J. Math. Sci. Univ. Tokyo
4 (1997), 551–593.

On Continuation of Gevrey Class Solutions of Linear Partial Differential Equations

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Dedicated to Professor Hikosaburo KOMATSU for his 60-th anniversary

Abstract. We give a sufficient condition for the removability of thin singularities of Gevrey class solutions of linear partial differential equations. In §1 we give a sufficient condition for the removability in the case of equations with constant coefficients. Then in §2 we discuss the necessity of the condition and construct non-trivial solutions with irremovable thin singularities for some class of equations. In §3 we give a sufficient condition for the removability of thin singularities of Gevrey class solutions in the case of equations with real analytic coefficients.

§0. Introduction

In this article, we gather results on continuation to thin singularity (or removability of thin singularities) of Gevrey class solutions to linear partial differential equations. Some of the results given here are easily derived from Grushin's pioneering works on continuation of C^∞ solutions and from the author's former works on continuation of regular solutions. But it will be worth gathering them all to an article, because they may not be obvious for the readers who are not specialized in this subject. Moreover it will be adequate to dedicate this to Professor Hikosaburo Komatsu, who devoted his half carreer to the study of ultra-differentiable functions and ultradistributions.

Here is a brief plan of the present article. The first two sections treat equations with constant coefficients. In §1 we give a sufficient condition for

1991 *Mathematics Subject Classification.* 35G05, 35B60, 35E20.
[1]Partially supported by GRANT-IN-AID FOR SCIENTIFIC RESEARCH No. 07404003.

the removability of thin singularities of the Gevrey class solutions. This is a translation of Grushin's work except for small details. In §2 we discuss the necessity of the condition given in §1. This is to construct non-trivial solutions with irremovable thin singularities under the condition opposite to §1. We generalize the construction of Grushin who gave such a few examples in his work [G2]. As an example, the precise Gevrey index for the threshold of existence of solutions with thin singularity is determined for the Schrödinger equation. In §3 we give a sufficient condition for the removability of thin singularities of Gevrey class solutions in the case of equations with real analytic coefficients. This is a modification of the author's work for the removability of thin singularities of real analytic solutions.

For a general survey on this subject, we refer to [Kn11] for results until 1992, and [Kn12], where a list of open problems is gathered. The present article treats some of them concerning Gevrey class solutions.

§1. Continuation of Gevrey class solutions to equations with constant coefficients

Let $P(D)$ be a linear partial differential operator with constant coefficients, where $P(\zeta)$ is a polynomial in n variables $\zeta = (\zeta_1, \dots, \zeta_n)$ and $D = (D_1, \dots, D_n)$ with $D_j = -i\partial/\partial x_j$, $j = 1, \dots, n$. We define the two spaces of Gevrey class functions of index s by

$$(1.1)\quad \mathcal{E}^{(s)}(\Omega) := \{f(x) \in C^\infty(\Omega);\ \forall K \subset\subset \Omega,\ \forall h > 0,\ \exists C_{K,h} > 0 \\ \sup_{x \in K} |D^\alpha f(x)| \le C_{K,h} h^{|\alpha|} \alpha!^s \ \text{ for } \forall \alpha\},$$

and

$$(1.2)\quad \mathcal{E}^{\{s\}}(\Omega) := \{f(x) \in C^\infty(\Omega);\ \forall K \subset\subset \Omega,\ \exists h = h(K) > 0, \\ \exists C = C(K) > 0, \\ \sup_{x \in K} |D^\alpha f(x)| \le C h^{|\alpha|} \alpha!^s \ \text{ for } \forall \alpha\}.$$

The first space has a simple topology of Fréchet space and is easier to treat, but it is a little less natural because when $s = 1$, this corresponds to the space of entire functions. The second space has a very complicated topological structure and does not allow the closed range theorem to hold.

Hence for this space we cannot utilize fundamental theorems such as the global surjectivity on convex open sets of linear partial differential operators with constant coefficients or the Fundamental Principle of Ehrenpreis-Palamodov. But it is more natural because for $s = 1$ this corresponds to the space of real analytic functions which is localizable along the real axis.

We also set

$$\mathcal{E}^{1+}(\Omega) := \bigcap_{s>1} \mathcal{E}^{(s)}(\Omega) = \bigcap_{s>1} \mathcal{E}^{\{s\}}(\Omega). \tag{1.3}$$

This is a very convenient space, still containing enough functions in non-quasianalytic ultra-differentiable class. Following the usage of Komatsu, we shall denote in the sequel by $\mathcal{E}^*(\Omega)$ either of the spaces $\mathcal{E}^{(s)}(\Omega)$, $\mathcal{E}^{\{s\}}(\Omega)$, $\mathcal{E}^{1+}(\Omega)$ when we can state something commonly to these spaces. Thus $\mathcal{E}^*$ denotes either of these function classes. In the same time, this symbol will denote the corresponding sheaf (that is, the localization) on $\boldsymbol{R}^n$. As usual we let $\mathcal{D}^*(\Omega)$ denote the functions of class $\mathcal{E}^*$ with compact support contained in Ω (together with the obvious topology if the dual space, that is, the space of ultradistributions of this class, is considered). For a general set L we let $\mathcal{E}^*(L)$ denote the functions of class $\mathcal{E}^*$ defined on a neighborhood of L, with the obvious identification in the sense of inductive limit with respect to the neighborhoods.

In general, we denote by $\mathcal{E}^*_P(\Omega)$ the space of solutions in Ω of the equation $P(D)u = 0$ of class $\mathcal{E}^*$. Let $K \subset \Omega$ denote a thin compact subset. Here "thin" means that the interior is void. We assume that it is contained in a hyperplane, say $\nu \cdot x = 0$. (This follows automatically for convex thin set, as we mainly consider in the sequel.) We study the continuation of solutions of $P(D)u = 0$ in $\mathcal{E}^*(\Omega \setminus K)$ to solutions in $\mathcal{E}^*(\Omega)$.

Proposition 1.1. *The canonical map induced by the canonical restriction from Ω to $\Omega \setminus K$:*

$$\mathcal{E}^*_P(\Omega) \to \mathcal{E}^*_P(\Omega \setminus K)$$

is injective. In other words, there are no solutions of $P(D)u = 0$ with compact support.

Actually, any element in the kernel of the above map would be a solution of $P(D)u = 0$ with compact support. But via the Fourier transform we would then obtain $P(\zeta)\widehat{u} = 0$, where $\widehat{u}$ is entire, whence $\widehat{u} \equiv 0$.

Thus the quotient space

$$\mathcal{E}_P^*(\Omega \setminus K)/\mathcal{E}_P^*(\Omega) \tag{1.4}$$

will represent the obstruction for continuation of solutions of this class to K. The reason why we restrict K to thin sets is obvious from the non-quasianalyticity of the function class under consideration: If K had a non-void interior, then choosing $f \in \mathcal{D}^*(\operatorname{Int} K) \setminus P(D)\mathcal{D}^*(\operatorname{Int} K)$ and a solution $u \in \mathcal{E}^*(\boldsymbol{R}^n)$ of $P(D)u = f$, $u|_{\Omega\setminus K}$ would present a non-trivial element of (1.4).

First we shall show that the obstruction space (1.4) depends only on K and not on Ω. For this purpose we recall the notion of local cohomology groups with coefficients in the solution sheaf $\mathcal{E}_P^*$ of class $\mathcal{E}^*$ of the equation $Pu = 0$.

Proposition 1.2. *We have the following isomorphism*

$$\mathcal{E}_P^*(\Omega \setminus K)/\mathcal{E}_P^*(\Omega) \stackrel{\sim}{=} H_K^1(\Omega, \mathcal{E}_P^*). \tag{1.5}$$

More generally, for any set L containing K in its interior, we have

$$\mathcal{E}_P^*(L \setminus K)/\mathcal{E}_P^*(L) \stackrel{\sim}{=} H_K^1(L, \mathcal{E}_P^*). \tag{1.5bis}$$

The quotient space in (1.5) *or* (1.5bis) *is determined by K only and does not depend on the choice of the neighborhoods.*

Proof. Recall the following fundamental exact sequence of local cohomology groups:

$$\begin{aligned} 0 &\to \Gamma_K(\Omega, \mathcal{E}_P^*) \to \Gamma(\Omega, \mathcal{E}_P^*) \to \Gamma(\Omega \setminus K, \mathcal{E}_P^*) \\ &\to H_K^1(\Omega, \mathcal{E}_P^*) \to H^1(\Omega, \mathcal{E}_P^*) \to H^1(\Omega \setminus K, \mathcal{E}_P^*) \\ &\to H_K^2(\Omega, \mathcal{E}_P^*) \to H^2(\Omega, \mathcal{E}_P^*) = 0. \end{aligned} \tag{1.6}$$

Here we have $\Gamma_K(\Omega, \mathcal{E}_P^*) = 0$ by Proposition 1.1. (The fact $H^2(\Omega, \mathcal{E}_P^*) = 0$ follows from the resolution (1.7) as will be discussed below.) We shall show that the mapping

$$H^1(\Omega, \mathcal{E}_P^*) \to H^1(\Omega \setminus K, \mathcal{E}_P^*)$$

is always injective. Then we will obtain the isomorphism (1.5). Incidentally, we obtain the exact sequence

$$0 \to H^1(\Omega, \mathcal{E}_P^*) \to H^1(\Omega \setminus K, \mathcal{E}_P^*) \to H_K^2(\Omega, \mathcal{E}_P^*) \to 0. \tag{1.7}$$

Recall now the following exact sequence of sheaves

$$0 \to \mathcal{E}_P^* \to \mathcal{E}^* \stackrel{P(D)}{\to} \mathcal{E}^* \to 0. \tag{1.8}$$

Here the surjectivity in the last arrow, that is, the local solvability in this class, is an easy consequence of the existence of a fundamental solution for a single linear partial differential operator P with constant coefficients. Taking the fundamental exact sequence of global cohomology groups on an open set Ω, we obtain

$$\begin{aligned} 0 \to \Gamma(\Omega, \mathcal{E}_P^*) \to \Gamma(\Omega, \mathcal{E}^*) &\xrightarrow{P(D)} \Gamma(\Omega, \mathcal{E}^*) \\ &\to H^1(\Omega, \mathcal{E}_P^*) \to H^1(\Omega, \mathcal{E}^*) = 0. \end{aligned} \tag{1.9}$$

The fact $H^1(\Omega, \mathcal{E}^*) = 0$ is obvious because the sheaf $\mathcal{E}^*$ is fine. (From this $H^2(\Omega, \mathcal{E}_P^*) = 0$ also follows.) Thus it suffices to show that the natural mapping induced from the restriction

$$\mathcal{E}^*(\Omega)/P(D)\mathcal{E}^*(\Omega) \to \mathcal{E}^*(\Omega \setminus K)/P(D)\mathcal{E}^*(\Omega \setminus K)$$

is injective. Suppose that $u \in \mathcal{E}^*(\Omega)$ represents an element mapped to 0. This implies that there exists $v \in \mathcal{E}^*(\Omega \setminus K)$ such that

$$u|_{\Omega \setminus K} = P(D)v.$$

Employing partitions of unity, construct $h \in \mathcal{E}^*(\Omega)$ and $w \in \mathcal{E}^*(\boldsymbol{R}^n \setminus K)$ such that $v = h - w$ on $\Omega \setminus K$. We can obviously choose w in such a way that $w \equiv 0$ outside a ball of some radius $R > 0$, by cutting v smoothly in the class $\mathcal{E}^*$ on a neighborhood of K. Put

$$g = \begin{cases} P(D)w & \text{on} \quad \boldsymbol{R}^n \setminus K, \\ P(D)h - u & \text{on} \quad \Omega. \end{cases}$$

This definition is consistent on the common domain:

$$P(D)w = P(D)h - P(D)v = P(D)h - u \qquad \text{on} \quad \Omega \setminus K.$$

Thus g becomes a well defined element of $\mathcal{E}^*(\boldsymbol{R}^n)$, which has compact support by the choice of w as above. Let $f \in \mathcal{E}^*(\boldsymbol{R}^n)$ be a solution of $P(D)f = g$. (We can simply obtain such f by convoluting g with the distributional fundamental solution of $P(D)$ which obviously preserves the Gevrey regularity.) Then, on Ω we have

$$u = P(D)(h - f).$$

Hence, it represents 0 in $\mathcal{E}^*(\Omega)/P(D)\mathcal{E}^*(\Omega)$.

The proof for general neighborhood L of K is just similar. Now that the isomorphism (1.5) is established, the final conclusion follows from the excision theorem of local cohomology groups. □

Remark. 1) We cannot expect $H^1(\Omega, \mathcal{E}_P^*) = 0$, unless we have the global surjectivity of $P(D)$ on Ω in this function class. This follows from the exact sequence (1.8). To have this surjectivity for open Ω, we first of all need to assume that Ω is convex. Then it is valid for the class $\mathcal{E}^{(s)}$ (see e.g. Björck [Bj1]), but still not in general for $\mathcal{E}^{\{s\}}$ (see e.g. Cattabriga [C1]). The above method of argument was first introduced by [Kn6] for the real analytic solutions, to which the global surjectivity is neither available. Note that the above Proposition (or the sequence (1.7)) implies that in such a situation, the obstruction for the global surjectivity is concentrated on the neighborhood of $\partial\Omega$.

2) We have an alternative choice of neighborhoods of K for which the global surjectivity holds. It is to take compact neighborhoods $L \supset\supset K$. In this case the surjectivity of $P(D) : \mathcal{E}^*(L) \to \mathcal{E}^*(L)$ holds irrespective of the convexity of L, because of the non-quasianalytic property of our class. This

fact was implicitly employed by some of the proofs for the corresponding assertions in the author's former publications.

In view of the above Proposition, we can always assume Ω to be convex, thus allowing to apply the Fourier analysis. Sometimes the choice of convex compact L simplifies the situation further.

Next, we shall show that the obstruction (1.4) can be decomposed via the irreducible components of $P(D)$:

Corollary 1.3. *Let Q be any factor of P. Then we have a canonical injection*

$$H_K^1(\Omega, \mathcal{E}_Q^*) \hookrightarrow H_K^1(\Omega, \mathcal{E}_P^*). \tag{1.10a}$$

Conversely, let $P = Q_1^{m_1} \cdots Q_N^{m_N}$ be the decomposition of $P(\zeta)$ into different irreducible components with their multiplicities counted. Then we have a (*non-canonical*) *injection*

$$H_K^1(\Omega, \mathcal{E}_P^*) \hookrightarrow \prod_{j=1}^{N} [H_K^1(\Omega, \mathcal{E}_{Q_j}^*)]^{m_j}. \tag{1.10b}$$

Hence we have $H_K^1(\Omega, \mathcal{E}_P^*) = 0$ if and only if $H_K^1(\Omega, \mathcal{E}_{Q_j}^*) = 0$ for $j = 1, \ldots, N$.

Proof. Let $P = QR$ be a decomposition of polynomial. (We do not assume that Q, R are mutually prime.) Note that we have the following exact sequence of sheaves similar to (1.8):

$$0 \to \mathcal{E}_Q^* \to \mathcal{E}_P^* \overset{Q(D)}{\to} \mathcal{E}_R^* \to 0. \tag{1.11}$$

As a matter of fact, the exactness is obvious except for the surjectivity of the last arrow. But any solution $u \in \mathcal{E}^*$ of $Q(D)u = f$ for $f \in \mathcal{E}_R^*$ will satisfy $P(D)u = R(D)(Q(D)u) = R(D)f = 0$. Taking the fundamental exact sequence of the relative cohomology groups for an open neighborhood $\Omega \supset K$, we obtain from (1.11) the following exact sequence:

$$0 \to \Gamma_K(\Omega, \mathcal{E}_Q^*) \to \Gamma_K(\Omega, \mathcal{E}_P^*) \to \Gamma_K(\Omega, \mathcal{E}_R^*) \tag{1.12}$$

$$\begin{aligned}&\to H^1_K(\Omega,\mathcal{E}^*_Q)\to H^1_K(\Omega,\mathcal{E}^*_P)\to H^1_K(\Omega,\mathcal{E}^*_R)\\&\to H^2_K(\Omega,\mathcal{E}^*_Q)\to H^2_K(\Omega,\mathcal{E}^*_P).\end{aligned}$$

Here the terms in the first row vanish because of the absence of solutions with compact support. Hence the existence of canonical inclusion mapping (1.10a) follows. Since $\mathcal{E}^*$ is not flabby, the second degree relative cohomology groups do not vanish even for single equations. But we have at least the injection mapping

$$H^1_K(\Omega,\mathcal{E}^*_P)/H^1_K(\Omega,\mathcal{E}^*_Q)\hookrightarrow H^1_K(\Omega,\mathcal{E}^*_R),$$

whence in view of the complete reducibility of the vector spaces, we have a (non-canonical) injection mapping

$$H^1_K(\Omega,\mathcal{E}^*_P)\hookrightarrow H^1_K(\Omega,\mathcal{E}^*_Q)\oplus H^1_K(\Omega,\mathcal{E}^*_R). \tag{1.13}$$

Repeating this argument for Q, R, we finally obtain an injection like (1.10a). $\square$

Remark. In the preprint version of this article, we gave a proof for the assertion that (1.10b) is an algebraic isomorphism, which was wrong as the referee kindly pointed out. Here we give another proof of the isomorphism for curiosity's sake, although it will not be useful because we cannot give a canonical mapping.

To prove an abstract isomorphism, it suffices to show that both sides of (1.10b) have algebraic dimension (over $\boldsymbol{C}$ always) of the same cardinality. Note that in view of Corollary 1.3 the algebraic dimension of each side of (1.10b) is estimated by a finite multiple of the other's. Thus it suffices to show that each $H^1_K(\Omega,\mathcal{E}^*_Q)$ is either 0 or is infinite dimensional. Suppose that it has a non-zero finite dimension, and let $u\in\mathcal{E}^*_Q(\Omega\setminus K)$ represent a non-trivial element. Choose R which is irreducible and not contained in the factors of Q. Then $R(D)^ju$, $j=0,1,2,\dots$ will define elements of $H^1_K(\Omega,\mathcal{E}^*_Q)$ of which a finite number are linearly dependent, say

$$S(D)u:=\sum_{j=0}^m c_jR(D)^ju=v,$$

$$v \in \mathcal{E}_P^*(\Omega), \quad c_j \in \boldsymbol{C}, j = 0, 1, 2, \dots, m, \quad c_m \neq 0.$$

The simultaneous equation

$$S(D)w = v, \qquad Q(D)w = 0$$

has a solution $w \in \mathcal{E}^*(\Omega)$ as long as we shrink Ω a little for the fear of the case of $* = \{s\}$ type space. (We neglect to introduce a new notation for the shrinked domain.) Then $u - w \in \mathcal{E}^*(\Omega \setminus K)$ will satisfy

$$Q(D)(u - w) = 0, \qquad S(D)(u - w) = 0.$$

Obviously, Q and S are primary to each other. Hence they define an overdetermined system, and by Ehrenpreis-Malgrange's classical theorem the solution $u - w$ can be continued to K. (Though the theorem may not have been written down for the class $\mathcal{E}^*$, it is easy to modify their theory to this case. A more easy-going way is that if ever we have a continuation as a C^∞-solution, we can show that it is in class $\mathcal{E}^*$ via the propagation of $\mathcal{E}^*$ regularity for solutions of, say, $Q(D)u = 0$ up to K. This propagation theorem can be shown by a standard argument employing a cut-off function in this class and a distribution fundamental solution of $Q(D)$ by which the convolution preserves the $\mathcal{E}^*$ regularity.) Thus $u - w$, hence u, can be continued to a solution of $Q(D)u = 0$ near K, and irrespective of the fact of shrinking Ω, we conclude that $u \in \mathcal{E}_Q^*(\Omega)$ for the original Ω, which is a contradiction.

Here we recall the notion of irregularity of a characteristic direction. We adopt the following definition. Let $P(\zeta)$ be an irreducible polynomial of order m such that $P_m(\nu) = 0$, where P_m denotes the principal part. Consider

$$Q(s, t) := P(t\xi + s\nu) = q_0(\xi)t^m + q_1(s; \xi)t^{m-1} + \cdots + q_m(s; \xi)$$

as a polynomial of the two variables (s, t). For generically fixed ξ, let κ be the minimum value of the leading powers of the Puiseux expansions of the roots of Q for t in terms of s representing irreducible germs of $N(Q)$ passing through the point $(\infty, 0)$ at infinity. Then we set $\mu = (1 - \kappa)^{-1}$

and call it the *multiplicity* of ν. A more exact definition may be given via the Newton polygon, transforming the point at infinity to the origin: Set $\sigma = 1/s$, $\tau = t/s$ and let

$$q(\sigma, \tau) = s^{-m} Q(s,t)\big|_{s=1/\sigma, t=\tau/\sigma}.$$

Factorize it as a polynomial of τ with coefficients in $\mathcal{O}_{\sigma,0}$ at 0. Then μ is the inverse of the minimum value with respect to the irreducible factors of the leading powers of the Puiseux expansions of the roots of them.

Notice that the irregularity employed here is the mildest one, in comparison with the strongest one which is usually used e.g. to define the hyperbolicity.

Our main result here is the following

THEOREM 1.4. *Let K be a compact set contained in a hyperplane $\nu \cdot x = 0$. Assume further that every irreducible component of $P(\zeta)$ has ν as characteristic direction of irregularity $\leq \mu$. Then*

$$\mathcal{E}_P^{(s)}(\Omega \setminus K)/\mathcal{E}_P^{(s)}(\Omega) = 0, \qquad \text{if } s \leq \mu/(\mu - 1),$$

$$\mathcal{E}_P^{\{s\}}(\Omega \setminus K)/\mathcal{E}_P^{\{s\}}(\Omega) = 0, \qquad \text{if } s < \mu/(\mu - 1).$$

COROLLARY 1.5. *Assume that every irreducible component of $P(\zeta)$ is non-elliptic. Then the isolated singularities of solutions of class $\mathcal{E}^{1+}$ are removable, that is,*

$$\mathcal{E}_P^{1+}(\Omega \setminus \{0\})/\mathcal{E}_P^{1+}(\Omega) = 0. \tag{1.14}$$

Remark that this sufficient condition on $P(D)$ is the same as the one for the removablilty of isolated singularities of real analytic solutions given in [Kn1]–[Kn2].

Although the proof of Theorem 1.4 is almost a literal translation of Grushin's original article [G2] for the removability of isolated singularities of C^∞ solutions, we shall reproduce it here in detail, because it is nevertheless important to indicate in which point the Gevrey regularity allows to simplify the result.

PROOF. In view of the Propositions prepared hitherto, we can assume without loss of generality that $P(D)$ is irreducible and Ω is convex. Now we need to recall the Grushin representation: Given $u(x) \in \mathcal{E}_P^*(\Omega \setminus K)$, choose an extension $[u]$ to an element of $\mathcal{E}^*(\Omega)$ with modification in the ε-neighborhood K_ε of K. Then $P(D)[u] \in \mathcal{D}^*(K_\varepsilon)$. Thus its Fourier transform $F(\zeta) = \widehat{P(D)[u]}$ becomes an entire function and in view of the Paley-Wiener type theorem (see e.g. Komatsu [Km2], Theorem 9.1) it satisfies

$$(1.15)\quad \forall h > 0 \qquad |F(\zeta)| \le C_h e^{-h|\operatorname{Re}\zeta|^{1/s} + H_K(\operatorname{Im}\zeta) + \varepsilon|\operatorname{Im}\zeta|}, \qquad \text{if } \{*\} = (s),$$

$$(1.16)\qquad \exists h = h(\varepsilon) > 0 \qquad |F(\zeta)| \le C e^{-h|\operatorname{Re}\zeta|^{1/s} + H_K(\operatorname{Im}\zeta) + \varepsilon|\operatorname{Im}\zeta|}, \qquad \text{if } \{*\} = \{s\}.$$

If we restrict $F(\zeta)$ to the complex characteristic variety

$$N(P) := \{\zeta \in \boldsymbol{C}^n;\ P(\zeta) = 0\}$$

of P, it defines a global holomorphic function on $N(P)$ with the same estimate. This does not depend on the choice of the modification $[u]$ of u nor of ε. For, the difference of any such two modifications has the form $P(D)v$ with $v \in \mathcal{D}^*(K_\varepsilon)$, where ε denotes the bigger one. Hence it vanishes by the Fourier transformation and restriction to $N(P)$. Thus we obtain the *Grushin representation*:

$$(1.17)\qquad H_K^1(\Omega, \mathcal{E}_P^*) \to \widehat{\mathcal{E}^*(K)}[N(P)].$$

Here obviously the right-hand side denotes the space of global holomorphic functions on $N(P)$ satisfying the estimate (1.15) or (1.16).

A variant of the so called Fundamental Principle asserts that (1.17) is a topological linear isomorphism. For our class (s) this really takes place, but for $\{s\}$ this holds only partially (see Proposition 1.6 below). To prove the continuation of solutions, however, we only need the injectivity of the mapping (1.17), which we shall show here: If $F(\zeta)|_{N(P)} = 0$, then $F(\zeta)$ is divisible by $P(\zeta)$ as an entire function. Then Malgrange's theorem guarantees that the quotient $G(\zeta) = F(\zeta)/P(\zeta)$ satisfies the same estimate as $F(\zeta)$. By the Paley-Wiener type theorem in the inverse direction, we can find $g \in \mathcal{D}^*(K_\varepsilon)$ such that $G(\zeta) = \widehat{g}$. Thus $P(D)[u] = P(D)g$, hence $v = [u] - g$ is another modification of u which satisfies $P(D)v = 0$ on the whole of Ω. This implies that our solution u can be continued to a solution on Ω if it is modified in K_ε. Such continuation with small modification is uniquely determined, because the difference of such two would again give a solution with compact support. Hence in view of the arbitrariness of ε this implies that u itself can be continued to K as a solution.

Thus (1.17) is injective. Therefore, to prove the theorem it suffices to show that the image of the obstruction $H^1_K(\Omega, \mathcal{E}^*_P)$ in (1.17) is trivial.

Until now the discussion was common to any $P(D)$ and any convex compact K. Now we employ the assumptions of our theorem. Choose the coordinate system in such a way that $\nu = (0, \dots, 0, 1)$ and that $(1, 0, \dots, 0)$ is a non-characteristic direction. In the sequel let us employ the abbreviation $\zeta'' = (\zeta_2, \dots, \zeta_{n-1})$. By the assumption on $P(D)$, for a generic choice of $\zeta'' = \zeta''_0$ and a small $\delta > 0$, the equation $P(\zeta_1, \zeta'', \zeta_n) = 0$ for ζ_1 has a solution $\zeta_1 = \tau(\zeta'', \zeta_n)$ which is multi-valued analytic in ζ_n in $|\zeta_n| \geq 1/\delta$ for each fixed ζ'' in $|\zeta'' - \zeta''_0| \leq \delta$, and which satisfies there the following estimate:

$$|\tau(\zeta'', \zeta_n)| \leq C|\zeta_n|^{(\mu-1)/\mu}.$$

For the present system of coordinates the assumption on K reads as $K \subset \{x_n = 0\}$, hence

$$H_K(\eta) \leq A|\eta_1| + A|\eta''|, \qquad H_{K_\varepsilon}(\eta) \leq (A+\varepsilon)|\eta_1| + (A+\varepsilon)|\eta''| + \varepsilon|\eta_n|.$$

Thus for each fixed ζ'' we obtain a function

$$G(z) := F(\tau(\zeta'', z), \zeta'', z)$$

of one variable z, which is multi-valued holomorphic in $|z| \geq 1/\delta$ and which satisfies there

$$|G(z)| \leq C_\varepsilon e^{-h|z|^{1/s} + (A+\varepsilon)C|z|^{(\mu-1)/\mu} + \varepsilon|\operatorname{Im} z|}. \tag{1.18}$$

Here, by the assumption, we have $1/s \geq (\mu-1)/\mu$ but we can choose $h > (A+1)C$ in case $* = (s)$, and we have $1/s > (\mu-1)/\mu$ in case

$* = \{s\}$. Thus in either case, in view of the arbitrariness of ε we can apply the Phragmén-Lindelöf principle to $G(z)$ to conclude that $G(z)$ is bounded on $\operatorname{Im} z \geq 1/\delta$, and similarly on $\operatorname{Im} z \leq -1/\delta$, hence on $|z| \geq 1/\delta$. Recall that the multivaluedness of $G(z)$ at $z = \infty$ comes from that of $\tau(\zeta'', z)$. Hence it can be represented by a Puiseux series of z, and we can find an integer $q \geq 1$ such that $H(w) := G(w^q)$ is single valued holomorphic at $w = \infty$. Therefore we can apply Riemann's theorem on removable isolated singularity to $H(w)$ to conclude that it is holomorphic at $w = \infty$. By the estimate (1.18), however, $H(w)$ decreases faster than any inverse power of $|w|$ along the real axis. Thus its Taylor expansion at $w = \infty$ should be trivial. Thus we conclude that $H(w) \equiv 0$, whence $G(z) \equiv 0$.

Now we have shown that $F(\zeta_1, \zeta'', \zeta_n) = 0$ on

$$N(P) \cap \{(\zeta_1, \zeta'', \zeta_n);\ |\zeta'' - \zeta_0''| \leq \delta,\ |\zeta_n| \geq 1/\delta,\ \zeta_1 = \tau(\zeta'', \zeta_n)\}.$$

Since this is an open subset of the irreducible algebraic variety $N(P)$, we conclude that $F(\zeta) \equiv 0$ on $N(P)$. This shows the continuation of this solution as is already remarked. □

In the next section, we need the surjectivity of (1.17), because in order to show the existence of a non-trivial solution with thin singularity, we construct a global holomorphic function on the variety $N(P)$ with the indicated growth condition. Therefore we prepare

Proposition 1.6. (1.17) *is an algebraic isomorphism for a general convex compact set K.*

It remains to show the surjectivity. It is proved in the standard way in the theory of Ehrenpreis-Palamodov on Fundamental Principle. Here we only sketch the outline. For a given $F(\zeta) \in \widehat{\mathcal{E}^*(K)}[N(P)]$, we first choose local extensions and make a 1-cochain $\{F_\lambda(\zeta)\}$ for a covering of $\boldsymbol{C}^n$. Then we make a 2-cocycle $\{F_{\lambda\mu}(\zeta) = (F_\lambda(\zeta) - F_\mu(\zeta))/P(\zeta)\}$. (The divisibility comes from the fact that each element of the 1-cochain vanishes on the multiplicity variety $N(P)$.) Finally, we prove the vanishing of degree 1 Čech cohomology group with growth condition corresponding to $\widehat{\mathcal{E}^*(K)}[N(P)]$. For $* = (s)$ the proof is similar to the case of $\widehat{\mathcal{B}[K]}$ of Fréchet type given in [Kn3]. For $* = \{s\}$, it is difficult to prove it directly because of the complicated topological structure. We can, however, prove the surjectivity under the growth condition

$$|F(\zeta)| \leq Ce^{-h|\zeta|^{1/s}+H_K(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|}.$$

with a *fixed* $\varepsilon > 0$. Then the problem becomes one for a DF type space and is treated by Palamodov [P1]. We obtain in this way a solution u_ε with singularity of size $\overline{K_\varepsilon}$, or more precisely, such that $P(D)u_\varepsilon \in \mathcal{D}^*(\Omega)$, with support in $\overline{K_\varepsilon}$. But the difference of two such $P(D)u_\varepsilon - P(D)u_{\varepsilon'}$ for $\varepsilon > \varepsilon'$, has the form of $P(D)w$ with $\operatorname{supp} w \subset \overline{K_\varepsilon}$, because its Fourier transform vanishes on $N(P)$. Thus

$$P(D)(u_\varepsilon - w) = P(D)u_{\varepsilon'}.$$

Hence we can modify $u_{\varepsilon'}$ by an element of $\mathcal{E}^*(\Omega)$ which is harmless, so that we can show the extendability of u_ε as a solution with singularity of smaller size $\overline{K_{\varepsilon'}}$. Continuing this process with a suitable choice of sequence $\varepsilon_k \to 0$, we can finally find a solution in class $\mathcal{E}^{\{s\}}$ with singularity in K. From this way of proof the topological isomorphism cannot be seen in this case (maybe false).

At the end of this section, we give an analogy of Grushin's result in [G1] concerning the removability of weak singularity. We shall say that an isolated singularity of a solution u of class $\mathcal{E}^*$ is weak if u is prolongeable to a neighborhood of the singularity as an ultradistribution of class $\mathcal{D}^{*\prime}$. Recall that $P(D)$ is called hypoelliptic in the class $\mathcal{E}^*$ if $P(D)$ admits a fundamental solution in $\mathcal{D}^{*\prime}$ which has $\mathcal{E}^*$ regularity outside the origin, or equivalently, if every local solution of $P(D)u = 0$ in $\mathcal{D}^{*\prime}$ becomes regular of class $\mathcal{E}^*$.

THEOREM 1.7. *The isolated weak singularity of any solution of $P(D)u = 0$ of class $\mathcal{E}^*$ is always removable if and only if $P(D)$ contains no irreducible factor which is hypoelliptic in the class $\mathcal{E}^*$.*

PROOF. We just copy Grushin's proof. The necessity is obvious because the fundamental solution of the hypoelliptic factor will provide a solution with irremovable weak singularity.

Conversely, assume that $P(D)$ admits a solution u in $\mathcal{E}^*$ with irremovable weak isolated singularity, say at the origin. We can find an irreducible factor $Q(D)$ of $P(D)$ and a factor $R(D)$ of $P(D)$ such that $v = R(D)u$ is a non-trivial solution of $Q(D)v = 0$ in $\mathcal{E}^*$ with irremovable weak isolated singularity. In fact, if $R(D)u$ becomes zero, we can discuss with $R(D)$ instead of $P(D)$. On the other hand, if the singularity of $v = R(D)u$ becomes removable as a solution of $Q(D)v = 0$, we can find a local solution w in $\mathcal{E}^*$ of $R(D)w = [v]$, where $[v]$ is the continuation of v in $\mathcal{E}^*$. Then $h = u - w$ will be a solution of $R(D)h = 0$ with irremovable weak isolated singularity, and we can continue again with $R(D)$.

Thus we can assume from the beginning that $P(D)$ is irreducible. Let $[u]$ be any prolongation of u to the singularity in the class $\mathcal{D}^{*\prime}$. Then by the structure theorem of elements of $\mathcal{D}^{*\prime}$ with isolated support, we can find an infinite order differential operator $J(D)$ adapted to this class such that

$$P(D)[u] = J(D)\delta.$$

Let $\chi(x)$ be a function in $\mathcal{D}^*(\boldsymbol{R}^n)$ with small support such that $\chi \equiv 1$ on a smaller neighborhood of the origin. Then

$$P(D)(\chi(x)[u]) = J(D)\delta + \varphi(x),$$

with $\varphi \in \mathcal{D}^*(\boldsymbol{R}^n)$. Employing this identity we can show that $P(D)$ is $J(D)$-hypoelliptic in $\mathcal{E}^*$, that is, for any solution f of $P(D)f = 0$ in class $\mathcal{D}^{*\prime}$ $J(D)f$ becomes regular of class $\mathcal{E}^*$:

$$J(D)f = J(D)\delta * f = P(D)(\chi[u]) * f - \varphi * f = -\varphi * f.$$

Note that $J(D)$ is not divisible by the irreducible polynomial $P(D)$, because if so, the singularity of u would be removable as is easily seen. From this fact, via standard argument we can show that the simultaneous equation

$$J(D)f = g, \qquad P(D)f = 0$$

has always a local solution in $\mathcal{D}^{*\prime}$ for any right-hand side g of class $\mathcal{D}^{*\prime}$ which itself satisfies $P(D)g = 0$. (This existence theorem is not trivial, because $J(D)$ is not a polynomial in general. See the Appendix.) Since as remarked above $J(D)f$ is in $\mathcal{E}^*$, so is g. This means that $P(D)$ is hypoelliptic in the class $\mathcal{E}^*$. This proves the sufficiency part of our theorem. $\square$

§2. Equations possessing Gevrey solutions with thin compact singularity

In this section we present some class of equations with constant coefficients for which the continuation of Gevrey solutions as discussed in the preceding section does not hold. Since the preparatory arguments there apply as well, we shall restrict ourselves to consideration of irreducible operators.

First of all, it is well known that hypoelliptic equations have C^∞ solutions with isolated singularity. In our case this corresponds to the following, which makes the necessity part of Theorem 1.7 more concrete:

PROPOSITION 2.1. *Assume that there exist positive constants $q \geq 1$ and δ such that*

$$|\operatorname{Im}\zeta| \geq \delta|\operatorname{Re}\zeta|^{1/q} \quad \text{on} \quad \zeta \in N(P) \cap \{|\zeta| \geq 1/\delta\}. \tag{2.1}$$

Then there exist solutions with isolated non-removable singularity of class $\mathcal{E}^{(s)}$ for $s > q$, and of class $\mathcal{E}^{\{s\}}$ for $s \geq q$.

In fact, a fundamental solution $E(x)$, that is a solution of $P(D)E = \delta$, presents a non-trivial example. The irremovability of the singularity is obvious because if there is such a continuation, say $[E]$, then the difference $v = E - [E]$ would be a distribution supported by the origin satisfying $P(D)v = \delta$, which is impossible. Thus it only remains to see the Gevrey regularity of E outside the origin. This is rather classical (see e.g. Palamodov[P1], Chapter 6, §5).

A proof based on the Grushin representation is as follows: Choose $F(z) \equiv 1$ as a function on $N(P)$. (Actually, this corresponds to the fundamental solution of P via the Grushin transform.) In view of (2.1) it will satisfy, for any $\varepsilon > 0$,

$$|F(z)| = 1 \leq Ce^{-\delta\epsilon|\operatorname{Re}\zeta|^{1/q}+\varepsilon|\operatorname{Im}\zeta|} \quad \text{on} \quad N(P).$$

Thus $F(z) \in \widehat{\mathcal{E}^*(K)}[N(P)]$ for $* = \{s\}$ with $s \geq q$, hence for $* = (s)$ with $s > q$. Thus in view of Proposition 1.6, this $F(\zeta)$ corresponds to a solution in the indicated class with isolated singularity.

Note that $q = 1$ corresponds to the elliptic equation.

A remarkable discovery of Grushin was that there exist equations which are non-hypoelliptic but nevertheless allow C^∞ solutions with isolated singularity. Grushin only gave a few examples. Here we shall try to generalize his idea of construction. The following generalizes Grushin's example $D_1^3 - D_2^2$, which is obviously non-hypoelliptic. We first give a result on thin singularity, which is more general than the case of isolated singularity as will be given later:

THEOREM 2.2. *Let $P(D)$ be an operator of two independent variables with the following form*

$$P(D) = D_1^m - aD_2^k + \cdots, \qquad k < m, \qquad a \neq 0 \tag{2.2}$$

where $\cdots$ denotes lower order terms in the sense of weighted homogeneity. Then for any prescribed $s > m/k$, $P(D)$ admits a solution in $\mathcal{E}^{\{s\}}$ with a compact thin singularity K contained in $x_2 = 0$.

Note that for this operator $(0, 1)$ is a characteristic direction of multiplicity m and irregularity $\mu = m/(m-k)$, hence $\mu/(\mu - 1) = m/k$.

PROOF. Let $\psi(t) \in \mathcal{D}^{\{q\}}(\boldsymbol{R}^1)$, with $q > 1$ which is supposed to be close to 1. We have

$$|\widehat{\psi}(\tau)| \leq Ce^{-A|\tau|^{1/q} + B|\operatorname{Im}\tau|}. \tag{2.3}$$

Set

$$F(\zeta_1, \zeta_2) = \widehat{\psi}(\zeta_1)|_{N(P)}.$$

We shall show that $F(\zeta)$ satisfies

$$|F(\zeta)| \leq C_\varepsilon e^{-A'|\zeta|^{1/s} + B|\operatorname{Im}\zeta|} \tag{2.4}$$

with any fixed $s > m/k$ and with some correspondingly defined $A' > 0$. Remark that on $N(P)$ we have

$$|\zeta_1| \sim |a|^{1/m} |\zeta_2|^{k/m}. \tag{2.5}$$

Hence we simply obtain

$$|F(\zeta)| \le Ce^{-(A/2)|\zeta_1|^{1/q}-(A/3)|a|^{1/qm}|\zeta_2|^{k/qm}+B|\operatorname{Im}\zeta_1|}.$$

This gives a solution in $\mathcal{E}^{\{s\}}$ for $s = qm/k$ and in $\mathcal{E}^{(s)}$ for $s > qm/k$ with singularity $|x_1| \le B$ on the line $x_2 = 0$. Since q can be chosen as close to 1 as we wish, we obtain the assertion. □

Remark that for $s < m/k$ such a solution does not exist because of Theorem 1.4. Notice that the above theorem is not a general nonsence in spite of the simplicity of its proof. Trivial thing is rather that for thin K contained in $a_1x_1 + a_2x_2 = 0$ with $a_1 \ne 0$, we can always show a solution for s in $1 < s < m/k$:

PROPOSITION 2.3. *Let $P(D)$ be an operator as in the preceding theorem. Then for any prescribed $s > 1$, $P(D)$ admits a solution in $\mathcal{E}^{(s)}$ or in $\mathcal{E}^{\{s\}}$ with a compact thin singularity K contained in $x_1 = bx_2$.*

PROOF. Our operator is weakly hyperbolic to the direction dx_1 and posseses a fundamental solution for the Cauchy problem in ultradistributions with the positive x_1-axis as the propagation cone. Hence it supplies the solution of the Cauchy problem for any Cauchy data on $x_1 = bx_2$ supported by K in the class $\mathcal{E}^{(s)}$ with $1 < s < m/k$. (See e.g. Bronshtein [Br1]. The present case may be verified directly via the Fourier analysis.) The solution u has support in the cylinder with base K and the generators parallel to the x_1-axis. If we put $u = 0$ outside this cylinder and also on the side $x_1 < bx_2$, then we obtain a solution $u \in \mathcal{E}^{(s)}(\boldsymbol{R}^n \setminus K)$ which obviously has K as non-removable singularity. □

Next we try to construct solutions with isolated singularity. We have rather a partial result on this. First we prepare the following lemma, which is implicitly contained in Theorems 4.1.1 and 4.1.8 of Boas [Bo1].

LEMMA 2.4. *For any $\delta > 0$ we can find $p_\delta > 1$ such that for any prescribed p with $1 < p \le p_\delta$, we can construct an entire function of one variable $G(\tau)$ which satisfies*

$$|G(\tau)| \le Ce^{-A|\tau|^{1/p}}, \qquad on \quad \operatorname{Re}\tau \ge \delta|\operatorname{Im}\tau|,$$

$$|G(\tau)| \le Ce^{B|\tau|^{1/p}}, \qquad \textit{on} \quad \operatorname{Re}\tau \le \delta|\operatorname{Im}\tau|,$$

with some constants $B > A > 0$.

PROOF. Put

$$G(\tau) = \prod_{n=1}^{\infty}\left(1 - \frac{\tau}{n^p}\right). \tag{2.6}$$

On the region $\operatorname{Re}\tau \le \delta|\operatorname{Im}\tau|$ (actually everywhere), we have

$$\begin{aligned}\prod_{n=1}^{\infty}\left|1 - \frac{\tau}{n^p}\right| \le \prod_{n=1}^{\infty}\left(1 + \frac{|\tau|}{n^p}\right) &\le \exp\left[\sum_{n=1}^{\infty}\log\left(1 + \frac{|\tau|}{n^p}\right)\right] \\ &\le \exp\left[\int_0^{\infty}\log\left(1 + \frac{|\tau|}{s^p}\right)ds\right].\end{aligned}$$

Here we utilized the fact that $\log(1 + |\tau|/s^p)$ is monotone decreasing in s for $0 < s < \infty$. Via change of variable $t = |\tau|/s^p$ we obtain

$$\le \exp\left[\frac{1}{p}|\tau|^{1/p}\int_0^{\infty}\log(1+t)t^{-1-1/p}dt\right] \le e^{B|\tau|^{1/p}},$$

where

$$B = \frac{1}{p}\int_0^{\infty} t^{-1-1/p}\log(1+t)dt.$$

Next, on the region $\operatorname{Re}\tau \ge \delta|\operatorname{Im}\tau|$ we have $|\operatorname{Re}\tau| \ge \delta|\tau|/\sqrt{1+\delta^2}$, hence,

$$\begin{aligned}\left|1 - \frac{\tau}{n^p}\right| = \left(1 - 2\frac{|\operatorname{Re}\tau|}{n^p} + \frac{|\tau|^2}{n^{2p}}\right)^{1/2} &\le \left(1 - 2\frac{\delta|\tau|}{\sqrt{1+\delta^2}n^p} + \frac{|\tau|^2}{n^{2p}}\right)^{1/2} \\ &\le \left(1 - \frac{\delta|\tau|}{2\sqrt{1+\delta^2}n^p}\right),\end{aligned}$$

provided

$$n^p \ge \frac{4+3\delta^2}{4\delta\sqrt{1+\delta^2}}|\tau|.$$

Set $c = 2\sqrt{1+\delta^2}/\delta$. Then if we choose $\lambda \geq c^{-1/p}$, we have $|\tau|/cn^p \leq 1$ for $n \geq \lambda|\tau|^{1/p}$, hence

$$\begin{aligned}\prod_{n\geq\lambda|\tau|^{1/p}}\left(1-\frac{|\tau|}{cn^p}\right) &\leq \exp\left[\sum_{n\geq\lambda|\tau|^{1/p}}\log\left(1-\frac{|\tau|}{cn^p}\right)\right]\\ &\leq \exp\left[-\sum_{n\geq\lambda|\tau|^{1/p}}\frac{|\tau|}{cn^p}\right]\\ &\leq \exp\left[-\int_{\lambda|\tau|^{1/p}}^{\infty}\frac{|\tau|}{cs^p}ds\right] \leq e^{-A|\tau|^{1/p}}\end{aligned}$$

with

$$A = \frac{1}{(p-1)\lambda^{p-1}c} = \frac{\delta}{2(p-1)\lambda^{p-1}\sqrt{1+\delta^2}}.$$

Note that this constant grows to ∞ as $p \to 1$ as long as λ remains bounded away from 0. Thus if we choose λ in such a way that

$$\lambda = \max\left\{\left(\frac{\delta}{2\sqrt{1+\delta^2}}\right)^{1/p},\ \left(\frac{4+3\delta^2}{4\delta\sqrt{1+\delta^2}}\right)^{1/p}\right\},$$

then we have

$$\prod_{n\geq\lambda|\tau|^{1/p}}\left|1-\frac{\tau}{n^p}\right| \leq e^{-A|\tau|^{1/p}}.$$

On the other hand, we have

$$\begin{aligned}\prod_{n\leq\lambda|\tau|^{1/p}}\left|1-\frac{\tau}{n^p}\right| &\leq \prod_{n\leq\lambda|\tau|^{1/p}}\left(1+\frac{|\tau|}{n^p}\right)\\ &\leq \exp\left[\sum_{n\leq\lambda|\tau|^{1/p}}\log\left(1+\frac{|\tau|}{n^p}\right)\right]\\ &\leq (1+|\tau|)\exp\left[\int_1^{\lambda|\tau|^{1/p}}\log\left(1+\frac{|\tau|}{s^p}\right)ds\right]\\ &\leq (1+|\tau|)\exp\left[\frac{1}{p}|\tau|^{1/p}\int_{\lambda^{-p}}^{|\tau|}t^{-1-1/p}\log(1+t)dt\right]\end{aligned}$$

$$\leq (1+|\tau|)e^{b|\tau|^{1/p}},$$

where

$$b = \frac{1}{p}\int_{\lambda^{-p}}^{\infty} t^{-1-1/p}\log(1+t)dt.$$

Note that if we let $p \to 1$, we have finally $A > b$. Hence if p is sufficiently close to 1, we obtain the desired decay estimate with another $A > 0$ in this region. □

REMARK. Via scaling of the variable τ, we can let B smaller than any prescribed positive constant or let A larger than any prescribed positive constant, although in each case the remaining constant A resp. B changes proportionally.

THEOREM 2.5. *Let $P(D)$ be an operator of two independent variables as in Theorem 2.2. Assume further that $a \neq 0$ is real, m is odd, and k is even. Then for any prescribed $s > m/k$, $P(D)$ admits a solution with isolated singularity in $\mathcal{E}^{\{s\}}$ and in $\mathcal{E}^{(s)}$.*

PROOF. Since m is odd, by the change of sign of the x_1-axis if necessary, we can assume without loss of generality that $a > 0$. Take an entire function of one variable $G(\tau)$ as is given by Lemma 2.4 with $p > 1$ close to 1. Then put

$$F(\zeta_1, \zeta_2) = G(\zeta_1).$$

Recall the asymptotic form (2.5). Thus on the region $|\operatorname{Im}\zeta_1| \leq \delta \operatorname{Re}\zeta_1$ (actually on $\operatorname{Re}\zeta_1 \geq \delta|\operatorname{Im}\zeta_1|$) we have

$$|F(\zeta)| \leq Ce^{-A|\zeta_1|^{1/p}} \leq C'e^{-(A/2)|\zeta_1|^{1/p}-(A/3)a^{k/m}|\zeta_2|^{k/pm}}.$$

Next, on the region $|\operatorname{Im}\zeta_1| \geq \delta|\operatorname{Re}\zeta_1|$, we have $|\zeta_1| \leq \sqrt{1+\delta^{-2}}|\operatorname{Im}\zeta_1|$, hence

$$\begin{aligned}|F(\zeta)| &\leq Ce^{B|\zeta_1|^{1/p}} \leq Ce^{-A|\zeta_1|^{1/p}+(A+B)\sqrt{1+\delta^{-2}}|\operatorname{Im}\zeta_1|^{1/p}} \\ &\leq C_\varepsilon e^{-(A/2)|\zeta_1|^{1/p}-(A/3)a^{k/m}|\zeta_2|^{k/pm}+\varepsilon|\operatorname{Im}\zeta_1|}.\end{aligned}$$

Finally, on the region $|\operatorname{Im}\zeta_1| \leq -\delta \operatorname{Re}\zeta_1$, we put

$$\zeta_1 = \rho e^{\pi i + \theta i}, \quad \text{where} \quad |\theta| \leq \operatorname{Arctan}\delta.$$

Then for $\rho \to \infty$ we have

$$\zeta_2 \sim a^{-m/k}\rho^{m/k}e^{(m/k)\theta i+((2\ell+1)/k)\pi i}, \qquad \ell = 0, \dots, k-1.$$

By the assumptions on a, m, k, we see from this that for δ sufficiently small, each branch satisfies

$$\operatorname{Im}\zeta_2 \sim a_\ell \rho^{m/k}, \qquad \text{with } \ a_\ell = a^{-m/k}\sin\left(\frac{m}{k}\theta + \frac{2\ell+1}{k}\pi\right) \neq 0.$$

Thus in this region we obtain, with some $a' > 0$

$$\begin{aligned}
|F(\zeta)| &\le Ce^{B|\zeta_1|^{1/p}} \le Ce^{-A|\zeta_1|^{1/p}+(A+B)|\zeta_1|^{1/p}} \\
&\le Ce^{-(A/2)|\zeta_1|^{1/p}-(A/3)a^{k/m}|\zeta_2|^{k/pm}+(A+B)\rho^{1/p}} \\
&\le C_\varepsilon e^{-(A/2)|\zeta_1|^{1/p}-(A/3)a^{k/m}|\zeta_2|^{k/pm}+\varepsilon\rho^{m/k}} \\
&\le C_\varepsilon e^{-(A/2)|\zeta_1|^{1/p}-(A/3)a^{k/m}|\zeta_2|^{k/pm}+\varepsilon|\operatorname{Im}\zeta_2|}.
\end{aligned}$$

The obtained estimate shows that $F(\zeta)$ is the Grushin transform of a solution with isolated singluarity at the origin in the class $\mathcal{E}^{\{s\}}$ for $s = pm/k$ and in the class $\mathcal{E}^{(s)}$ for $s > pm/k$. □

The above proof is adopted from the original example of Grushin $D_1^3 - D_2^2$ (although he did not discuss the threshold Gevrey index and only presented a C^∞ solution). We hope that the additional assumption which we posed in Theorem 2.5 (in comparison to Theorem 2.2) is only technical. Note, however, that for imaginary a or for m, k even and $a < 0$ our operator becomes hypoelliptic and the existence of solutions with weak isolated singularity becomes trivial. In view of Theorem 1.7 the singularity of such an example of solution is never weak for non-hypoelliptic case.

We also believe that we will be able to construct solutions with thin singularity in $\mathcal{E}^{(s)}$ with $s = m/k$, but our method does not work for this class. (We can improve e.g. Lemma 2.4 to replace $|\tau|^{1/p}$ in the decay/growth condition by $|\tau|/(\log|\tau|)^2$. But this does not improve the result for the threshold value of s. In order to do this by our argument, we need to replace $|\tau|^{1/p}$ by $|\tau|$, which is of course impossible.)

We conclude this section by showing a solution with thin singularity for the Schrödinger equation for general space dimension. A consideration of general equations with more than two variables need the study of cohomology groups with bound on the variety $N(P)$, and is left for the future.

PROPOSITION 2.6. *Consider the operator* $P(D) = D_1^2 + \cdots + D_{n-1}^2 - D_n$ *corresponding to the Schrödinger equation. This admits a solution of class* $\mathcal{E}^*$ *with* $* = \{s\}$ *or* $* = (s)$ *for* $s > 2$ *with compact irremovable singularity contained in* $x_n = 0$.

Note that for this operator $\nu = (0, \ldots, 0, 1)$ is a characteristic direction of multiplicity and irregularity equal to 2, hence the threshold value is $\mu/(\mu - 1) = 2$.

PROOF. Let $q > 1$ be close to 1 and let $\psi(t)$ be a function in $\mathcal{D}^{\{q\}}(\boldsymbol{R}^1)$ which is even. Then its Fourier transform $\widehat{\psi}(\tau)$ is also even and satisfies the estimate (2.3). Set

$$F(\zeta) = \widehat{\psi}\left(\sqrt{\zeta_1^2 + \cdots + \zeta_{n-1}^2}\right).$$

We shall estimate this in various regions. In the sequel we set $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$, and we let $|\zeta'|$ denote the (complex) Euclidean norm of ζ'. We also set $\xi' = \operatorname{Re}\zeta'$, $\eta' = \operatorname{Im}\zeta'$, use similar symbols for their norms, and abbreviate their Euclidean inner product as $\xi'\eta'$.

Fix δ such that $0 < \delta < 1$. First, consider the region $|\eta'| \leq \delta|\xi'|$. Here we have $\xi'^2 - \eta'^2 \geq (1 - \delta^2)|\xi'|^2 \geq 0$. Hence recalling the estimate

$$|\operatorname{Im}\sqrt{a + bi}| = \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{|b|}{\sqrt{\sqrt{a^2 + b^2} + a}} \leq \frac{|b|}{2\sqrt{a}}$$

which is valid for $a \geq 0$, we obtain

$$\left|\operatorname{Im}\sqrt{\zeta_1^2 + \cdots + \zeta_{n-1}^2}\right| \leq \frac{2|\xi'||\eta'|}{2\sqrt{1 - \delta^2}|\xi'|} \leq \frac{1}{\sqrt{1 - \delta^2}}|\eta'|.$$

Thus taking account of the fact

$$|\zeta_n| = |\zeta_1^2 + \cdots + \zeta_{n-1}^2| = \left\{(|\xi'|^2 - |\eta'|^2)^2 + 4(\xi'\eta')^2\right\}^{1/2} \geq \frac{1 - \delta^2}{1 + \delta^2}|\zeta'|^2$$

on $N(P)$, we obtain in this region

$$(2.7)\quad \begin{aligned} |F(\zeta)| & \\ &\le Ce^{-A|\zeta'|^{1/q}+B|\operatorname{Im}\sqrt{\zeta_1^2+\cdots+\zeta_{n-1}^2}|} \\ &\le Ce^{-(A/2)|\zeta'|^{1/q}-(A/2)(1+\delta^2)^{1/2q}(1-\delta^2)^{-1/2q}|\zeta_n|^{1/2q}+(1-\delta^2)^{-1/2}B|\operatorname{Im}\zeta'|}. \end{aligned}$$

Next, on the region $|\eta'| \ge \delta|\xi'|$, we employ $|\zeta_n| \le |\zeta'|^2$ and

$$\operatorname{Im}\sqrt{\zeta_1^2+\cdots+\zeta_{n-1}^2} \le |\zeta'| \le (1+\delta^{-2})^{1/2}|\eta'|.$$

Then we have

$$(2.8)\quad \begin{aligned} |F(\zeta)| &\le Ce^{B|\zeta'|} \\ &\le Ce^{-(A/2)|\zeta'|^{1/q}-(A/2)|\zeta_n|^{1/2q}+A|\zeta'|^{1/q}+B|\zeta'|} \\ &\le Ce^{-(A/2)|\zeta'|^{1/q}-(A/2)|\zeta_n|^{1/2q}+A(1+\delta^{-2})^{1/2q}|\eta'|^{1/q}+B(1+\delta^{-2})^{1/2}|\eta'|}. \end{aligned}$$

Combinig (2.7) and (2.8), we conclude that we can find a universal constant $\lambda > 0$ such that on $N(P)$ we have

$$|F(\zeta)| \le Ce^{-A|\zeta|^{1/2q}+\lambda B|\operatorname{Im}\zeta'|}.$$

This implies that $F(\zeta)$ gives a solution in $\mathcal{E}^{\{2q\}}$ with compact thin singularity contained in $x_n = 0$. Since $q > 1$ is arbitrary, we obtain the desired conclusion. □

It is an interesting problem to know if we can construct similar solution with *isolated singularity.* In the above proof we can let the “horizontal” size B of the singularity as small as we like, but never to 0.

§3. Continuation of Gevrey solutions for equations with real analytic coefficients

Now we consider an operator $P(x, D)$ with real analytic coefficients defined on a neighborhood of the origin of $\boldsymbol{R}^n$. We assume that the thin singularity K is contained in the hyperplane $x_1 = 0$ which is non-characteristic with respect to P.

To state the result we first recall the set of boundary characteristic points for the real analytic solutions from the sides $\pm x_1 > 0$, introudced in [Kn8]:

$$
\begin{aligned}
(3.1)\ V^{\pm}_{S,A}(P)& \\
:= \{(x', \xi');\ & \text{there exists a sequence } (x^{(k)}, \xi^{(k)}) \in \boldsymbol{R}^n \times \boldsymbol{C} \times \boldsymbol{R}^{n-1} \\
& \text{such that } \ P_m(x^{(k)}, \xi^{(k)}) = 0,\ \pm x_1^{(k)} > 0,\ \operatorname{Im} \xi_1^{(k)} > 0 \\
& \text{and } \ (x^{(k)\prime}, \xi^{(k)\prime}) \to (x', \xi')\}.
\end{aligned}
$$

Here the suffix S represents the hypersurface $x_1 = 0$ under consideration. We also set

$$
V_{S,A}(P) := V^{+}_{S,A}(P) \cup V^{-}_{S,A}(P). \tag{3.2}
$$

When we discussed the continuation of real analytic solutions in [Kn8], it was fundamental that the analytic wavefront set of the boundary values of real analytic solutions of $P(x, D)u = 0$ on $\pm x_1 > 0$ is contained in $V^{\pm}_{S,A}(P)$, respectively, hence the analytic wavefront set of their gap in $V_{S,A}(P)$. We can formulate an analogous assertion that the same holds for Gevrey class solutions of suitable index. But the boundary values of Gevrey class solutions need not be contained in the corresponding ultradistributions and in general become hyperfunctions, hence the notion of this wavefront set is not classical. We first give a result which does not utilize the notion of such a wavefront set, and will discuss this problem after that.

Theorem 3.1. *Let $P(x, D)$ be an m-th order linear partial differential operator with real analytic coefficients defined on a neighborhood of the origin. Assume that $x_1 = 0$ is non-characteristic with respect to P, and let K be a compact subset of $x_1 = 0$. Assume further that there exists some direction $\nu' \in \boldsymbol{R}^{n-1}$ such that $K \subset \{\nu' x' = 0\}$ and that $(K' \times \{\pm\nu'\}) \cap V_{S,A}(P) = \emptyset$, where K' denotes the set K considered as the $n-1$ dimensional one. Let μ be the maximum value of the multiplicity of the characteristic roots for the Cauchy problem to the directions $\pm dx_1$. Then every solution of class $\mathcal{E}^*$ for $* = \{s\}$ with $s < \mu/(\mu-1)$ or for $* = (s)$ with $s \leq \mu/(\mu-1)$ defined outside K can be continued as a hyperfunction solution to a neighborhood of K.*

Proof. Let $W(x', \omega')$ denote the component of Kashiwara's twisted Radon decomposition of $\delta(x')$. The assumption implies that our operator is micro-locally semihyperbolic in the sense of [Kn8] to both sides of $x_1 = 0$ near the directions $\pm\nu'$, hence so is ${}^tP$ near $\mp\nu'$. There we have shown that under this condition there exists a complete set of local hyperfunction fundamental solutions $E_k(x, y')$, where $x \in \boldsymbol{R}^n$, $y' \in \boldsymbol{R}^{n-1}$, which gives a micro-local solution near $\pm\nu'$ to the following Cauchy problem with respect to the initial hyperplane $x_1 = 0$:

$$
\begin{aligned}
&{}^tP(x, D)E_k = 0 \qquad \text{(that is, is micro-analytic near the directions } \pm\nu'),\\
&\left(-\frac{\partial}{\partial x_1}\right)^j E_k\big|_{x_1=0} = \delta_{j,m-k-1} W(y' - x', \omega'), \qquad 0 \le j \le m-1.
\end{aligned}
$$

such that $\mathrm{WF}_A E_k(x, y', \omega')$ on $|x_1| < \varepsilon$ is contained in some $c(\varepsilon)$-neighborhood of $\{(0, x', y'; i\xi, i\eta');\ x' = y',\ \xi' = -\eta' = \omega'\}$, where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Namely, if $\{u_k(x')\}_{k=0}^{m-1}$ are hyperfunctions of which the analytic wavefront sets are contained in a small neighborhood of $K' \times \{\pm\nu'\} \subset \boldsymbol{R}^{n-1} \times \boldsymbol{S}^{n-2}$, then

$$
u(x) = \sum_{k=0}^{m-1} \int E_k(x, y') u_k(y') dy' \tag{3.3}
$$

becomes a hyperfunction solution of $P(x, D)u = 0$ with the given Cauchy data. Here we need to show that $E_k(x, y')$ are not only mere hyperfunctions but also ultradistributions dual to $\mathcal{E}^{\{q\}}$ with $q = \mu/(\mu - 1)$. It is rather a hard task to show this via the estimation of $E(z, y')$ when $\operatorname{Im} z$ tends to zero. We can, however, use a result of Kajitani-Wakabayashi [KW1] who showed the solvability of this micro-local Cauchy problem in $\mathcal{E}^{\{q\}'}$ (see [KW1], Theorem 4.11; see also Kajitani [Kj1] of which the discussion is obviously micro-localizable). From their result, we can see, in view of the uniqueness of the solution of the Cauchy problem in hyperfunctions, that if $\{u_k(x')\}$ are in $\mathcal{E}^{\{q\}}$, then (3.3) gives a solution in $\mathcal{E}^{\{q\}}$. Hence applying the kernel theorem (see [Km3], Theorem 2.3), we conclude that the fundamental solutions E_k are in $\mathcal{E}^{\{q\}'}$.

Now let $u(x)$ be a solution in $\mathcal{E}^*$ defined outside K. By the boundary value theory of Komatsu-Kawai-Schapira, we can define the boundary values $u_k^\pm(x')$ of u to $x_1 = 0$ from the sides $\pm x_1 > 0$ by the identity:

$$P(x, D)[u]^\pm = \pm \sum_{k=0}^{m-1} u_k^\pm(x')\delta^{(m-1-k)}(x_1), \tag{3.4}$$

where $[u]^\pm$ denotes the canonical extension of u from the side $\pm x_1 > 0$, respectively, with $\mathrm{supp}[u]^\pm \subset \{\pm x_1 \geq 0\}$. The double sign on the top of the right-hand side is adopted for the coordinate invariance. In view of the uniqueness of the expression of the form (3.4), it is easily seen that if u is a classical solution of class C^m on a neighborhood of $x_1 = 0$, we have $[u]^\pm = u(x)Y(\pm x_1)$, where Y denotes the Heaviside function. Correspondingly, in such a case the coefficients are given by

$$u_k^\pm = B_k(x, D)u\big|_{x_1 \to \pm 0},$$

where $\{B_k(x, D)\}_{k=0}^{m-1}$ denotes the system of boundary trace operators dual to the natural one $\{(\partial/\partial x_1)^k\}_{k=0}^{m-1}$. Thus we see that $[u]^\pm$ agree on $x_1 = 0$ outside the set K, and

$$[u] := [u]^+ + [u]^- = \sum_{k=0}^{m-1} u_k(x')\delta^{(m-1-k)}(x_1),$$

where $u_k(x') := u_k^+(x') - u_k^-(x')$ also satisfy $\mathrm{supp}\, u_k \subset K'$. (Note that $[u]$ is among the extensions of u to K figuring in the Grushin representation, although it is not in $\mathcal{E}^*$. In general, the extension is not unique even if we do not employ the modification in the ε-neighborhood and allow it to be a hyperfunction. But in case when K is contained in a non-characteristic hypersurface, we have this canonical choice.) Choose $\varepsilon > 0$ and a neighborhood U' of K', such that $[u]$ and E_k are defined on a neighborhood of $\{|x_1| \leq \varepsilon\} \times \overline{U'}$. Then for a choice of smaller neighborhood $V' \subset\subset U'$ of K', we can choose a modification $[[u]]$ of u as a hyperfunction, but obtained via cut-off function of x' in our Gevrey class, such that $\mathrm{supp}[[u]] \subset \boldsymbol{R} \times \overline{U'}$, $[[u]] \equiv u$ in $\boldsymbol{R} \times V'$. Thus we can find a Gevrey class function v in our class, with support contained in $\boldsymbol{R} \times (\overline{U'} \setminus V')$, such that

$$P(x, D)[[u]] + v = \sum_{k=0}^{m-1} u_k(x')\delta^{(m-k-1)}(x_1).$$

Now we employ the Green formula

$$
\begin{aligned}
(3.5)\quad &\int u_k(x')W(y'-x',\omega')dx' \\
&= \int \sum_{j=0}^{m-1} B_j(x,D)[[u]]\big|_{x_1=\varepsilon}(-\partial/\partial x_1)^{m-j-1}E_k(x,y',\omega')\big|_{x_1=\varepsilon}dx' \\
&- \int \sum_{j=0}^{m-1} B_j(x,D)[[u]]\big|_{x_1=-\varepsilon}(-\partial/\partial x_1)^{m-j-1}E_k(x,y',\omega')\big|_{x_1=-\varepsilon}dx' \\
&+ \int vE_k(x,y',\omega')Y(\varepsilon^2-x_1^2)dx.
\end{aligned}
$$

This formula can be formally derived via integration by parts from

$$\int P(x,D)[[u]]E_k(x,y',\omega')dx.$$

The only necessary observation is the interpretation of the products appearing in the formal calculus which are not justified by the mere product rule based on the wavefront sets. It was first given in [Kn8] but the proof contained some trivial error and its correction was given in Appendix B of [Kn10]. It was extended to a powerful abstract theorem by Kataoka [Kt1]. But his formulation contains no right-hand side, which is very important for our argument.

From this identity, we can find that for a small neighborhood Δ' of $\pm\nu'$, the term

$$\int_{\Delta'} d\omega' \int u_k(x')W(y'-x',\omega')dx'$$

is micro-locally in $\mathcal{E}^*$ at the origin, as is seen from the right-hand side of (3.5). On the other hand, the term

$$\int_{\boldsymbol{S}^{n-2}\setminus\Delta'} d\omega' \int u_k(x')W(y'-x',\omega')dx'$$

is micro-analytic to the directions $\pm\nu'$. These sum $u_k(x')$ has support in K'. Thus we are led to the situation of Lemma 3.2 below, with $n-1$ variables instead of n, and $\nu'=(0,\dots,0,1)$, and we can conclude that $u_k\equiv 0$. Hence

$[[u]]$ satisfies $P[[u]] = 0$ near K, and $[[u]]$ serves as a continuation of u as a hyperfunction solution to K. $\square$

REMARK. Assume that ${}^tP(x, D)$ possesses a fundamental solution in $\mathcal{D}^{*\prime}$ near K such that $WF_{\mathcal{E}^*}E(x, y) \subset \{\xi dx + \eta dy; \xi \neq 0, \eta \neq 0\}$ (a condition which E usually satisfies if ever it exists). Then we can show that the continued solution is in the same Gevrey class also on K. This follows from the standard argument on the propagation of regularity: Let us write u for $[[u]]$ to simplify the notation. Let $\chi(x) \in \mathcal{D}^*(\boldsymbol{R}^n)$ be such that $\chi \equiv 1$ on a neighborhood of K. Then on this neighborhood we have

$$\begin{aligned} u(x) = \chi(x)u(x) &= \int \delta(x-y)\chi(y)u(y)dy \\ &= \int ({}^tP(y, D)E(x, y))\chi(y)u(y)dy \\ &= \int E(x, y)P(y, D)(\chi(y)u(y))dy. \end{aligned}$$

Here the last term is in $\mathcal{E}^*$ because $P(x, D)(\chi(x)u(x))$ is zero on a neighborhood of K where u may have hyperfunction singularity. In general P may not be locally solvable, hence we cannot expect the Gevrey regularity of the continued solution.

LEMMA 3.2. *Let $f(x)$ be a hyperfunction with support in a compact subset K of $x_n = 0$. Let $g(x)$ be a continuous function defined on a neighborhood of K. If $f + g$ is micro-analytic to the direction $\pm dx_n$ on a neighborhood of K, then $f \equiv 0$.*

PROOF. First consider the case of one variable x_n only. Then the assumption implies that f is supported by the origin, g is continuous and the sum $f + g$ is real analytic at the origin. But this will imply that f is continuous at the origin. Hence $f \equiv 0$.

In the general case we prove this lemma by reducing it to the case of one variable x_n via the definite integration coupled with the real analytic test functions of the variables $x' := (x_1, \dots, x_{n-1})$: Let $\varphi(x')$ be any polynomial. Then

$$v(x_n) := \int_{\boldsymbol{R}^{n-1}} f(x)\varphi(x')dx'$$

is well defined and gives a hyperfunction of one variable supported by one point $x_n = 0$. On the other hand, without loss of generality we can assume that g has support in a small neighbrohood of K, or at least decreases exponentially. Then

$$w(x_n) := \int_{\boldsymbol{R}^{n-1}} g(x)\varphi(x')dx'$$

is obviously continuous. We shall show below that for a suitable choice of g their sum $v + w$ becomes real analytic at $x_n = 0$. Then as is shown above we will obtain $v(x_n) \equiv 0$, hence

$$\langle f(x), \varphi(x')\chi(x_n)\rangle = \int_{\boldsymbol{R}} \chi(x_n)dx_n \int_{\boldsymbol{R}^{n-1}} f(x)\varphi(x')dx' = 0$$

for any polynomial $\chi(x_n)$ of one variable. Since the linear combination of functions of the form $\varphi(x')\chi(x_n)$ is dense in $\mathcal{A}(K)$, we can then conclude that $f(x) \equiv 0$.

Notice that $f + g$ is micro-analytic to $\pm dx_n$ only on a smaller neighborhood, say U of K. (We cannot cut the support of g preserving this condition!) Thus $v + w$ need not be real analytic at $x_n = 0$ in general, and we have to modify g in a suitable way. This argument is routine:

Choose a sufficiently small neighborhood $\varDelta$ of $\pm dx_n$ in $\boldsymbol{S}^{n-1}$. Let $\underset{\sim}{W}(x, \omega)$ denote the exponentially decaying variant of Kashiwara's twisted Radon decomposition introduced in [Kn7], and let

$$\underset{\sim}{W}(x, \varDelta) = \int_{\varDelta} \underset{\sim}{W}(x, \omega)d\omega.$$

Then

$$(f + g) * \underset{\sim}{W}(x, \varDelta)$$

will be real analytic in U because it contains no wavefront there. Hence $h = g * \underset{\sim}{W}(x, \varDelta)$ becomes real analytic in $U \setminus K$. Choose a continuous function $\chi(x')$ with support in the $(n-1)$-dimensional set corresponding to $U \cap \{x_n = 0\}$, such that $\chi(x') \equiv 1$ on a neighborhood of K', and set

$$g_1 = \chi(x')h(x).$$

On the other hand,

$$g_2 := g * \underset{\sim}{W}(x, \boldsymbol{S}^{n-1} \setminus \varDelta)$$

is obviously micro-analytic to the directions $\pm\nu'$ everywhere on $\boldsymbol{R}^n$. Thus if we replace g by $g_1 + g_2$ constructed above, the assumption of our lemma is preserved for this new g. Now it is a simple calculation to show that

$$\int_{\boldsymbol{R}^{n-1}} \{f(x) + g_1(x)\}\varphi(x')dx', \quad \text{and} \quad \int_{\boldsymbol{R}^{n-1}} g_2(x)\varphi(x')dx'$$

become real analytic on a smaller neighborhood of $x_n = 0$. □

Remark. The above lemma may be regarded as a variant of Holmgren type theorem of Kashiwara-Kawai. One may therefore think that the assumption of the above lemma could be weakened to the one "$f + g$ is micro-analytic in *either* of the directions $\pm dx_n$" instead of *both*. But this is false even in the case of one variable. In fact, $f(x) = [ze^{i/z}]$ is a hyperfunction supported at the origin (which is actually in ultradistribution $\mathcal{E}^{(2)'}$). Since the boundary value from the lower half plane is a continuous function $g(x) = xe^{i/x}$, the difference $f - g$ becomes micro-analytic to $-dx_n$. This situation cannot be improved even if we require $g(x)$ to be in some $\mathcal{E}^*$ for $* = \{s\}$ or $* = (s)$ with $s > 1$ (except for the case $s = 1$ for which the assertion becomes trivial). In fact, for any prescribed $q > 1$, there exists an entire function $G(\tau)$ which satisfies

$$|G(\tau)| \le Ce^{-|\tau|^{1/q}} \qquad \text{on} \quad \operatorname{Im}\tau \ge 0.$$

To obtain such a function, we can employ e.g. Theorem 3 of Arakeljan [A1] which shows the existence of an entire function $\varPhi(\tau)$ satisfying $|\varPhi(\tau)| \le Ce^{-|\tau|^{1/2q}}$ outside a narrow region surrounding the positive real axis, say $\{z = x + iy;\ x \ge 1,\ |y| \le 1/x\}$. Then $G(\tau) := \varPhi((\tau + i)^2)$ will be a desired one. Now $f(x) := [G(1/z)]$ is again a hyperfunction supported by the origin, and the boundary value $g(x)$ of $G(1/z)$ from the lower half plane is now in $\mathcal{E}^q$, as is seen from Theorem 11.5 of [Km2].

Notice that if we assume $f(x)$ to be in $\mathcal{D}^{s'}$ for some $s > 1$, the situation is completely different. In fact, we have the following result corresponding to this situation:

Theorem 3.3. *Let $P(x,D)$ be an m-th order linear partial differential operator with coefficients of class $\mathcal{E}^*$ defined on a neighborhood of the origin. Assume that $x_1=0$ is non-characteristic with respect to P, and let K be a compact subset of $x_1=0$. Assume further that there exists some direction $\nu'\in\boldsymbol{R}^{n-1}$ such that $K\subset\{\nu'x'=0\}$ and that $(K'\times\{\nu'\})\cap V_{S,A}(P)=\emptyset$, where K' denotes the set K considered as the $n-1$ dimensional one. Let μ be the maximum value of the multiplicity of the characteristic roots for the Cauchy problem to the directions $\pm dx_1$. Then every solution of class $\mathcal{E}^*$ for $*=\{s\}$ with $s<\mu/(\mu-1)$ or for $*=(s)$ with $s\le\mu/(\mu-1)$ defined outside K and prolongeable to K as an ultradistribution of class $\mathcal{D}^{*\prime}$ with the same value of $*$, can be continued as an ultradistribution solution of this class to a neighborhood of K.*

This time we can formulate the boundary value theory in the framework of ultradistributions, hence the regularity of the coefficients are required only in the corresponding Gevrey class. That is, if we choose an extension v of $u|_{\pm x_1>0}$ with $\operatorname{supp} v\subset\{\pm x_1\ge 0\}$, then $P(x,D)v$ becomes an ultradistribution supported by $x_1=0$. By means of division, that is, by modifying v by elements of the form $P(x,D)w$ with an ultradistribution w supported by $\{x_1=0\}$, we can choose a canonical extension $[u]^\pm$ of u such that a formula like (3.4) holds with ultradistribution coefficients $u_k(x')$ (see Komatsu [Km4]). The remaining calculus is the same. The essential difference hereafter is that in the case of weak singularity, we only need either of the directions $\pm\nu'$ in the assumption. This is due to the following variant of Lemma 3.2:

Lemma 3.4. *Let $f(x)$ be an ultradistribution of class $\mathcal{D}^{*\prime}$ with support in a compact subset K of $x_n=0$. If f is micro-locally in $\mathcal{E}^*$ to the direction dx_n on a neighborhood of K, then $f\equiv 0$.*

If $*=\{s\}$ with $s\ge 2$ or $*=(s)$ with $s>2$, then we have the following stronger assertion: If there exists a distribution defined on a neighborhood of K such that $f+g$ is micro-locally in $\mathcal{E}^*$ to the direction dx_n on a neighborhood of K, then f itself becomes a distribution. Hence in particular, if we can choose such g as only in L^2_{loc}, then we have the same conclusion $f\equiv 0$ as before.

Proof. As in the proof of Lemma 3.2, we only have to consider the case of one variable, and $K=\{0\}$. We first consider the latter assertion.

Let $f(x) = F(x + i0) - F(x - i0)$ be the boundary value representation by a defining function $F(z)$ holomorphic in $\boldsymbol{P}^1 \setminus \{0\}$ and zero at ∞. Let $g(x) = G_+(x + i0) - G_-(x - i0)$ be a boundary value representation such that $G_\pm(z)$ has a distribution limit individually when $\operatorname{Im} z \to 0$. (Actually this is true for *any* representation in the case of one variable.) By the assumption there exists an ultradistribution of the form $H(x - i0)$ such that $f(x) + g(x) - H(x - i0)$ is regular of class $\mathcal{E}^*$. Thus by the generalized Painlevé theorem $F + G_+$ resp. $F + G_- - H$ is individually prolongeable up to the real axis in the class $\mathcal{E}^*$. Hence $F(z)$ is prolongeable in distribution and therefore has tempered growth when $\operatorname{Im} z \to 0$ from the side $\operatorname{Im} z > 0$. Consider the entire function $\Phi(z) := F(i/z)$. This satisfies

$$\begin{aligned} |\Phi(z)| &\le Ce^{h|z|^{1/(s-1)}} && \text{everywhere}, \\ |\Phi(z)| &\le C(1 + |z|)^M && \text{on} \quad \operatorname{Re} z < 0, \end{aligned}$$

where s denotes the (dual) Gevrey index of f. Thus if we assume the restriction for the value of s, then we have $1/(s-1) < 1$ or $1/(s-1) = 1$ but then h may be arbitrary. Thus we can apply the Phragmén-Lindelöf theorem to $\Psi(z) = \Phi(z^q)(1 + z^q)^{-M}$ on $\operatorname{Re} z \ge 0$ to conclude that $\Psi(z)$ is bounded there. Hence $\Phi(z)$ is of $O((1 + |z|)^M)$ everywhere, whence $F(z)$ defines a distribution at the origin.

If further g is in L^2_{loc}, then $F(x + i0)$ must also be in L^2_{loc}. But this is impossible for $F(z)$ which has pole of finite order at 0.

Now we consider the general case. The Fourier transform of $f(x)$ becomes an entire function satisfying

$$|\widehat{f}(\zeta)| \le \begin{cases} Ce^{a|\zeta|^{1/s}} & \text{everywhere}, \\ Ce^{-b|\zeta|^{1/s}} & \text{on the negative imaginary axis}. \end{cases}$$

Here a may be arbitrarily small while b is fixed in case $* = \{s\}$, and a is fixed but b may be arbitrary large in case $* = (s)$. (In both cases the constant C should depend on the arbitrarily chosen constant.) In view of the minimum modulus theorem (Boas [Bo], Theorem 3.2.11), this is impossible unless $\widehat{f}$ is trivial. □

The restriction posed on s for the latter part of Lemma 3.4 is optimal. In fact, the example $F(z) = ze^{i/z}$ given in Remark before Theorem 3.3 provides a counter-example for $\mathcal{D}^{(2)\prime}(\boldsymbol{R})$.

We are tempted to introduce the following definition of micro-local differentiability for general hyperfunctions:

DEFINITION. We shall say that a hyperfunction $f(x)$ is micro-C^∞ at (x_0, ξ_0) if it can be written in the form

$$f(x) = \varphi(x) + g(x),$$

where $\varphi(x)$ is C^∞ at x_0 and $g(x)$ is micro-analytic at (x_0, ξ_0) in the usual sense. $\mathrm{WF}_{C^\infty} f$, the C^∞-wavefront set of f, is just the complement of where f is micro-C^∞.

We shall define the micro-local $\mathcal{E}^*$ regularity and the corresponding notion of wavefront sets $\mathrm{WF}_{\mathcal{E}^*}$ for hyperfunctions just by the same manner, by replacing the regularity of $\varphi(x)$ by that of the corresponding class.

If we review the proof of Theorem 3.1 for separate direction ν', we can easily obtain the following:

COROLLARY 3.5. *Assume that $x_1 = 0$ is non-characteristic with respect to P. Let μ be the maximum value of the multiplicity of the characteristic roots for the Cauchy problem to the direction dx_1 (resp. $-dx_1$). Then for every solution of class $\mathcal{E}^*$ for $* = \{s\}$ with $s < \mu/(\mu-1)$ or for $* = (s)$ with $s \leq \mu/(\mu-1)$ defined on the side $x_1 > 0$ (resp. $x_1 < 0$), the $\mathcal{E}^*$-wavefront set of its boundrary values in the sense of above definition is contained in the set $V^+_{S,A}$ (resp. $V^-_{S,A}$).*

The assertions of Theorems 3.1, 3.3 and Corollary 3.5 may be strengthened by the introduction of micro-localization of s-semihyperbolic operators of [Kj1] or by consideration of lower order terms as is discussed by Komatsu [Km5] for the case of constant multiplicity. Especially, we can obviously take μ to be the irregularity in the sense of [Km5] instead of multiplicity as above, if P has constant multiplicity on a micro-local neighborhood of $(0, \nu')$. (We cannot employ, however, the milder version of irregularity as in the case of constant coefficients, as long as the irreducibility problem for operators with variable coefficients is not well settled.) There seems to exist similar results in the case of variable multiplicity. But we could not find reference giving a definitive result. Since we cannot infer the ultimate form for the moment, it will be adequate to define

$$V^{\pm}_{S,\mathcal{E}^*}(P) := \{(x',\xi');\ \text{there exists a solution of class } \mathcal{E}^* \text{ on } \pm x_1 > 0 \text{ such that } (x',\xi') \text{ is contained in } \mathrm{WF}_{\mathcal{E}^*} \text{ of one of its boundary values to } x_1 = 0\}$$

Then the above Corollary implies that $V^{\pm}_{S,\mathcal{E}^*}(P) = V^{\pm}_{S,A}(P)$ provided the Gevrey index $*$ is related with the multiplicity of the characteristic roots as above.

Concerning the analogy of the Holmgren type theorem, we have the following conjecture much far strengthening the above lemma:

CONJECTURE. *If* $\operatorname{supp} f \subset \{x_n = 0\}$ *and if* f *is micro-*C^∞ *at* $(0, \pm dx_n)$*, then* $f \equiv 0$ *at* 0.

The validity of this conjecture is crucial in improving the results of continuation of C^∞ or Gevrey class solutions. Actually, if this is true, then in the hypothesis of Theorem 3.1 K may not be compact but simply contained in $x_1 = x'\nu' = 0$.

We have still attractive problem of generalizing our discussion to equations with C^∞ or Gevrey coefficients. For solutions with "weak singularities", we can apply a variant of the boundary value theory of Komatsu-Kawai-Schapira adapted to these classes as above. But for general solutions of equations with C^∞ or Gevrey coefficients, we have no available tool for the moment.

Appendix. Proof of solvability for some simultaneous equations

We give here a proof of solvability of simultaneous equations with constant coefficients consisting of one infinite order local operator and one finite order operator employed in the proof of Theorem 1.7. We first discuss the result in the real analytic category which was formerly given as Theorem 4.1 in [Kn4], but the proof of which contained some error as was pointed out by Prof. Komatsu. Here for the sake of simplicity we assume that $P(D)$ is a single operator. A different proof has been given in Supplement of [Kn5] in case $P(D)$ is a general system but $J(D)$ has a special form (function of the n-dimensional Laplacian), which was enough for applications in [Kn4].

Proposition A.1. *Let $K \subset \boldsymbol{R}^n$ be a convex compact set and let $J(D)$ be a local operator, $P(D)$ be a linear partial differential operator with constant coefficients. Assume that the symbols of these operators have no common factor. Then the simultaneous equation*

$$J(D)u = f, \qquad P(D)u = g$$

has a solution $u \in \mathcal{A}(K)$ for $f, g \in \mathcal{A}(K)$ satisfying the compatibility condition $J(D)g = P(D)f$.

Proof. Since $P(D)$ is surjective in the space $\mathcal{A}(K)$ (see e.g. Komatsu [Km1]), we can find $v \in \mathcal{A}(K)$ such that $P(D)v = g$. Then by means of $w := u - v$, the above system is transformed to

$$J(D)w = h := f - J(D)v, \qquad P(D)w = 0.$$

By the duality argument, the surjectivity of $J(D)$ in the space $\mathcal{A}_P(K)$ is translated into the injectivity and the closed range property of the mapping

$${}^tJ(D) : \mathcal{B}[K]/{}^tP(D)\mathcal{B}[K] \to \mathcal{B}[K]/{}^tP(D)\mathcal{B}[K],$$

hence of the mapping

$$J(-\zeta)\cdot : \widehat{\mathcal{B}[K]}/P(-\zeta)\widehat{\mathcal{B}[K]} \to \widehat{\mathcal{B}[K]}/P(-\zeta)\widehat{\mathcal{B}[K]}$$

by means of the Fourier transform. In order to simplify the notation we shall omit the minus sign in the sequel and write P for ${}^tP$ etc. The injectivity of this mapping is not difficult to see: If $\widehat{u} \in \widehat{\mathcal{B}[K]}$ satisfies $J(\zeta)\widehat{u} = P(\zeta)\widehat{v}$ for some $\widehat{v} \in \widehat{\mathcal{B}[K]}$, then putting $P(\zeta) = 0$, we will obtain $\widehat{u} \equiv 0$ on $N(P)$, in view of the assumption of relatively prime property. Thus we can write as $\widehat{u} = P(\zeta)\widehat{w}$ for some $\widehat{w} \in \widehat{\mathcal{B}[K]}$. This implies the injectivity.

To prove the closed range property, recall the Fundamental Principle ([Kn2], Theorem 3.8)

$$\widehat{\mathcal{B}[K]}/P(\zeta)\widehat{\mathcal{B}[K]} \cong \widehat{\mathcal{B}[K]}[N(P), d],$$

where $\widehat{\mathcal{B}[K]}[N(P),d]$ denotes the space of global holomorphic functions on $N(P)$ which are locally in the range of the Noetherian operator d and which satisfy the same growth condition as $\widehat{\mathcal{B}[K]}$. (Noetherian operator is a generalization of restriction describing the local condition of the image of the multiplication operator $P(\zeta)$. d is a simple restriction if P does not contain multiple factors, but if it does, d becomes a vector of transversal derivatives composed with restriction.) Thus the problem is translated into the closed range property of the mapping

$$J(\zeta)\cdot : \widehat{\mathcal{B}[K]}[N(P),d] \to \widehat{\mathcal{B}[K]}[N(P),d]. \tag{A.1}$$

In the proof of Theorem 4.1 in [Kn4], we proved the following minimum modulus theorem for $J(\zeta)$: There exists a constant $H > 0$ such that for any $\varepsilon > 0$ and for any $\zeta \in \boldsymbol{C}^n$, we can find $\tau \in \boldsymbol{C}$ with $||\tau| - |\zeta|| \leq 8\varepsilon|\zeta|$ such that

$$|J(\tau\zeta/|\zeta|)| \geq e^{-H\varepsilon|\zeta|}.$$

Employing this, we proved the closed range property of the multiplication operator $J(\zeta)$ on the space $\widehat{\mathcal{B}[K]}$. It is, however, not enough to show the closed range property of (A.1). Without loss of generality, we can suppose that $P(\zeta)$ is of degree m and has the form

$$P(\zeta) = \zeta_1^m + P_1(\zeta')\zeta_1^{m-1} + \cdots + P_m(\zeta'). \tag{A.2}$$

Then $J(\zeta)$ can be written in the form

$$J(\zeta) = \sum_{k=0}^{m-1} J_{m-k-1}(\zeta')\zeta_1^k + Q(\zeta)P(\zeta), \tag{A.3}$$

where $J_k(\zeta')$ are symbols of local operators of $n-1$ variables. Obviously we can replace J by the first term of (A.3), or equivalently, $Q(\zeta) \equiv 0$ here. Thus we shall assume this from now on. The problem is to estimate this new $J(\zeta)$ from below on $N(P)$. Note that on $N(P)$, ζ_1 agrees with one of the roots $\tau_k(\zeta')$, $k = 1, \dots, m$ of the polynomial (A.2) of ζ_1. Therefore on $N(P)$ the value of J is equal to either of $J(\tau_k(\zeta'), \zeta')$, $k = 1, \dots, m$. Let $\sigma_j(\zeta')$ denote the j-th fundamental symmetric function of $J(\tau_k(\zeta'), \zeta')$, $k = 1, \dots, m$. These are polynomials of the coefficients $J_k(\zeta')$ and $P_k(\zeta')$, hence entire functions of infra-exponential growth as well. The value of J is among the roots of the equation

$$\lambda^m - \sigma_1(\zeta')\lambda^{m-1} + \cdots + (-1)^m \sigma_m(\zeta') = 0. \tag{A.4}$$

We have $\sigma_m(\zeta') \not\equiv 0$ by the assumption of relatively prime property. Thus by applying the above minimum modulus theorem to $\sigma_m(\zeta')$ of $n-1$ variables, we see that there exists $H > 0$ such that for any $\varepsilon > 0$ and for any $\zeta' \in \boldsymbol{C}^{n-1}$, we can find $\tau \in \boldsymbol{C}$ with $||\tau| - |\zeta'|| \leq 8\varepsilon|\zeta'|$ such that

$$|1/\sigma_m(\tau\zeta'/|\zeta'|)| \leq e^{H\varepsilon|\zeta'|}.$$

Since the value of $1/J(\zeta)$ agrees with a root of the equation obtained from (A.4) via the substitution $\lambda \to 1/\lambda$, by a theorem on the estimation of the roots from the coefficients we conclude that

$$J(\tau_k(\tau\zeta'/|\zeta'|), \tau\zeta'/|\zeta'|) \geq e^{-H\varepsilon|\zeta'|}$$

for such τ. Employing this as in the proof of Theorem 4.1 in [Kn4], we can show that for $F(\zeta) \in \widehat{\mathcal{B}[K]}[N(P), d]$,

$$|F(\zeta)| \leq C_\varepsilon e^{(8A+Hd)\varepsilon|\zeta|} \sup_{|z-\zeta| \leq 16\varepsilon|\zeta|} |J(z)F(z)|, \tag{A.5}$$

with some universal constants A, d. This implies the closed range property of (A.1). $\square$

Similar proof gives the following

PROPOSITION A.2. *Let $U \subset \boldsymbol{C}^n$ be a convex domain and let $J(D)$ be a local operator, $P(D)$ be a linear partial differential operator with constant coefficients. Assume that the symbols of these operators have no common factor. Then the simultaneous equation*

$$J(D)W = F, \qquad P(D)W = G \tag{A.6}$$

has a solution $W \in \mathcal{O}(U)$ for $F, G \in \mathcal{O}(U)$ satisfying the compatibility condition $J(D)G = P(D)F$.

This is proved via the duality argument, by showing the injectivity and closed range property of $J(\zeta)\cdot : \widehat{\mathcal{O}'(U)} \to \widehat{\mathcal{O}'(U)}$ just in the same way as above.

Corollary A.3. *Let $\Omega \subset \boldsymbol{R}^n$ be a convex open set and let $J(D)$ be a local operator, $P(D)$ be a linear partial differential operator with constant coefficients. Assume that the symbols of these operators have no common factor. Then the simultaneous equation*

$$J(D)u = f, \qquad P(D)u = g$$

has a solution $u \in \mathcal{B}(\Omega)$ for $f, g \in \mathcal{B}(\Omega)$ satisfying the compatibility condition $J(D)g = P(D)f$.

This solvability follows from the one for holomorphic functions, i.e. Proposition A.2, by applying the latter to the defining functions of the hyperfunctions. Though the argument is standard, we sketch it for completeness: Let U be a convex complex neighborhood of Ω, and set

$$U\#\Omega = U \cap \bigcap_{j=1}^{n} \{\operatorname{Im} z_j \neq 0\}, \qquad U\#_j\Omega = U \cap \bigcap_{k\neq j} \{\operatorname{Im} z_k \neq 0\}.$$

These are disconnected complex open sets with convex components. Then the space of hyperfunctions on Ω is represented as

$$\mathcal{B}(\Omega) = \mathcal{O}(U\#\Omega) / \sum_{j=1}^{n} \mathcal{O}(U\#_j\Omega).$$

Thus let $F, G \in \mathcal{O}(U\#\Omega)$ be representatives of f, g, respectively. Then the compatibility condition implies that there exists $H_j \in \mathcal{O}(U\#_j\Omega)$, $j = 1, \dots, n$ such that

$$J(D)G = P(D)F + \sum_{j=1}^{n} H_j.$$

We can solve $P(D)F_j = H_j$ on the convex components of $U\#_j\Omega$ by the classical existence theorem. Then replacing F by $F + \sum_{j=1}^{n} F_j$, which does not change the represented hyperfunction f, we can assume from the beginning that the compatibility condition $J(D)G = P(D)F$ holds in the level of defining functions. Thus we can find a holomorphic solution $W \in \mathcal{O}(U\#\Omega)$ of (A.6) by applying Proposition A.2 componentwise. Then the hyperfunction u defined by W is a desired solution.

Now we discuss the case of Gevrey category which is really needed here.

Proposition A.4. *Let K be a convex compact set and let $J(D)$ be a local operator in the class $\mathcal{E}^*$, $P(D)$ be a linear partial differential operator with constant coefficients. Assume that the symbols of these operators have no common factor. Then the simultaneous equation*

$$J(D)u = f, \qquad P(D)u = g$$

has a solution $u \in \mathcal{E}^*(K)$ for $f, g \in \mathcal{E}^*(K)$ satisfying the compatibility condition $J(D)g = P(D)f$. The same assertion holds for $\mathcal{D}^{*\prime}(K)$ instead of $\mathcal{E}^*(K)$ everywhere.

The proof is similar. The only difference is the minimum modulus theorem. This time, $J(\zeta)$ satisfies the following growth condition

$$|J(\zeta)| \le C_h e^{h|\zeta|^{1/s}} \quad \text{for } \forall\ h > 0, \qquad \text{if } * = \{s\},$$
$$|J(\zeta)| \le C e^{h|\zeta|^{1/s}} \quad \text{for } \exists\ h > 0, \qquad \text{if } * = (s).$$

In view of Theorem 3.2.11 in [Bo1], for an entire function $F(\tau)$ of one variable of order ρ with $0 < \rho < 1$ there exists a sequence $R_k \to \infty$ such that for $R_k \le r \le R_k + R_k^{1-\rho-\varepsilon}$ we have

$$\log m(r) > (\cos \pi\rho - \varepsilon) \log M(r),$$

where $m(r)$ resp. $M(r)$ denotes the minimum modulus resp. maximum modulus of $F(\tau)$ on $|\tau| = r$. Applying this to $J(\tau\zeta)$ as above, we can show an estimate of the form

$$|F(\zeta)| \le C_\varepsilon e^{Ah|\zeta|^{1/s}} \sup_{|z-\zeta| \le 16\varepsilon|\zeta|} |J(z)F(z)|$$

with a universal constant $A > 0$. This implies the desired closed range property.

For the case of $* = (s)$, we have the global solvability on convex open sets, although we do not need this in this article:

PROPOSITION A.5. *Let Ω be a convex open set and let $J(D)$ be a local operator in the class $\mathcal{E}^{(s)}$, $P(D)$ be a linear partial differential operator with constant coefficients. Assume that the symbols of these operators have no common factor. Then the simultaneous equation*

$$J(D)u = f, \qquad P(D)u = g$$

has a solution $u \in \mathcal{E}^{(s)}(\Omega)$ for $f, g \in \mathcal{E}^{(s)}(\Omega)$ satisfying the compatibility condition $J(D)g = P(D)f$. The same assertion holds for $\mathcal{D}^{(s)'}(\Omega)$ instead of $\mathcal{E}^{(s)}(\Omega)$ everywhere.

The proof is similar, because $\mathcal{E}^{(s)}(\Omega)$ resp. $\mathcal{D}^{(s)'}(\Omega)$ is FS resp. DFS space and the closed range theorem holds.

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(Received April 11, 1996)
(Revised April 2, 1997)

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