, Y^t_{m,n}] \le 0$. Then, if the initial values are non-negative, it follows that $Q^t_{m,n}, W^t_{m,n} \ge 0$ for all n, m, t. ■

Lemma 4.2.2 $\begin{cases} \text{(i)} \; Q^t_{m,n} = 0 \;\; (\forall n) \;\Rightarrow\; Q^t_{m-1,n} = 0,\, W^{t+1}_{m,n} = W^t_{m,n} \;\; (\forall n) \\ \text{(ii)} \; W^t_{m,n} = 0 \;\; (\forall n) \;\Rightarrow\; W^{t+1}_{m,n} = 0,\, Q^t_{m-1,n} = Q^t_{m,n} \;\; (\forall n) \end{cases}$.

Proof. The part (i) follows from $\min[0, X^t_{m,n}] = -W^t_{m,n}$. The part (ii) follows from $\min[0, Y^t_{m,n}] = -Q^t_{m,n}$. ■

Corollary 4.2.3 $\begin{cases} \text{(i)} \; Q^t_{0,n} = 0 \;\; (\forall n) \;\Rightarrow\; Q^t_{m,n} = 0,\, W^{t+1}_{m,n} = W^t_{m,n} \;\; (\forall m \le 0, \forall n) \\ \text{(ii)} \; W^t_{0,n} = 0 \;\; (\forall n) \;\Rightarrow\; W^{t+1}_{m,n} = 0,\, Q^t_{m,n} = Q^t_{m,n} \;\; (\forall m \le 0, \forall n) \end{cases}$.■

Suppose $E < D$. If $Q^0_{0,n} = Q^1_{0,n} = \cdots = Q^{E-1}_{0,n} = 0$ for all n, it follows that $Q^0_{m,n} = Q^1_{m,n} = \cdots = Q^{E-1}_{m,n} = 0$ for all $m \le 0$, n, or equivalently, $Q^{kD}_{0,n} = Q^{kD+1}_{0,n} = \cdots = Q^{kD+E-1}_{0,n} = 0$ for all $k \ge 0$, n, by corollary 4.2.3 (i). Again using corollary 4.2.3 (i), we have $W^{kD}_{m,n} = W^{kD+1}_{m,n} = \cdots = W^{kD+E}_{m,n}$ for all $k \ge 0$, $m \le 0$, n.

Define the new valuables $\widehat{Q}^t_{m,n}$, $\widehat{W}^t_{m,n}$ by the formula:

$$\widehat{Q}^j_{m,n} := Q^{j+E}_{m,n}, \quad (j = 0, 1, \ldots, D-E-1,\ \forall m, n), \qquad \widehat{Q}^{t-(D-E)}_{m-E,n-F} := \widehat{Q}^t_{m,n},$$
$$\widehat{W}^j_{m,n} := W^{j+E}_{m,n}, \quad (j = 0, 1, \ldots, D-E-1,\ \forall m, n), \qquad \widehat{W}^{t-(D-E)}_{m-E,n-F} := \widehat{W}^t_{m,n}.$$

Proposition 4.2.4 *The new valuables $\{\widehat{Q}^t_{m,n}, \widehat{W}^t_{m,n}\}$ again satisfy the pBBS (4.1.8–4.1.9) with constraint*

$$\widehat{Q}^{t-(D-E)}_{m-E,n-F} \equiv \widehat{Q}^t_{m,n}, \qquad \widehat{W}^{t-(D-E)}_{m-E,n-F} \equiv \widehat{W}^t_{m,n}.$$

Proof. First we prove $\widehat{W}^t_{m,n} = W^s_{m,n} \;\Rightarrow\; \widehat{W}^{t+1}_{m,n} = W^{s+1}_{m,n}$. Let $t = k(D-E) + j$ $(k \in \mathbb{Z}, 0 \le j < D-E)$. Then it follows that

$$\widehat{W}^t_{m,n} = \widehat{W}^{k(D-E)+j}_{m,n} = \widehat{W}^j_{m-kE,n-kF} = W^{j+E}_{m-kE,n-kF} = W^{kD+E+j}_{m,n}. \tag{4.2.2}$$

If $j \ne D-E-1$, there is nothing to prove. If $j = D-E-1$, $\widehat{W}^{t+1}_{m,n} = W^{(k+1)D+E}_{m,n} = W^{(k+1)D}_{m,n} = W^{kD+E+(D-E-1)+1}_{m,n} = W^{s+1}_{m,n}$, where $\widehat{W}^t_{m,n} = W^s_{m,n}$.

Next we proceed for (4.1.8–4.1.9). It is enough to prove $\widehat{W}^t_{m+1,n} = W^s_{m+1,n} \;\Rightarrow\; \widehat{W}^t_{m+1,n} = W^s_{m+1,n}$ and $\widehat{Q}^t_{m,n} = Q^s_{m,n} \;\Rightarrow\; \widehat{Q}^t_{m+1,n+1} = Q^s_{m,n}$ *etc.* These relations follow from (4.2.2) ■

Interchanging the roles of Q and W, we soon obtain an analogous statement to proposition 4.2.4. Assume $D < E$ and $W^0_{-E+1,n} = W^0_{-E+2,n} = \cdots = W^0_{-E+D,n} = 0$ for all n. Define the new variables $\widetilde{Q}^t_{m,n}$ and $\widetilde{W}^t_{m,n}$ by the formula:

$$\widetilde{Q}^t_{j,n} := Q^t_{j+D,n}, \quad (j = 0, 1, \ldots, E-D-1,\ \forall t, n), \qquad \widetilde{Q}^{t-D}_{m-(E-D),n-F} := \widetilde{Q}^t_{m,n},$$
$$\widetilde{W}^t_{j,n} := W^t_{j+D,n}, \quad (j = 0, 1, \ldots, E-D-1,\ \forall t, n), \qquad \widetilde{W}^{t-D}_{m-(E-D),n-F} := \widetilde{W}^t_{m,n}.$$

Proposition 4.2.5 *The new valuables* $\{\widetilde{Q}^t_{m,n}, \widetilde{W}^t_{m,n}\}$ *again satisfy the pBBS* (4.1.8–4.1.9) *with constraint*

$$\widetilde{Q}^{t-D}_{m-(E-D),n-F} \equiv \widetilde{Q}^t_{m,n}, \quad \widetilde{W}^{t-D}_{m-(E-D),n-F} \equiv \widetilde{W}^t_{m,n}. \quad \blacksquare$$

Propositions 4.2.4, 4.2.5 state that the pBBS with reduction $Q^t_{m,n} = Q^{t-D}_{m-E,n-F}$, $W^t_{m,n} = W^{t-D}_{m-E,n-F}$ can be embedded into another pBBS with reduction

(i) $Q^t_{m,n} = Q^{t-(D+E)}_{m-E,n-F}$, $W^t_{m,n} = W^{t-(D+E)}_{m-E,n-F}$ or

(ii) $Q^t_{m,n} = Q^{t-D}_{m-(E+D),n-F}$, $W^t_{m,n} = W^{t-D}_{m-(E+D),n-F}$

due to "dummy" zero-initial values. Repeating this procedure, we can assume g.c.d.$(D + E + F, N) = 1$ without loss of generality because g.c.d.$(D, E) = 1$.

In the rest of this thesis, we assume g.c.d.$(D + E + F, N) = 1$.

4.2.1 Ultradiscrete theta function

For g-vectors $\boldsymbol{v} = (v_1, v_2, \ldots, v_g)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_g)$, define $\langle \boldsymbol{v}, \boldsymbol{w} \rangle := \sum_{i=1}^g v_i w_i$.

The main tool for expressing the general solution of pBBS is the *ultradiscrete theta function* (ud-theta function), which is defined by the following formal series over the tropical semifield $\mathbb{T}$:

$$\Theta(\boldsymbol{Z}; B) := \min_{\boldsymbol{n} \in \mathbb{Z}^g} \left[\frac{1}{2} \langle \boldsymbol{n}, B\boldsymbol{n} \rangle + \langle \boldsymbol{m}, \boldsymbol{Z} \rangle \right], \qquad \boldsymbol{Z} \in \mathbb{T}^g, \tag{4.2.3}$$

where $B \in \mathrm{Mat}_g(\mathbb{T})$ is a positive definite symmetric matrix. This function is defined to be an analogue of the Riemann theta function over complex g-space []. Moreover, by virtue of theorem 2.4.3 (Chapter 2), we have an explicit relation between them. Let C_ε be a plane affine curve defined by a pL-polynomial (section 2.2.2) and B_ε be the period matrix of C_ε. Denote the associated tropical curve by $\mathrm{Trop}\, C$ and tropical period matrix by B_T.

Lemma 4.2.6 *Let* $\boldsymbol{z} = \dfrac{-1}{2\pi i \varepsilon} \cdot \boldsymbol{Z}$. *If* C_ε *has a good tropicalisation* (*section* 2.4.2), *the Riemann theta function* $\theta(\boldsymbol{z}; B_\varepsilon)$ *satisfies*

$$-\lim_{\varepsilon \to 0^+} \varepsilon \log \theta(\boldsymbol{z}; B_\varepsilon) = \Theta(\boldsymbol{Z}; B_T).$$

Proof.

$$\begin{aligned}
-\varepsilon \log \theta(\boldsymbol{z}; B_\varepsilon) &= -\varepsilon \log \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp\left(\pi i \langle \boldsymbol{n}, B_\varepsilon \boldsymbol{n} \rangle + 2\pi i \langle \boldsymbol{n}, \boldsymbol{z} \rangle\right) \\
&\sim -\varepsilon \log \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \exp\left(-\frac{1}{2\varepsilon} \langle \boldsymbol{n}, B_T \boldsymbol{n} \rangle - \frac{1}{\varepsilon} \langle \boldsymbol{n}, \boldsymbol{Z} \rangle\right) \qquad (\because \text{ Theorem 2.4.3}) \\
&\overset{\varepsilon \to 0^+}{\longrightarrow} \Theta(\boldsymbol{Z}; B_T). \qquad \blacksquare
\end{aligned}$$

4.2.2 Tropical Abel mapping

Let $f_\varepsilon(x, y)$ be a pL-polynomial and $\mathrm{Trop}\, C = TV_{\mathbb{R}^2}(f_\varepsilon)$, $C_\varepsilon = V_{\mathbb{C}^2}(f_\varepsilon)$. Fix an arbitrary base point $\mathfrak{p}_0$ of $\mathrm{Trop}\, C$. The *universal covering* $\mathrm{Trop}\, \mathfrak{U}$ *of* $\mathrm{Trop}\, C$ is a topological set defined by:

$$\mathrm{Trop}\, \mathfrak{U} := \{(\mathfrak{p}, \Gamma) \mid \mathfrak{p} \in \mathrm{Trop}\, C,\ \Gamma \subset \mathrm{Trop}\, C \text{ is a homotopy class of paths connecting } \mathfrak{p}_0 \text{ and } \mathfrak{p}\}.$$

The *tropical Abel mapping* $T\widetilde{\boldsymbol{A}}$ is a mapping from Trop $\mathfrak{U}$ to $\mathbb{R}^g$ defined by:

$$T\widetilde{\boldsymbol{A}}(\mathfrak{p},\Gamma) := (\ell_T(T_{\beta_1},\Gamma), \ell_T(T_{\beta_2},\Gamma), \ldots, \ell_T(T_{\beta_g},\Gamma)) \in \mathbb{R}^g.$$

By the result in Chapter 2, the relationship between the tropical Abel mapping $T\widetilde{\boldsymbol{A}}$ and the complex Abel mapping $\widetilde{\boldsymbol{A}}$ (Chapter 3) is expressed explicitly.

Lemma 4.2.7 *Assume that C_ε has a good tropicalisation. Let $\gamma = \gamma_\varepsilon$ be a closed path on C_ε which converges to the path $\Gamma \subset \mathrm{Trop}\, C$ in the sense of* Preliminaries for an integration theory *in page* 23. *Then it follows that*

$$\widetilde{\boldsymbol{A}}(\gamma) \sim \frac{-1}{2\pi \mathrm{i}\varepsilon} \cdot T\widetilde{\boldsymbol{A}}(\Gamma), \quad (\varepsilon \to 0^+). \quad \blacksquare$$

4.2.3 Ultradiscretisation of theta function solutions of KP

Now we proceed for the initial value problem. Let $Q^0_{0,n}, \ldots, Q^{D-1}_{0,n}, W^0_{0,n}, \ldots, W^0_{-E+1,n}$ $(E > 0)$ (resp. $Q^0_{0,n}, \ldots, Q^{D-1}_{0,n}$, $W^D_{0,n}, \ldots,$ $W^D_{E,n} (E < 0)$) be the initial value of pBBS.

Let K be the field of convergent Puiseux series with respected to e (Chapter 2). Using the initial values of pBBS, we define the lifted initial value of KP by the formula

$$Q^t_{m,n} = -\lim_{\varepsilon \to 0^+} \varepsilon \log I^t_{m,n}, \quad W^t_{m,n} = -\lim_{\varepsilon\to 0^+} \varepsilon \log V^t_{m,n}, \quad I^t_{m,n}, V^t_{m,n} \in K.$$

Of course, we have a lot of freedom in the choice of $I^t_{m,n}$ and $V^t_{m,n}$. We should choose generic I, Vs to make the spectral curve $C = C_\varepsilon$ (Chapter 3) generic.

By the analytical method in Chapter 3, the general solution of the initial value problem of KP is expressed as follows:

$$I^t_{m,n} = \epsilon^t \times \frac{\tau^t_{m+1,n}\tau^{t+1}_{m+1,n+1}}{\tau^t_{m+1,n+1}\tau^{t+1}_{m+1,n}}, \qquad V^t_{m,n} = \gamma_m \times \frac{\tau^t_{m,n}\tau^t_{m+1,n+1}}{\tau^t_{m+1,n}\tau^t_{m,n+1}}. \tag{3.2.32}$$

$$\epsilon^t = \{I_t\}^{1/N} \cdot \exp(2\mathrm{i}\, N^{-1}\pi \cdot \langle \boldsymbol{m}, \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{A}_t)\rangle), \tag{3.2.35}$$

$$\gamma_m = \{V_m\}^{1/N} \cdot \exp(2\mathrm{i}\, N^{-1}\pi \cdot \langle \boldsymbol{m}, \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{B}_m)\rangle), \tag{3.2.36}$$

$$\tau^t_{m,n} = \theta\left(\widetilde{\boldsymbol{A}}\{\mathfrak{D}(\sigma^{n-1}X^t_{m-1}) - \widetilde{P} - \Delta\}\right), \tag{4.2.4}$$

where $N \cdot \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{Q}) = \boldsymbol{n} + B_\varepsilon \boldsymbol{m}$. Note that $\boldsymbol{m} \sim -(2\pi \mathrm{i}\varepsilon N)B_T^{-1} \cdot \widetilde{\boldsymbol{A}}(\widetilde{P} - \widetilde{Q})$. By lemma 4.2.7, the ultradiscrete limits $\mathfrak{E}^t = -\lim \varepsilon \log \epsilon^t$ and $\mathfrak{B}_m = -\lim \varepsilon \log \gamma_m$ satisfy

$$\mathfrak{E}^t = \frac{Q^t}{N} + \left\langle B_T^{-1} T\widetilde{\boldsymbol{A}}(\Gamma_{Q\to P}), T\widetilde{\boldsymbol{A}}(\Gamma_{A_t \to P})\right\rangle, \quad Q^t = Q^t_{m,1} + \cdots + Q^t_{m,N}, \tag{4.2.5}$$

$$\mathfrak{B}_m = \frac{W_m}{N} + \left\langle B_T^{-1} T\widetilde{\boldsymbol{A}}(\Gamma_{Q\to P}), T\widetilde{\boldsymbol{A}}(\Gamma_{B_m \to P})\right\rangle, \quad W_m = W^t_{m,1} + \cdots + W^t_{m,N}. \tag{4.2.6}$$

Denote $\mathfrak{v}_0 := T\widetilde{\boldsymbol{A}}(\mathfrak{D}(X^0_0) - \widetilde{P} - \Delta)$, $\mathfrak{a}(t) := T\widetilde{\boldsymbol{A}}(\Gamma_{A_t \to P})$, $\mathfrak{b}(m) := T\widetilde{\boldsymbol{A}}(\Gamma_{B_m \to P})$, $\mathfrak{c} := T\widetilde{\boldsymbol{A}}(\Gamma_{Q \to P})$, and define $\mathfrak{v}^t_{m,n} := \mathfrak{v}_0 + \sum_{i=0}^{t-1} \mathfrak{a}(i) + \sum_{j=1}^{m-1} \mathfrak{b}(j) + (n-1) \cdot \mathfrak{c}$. By propositions 4.2.6 and 4.2.7, the ultradiscrete limit $\mathfrak{T}^t_{m,n} := -\lim_{\varepsilon \to 0^+} \varepsilon \log \tau^t_{m,n}$ is expressed as

$$\mathfrak{T}^t_{m,n} = \Theta(\mathfrak{v}^t_{m,n}; B_T).$$

Finally, we have the following:

Theorem 4.2.8 *pBBS* (4.1.8–4.1.9) *is solved by the formula:*

$$Q^t_{m,n} = \mathfrak{E}^t + (\mathfrak{T}^t_{m+1,n} + \mathfrak{T}^{t+1}_{m+1,n+1} - \mathfrak{T}^t_{m+1,n+1} - \mathfrak{T}^{t+1}_{m+1,n}), \tag{4.2.7}$$

$$W^t_{m,n} = \mathfrak{B}_m + (\mathfrak{T}^t_{m,n} + \mathfrak{T}^t_{m+1,n+1} - \mathfrak{T}^t_{m+1,n} - \mathfrak{T}^t_{m,n+1}). \tag{4.2.8}$$

4.3 Examples

In this section, we introduce examples of concrete calculations for the initial value problem. Throughout this section, we consider the following 4-soliton system:

```
t=0     11...1112....111...11111......
  1     ..11....2111....111.....11111.
  2     ..11.....1112...111.....11111.
  3     11..1111....2111...111.......1
  4     11..1111.....1112..111.......1
```

4.3.1 How to calculate?

add dummy zeros

We start with the initial value

$$(Q_1^0,\, Q_1^1,\, Q_2^0,\, Q_2^1, Q_3^0,\, Q_3^1,\, Q_4^0,\, Q_4^1) = (2,0,3,1,3,0,5,0),$$
$$(W_1^0, W_2^0, W_3^0, W_4^0) = (3,4,3,6).$$

Unfortunetely, the 4 soliton systems with 2 kinds of balls do not setisfy the condition g.c.d.$(D+E+F, N) = 1$. (In fact, $D = 2, E = 1, F = -1, N = 4$.) Therfore, we add *dummy zeros* to the initial value.

$$(Q_1^0,\, Q_1^1, Q_1^2,\, Q_2^0,\, Q_2^1, Q_2^2, Q_3^0,\, Q_3^1, Q_3^2\, Q_4^0,\, Q_4^1, Q_4^2) = (2,0,0,3,1,0,3,0,0,5,0,0),$$
$$(W_1^0, W_2^0, W_3^0, W_4^0) = (3,4,3,6).$$

Here, $Q_1^2, Q_2^2, Q_3^2, Q_4^2$ are the dummy zeros.

define the tropical spectral curve

Let val : $K \to \mathbb{T}$ be the canonical valuation and $R^\times \subset K$ be the multiplicative group $\{x \in K \,|\, \mathrm{val}\,(x) = 0\}$. (Chapter 2).

According to the initial value, define $I_1^0 = \kappa_{1,0}\, e^2$, $I_1^1 = \kappa_{1,1}\, e^0$; $I_2^0 = \kappa_{2,0}\, e^3, \ldots$; $V_1^0 = \lambda_1 e^3, V_1^0 = \lambda_2 e^4, \ldots$ *etc.* $(\kappa_{n,0}, \kappa_{n,1}, \lambda_n \in R^\times)$.

Therefore, the determinant $\Phi_\varepsilon(x,y) = \det(X(y) - xE)$ (Chapter 3) can be expanded:

$$\Phi_\varepsilon = -y^3 + (r_{3,1}e^0 x + r_{3,0}e^0)y^2 + (r_{2,2}e^0 x^2 + r_{2,1}e^0 x + r_{2,0}e^1)y$$
$$+ (x^4 + r_{1,3}e^2 x^3 + r_{1,2}e^5 x^2 + r_{1,1}e^9 x + r_{1,0}e^{14}) - re^{30}y^{-1}$$

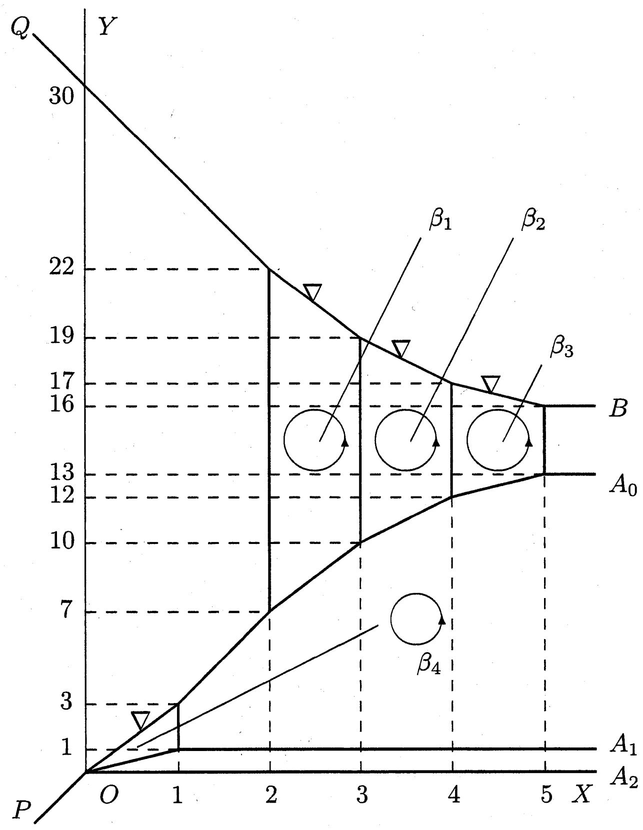
$(r_{i,j} \in R^\times,\ \forall i,j)$. Recall that Φ_ε is the defining pL-polynomial of the spectral curve C_ε.

The picture of the tropical curve $\mathrm{Trop}\, C = TV(\Phi_\varepsilon)$ is given in page 55. On C_ε, there exist 6 points $A_0 : (x,y) = \left(0, \prod_{n=1}^4 I_n^0\right)$, $A_1 : (x,y) = \left(0, \prod_{n=1}^4 I_n^1\right)$, $B : (x,y) = \left(0, \prod_{n=1}^4 V_n^0\right)$, $Q : (x,y) = (\infty, 0)$ and $P : (x,y) = (\infty,\infty)$ (proposition 3.2.1). The associated points on $\mathrm{Trop}\, C$ are given by:

(i) $A_0 : (X,Y) = (\mathrm{val}\,(0),\ \mathrm{val}\,(\prod_{n=1}^4 I_n^0)) = (+\infty, 13)$,

(ii) $A_1 : (X,Y) = (\mathrm{val}\,(0),\ \mathrm{val}\,(\prod_{n=1}^4 I_n^1)) = (+\infty, 1)$,

(iii) $A_2 : (X,Y) = (\mathrm{val}\,(0),\ \mathrm{val}\,(\prod_{n=1}^4 I_n^2)) = (+\infty, 0)$,



Q
Y
30
22
19
17
16
13
12
10
7
3
1
β_1
β_2
β_3
β_4
B
A_0
A_1
A_2
P
O
1
2
3
4
5
X

(iv) $B : (X, Y) = (\mathrm{val}\,(0), \mathrm{val}\,(\prod_{n=1}^{4} V_n^0)) = (+\infty, 16)$,

(v) $Q : (X, Y) = (-\infty, +\infty)$,

(vi) $P : (X, Y) = (-\infty, -\infty)$.

We will slightly abuse the symbols A_0, A_1,... *etc.* to mean the points both on the complex curve and on the associated tropical curve.

define the paths

Let $\mathrm{Trop}\,\mathfrak{U}$ by the universal covering of $\mathrm{Trop}\,C$. To define the fundamental domain $D_0 \hookrightarrow \mathrm{Trop}\,\mathfrak{U}$, we fix the 4 cuts on $\mathrm{Trop}\,C$ (the symbols '∇'). Denote $D_0 := \mathrm{Trop}\,C \setminus \{\text{four } \nabla \text{ s}\}$. D_0 is a simply connected domain.

Then define the oriented paths $\gamma_{Q\to P}, \gamma_{A_i\to P}, \gamma_{B\to P}, \subset D_0$ $(i = 0, 1, 2)$ which connect two points $Q \to P$, $A_i \to P$, $B \to P$ respectively.

define the lifted paths

Define $\beta_1, \beta_2, \ldots, \beta_4 \in H_1(\mathrm{Trop}\,C, \mathbb{Z})$ as in the figure of $\mathrm{Trop}\,C$. The following data are calculated graphically (section 2.4.1).

$$B_T = \begin{pmatrix} 26 & -9 & 0 & 0 \\ -9 & 16 & -5 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$T\widetilde{\boldsymbol{A}}(\gamma_{Q\to P}) = (15, 0, 0, -3), \quad T\widetilde{\boldsymbol{A}}(\gamma_{A_0\to P}) = (-1, -1, -1, -3),$$
$$T\widetilde{\boldsymbol{A}}(\gamma_{A_1\to P}) = (0, 0, 0, -1), \quad T\widetilde{\boldsymbol{A}}(\gamma_{A_2\to P}) = (0, 0, 0, 0), \quad T\widetilde{\boldsymbol{A}}(\gamma_{B\to P}) = (-1, -1, -4, -3).$$

One will find that the path $\gamma_{A_2\to P}$ can be ignored because $T\widetilde{\boldsymbol{A}}(\gamma_{A_2\to P})$ is zero. This preperty refrects the fact that the path $\gamma_{A_2\to P}$ comes from the dummy zero valuables.

Now we define the lifted paths $\Gamma_{Q\to P}$, $\Gamma_{A_0\to P}, \ldots$ of $\gamma_{Q\to P}$, $\gamma_{A_0\to P}, \ldots$ on $\mathrm{Trop}\,\mathfrak{U}$ to satisfy the relation (3.2.25):

$$-T\widetilde{\boldsymbol{A}}(\Gamma_{Q\to P}) + T\widetilde{\boldsymbol{A}}(\Gamma_{A_0\to P}) + T\widetilde{\boldsymbol{A}}(\Gamma_{A_1\to P}) + T\widetilde{\boldsymbol{A}}(\Gamma_{B\to P}) = 0.$$

For example, let

$$\Gamma_{Q\to P} := \gamma_{Q\to P}, \quad \Gamma_{A_0\to P} := \gamma_{A_0\to P},$$
$$\Gamma_{A_1\to P} := \gamma_{A_1\to P}, \quad \Gamma_{B\to P} := \gamma_{B\to P} + \beta_1 + \beta_2 + \beta_3 + \beta_4.$$

construct the solution

It is easy to calculate the parameters $\mathfrak{C}^t$, $\mathfrak{B}_m$ (Section 4.2.3):

$$\begin{aligned}\mathfrak{C}^0 &= 13/4 + \langle B_T^{-1}(15, 0, 0, -3), (-1, -1, -1, -3)\rangle \\ &= 13/4 + \langle (3/4, 2/4, 1/4, -3/4), (-1, -1, -1, -3)\rangle = 4, \\ \mathfrak{C}^1 &= 1/4 + \langle (3/4, 2/4, 1/4, -3/4), (0, 0, 0, -1)\rangle = 1, \\ \mathfrak{B} &= 16/4 + \langle (3/4, 2/4, 1/4, -3/4), (16, 1, 1, 1)\rangle = 16.\end{aligned}$$

Therefore, the solution of the pBBS is:

$$Q_n^t = 4 + (\mathfrak{T}_{n+1}^{t-2} + \mathfrak{T}_{n+2}^{t-1} - \mathfrak{T}_{n+2}^{t-2} - \mathfrak{T}_{n+1}^{t-1}), \quad (t : \text{even}) \tag{4.3.1}$$

$$Q_n^t = 1 + (\mathfrak{T}_{n+1}^{t-2} + \mathfrak{T}_{n+2}^{t-1} - \mathfrak{T}_{n+2}^{t-2} - \mathfrak{T}_{n+1}^{t-1}), \quad (t : \text{odd}) \tag{4.3.2}$$

$$W_n^t = 16 + (\mathfrak{T}_{n-1}^{t} + \mathfrak{T}_{n+1}^{t-2} - \mathfrak{T}_{n}^{t-2} - \mathfrak{T}_{n}^{t}), \tag{4.3.3}$$

where $\mathfrak{T}_n^t = \Theta(\mathfrak{v}_n^t; B_t)$,

$$\mathfrak{v}_n^t = \mathfrak{v}_0 + \left[\frac{t+1}{2}\right](-1,-1,-1,-3) + \left[\frac{t}{2}\right](0,0,0,-1) + (n-1)\cdot(15,0,0,3), \tag{4.3.4}$$

$[x] = \max\{r \in \mathbb{Z} \,|\, r \leq x\}$, $\mathfrak{v}_0 \in \mathbb{R}^4$ is a initial value.

4.3.2 Periods

As an application of the present method, we demonstrate the calculation of the *relative period* [24] of pBBS.

The relative period of pBBS is the minimal positive integer $T > 0$ such that:

there exists an integer α such that $Q_{n+\alpha}^{t+T} = Q_n^t$, $W_{n+\alpha}^{t+T} = W_n^t$ for all n, t.

In other words, $T = \min\{t > 0 \,|\, \mathfrak{v}_n^t \equiv \mathfrak{v}_0 \pmod{B_T\mathbb{Z}^4}, \exists n\}$. Then, we have

$$\begin{aligned} &T = 2k \text{ is the relative period} \\ &\quad\Leftrightarrow \exists n,\ \ k\cdot(-1,-1,-1,-4) + n\cdot(15,0,0,-3) \in B_T\mathbb{Z}^4 \\ &\quad\Leftrightarrow \exists n,\ \ k\cdot B_T^{-1}(-1,-1,-1,-4) + n\cdot B_T^{-1}(15,0,0,-3) \in \mathbb{Z}^4 \\ &\quad\Leftrightarrow \exists n,\ \ k\cdot(-1/10,-8/45,-17/90,-1) + n\cdot(3/4,2/4,1/4,-3/4) \in \mathbb{Z}^4. \end{aligned} \tag{4.3.5}$$

From this, we conclude that the relative period of given state of pBBS is $90 \times 2 = 180$ (see *Example picture* in page 57).

Example picture

```
t= 0    11...1112....111...11111......
   2    ..11.....1112...111.....11111.
   4    11..1111.....1112..111.......1
   6    ..11....11111....112..1111....
   8    1...11.......1111...112...1111
  10    .111..1111.......111...1112...
  12    2...11....11111.....111....111
  14    .1112.11.......11111...111....
  16    1....12.1111........111...1111
  18    .1111..12...11111......111....
t=20    .....11..1112....11111....111.
        111....11....1112.....1111...1
        ...1111..111.....1112.....111.
        11.....11...11111....1112....1
        ..111....11......1111....11112
        112..1111..111.......1111.....
        ...112...11...11111......1111.
```

```
          111...112..11......11111.....1
          ...111...112.111........11111.
          111...111...12..11111........1
t=40      ...111...111..112....11111....
          1.....111...11...1112.....1111
          .11111...111..11.....1112.....
          ......111...11..11111....1112.
          1112.....111..11.....1111....1
          ....11112...11..111......1111.
          111......1112.11...1111......1
          ...1111......12.111....11111..
          111....1111....12..111......11
          ...1111....1111..112..111.....
t=60      .......1111....11...112..11111
          11111......1111..11....112....
          .....11111.....11..1111...112.
          112.......11111..11....111...1
          ...11112.......11..1111...111.
          111.....11112....11....111...1
          ...1111......11112.11.....111.
          11.....1111.......12.11111...1
          ..11111....1111.....12....111.
          11.....1111....11111..12.....1
t=80      ..111......1111.....11..11112.
          112..1111......1111...11.....1
          ...112...11111.....111..111...
          1.....112.....11111...11...111
          .11111...112.......111..111...
          1.....111...11112.....11...111
          .1111....111.....11112..11....
          .....1111...111.......112.1111
          11111....111...1111......12...
          .....1111...111....11111...12.
t=100     112......111...1111.....111..1
          ...11112....111....1111....11.
          1.......11112..111.....1111..1
          .1111........112..11111....11.
          1....11111......112....1111..1
          .1111.....11111....112.....11.
          1....1111......1111...1112...1
          .111.....1111......111....1112
          12..11111....1111.....111.....
          ..12.....1111....11111...111..
t=120     11..112......1111.....111...11
          ..11...11112.....1111....111..
          1...11......11112....1111...11
          .111..111........11112...111..
          1...11...11111........1112..11
          .111..111.....11111.......112.
          12..11...1111......11111.....1
```

```
        ..112.11.....1111.......11111.
        11...12.11111....1111........1
        ..111..12....1111....11111....
t=140   1....11..112.....1111.....1111
        .1111..11...1112.....1111.....
        .....11..111....11112....1111.
        111....11...111......11112...1
        ...1111..111...111........1112
        1112...11...111...11111.......
        ....1112.11....111.....11111..
        111.....12.1111...111.......11
        ...11111..12...111...1111.....
        ........11..1112..111....11111
t=160   11111.....11....112..1111.....
        .....11111..11.....112...1111.
        111.......11..11111...112....1
        ...1111.....11.....111...11112
        1112...11111..11......111.....
        ....1112....11..11111....111..
        11......1112..11.....1111...11
        ..11111.....112.111......111..
        1......11111...12..1111.....11
        .111........111..112...11111..
t=180   1...11111......11...1112....11
```

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